Knots and algebras

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Abstract.

Starting from the existence of the 2-variable polynomial P for oriented links we develop the linear skein theory approach to give a geometric realisation of the Hecke algebras. A careful definition of the ring Λ in which P takes values makes it easier than usual to study the representations of pieces of a knot diagram in the Hecke algebras and in specialisations of them such as the group algebras $\mathbf{Z}[S_n]$.

An analogous method is used to construct algebras based on Kauffman's polynomial. Here a similarly careful choice of ring, coupled with the use of the Dubrovnik variant of the polynomial allows a natural specialisation to Brauer's algebras.

Introduction.

The 2-variable polynomial $P(K)$ of an oriented link K was developed from two different starting points. One approach was through the Hecke algebras H_n [**J1,J2,O**], while the other combinatorial approach was through knot diagrams and eventually linear skein theory, [**FYHLMO,PT**]. We use here the linear skein theory approach based on tangles (pieces of a knot diagram) to give a geometric realisation of the Hecke algebras, assuming the existence of $P(K)$. One important feature in the construction as presented here is the definition of the ring Λ in which $P(K)$ takes values, to be a subring of the more usual Laurent polynomial ring. This allows various specialisations of the algebras, which result from specialising Λ, to be readily constructed.

We use similar methods, starting with Kauffman's polynomial in its Dubrovnik form [**K**], again keeping a close watch on the ring used, to construct algebras whose specialisations can be identified with Brauer's algebras [**B**] in a very natural way. Such algebras have been studied in more detail by Birman and Wenzl [**BW**], and it was from them that we got to hear of Brauer's algebras. While their algebras are essentially isomorphic to the ones constructed here, our choice of ring and the use of the Dubrovnik variant of the polynomial allow us to make their conjectured connection with Brauer's algebras very directly both algebraically and geometrically. For this reason, and also because of the form of recently discovered connections of both polynomials with invariants derived from Lie algebras [**T**], we feel that the Dubrovnik version, and the version of P which we use, both have a particularly appropriate choice of variables.

* Appears in 'Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond', ed. E. Martin-Peinador and A. Rodez Usan, University of Zaragoza, (1990), 201-220.

While this method of geometrically constructing such algebras has been known to a number of workers in the area, we feel that it is worth expounding here as it underlines the general strategy for computing invariants by working systematically with tangles.

This paper is a revised version of one originally called 'Knots, skeins and algebras' in which we also described how the direct connections between algebra and geometry had provided the original basis for our work **[MT]** on satellites around mutants.

1. The invariant P.

The polynomial $P(K)$ for an oriented link K is normally regarded as lying in $Z[v^{\pm 1}, z^{\pm 1}]$. We shall take the defining skein relation to be $v^{-1}P(K^+) - vP(K^-) = zP(K^0)$, as in **[M]**, rather than the version with all signs positive.

It can be seen from the construction of P, either from the Hecke algebras **[M]**, or via the skein relation, as in **[LM]** or **[PT]**, that P always lies in a subring of $\mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ isomorphic to the ring $\Lambda = \mathbf{Z}[v^{\pm 1}, z, \delta]/ < v^{-1} - v = z\delta >$. This quotient ring Λ is mapped injectively to $\mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ by the assignment $\delta = (v^{-1} - v)/z$.

Recall that when K is the unlink with n components we have $P(K) = \delta^{n-1}$. The combinatorial definition of P using the skein relation then determines P very explicitly as an element of Λ in the first instance rather than $\mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ directly, underlining the considerable restrictions on the appearance of z^{-1} in P.

We may exploit the fact that P lies in Λ by considering the effect of the homomorphism $e : \Lambda \to \mathbf{Z}[\delta]$ defined by $e(v) = 1, e(z) = 0$ and $e(\delta) = \delta$. From the skein relation we have $e(P(K^+)) = e(P(K^-))$, so that the evaluation $e(P(K)) \in Z[\delta]$ is unchanged under any crossing switches in a diagram of K. Since any K can be altered in this way to the unlink on the same number of components, it follows at once that $e(P(K)) = \delta^{|K|-1}$, where $|K|$ is the number of components of K.

This gives a quick confirmation of a result of Yetter.

Theorem. (Yetter)

$$P(K)(1 + \alpha t, \beta t) \to (-2\alpha/\beta)^{|K|-1} \text{ as } t \to 0.$$

Proof: In $\Lambda \subset \mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$ the element $P - e(P) \to 0$ as $(v, z) \to (1, 0)$, while $\delta = ((1 + \alpha t)^{-1} - (1 + \alpha t))/\beta t \to -2\alpha/\beta$ as $t \to 0$. □

Remark. The homomorphism e, which is not defined on the whole of the ring $\mathbf{Z}[v^{\pm 1}, z^{\pm 1}]$, has the effect of passing from a knot diagram to its projection where under- and over-crossings are not distinguished. In the version of P with positive signs in the skein relation this process requires the use of the complex number i in the specialisation.

2. Tangles.

We shall use the language of tangles, slightly altered from Conway's original descriptions, to denote pieces of knot diagram lying inside a rectangle in the plane and meeting its boundary in a prescribed way.

Definition. An (m,n)-tangle is a piece of knot diagram in a rectangle R in the plane, consisting of arcs and closed curves, so that the end points of the arcs consist of m points at the top of the rectangle and n points at the bottom, in some standard position.

An example of a $(4,2)$-tangle is shown in figure 1.

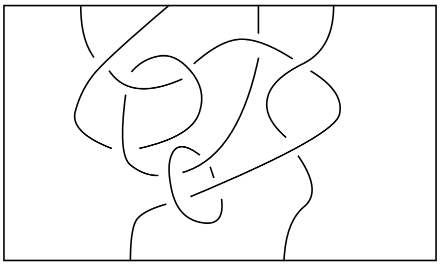


Figure 1

Definition. Two tangles are *ambient isotopic* if they are related by a sequence of Reidemeister's moves I, II and III, (see figure 2), together with isotopies of R fixing its boundary.
They are *regularly isotopic* if Reidemeister move I is not used.

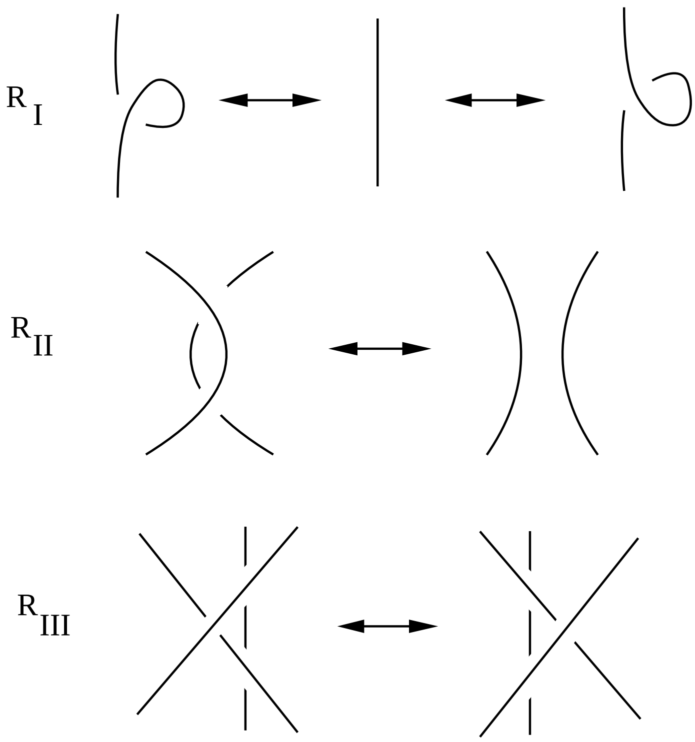


Figure 2

Notation. Write $\mathcal{U}_n^m$ for the set of (m,n)-tangles up to regular isotopy, and $\bar{\mathcal{U}}_n^m$ up to ambient isotopy.

Remark. A tangle may be regarded as a sideways view of part of a knot lying in a cylinder $D^2 \times I$, meeting only its top and bottom. Ambient isotopy of tangles corresponds to ambient isotopy of the knot within $D^2 \times I$.

We shall be concerned here with two classes of tangles.
The first, studied in connection with Kauffman's polynomial, is simply the set $\mathcal{U}_n^n$ of all (n,n)-tangles up to regular isotopy.

In working with P we need to handle *oriented* diagrams. We shall confine our attention to those (n,n)-tangles in which each arc joins a point at the top of R to a point at the bottom. We shall further suppose that the strings have been oriented to run from top to bottom.

Notation. Write $\mathcal{T}_n$ for the set of such oriented tangles up to regular isotopy, and $\overline{\mathcal{T}}_n$ up to ambient isotopy.

Remark. $\mathcal{T}_n$ naturally defines a subset of $\mathcal{U}_n^n$ by ignoring orientation. Two elements of $\mathcal{T}_n$ may yield the same element of $\mathcal{U}_n^n$, since altering the orientation on a closed component of an oriented tangle will in general give a different element of $\mathcal{T}_n$.

Each of the four classes of tangles $\mathcal{U}_n^n$, $\bar{\mathcal{U}}_n^n$, $\mathcal{T}_n$ and $\overline{\mathcal{T}}_n$ admits an *associative multiplication*, defined by placing representative tangles one below the other.

A well-known subset $B_n \subset \overline{\mathcal{T}}_n$ consists of geometric braids - in this context represented by tangles (necessarily without closed components) where the height coordinate in R increases monotonically on each component. It can be shown that B_n is the full group of units in $\overline{\mathcal{T}}_n$ under the multiplication.

The *closure*, $\widehat{T}$, of an (n,n)-tangle T, is defined, by analogy with the closure of a braid, to be the link diagram (or $(0,0)$-tangle) given from T by joining the points on the top of R to those on the bottom by arcs lying outside R with no further crossings.

We shall also write $\wedge(T)$ for the closure $\widehat{T}$, defining a closure map $\wedge : \mathcal{U}_n^n \to \mathcal{U}_0^0$ etc. for each of the classes above.

3. The Dubrovnik invariant.

The invariant P described above gives a map $P : \overline{\mathcal{T}}_0 \to \Lambda$.

Kauffman's polynomial, in its Dubrovnik form, comes from a map $\mathcal{D} : \mathcal{U}_0^0 \to \Lambda'$, for a ring Λ', i.e. a function on diagrams which is unaltered by *regular* isotopy.

This function $\mathcal{D}$ has the basic properties:

(1) $\quad \mathcal{D}(K^+) - \mathcal{D}(K^-) = z(\mathcal{D}(K^0) - \mathcal{D}(K^\infty))$

where the diagrams $K^{\pm}$, K^0 and K^{∞} differ only as in figure 3, and

(2) $\mathcal{D}(K^{\text{left}}) = \lambda\mathcal{D}(K), \quad \mathcal{D}(K^{\text{right}}) = \lambda^{-1}\mathcal{D}(K),$

where K^{left} and K^{right} are given from K by adding a left or right hand curl as in figure 4.

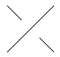
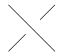
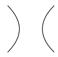
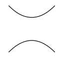
$K^+ =$ , $K^- =$ , $K^0 =$ , $K^{\infty} =$ .

Figure 3

right hand curl = , left hand curl = .

Figure 4

It is usually normalised by

(3) $\mathcal{D}(O) = 1,$

where O is the diagram of the unknot without any crossings.

It also satisfies

(4) $\mathcal{D}(K \amalg O) = \delta\mathcal{D}(K),$

where $K \amalg O$ is the union of K and a circle having no crossings with K or with itself, and $\delta \in \Lambda'$ satisfies $\lambda^{-1} - \lambda = z(\delta - 1)$.

We shall take Λ' to be the ring $\Lambda' = \mathbf{Z}[\lambda^{\pm 1}, z, \delta]/ < \lambda^{-1} - \lambda = z(\delta - 1) >$.

An invariant D of oriented links up to isotopy, also lying in Λ', is readily constructed from $\mathcal{D}$ as follows. Suppose that an oriented diagram K is given. We put $D(K) = \lambda^{w(K)}\mathcal{D}(K)$, where $w(K)$ is the number of crossings in K counted with sign, to get an invariant under Reidemeister moves I, II and III. The invariant does depend on the choice of orientations of the components of K, but only marginally so. A 'neutral' choice could be made by counting only 'pure' crossings in the signed crossing number, that is crossings of each component with itself only, to give an isotopy invariant which is independent of string orientations.

The ring Λ' is isomorphic to a subring of $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$ but as with Λ above it admits a homomorphism $e : \Lambda' \to \mathbf{Z}[\delta]$ with $e(z) = 0$, $e(\lambda) = 1$ and $e(\delta) = \delta$.

Proposition. $e(D(K)) = e(\mathcal{D}(K)) = \delta^{|K|-1}$.

Proof: As in the previous case. □

Corollary. (Wenzl)
$D(K)(1 + \gamma t, \beta t) \to (1 - 2\gamma/\beta)^{|K|-1}$ as $t \to 0$. □

Remark. The invariant D is equivalent to Kauffman's original invariant F. This was proved by Lickorish shortly after Kauffman proposed the invariant D.

Theorem. (Lickorish [**L**])
$D(K)(\lambda, z) = (-1)^{|K|-1} F(K)(i\lambda, iz)$. □

While this theorem effectively removed the need to consider D, it now appears that D is more natural than F in a number of contexts, in that the use of i can frequently be avoided. The regular isotopy invariant $\mathcal{D}$ behaves particularly well [**T,Y**], especially when the alternative normalisation $\mathcal{D}(0) = \delta$ is used, i.e. $\mathcal{D}(\emptyset) = 1$ by use of (4).

Lickorish uses the notations Λ^* for $\mathcal{D}$ and F^* for D, while Turaev uses Q_- for $\mathcal{D}$.

4. Algebras.

From P and $\overline{\mathcal{T}}_n$ we show how to construct an algebra L_n isomorphic to the Hecke algebra H_n. A similar approach, using $\mathcal{D}$ and $\mathcal{U}_n^n$ gives an algebra M_n which we call Kauffman's algebra. It is isomorphic to the algebra produced by Birman and Wenzl [**BW**], but certain features, for example its dimension, and its relation to Brauer's algebra [**B**], which were not directly proved in their original approach, appear here very simply, by use of the homomorphism e and the Dubrovnik invariant $\mathcal{D}$.

The two constructions and their properties follow very similar lines. We treat the case of P in more detail first, but eventually concentrate on the less familiar ideas related to $\mathcal{D}$.

Factor out two types of relation from $\Lambda[\overline{\mathcal{T}}_n]$, the set of all Λ-linear combinations of tangles in $\overline{\mathcal{T}}_n$, to give a Λ-module L_n.

These relations are:

(1) $\quad v^{-1}T^+ - vT^- = zT^0$,

where $T^{\pm}$ and T^0 are represented by tangles differing only as in figure 5.

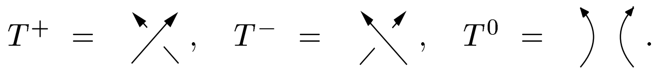


Figure 5

(2) $\quad T \amalg O = \delta T$,

where $T \amalg O$ consists of T and a disjointly embedded unknotted component.

Proposition 4.1. The map $P \circ \wedge : \overline{\mathcal{T}}_n \to \Lambda$ calculating the polynomial of the closure of a tangle induces a Λ-linear map $L_n \to \Lambda$.

Proof: Immediate from (1) and (2). □

Proposition 4.2. Composition of tangles induces a Λ-bilinear multiplication on L_n making L_n an algebra over Λ.

Proof: Check that the relations carry down under the multiplication in $\Lambda[\overline{\mathcal{T}}_n]$. $\square$

We now show how to find a free basis for L_n with $n!$ elements corresponding to permutations in S_n. The proof that the elements selected form a generating set for L_n is an easy consequence of the relations (1) and (2), using the methods of Lickorish and Millett. To prove independence we need to use the existence of P; we use the homomorphism e to finish the proof very quickly.

We develop the notation and techniques to handle this case and the other algebra M_n at the same time.

Definition. Given a tangle T, choose a sequence of base-points, consisting firstly of one end point of each arc, and then one point on each closed component. Say that T is *totally descending* (with this choice of base points) if on traversing all the strands of T, starting from the base point of each component in order, each crossing is first met as an overcrossing.

We shall make a convention about the order of base-points for the arcs of an (n, n)-tangle as follows. Order the $2n$ end points of the arcs, starting with the bottom left point on the boundary of the rectangle, and reading anticlockwise round the boundary. Assign base-points successively in this order, skipping any end point whose arc has already been numbered.

An example of a totally descending $(3, 3)$-tangle is shown in figure 6, with base-points numbered according to this convention.

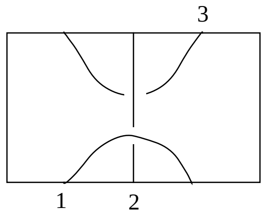


Figure 6

Remark. With this convention the base-points for the tangles used in $\overline{\mathcal{T}}_n$ will start with the n 'inputs' on the bottom taken in order.

Theorem 1. L_n is linearly generated by totally descending tangles.

Proof: Use the techniques of Lickorish and Millett [**LM**]. Let T be a tangle representing an element of L_n. Choose base points for T as above. Use relation (1) at the first non-descending crossing of T to write T as a linear combination of two tangles, one with fewer crossings, the other with fewer non-descending crossings. The theorem follows by induction, firstly on the number of crossings, then on the number of non-descending crossings. $\square$

Corollary. L_n is linearly generated by totally descending tangles without closed components.

Proof: If T is totally descending, with r closed components, then these components are unknotted curves stacked below the arcs of T, so that by (2), $T = \delta^r T'$ in L_n, where T' consists simply of the arcs of T. $\square$

5. Permutations and connectors. To each tangle T in $\mathcal{T}_n$ we can associate a permutation $\mathrm{perm}(T) \in S_n$ by comparing the bottom and top points of the n arcs.

For a general tangle we extend the idea of a permutation to that of an $n-$ *connector*, defined to be a pairing of $2n$ points into n pairs. The set C_n of n-connectors has $(2n)!/2^n n!$ elements, the product of the first n odd integers.

Take the set of $2n$ points to be the end points of (n,n)-tangles. The arcs of any $T \in \mathcal{U}_n^n$ pair these end points to give a connector, which we write as $\mathrm{conn}(T) \in C_n$.

Brauer's algebra. Brauer [**B**] uses C_n as the basis for an algebra over $\mathbf{Z}[\delta]$, (writing n in place of δ and f in place of n). He divides the $2n$ points to be connected into two subsets $t_1, \ldots, t_n$ and $b_1, \ldots, b_n$, arranged along the top and bottom of a rectangle, and views a connector c as a set of n intervals with these $2n$ points as endpoints, which join the points paired by c. Two connectors c_1 and c_2 are composed by placing one rectangle above the other, giving n arcs whose endpoints are the new top and bottom points, together with some number $r \geq 0$ of closed curves.
Brauer sets $c_1 c_2 = \delta^r d$, where d is the connector defined by the new arcs. This defines an associative multiplication on $\mathbf{Z}[\delta][C_n] = A_n$ making it an algebra over $\mathbf{Z}[\delta]$, called *Brauer's algebra.*

Having divided the $2n$ points in this way there is a natural embedding $S_n \subset C_n$.

Theorem 2. Let S and T be totally descending (n,n)-tangles, without closed components, such that $\mathrm{conn}(S) = \mathrm{conn}(T)$. Then S and T are ambient isotopic.

Proof: Number the arcs of S and T according to the order of their base points. Since $\mathrm{conn}(S) = \mathrm{conn}(T)$, the ith arc in each tangle joins the same pair of end points. The arcs can be arranged to lie in disjoint levels 1 to n above the plane of R, since arc i lies above arc j at every crossing when $i < j$. Each individual arc is unknotted, because the tangle is descending, so it can be isotoped to an arc without self-crossings in its level. The resulting tangles are then isotopic by level-preserving isotopy. $\square$

Remark. If the arcs of S and T have no self-crossings initially then S and T are regularly isotopic.

Construction. For each connector $c \in C_n$, construct a totally descending tangle with connector c such that any two arcs cross at most once. (Start from such a diagram of the connector, and make it descending, by choosing the sense of each crossing.) The element $T_c \in \mathcal{U}_n^n$ represented by this tangle then depends only on c by Theorem 2.

Remark. For $c \in S_n$ the resulting tangles T_c have been studied, [**E**], under the name 'positive permutation braids'. They can be represented by a braid in B_n with positive crossings and permutation c in which any two strings cross at most once.

These braids have also been used in [**MS1,2**], to give easily handled generators for the Hecke algebra H_n.

6. The Kauffman algebra.

Definition. The *Kauffman algebra*, M_n, is constructed from $\Lambda'[\mathcal{U}_n^n]$ by factoring out three sets of relations:

(1) $\quad T^+ - T^- = z(T^0 - T^\infty)$

(2) $\quad T^{\text{right}} = \lambda^{-1}T,\ T^{\text{left}} = \lambda T$

(3) $\quad T \amalg O = \delta T,$

where $T^\pm$, T^0, T^∞ and $T \amalg O$ are related as for link diagrams.

As in the case of L_n, we have a Λ'-linear map $M_n \to \Lambda'$ induced by $\mathcal{D} \circ \wedge : \mathcal{U}_n^n \to \Lambda'$ and a bilinear multiplication induced from tangle product making M_n an algebra over Λ'.

Theorem 3. M_n is linearly generated by the finite set $\{T_c\}$, $c \in C_n$.

Proof: A direct analogue of theorem 1 and its corollary shows that M_n is generated by tangles which, using theorem 2, are ambient isotopic to T_c, for various c. By use of relation (2), any tangle ambient isotopic to T_c represents $\lambda^k T_c$ in M_n, for some k. $\square$

Theorem $3'$**.** L_n is linearly generated by $\{T_c\}$, $c \in S_n$.

Proof: Immediate. $\square$

Theorem 4. The set $\{T_c\}$, $c \in C_n$ forms a free Λ'-basis for M_n.

Proof: Write $b : M_n \times M_n \to \Lambda'$ for the bilinear form defined on tangles by $b(S,T) = \mathcal{D}(\wedge(ST))$. Represent b in terms of the generating set $\{T_c\}$ by a $|C_n| \times |C_n|$ matrix A, with entries $a_{cd} = b(T_c, T_d)$. We show that b is non-degenerate, and in addition, since Λ' has no zero-divisors, that $\{T_c\}$ is a free basis of M_n, by proving that $\det A \neq 0$.

The link $\wedge(T_c T_d)$ has r components, say. Each component contains at least one arc from each of T_c and T_d, so $r \leq n$. When $r = n$ each component has exactly one arc from each, so that the connector d is the 'mirror image' of c, given by interchanging the roles of the top and bottom points. Set $\bar{c} = d$ in this case, so that we have $r = n$ if and only if $d = \bar{c}$.

Now apply the homomorphism e to the entries in A. Then $e(a_{cd}) = \delta^{r-1}$, $r \leq n$, and $r = n$ if and only if $d = \bar{c}$. The matrix $e(A)$ has then one entry δ^{n-1} in each row and column, so $e(\det A) = \det(e(A)) \in \mathbf{Z}[\delta]$ has a non-zero coefficient for $\delta^{n(n-1)}$. Thus $e(\det A) \neq 0$, so $\det A \neq 0$. $\square$

Theorem $4'$**.** The set $\{T_c\}$, $c \in S_n$ forms a free Λ-basis for L_n.

Proof: The Λ-bilinear form on L_n constructed analogously using P is non-degenerate, by a similar proof using $e : \Lambda \to \mathbf{Z}[\delta]$. $\square$

Remark. The actual entries in the matrix A can be quite complicated. Use of e makes an enormous simplification.

We now look at M_n and L_n as algebras, and compare them with Brauer's algebra A_n or with $\mathbf{Z}[S_n]$ respectively.

We can modify the map $\mathrm{conn} : \mathcal{U}_n^n \to C_n$ to give a multiplicative homomorphism $c : \mathcal{U}_n^n \to A_n$, which extends to $c : M_n \to A_n$ as follows. For $T \in \mathcal{U}_n^n$ set $c(T) = \delta^{|T|}\mathrm{conn}(T) \in A_n$, where $|T|$ is the number of closed components of T. Now extend to $c : \Lambda'[\mathcal{U}_n^n] \to A_n$ by setting $c(\sum \lambda_i T_i) = \sum e(\lambda_i) c(T_i)$.

Theorem 5. There is an induced homomorphism $c : M_n \to A_n$.

Proof: The relations (1)-(3) defining M_n are respected. $\square$

Remark. In fact A_n is exactly the algebra $M_n \underset{\Lambda'}{\otimes} \mathbf{Z}[\delta]$ given from M_n by replacing the coefficients Λ' with $\mathbf{Z}[\delta]$, using the homomorphism e.

Theorem 6. There is an isomorphism of $\mathbf{Z}[\delta]$-algebras induced by c between $M_n \underset{\Lambda'}{\otimes} \mathbf{Z}[\delta]$ and A_n.

Proof: The map $c : M_n \to A_n$ factors through a $\mathbf{Z}[\delta]$-homomorphism $M_n \underset{\Lambda'}{\otimes} \mathbf{Z}[\delta] \to A_n$. Since $M_n \underset{\Lambda'}{\otimes} \mathbf{Z}[\delta]$ is generated over $\mathbf{Z}[\delta]$ by $\{T_c\}$ which maps onto a *basis* of A_n of the same cardinality, this set must be a $\mathbf{Z}[\delta]$-basis in the specialisation, and the map is hence an isomorphism. $\square$

Remark. The existence of $c : M_n \to A_n$ can be viewed as the consequence of specialising the coefficients so that the relations no longer distinguish under- from over-crossings. Then tangles pass to their projections, retaining only the information of their connectors. The crucial technical feature here is that we can specialise Λ' so as to retain δ, while fixing λ and z. Complications arise if we simply work with $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$.

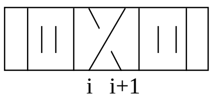
The algebra L_n admits a similar map $c : L_n \to \mathbf{Z}[\delta][S_n]$ taking tangles to their permutations, which similarly gives an isomorphism of $L_n \underset{\Lambda}{\otimes} \mathbf{Z}[\delta]$ with $Z[\delta][S_n]$, and further, putting $\delta = 1$, and isomorphism of $L_n \underset{\Lambda}{\otimes} \mathbf{Z}$ with $\mathbf{Z}[S_n]$.

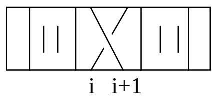
7. Algebra generators.

As an algebra, L_n can be generated by the elementary braids $\sigma_i \in \overline{\mathcal{T}}_n$, since each T_c, $c \in S_n$ is a composite of these. It is immediate that the generators σ_i satisfy the braid relations $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. In addition, the relation $v^{-1}\sigma_i - v\sigma_i^{-1} = z$ comes from applying the relation (1) in L_n to the only crossing in $\sigma_i = T^+$, when we have $T^- = \sigma_i^{-1}$ and $T^0 =$ identity tangle.

The algebra defined abstractly by these generators and relations is known as a Hecke algebra, H_n, so that L_n is automatically a quotient of H_n. It is not difficult to establish that H_n has a set of $n!$ linear generators which map to $\{T_c\}$; see [**M**] for a description of these generators in terms of $c_i = v^{-1}\sigma_i$. The independence of $\{T_c\}$ already established in L_n then ensures that the quotient map is an isomorphism, without having to show directly that these generators of H_n are independent. Thus L_n may be regarded as a concrete realisation of H_n. From this point of view the trace funtion on the algebra H_n used in [**M**] can be recovered from the evaluation map $L_n \to \Lambda$ by taking $Tr(S) = P(\wedge(S))/\delta^{n-1}$.

In the case of the Kauffman algebras we can similarly find generators and relations, as in [**BW**], although this task is simplified by knowing a linear basis already. The tangles s_i, s_i^{-1} and h_i shown in figure 7 satisfy the relations $s_i - s_i^{-1} = z(1 - h_i)$ in M_n.

$s_i \ =$ 

, $s_i^{-1} \ =$ 

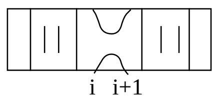
, $h_i \ =$ 

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Figure 7

Since the basis elements $\{T_c\}$ of M_n can be written as monomials in $s_i^{\pm 1}$ and h_i then M_n can be generated by s_i and h_i as an algebra over Λ'. The elements s_i satisfy the braid relations while the elements h_i satisfy Kauffman's bracket relations, $h_i^2 = \delta h_i$, $h_ih_j = h_jh_i$, $|i-j| > 1$, and $h_ih_{i+1}h_i = h_i$. They also satisfy further relations between h_i and s_j, in particular $s_ih_i = \lambda h_i$. Inclusion of enough of these to allow all products T_ds_i and T_dh_i to be written as Λ'-combinations of $\{T_c\}$ will then give an explicit presentation of M_n as an algebra.

The braid group B_n appears to play a slightly ambivalent role in M_n. It can clearly be represented in M_n by taking σ_i to λs_i. Its image does not linearly generate the whole of M_n, since on composition with $c : M_n \to A_n$ the image only contains combinations of permutations $S_n \subset C_n$. However it *is* possible to write all tangles in M_n as linear combinations of braids if the coefficient ring is extended to include z^{-1}. (In this setting the algebra can be generated linearly by $s_i^{\pm 1}$ by solving the relations for h_i.) Birman and Wenzl make use of this representation of B_n in describing the algebra over the ring $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$.

Remark. This representation entails a cubic polynomial relation for s_i, since $s_i^2-1 = z(s_i - s_i h_i) = z(s_i - \lambda h_i) = zs_i + \lambda(s_i - s_i^{-1} - z)$. Then s_i satisfies the polynomial equation $s^3-(z+\lambda)s^2+(\lambda z-1)s+\lambda = 0$, with roots λ, q^{-1} and $-q$, where $z = q^{-1}-q$. While it is now possible to construct a linear basis of M_n over $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$ consisting of braids, it does not seem to be straightforward to find any particularly natural choice of $|C_n|$ braids which will work.

For calculations M_n can be used to find $\mathcal{D}$, and hence Kauffman's polynomial, for a link presented as the closure of an (n,n)-tangle. The tangle must be written as a linear combination of generators during the course of the calculation; this will be helped if it can be presented as a monomial in $s_i^{\pm 1}$ or $s_i^{\pm 1}$ and h_i. A recursive calculation, knowing only the products of the linear generators with $s_i^{\pm 1}$ or with s_i and h_i will then be enough to allow quick mechanical computation. The tangles T_c lend themselves to reasonably simple expressions when composed with s_i or h_i, although not so simple as for the positive permutation braids used in L_n for calculating P, [**MS2**].

8. Related ideas.

We close with some remarks about a very satisfying relation between tangles and algebras. In his beautiful work on the bracket polynomial, a near equivalent of the original Jones polynomial, Kauffman describes an algebra in terms of diagrams. This is the subalgebra of M_n generated (as an algebra) by the elements h_i, or equally, generated linearly by those T_c without any crossings (of which there are $\binom{2n}{n}/(n+1)$, the nth Catalan number). This algebra is essentially isomorphic to its image in A_n, and to the original algebra used by Jones. It depends on n and δ, and it gave rise to an early connection with ideas from theoretical physics, where it has been known as a Temperley-Lieb algebra.

The geometrical view described here can be used for example in finding natural ideals of the algebras.

Definition. The *rank* of an (m,n)-tangle T is the least k for which T is the product of an (m,k) and a (k,n) tangle. (Cf. rank of $m \times n$ matrices.)

Clearly, $\operatorname{rank}(ST) \leq \operatorname{rank}(S),\ \operatorname{rank}(T)$.

Corollary. The submodule $M_n^{(k)}$ of M_n generated by tangles of rank $\leq k$ is an ideal. $\square$

This construction, applied to the Temperley-Lieb algebras, for example, gives all their ideals, except when the ring is specialised with certain choices of δ. See also [**HW**], where such ideas are used for A_n. It underlines the benefits when working with an algebra of having some geometrical model available.

Acknowledgments. The second author was supported during this work by SERC grant GR/C/48974. The first author acknowledges the hospitality of the University of Zaragoza during the revision of the manuscript, and support from the Spanish Ministry of Education and Science.

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Version 2.2 August 1988, references updated March 1991, pictures redrawn August 1999.

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