Mutant knots with symmetry

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Abstract

Mutant knots, in the sense of Conway, are known to share the same Homfly polynomial. Their 2-string satellites also share the same Homfly polynomial, but in general their m-string satellites can have different Homfly polynomials for $m > 2$. We show that, under conditions of extra symmetry on the constituent 2-tangles, the directed m-string satellites of mutants share the same Homfly polynomial for $m < 6$ in general, and for all choices of m when the satellite is based on a cable knot pattern.

We give examples of mutants with extra symmetry whose Homfly polynomials of some 6-string satellites are different, by comparing their quantum $sl(3)$ invariants.

1. *Introduction*

This paper has been inspired by recent observations of Ochiai and Jun Murakami about the Homfly skein theory of m-parallels of certain symmetrical 2-tangles. In [**8**] Ochiai remarks that the 3-parallels of the tangle AB in figure 1 and its mirror image $\overline{AB} = BA$ are equal in the Homfly skein of 6-tangles, in other words, in the Hecke algebra H_6, [**1**].

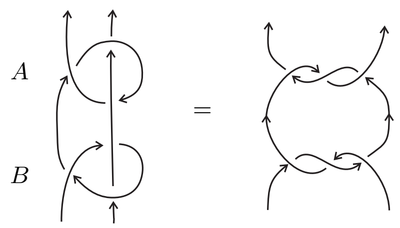


Fig. 1. The 2-tangle AB used by Ochiai

As a consequence, the 3-parallels of any mutant pair of knots given by composing the 2-tangles AB and BA with any other 2-tangle C and then closing, as in figure 3 will share the same Homfly polynomial. This is in contrast with the known fact that 3-parallels of mutant knots in general can have different Homfly polynomials, [**7**, **4**].

There is interest in the extent to which the Homfly polynomial of m-parallels or other m-string satellites can distinguish mutants which are closures of ABC and BAC with A and B as above. Ochiai has found that the 4-parallels of AB and BA are different in the skein H_8.

The purpose of this paper is to show that if A and B are any two oriented 2-tangles which have the symmetry shown in figure 2 then the m-parallels, and indeed any directed m-string satellite, of knots $\widehat{ABC}$ and $\widehat{BAC}$ shown in figure 3 share the same Homfly polynomial provided that $m < 6$. In contrast there exist examples of A, B and C where A and B have the symmetry shown in figure 2, including Ochiai's case with

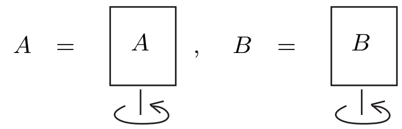


Fig. 2. The symmetry imposed on the tangles A and B

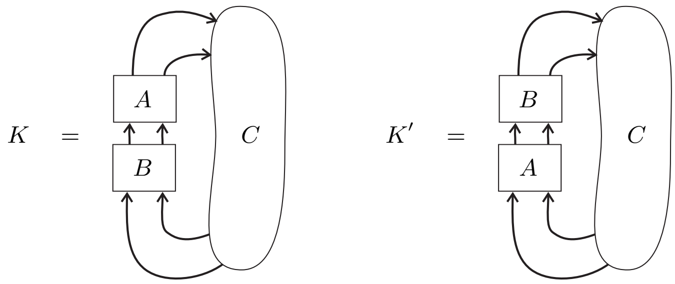
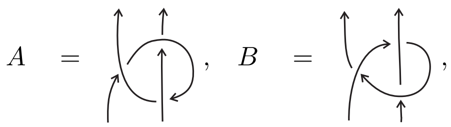


Fig. 3. Tangle interchange



for which the Homfly polynomials of the 6-fold parallel of K and K' are different. The simplest such example, the pretzel knots shown in figure 8, uses Ochiai's choice of symmetric tangles A and B.

In an unexpected extension of the main result we show that the Homfly polynomial of a genuine connected cable, based on the (m, n) torus knot pattern, with m and n coprime, will not distinguish mutants with symmetry above, for *any* number of strings, m, although a more general connected satellite pattern can do so.

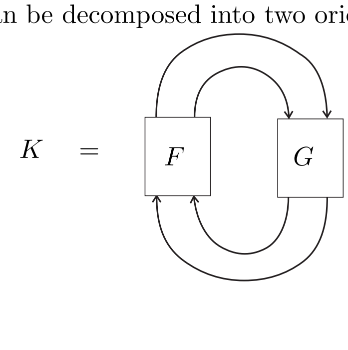
The examples which exhibit differences for the directly oriented 6-parallel can also be used to show that the 4-parallels with two pairs of reverse strands have distinct Homfly polynomials.

The proofs are based on the relation of the Homfly satellite invariants to quantum $sl(N)$ invariants, and the techniques are an extension of work with Cromwell [**4**] and with H. Ryder [**6**]. The eventual calculations that exhibit the difference of invariants in the specific example depend on the 27 dimensional irreducible module over $sl(3)$ corresponding to the partition $4, 2$, and some Maple calculations following similar lines to those in [**6**].

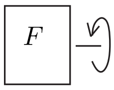
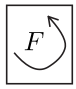
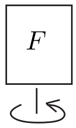
2. *Shared invariants of mutants*

The term *mutant* was coined by Conway, and refers to the following general construction.

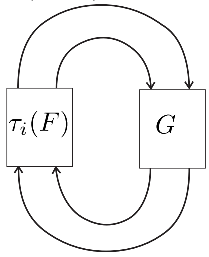
Suppose that a knot K can be decomposed into two oriented 2-tangles F and G



A new knot K' can be formed by replacing the tangle F with the tangle $F' = \tau_i(F)$ given by rotating F through π in one of three ways,

$\tau_1(F) =$ , $\tau_2(F) =$ , $\tau_3(F) =$ ,

reversing its string orientations if necessary. Any of the three knots

$K' =$ 

is called a *mutant* of K.

The two 11-crossing knots, C and KT, found by Conway and Kinoshita-Teresaka are the best-known example of mutant knots. These two knots are shown in figure 4.

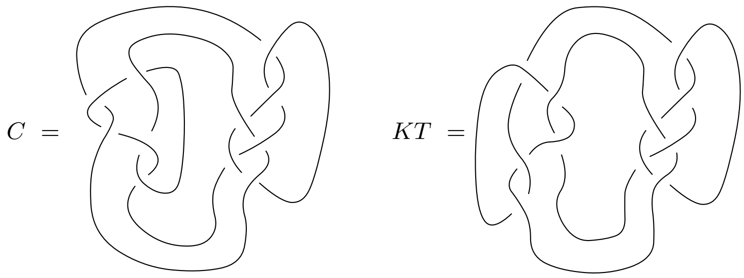


Fig. 4. The Conway and Kinoshita-Teresaka mutant pair

2·1. *Satellites*

A satellite of K is determined by choosing a diagram Q in the standard annulus, and then drawing Q on the annular neighbourhood of K determined by the framing, to give the satellite knot $K * Q$. We refer to this construction as *decorating* K *with the pattern* Q, as shown in figure 5.

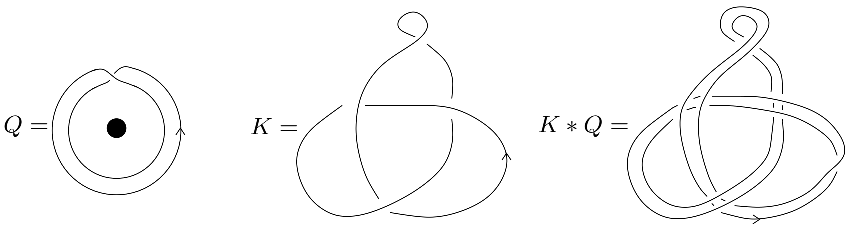
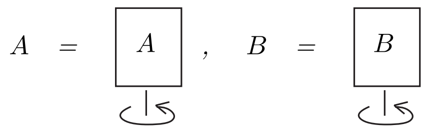


Fig. 5. Satellite construction

For fixed Q the Homfly polynomial $P(K * Q)$ of the satellite is an invariant of the framed knot K. The invariants $P(K * Q)$ as Q varies make up the *Homfly satellite invariants* of K. We use the alternate notation $P(K; Q)$ in place of $P(K * Q)$ when we want to emphasise the dependence on K.

The general symmetry result compares the invariants of two knots K and K' made up of 2-tangles A, B and C, by interchanging A and B as in figure 3.

THEOREM 1. *Suppose that A and B are both symmetric under the half-twist τ_3, so that*



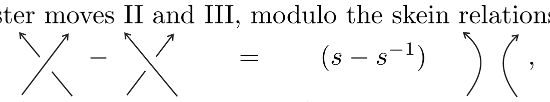
Let K and K' be knots which are the closure of ABC and BAC respectively for any tangle C, as in figure 3. Then $P(K * Q) = P(K' * Q)$ for every closed braid pattern Q on $m < 6$ strings.

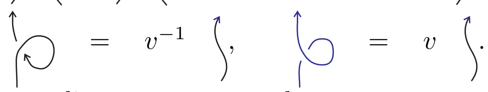
REMARK 1. *The proof applies equally to the case where Q is the closure of any directly oriented m-tangle with $m < 6$.*

In order to prove the theorem we must rewrite the Homfly satellite invariants in terms of quantum $sl(N)$ invariants, so we now give a brief summary of the relations between these invariants, originally established by Wenzl. Further details can be found in [**1**] and the thesis of Lukac, [**3**], including details of variant Homfly skeins with a framing correction factor, x. These are isomorphic to the skeins used here but the parameter allows a careful adjustment of the quadratic skein relation to agree directly with the natural relation arising from use of the quantum groups $sl(N)$.

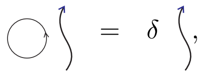
2·2. *Homfly skeins*

For a surface F with some designated input and output boundary points the (linear) Homfly skein of F is defined as linear combinations of oriented diagrams in F, up to Reidemeister moves II and III, modulo the skein relations

(i) 

(ii) 

It is an immediate consequence that



where $\delta = \dfrac{v^{-1} - v}{s - s^{-1}} \in \Lambda$. The coefficient ring Λ is taken as $Z[v^{\pm 1}, s^{\pm 1}]$, with denominators $s^r - s^{-r}, r \geq 1$.

The skein of the annulus is denoted by $\mathcal{C}$. It becomes a commutative algebra with a product induced by placing one annulus outside another.

The skein of the rectangle with m inputs at the top and m outputs at the bottom is denoted by H_m. We define a product in H_m by stacking one rectangle above the other, obtaining the Hecke algebra $H_m(z)$, when $z = s - s^{-1}$ and the coefficients are extended to Λ. The Hecke algebra H_m can also be regarded as the group algebra of Artin's braid group B_m generated by the elementary braids σ_i, $i = 1, \ldots, m-1$, modulo the further quadratic relation $\sigma_i^2 = z\sigma_i + 1$.

The closure map from H_m to $\mathcal{C}$ is the Λ-linear map induced by mapping a tangle T to its closure $\widehat{T}$ in the annulus (see figure 6). We refer to a diagram $Q = \widehat{T}$ as a *directly oriented* pattern.

The image of this map is denoted by $\mathcal{C}_m$, which has a useful interpretation as the space of symmetric polynomials of degree m in variables $x_1, \ldots, x_N$ for large enough N.

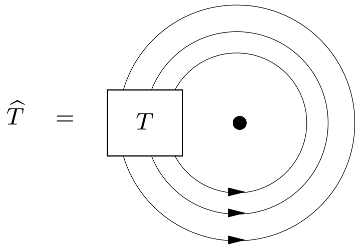


Fig. 6. The closure map

Moreover, the submodule $\mathcal{C}_+ \subset \mathcal{C}$ spanned by the union $\cup_{m\geq 0}\mathcal{C}_m$ is a subalgebra of $\mathcal{C}$ isomorphic to the algebra of the symmetric functions.

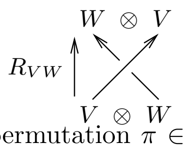
2·3. *Quantum invariants*

A quantum group $\mathcal{G}$ is an algebra over a formal power series ring $\mathbf{Q}[[h]]$, typically a deformed version of a classical Lie algebra. We write $q = e^h, s = e^{h/2}$ when working in $sl(N)_q$. A finite dimensional module over $\mathcal{G}$ is a linear space on which $\mathcal{G}$ acts.

Crucially, $\mathcal{G}$ has a coproduct Δ which ensures that the tensor product $V \otimes W$ of two modules is also a module. It also has a *universal R-matrix* (in a completion of $\mathcal{G} \otimes \mathcal{G}$) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \to W \otimes V.$$

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.



A braid β on m strings with permutation $\pi \in S_m$ and a colouring of the strings by modules $V_1, \ldots, V_m$ leads to a module homomorphism

$$J_\beta : V_1 \otimes \cdots \otimes V_m \to V_{\pi(1)} \otimes \cdots \otimes V_{\pi(m)}$$

using $R^{\pm 1}_{V_i,V_j}$ at each elementary braid crossing. The homomorphism J_β depends *only on the braid* β itself, not its decomposition into crossings, by the Yang-Baxter relation for the universal R-matrix.

When $V_i = V$ for all i we get a module homomorphism $J_\beta : W \to W$, where $W = V^{\otimes m}$. Equally, a directed m-tangle T determines an endomorphism J_T of $W = V^{\otimes m}$. Now any $sl(N)$ module W decomposes as a direct sum $\bigoplus (W_\mu \otimes V_\mu^{(N)})$, where W_μ is the linear subspace consisting of the *highest weight vectors* of type μ associated to the module $V_\mu^{(N)}$. Highest weight subspaces of each type are preserved by module homomorphisms, and so J_T determines (and is determined by) the restrictions $J_T(\mu) : W_\mu \to W_\mu$ for each μ.

If a knot K is decorated by a pattern Q which is the closure of an m-tangle T then its quantum invariant $J(K * Q; V)$ can be found from the endomorphism J_T of $W = V^{\otimes m}$ in terms of the quantum invariants of K and the highest weight maps $J_T(\mu) : W_\mu \to W_\mu$ by the formula

$$J(K * Q; V) = \sum c_\mu J(K; V_\mu^{(N)}) \tag{2·1}$$

with $c_\mu = \operatorname{tr} J_T(\mu)$. This formula follows from lemma II.4.4 in Turaev's book [**11**]. Here μ runs over partitions with at most N parts when we are working with $sl(N)$, and we set $c_\mu = 0$ when W has no highest weight vectors of type μ.

Proof of theorem 1 Take $V = V^{(N)}$ as the fundamental module of dimension N for $sl(N)$. Then the only highest weight types μ which occur in equation (2·1) are partitions of m with at most N rows. Because $J(K*Q;V^{(N)}) = P(K*Q)$ when $v = s^{-N}$ we can show that $P(K*Q) = P(K'*Q)$ by showing that $J(K*Q;V^{(N)}) = J(K'*Q;V^{(N)})$ for all N. By equation 2·1 it is then enough to show that $J(K;V_\mu^{(N)}) = J(K';V_\mu^{(N)})$ for all N and all partitions $\mu \vdash m$.

Now each tangle A and B determines an endomorphism J_A, J_B of $V_\mu \otimes V_\mu$. If J_A and J_B commute then $J(K;V_\mu) = J(K';V_\mu)$. The endomorphisms J_A and J_B are determined by their restriction $J_A(\nu), J_B(\nu)$ to the highest weight subspaces W_ν in the decomposition $V_\mu \otimes V_\mu = \sum W_\nu \otimes V_\nu$, so it is enough to show that $J_A(\nu)$ and $J_B(\nu)$ commute where V_ν is a summand of $V_\mu \otimes V_\mu$. This is certainly the case for all ν where W_ν is 1-dimensional, which includes the case of single row or column partitions μ, [**4**].

As a special case of the work of Rosso and Jones, [**9**, **5**], we know that the endomorphism of $V_\mu \otimes V_\mu$ for the full twist Δ^2 on two strings operates as a scalar $e^{f(\nu)}$ on each highest weight space W_ν, while the half twist Δ, represented by the R-matrix $R_{V_\mu V_\mu}$, operates on W_ν with two eigenvalues $\pm e^{\frac{1}{2}f(\nu)}$.

The positive and negative eigenspaces correspond to the classical decomposition of the Schur function $(s_\mu)^2$ into symmetric and skew-symmetric parts, $h_2(s_\mu)$ and $e_2(s_\mu)$, and the dimension of each eigenspace of W_ν is the multiplicity of s_ν in $h_2(s_\mu)$ and $e_2(s_\mu)$ respectively.

Now $A = \tau_3(A)$, so that $A\Delta = \Delta A$. Hence the endomorphism J_A, and similarly J_B, preserves the positive and negative eigenspaces of each W_ν. If these eigenspaces have dimension 1 or 0 then J_A and J_B will commute on W_ν.

The theorem is then established by checking that no s_ν occurs in $h_2(s_\mu)$ or $e_2(s_\mu)$ with multiplicity > 1 for any μ with $|\mu| \le 5$. The decomposition of all of these can be quickly confirmed using the Maple program SF of Stembridge [**10**]. □

COROLLARY 2. *Examples include the k-pretzel knots $K(a_1, \ldots, a_k)$ with odd a_i shown in figure 7, where the numbers of half-twists a_i can be permuted without changing the Homfly polynomial of any satellite with ≤ 5-strings.*

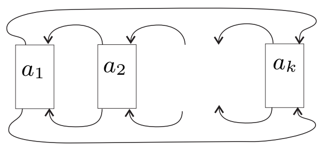


Fig. 7. The pretzel knot $K(a_1, \ldots, a_k)$

3. *Satellites with different Homfly polynomials*

A further check with the program SF when $|\mu| = 6$ shows that there are just three partitions, $\mu = 4,2$, its conjugate $\mu = 2,2,1,1$ and $\mu = 3,2,1$ whose symmetric square $h_2[s_\mu]$ contains summands with multiplicity > 1, as does the exterior squares of $\mu = 3,2,1$. Explicitly $h_2[s_{4,2}] = s_{8,4} + s_{8,2,2} + s_{7,4,1} + s_{7,3,2} + s_{7,3,1,1} + s_{6,6} + s_{6,5,1} + 2\,s_{6,4,2} + s_{6,3,2,1} + s_{6,2,2,2} + s_{5,5,1,1} + s_{5,4,3} + s_{5,4,2,1} + s_{5,3,3,1} + s_{4,4,4} + s_{4,4,2,2}$. This means that, although m-string satellites of K and K' must share the Homfly polynomial when $m \le 5$, it is possible for the Homfly polynomials of some 6-string satellites to differ.

We give an example now where this does indeed happen.

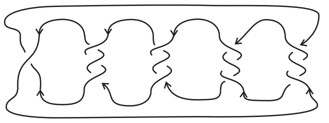
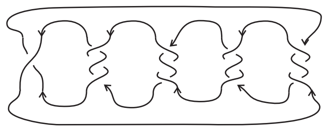
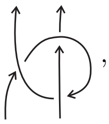
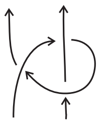
$K =$  , $K' =$ 

Fig. 8. The pretzel knots $K = K(1,3,3,-3,-3)$ and $K' = K(1,3,-3,3,-3)$

THEOREM 3. *Let K and K' be the pretzel knots $K = K(1,3,3,-3,-3)$ and $K' = K(1,3,-3,3,-3)$ shown in figure 8. The 6-fold parallels $K * Q$ and $K' * Q$, where Q is the closure of the identity braid on 6 strings, have different Homfly polynomials.*

Proof. Write K and K' as the closure of the products $\Delta ABAB$ and $\Delta BAAB$ respectively, where

$A =$  , $B =$  ,

are the partially closed 3-braids shown, and Δ is the positive half-twist. We show that $P(K * Q) \neq P(K' * Q)$ when $v = s^{-3}$. These values are given by the $sl(3)$ quantum invariants $J(K*Q; V^{(3)})$ and $J(K'*Q; V^{(3)})$, where $V^{(3)}$ is the fundamental 3-dimensional module for $sl(3)$. Since Q is the closure of the identity braid on 6 strings it induces the identity endomorphism on the module $(V^{(3)})^{\otimes 6}$. This module decomposes as $\bigoplus W_\mu \otimes V_\mu^{(3)}$ where μ runs through partitions of 6 with at most 3 rows. The trace of the identity on W_μ is just $d_\mu = \dim W_\mu$, giving

$$J(K * Q; V^{(3)}) = \sum d_\mu J(K; V_\mu^{(3)}).$$

The only partition μ in this range for which the exterior or symmetric square contains highest weight vectors of multiplicity > 1 is the partition $\mu = 4,2$, since the partition $\mu = 2,2,1,1$ has 4 rows and the repeated factors for $\mu = 3,2,1$ occur for partitions with more than 3 rows. Now $J_A(\mu)J_B(\mu) = J_B(\mu)J_A(\mu)$ for all other μ since A and B are symmetric up to altering the framing on both strings, while maintaining the writhe. Then

$$P(K * Q) - P(K' * Q) = d_\mu (J(K; V_\mu^{(3)}) - J(K'; V_\mu^{(3)}))$$

when $v = s^{-3}$ and $\mu = 4,2$. Since $d_\mu \neq 0$ it is enough to show that $J(K; V_\mu^{(3)}) \neq J(K'; V_\mu^{(3)})$. The module $V_\mu^{(3)}$ has dimension 27.

We now work in the quantum group $sl(3)$ and drop the superscript (3) from the irreducible modules.

Decompose the module $V_\mu \otimes V_\mu$ as $\sum W_\nu \otimes V_\nu$ and compare the endomorphisms given by the tangles $T = ABAB\Delta$ and $T' = BAAB\Delta$.

In this case just one of the invariant subspaces of highest weight vectors has dimension > 1. It can be shown that the corresponding 2×2 matrices A_μ and B_μ arising from the two mirror-image tangles A and B with 3 crossings satisfy $\mathrm{tr}(A_\mu B_\mu A_\mu B_\mu - A_\mu A_\mu B_\mu B_\mu) \neq 0$, which results in a difference in their $sl(3)$ invariants $J(K; V_\lambda)$.

None of the other 6-cell invariants differ on the two knots. Consequently the 6-parallels have different $sl(3)$ invariants. The $sl(3)$ invariant of the 6-parallels of the two pretzel knots coloured with the fundamental module, and thus their Homfly polynomials, are then different. □

3·1. *Use of the quantum group $sl(3)_q$*

The calculation of the 2×2 matrices A_ν and B_ν giving the effect of the two tangles on the highest weight vectors where there is a 2-dimensional highest weight subspace of the symmetric part of the module depends on finding the explicit action of the quantum group on the 27-dimensional module $V_\mu^{(3)}$ with $\mu = 4, 2$ and its tensor square, as well as the homomorphism representing its R-matrix. I used the linear algebra packages in Maple to handle the matrix working and subsequent polynomial factorisation, following fairly closely the techniques developed with H. Ryder in the paper [**6**].

In the interests of reproducibility I give an account of the methods used, and some of the checks applied during the calculations, to test against known properties.

We start from a presentation of the quantum group $sl(3)_q$ as an algebra with six generators, $X_1^\pm$, $X_2^\pm$, H_1, H_2, and a description of the comultiplication and antipode.

Let M be any finite-dimensional left module over $sl(3)_q$. The action of any one of these six generators Y will determine a linear endomorphism Y_M of M. We build up explicit matrices for these endomorphisms on a selection of low-dimensional modules, using the comultiplication to deal with the tensor product of two known modules, and the antipode to construct the action on the linear dual of a known module. We must eventually determine the matrices Y_M for our module $M = V_{\yng(3,1)}$, and find the 729×729 R-matrix, R_{MM} which represents the endomorphism of $M \otimes M$ needed for crossings.

We follow Kassel in the basic description of the quantum group from using generators H_1 and H_2 for the Cartan sub-algebra, but with generators $X_i^\pm$ in place of X_i and Y_i. We use the notation $K_i = \exp(hH_i/4)$, and set $a = \exp(h/4)$, $s = \exp(h/2) = a^2$ and $q = \exp(h) = s^2$, unlike Kassel. The generators satisfy the commutation relations

$$[H_i, H_j] = 0,\ [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm,\ [X_i^+, X_i^-] = (K_i^2 - K_i^{-2})/(s - s^{-1}),$$

where $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix for $SU(3)$ (and also the Serre relations of degree 3 between $X_1^\pm$ and $X_2^\pm$).

Comultiplication is given by

$$\begin{aligned} \Delta(H_i) &= H_i \otimes I + I \otimes H_i, \\ (\text{so } \Delta(K_i) &= K_i \otimes K_i,) \\ \Delta(X_i^\pm) &= X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm, \end{aligned}$$

and the antipode S by $S(X_i^\pm) = -s^{\pm 1} X_i^\pm$, $S(H_i) = -H_i$, $S(K_i) = K_i^{-1}$.

The fundamental 3-dimensional module, which we denote by E, has a basis in which the quantum group generators are represented by the matrices Y_E as listed here.

$$X_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For calculations we keep track of the elements K_i rather than H_i, represented by

$$K_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$$

for the module E.

We can then write down the elements Y_{EE} for the actions of the generators Y on the module $E \otimes E$, from the comultiplication formulae. The R-matrix R_{EE} can be given, up to a scalar, by the prescription

$$\begin{aligned} R_{EE}(e_i \otimes e_j) \quad &= e_j \otimes e_i, \text{ if } i > j, \\ &= s\, e_i \otimes e_i, \text{ if } i = j, \\ &= e_j \otimes e_i + (s - s^{-1}) e_i \otimes e_j, \text{ if } i < j, \end{aligned}$$

for basis elements $\{e_i\}$ of E.

The linear dual M^* of a module M becomes a module when the action of a generator Y on $f \in M^*$ is defined by $< Y_{M^*} f, v >=< f, S(Y_M) v >$, for $v \in M$. For the dual module $F = E^*$ we then have matrices for Y_F, relative to the dual basis, as follows.

$$X_1^+ = \begin{pmatrix} 0 & 0 & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix}$$

$$X_1^- = \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{pmatrix}.$$

The most reliable way to work out the R-matrices R_{EF}, R_{FE} and R_{FF} is to combine R_{EE} with module homomorphisms cup_{EF}, cup_{FE}, cap_{EF} and cap_{FE} between the modules $E \otimes F$, $F \otimes E$ and the trivial 1-dimensional module, I, on which $X_i^{\pm}$ acts as zero and K_i as the identity. The matrices are determined up to a scalar by such considerations; a choice for one dictates the rest.

Once these matrices have been found they can be combined with the matrix R_{EE}^{-1} to construct the R-matrices R_{EF}, R_{FE}, R_{FF}, using the diagram shown in figure 9, for example, to determine R_{EF} as

$$R_{EF} = (1_F \otimes 1_E \otimes \mathrm{cap}_{EF}) \circ (1_F \otimes R_{EE}^{-1} \otimes 1_F) \circ (\mathrm{cup}_{FE} \otimes 1_E \otimes 1_F).$$

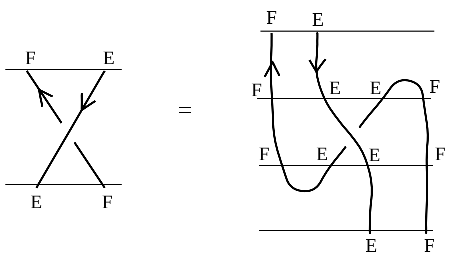


Fig. 9. Construction of the R-matrix R_{EF}

The module structure of $M = V_{\yng(4,2)}$ can be found by identifying M as a 27-dimensional submodule of $V_{\yng(2,2)} \otimes V_{\yng(2)}$, while the two 6-dimensional modules $V_{\yng(2)}$ and $V_{\yng(2,2)}$ are themselves submodules of $E \otimes E$ and $F \otimes F$ respectively.

We know, by the Pieri formula, that there is a direct sum decomposition of $V_{\yng(2,2)} \otimes V_{\yng(2)}$ as $M \oplus N$, where $M = V_{\yng(4,2)}$ and N is the sum of the 8-dimensional module $V_{\yng(3,2,1)}$ and the 1-dimensional trivial module.

We first identify the module $V_{\yng(2)}$ as a submodule of $E \otimes E$, knowing that $E \otimes E$ is isomorphic to $V_{\yng(2)} \otimes F$. The full twist element on the two strings both coloured by E is represented by R^2_{EE} which acts on $E \otimes E$ as a scalar on each of the two irreducible submodules $V_{\yng(2)}$ and F.

Use Maple to find bases for the two eigenspaces of R^2_{EE}. Then we can identify $V_{\yng(2)}$ with the 6-dimensional one, and write P and Q for the 9×6 and 9×3 matrices whose columns are these bases. The partitioned matrix $(P|Q)$ is invertible, and its inverse, found by Maple, can be written as $\begin{pmatrix} R \\ S \end{pmatrix}$, where R is a 6×9 matrix with $RP = I_6$ and $RQ = 0$.

Regard $P = \mathrm{inj}M_1EE$ as the matrix representing the inclusion of the module $V_{\yng(2)}$ into $E \otimes E$. Then $R = \mathrm{proj}EEM_1$ is the matrix, in the same basis, of the projection from $E \otimes E$ to $V_{\yng(2)}$. For $M_1 = V_{\yng(2)}$ the module generators Y_{M_1} are given by $Y_{M_1} = R\,Y_{EE}\,P$, giving the explicit action of the quantum group on $V_{\yng(2)}$.

We perform a similar calculation on $F \otimes F$ to identify the module $M_2 = V_{\yng(2,2)}$ and the matrices $\mathrm{inj}M_2FF$ and $\mathrm{proj}FFM_2$, giving the action of the quantum group on $M_2 = V_{\yng(2,2)}$ in a similar way.

We use inclusion and projection further to find the four $6^2 \times 6^2$ R-matrices $R_{M_iM_j}$. For example, to construct $R_{M_1M_2} : M_1 \otimes M_2 \to M_2 \otimes M_1$, first map $M_1 \otimes M_2$ to $E \otimes E \otimes F \otimes F$ by $\mathrm{inj}M_1EE \otimes \mathrm{inj}M_2FF$. Then construct the R-matrix crossing two strings with $E \otimes E$ and two with $F \otimes F$ as the composite of $1 \otimes R_{EF} \otimes 1$, $R_{EF} \otimes R_{FE}$ and $1 \otimes R_{FF} \otimes 1$, and finally compose with the projections $\mathrm{proj}FFM_2 \otimes \mathrm{proj}EEM_1$, as indicated in figure 10. A similar calculation on the module $M_1 \otimes M_2$ yields the submodule

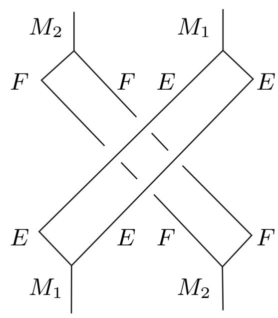


Fig. 10. Construction of the R-matrix $R_{M_1M_2}$

$M = V_{\yng(4,2)}$. The full twist on two strings, one coloured by M_1 and one by M_2, is represented by the product $R_{M_2M_1}R_{M_1M_2}$ and will have one 27-dimensional eigenspace M complemented by two other eigenspaces. Taking the bases of these eigenspaces in a partitioned 36×36 matrix as above will determine a 36×27 matrix $P = \mathrm{inj}MM_1M_2$ and a 27×36 matrix $R = \mathrm{proj}M_1M_2M$. The quantum group actions $Y_{M_1M_2}$ on the tensor product are determined by the coproduct formulae, and the actions Y_M are then given from these using P and R. These in turn give rise to the quantum group actions Y_{MM} on $M \otimes M$.

We are also able to construct the $27^2 \times 27^2$ R-matrix R_{MM} using the same inclusion and projection to map $M \otimes M$ into $M_1 \otimes M_2 \otimes M_1 \otimes M_2$, followed by the matrix for crossing four strands, built up from the R-matrices $R_{M_i M_j}$ and then the projections back to $M \otimes M$.

3·2. *Completing the calculations*

REMARK 2. *We can reach this stage directly if we know the six module generators Y_M and the R-matrix R_{MM} for the module $M = V_{4,2}$. We can then calculate the module generators Y_{MM} using the coproduct, and the twisting element $T_M = (K_{1M})^4(K_{2M})^4$.*

Knowing the module generators Y_{MM} gives an immediate means of finding the highest weight vectors as common null-vectors of X^+_{iMM}, and their weights can be identified. All the submodules of $M \otimes M$ occur with multiplicity 1 except V_ν with partition $\nu = 6,4,2$ whose highest weights are $2,2$. The 3-dimensional space W_ν of highest weight vectors for ν is found by solving the linear equations $X^+_{1MM}v = 0$, $X^+_{2MM}v = 0$, $K_{1MM}v = a^2 v$ and $K_{2MM}v = a^2 v$ for v. We then find the 2-dimensional positive eigenspace for R_{MM} on W_ν. The endomorphisms J_A and J_B will preserve this eigenspace.

Represent the 3-braid $\sigma_2\sigma_1^{-1}\sigma_2$ in the 2-tangle A by an endomorphism F_A of $M \otimes M \otimes M$, using R_{MM} and its inverse. Then use T_M and the partial trace to close off one string, hence giving the endomorphism J_A of $M \otimes M$ determined by A. Explicitly, choose a basis $\{e_i\}$ of M and write

$$F_A(v \otimes T_M(e_i)) = \sum_j f_{ij}(v) \otimes e_j$$

with $f_{ij}(v) \in M \otimes M$. Then $J_A(v) = \sum_i f_{ii}(v)$. Applied to each of the two vectors in the highest weight space this determines a 2×2 matrix A_ν representing the restriction of J_A to this subspace. Similarly B_ν is found using the mirror image braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

We know that R_{MM} acts as a scalar on the 2-dimensional space so $J(K;V_\mu) - J(K';V_\mu)$ is a non-zero scalar multiple of $\mathrm{tr}(A_\nu B_\nu A_\nu B_\nu - B_\nu A_\nu A_\nu B_\nu)$.

This difference is $2(q^6+q^5+q^4+q^3+q^2+q+1)(q^4+1)(q^6+q^3+1)^2(q^4-q^2+1)^2(q^4+q^3+q^2+q+1)^3(q^2+1)^4(q^2+q+1)^4(q^2-q+1)^4(q+1)^{10}(q-1)^{18}$, up to a power of $q = s^2$ and the quantum dimension of V_ν.

3·3. *Further examples of difference*

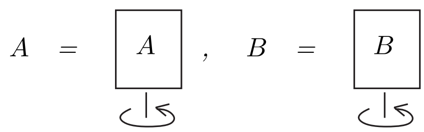
Using the same matrices A_ν and B_ν it is possible to find further pretzel knot examples based on sequences of the tangles A and B where the 6-parallels have different Homfly polynomial, such as the knots $K(3,3,3,-3,-3)$ and $K(3,3,-3,3,-3)$. The difference here is the same as for the first example multiplied by the factor $2q^{32} - q^{31} - 3q^{30} + 5q^{29} + 3q^{28} - 10q^{27} + q^{26} + 14q^{25} - 6q^{24} - 19q^{23} + 21q^{22} + 20q^{21} - 46q^{20} + 2q^{19} + 61q^{18} - 48q^{17} - 35q^{16} + 83q^{15} - 27q^{14} - 66q^{13} + 72q^{12} + 3q^{11} - 57q^{10} + 40q^9 + 10q^8 - 33q^7 + 16q^6 + 7q^5 - 12q^4 + 7q^3 - 4q + 2$. The same calculations guarantee that satellites based on any closed 6-tangle $Q = \widehat{T}$ will have different Homfly polynomial, provided that the trace c_μ of the endomorphism $J_{\widehat{T}}$ on the highest weight space W_μ of $V^{\otimes 6}$ is non-zero, where μ is the partition $4,2$. This will be the case for most, but not all, patterns Q, and certainly will be the case for many satellites which are knots rather than links.

The calculations in section 3·2 also show that the 4-parallels of the two pretzel knots $K(1,3,3,-3,-3)$ and $K(1,3,-3,3,-3)$ with two strings oriented in one direction and two in the opposite direction will have different Homfly polynomials, by using the decomposition of the corresponding $sl(3)_q$ module $W = V \otimes V \otimes V_{1,1} \otimes V_{1,1}$ into a sum of irreducible $sl(3)_q$ modules. The only module to figure in this decomposition with any multiplicity in its symmetric or exterior square is again $V_{(4,2)}$. The calculations above, using the fact that Homfly with $v = s^{-3}$ can be calculated by colouring strings with reverse orientation by the dual module V^* to the fundamental module, and that this is $V_{(1,1)}$ for $sl(3)_q$.

4. *Cable patterns*

By way of contrast, if the pattern Q is a cable on any number of strings then $K * Q$ and $K' * Q$ share the same Homfly polynomial, where K and K' have the same symmetry as in theorem 1.

THEOREM 4. *Suppose that A and B are both symmetric under the half-twist τ_3, so that*



Let K and K' be knots which are the closure of ABC and BAC respectively for any tangle C, as in figure 3. Then $P(K * Q) = P(K' * Q)$ for every (m, n) cable pattern Q where m and n are coprime.

Proof. As in the proof of theorem 1 we show that $J(K * Q; V^{(N)}) = J(K' * Q; V^{(N)})$ for all N. By equation 2·1 it is then enough to show that $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$ for all N and all partitions $\mu \vdash m$ for which the coefficient $c_\mu \neq 0$. The coefficients c_μ depend on the pattern Q and arise as the trace of the endomorphism J_T when restricted to the highest weight space $W_\mu \subset V^{\otimes m}$, where Q is the closure of the m-braid $T = (\sigma_1 \sigma_2 \cdots \sigma_{m-1})^n$.

It is shown in [**9**], (see also [**5**]), that for any such cable Q the only non-zero coefficients c_μ occur when the partition μ is a *hook*, if m and n are coprime . It is then enough to show that $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$ for all hook partitions μ.

Using the same argument as in theorem 1 it remains to check that no Schur function s_ν occurs with multiplicity > 1 in the decomposition of either the symmetric or exterior squares, $h_2(s_\mu)$ or $e_2(s_\mu)$, for any hook partition μ. This fact has been established by Carbonara, Remmel and Yang in theorem 3 of [**2**], and so the proof is complete. □

REMARK 3. *Theorem 4 highlights the importance of a precise terminology for different types of satellite. The term* cable *is sometimes used to mean any satellite, while there is a clear distiction here between the behaviour of cables and of parallels or other satellites, which is not primarily a matter of the number of components of the satellite.*

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