The Multivariable Alexander Polynomial for a Closed Braid

H. R. Morton

Abstract. A simple multivariable version of the reduced Burau matrix is constructed for any braid. It is shown how the multivariable Alexander polynomial for the closure of the braid can be found directly from this matrix.

1. Introduction

It has been known for some time that the Alexander polynomial of a closed braid $\hat{\beta}$ can be found from the Burau matrix β, [**1**]. This relation was extended in [**3**] to present the 2-variable Alexander polynomial $\Delta_{\hat{\beta}\cup A}(t,x)$ of the link consisting of the closed braid $\hat{\beta}$ together with its axis A as the characteristic polynomial, $\det(I - x\overline{B}_\beta(t))$, of the reduced $(n-1)\times(n-1)$ Burau matrix $\overline{B}_\beta(t)$ of the braid β. The Alexander polynomial of $\hat{\beta}$ can be recovered by applying the Torres-Fox formula to the 2-variable polynomial to get the equation

$$\frac{\Delta_{\hat{\beta}}(t)}{1-t} = \frac{\Delta_{\hat{\beta}\cup A}(t,1)}{1-t^n}.$$

In this paper I give a similar method for finding the multivariable Alexander polynomial of a link L presented as the closure of a braid β. The main ingredient is a readily constructed multivariable version of the reduced Burau matrices. Other versions of 'coloured' Burau matrices have been developed, for example by Penne, [**4**], which can be interpreted as determining linear presentations of suitably extended versions of the braid group.

The most useful feature of the matrices which are used here is that they are extremely simple to remember and they give an immediate and very straightforward construction of the Alexander polynomial of a closed braid and axis, leading at once to the polynomial of the closed braid. For a pure braid the resulting matrix is conjugate to a reduced version of the Gassner matrix; the construction given here has the advantage that it applies to any braid which presents the link, and does not require the braid to be rewritten in any special form.

An implementation of this calculation by a Maple procedure, which returns the multivariable Alexander polynomial of $\hat{\beta}$ given the braid β, was made in early

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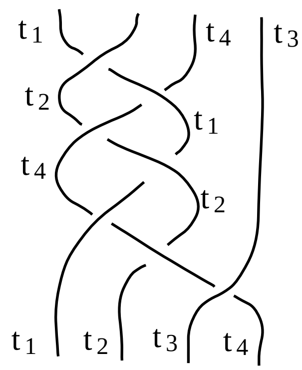

FIGURE 1. The labelled braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3$.

1996 by a Liverpool MSc student, Julian Hodgson, and is usually available from the Liverpool knot-theory website,`\http:\\www.liv.ac.uk\PureMaths\knots.html`.

2. The multivariable Alexander polynomial

Given a homomorphism $\varphi : G_L \to H$ from the group G_L of an oriented link L to an abelian group H we can use Fox's free differential calculus on a presentation of G_L to find an invariant of L which lies in the group ring $\mathbf{Z}[H]$. The full multivariable Alexander polynomial Δ_L arises when H is the abelianisation G_L/G'_L and φ is the natural projection. In this case H is the free abelian group on k generators, $t_1, \ldots, t_k$ say, where L has k components $L_1, \ldots, L_k$, and $\varphi(x_i) = t_i$ for every oriented meridian x_i of the component L_i. Then $\Delta_L \in \mathbf{Z}[G_L/G'_L]$ is a Laurent polynomial in $t_1, \ldots, t_k$. Any other homomorphism $\varphi : G_L \to H$ factors through a homomorphism $\overline{\varphi} : G_L/G'_L \to H$ and the resulting invariant is given by substituting the images of $\overline{\varphi}(t_i)$ in Δ_L, [**2**].

2.1. A coloured Burau matrix for β. Label the individual strings of $\beta \in B_n$ by $t_1, \ldots, t_n$, putting the label t_j on the string which starts from the point j at the bottom. Figure 1 shows this labelling for the braid

$$\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3 \in B_4.$$

Write $\overline{C}_i(a)$ for the $(n-1) \times (n-1)$ matrix which differs from the unit matrix only in the three places shown on row i, for $1 \le i \le n-1$.

$$\overline{C}_i(a) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & a & -a & 1 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

When $i = 1$ or $i = n-1$ the matrix is truncated appropriately to give two non-zero entries in row i.

Now construct the *coloured reduced Burau matrix* $\overline{B}_\beta(t_1,\ldots,t_n)$ of the general braid

$$\beta = \prod_{r=1}^{l} \sigma_{i_r}^{\varepsilon_r}$$

as a product of matrices $\overline{C}_i(a)$, in which a is the label of the current undercrossing string. This gives

$$\overline{B}_\beta(t_1,\ldots,t_n) = \prod_{r=1}^{l} (\overline{C}_{i_r}(a_r))^{\varepsilon_r},$$

where a_r is the label of the undercrossing string at crossing r, counted from the top of the braid.

In the example shown, where $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3$, the labels $a_1,\ldots,a_7$ are t_1, t_4, t_2, t_1, t_4, t_2, t_4 respectively and $\overline{B}_\beta$ is the 3×3 matrix product

$$\overline{C}_1(t_1)\overline{C}_2(t_4)^{-1}\overline{C}_1(t_2)\overline{C}_2(t_1)^{-1}\overline{C}_1(t_4)\overline{C}_2(t_2)^{-1}\overline{C}_3(t_4).$$

Each braid β determines a permutation $\pi \in S_n$ by the representation of B_n on S_n in which a string connects position j at the bottom to position $\pi(j)$ at the top. In the example above $\pi(1)=1, \pi(2)=2, \pi(3)=4, \pi(4)=3$.

THEOREM 1. *The multivariable Alexander polynomial $\Delta_{\hat\beta\cup A}$, where A is the axis of the closed n-braid $\hat\beta$, is given by the characteristic polynomial* $\det(I - x\overline{B}_\beta(t_1,\ldots,t_n))$ *with the identifications $t_{\pi(j)} = t_j$.*

Remarks: (1) Suppose that a link L is presented as the closure of a braid β on n strings and that some homomorphism $\varphi : G_L \to H$ is given. Look at the part of the diagram of L which consists of β. The oriented meridians $x_1,\ldots,x_n$ for the strings at the bottom of β determine elements $\varphi(x_j)\in H$. At each point further up the braid the meridian of a string which starts at the bottom as string j will be mapped to the same element in H. Furthermore, when the braid is closed to form L the strings j and $\pi(j)$ are identified, so that $\varphi(x_j) = \varphi(x_{\pi(j)})$.

When π is the product of k disjoint cycles then the link $L=\hat\beta$ has k components. The variables in the resulting polynomial are x and a k-element subset of $t_1,\ldots,t_n$ after the identifications have been made. In the case $k=1$ the substitution $t_1 = \cdots = t_n = t$ in the matrix $\overline{B}_\beta$ gives the standard reduced Burau matrix $\overline{B}(t)$ for β.

(2) The discussion of the Alexander polynomials of a link with k components can be done most uniformly in terms of the *Alexander invariant* D_L of the link, defined by

$$D_L = \begin{cases} \Delta_L & \text{for } k>1, \\ \dfrac{\Delta_L(t)}{1-t} & \text{for } k=1. \end{cases}$$

The Torres-Fox formula gives the Alexander invariant of a sublink L of a link $L\cup C$ in terms of the invariant for $L\cup C$. It says that

$$D_L(\mathbf{t}) = \frac{\Delta_{L\cup C}(\mathbf{t},1)}{1-\varphi(c)},$$

where the meridian of the curve C to be suppressed is replaced by 1 and $\varphi(c)$ is the element represented by the curve C in the complement of L, abelianised appropriately. Thus we can calculate the Alexander invariant of $L=\hat\beta$ from theorem 1

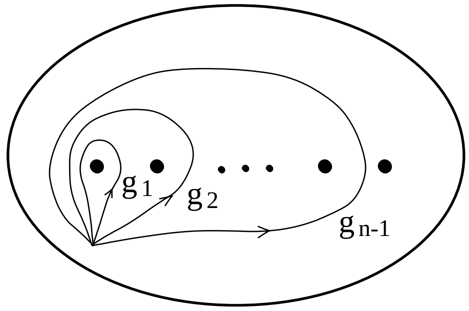

FIGURE 2. The generators in the punctured disk.

by suppressing the axis and applying the Torres-Fox formula. Put $x = 1$ and note that $\varphi(A) = t_1 t_2 \cdots t_n$ in the complement of L, to get

$$D_L = \frac{\det(I - \overline{B}_\beta(t_1, \ldots, t_n))}{1 - t_1 t_2 \cdots t_n},$$

making any identifications $t_{\pi(i)} = t_i$ required where strings of β belong to the same component of L. In the case when L has one component this gives the well-known formula $\Delta_L(t) = \dfrac{\det(I - \overline{B}_\beta(t))}{1 - t^n}(1 - t)$ quoted earlier.

PROOF OF THEOREM 1. Apply Fox's free differential calculus to a presentation of the fundamental group of the closed braid and axis, very much as in [**1**] or [**3**]. The heart of the proof lies in relating the fundamental group of the n-punctured disk spanning the axis A which meets $\hat\beta$ at the bottom of β to the corresponding group for the disk at the top of β. Write $x_1, \ldots, x_n$ for the generators of the group at the bottom represented by meridian loops around the punctures, and $X_1, \ldots, X_n$ for the meridian loops around the punctures at the top. It is well known that $X_1, \ldots, X_n$ can be expressed in terms of $x_1, \ldots, x_n$ as $X_i = F_\beta(x_i)$, where $F_\beta : F_n \to F_n$ is an automorphism of the free group F_n determined by the braid β. Furthermore the map $\beta \mapsto F_\beta$ from the braid group B_n to $\operatorname{Aut} F_n$ is itself a group homomorphism. Then F_β is the composite of elementary automorphisms corresponding to the elementary braids making up β, which are given explicitly by $F_{\sigma_i}(x_i) = x_{i+1}$, $F_{\sigma_i}(x_{i+1}) = x_{i+1} x_i x_{i+1}^{-1}$ and $F_{\sigma_i}(x_j) = x_j$, $j \neq i, i+1$.

The group of the link $\hat\beta \cup A$ is presented by a generator x, arising from a meridian of A, and generators $x_1, \ldots, x_n$ as above, with n relations $F_\beta(x_i) = x^{-1} x_i x$. The reduced Burau matrix shows up most naturally by using a different system of generators $g_1, \ldots, g_n$ for F_n, defined recursively by $g_1 = x_1$, $g_{i+1} = x_{i+1} g_i$. The element g_i can be represented by a loop in the disk which encircles the first i punctures as indicated in figure 2. The automorphism F_{σ_i} then satisfies $F_{\sigma_i}(g_i) = F_{\sigma_i}(x_i g_{i-1}) = x_{i+1} g_{i-1} = g_{i+1} g_i^{-1} g_{i-1}$ and $F_{\sigma_i}(g_j) = g_j$, $j \neq i$.

The standard method for finding the Alexander invariant of a link (or knot) L evaluated in $\mathbf{Z}[H]$ using a homomorphism $\varphi : G_L \to H$ starting from a presentation of G_L by $n+1$ generators $\{g_j\}$ and n relations $\{r_i\}$ is the following. Calculate the $n \times (n+1)$ matrix $\dfrac{\partial r_i}{\partial g_j}$ of free derivatives, and evaluate the entries in $\mathbf{Z}[H]$ by applying φ. Then delete one column corresponding to a generator c, say, with

$\varphi(c) \neq 1$, and divide the determinant of the remaining matrix by $1-\varphi(c)$. The result is the evaluation of the Alexander invariant, that is, the multi-variable Alexander polynomial when L has more than one component, or $\Delta_L(t)/(1-t)$ when L is a knot. Where a relation is written in the form $r = s$ the entries $\dfrac{\partial(r-s)}{\partial g_j}$ work equally well in the matrix.

The presentation of G_L by generators $g_1, \ldots, g_n$ and x above will give an immediate calculation of the Alexander invariant, once we know the entries $\dfrac{\partial F_\beta(g_i)}{\partial g_j}$.

The chain rule for free derivatives [**1**] shows that if we write $\beta = \beta_1\beta_2$ and set $G_i = F_{\beta_2}(g_i)$ then $F_\beta(g_i) = F_{\beta_1}(G_i)$ and

$$\frac{\partial F_\beta(g_i)}{\partial g_j} = \sum_{k=1}^{n} \frac{\partial F_{\beta_1}(G_i)}{\partial G_k}\frac{\partial G_k}{\partial g_j}.$$

We can then compute the matrix $\dfrac{\partial F_\beta(g_i)}{\partial g_j}$ as the product of matrices $\dfrac{\partial F_\gamma(G_k)}{\partial G_j}$, where γ runs through the elementary braids $\sigma_i^{\pm 1}$ in β.

Now when $\gamma = \sigma_i$ we have

$$\begin{aligned} F_\gamma(G_i) &= G_{i+1}G_i^{-1}G_{i-1} \\ F_\gamma(G_j) &= G_j,\ j \neq i. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial F_\gamma(G_i)}{\partial G_{i-1}} &= G_{i+1}G_i^{-1} \\ \frac{\partial F_\gamma(G_i)}{\partial G_i} &= -G_{i+1}G_i^{-1} \\ \frac{\partial F_\gamma(G_i)}{\partial G_{i+1}} &= 1 \\ \frac{\partial F_\gamma(G_k)}{\partial G_j} &= \delta_{j\,k},\ \text{otherwise.} \end{aligned}$$

Apply φ to the matrix to get

$$\varphi\left(\frac{\partial F_\gamma(G_k)}{\partial G_j}\right) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & a & -a & 1 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where $a = \varphi(G_{i+1}G_i^{-1})$. This is the value of φ on the meridian at the undercrossing in the current elementary braid $\gamma = \sigma_i$. Similarly, when $\gamma = \sigma_i^{-1}$ we get the matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & 1 & -a^{-1} & a^{-1} & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

with $a = \varphi(G_i G_{i-1}^{-1})$ which is again the value of φ on the meridian of the undercrossing string.

The generator g_n is clearly unchanged by any automorphism F_β, so the last row in each of the matrices of partial derivatives is $(0 \cdots 0\ 1)$. The leading $(n-1)\times(n-1)$ submatrix for the partial derivatives of $F_\beta(g_i)$ is then the product of the leading submatrices from the constituent elementary braids, and so is the reduced colured Burau matrix of β introduced earlier, with $t_j = \varphi(x_j)$. Thus

$$\varphi\left(\frac{\partial F_\beta(g_i)}{\partial g_j}\right) = \begin{pmatrix} \overline{B}_\beta(t_1,\ldots,t_n) & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} = \tilde{B}_\beta(t_1,\ldots,t_n), \text{ say,}$$

for some column $\mathbf{v}$. The matrix of free derivatives arising from the presentation of the group by generators $g_1,\ldots,g_n$ and x given above is the $n \times (n+1)$ matrix whose first n columns, corresponding to the generators $g_1,\ldots,g_n$, are $\varphi(\frac{\partial}{\partial g_j}(F_\beta(g_i) - x^{-1}g_i x))$. These columns form the matrix $\tilde{B}_\beta(t_1,\ldots,t_n) - x^{-1}I_n$. They arise from the full matrix of free derivatives by deleting the column corresponding to the generator x, so the Alexander invariant for the link is the determinant of this matrix divided by $1-x$. Now clearly $\det(\tilde{B}_\beta(t_1,\ldots,t_n) - x^{-1}I_n) = x^{-n}\det(x\overline{B}_\beta(t_1,\ldots,t_n) - I)(1-x)$. Then $\det(I - x\overline{B}_\beta(t_1,\ldots,t_n))$, making the identifications of variables $t_j = t_{\pi(j)}$ forced by matching the strings at the top and bottom, is the Alexander polynomial of the closed braid and its axis, up to a power of x. □

The Alexander polynomial of the closed braid itself can then be found from the Torres-Fox formula as shown above.

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Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, England.

E-mail address: morton@liv.ac.uk

URL: http://www.liv.ac.uk/~su14