ASYMPTOTIC ANALYSIS OF DEPENDENT RISKS AND EXTREMES IN INSURANCE AND FINANCE

by

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Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor in Philosophy

July, 2015

Abstract

In this thesis, we are interested in the asymptotic analysis of extremes and risks. The heavy-tailed distribution function is used to model the extreme risks, which is widely applied in insurance and is gradually penetrating in finance as well. We also use various tools such as copula, to model dependence structures, and extreme value theorem, to model rare events. We focus on modelling and analysing of extreme risks as well as demonstrate how the derived results that can be used in practice.

We start from a discrete-time risk model. More concretely, consider a discrete-time annuity-immediate risk model in which the insurer is allowed to invest its wealth into a risk-free or a risky portfolio under a certain regulation. Then the insurer is said to be exposed to a stochastic economic environment that contains two kinds of risk, the insurance risk and financial risk. The former is traditional liability risk caused by insurance loss while the later is the asset risk resulting from investment. Within each period, the insurance risk is denoted by a real-valued random variable X, and the financial risk Y as a positive random variable fulfils some constraints. We are interested in the ruin probability and the tail behaviour of maximum of the stochastic present values of aggregate net loss with Sarmanov or Farlie-Gumbel-Morgenstern (FGM) dependent insurance and financial risks. We derive asymptotic formulas for the finite-ruin probability with lighted-tailed or moderately heavy-tailed insurance risk for both risk-free investment and risky investment. As an extension, we improve the result for extreme risks arising from a rare event, combining simulation with asymptotics, to compute the ruin probability more efficiently.

Next, we consider a similar risk model but a special case that insurance and financial risks following the least risky FGM dependence structure with heavy-tailed distribution. We follow the study of Chen (2011) that the finite-time ruin probability in a discrete-time risk model in which insurance and financial risks form a sequence of independent and identically distributed random pairs following a common bivariate FGM distribution function with parameter $-1 \leq \theta \leq 1$ governing the strength of dependence. For the subexponential case, when $-1 < \theta \leq 1$, a general asymptotic formula for the finite-time ruin probability was

derived. However, the derivation there is not valid for $\theta = -1$. In this thesis, we complete the study by extending Chen's work to $\theta = -1$ that the insurance risk and financial risk are negatively dependent. We refer this situation as the least risky FGM dependent insurance risk and financial risk. The new formulas for $\theta = 1$ look very different from, but are intrinsically consistent with, the existing one for $-1 < \theta \leq 1$, and they offer a quantitative understanding on how significantly the asymptotic ruin probability decreases when θ switches from its normal range to its negative extremum.

Finally, we study a continuous-time risk model. Specifically, we consider a renewal risk model with a constant premium and a constant force of interest rate, where the claim sizes and inter-arrival times follow certain dependence structures via some restriction on their copula function. The infinite-time absolute ruin probabilities are studied instead of the traditional infinite-time ruin probability with light-tailed or moderately heavy-tailed claim-size. Under the assumption that the distribution of the claim-size belongs to the intersection of the convolution-equivalent class and the rapid-varying tailed class, or a larger intersection class of O-subexponential distribution, the generalized exponential class and the rapid-varying tailed class, the infinite-time absolute ruin probabilities are derived.

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Acknowledgement

This thesis is the achievement of 3 years of academic research, which contains a section of research papers that I completed during my Ph.D. at the University of Liverpool. A Ph.D. can never be undertaken on your own and I am indebted to a great number of special people.

First and foremost, I would like to thank my first supervisor Dr. Yiqing Chen (Univeristy of Liverpoool & Drake University). It is she who firstly led me to research on actuarial science and guided me to find my own research interest. I really appreciate Dr Yiqing Chen's advice and encouragement from both academic side and non-academic side in the past years. Without her help, this work would not have been possible.

Next, my thankfulness should also go to supervisor Dr. Yi Zhang (University of Liverpool). I would like to thank him for his kind encouragement and outstanding remarks on an earlier version of this thesis in my third year. His stimulating and rigorous academic attitude has been an important effect for me to perform research at the early age.

Special acknowledgements are also given to Prof. Yang Yang (Nanjing Audit University). His cooperation, helpful remarks and sustained precision have increased the quality of my work.

I am grateful to Prof. Qihe Tang for the fruitful and comprehensive discussions and providing me the chance to visit the University of Iowa in 2015.

I am also very grateful to many of my colleagues and students who shared their thoughts and knowledge with me during my Ph.D. study.

Finally, I want to thank my family members. Your unlimited and unconditional love has always been a driving force that keeps me moving forward.

Part I

Introduction

Chapter 1

General Introduction

This chapter plays a role to induce the problems studied in this thesis. It will explain its motivation, background and how it contributes to the literature for each part. Furthermore, the interconnection of each part will also be expounded.

1.1 Motivations

The prevalence of rare events such as earthquake, flood, wind-storm, or terrorism which are accompanied by disastrous economic and social consequence is the so-called Black-Swan phenomenon that make today's world far different from decades ago. In recent years, some frequent occurrences of catastrophes include: the 2008 Sichuan Earthquake in China which costs over \$148 billion, the 2008 financial crisis that directly result in the 2008-2012 global recession, the 2010 Haiti Earthquake with estimated cost between \$7.2–13.2 billion, 2011 Japan Earthquake, Tsunami and Nuclear Crisis with loss over \$14.5-34.6 billion and world Banks estimated economic cost over \$235 billion, the 2013 Typhoon Haiyan with damage over \$1.5 billion. These catastrophes can lead to extremely large insurance and financial losses, which can be followed by ruin of insurance industries or bankruptcy of financial institutions that are suffering such losses. These natural or man-made catastrophes, which make extreme losses or outliers" in statistical data, are rare events which make them particularly difficult to prognosticate; see Embrechts et al. (1997), Section 1.4 of McNeil et al. (2005), among many others. These extremes and risks have increased awareness of the need to quantify probabilities of large losses, and for helping risk management systems to control such rare events.

Considering the grim consequences of extreme events, we carry out analysis of extremes and risks in finance and insurance in this thesis. It is important to realize that extreme losses and risks are controlled by the same economic factors (such as global, national or regional economic growth), or affected by a common external event (such as flood, windstorm, forest fire, earthquake or terrorism). Therefore, they should be strongly dependent on each other. Moreover, a strong dependence among the losses can tend to make the losses jointly large. For example, properties may be damaged in a hurricane, resulting in large claims to an insurer, while they tend to be destroyed all together causes even more concern to the insurer. In extreme risk analysis, it is of particular significance to model both large individual losses and the dependence among them. To model such extreme losses and risks that are potentially large, we use light-tailed distribution and heavy-tailed distribution classes, while to model dependence, we use various tools such as copulas and joint cumulative distribution functions.

When modeling extreme losses and risks, we focus on both discrete-time and continuous-time risk models. In the first part of this thesis, we consider the ruin probabilities in a discrete time risk model with dependent risks.

The insurance business is always described by a discrete-time risk model in which the two risks that are usually called the insurance risk and the financial risk are quantified by concrete random variables. The study of the probability of ruin has become specially relevant for insurance business because of modern regulatory frameworks (such as EU Solvency II) that require insurance company to hold solvency capital so that the ruin probability can be taken control of. This risk model was proposed by Nyrhinen (1990, 2001). There has been a vast amount of literature in this aspect, including Norberg (1999), Tang and Tsitsiashvili (2003, 2004) and Goovaerts et al. (2005) among others. As discussed above, the insurance risk is referred to the risk resulting from insurance claims and the financial risk is referred to the risk of large losses from financial investments, classifying as credit risk, market risk, operational risk, and so on. It has been commonly assumed that the insurance and financial risks are independent. However, certain risks from the same policy of insurance during the successive periods or from the different policies during the same period take place in a similar environment, so that these two risks in the same time horizon should be dependent on each other. Moreover, a large number of people start to securitize their insurance risk and export it to capital market using insurance-linked securities, such as the catastrophe bounds, to hedge against catastrophic risk. Thus the insurer who invests in the capital market would like to re-insure its insurance risk, which has interconnection between the insurance risk and financial risk.

Research also involves interesting dependent risk models. Goovaerts et al. (2005) studied the problem of approximating the tail probability of random weighted sum in the case when the losses have Pareto-like distributions and the discount factors are mutually dependent. Laeven et al. (2005) investigated asymptotic results for sums of dependent random variables, in the presence of heavy-tailedness conditions. Tang and Vernic (2007) established an exact asymptotic formula for the ruin probability, in which the financial risks constitute a stationary process with finite dimensional distributions of FGM copula. Chen and Ng (2007) obtained a formula for the ruin probability of the renewal risk model with constant interest force, in which the claims are pairwise negatively dependent and extended regularly varying tailed. Zhang et al. (2009) considered the problem of approximating the tail probability of randomly weighted sums with assumptions that claims has no bivariate upper tail dependence along with some other mild conditions. Weng et al. (2009) obtained asymptotic results for both finite and infinite ruin probabilities in a discrete time risk model with constant interest rates, in which the individual net losses have zero index of upper tail dependence. Recently, Chen (2011) worked out a asymptotic formula for the finite time ruin probability in a discrete time risk model with FGM dependent insurance risk and financial risks and Yang and Wang (2013) extended the result to a more general case that insurance risk and financial risk follow a bivariate Sarmanov distribution, among many others.

In the second part of this thesis, we study the ruin probabilities in a continuous-time dependent risk model. Thus far, the finite-ruin or infinite-time ruin probability in a continuous-time risk model has been widely investigated by many researchers, see Asmussen (1998), Klüppelberg and Stadtmüller (1998), Konstantinides et al. (2002) and Tang (2005, 2007), among many others. The standard renewal risk model was first proposed by Andersen (1957), assuming that the claim size and inter-arrival time were mutually independent, which is particularly paid attention to by many other researchers. In the actuarial literature, the probability of infinite-time ruin is defined to be the probability that the surplus falls below zero, which plays an unrealistic role. Compared with the research on the ordinary sense ruin probabilities, the absolute ruin probabilities have received less attention than they deserve. Moreover, the independent and identically distributed (i.i.d.) assumption is for mathematical convenience but far away from reality. It is obvious that the waiting time for a claim is dependent on the claim size. Various dependence structures were introduced to the renewal risk model by many researchers. For related discussions, we refer the reader to Chen and Ng (2007), Yang and Wang (2010), Yang et al. (2013).

Apart from the proper choice of risk models, it is required to assess ruin probabilities with sufficiently large risk reserve that is required by certain risk reserve regulations such as EU Solvency II. However, the computation of ruin probabilities for extreme risks with large initial capital is very difficult. It is not possible to calculate the exact value of the ruin probabilities because of the complex stochastic structures that we choose. The most common way is to do the

Monte Carlo simulations. Then computation issues arise. It is quite often that these simulations are not quite efficient when dealing with the extremely small ruin probabilities; see e.g. Tang and Yuan (2012). In addition, these simulations can not help us quantitatively understand tail behaviors of extreme losses and risks. In this thesis, we study asymptotic behaviors for extremes and risks as an alternative way to simulations. With asymptotic expressions, one can easily compute the results and it takes almost no time to get such results. Furthermore, the asymptotic expressions offer us insights that one can easily see the asymptotic behaviors of risks and how they increase comparing to some well-known quantity. For these reasons, we conduct the asymptotic tail probabilities of extreme and risks.

1.2 Structure of the Thesis

In this thesis, we study the asymptotic tail probabilities of quantities of interest in various risk models, from a discrete-time annuity-immediate risk model to a renewal risk model with constant force of interest. We investigate the impact of insurance and financial losses and how the regulations of investment affect them. We consider both light-tailed and heavy-tailed losses in our models. It turns out that the tail behavior of the loss distribution may vary in different types of insurance business.

Chapter 2 serves as a brief introduction to the theory and tools we use in this thesis.It includes notations and conventions, heavy-tailed and light-tailed distribution classes, extreme value theory, and dependence structure.

In Chapter 3, we study a discrete-time risk model. More concretely, consider a discrete-time annuity-immediate risk model in which the insurer is allowed to invest its wealth into a risk-free or a risky portfolio under a certain regulation. Then the insurer is said to be exposed to a stochastic economic environment that contains two kinds of risks, the insurance risk and financial risk. The former is the traditional liability risk caused by insurance losses, while the later is the asset risk resulting from investments. Within each period, the insurance risk is denoted by a real-valued random variable X, and the financial risk Y is a positive random variable fulfilling some constraints. Sarmanov distribution which is built from given marginals demonstrates the advantage of having a flexible structure which can be widely used in modelling data dependencies. Particularly, it contains the famous FGM distribution, whose correlation coefficients cannot exceed 1/3. N-evertheless, this restriction is not specific to the general Sarmanov distribution, see, e.g., Bairamov et al. (2001), Shubina and Lee (2004). We are interested in the ruin probability and the tail behaviour of maximum of the stochastic discounted values of aggregate net loss with Sarmanov or FGM dependent insurance and financial risks. We derive some asymptotic formulas for the finite-time ruin probability with lighted-tailed or moderately heavy-tailed insurance risk in a discrete-time risk model with a risk-free or risky investment. As an extension, we improve our results to the case of extreme risks, which arise from rare events, by combining some simulation with asymptotics, to compute the ruin probabilities more efficiently.

In Chapter 4, we consider a similar discrete-time risk model but a special case that insurance and financial risks following the least risky FGM dependence structure with heavy-tailed distribution. We follow the study of Chen (2011) that the finite-time ruin probability in a discrete-time risk model in which insurance and financial risks form a sequence of independent and identically distributed random pairs following a common bivariate FGM distribution function with parameter $-1 \leq \theta \leq 1$ governing the strength of dependence. For the subexponential case, when $-1 < \theta \leq 1$, a general asymptotic formula for the finite-time ruin probability was derived by Chen (2011). However, the derivation there is not valid for $\theta = -1$. In this thesis, we complete the study by extending Chen's work to $\theta = -1$ that the insurance risk and financial risk are negatively dependent. We refer to this situation as the least risky FGM dependent insurance risk and financial risk. It turns out that the finite-time ruin probability behaves essentially differently for $-1 < \theta \leq 1$ and $\theta = -1$.

In Chapter 5, we consider a renewal risk model with a constant premium and a constant force of interest rate, where the claim sizes and inter-arrival times follow certain dependence structures via some restriction on their copula function. The infinite-time absolute ruin probabilities are studied instead of the traditional infinite-time ruin probability with light-tailed or moderately heavy-tailed claim-size. Many popular distributions such as the lognormal-like, the Weibull-like, the exponential-like, and the generalized inverse Gaussian distributions are often applied to model the claim size distributions in ruin theory, see, for example, Asmussen (1998). These popular distributions are belong to the light-tailed or moderately heavy-tailed distribution classes, such as the generalized exponential class, convolution-equivalent class and the rapid-varying tailed class. Under the assumption that the distribution of the claim-size belongs to the intersection of the convolution-equivalent class and the rapid-varying tailed class, or a larger intersection class of O-subexponential distribution, the generalized exponential class and the rapid-varying tailed class, the infinite-time absolute ruin probabilities are derived.

1.2.1 Publications

Chen, Y.; Liu, J.; Liu, F. Ruin with insurance and financial risks following the least risky FGM dependence structure. Insurance: Mathematics and Economics, 62 (2015), 98–106.

Chen, Y.; Liu, J.; Yang, Y. Ruin with dependent insurance and financial risks in a discrete-time annuity-immediate risk model with a risk-free or risky investment (2015). Submitted.

Liu, J.; Yang, Y. Infinite-time absolute ruin in dependent renewal risk models with constant force of interest. Submitted.

1.2.2 Conference presentations

The workshop on New Direction in Risk Theory, Nanjing, China, November 2015, Contributed talk: "Ruin with dependent insurance and financial risks in a Discrete-time annuity-immediate risk model".

The 19th International Congress on Insurance: Mathematics and Economics, Liverpool, UK, June 2015, Contributed talk: "Infinite-time Absolute Ruin in Dependent Renewal Risk Model with Constant Force of Interest".

The 18th International Congress on Insurance: Mathematics and Economics, Shanghai, China, July 2014, Contributed talk: "Ruin with Dependent Insurance and Financial Risks From light tails to Heavy tails".

Conference in Actuarial Science and Finance on Samos, Greece, May 2014, Contributed talk: "Ruin with Insurance and Financial Risks Following a Special Dependence Structure".

The Second Workshop on Heavy-Tailed Models and Their Applications, Soochow University, China, March 2014, Contributed talk: "Ruin with Insurance and Financial Risks Following the Least Risky Dependence Structure".

Chapter 2

Preliminaries

In this chapter, some mathematical concepts and tools based on which the thesis is built are provided.

2.1 Notations and Conventions

Throughout this thesis, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f_1(\cdot)$ and $f_2(\cdot)$, we write $f_1(x) \lesssim f_2(x)$ or $f_2(x) \gtrsim f_1(x)$ if $\limsup f_1(x)/f_2(x) \leq 1$ and write $f_1(x) \sim f_2(x)$ if $\lim f_1(x)/f_2(x) = 1$. We also write $f_1(x) \asymp f_2(x)$ if $0 < \liminf f_1(x)/f_2(x) \leq \limsup f_1(x)/f_2(x) < \infty$. For convenience, we introduce a real function $a(\cdot)$ defined on $\mathbb{R}^+$ as an auxiliary function if it satisfies $0 \leq a(x) < x/2$, $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$.

Furthermore, for two positive bivariate functions $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$, we say the asymptotic relation $f_1(\cdot,\cdot) \sim f_2(\cdot,\cdot)$ holds uniformly for t and a non-empty set Δ if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \left| \frac{f_1(x,t)}{f_2(x,t)} - 1 \right| = 0.$$

Clearly, the asymptotic relation $f_1(\cdot,\cdot) \sim f_2(\cdot,\cdot)$ holds uniformly for $t \in \Delta$ if and only if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \frac{f_1(x,t)}{f_2(x,t)} \leq 1 \quad \text{and} \lim_{x\to\infty} \sup_{t\in\Delta} \frac{f_1(x,t)}{f_2(x,t)} \geq 1,$$

which mean that $f_1(x) \lesssim f_2(x)$ or $f_1(x) \gtrsim f_2(x)$ holds uniformly for $t \in \Delta$, respectively.

Let F denote the distribution function of a random variable X, whose tail is denoted by $\overline{F}(x) = 1 - F(x) = \Pr(X > x)$.

For two independent random variables X^* and Y^* with distributions F and G, respectively, denote by $F * G$ the distribution of the sum $X^* + Y^*$ and by $F \otimes G$ the distribution of the product X^*Y^*. The former is often referred to as the sum convolution of F and G, while the latter as the product convolution (or sometimes as the Mellin–Stieltjes convolution) of the distributions F and G. Clearly, if F is supported on $\mathbb{R}$ and G on $\mathbb{R}$, then the convolution of F and G is defined as

$$F * G(x) = \int_{-\infty}^{\infty} F(x-y)G(\mathrm{d}y) = \int_{-\infty}^{\infty} G(x-y)F(\mathrm{d}y),$$

which is the distribution function of the sum $X+Y$. In particular, we write F^{0*} as a distribution degenerate at 0, $F^{1*}=F$, and $F^{n*}=F^{(n-1)*}*F$ for $n=2,3,....$ Moreover, if F is supported on $\mathbb{R}$ and G on $\mathbb{R}_+$, then

$$\overline{F \otimes G}(x) = \int_0^{\infty} \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y), \qquad x > 0. \tag{2.1.1}$$

2.2 Heavy-tailed and light-tailed distribution classes

2.2.1 Heavy-tailed distribution classes and related

In this thesis, we follow the style of Embrechts et al. (1997) to categorize heavy-tailed distribution, although there are many criteria to define heavy-tailed distribution functions in the literature. That is, a distribution function F on $\mathbb{R}$ is said to belong to the (right) heavy tailed distribution class $\mathcal{K}$, if it holds for every $\epsilon > 0$ that

$$\int_0^{\infty} \mathrm{e}^{\epsilon x} \mathrm{d}F(x) = \infty.$$

All distribution functions not in $\mathcal{K}$ are known as light-tailed distributions.

In actuarial mathematics, distributions of risks or claim-size are often assumed to belong to some subclass of the heavy-tailed distribution class $\mathcal{K}$. Next we introduce some distribution classes that will be mainly used, most being subclasses of the class $\mathcal{K}$.

A distribution function F on $\mathbb{R}$ is said to be long tailed, written as $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(x+y) \sim \overline{F}(x) \tag{2.2.1}$$

holds for some (or, equivalently, for all) $y \neq 0$. For $F \in \mathcal{L}$, automatically there is some real function $l(\cdot)$ with $0 < l(x) \leq x/2$ and $l(x) \uparrow \infty$ such that relation (2.2.1) holds uniformly for $y \in [x-l(x), x+l(x)]$; that is,

$$\lim_{x\to\infty} \sup_{x-l(x)\leq y\leq x+l(x)} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| = 0.$$

We then introduce the class of subexponential distribution functions, which contains a lot of important distributions such as lognormal, heavy-tailed Weibull and Pareto distributions.

A distribution function F on $\mathbb{R}^+ = [0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x),$$

where F^{2*} denotes the two-fold convolution of F. More generally, a distribution function F on $\mathbb{R}$ is still said to be subexponential if the distribution function $F_+(x) = F(x)1_{(x\geq 0)}$ is subexponential. It is well known that $\mathcal{S} \subset \mathcal{L}$; see, for example, Lemma 1.3.5(a) of Embrechts et al. (1997). Subexponential distribution functions follow the principle of a single big jump which underlies the probabilistic behaviour of sums of independent subexponential distribution random variables. For example, for real-valued i.i.d. random variables $X_1, ..., X_n$ following a subexponential distribution, we have

$$\Pr\left(\sum_{i=1}^{n} X_i > x\right) \sim n\overline{F}(x) \sim \Pr\left(\max_{1\leq i\leq n} X_i > x\right).$$

This feature explicates why subexponential distributions have become popular in modelling heavy-tailed phenomena in insurance and finance. Let F denote a distribution function and f denote its density function. We list the following well-known subexponential distributions (see Table 1.2.6 of Embrechts et al. (1997)):

- Benktander-type I: for $\alpha > 0$, $\beta > 0$,

$$\overline{F}(x) = \left(1 + \frac{2\beta}{\alpha}\ln x\right)\exp\{-\beta(\ln x)^2 - (\alpha+1)\ln x\}, x > 1;$$

- Benktander-type II: for $\alpha > 0$, $0 < \beta < 1$,

$$\overline{F}(x) = \mathrm{e}^{\alpha/\beta}x^{-(1-\beta)}\exp\left\{-\frac{\alpha x^{\beta}}{\beta}\right\}, x > 1;$$

- Burr: for $\alpha > 0$, $\kappa > 0$, $\tau > 0$,

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x^{\tau}}\right)^{\alpha}, x > 0;$$

- Loggamma: for $\alpha > 0$, $\beta > 0$,

$$f(x) = \frac{\alpha^{\beta}}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}, x > 1$$

- Lognormal: for $-\infty < \mu < \infty$ and $\sigma > 0$,

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, x > 0;$$

- Pareto: for $\alpha > 0$, $\kappa > 0$,

$$\overline{F} = \left(\frac{\kappa}{\kappa + x} \right)^{\alpha}, x > 0;$$

- Weibull: for $a > 0$, $0 < b < 1$,

$$\overline{F}(x) = \exp\left\{-ax^b\right\}, x > 0.$$

A distribution function F on $\mathbb{R}$ is said to be dominatedly-varying tailed, written as $F \in \mathcal{D}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(xy) = O\left(\overline{F}(x)\right)$$

holds for some (or, equivalently, for all) $0 < y < 1$. The intersection $\mathcal{L} \cap \mathcal{D}$ forms a useful subclass of $\mathcal{S}$; see Proposition 1.4.4(a) of Embrechts et al. (1997).

The intersection $\mathcal{L} \cap \mathcal{D}$ covers the class $\mathcal{C}$ of distributions with a consistently-varying tail. By definition, for a distribution function F on $\mathbb{R}$, we write $F \in \mathcal{C}$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Clearly, for $F \in \mathcal{C}$ it holds for every $o(x)$ function that $\overline{F}(x + o(x)) \sim \overline{F}(x)$.

A slightly smaller is the class $\mathcal{R}$ of distributions with a regularly-varying tail. By definition, for a distribution function F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $0 \leq \alpha < \infty$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and the relation

$$\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$$

holds for all $y > 0$, and we write $\mathcal{R}$ the union of $\mathcal{R}_{-\alpha}$ over $0 \leq \alpha < \infty$.

In summary, we have

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

Moreover, a distribution function F on $\mathbb{R}$ is said to be rapidly-varying tailed, written as $F \in \mathcal{R}_{-\infty}$, if the relation

$$\overline{F}(xy) = o\left(\overline{F}(x)\right)$$

holds for all $y > 1$. This is a very broad class containing both heavy-tailed and light-tailed distributions.

Matuszewska Indices

For a distribution F with $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$, its upper and lower Matuszewska indices are defined as

$$M^*(F) = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\} \quad \text{and} \quad M_*(F) = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\},$$

respectively, where $\overline{F}_*(y) = \liminf \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup \overline{F}(xy)/\overline{F}(x)$. It is clear that $F \in \mathcal{D}$ if and only if $0 \le M^*(F) < \infty$, while if $F \in \mathcal{R}_{-\alpha}$ for $0 \le \alpha \le \infty$ then $M^*(F) = M_*(F) = \alpha$.

2.2.2 Light-tailed Distribution Classes

A important subclass of $\mathcal{R}_{-\infty}$ is the generalized exponential class $\mathcal{L}(\gamma)$ with $\gamma > 0$, as defined below.

A distribution F on $\mathbb{R}^+$ is said to be convolution-equivalent, denoted by $F \in \mathcal{S}(\gamma)$ for some $\gamma \ge 0$ if

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = \mathrm{e}^{\gamma y} \tag{2.2.2}$$

for every real number y and the limit

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) \tag{2.2.3}$$

exists and is finite. A larger class $\mathcal{L}(\gamma)$ is defined by relation (2.2.2) alone. If $\gamma = 0$, the classes $\mathcal{L}(0) = \mathcal{L}$ and $\mathcal{S}(0) = \mathcal{S}$ are defined in Section 2.2.1. A large amount of distributions in the class $\mathcal{S}(0)$ such as the lognormal and the Weibull distributions also belong to the class $\mathcal{R}_{-\infty}$. We remark that if $\gamma > 0$, then all distributions in $\mathcal{L}(\gamma)$ are light-tailed.

There are some easy-verified statements which can also be found in Tang and Tsitsiashvili (2004).

(1) For a r.v. for any random variable X with its distribution belonging to the class $\mathcal{S}(\gamma)$ with $\gamma \ge 0$, the distribution of cX with any positive constant c belongs to the class $\mathcal{S}(\gamma/c)$;

(2) For a r.v. for two distributions $F_1 \in \mathcal{L}(\gamma_1)$ and $F_2 \in \mathcal{L}(\gamma_2)$ with some $0 \le \gamma_1 \le \gamma_2 < \infty$, we have $\overline{F}_2(x) = o(\overline{F}_1(x))$;

(3) For a r.v. for two distribution F_1 and F_2 satisfying $\overline{F}_1(x) \sim c\overline{F}_2(x)$ for some positive constant c, then F_1 is said belong to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ with $\gamma \ge 0$ whenever F_2 belongs to this class;

(4) For a r.v. For two distributions, $F_1 \in \mathcal{L}(\gamma)$ and $F_2 \in \mathcal{L}(\gamma)$, satisfying

$$0 < \liminf \overline{F_1}(x)/\overline{F_2}(x) \le \limsup F_1(x)/\overline{F_2}(x) < \infty,$$

it is known that $F_1 \in \mathcal{S}(\gamma)$ if and only if $F_2 \in \mathcal{S}(\gamma)$, see Klüppelberg (1988).

A distribution class wider than the one of all convolution-equivalent distributions is the class of O-subexponential distributions, which was firstly introduced by Shimura and Watanabe (2005). A distribution F is said to belong the class $\mathcal{OS}$, if $\overline{F}(x) > 0$ for $x \in \mathbb{R}$ and

$$\overline{F^{2*}}(x) = O\left(\overline{F}(x)\right). \tag{2.2.4}$$

Klüppelberg and Villasenor (1991) presented examples to show that the inclusion $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma) \cap \mathcal{OS}$ is strict. Moreover, Lin and Wang (2012), Leslie (1989) have constructed some new distributions which belong to $\mathcal{L}(\gamma) \cap \mathcal{OS}$ but not to $\mathcal{S}(\gamma)$ for $\gamma > 0$ and for $\gamma = 0$, respectively.

2.3 Extreme Value Theory in Insurance and Finance

The prevalence of rare events such as earthquake, flood, windstorm, or terrorism which are accompanied by disastrous economic and social consequence, is so-called Black-Swan events that make today's world far different from decades ago. Apart from the mutual exterior event, a number of highly publicised catastrophic incidents involving barings, orange county, daiwa bank, and long term capital management, have made today's financial systems more sophisticated. These extremes and risks have increased awareness of the need to quantify probabilities of large losses, and for helping risk management systems to control such rare events. There is a much longer history of using Extreme Value Theory in the insurance industry, which provides the approach to characterise the tail behaviour of the distribution without trying the analysis down to a parametric family fitted to the whole distribution. It was first derived heuristically by Fisher and Tippett (1928), which classifies distributions according to their maximum domain of attraction. The distribution function F is said to belong to the max-domain of attraction of a univariate extreme value distribution function F_0, written $F \in \mathrm{MDA}(F_0)$, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - F_0(x)| = 0 \tag{2.3.1}$$

holds for some suitable normalising constants $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$. For more details on univariate max-domains of attraction, see e.g., Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), De Haan and Ferreira (2006), or Resnick (2008). A single generalised extreme value distribution was first proposed by von Miss (1936) of the form

$$F_{0\xi}(x) = \exp\left\{-(1+\xi x)_+^{-1/\xi}\right\}$$

where $y_+ = \max\{y, 0\}$ and the right hand side is interpreted as $\exp\{-\mathrm{e}^{-x}\}$ when $\xi = 0$. The regions $\xi = 1/\alpha > 0$, $\xi = 0$ and $\xi = -1/\alpha < 0$ correspond to the Fréchet , Gumbel and Weibull cases, respectively.

The Fréchet distribution has the cumulative distribution function

$$\Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\}, \qquad \alpha, x > 0$$

A distribution function F belongs to MDA(Φ_α) if only if $\overline{F}(\cdot)$ is regularly varying at infinity with index $-\alpha$, that is,

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

holds for all $y > 0$; see Theorem 3.3.7 of Embrechts et al. (1997). We list the following distributions of MDA(Φ_α) from Table 3.4.2 of Embrechts et al.(1997).

- Burr: for $\alpha > 0$, $\kappa > 0$, $\tau > 0$,

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x^\tau}\right)^\alpha, x > 0;$$

- Cauchy: for $x \in \mathbb{R}$,

$$f(x) = \left(\pi(1 + x^2)\right)^{-1}, x \in \mathbb{R};$$

- F-distribution: for $d_1 > 0$ and $d_2 > 0$,

$$f(x) = \frac{1}{B(d_1/2, d_2/2)} \left(\frac{d_1}{d_2}\right)^{d_1/2} x^{\frac{d_1}{2}-1} \left(1 + \frac{d_1}{d_2}x\right)^{-\frac{d_1+d_2}{2}}, x \geq 0;$$

- Loggamma: for $\alpha > 0$, $\beta > 0$,

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1} x^{-\alpha-1}, x > 1;$$

- Pareto: for $\alpha > 0$, $\kappa > 0$,

$$\overline{F} = \left(\frac{\kappa}{\kappa + x}\right)^\alpha, x > 0;$$

- Student's t: for $v > 0$,

$$f(x) = \frac{\Gamma\left((v+1)/2\right)}{\sqrt{v\pi}\Gamma(v/2)} \left(1 + \frac{x^2}{v}\right)^{-(v+1)/2}, x \in \mathbb{R}.$$

The Gumbel distribution has the cumulative distribution function

$$\Lambda(x) = \exp\left\{-\mathrm{e}^{-x}\right\}, \quad x \in \mathbb{R}.$$

A distribution function F with an upper endpoint $\hat{x} = \sup\{x \in \mathbb{R} : F(x) < 1\} \leq \infty$ belongs to MDA(Λ) if and only if the relation

$$\lim_{x \uparrow \hat{x}} \frac{\overline{F}(x + b(x)y)}{\overline{F}(x)} = \mathrm{e}^{-y}$$

holds for some positive auxiliary function $b(\cdot)$ and all $y \in \mathbb{R}$, where the auxiliary function $b(\cdot)$ can be chosen to be the mean excess loss function:

$$b(x) = \mathrm{E}(X - x | X > x).$$

Moreover, every distribution F from MDA(Λ) with an infinite upper endpoint also has a rapidly varying tail (see Embrechts et al. (1997), page 148). We provide some examples of MDA(Λ) from Table 3.4.2 of Embrechts et al.(1997):

- Benktander-type I: for $\alpha > 0$, $\beta > 0$,

$$\overline{F}(x) = \left(1 + \frac{2\beta}{\alpha} \ln x\right) \exp\{-\beta(\ln x)^2 - (\alpha + 1)\ln x\}, x > 1;$$

- Benktander-type II: for $\alpha > 0$, $0 < \beta < 1$,

$$\overline{F}(x) = \mathrm{e}^{\alpha/\beta} x^{-(1-\beta)} \exp\left\{-\frac{\alpha x^{\beta}}{\beta}\right\}, x > 1;$$

- Gamma: for $\alpha > 0$ and $\beta > 0$,

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} \mathrm{e}^{-\beta x}, x > 0;$$

- Lognormal: for $-\infty < \mu < \infty$ and $\sigma > 0$,

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, x > 0;$$

- Weibull-like: for $K > 0$, $\alpha > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$

$$\overline{F}(x) \sim K x^{\gamma} \exp\left\{-\alpha x^{\beta}\right\}.$$

The Weibull distribution is in the form of

$$\Psi_{\alpha}(x) = \exp\left\{-|x|^{\alpha}\right\}, \quad \alpha > 0, x \leq 0.$$

Furthermore, a distribution function F belongs to MDA(Ψ_α) if only if its endpoint $\hat{x}$ is finite and the relation

$$\lim_{x\to\infty} \frac{\overline{F}(\hat{x}-y/x)}{\overline{F}(\hat{x}-1/x)} = y^\alpha$$

holds for all $y > 0$; see Theorem 3.3.12 of Embrechts et al. (1997). However, MDA(Ψ_α) can only be applied to model the bounded risk variables. The following examples are for MDA(Ψ_α):

- Beta: for $a > 0$ and $b > 0$,

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, 0 < x < 1;$$

- Uniform: on (a, b) for $0 < a < b$.

2.4 Dependence structure

Recently, a new trend of research has been to introduce various dependence structures to the risk model. In this direction, we refer the reader to Goovaerts et al. (2005), Tang (2006a), Zhang et al. (2009), Weng et al. (2009), Yi et al. (2011), Chen (2011) and Yang and Wang (2013), among many others.

2.4.1 Copulas

It is very important to model dependence structures in insurance and finance. The copula plays a significant role in measuring dependence. We refer the reader to Chapter 5 of McNeil et al. (2005) or the monograph Nelsen (2006) for comprehensive applications of copulas, and see also Frees and Valdez (1998) for more details in actuarial applications.

A copula is the joint distribution function that couple the multivariate distribution to standard uniform marginal distributions taking values in $[0, 1]$. By Sklar's theorem (Sklar (1959)), let F denote a joint distribution function with margin distribution functions $F_1, ..., F_n$, there exists a copula C: $[0, 1]^n \to [0, 1]$ such that for all $x_i \in \mathbb{R}, i = 1, ..., n$,

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)).$$

If the marginal distribution is continuous, then copula is unique; otherwise, it is unique only on $\prod_{i=1}^n Ran(F_i)$, where $Ran(F_i)$ denotes the range of F_i. More details can be found in Section 5.1 of McNeil et al.(2005). On the other hand, copulas draw the information of dependence from a joint distribution function.

There are some commonly-used copulas such as t-copulas, FGM copulas, and Archimedean copulas.

In the rest of this section, we introduce some dependence structure that mainly used in this thesis.

A Bivariate Sarmanov Distribution

Sarmanov's family of multivariate distributions presents the advantage of having a flexible structure that is widely used to model a vast range of data dependencies. A bivarite Sarmanov distribution was firstly introduced by Sarmanov (1966) and then applied in physics by Cohen (1984) under a more general form. Lee (1996) offered a multivariate version and also discussed several applications in medicine afterwards. A more general class of bivariate distributions which includes Sarmanovs distribution was introduced by Bairamov et al. (2001). Recently, a new trend of the study is to incorporate Sarmanovs distribution in various models. Schweidel et al. (2008) used the Sarmonov family of bivariate distributions to capture the relationship between acquisition and retention. Hernández-Bastida et al. (2009) focused on the collective and Bayes net premiums for the aggregate amount of claims under a compound model with Sarmanov dependence between the risk profiles. Danaher and Smith (2011) demonstrated how copula models can be constructed and used in a variety of marketing applications. Hernández-Bastida and Fernández-Sánchez (2012) developed a Sarmanov-Lee family with gamma and beta marginal distributions to apply this family in a calculating Bayes premium problem. Pelican and Vernic (2013) studied maximum-likelihood procedures for estimating Sarmanovs distribution parameters for two different models. Particularly, Vernic (2015) derived exact expressions for sums Sarmanov distributed random variables, which are useful in solving, e.g., financial and actuarial problems.

Sarmanov distribution contains the popular FGM distribution. whose correlation coefficients cannot excess 1/3. Nevertheless, this restriction is not specific to the general Sarmanov distribution, see, e.g., Bairamov et al. (2001), Shubina and Lee (2004). Tang and Vernic (2006a) derive the tail asymptotics for the product of two random variables X and Y following a bivariate FGM distribution and the distribution of X is regularly varying or rapidly-varying tailed. A more general dependence structure is studied by Yang and Wang (2013), in which they assume that (X, Y) jointly follows a bivariate Sarmanov distribution of the form

$$Pr(X \in \mathrm{d}x, Y \in \mathrm{d}y) = (1 + \theta\phi_1(x)\phi_2(y))F(\mathrm{d}x)G(\mathrm{d}y), \quad x \in \mathbb{R}, y \geq 0, \quad (2.4.1)$$

where F and G are corresponding marginal distribution functions, ϕ_1 and ϕ_2 are kernels, and the parameter θ is a real constant, such that

$$E\phi_1(X) = E\phi_2(Y) = 0, \tag{2.4.2}$$

and

$$1 + \theta\phi_1(x)\phi_2(t) \geq 0 \quad \text{for all } x \in D_x, y \in D_y, \tag{2.4.3}$$

where $D_x = \{x \in \mathbb{R} : \Pr(X \in (x-\delta, x+\delta)) > 0 \text{ for all } \delta > 0\}$ and $D_y = \{y \in \mathbb{R} : \Pr(Y \in (y-\delta, y+\delta)) > 0 \text{ for all } \delta > 0\}$. For more details on multivariate Sarmanov distribution, see Lee (1996) and Kotz et al. (2000). An advantage of the form (2.4.1) is that it unifies both continuous and discrete Sarmanov distributions. Note that if $\theta = 0$ or $\phi_1(x) \equiv 0, x \in D_x$ or $\phi_2(y) \equiv 0, y \in D_y$, then X and Y are independent. Therefore, if $\theta \neq 0$, and the kernels ϕ_1 and ϕ_2 are not identical to 0 in D_X and D_Y, we say that (X, Y) follows a proper bivariate Sarmanov distribution. There are some common choices for kernels ϕ_1 and ϕ_2:

(a) $\phi_1(x) = 1 - 2F(x)$ and $\phi_2(y) = 1 - 2G(y)$ for all $x \in D_X$ and $y \in D_Y$, leading to the well-known Farlie-Gumbel-Morgenstern distribution (2.4.8);

(b) $\phi_1(x) = (\mathrm{e}^{-x} - c_1)1_{\{x \geq 0\}}$ with $c_1 = E\mathrm{e}^{-X}1_{\{X \geq 0\}}/\Pr(X \geq 0)$ and $\phi_2(y) = \mathrm{e}^{-y} - E\mathrm{e}^{-Y}$ for all $x \in D_X$ and $y \in D_Y$, here, 1_A is the indicator function of an event A;

(c) $\phi_1(x) = x^\alpha$ and $\phi_2(y) = y^\alpha - EY^\alpha$ for some $\alpha > 0$ and all $x \in D_X$ and $y \in D_Y$.

The following lemma comes from Wang and Yang (2013), which claims that the kernels for any proper bivariate Sarmanov distribution are bounded.

Lemma 2.4.1 *Assume that (X, Y) follows a proper bivariate Sarmanov distribution of the form (2.4.1). Then there exist two positive constants b_1 and b_2 such that $|\phi_1(x)| \leq b_1$ for all $x \in D_X$ and $|\phi_2(y)| \leq b_2$ for all $y \in D_Y$.*

Hence, by Lemma 2.4.1, there exist two constant $b_1 > 1$ and $b_2 > 1$ such that $|\phi_1(x)| \leq b_1 - 1$, $|\phi_2(y)| \leq b_2 - 1$ for all $x \in D_X$ and $y \in D_Y$. It is obvious that $d_1 = \lim_{x \uparrow \hat{x}} \phi_1(x) < b_1$ with X's upper endpoint $\hat{x}$. As in Yang and Wang (2013), introduce two independent r.v.s $\tilde{X}^*$ and $\tilde{Y}^*$, which are also independent of X^*,Y^*, with distributions $\tilde{F}$ and $\tilde{G}$, respectively, defined by

$$\tilde{F}(\mathrm{d}x) = \left(1 - \frac{\phi_1(x)}{b_1}\right) F(\mathrm{d}x),$$

and

$$\tilde{G}(\mathrm{d}y) = \left(1 - \frac{\phi_2(y)}{b_2}\right) G(\mathrm{d}y), \quad x \in D_X, y \in D_Y, \tag{2.4.4}$$

here, X^* and Y^* are two independent r.v.s with distributions F and G, respectively.

Note that Lemma 2.4.1 motivates the following assumptions on the kernel functions,

$$\lim_{x\uparrow\hat{x}} \phi_1(x) = d_1 < \infty \quad \text{and} \quad \lim_{\mathrm{y}\uparrow\hat{\mathrm{y}}} \phi_2(\mathrm{y}) = \mathrm{d}_2 < \infty, \tag{2.4.5}$$

with $\hat{x}$ and $\hat{y}$ the upper endpoints of the distributions F and G, respectively. This implies that for any $b_1 > d_1$ and $b_2 > d_2$,

$$\overline{\tilde{F}}(x) \sim \left(1 - \frac{b_1}{d_1}\right) \overline{F}(x), \qquad x \uparrow \hat{x}, \tag{2.4.6}$$

and

$$\overline{\tilde{G}}(y) \sim \left(1 - \frac{b_2}{d_2}\right) \overline{G}(x), \qquad y \uparrow \hat{y}. \tag{2.4.7}$$

A Bivariate Farlie-Gumbel-Morgenstern (FGM) Distribution

A brivariate Sarmanov distribution leads to the popular FGM distribution when $\phi_1(x) = 1 - 2F(x)$ and $\phi_2(y) = 1 - 2G(y)$ are chosen with F on $\mathbb{R}$ and G on $\mathbb{R}^+$. Even if there are a large number of copula families, the FGM copula is chosen since it provides the advantage of being mathematically tractable as exemplified in Cossette et al. (2009). Although the FGM copula only governs light dependence, it covers some positive as well as negative dependence structures between two r.v.s, and includes the independence when $\theta = 0$. Moreover, the FGM copula is a Taylor approximation of order one of the Frank copula, the Ali-Milkhail-Haq copula and Plackett copula, see Nelsen (2006). A brivariate FGM distribution was firstly introduced by Morgenstern (1956) and then investigated by Gumbel (1960) for exponential marginal distributions. The subsequent generalization to the current form (2.4.8) is due to Farlie (1960). For more recent discussions on FGM distributions, The reader is referred to Huang and Kotz (1999), Drouet Mari and Kotz (2001), Bairamov and Kotz (2002), Amblard and Girard (2002), and RodrÍguez-Lallena and úbeda-Flores (2004), among many others. Recently, Cossette et al. (2008) offered an application of the generalized FGM distributions in actuarial science. Bargés et al. (2009) carried out the problem of insurance capital allocation by assuming that the insurance risks are exponentially distributed and joined by a multivariate FGM distribution. The asymptotic behaviour of the ruin probability $\psi(x; n)$ in (3.1.1) was studied by Chen (2011) for the case with FGM dependent insurance and financial risks. Concretely, assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with a generic random pair (X, Y) whose components are however dependent. A bivariate FGM distribution is used to dissolve the dependence of (X, Y), which is of the form

$$\Pi\left(x, y\right) = F(x)G(y)\left(1 + \theta\overline{F}(x)\overline{G}(y)\right) \tag{2.4.8}$$

where F on $\mathbb{R}$ and G on $\mathbb{R}^+$ are marginal distributions, and $\theta \in [-1, 1]$ is a parameter governing the strength of the dependence, which is usually used to control the correlation coefficient of X and Y. In the study of Chen (2011), a general asymptotic result for $\psi(x; n)$ was derived for the case $\theta \in (-1, 1]$ with the assumptions that F is subexponential, and G fulfills some constraints, in order that the product convolution of F and G is subexponential as well. However, the derivation there is not valid for the case $\theta = -1$. This difficulty has been solved by Chen et al. (2015), showing that the finite-time ruin probability behaves essentially differently for $-1 < \theta \leq 1$ and $\theta = -1$. In this thesis, we consider some results with $\theta \in [-1, 1]$ for the light-tailed or heavy-tailed cases under a certain regulation.

As in Chen et al. (2014), see also Yang et al. (2011), for a r.v. X, introduce two independent r.v.s $X^*_\vee$, identically distributed as $X^*_1 \vee X^*_2$, and $X^*_\wedge$, identically distributed as $X^*_1 \wedge X^*_2$, which are independent of all other sources of randomness, where X^*_1 and X^*_2 are two i.i.d. copies of X. Trivially, if X is distributed by F, then $X^*_\vee$ is distributed by F^2 and the tail of $X^*_\wedge$ is $\overline{F}^2$.

Part II

Ruin Probabilities in a Discrete Time Risk Model with Dependent Risks

Chapter 3

Ruin with Dependent Insurance and Financial Risks in a Discrete-time annuity-immediate Risk Model with a Risk-free or Risky investment [1]

3.1 Introduction

An abundance of relevant research in actuarial science involves sums of dependent random variables. For instance, one may consider a renewal risk model with dependent claim-sizes and claim arrival times, or a discrete-time risk model with dependent insurance risks and financial risks, or the value-at-risk of a stochastically discounted life annuity. Whatever kind of the risk model it is, in actuarial applications the tail distribution functions of the sums of some dependent r.v.s are particularly paid attention to. By use of asymptotic relations, the tail asymptotic behaviour of such sums can be derived in the case that the distribution of the increment is light-tailed or moderately heavy-tailed. The asymptotic estimate always performs well, while some other approximations might perform worse when the heavy-tailedness increases; see e.g., Section 4.3 of Tang and Yuan (2012).

A rich amount of literature is available on the use of asymptotics in a discrete-time risk model. Nyrhinen (1999) introduced a discrete-time model and established large-deviation type estimates for the ruin probabilities. As concluded by Norberg (1999), an insurer who invests his wealth in a financial market is exposed to two kinds of risks, the insurance risk and financial risk. The former one is the traditional liability risk caused by insurance claims while the later is the asset risk

[1]This chapter is based on Chen, Liu and Yang (2015).

related to risky investments. Tang and Tsitsiashvili (2003, 2004) investigated the finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments under the assumption that the claim-size r.v.s are i.i.d.. It is obvious that the i.i.d. assumption is far unrealistic, for example, some certain risks from the same policy of insurance during the successive periods or from the different policies during the same period take place in a similar environment. Therefore, these risks in the same time horizon should be dependent on each other, which are dominated by the same economic factors (such as regional, national or global economic growth), or impacted by a mutual exterior event (such as earthquake, flood, windstorm, forest fire, or terrorism). These rare events have increased awareness of the necessity to investigate ruin probabilities of large losses, and for risk management systems to get rid of such events. Embrechts et al. (1997) provided an exhaustive study of modelling extreme events via Extreme Value Theory (EVT) to both insurance and financial risk management. Using EVT to characterize the tail behaviour in either light-tailed or heavy-tailed case exposed to an extreme environment can make the asymptotic result more exact, see e.g. Tang et al. (2011) and Hashorva et al. (2010).

Specifically, consider a discrete-time annuity-immediate insurance risk model with insurance and financial risks in an investment portfolio as in Nyrhinen (1990) and Tang and Tsitsiashvili (2003). Within each period i, the total premium income of an insurance company is denoted by A_i and the total claim amount plus other daily costs is denoted by B_i. Both A_i and B_i are non-negative random variables. We assume that the premiums are received at end of each period. Suppose that the insurer positions himself in a stochastic economic environment. To be specific, such a stochastic economic environment is referred to the financial market constituted by a risk-free investment with a constant interest rate $r > 0$ and a risky investment with a stochastic periodic return rate $R_i \geq -1$ during each period i, which leads to an overall stochastic accumulation factor Z_i over each period i. Usually, in the beginning of each period i, the insurer invests a proportion $\pi \in [0, 1]$ of his/her current wealth in stock and keeps the remaining in the bond. Thus, with the initial wealth $W_0 = x$, the current wealth of insurer at time n satisfies

$$W_n = x \prod_{j=1}^{n} Z_j + \sum_{i=1}^{n} (A_i - B_i) \prod_{j=i+1}^{n} Z_j,$$

where the stochastic accumulation factor Z_i is

$$Z_i = ((1 - \pi)(1 + r) + \pi(1 + R_i)).$$

Denote by a real-valued r.v. $X_i = B_i - A_i$, $i \in \mathbb{N}$, the net loss (the total amount of claims less premiums) within each period i, while the overall stochastic discount factor (the reciprocal of the stochastic accumulation factor) is denoted by a positive r.v. Y_i. In the terminology of Tang and Tsitsiashvili (2003, 2004), $\{X_i, i \in \mathbb{N}\}$ and $\{Y_i, i \in \mathbb{N}\}$ are called the insurance risks and financial risks, respectively. In practice, under certain financial regulation, Y_i takes value in the interval $(0, \hat{y})$ with $\hat{y} = [1 + (1 - \pi)r]^{-1}$ in a risk-free investment strategy (i.e. $R_i \geq 0$) or with $\hat{y} = (1 - \pi)^{-1}(1 + r)^{-1}$ involving a risky investment strategy (i.e. $R_i > -1$). In this framework, the probability of ruin with finite time is defined

$$\psi(x; n) = \Pr\left(\max_{1 \leq m \leq n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x\right), \quad n \in \mathbb{N}. \tag{3.1.1}$$

In insurance and finance applications, there are two contrary phenomena: accumulating and discounting that we are often confronted with. The sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}, \tag{3.1.2}$$

represents the stochastic present value of aggregate net losses up to time n, which are closely related to stochastic recurrence equations. Apart form the discrete-time risk model, there are many other examples leading sum-product stochastic structure. Theobald and Price (1984) presented the impact of nontrading effects upon the measured means, variances, and autocovariances of the returns on an index' within a sum-product stochastic framework. Brealey and Myers (1988) offered an introduction on this sum-product stochastic structure within the realm of finance. Gerber (1990), Section 2.6, provided a brief introduction to perpetuities on stochastic recurrence equations from a life insurance point of view. Large deviations and ruin probabilities to stochastic recurrence equations with heavy-tailed innovations were investigated by Konstantinides and Mikosch (2004), in which they also claims that the stochastic recurrence equation approach has also proved useful for the estimation of GARCH and related models. Then Straumann and Mikosch (2006) studied the quasi-maximum-likelihood estimator in a general conditionally heteroscedastic time series model, in which they give conditions for existence and uniqueness of a strict stationary solution to the stochastic recurrence equation. The random wetting transition on the Cayley tree was studied by Monthus and Garel(2009), which contains the specific structure of a Kesten variable consisting in a sum of products of random variables. Blanchet et al. (2013) focused on rare-event simulation for stochastic recurrence equations with heavy-tailed innovations.

In this thesis, we aim to investigate the maximum of the stochastic discounted values of aggregate net losses up to time n, specified as

$$U_n = \max_{1 \leq m \leq n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}. \tag{3.1.3}$$

The insurer has to hold enough risk reserve x to avoid risk that one can not pay its future liabilities, to be consistent with certain risk reserve regulations such as EU Solvency II, which makes the research of ruin theory particularly relevant.

Inspired by the work of Tang and Tsitsiashvili (2003), there have been a large number of papers studying the asymptotic behaviour of the ruin probability of such a discrete-time risk model in the presence of heavy-tailed insurance and financial risks. However, there is an obvious disadvantage that heavy-tailedness restriction excludes many popular distributions such as the Weibull-like, the exponential-like, the lognormal-like and the generalized inverse Gaussian distributions, which have been widely used to model the risk or claim size distribution in ruin theory, e.g. see Asmussen (1998). Furthermore, the distribution of the product of two light-tailed r.v.s can also belong to the class of heavy-tailed distributions, e.g. see Tang (2008), Liu and Tang (2010), which is a basic element of modelling in applied areas. Since the tail behaviour of a certain complicated risk process can always be reduced to the tail behaviour of sums and products, the distribution of U_n in (3.1.3) can also belong to the class of heavy-tailed distributions, although each component is light-tailed. As remarked by Embrechts et al. (1997), ruin is mainly caused by one large claim, so, corresponding to this situation, it is almost dominated by one large product of a pair of insurance and financial risks. In such circumstance, the financial risk Y builds a bridge between light tails and heavy tails. Last but not least, light-tailed or moderately heavy-tailed distributions are broadly used to modelling rare events, which merit some further investigation.

In this thesis, we shall incorporate some dependence structures to this discrete-time risk model where the distribution of insurance risk X belongs to a class of light-tailed or moderately heavy-tailed distributions. At the same time, some estimators of the ultimate ruin probability are derived for the case where the insurance risk belongs to some selected class of heavy-tailed distributions. We assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with generic random pair (X, Y), whose components are however dependent. We firstly study a more general situation where the insurance and financial risks are dependent according to a bivariate Sarmanov distribution. Then a special case that the two risks follow a bivariate FGM distribution, will be taken into consideration. For both cases, we consider the situation that the insurer invests all his surplus into a risk-free asset, or invests partly into a risk-free asset but the remaining into a risky asset, of which the tail behaviours are essentially different in the light-tailed and moderately heavy-tailed cases. Finally, we present an extension for extreme risks through EVT to characterize the tail behaviour more exactly.

We are interested in the question how a regulation of investment affects the tail behaviour of the maximum of the stochastic discounted values of aggregate net loss. We are particularly interested in the issue how dependence structure affects the finite-time ruin probabilities. Section 3.2 contains the main results of this chapter, immediately following a section on an extension of extreme risks in insurance and finance. Section 3.3 performs some simulation studies to verify the approximate relationships in our main results. Finally, all of the proofs are given in Section 3.4.

3.2 Main Results

3.2.1 A bivariate Sarmanov distribution

We continue to use the notions and notation in Section 2.3. Furthermore, suppose that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with generic random pair (X, Y). The components of (X, Y) are dependent and follow a joint bivariate Sarmanov distribution (2.4.1). It is easy to see that the distribution H of XY satisfies

$$\begin{aligned}\overline{H}(x) &= \int_0^\infty \int_{x/v}^\infty (1+\theta\phi_1(u)\phi_2(v))F(\mathrm{d}u)G(\mathrm{d}v) \\ &= (1+\theta b_1 b_2)\Pr(X^*Y^* > x) - \theta b_1 b_2 \Pr(\tilde{X}^*Y^* > x) \\ &\quad -\theta b_1 b_2 \Pr(X^*\tilde{Y}^* > x) + \theta b_1 b_2 \Pr(\tilde{X}^*\tilde{Y}^* > x). \end{aligned} \tag{3.2.1}$$

Recall $U_n,\ n \in \mathbb{N}$, in (3.1.3). Alternatively, denote

$$V_n = \max_{1\le m\le n} \sum_{i=1}^m X_i \prod_{j=i}^m Y_j, \quad n \in \mathbb{N}. \tag{3.2.2}$$

Due to the i.i.d. assumption on the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$, V_n defined in (3.2.2) is identically distributed as U_n in (3.1.3), which fulfills the recursive formula

$$V_0 = 0, \quad V_{n+1} = (V_n + X_{n+1})_+ Y_{n+1}, \quad n \in \mathbb{N}.$$

Similarly to (3.2.1), we can write $\Pr(V_{n+1} > x)$ as

$$\begin{aligned} &\Pr(V_{n+1} > x) \\ =\ & (1+\theta b_1 b_2)\Pr\left((V_n + X^*)_+ Y^* > x\right) - \theta b_1 b_2 \Pr\left(\left(V_n + \tilde{X}^*\right)_+ Y^* > x\right) \\ & -\theta b_1 b_2 \Pr\left((V_n + X^*)_+ \tilde{Y}^* > x\right) + \theta b_1 b_2 \Pr\left(\left(V_n + \tilde{X}^*\right)_+ \tilde{Y}^* > x\right) \\ =\ & (1+\theta b_1 b_2) J_1 - \theta b_1 b_2 J_2 - \theta b_1 b_2 J_3 + \theta b_1 b_2 J_4. \end{aligned} \tag{3.2.3}$$

3.2.2 A bivariate FGM distribution

Assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with generic random pair (X, Y). However, the components of (X, Y) are dependent and follow a joint bivariate FGM distribution of the form (2.4.8). The general derivations will be mainly used in following proofs. Since the decomposition

$$\Pi = (1+\theta)\, FG - \theta F^2 G - \theta F G^2 + \theta F^2 G^2,$$

or, equivalently,

$$\Pi = (1+\theta)\, FG - \theta(1-\overline{F}^2)G - \theta F(1-\overline{G}^2) + \theta(1-\overline{F}^2)(1-\overline{G}^2), \qquad (3.2.4)$$

with the fact that $X_\wedge^*$ is tail distributed by $\overline{F}^2$ and $Y_\wedge^*$ by $\overline{G}^2$, it follows that

$$\begin{aligned} \overline{H}(x) &= (1+\theta)\Pr(X^*Y^* > x) - \theta \Pr(X^*Y_\wedge^* > x) \\ &\quad -\theta \Pr(X_\wedge^* Y^* > x) + \theta \Pr(X_\wedge^* Y_\wedge^* > x)\,. \end{aligned} \qquad (3.2.5)$$

Applying the decomposition in (5.2.3), we have

$$\begin{aligned} & \Pr(V_{n+1} > x) \\ =\ & (1+\theta)\Pr\big((V_n + X^*)_+ Y^* > x\big) - \theta \Pr\big((V_n + X_\wedge^*)_+ Y^* > x\big) \\ & -\theta \Pr\big((V_n + X^*)_+ Y_\wedge^* > x\big) + \theta \Pr\big((V_n + X_\wedge^*)_+ Y_\wedge^* > x\big) \\ =\ & (1+\theta)\, I_1 - \theta I_2 - \theta I_3 + \theta I_4. \end{aligned} \qquad (3.2.6)$$

3.2.3 Finite-time ruin with a risk-free investment

We recall that the financial risks $Y_i = ((1-\pi)(1+r) + \pi(1+R_i))^{-1},\ i \in \mathbb{N}$. In this subsection, we considered that the insurer as a risk-averse individual invests all his surplus into an almost risk-free portfolio including a fixed proportion $1-\pi$ of a risk-free asset with a constant interest rate r and a fixed proportion π of a risk-free bond with the non-negative stochastic returns $R_i \geq 0$, which indicates that Y_i takes value in the interval $(0, \hat{y})$ with $\hat{y} = [1 + (1-\pi)r]^{-1}$. This implies that $\hat{y} \in (0, 1]$.

In the first result, we consider a more general risk model with Sarmanov dependent insurance and financial risks.

Theorem 3.2.1 *Consider a discrete-time annuity-immediate risk model with a risk-free investment. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, and G with $\hat{y} = [1 + (1-\pi)r]^{-1}$ satisfying* (2.4.5)*. If $1 + \theta d_1 d_2 > 0$, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x; n) \sim \mathrm{E}e^{\gamma V_{n-1}} \Pr(X^* Y_r^* > x) \sim \mathrm{E}e^{\gamma V_{n-1}} \Pr(XY > x), \qquad (3.2.7)$$

where Y_r^* is distributed by G_r with $G_r(\mathrm{d}y) = (1+\theta d_1\phi_2(y))G(\mathrm{d}y)$, and independent of X^*.

Remark 3.2.1 *Note that one can extend (3.2.7) by directly applying Lemmas 3.4.4 and 3.4.3, which benefits the further numerical studies a lot, that is, $\psi(x;n) \sim (1+\theta d_1 d_2)\mathrm{E}e^{\gamma V_{n-1}} \Pr(X^*Y^* > x)$ holds for each $n \in \mathbb{N}$.*

Remark 3.2.2 *It seems that the above result for the Sarmanov dependence case is more general, which almost covers the FGM dependence with $\theta \in (-1,1]$. However, one needs to be aware that the right hand side of the first asymptotic relation in (3.2.7) may reduce to $o(\Pr(X^*Y^* > x))$. Once the distribution of the insurance risk X has a rapidly-varying tail and relation (2.4.5) is satisfied, there is a possibility that the limit in (3.2.7) can sometimes correspond to a probability measure degenerate to 0 (in this case, $1+\theta d_1 d_2 = 0$), see Remark 2.1 of Jiang and Tang (2011), which is overcomed by the assumption of a proper bivariate Sarmanov distribution. Actually, observe that in the FGM case, which is a special Sarmanov case, the FGM parameter $\theta = -1$ is excluded in Theorem 3.2.1.*

In the next result, we consider the pair of dependent insurance and financial risks following a bivariate FGM distribution with parameter θ in the whole interval $[-1,1]$.

Theorem 3.2.2 *In a discrete-time annuity-immediate risk model with a risk-free investment, assume that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X,Y) following a bivariate FGM distribution (2.4.8) with $\theta \in [-1,1]$. If $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$ and $\hat{y} = [1+(1-\pi)r]^{-1} > \frac{1}{2}$, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x;n) \sim \mathrm{E}e^{\gamma V_{n-1}} \Pr(X^*Y_\theta^* > x) \sim \mathrm{E}e^{\gamma V_{n-1}} \Pr(XY > x), \tag{3.2.8}$$

where Y_θ^*, independent of X^*, is distributed by G_θ with $G_\theta(y) = (1-\theta)G(y) + \theta G^2(y)$.

Remark 3.2.3 *Note that $\hat{y} = [1+(1-\pi)r]^{-1} > \frac{1}{2}$ is easily satisfied since the interest rate r is always below 1 in practice. In case of a better result for numerical studies, we can rewrite (3.2.8) by applying Lemmas 3.4.5 and 3.4.6 as $\psi(x;n) \sim (1+\theta(1-\hat{p}))\mathrm{E}e^{\gamma V_{n-1}} \Pr(X^*Y^* > x)$ for $\theta \in (-1,1]$ and $\psi(x;n) \sim \mathrm{E}e^{\gamma V_{n-1}} \Pr(X^*Y_\wedge^* > x)$ for $\theta = -1$. In this section, the risk-averse investor only invests all his wealth into a risk-free portfolio, so that there is no need to hedge the downside risks. Therefore, we reasonably assume that the insurance risk Y has no mass at its endpoint $\hat{y} = [1+(1-\pi)r]^{-1}$ in practice, that is, $\hat{p} = \Pr(Y = \hat{y}) = 0$, leading to $\psi(x;n) \sim (1+\theta)\mathrm{E}e^{\gamma V_{n-1}} \Pr(X^*Y^* > x)$ for $\theta \in (-1,1]$. However, we refer $\theta = -1$ as a least risky situation. This also explains why Theorem 3.2.1 can not fully cover Theorem 3.2.2.*

In the following results, we generalize the above two results to the case that the distribution of insurance risk belongs to a larger intersection class $\mathcal{OS}\cap\mathcal{R}_{-\infty}\cap\mathcal{L}(\gamma)$.

Theorem 3.2.3 *Consider a discrete-time annuity-immediate risk model with a risk-free investment. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F \in \mathcal{OS}\cap\mathcal{L}(\gamma)\cap\mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, and G with $\hat{y} = [1+(1-\pi)r]^{-1}$ satisfying* (2.4.5)*. If $1+\theta d_1 d_2 > 0$, then relation* (3.2.7) *holds for each $n \in \mathbb{N}$.*

For the special case of FGM distributions, Theorem 3.2.3 yields that

Corollary 3.2.1 *In a discrete-time annuity-immediate risk model with a risk-free investment, assume that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate FGM distribution (2.4.8) with $\theta \in (-1, 1]$. If $F \in \mathcal{OS}\cap\mathcal{L}(\gamma)\cap\mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, and $\hat{y} = [1 + (1-\pi)r]^{-1}$, then relation* (3.2.8) *holds for each $n \in \mathbb{N}$.*

3.2.4 Finite-time ruin with a moderately risky investment

In this subsection, we consider a more realistic case that some negative investment returns may be earned. The insurer keeps his wealth partly in a risk-free asset and invests the remaining into a risky asset, which roughly means $\pi \in (0, 1)$. It is reasonable that the financial risk Y is modelled by a positive and bounded r.v. Hence, assume that the underlying financial risk in the economic environment is $\bar{Y} \in (0, \infty)$. In the meantime, in order to hedge the downside risks, the insurer always considers to buy an option when he invests his surplus into a risky asset. Therefore, the financial risk is revised by such a strategy as

$$Y = \bar{Y}1_{\{0<\bar{Y}<\hat{y}\}} + \hat{y}1_{\{\hat{y}\leq\bar{Y}<\infty\}}$$

for some $\hat{y} > 1$. In this strategy, the insurance risk Y has a positive mass $\Pr(\hat{y} \leq \bar{Y} < \infty)$ at its endpoint $\hat{y}$. In particular, there is a economic regulation that the insurer can only partly invest his surplus into a risky asset as a matter of fact, which leads to a stochastic return rate $R_i \in [-1, \infty)$ with $\Pr(R_i = -1) = \hat{p} \geq 0$. Thus, the overall return rate $\tilde{R}_i$ is in the form of

$$\tilde{R}_i = (1-\pi)(1+r) + \pi(1+R_i) - 1.$$

However, a small proportion $\pi \leq r/(1 + r)$ with a low constant rate r invested in a risky asset indicates $\tilde{R}_i \geq 0$. This is equivalent to a risk-free investment strategy, which has already been discussed in the previous subsection. Therefore, in order to make it possible to obtain some negative overall investment returns, the proportion π above the level $r/(1+r)$ is required in this subsection. Hence, if the insurer losses all the money invested into the risky asset, then the default risk is positive. Clearly, the financial risks are described as $Y_i = ((1-\pi)(1+r) + \pi(1+R_i))^{-1}$, $i \in \mathbb{N}$, which are bounded from above by $\hat{y} = (1-\pi)^{-1}(1+r)^{-1}$ with $\Pr(Y = \hat{y}) = \hat{p} \geq 0$. Consequently, the financial risk Y has a finite endpoint $\hat{y} \in (1, \infty)$ and might have a positive mass.

Now we consider a more realistic case when the insurance and financial risks are Sarmanov dependent.

Theorem 3.2.4 *Consider a discrete-time annuity-immediate risk model with a moderately risky investment. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, and G, with a finite upper endpoint $\hat{y} = (1-\pi)^{-1}(1+r)^{-1} > 1$, satisfying* (2.4.5)*, respectively. If $1 + \theta d_1 d_2 > 0$, then it holds that for each $n \in \mathbb{N}$,*

$$\begin{aligned} \psi(x; n) &\sim B_{n-1}(\gamma) \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right) \\ &\sim (1+\theta d_1 d_2) B_{n-1}(\gamma) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x\right), \end{aligned} \tag{3.2.9}$$

with

$$B_{n-1}(\gamma) = \prod_{i=1}^{n-1} \mathrm{E}\left\{(1+\theta d_2 \phi_1(X^*))\mathrm{e}^{\gamma \hat{y}^{-i} X^*}\right\}. \tag{3.2.10}$$

Recall that the FGM copula is satisfied by the Sarmanov distribution as described in the previous subsection. However, the case $\theta = -1$ is still excluded in Theorem 3.2.4 by a similar discussion as in Remark 3.2.2. Next we consider an extension for the FGM case with parameter θ in the whole interval $[-1, 1]$.

Theorem 3.2.5 *In a discrete-time annuity-immediate risk model with a moderately risky investment, assume that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate FGM distribution (2.4.8) with $\theta \in [-1, 1]$. If $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, and G has a finite upper endpoint $\hat{y} = (1-\pi)^{-1}(1+r)^{-1} > 1$, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x; n) \sim C_{n-1}(\gamma) \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right), \tag{3.2.11}$$

with

$$
\begin{aligned}
C_{n-1}(\gamma) &= \prod_{i=1}^{n-1}\left((1+\theta-\theta\hat{p})\,\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}X^*}-\theta\,(1-\hat{p})\,\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}X^*_\wedge}\right)\\
&= \prod_{i=1}^{n-1}\mathrm{E}\left\{(1+\theta(1-\hat{p})(1-2F(X^*)))\mathrm{e}^{\gamma\hat{y}^{-i}X^*}\right\}. \qquad (3.2.12)
\end{aligned}
$$

Remark 3.2.4 *By Lemmas 3.4.5 and 3.4.6, we can further refine the probability on the right hand side of relation* (3.2.11) *as*

$$
\Pr\left(X_1\prod_{j=1}^{n}Y_j>x\right)\sim(1+\theta(1-\hat{p}))\Pr\left(X^*\prod_{j=1}^{n}Y_j^*>x\right),\qquad \theta\in(-1,1],
$$

or

$$
\Pr\left(X_1\prod_{j=1}^{n}Y_j>x\right)\sim\Pr\left(X^*Y^*_\wedge\prod_{j=2}^{n}Y_j^*>x\right),\qquad \theta=-1.
$$

Next, we generalize the above two results under the condition that the distribution of insurance risk belongs to a larger intersection.

Theorem 3.2.6 *In a discrete-time annuity-immediate risk model with a moderately risky investment, assume that (X_i,Y_i), $i\in\mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X,Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F\in\mathcal{L}(\gamma)\cap\mathcal{OS}\cap\mathcal{R}_{-\infty}$, and G, with a finite upper endpoint $\hat{y}=(1-\pi)^{-1}(1+r)^{-1}>1$, satisfying* (2.4.5)*, respectively. If $1+\theta d_1d_2>0$, then for each $n\in\mathbb{N}$, relation* (3.2.9) *holds.*

Specially, the following result is a direct corollary of Theorem 3.2.6.

Corollary 3.2.2 *In a discrete-time annuity-immediate risk model with a moderately risky investment, assume that (X_i,Y_i), $i\in\mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X,Y) following a bivariate FGM distribution (2.4.8) with $\theta\in(-1,1]$. If $F\in\mathcal{L}(\gamma)\cap\mathcal{OS}\cap\mathcal{R}_{-\infty}$ for some $\gamma\geq 0$, and G has a finite upper endpoint $\hat{y}>1$, then relation* (3.2.11) *holds for each $n\in\mathbb{N}$.*

3.2.5 Finite-time ruin with a most risky investment

Consider that the insurer as a risk-seeking investor invests all his surplus into a risky asset, which leads to a most risky case that the financial risk Y has an infinite upper endpoint. It is unwise that insurer exposes himself to a most dangerous environment. In such a case, the financial risk Y builds a bridge between light tails and heavy tails, which leads to the heavy-tailedness of the distribution of the maximum stochastic sum in (3.1.3), even if the insurance and financial risks are both light-tailed.

For simplicity, we assume that there exists an auxiliary function $a(\cdot)$ defined on $\mathbb{R}^+$ such that $a(x) \uparrow \infty$, $a(x)/x \downarrow 0$ and

$$\overline{G}(a(x)) = o(\overline{H^*}(x)), \tag{3.2.13}$$

where H^* is the distribution of the product of two independent r.v.s X^* and Y^*.

Theorem 3.2.7 *Consider a discrete-time annuity-immediate risk model with a most risky investment. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F \in \mathcal{L}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma > 0$, and G, with the upper endpoint $\hat{y} = \infty$, satisfying* (2.4.5)*. If $1 + \theta d_1 d_2 > 0$ and* (3.2.13) *is satisfied, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x; n) \sim \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right) \sim (1 + \theta d_1 d_2) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x\right). \tag{3.2.14}$$

Corollary 3.2.3 *In a discrete-time annuity-immediate risk model with a risky investment, assume that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate FGM distribution (2.4.8) with $\theta \in (-1, 1]$. If $F \in \mathcal{L}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma > 0$ and* (3.2.13) *is satisfied, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x; n) \sim \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right) \sim (1 + \theta) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x\right). \tag{3.2.15}$$

The following result compliments Corollary 3.2.3 for $\theta = -1$, and also provides an example showing that (3.2.13) holds automatically.

Theorem 3.2.8 *Consider a discrete-time annuity-immediate risk model with a risky investment. Assume that (X_i, Y_i), $i \in \mathbb{N}$, constitute a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate FGM distribution (2.4.8) with $\theta = -1$. If $F \in \mathcal{L}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma > 0$ and $\overline{F}(x) \sim c\overline{G}(x)$ for some $c > 0$, then it holds that for each $n \in \mathbb{N}$,*

$$\psi(x; n) \sim (1 + c^{-1}) \Pr\left(X_\wedge^* \prod_{j=1}^{n} Y_j^* > x\right). \tag{3.2.16}$$

Remark 3.2.5 *Note that the condition $\overline{F}(x) \sim c\overline{G}(x)$ indicates that the tail of the insurance risk is almost as heavy as that of the financial risk. In practice, it is hard to distinguish which one of the insurance risk and the financial risk dominates the other. The finite-time ruin probability can be mainly determined by any one of such two kinds of risks. Tang and Tsitsiashvili (2003) gave two examples illustrating as anticipated, the finite-time ruin probability is mainly determined by the financial risk. Recently, Li and Tang (2014) provided a further treatment that no dominating relationship exists between the insurance and financial risks.*

3.2.6 An extension: extreme risks in insurance and finance

Convolution equivalent distributions can be in two different maximum domains of attraction MDA(Φ_α) and MDA(Λ), but MDA(Ψ_α) is excluded, since all distributions in MDA(Ψ_α) have bounded supports to the right. All distributions in MDA(Φ_α) are subexponential, see Lemma 1.3.1 of Embrechts et al. (1997). Other convolution equivalent distributions may belong to MDA(Λ). A typical example in $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$ is the distribution with density function

$$f(x) \sim kx^{\beta}\mathrm{e}^{-x^{\alpha}}, \quad x \to \infty,$$

for some $k, \beta \in \mathbb{R}, \alpha \in (0, 1)$, like the heavy-tailed Weibull distributions. The distribution, whose probability density satisfies

$$f(x) \sim kx^{\beta-1}\mathrm{e}^{-\gamma x}, \quad x \to \infty,$$

for $\beta < 0$, belongs to the important subclass of $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$, like the generalized inverse Gaussian distribution, the normal inverse Gaussian distribution and the generalized hyperbolic distribution. For more details, see Cline (1986) and Goldie and Resnick (1988).

The use of EVT to characterize the tail behaviour is a fairly recent innovation, but there is a rich amount of investigations in the actuarial literature. Hashorva et al. (2010) presented a result for ruin in presence of risky investment with $F \in \mathrm{MDA}(\Phi_\alpha)$ or $F \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ under the assumption of complete independence, which is far unrealistic. Similar results can be found in Chen (2011) and Yang and Wang (2013) with dependent insurance and financial risks. In this section, we present a more general result when the insurance risk $F \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$.

Corollary 3.2.4 *Consider a discrete-time annuity-immediate risk model. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate Sarmanov distribution of the form* (2.4.1) *with marginal distributions $F \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ with an auxiliary function $b(\cdot)$, and $G \in \mathrm{MDA}(\Psi_\alpha)$ for some $\gamma \geq 0$ and $\alpha > 0$.*

(a) Under the conditions of Theorem 3.2.1, it holds that for each $n \in \mathbb{N}$,

$$\psi(x; n) \sim (1 + \theta d_1 d_2)\mathrm{E}\mathrm{e}^{\gamma V_{n-1}}\overline{F}\left(\frac{x}{\hat{y}}\right)\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^2}{x}b\left(\frac{x}{\hat{y}}\right)\right). \quad (3.2.17)$$

(b) Under the conditions of Theorem 3.2.4, it holds that for each $n \in \mathbb{N}$,

$$\psi(x; n) \sim B_{n-1}(\gamma)(1+\theta d_1 d_2))\overline{F}\left(\frac{x}{\hat{y}^n}\right)\left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{n+1}}{x}b\left(\frac{x}{\hat{y}^n}\right)\right)\right)^n. \quad (3.2.18)$$

Corollary 3.2.5 *Consider a discrete-time annuity-immediate risk model. Suppose that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) following a bivariate FGM distribution (2.4.8) with $\theta \in [-1, 1]$. Assume that $F \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ with an auxiliary function $b(\cdot)$, and $G \in \mathrm{MDA}(\Psi_\alpha)$ for some $\gamma \geq 0$ and $\alpha > 0$ with an upper endpoint $\hat{y} < \infty$.*

(a) Under the conditions of Theorem 3.2.2, it holds that for each $n \in \mathbb{N}$,

$$\psi(x; n) \sim (1+\theta)\mathrm{E}e^{\gamma V_{n-1}}\overline{F}\left(\frac{x}{\hat{y}}\right)\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^2}{x}b\left(\frac{x}{\hat{y}}\right)\right), \qquad (3.2.19)$$

or

$$\psi(x; n) \sim \mathrm{E}e^{\gamma V_{n-1}}\overline{F}\left(\frac{x}{\hat{y}}\right)\Gamma(\alpha+1)\overline{G}^2\left(\hat{y} - \frac{\hat{y}^2}{x}b\left(\frac{x}{\hat{y}}\right)\right), \qquad (3.2.20)$$

for $\theta \in (-1, 1]$ or $\theta = -1$, respectively.

(b) Under the conditions of Theorem 5.2.4, it holds that for each $n \in \mathbb{N}$,

$$\begin{aligned}\psi(x; n) \quad \sim \quad & C_{n-1}(\gamma)(1+\theta(1-\hat{p}))\overline{F}\left(\frac{x}{\hat{y}^n}\right) \\ & \left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{n+1}}{x}b\left(\frac{x}{\hat{y}^n}\right)\right)\right)^n, \end{aligned} \qquad (3.2.21)$$

or

$$\begin{aligned}\psi(x; n) \quad \sim \quad & C_{n-1}(\gamma)\overline{F}\left(\frac{x}{\hat{y}^n}\right)(\Gamma(\alpha+1))^n\left(\overline{G}\left(\hat{y} - \frac{\hat{y}^n}{x}b\left(\frac{x}{\hat{y}^{n-1}}\right)\right)\right)^{n-1} \\ & \overline{G}^2\left(\hat{y} - \frac{\hat{y}^2}{x}b\left(\frac{x}{\hat{y}}\right)\right), \end{aligned} \qquad (3.2.22)$$

for $\theta \in (-1, 1]$ or $\theta = -1$, respectively.

3.3 Numerical Studies

In this section, we conduct some numerical studies by using Matlab to estimate the ruin probability in (3.1.1). Because of the complex stochastic structures among underlying risk variables, it is not possible to obtain the exact value of the ruin probability $\psi(x; n)$ of (3.1.1) and an estimate is required. The most common way to estimate the ruin probability $\psi(x; n)$ is by using the crude Monte Carlo (CMC) simulation, which gives an estimate as

$$\psi_1(x; n) = \frac{1}{N}\sum_{k=1}^{N} 1_{\{L_k > x\}},$$

where $L_k, k = 1, ..., N$, are i.i.d. samples from

$$L = \max_{1 \le m \le n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j.$$

The CMC simulation is considered as one of the most powerful methods widely used in many computational problems in insurance and finance. However, the estimate $\psi_1(x; n)$ performs efficiently only when the true value of the ruin probability $\psi(x; n)$ is not too small, or say, the initial capital x is not too large. Nevertheless, the large initial capital x is required by certain risk reserve regulations such as EU Solvency II, which makes the value of $\psi(x; n)$ extremely small. For simulations of such a case, the extremely large sample size N is chosen to offset the negative effect of $\psi(x; n)$. As mentioned by Tang and Yuan (2012), for some $h > 0$ close to 0 and $0 < p < 1$ close to 1, in order to keep the simulated estimate $\psi_1(x; n)$ within $100h\%$ of the true value of the $\psi(x; n)$ with the probability not smaller than p, i.e.

$$\Pr\left(|\psi_1(x; n) - \psi(x; n)| \le h\psi(x; n)\right) \ge p,$$

the sample size N needed for the required accuracy is

$$N \ge \left(\frac{z_p}{h}\right)^2 \frac{1 - \psi(x; n)}{\psi(x; n)},$$

where $z_p = \Phi^{-1}((1 + p)/2)$ is the quantile of the standard normal distribution at $(1 + p)/2$. In this section, we choose $h = 0.05$ and $p = 0.95$. While for the extremely large sample size N to obtain the required accuracy, some big issues arise for both computational time and memory allocation. More details can be found in Asmussen and Glynn (2007). Therefore, when the CMC method breaks down for the large initial capital x, our asymptotic method might be an accurate estimate for $\psi(x; n)$ such that

$$\lim_{x \to \infty} \frac{\psi_2(x; n)}{\psi(x; n)} = 1,$$

here, our asymptotic estimate $\psi_2(x; n)$ is defined in (3.3.2) below.

Throughout the rest of this chapter, $|\psi_2(x; n)/\psi(x; n) - 1| \le 0.05$ is stratified when we work out the numerical studies to examine its accuracy by testing how close to 1 the ratio $\psi_2(x; n)/\psi_1(x; n)$ is by Matlab.

Starting from the case of heavy-tailed distributions, we assume that the insurance risk X follows the heavy-tailed Weibull distribution that belongs to the class of the subexponential distributions,

$$F(x) = 1 - \mathrm{e}^{-(x/a)^b}, \quad x \in \mathbb{R}, a > 0, 0 < b < 1,$$

and that the financial risk Y follows the uniform distribution with parameters c and d. Then the various parameters are set to be:

$$\begin{aligned}
&n = 4, \\
&r = 1.25\%, \\
&\pi = 0.8, \\
&\hat{p} = 0.05, \\
&\theta = 0.6 \\
&a = 1, b = 0.5, c = 0, d = \hat{y} = 4.9383.
\end{aligned}$$

The following algorithm (see Johnson(1986)) is used to generate r.v.s X and Y fulfilling the FGM copula:

a. Generate two independent uniform $(0,1)$ variables v_1, v_2;

b. Set $a_1 = 1 + \theta(1 - 2v_1)$, $a_2 = \sqrt{a_1^2 - 4(a_1 - 1)v_2}$;

c. Set $u_1 = v_1$, $u_2 = v_2/(a_1 + a_2)$;

d. Then (u_1, u_2) returns the outcome of two dependent variables following the FGM copula.

It is easy to calculate that $\hat{y} = 4.9383$ by relation $\hat{y} = (1-\pi)^{-1}(1+r)^{-1}$, which indicates that risky investments are taken into consideration. Thus, by Theorem 5.2.4 and Remark 3.2.4, the asymptotic formula for the heavy-tailed case $\gamma = 0$ can certainly be simplified to

$$\psi(x; n) \sim (1 + \theta(1 - \hat{p})) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x \right). \tag{3.3.1}$$

The probability on the right hand side of (3.3.2) is still estimated by the CMC method, that is,

$$\psi_2(x; n) = (1 + \theta(1 - \hat{p})) \frac{1}{N} \sum_{k=1}^{N} 1_{\{M_k > x\}}, \tag{3.3.2}$$

where $M_k, k = 1, ..., N$, are i.i.d. samples from

$$M = X^* \prod_{j=1}^{n} Y_j^*.$$

Both of the two estimates $\psi_1(x; n)$ and $\psi_2(x; n)$ have their advantage and disadvantage: the CMC simulated estimate $\psi_1(x; n)$ meets the accuracy for relative

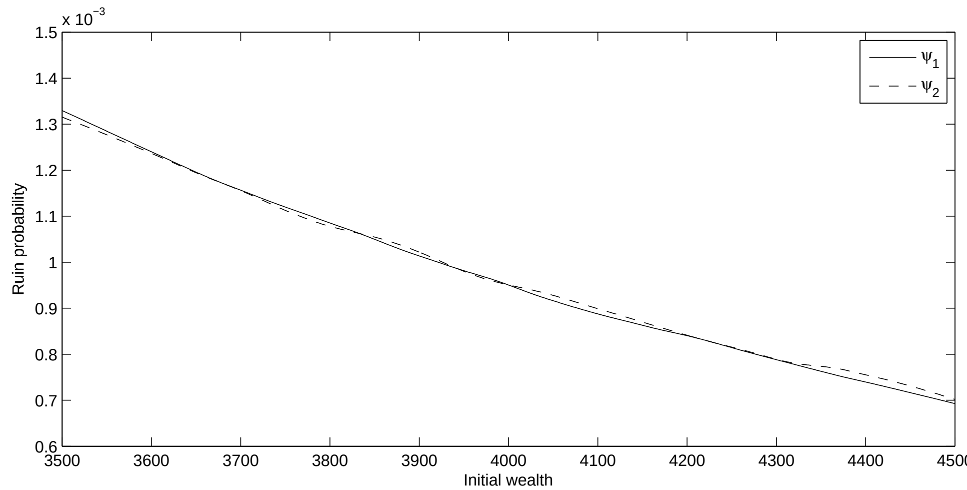

Figure 3.1: The values of $\psi_1(x; n)$ and $\psi_2(x; n)$ $(N = 10^6)$

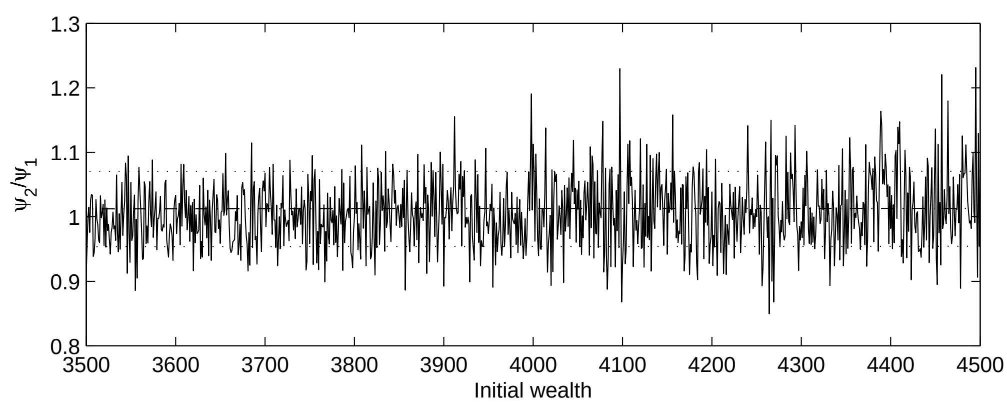

Figure 3.2: The ratio of $\psi_2(x; n)/\psi_1(x; n)$ $(N = 10^6)$

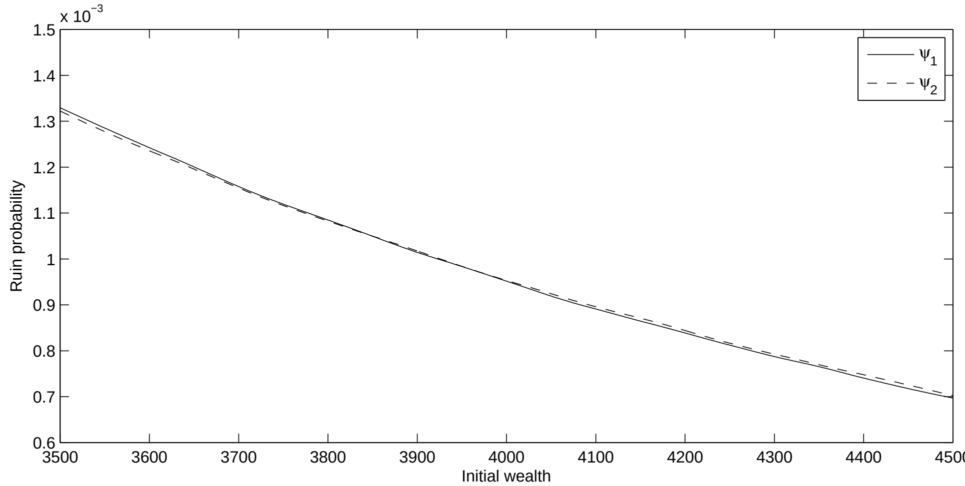

Figure 3.3: The values of $\psi_1(x;n)$ and $\psi_2(x;n)$ $(N = 10^7)$

large values of $\psi(x;n)$, while the asymptotic estimate $\psi_2(x;n)$ performs well for small values of $\psi(x;n)$.

The sample size is chosen as $N = 1,000,000$, then the values of $\psi_1(x;n)$ and $\psi_2(x;n)$ are shown in the Figures 3.1,3.2, in which we compare the two estimates on the left and present their ratio on the right. These figures show that, although $\psi_2(x;n)$ seems to converge to $\psi_1(x;n)$ as the initial wealth x increases on the left, the ratio $\psi_2(x;n)/\psi_1(x;n)$ fluctuates around 1 and fluctuates more as well. Without surprise, the larger the initial capital is, the smaller the ruin probability $\psi(x;n)$ becomes and the more fluctuation the ratio $\psi_2(x;n)/\psi_1(x;n)$ exhibits. Actually, this is due to the poor performance of CMC, since the sample size is not large enough. Hence, we repeat the simulation with the sample size N increasing from $1,000,000$ to $10,000,000$ and draw Figures 3.3,3.4. Then we observe a much improved convergence of ratio $\psi_2(x;n)/\psi_1(x;n)$. Furthermore, we compare the FGM parameter $\theta = 0.4,\ 0.5,\ 0.6$, with sample size $N = 10,000,000$ in the next Figures 3.5,3.6, in order to see how the parameter θ affects the convergence of the ratio. It is easy to see that the two estimates $\psi_1(x;n)$ and $\psi_2(x;n)$ increase as θ increases from 0.4 to 0.6. Moreover, for $\theta = 0.4$, the ratio $\psi_2(x;n)/\psi_1(x,n)$ fluctuates dramatically and it does not seem to converge to 1. One possible reason is that the ratio $\psi_2(x;n)/\psi_1(x;n)$ converges with respect to the large initial wealth. Another reason of accuracy might be due to the small value of the ruin probability $\psi(x;n)$, when the ratio $\psi_2(x;n)/\psi_1(x;n)$ converges to 1. The ratio $\psi_2(x;n)/\psi_1(x;n)$ converges to 1 well for both the cases $\theta = 0.5$ and $\theta = 0.6$, even if there is a little bit fluctuation that the ratio $\psi_2(x;n)/\psi_1(x;n)$ does not satisfy $|\psi_2(x;n)/\psi(x;n) - 1| \leq 0.05$ in the far right area.

Next, we assume that insurance risk X follows the light-tailed inverse Gaussian

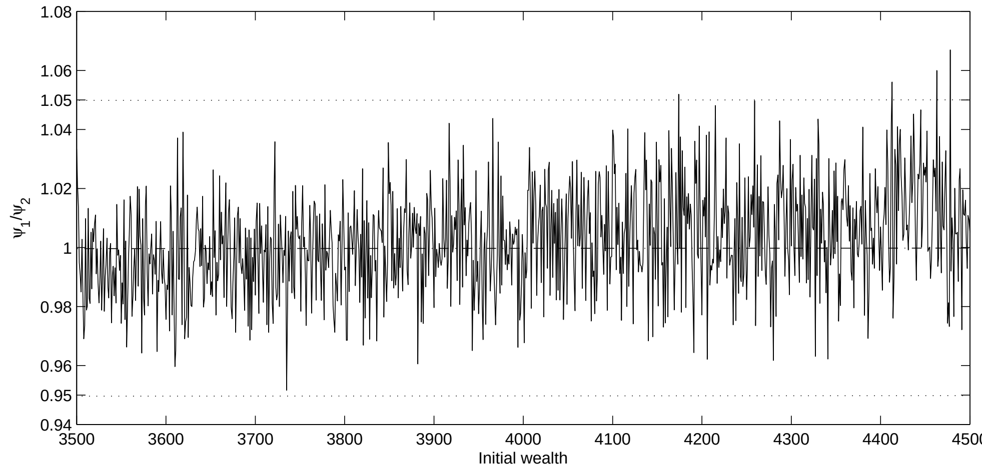

Figure 3.4: The ratio of $\psi_2(x;n)/\psi_1(x;n)$ $(N = 10^7)$

distribution, which belongs to the class $\mathcal{S}(\gamma)$ with $\gamma > 0$,

$$F(x) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left\{\frac{2\lambda}{\mu}\right\}\Phi\left(-\sqrt{\frac{\mu}{x}}\left(\frac{x}{\mu} + 1\right)\right), \tag{3.3.3}$$

for $x > 0$, where $\Phi(\cdot)$ is the standard normal distribution, $\mu > 0$ is the expectation, and $\lambda > 0$ is the shape parameter. As presented by the main theorem of Embrechts (1983), the inverse Gaussian distribution denoted by $IG(\mu, \lambda)$ belongs to the class $\mathcal{S}(\gamma)$ with $\gamma = \lambda/(2\mu^2)$. We still assume that the financial risk Y follows the uniform distribution with parameters c and d. For ease of computation, we set the parameters as follows:

$$\begin{aligned}
&n = 4,\\
&r = 1.25\%,\\
&\pi = 0.8,\\
&\hat{p} = 0.05,\\
&\theta = 0.8,\\
&\lambda = \mu = 1, c = 0, d = \hat{y} = 4.9383.
\end{aligned}$$

Therefore, by Theorem 5.2.4 and Remark 3.2.4, the asymptotic formula for the lighted-tailed case $\gamma > 0$ can be simplified to

$$\psi(x;n) \sim (1 + \theta(1 - \hat{p}))\, C_{n-1}(\gamma)\, \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x\right), \tag{3.3.4}$$

with $C_{n-1}(\gamma)$ defined in (3.2.12). Similarly to (3.3.2), the probability on the

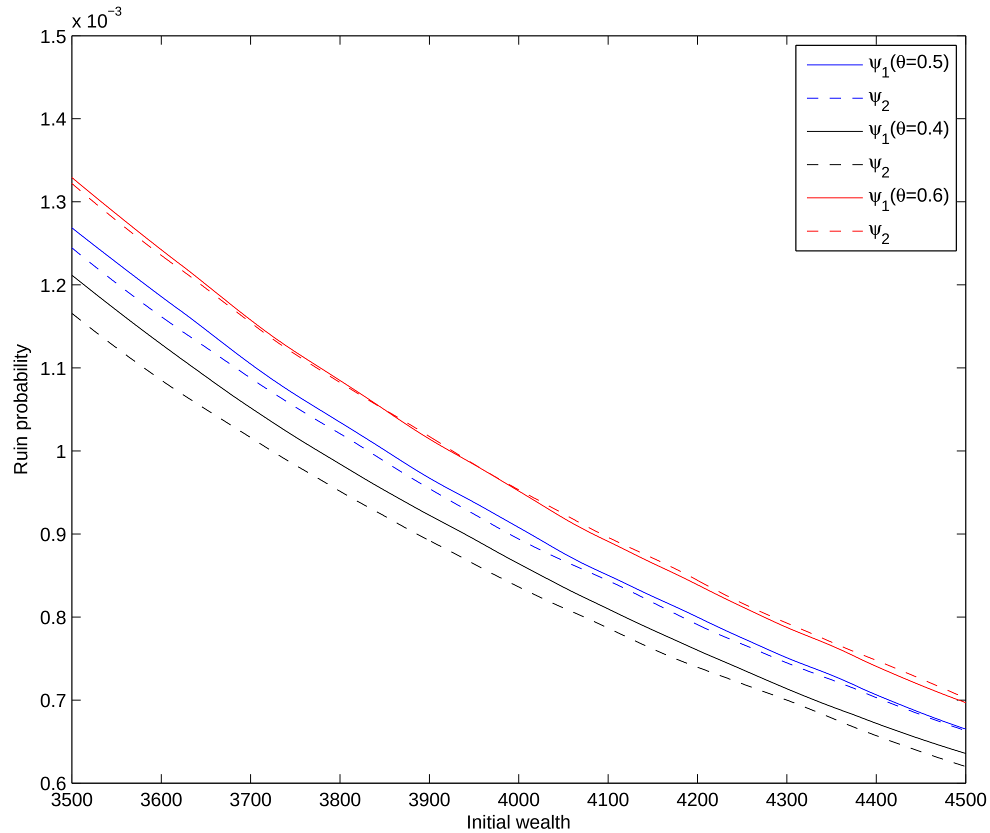

Figure 3.5: The values of $\psi_1(x;n)$ and $\psi_2(x;n)$ $(N = 10^7)$

right hand side of (3.3.4) is still estimated by the CMC method, that is,

$$\psi_2(x;n) = (1+\theta(1-\hat{p}))C_{n-1}(\gamma)\frac{1}{N}\sum_{k=1}^{N} 1_{\{M_k>x\}}. \tag{3.3.5}$$

For case of light-tailed distributions, we simulate with sample size $N = 10,000,000$ for both $\psi_1(x;n)$ and $\psi_2(x;n)$. We present the two estimates on the left and show the ratio on the right in Figures 3.7,3.8. In this case, the figures show that the ratio $\psi_2(x;n)/\psi_1(x;n)$ converges to 1 quickly as the initial wealth increases. After staying around 1 for a while, the ratio starts to fluctuate a lot and out of the accuracy we set in the far right area. As we mentioned before, the ruin probability decreases very fast as the initial wealth becomes large, leading to the bad performance of the CMC method, so that the ratio $\psi_2(x;n)/\psi_1(x;n)$ fluctuates dramatically. One way to improve this situation is that we repeat the simulation with sample size N increasing from $10,000,000$ to $100,000,000$, see Figures 3.9,3.10.

For both the light-tailed and heavy-tailed cases, the true value of $\psi(x;n)$

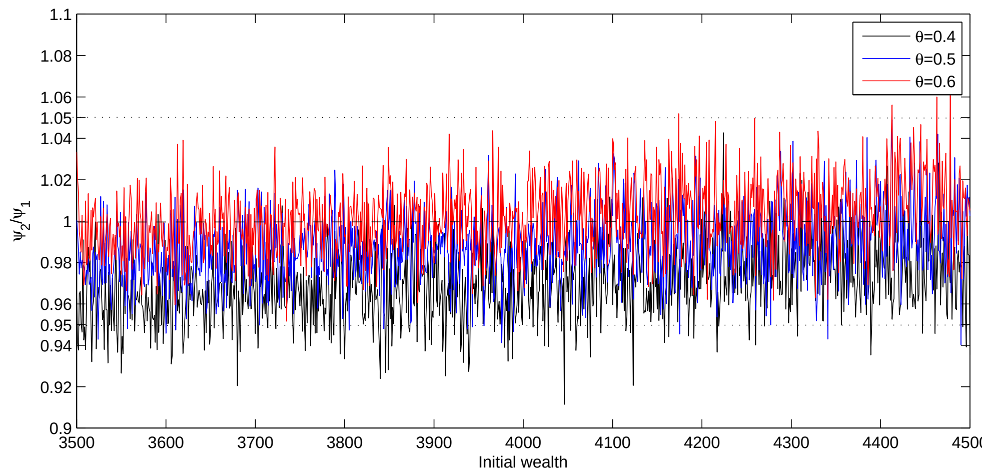

Figure 3.6: The ratio of $\psi_2(x;n)/\psi_1(x;n)$ $(N = 10^7)$

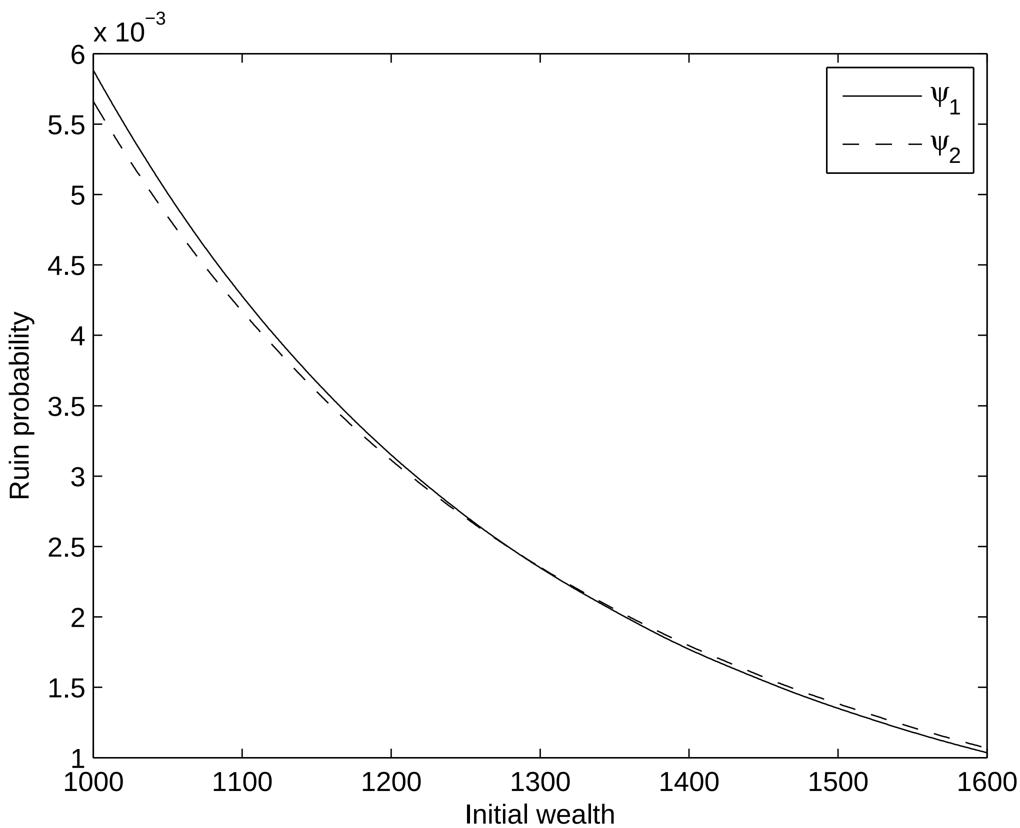

Figure 3.7: The values of $\psi_1(x;n)$ and $\psi_2(x;n)$ $(N = 10^7)$

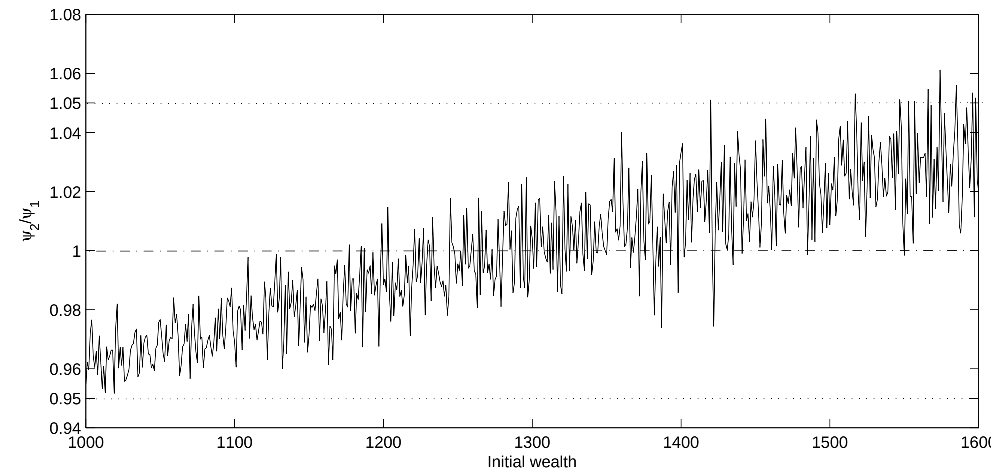

Figure 3.8: The ratio of $\psi_2(x;n)/\psi_1(x;n)$ $(N = 10^7)$

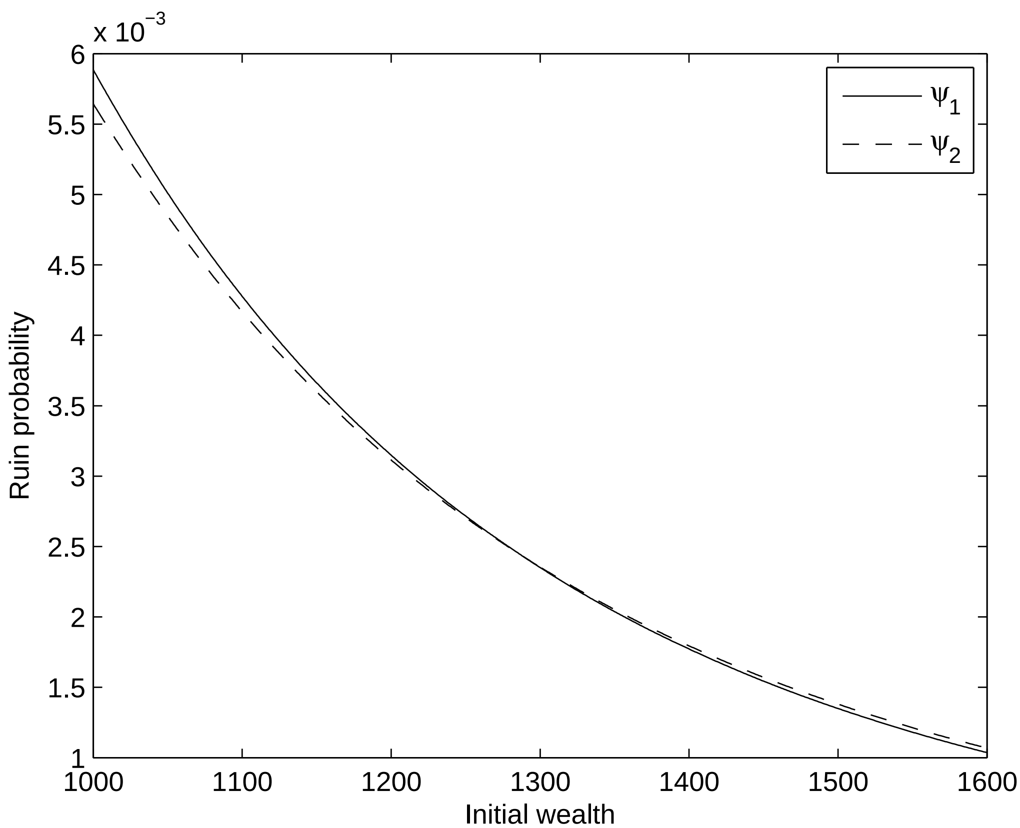

Figure 3.9: The values of $\psi_1(x;n)$ and $\psi_2(x;n)$ $(N = 10^8)$

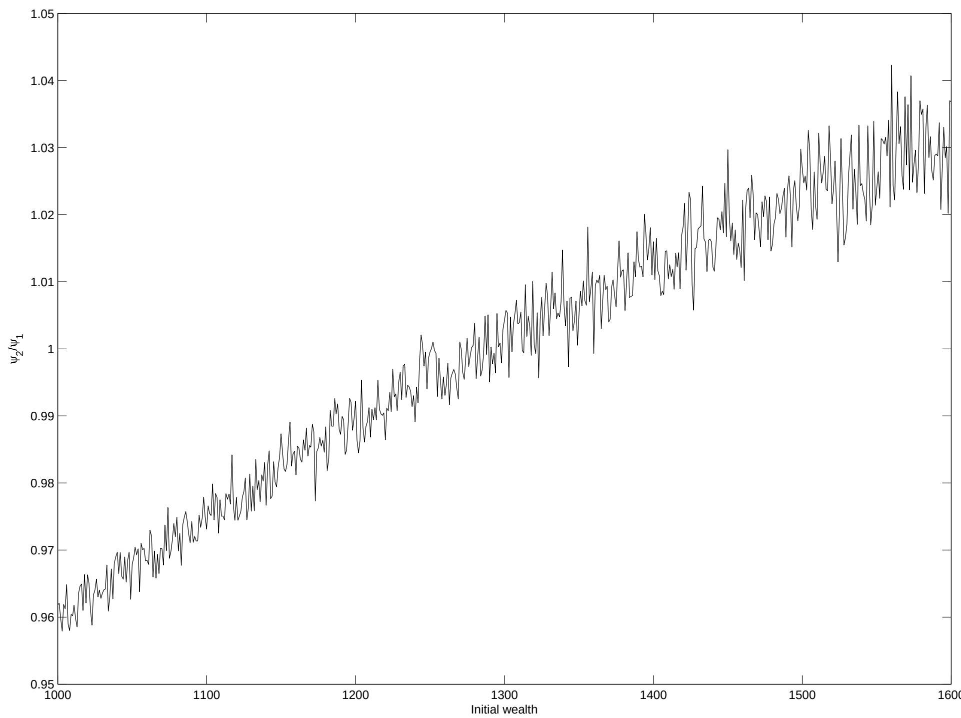

Figure 3.10: The ratio of $\psi_2(x;n)/\psi_1(x;n)$ ($N = 10^8$)

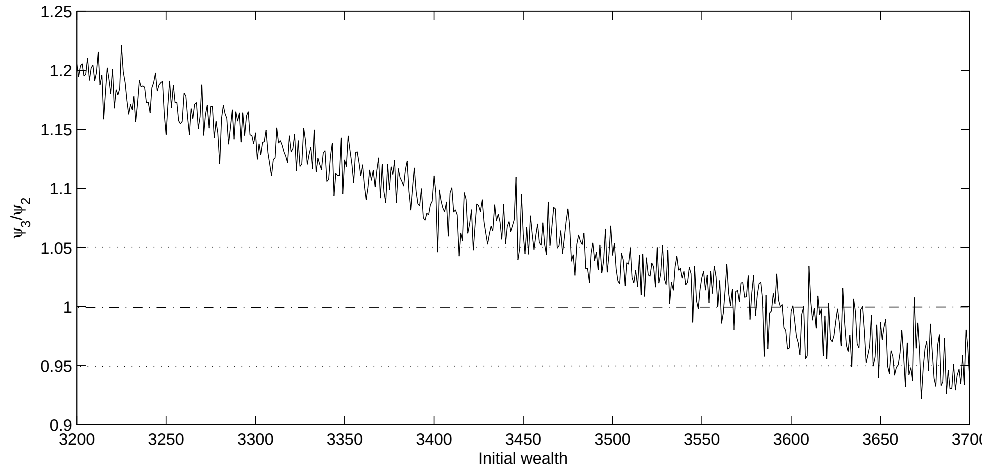

Figure 3.11: The ratio of $\psi_3(x;n)/\psi_2(x;n)$

becomes too small as the initial wealth x increases. An extremely large sample size N is needed to offset the negative effect of $\psi(x;n)$. However, for the simulation of such a scale, this will be a big issue for both computational time and memory allocation despite a little improvement of the $\psi_2(x;n)$. One might consider to use extreme value theorem to improve the asymptotic estimate $\psi_2(x;n)$, using an example of the lighted-tailed distribution. Under the conditions of Corollary 3.2.5 (b), if $\theta \in (-1,1]$, then (3.2.21) holds. By Theorem 3.3.12 of Embrechts et al. (1997), a possible choice for the function $b(x)$ is

$$b(x) = \int_x^\infty \frac{\overline{F}(t)}{\overline{F}(x)} \mathrm{d}t.$$

Thus, we have an improved asymptotic estimate

$$\psi_3(x;n) = C_{n-1}(\gamma)(1+\theta(1-\hat{p}))\overline{F}\left(\frac{x}{\hat{y}^n}\right)\left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{n+1}}{x} b\left(\frac{x}{\hat{y}^n}\right)\right)\right)^n, \tag{3.3.6}$$

which is the best estimate that one can desire for in rare-event simulation. A ratio test of $\psi_3(x;n)/\psi_2(x;n)$ is provided in Figure 3.11 with sample size $N = 100,000,000$. In practice, we can choose a reasonably large simple size N to fulfil the accuracy so that the ratio $\psi_2(x;n)/\psi_1(x;n)$ or $\psi_3(x;n)/\psi_1(x;n)$ starts to fall into the strip between 0.95 and 1.05 with such a value as the threshold u_0 or u_0'. One can define $\tilde{\psi}(x;n)$ as a globally accurate estimate for $\psi(x;n)$,

$$\tilde{\psi}(x;n) = \psi_1(x;n)1_{\{\psi_1(x;n)\geq u_0\}} + \psi_2(x;n)1_{\{\psi_1(x;n)<u_0\}},$$

or

$$\tilde{\psi}(x;n) = \psi_1(x;n)1_{\{\psi_1(x;n)\geq u_0'\}} + \psi_3(x;n)1_{\{\psi_1(x;n)<u_0'\}},$$

which is also called as the hybrid estimate for $\psi(x;n)$ in Tang and Yuan (2012). Hence, with this hybrid method, the insurer can make decision to optimize the portfolio easily.

3.4 Proofs

The following result is a restatement of an important result in Rogozin and Sgibnev (1999).

Lemma 3.4.1 *Let F,F_1, and F_2 be three distributions. If $F \in \mathcal{S}(\gamma)$ with $\gamma \geq 0$ and the limit*

$$\kappa_i = \lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}(x)}$$

exists and is finite for $i = 1, 2$, then it holds that

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = \kappa_1 \int_{-\infty}^{\infty} \mathrm{e}^{\gamma x} F_2(\mathrm{d}x) + \kappa_2 \int_{-\infty}^{\infty} \mathrm{e}^{\gamma x} F_1(\mathrm{d}x)$$

Motivated by Lemma 5.1 of Cai and Tang (2004), we obtain the following lemma.

Lemma 3.4.2 *Let $F \in \mathcal{S}(\gamma)$ be a distribution on $[0, \infty)$ with $\gamma \geq 0$. Then it holds that for any positive integer $k \geq 2$,*

$$\lim_{x\to\infty} \frac{\left(\overline{F}(x)\right)^k}{\overline{F}(kx)} = 0. \tag{3.4.1}$$

Proof. Let $X_i,\ i = 1, ..., k$, be i.i.d. r.v.s with common distribution F. Then, it is easy to see that for any $x > 0$,

$$\begin{aligned}\left(\sum_{i=1}^{k} X_i > kx\right) \supset & \left(\sum_{i=1}^{k} X_i > kx, X_i > x \text{ for only one } i = 1, ..., k\right) \cup \\ & (X_i > x \text{ for all } i = 1, ..., k)\,.\end{aligned}$$

For an arbitrary positive constant c and all $x > c$, it holds that

$$\begin{aligned}\Pr\left(\sum_{i=1}^{k} X_i > kx\right) &\geq k\Pr\left(\sum_{i=1}^{k} X_i > kx, X_1 > x, X_2 \leq x, ..., X_k \leq x\right) + \left(\overline{F}(x)\right)^k \\ &\geq k\Pr\left(\sum_{i=1}^{k} X_i > kx, X_2 \leq c, ..., X_k \leq c\right) + \left(\overline{F}(x)\right)^k \\ &= k \underset{0<x_j\leq c, j=2,...,k}{\int\cdots\int} \overline{F}\left(kx - \sum_{j=2}^{k} x_j\right) \prod_{j=2}^{k} F(\mathrm{d}x_j) + \left(\overline{F}(x)\right)^k \\ &\sim k\overline{F}(kx)\left(\mathrm{E}\mathrm{e}^{\gamma X} 1_{\{X\leq c\}}\right)^{k-1} + \left(\overline{F}(x)\right)^k.\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{x\to\infty}\frac{\left(\overline{F}(x)\right)^k}{\overline{F}(kx)} &\le \limsup_{x\to\infty}\frac{\Pr\left(\sum_{i=1}^k X_i>kx\right)}{\overline{F}(kx)}-k\left(\mathrm{E}\mathrm{e}^{\gamma X}1_{\{X\le c\}}\right)^{k-1}\\
&= k\left(\mathrm{E}\mathrm{e}^{\gamma X}\right)^{k-1}-k\left(\mathrm{E}\mathrm{e}^{\gamma X}1_{\{X\le c\}}\right)^{k-1}.
\end{aligned}$$

The desired result follows by the arbitrariness of c. ■

3.4.1 Proof of Theorem 3.2.1

Lemma 3.4.3 *Let X^*, Y^* and $\tilde{Y}^*$ be three independent r.v.s with distributions F, G and $\tilde{G}$, respectively. Assume that Y^* is nonnegative with its upper endpoint $0<\hat{y}\le\infty$ and $\tilde{G}$ is defined by* (2.4.4)*. If $F\in\mathcal{R}_{-\infty}$ and the second relation in* (2.4.5) *is satisfied, then*

$$\lim_{x\to\infty}\frac{\Pr(X^*\tilde{Y}^*>x)}{\Pr(X^*Y^*>x)}=1-\frac{d_2}{b_2}. \tag{3.4.2}$$

Proof. By $F\in\mathcal{R}_{-\infty}$ and Lemma 3.1 (ii) of Tang (2006), we have that for every $y_0\in(0,\hat{y})$,

$$\Pr(X^*\tilde{Y}^*>x)\sim\int_{x/\hat{y}}^{x/y_0}\overline{\tilde{G}}\left(\frac{x}{u}\right)F(\mathrm{d}u),$$

and

$$\Pr(X^*Y^*>x)\sim\int_{x/\hat{y}}^{x/y_0}\overline{G}\left(\frac{x}{u}\right)F(\mathrm{d}u).$$

Hence, we derive that

$$\frac{\Pr(X^*\tilde{Y}^*>x)}{\Pr(X^*Y^*>x)}\sim\frac{\int_{x/\hat{y}}^{x/y_0}\overline{\tilde{G}}(x/u)F(\mathrm{d}u)}{\int_{x/\hat{y}}^{x/y_0}\overline{G}(x/u)F(\mathrm{d}u)}=\frac{\int_{x/\hat{y}}^{x/y_0}\int_{x/u}^{\hat{y}}(1-\frac{\phi_2(u)}{b_2})G(\mathrm{d}u)F(\mathrm{d}u)}{\int_{x/\hat{y}}^{x/y_0}\overline{G}(x/u)F(\mathrm{d}u)},$$

which leads to

$$\inf_{y_0\le y<\hat{y}}\frac{\int_y^{\hat{y}}(1-\frac{\phi_2(u)}{b_2})G(\mathrm{d}u)}{\overline{G}(y)}\lesssim\frac{\Pr(X^*\tilde{Y}^*>x)}{\Pr(X^*Y^*>x)}\lesssim\sup_{y_0\le y<\hat{y}}\frac{\int_y^{\hat{y}}(1-\frac{\phi_2(u)}{b_2})G(\mathrm{d}u)}{\overline{G}(y)}.$$

For any $\epsilon>0$, there exists y_0 close enough to $\hat{y}$ such that for any $u\in[y_0,\hat{y}]$, by the second relation of (2.4.5),

$$(1-\epsilon)d_2\le\phi_2(u)\le(1+\epsilon)d_2,$$

which implies

$$(1-\epsilon)\left(1-\frac{d_2}{b_2}\right)\overline{G}(y)\le\int_y^{\hat{y}}(1-\frac{\phi_2(u)}{b_2})G(\mathrm{d}u)\le(1+\epsilon)\left(1-\frac{d_2}{b_2}\right)\overline{G}(y).$$

Letting $y_0\nearrow\hat{y}$, the desired result follows by the arbitrariness of ϵ. ■

Lemma 3.4.4 *Assume that (X,Y) follows a bivariate Sarmanov distribution of the form* (2.4.1). *If* (3.2.13), (2.4.5) *and* $1+\theta d_1 d_2 > 0$ *are satisfied, then*

$$\Pr(XY > x) \sim (1+\theta b_2 d_1)\Pr(X^*Y^* > x) - \theta b_2 d_1 \Pr(X^*\tilde{Y}^* > x) \tag{3.4.3}$$

or, equivalently,

$$\begin{aligned}\Pr(XY > x) &\sim \int_0^\infty (1+\theta d_1\phi_2(y))\,\overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y)\\ &= \Pr(X^*Y_r^* > x),\end{aligned}$$

where Y_r^* is defined in Theorem 3.2.1.

Proof. In terms of the function $a(\cdot)$ defined in (3.2.13), and by (2.4.6) we have that

$$\Pr(\tilde{X}^*Y^* > x) = \int_0^{a(x)} \overline{\tilde{F}}\left(\frac{x}{y}\right) G(\mathrm{d}y) + O\left(\overline{G}(a(x))\right) = \left(1-\frac{d_1}{b_1}+o(1)\right)\overline{H^*}(x). \tag{3.4.4}$$

Similarly, we can obtain that

$$\begin{aligned}\Pr(\tilde{X}^*\tilde{Y}^* > x) &= \int_0^{a(x)} \overline{\tilde{F}}\left(\frac{x}{y}\right)\tilde{G}(\mathrm{d}y) + O\left(\overline{\tilde{G}}\left(a(x)\right)\right)\\ &= \left(1-\frac{d_1}{b_1}+o(1)\right)\Pr(X^*\tilde{Y}^* > x) + o\left(\overline{H^*}(x)\right),\end{aligned} \tag{3.4.5}$$

where the last step holds due to

$$\overline{\tilde{G}}(a(x)) = \int_{a(x)}^\infty \left(1-\frac{\phi_2(u)}{b_2}\right) G(\mathrm{d}u) = O(\overline{G}(a(x))) = o(\overline{H^*}(x)).$$

Plugging the above (3.4.4) and (3.4.5) into relation (3.2.1), and by noting the second relation of (2.4.5) and $1+\theta d_1 d_2 > 0$, we derive that

$$\begin{aligned}\overline{H}(x) &= (1+\theta b_2 d_1 + o(1))\,\overline{H^*}(x) - (\theta b_2 d_1 + o(1))\Pr(X^*\tilde{Y}^* > x)\\ &= (1+o(1))\int_0^\infty (1+\theta d_1\phi_2(y))\,\overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y).\end{aligned}$$

It ends the proof of the lemma. ■

Now we turn to the proof of Theorem 3.2.1. For $n=1$, it follows that

$$\psi(x;1) = \Pr(V_1 > x) = \Pr(X_{1+}Y_1 > x) = \Pr(X_1Y_1 > x). \tag{3.4.6}$$

Note $\hat{y} = [1+(1-\pi)r]^{-1} < 1$, then relation (3.2.13) holds automatically. By applying Lemma 3.4.4, relation (3.2.7) holds for $n=1$. Arguing inductively, assume that (3.2.7) holds for n and we are going to prove it for $n+1$. Thus, by the inductive assumption, we have that

$$\begin{aligned}
\Pr(V_n > x) &\sim \mathrm{E}\mathrm{e}^{\gamma V_{n-1}} \int_0^1 (1+\theta d_1 \phi_2(y))\, \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) \\
&\leq \mathrm{E}\mathrm{e}^{\gamma V_{n-1}} (1+|\theta d_1| b_2) \int_0^1 \overline{F}\left(\frac{x}{y}\right) G(\mathrm{d}y) \\
&= o(\overline{F}(x)), \qquad (3.4.7)
\end{aligned}$$

where the last step holds due to $F \in \mathcal{R}_{-\infty}$. Recalling the decomposition relation (3.2.3), by Lemma 5.3.1 we derive that

$$\begin{aligned}
J_1(x) &= \int_0^1 \Pr\left(V_n + X^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\
&= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \int_0^1 \Pr\left(X^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\
&= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(X^* Y^* > x). \qquad (3.4.8)
\end{aligned}$$

Similarly, we have

$$J_3(x) = (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(X^* \tilde{Y}^* > x). \qquad (3.4.9)$$

For $J_2(x)$, by (2.4.6) and (3.4.7), $\Pr(V_n > x) = o(\overline{\tilde{F}}(x))$ holds. Then, by Lemma 5.3.1, we get that

$$\begin{aligned}
J_2(x) &= \int_0^1 \Pr\left(V_n + \tilde{X}^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\
&= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \int_0^1 \Pr\left(\tilde{X}^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\
&= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(\tilde{X}^* Y^* > x), \qquad (3.4.10)
\end{aligned}$$

and

$$J_4(x) = (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(\tilde{X}^* \tilde{Y}^* > x). \qquad (3.4.11)$$

Plugging (3.4.8)–(3.4.11) into (3.2.3), it follows that

$$\begin{aligned}
\Pr(V_{n+1} > x) &= (1+\theta b_2 d_1 + o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(X^* Y^* > x) - (\theta b_2 d_1 + o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(X^* \tilde{Y}^* > x) \\
&= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma V_n} \Pr(X^* Y_r^* > x).
\end{aligned}$$

This completes the proof.

3.4.2 Proof of Theorem 3.2.2

Lemma 3.4.5 *Assume that (X, Y) follows a bivariate FGM distribution (2.4.8) with marginal distributions F and G.*

(a) *In the case $\theta \in (-1, 1]$, if* (3.2.13) *holds for some auxiliary function $a(x)$ with $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$, then it holds that*

$$\Pr(XY > x) \sim (1+\theta) \Pr\left(X^* Y^* > x\right) - \theta \Pr\left(X^* Y^*_\wedge > x\right). \tag{3.4.12}$$

(b) *In the case $\theta = -1$, if $F \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$ and G has a finite upper endpoint $0 < \hat{y} < \infty$, then* (3.4.12) *holds.*

Proof. Recall the decomposition (4.3.2) in Section 3.2.

In the case $\theta \in (-1, 1]$, by $\overline{F_{X^*_\wedge}}(x) = (\overline{F}(x))^2$ and (3.2.13), we obtain that

$$\Pr(X^*_\wedge Y^* > x) = \int_0^{a(x)} \Pr\left(X^*_\wedge > \frac{x}{y}\right) G(\mathrm{d}y) + \overline{G}(a(x)) = o(1) \Pr(X^* Y^* > x).$$

Similarly, it follows that $\Pr(X^*_\wedge Y^*_\wedge > x) = o(1)(\Pr(X^* Y^*_\wedge > x) + \Pr(X^* Y^* > x))$. Plugging these two estimates into (4.3.2), we have that

$$\begin{aligned} \Pr(XY > x) &= (1+\theta) \Pr(X^* Y^* > x) - \theta \Pr(X^* Y^*_\wedge > x) \\ &\quad + o(1)(\Pr(X^* Y^* > x) + \Pr(X^* Y^*_\wedge > x)). \end{aligned} \tag{3.4.13}$$

By $0 \leq \Pr(X^* Y^*_\wedge > x) \leq \Pr(X^* Y^* > x)$,

$$(1+\theta) \Pr(X^* Y^* > x) - \theta \Pr(X^* Y^*_\wedge > x) \asymp \Pr(X^* Y^* > x),$$

which implies that

$$\begin{aligned} &\Pr(X^* Y^* > x) + \Pr(X^* Y^*_\wedge > x) \leq 2 \Pr(X^* Y^* > x) \\ &\asymp (1+\theta) \Pr(X^* Y^* > x) - \theta \Pr(X^* Y^*_\wedge > x). \end{aligned} \tag{3.4.14}$$

Thus, (3.4.13) and (3.4.14) lead to the desired (3.4.12).

In the case $\theta = -1$, relation (4.3.2) reduces to

$$\Pr(XY > x) = \Pr\left(X^* Y^*_\wedge > x\right) + \Pr\left(X^*_\wedge Y^* > x\right) - \Pr\left(X^*_\wedge Y^*_\wedge > x\right).$$

Without loss of generality, assume $\hat{y} = 1$. Clearly, by Lemma 3.4.2,

$$\frac{\Pr(X^*_\wedge Y^* > x)}{\Pr(X^* Y^*_\wedge > x)} \leq \frac{(\overline{F}(x))^2}{\Pr(X^* > 2x, Y^*_\wedge > 1/2)} = \frac{1}{(\overline{G}(1/2))^2} \frac{(\overline{F}(x))^2}{\overline{F}(2x)} \to 0,$$

which indicates that $\Pr\left(X^*_\wedge Y^* > x\right) = o(1) \Pr\left(X^* Y^*_\wedge > x\right)$. Similarly, $\Pr\left(X^*_\wedge Y^*_\wedge > x\right) = o(1) \Pr\left(X^* Y^*_\wedge > x\right)$. Hence, it follows that

$$\Pr(XY > x) \sim \Pr\left(X^* Y^*_\wedge > x\right).$$

The desired result (3.4.12) holds for all $\theta \in [-1, 1]$. ∎

Furthermore, this lemma provides the following insight. For $\theta \in [-1,1]$, introduce a positive r.v. Y_θ^*, independent of X^*, distributed by $G_\theta(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ as

$$G_\theta(y) = (1-\theta)G(y) + \theta G^2(y). \tag{3.4.15}$$

Then relation (3.4.12) can be rewritten as

$$\Pr(XY > x) \sim \Pr(X^* Y_\theta^* > x). \tag{3.4.16}$$

In this way, the dependence structure of (X, Y) is dissolved.

Motivated by Lemma 3.3 of Tang (2006), we obtain the following lemma.

Lemma 3.4.6 *Assume that X^* and Y^* are two independent r.v.s with distributions F on $\mathbb{R}$ and G on $\mathbb{R}^+$. If $F \in \mathcal{R}_{-\infty}$ and G has an upper endpoint $0 < \hat{y} \le \infty$, then*

$$\lim_{x\to\infty} \frac{\Pr(X^* Y_\wedge^* > x)}{\Pr(X^* Y^* > x)} = \Pr\left(Y^* = \hat{y}\right). \tag{3.4.17}$$

By convention, $\Pr(Y^* = \hat{y}) = 0$ *if* $\hat{y} = \infty$.

Proof. By Lemma 3.1(ii) of Tang(2006), for every $y_0 \in (0, \hat{y})$, we have

$$\Pr\left(X^* Y_\wedge^* > x\right) \sim \int_{x/\hat{y}}^{x/y_0} \left(\overline{G}\left(\frac{x}{u}\right)\right)^2 F(\mathrm{d}u),$$

and

$$\Pr(X^* Y^* > x) \sim \int_{x/\hat{y}}^{x/y_0} \overline{G}\left(\frac{x}{u}\right) F(\mathrm{d}u).$$

Hence,

$$\frac{\Pr(X^* Y_\wedge^* > x)}{\Pr(X^* Y^* > x)} \sim \frac{\int_{x/\hat{y}}^{x/y_0} \overline{G}^2(x/u) F(\mathrm{d}u)}{\int_{x/\hat{y}}^{x/y_0} \overline{G}(x/u) F(\mathrm{d}u)},$$

which yields that

$$\inf_{y_0 \le y < \hat{y}} \overline{G}(y) \lesssim \frac{\Pr(X^* Y_\wedge^* > x)}{\Pr(X^* Y^* > x)} \lesssim \sup_{y_0 \le y < \hat{y}} \overline{G}(y).$$

Letting $y_0 \nearrow \hat{y}$ yields the desired result. ■

Now we start to prove Theorem 3.2.2. It is easy to see that the FGM copula is satisfied by Theorem 3.2.1 when $\theta \in (-1, 1]$. Then, in this subsection we only prove the case $\theta = -1$. By (3.4.6), Lemma 3.4.5 and (3.4.15), relation (3.2.8) holds for $n = 1$. Now we inductively assume that (3.2.8) holds for n and we are going to prove it for $n + 1$.

Recall the decomposition relation (3.2.6), since $F \in \mathcal{S}(\gamma)$ and G has a finite upper endpoint $\hat{y} \le 1$, it is easy to verify that G_θ also has a finite upper endpoint $\hat{y}$. By relation (3.2.8), Lemma 2.2 of Tang and Tsitsiashvili (2004) and Theorem

1.1 of Tang (2006), the distribution of V_n belongs to $\mathcal{R}_{-\infty} \cap \mathcal{S}(\gamma/\hat{y})$. By the inductive assumption and $F \in \mathcal{R}_{-\infty}$, it follows that $\Pr(V_n > x) = o(\overline{F}(x))$. Hence, by Lemma 5.3.1, we derive that

$$I_3(x) = (1 + o(1))\mathrm{E}e^{\gamma V_n} \Pr\left(X^* Y_\wedge^* > x\right). \tag{3.4.18}$$

By $\hat{y} = [1 + (1-\pi)r]^{-1} > \frac{1}{2}$, we know $\overline{G}(\frac{1}{2}) > 0$. Then, by (3.2.8) and Lemma 3.4.5, in the case $\theta = -1$ it follows that

$$\frac{\Pr\left(X_\wedge^* > x\right)}{\Pr\left(V_n > x\right)} \leq \frac{\overline{F}^2(x)}{\Pr(XY > x)} \sim \frac{\overline{F}^2(x)}{\Pr(X^* Y_\wedge^* > x)} \leq \frac{\overline{F}^2(x)}{(\overline{G}(1/2))^2 \overline{F}(2x)},$$

which converges to 0 as $x \to \infty$ by Lemma 3.4.2. This yields $\Pr(X_\wedge^* > x) = o(\Pr(V_n > x))$. Hence, by $F_{V_n} \in \mathcal{S}(\gamma/\hat{y})$, Lemma 5.3.1 and the inductive assumption, we have that

$$\Pr(V_n + X_\wedge^* > x) \sim \mathrm{E}e^{\gamma/\hat{y} X_\wedge^*} \Pr(V_n > x) \sim \mathrm{E}e^{\gamma/\hat{y} X_\wedge^*} \mathrm{E}e^{\gamma V_{n-1}} \Pr(X^* Y_\wedge^* > x),$$

which indicates that

$$\begin{aligned} I_2(x) &= \int_0^1 \Pr(V_n + X_\wedge^* > xy^{-1}) G(\mathrm{d}y) \\ &\sim \mathrm{E}e^{\gamma/\hat{y} X_\wedge^*} \mathrm{E}e^{\gamma V_{n-1}} \Pr(X^* Y_\wedge^* Y^* > x) \\ &= o(\Pr(X^* Y_\wedge^* > x)), \end{aligned} \tag{3.4.19}$$

where that last step holds due to the fact that the distribution of $X^* Y_\wedge^*$ belongs to $\mathcal{R}_{-\infty}$, see Lemma 2.2 of Tang and Tsitsiashvili (2004). Similarly, we have

$$I_4(x) = o(\Pr(X^* Y_\wedge^* > x)). \tag{3.4.20}$$

Plugging (3.4.18)–(3.4.20) into (3.2.6), results in (3.2.8) holding for $n + 1$ in the case $\theta = -1$.

3.4.3 Proofs of Theorem 3.2.3 and Corollary 3.2.1

The following lemma comes from Cheng et al. (2012), which is mainly used in the following proofs.

Lemma 3.4.7 *Let F_1 and F_2 be two distributions on $\mathbb{R}$ with $F_1 \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma \geq 0$. If $\overline{F_2}(x) = o\left(\overline{F_1}(x)\right)$, then*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) \int_0^\infty e^{\gamma u} F_2(\mathrm{d}u).$$

Now we prove Theorem 3.2.3. Clearly, by (3.4.6) and Lemma 3.4.4, we know (3.2.7) holds for $n = 1$. Arguing inductively, suppose that (3.2.7) holds for n and we are going to prove it for $n+1$. Thus, we have

$$\Pr(V_n > x) \sim \mathrm{E}e^{\gamma V_{n-1}} \int_0^1 \Pr\left(X > \frac{x}{y}\right) \Pr(Y_r^* \in \mathrm{d}y) = o(\overline{F}(x)),$$

where the last step is due to $F \in \mathcal{R}_{-\infty}$. Note that Y_r^* also has a finite upper endpoint, which indicates that (3.2.13) is satisfied automatically. Applying Lemma 3.4.7, we obtain (3.4.8). By similar arguments to the proof of Theorem 3.2.1, we can derive (3.4.9),(3.4.10) and (3.4.11). Plugging them into (3.2.3), relation (3.2.7) holds for $n+1$.

Moreover, choosing $\phi_1(x) = 1 - 2F(x)$ and $\phi_2(y) = 1 - 2G(y)$, and applying Theorem 3.2.2, one can easily prove Corollary 3.2.1.

3.4.4 Proof of Theorem 3.2.4

Consider the model with a risky investment, which implies that the distribution of financial risk Y has a finite upper endpoint $1 < \hat{y} < \infty$ and relation (3.2.13) is satisfied. By $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$, and applying Lemmas 3.4.3 and 3.4.4, we derive that

$$\overline{H}(x) \sim (1 + \theta d_1 d_2) \Pr(X^* Y^* > x) \tag{3.4.21}$$

Thus, by Lemma 2.2 of Tang and Tsitsiashvili (2004) and Theorem 1.1 of Tang (2006), we have $H \in \mathcal{S}(\gamma/\hat{y}) \cap \mathcal{R}_{-\infty}$. By (3.4.6) and (3.4.21), relation (3.2.9) holds for $n = 1$. Arguing inductively, assume that (3.2.9) holds for n and we aim to prove it for $n+1$. By the inductive assumption, we have that

$$\Pr(V_n > x) \sim B_{n-1}(\gamma) \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right) \sim (1+\theta d_1 d_2) B_{n-1}(\gamma) \Pr(X^* \prod_{j=1}^{n} Y_j^* > x).$$

Applying Lemma 2.2 of Tang and Tsitsiashvili (2004) and Theorem 1.1 of Tang (2006) again, we have $F_{V_n} \in \mathcal{S}(\gamma \hat{y}^{-n}) \cap \mathcal{R}_{-\infty}$. Hence, by the right continuity of the distribution G, the condition $\overline{G}(1) > 0$ implies that there is some $y_0 > 1$ such that $\overline{G}(y_0) > 0$. By $F \in \mathcal{R}_{-\infty}$ we have that

$$\frac{\overline{F}(x)}{\Pr(X^* \prod_{i=1}^{n} Y^* > x)} \leq \frac{\overline{F}(x)}{\overline{F}(x/y_0)(\overline{G}(1))^{n-1}\overline{G}(y_0)} \to 0,$$

which indicates that $\overline{F}(x) = o(\Pr(V_n > x))$. Recall the decomposition (3.2.3). Then, by Lemma 5.3.1 and $\overline{F}(x) = o(\Pr(V_n > x))$, we have

$$\begin{aligned} J_1(x) &= \int_0^{\hat{y}} \Pr\left(V_n + X^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\ &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} X^*} \int_0^{\hat{y}} \Pr\left(V_n > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\ &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} X^*} \Pr(V_n Y^* > x). \end{aligned} \tag{3.4.22}$$

Moreover, $\Pr(\tilde{X}^* > x) = o(\Pr(V_n > x))$ leads to

$$\begin{aligned} J_2(x) &= \int_0^{\hat{y}} \Pr\left(V_n + \tilde{X}^* > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\ &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} \tilde{X}^*} \int_0^{\hat{y}} \Pr\left(V_n > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\ &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} \tilde{X}^*} \Pr(V_n Y^* > x). \end{aligned} \tag{3.4.23}$$

Similarly, by $F_{V_n} \in \mathcal{R}_{-\infty}$ and Lemma 3.4.3, we derive that

$$\begin{aligned} J_3(x) &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} X^*} \Pr(V_n \tilde{Y}^* > x) \\ &= (1+o(1))\left(1 - \frac{d_2}{b_2}\right) \mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} X^*} \Pr(V_n Y^* > x), \end{aligned} \tag{3.4.24}$$

and

$$\begin{aligned} J_4(x) &= (1+o(1))\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} \tilde{X}^*} \Pr(V_n \tilde{Y}^* > x) \\ &= (1+o(1))\left(1 - \frac{d_2}{b_2}\right) \mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} \tilde{X}^*} \Pr(V_n Y^* > x). \end{aligned} \tag{3.4.25}$$

Plugging (3.4.22)–(3.4.25) into (3.2.3), we derive that

$$\begin{aligned} \Pr(V_{n+1} > x) &= \left((1+\theta b_1 d_2)\mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} X^*} - \theta b_1 d_2 \mathrm{E}\mathrm{e}^{\gamma \hat{y}^{-n} \tilde{X}^*} + o(1)\right) B_{n-1}(\gamma) \\ &\qquad \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right) \\ &\sim B_n(\gamma) \Pr\left(X_1 \prod_{j=1}^{n} Y_j > x\right), \end{aligned}$$

with

$$\begin{aligned}
B_n(\gamma) &= \prod_{i=1}^{n}\left((1+\theta b_1 d_2)\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}X^*} - \theta b_1 d_2 \mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}\tilde{X}^*}\right) \\
&= \prod_{i=1}^{n}\left((1+\theta b_1 d_2)\int_{-\infty}^{\infty} \mathrm{e}^{\gamma\hat{y}^{-i}x} F(\mathrm{d}x)\right. \\
&\quad \left. -\theta b_1 d_2 \int_{-\infty}^{\infty} \mathrm{e}^{\gamma\hat{y}^{\gamma-i}x}\left(1-\frac{\phi_1(x)}{b_1}\right) F(\mathrm{d}x)\right) \\
&= \prod_{i=1}^{n} \mathrm{E}\left\{(1+\theta d_2\phi_1(X^*))\mathrm{e}^{\gamma\hat{y}^{-i}X^*}\right\}.
\end{aligned}$$

This ends the proof of Theorem 3.2.4.

3.4.5 Proof of Theorem 5.2.4

Clearly, by Theorem 3.2.4, the desired Theorem 5.2.4 holds for $\theta \in (-1, 1]$. Thus, we next consider the case $\theta = -1$. By (3.4.6) and Lemma 3.4.5, relation (3.2.11) holds for $n = 1$.

Now we inductively assume that (3.2.11) holds for n and we aim to prove it for $n+1$. Since $F \in \mathcal{S}(\gamma)$, by relation (3.2.8), Lemma 2.2 of Tang and Tsitsiashvili (2004) and Theorem 1.1 of Tang (2006), the distribution of V_n belongs to $\mathcal{R}_{-\infty} \cap \mathcal{S}(\gamma\hat{y}^{-n})$. By the similar arguments as (3.4.22)–(3.4.25), and using Lemma 3.4.6 and the inductive assumption, we can obtain that

$$\begin{aligned}
\Pr(V_{n+1} > x) &\sim \left((1+\theta-\theta\hat{p})\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-n}X^*} + (\theta\hat{p}-\theta)\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-n}X^*_{\wedge}}\right)\Pr(V_n Y^* > x) \\
&\sim C_n(\gamma)\Pr\left(X_1\prod_{j=1}^{n+1} Y_j > x\right),
\end{aligned}$$

with

$$\begin{aligned}
C_n(\gamma) &= \prod_{i=1}^{n}\left((1+\theta-\theta\hat{p})\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}X^*} + (\theta\hat{p}-\theta)\mathrm{E}\mathrm{e}^{\gamma\hat{y}^{-i}X^*_{\wedge}}\right) \\
&= \prod_{i=1}^{n}\left\{(1+\theta-\theta\hat{p})\int_{-\infty}^{\infty}\mathrm{e}^{\gamma\hat{y}^{-i}x}F(\mathrm{d}x) + (\theta\hat{p}-\theta)\int_{-\infty}^{\infty}\mathrm{e}^{\gamma\hat{y}^{-i}x}\Pr(X^*_{\wedge}\in \mathrm{d}x)\right\} \\
&= \prod_{i=1}^{n-1}\mathrm{E}\left\{(1+\theta(1-\hat{p})(1-2F(X^*)))\mathrm{e}^{\gamma\hat{y}^{-i}X^*}\right\}.
\end{aligned}$$

Thus, relation (3.2.11) holds for $n+1$.

3.4.6 Proof of Theorem 3.2.6

We firstly cite a lemma, which can be found in Cheng et al. (2012).

Lemma 3.4.8 *Assume that X^* and Y^* are two independent r.v.s with distributions F on $\mathbb{R}$ and G on $\mathbb{R}^+$. If $F \in \mathcal{OS}$ and relation* (3.2.13) *is satisfied, then $H^* \in \mathcal{OS}$.*

Now we prove Theorem 3.2.6. By (3.4.6) and Lemma 3.4.4, relation (3.2.9) holds for $n = 1$. Arguing inductively, assume that (3.2.9) holds for n and we aim to prove it for $n + 1$. By the inductive assumption,

$$\Pr(V_n > x) \sim (1 + \theta d_1 d_2) B_{n-1}(\gamma) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x \right),$$

which implies that $\overline{F}(x) = o(\Pr(V_n > x))$ by $\hat{y} > 1$ and $F \in \mathcal{R}_{-\infty}$. Clearly, $\hat{y} = (1 - \pi)^{-1}(1 + r)^{-1} < \infty$ indicates that (3.2.13) is satisfied automatically. Hence, by Lemma 2.2, Lemma A.4 of Tang and Tsitsiashvili (2004) and Lemma 3.4.8, we have $F_{V_n} \in \mathcal{R}_{-\infty} \cap \mathcal{L}(\gamma/\hat{y}) \cap \mathcal{OS}$ by arguing inductively. Then, applying Lemma 3.4.7, we obtain (3.4.22). By the similar arguments to the proof of Theorem 3.2.4, we can obtain (3.4.23)–(3.4.25). Plugging them into (3.2.3), relation (3.2.9) holds for $n + 1$.

3.4.7 Proof of Theorem 3.2.7

The following lemma comes from Pake (2004).

Lemma 3.4.9 *Let F_1 and F_2 be two distributions on $\mathbb{R}$. If $F_1 \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$, satisfying $\overline{F_2}(x) = o(\overline{F_1}(x))$ and $\int_{-\infty}^{\infty} \mathrm{e}^{\upsilon x} F_2(\mathrm{d}x) < \infty$ for some $\upsilon > \gamma$, then $F_1 * F_2 \in \mathcal{L}(\gamma)$ and*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) \int_{-\infty}^{\infty} \mathrm{e}^{\gamma x} F_2(\mathrm{d}x).$$

Consider the strategy that the insurer invests all his surplus into a risky asset, which leads to the financial risk Y has an infinite upper endpoint. By (3.4.6) and Lemmas 3.4.3, 3.4.4, we derive that

$$\psi(x; 1) \sim \Pr(X_1 Y_1 > x) \sim (1 + \theta d_1 d_2) \Pr(X^* Y^* > x).$$

Thus, relation (3.2.14) holds for $n = 1$. Hence, by relation (3.2.13), Corollary 1.1 of Tang (2008) and Lemma 2.2. of Tang and Tsitsiashvili (2004), the distribution of V_1 belongs to $\mathcal{L} \cap \mathcal{R}_{-\infty}$, and the relation

$$\overline{G}(a(x)) = o(\Pr(V_1 > x))$$

holds. Now we inductively assume that (3.2.14) holds for n and $F_{V_n} \in \mathcal{L} \cap \mathcal{R}_{-\infty}$. Then we are going to prove (3.2.14) for $n+1$. Thus, by the inductive assumption,

$F \in \mathcal{R}_{-\infty}$ and Fatou's lemma, we have $\overline{F}(x) = o(\Pr(V_n > x))$, so by (2.4.6) it holds that $\overline{\tilde{F}}(x) = o(\Pr(V_n > x))$. Applying Lemma 3.4.9, and by $\gamma > 0$ (implying F is light-tailed, so $\mathrm{E}e^{vX^*} < \infty$ for some $v > 0$), it follows that

$$\Pr(V_n + X^* > x) \sim \Pr(V_n > x),$$

and by (2.4.6),

$$\Pr(V_n + \tilde{X}^* > x) \sim \Pr(V_n > x).$$

By $\hat{y} = \infty$, we have $\overline{G}(1) > 0$, then by the inductive assumption,

$$\begin{aligned}
\Pr(V_n Y^* > x) &\geq \Pr(V_n > x)\overline{G}(1) \\
&\sim (1 + \theta d_1 d_2) \Pr\left(X^* \prod_{j=1}^{n} Y_j^* > x\right) \overline{G}(1) \\
&\geq (1 + \theta d_1 d_2)(\overline{G}(1))^n \overline{H^*}(x),
\end{aligned}$$

which, together with (3.2.13), leads to

$$\overline{G}(a(x)) = o(\Pr(V_n Y^* > x)). \tag{3.4.26}$$

By (3.4.26), for $J_1(x)$, we have

$$\begin{aligned}
J_1(x) &= \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \Pr\left(X^* + V_n > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \\
&= \int_0^{a(x)} \Pr\left(X^* + V_n > \frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) + O\left(\overline{G}(a(x))\right) \\
&= (1 + o(1)) \Pr(V_n Y^* > x). \qquad (3.4.27)
\end{aligned}$$

Similarly, by (2.4.7) we have

$$\begin{aligned}
J_3(x) &= \int_0^{a(x)} \Pr\left(X^* + V_n > \frac{x}{y}\right) \Pr(\tilde{Y}^* \in \mathrm{d}y) + O\left(\overline{G}(a(x))\right) \\
&= (1 + o(1)) \Pr(V_n \tilde{Y}^* > x) + o(\Pr(V_n Y^* > x)). \qquad (3.4.28)
\end{aligned}$$

In the same way,

$$J_2(x) = (1 + o(1)) \Pr(V_n Y^* > x), \tag{3.4.29}$$

and

$$J_4(x) = (1 + o(1)) \Pr(V_n \tilde{Y}^* > x). \tag{3.4.30}$$

Note that by Lemma 3.4.3 with $F_{V_n} \in \mathcal{R}_{-\infty}$,

$$\lim_{x \to \infty} \frac{\Pr(V_n \tilde{Y}^* > x)}{\Pr(V_n Y^* > x)} = 1 - \frac{d_2}{b_2}.$$

Plugging (3.4.27)–(3.4.30) into (3.2.3), and by the inductive assumption and (3.4.26), we derive that

$$\begin{aligned}
\Pr(V_{n+1} > x) &\sim \Pr(V_n Y^* > x) \\
&= \int_0^{a(x)} \Pr\left(V_n > \frac{x}{y}\right) G(\mathrm{d}y) + O(\overline{G}(a(x))) \\
&= (1+o(1)) \int_0^{a(x)} \Pr\left(X_1 \prod_{j=1}^{n} Y_j > \frac{x}{y}\right) G(\mathrm{d}y) + o(\Pr(V_n Y^* > x)) \\
&= (1+o(1)) \Pr\left(X_1 \prod_{j=1}^{n+1} Y_j > x\right) + o(\Pr(V_n Y^* > x)),
\end{aligned}$$

which implies

$$\Pr(V_{n+1} > x) \sim \Pr(V_n Y^* > x) \sim \Pr\left(X_1 \prod_{j=1}^{n+1} Y_j > x\right). \tag{3.4.31}$$

Therefore, relation (3.2.14) holds for $n+1$.

By $F_{V_n} \in \mathcal{L}(\gamma) \cap \mathcal{R}_{-\infty}$, (3.4.31) and (3.4.26), applying Corollary 1.1 of Tang (2008) and Lemma 2.2 of Tang and Tsitsiashvili (2004), we obtain $F_{V_{n+1}} \in \mathcal{L} \cap \mathcal{R}_{-\infty}$. This completes the proof of Theorem 3.2.7.

3.4.8 Proof of Theorem 3.2.8

The following Lemma is a restatement of Lemma 4.5 of Chen et al. (2014).

Lemma 3.4.10 *Assume that (X, Y) follows a bivariate FGM distribution of the form* (2.4.8) *with $\theta = -1$. If $\hat{y} = \infty$ and relation* (3.2.13) *is satisfied, then it holds that*

$$\Pr(XY > x) \sim \Pr(X^* Y^*_\wedge > x) + \Pr(X^*_\wedge Y^* > x). \tag{3.4.32}$$

Now we prove Theorem 3.2.8. By $F \in \mathcal{R}_{-\infty}$ and $\overline{F}(x) \sim c\overline{G}(x)$, it holds that for every $b > 0$,

$$\frac{\overline{G}(bx)}{\overline{H^*}(x)} \le \frac{\overline{G}(bx)}{\overline{F}(2/b)\overline{G}(bx/2)} \to 0, \tag{3.4.33}$$

which is equivalent to (3.2.13). Furthermore, it is easy to show for every $y > 1$ that $\overline{H^*}(xy) = o(\overline{H^*}(x))$, implying $H^* \in \mathcal{R}_{-\infty}$. Moreover, by Lemma 3.4.10, relation (3.4.32) holds. By $\overline{F}(x) \sim c\overline{G}(x)$, Lemma A.5 of Tang and Tsitsiashvili (2004), we derive that

$$\Pr(XY > x) \sim \Pr(X^* Y^*_\wedge > x) + \Pr(X^*_\wedge Y^* > x) \sim (1 + c^{-1}) \Pr(X^*_\wedge Y^* > x).$$

Similarly to (3.4.33), for any $b > 0$,

$$\overline{G}(bx) = o(\Pr(X^*_\wedge Y^* > x) = o(\overline{H}(x)),$$

which is equivalent to

$$\overline{G}(a(x)) = o(\overline{H}(x)), \tag{3.4.34}$$

for some auxiliary function $a(x)$ satisfying $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$. Note that (3.2.13) is slightly weaker than (3.4.34), since, by (3.4.32), we have $\overline{H}(x) \lesssim 2\overline{H^*}(x)$. Noting that $F_{X^*_\wedge} \in \mathcal{L}(2\gamma) \cap \mathcal{R}_{-\infty}$, then by (3.4.34), Corollary 1.1 of Tang (2008) and Lemma 2.2 of Tang and Tsitsiashvili (2004), we get $H \in \mathcal{L} \cap \mathcal{R}_{-\infty}$. By (3.4.6), it is easy to see that relation (3.2.16) holds for $n = 1$. Now we assume that (3.2.16) holds for n and we aim to prove it holding for $n + 1$. Furthermore, by the right continuity of the distribution G, the condition $\overline{G}(1) > 0$ leads to some $y_0 > 1$ such that $\overline{G}(y_0) > 0$. It follows from $F \in \mathcal{R}_{-\infty}$ that

$$\frac{\Pr(X^*_\wedge > x)}{\Pr\left(X^*_\wedge \prod_{i=1}^n Y^* > x\right)} \leq \frac{(\overline{F}(x))^2}{(\overline{F}(x/y_0))^2 (\overline{G}(1))^{n-1} \overline{G}(y_0)} \to 0,$$

which, together with the inductive assumption, indicates $\overline{F_{X^*_\wedge}}(x) = o(\Pr(V_n > x))$. Then the rest of the proof can be done along the lines of the proof of Theorem 3.2.7.

3.4.9 Proof of Corollary 3.2.4 and Corollary 3.2.5

Note that the Corollary 3.2.4 and Corollary 3.2.5 are direct applications of Theorem 3.2.2, Theorem 5.2.4 and Theorem 3.2.1, Theorem 3.2.4. Applying Theorem 3.2.2, Theorem 5.2.4 and Theorem 3.1 (a) of Hashorva et al. (2010), we can easily prove the Corollary 3.2.4. The Corollary 3.2.5 can be proved similarly.

Chapter 4

Ruin with Insurance and Financial Risks Following the Least Risky FGM Dependence Structure [2]

4.1 Introduction

Recently, Chen (2011) studied the asymptotic behavior of the ruin probability $\psi(x;n)$ in (3.1.1) for the case with dependent insurance and financial risks. Precisely, it was assumed that (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of i.i.d. copies of a generic random pair (X, Y) whose components are however dependent. The dependence between X and Y was realized via a bivariate FGM distribution of the form

$$\Pi(x,y) = F(x)G(y)\left(1 + \theta\overline{F}(x)\overline{G}(y)\right),$$

where $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$ and $G = 1 - \overline{G}$ on $\mathbb{R}_+ = [0, \infty)$ are marginal distributions of (X, Y), and $\theta \in [-1, 1]$ is a parameter governing the strength of dependence. Under the assumptions that F is a subexponential distribution, G fulfills some constraints in order for the product convolution of F and G (see (2.1.1) below) to be a subexponential distribution too, and $\theta \in (-1, 1]$, Chen (2011) derived a general asymptotic formula for $\psi(x;n)$. Note that the assumption $\theta \neq -1$ was essentially applied there; see related discussions on Page 1041 of Chen (2011). Hence, the derivation of Chen (2011) is not valid for $\theta = -1$.

The FGM distribution (2.4.8) describes an asymptotically independent situation. Recall that, for a copula function $C(\cdot,\cdot)$ on $(0,1)^2$, its survival copula is defined as $\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. For the FGM case, we have

$$\overline{C}(u,v) = C(u,v) = uv\left(1 + \theta(1-u)(1-v)\right), \qquad (u,v) \in (0,1)^2.$$

[2] This chapter is based on Chen, Liu and Liu (2015).

For every $\theta \in [-1, 1]$, the coefficient of upper tail dependence is

$$\chi = \lim_{u \downarrow 0} \frac{\overline{C}(u,u)}{u} = 0.$$

See Section 5.2 of McNeil et al. (2005) for details of the concepts used here. Nevertheless, asymptotically independent random variables may still exhibit different degrees of dependence. In this regard, Coles et al. (1999) proposed to use

$$\hat{\chi} = \lim_{u \downarrow 0} \frac{2 \log u}{\log \overline{C}(u,u)} - 1$$

to measure more subtly the strength of dependence in the asymptotic independence case. There are many other measures, e.g. see Nadarajah (2015). With a bit of calculation, we see that $\hat{\chi} = 0$ for $\theta \in (-1, 1]$ while $\hat{\chi} = -1/3$ for $\theta = -1$. This illustrates the essential difference between the cases $-1 < \theta \leq 1$ and $\theta = -1$.

In this Chapter, we still look at the ruin probability (3.1.1) but for the case $\theta = -1$. It turns out that, not surprisingly though, the asymptotic behavior of $\psi(x; n)$ in the case $\theta = -1$ is very different from that in the case $-1 < \theta \leq 1$. Due to the distinction between the two cases, in the present study new technicalities will be needed and more precise asymptotic analysis will be conducted. The main difficulty exists in dealing with the tail behavior of the product of X and Y following the FGM structure (2.4.8) with $\theta = -1$. Recent related discussions on the product of heavy-tailed (and dependent) random variables can be found in Hashorva et al. (2010), Jiang and Tang (2011), Yang et al. (2011), Yang and Hashorva (2013), and Yang and Wang (2013), among others.

While the scientific value of the present study is revealed during solving a series of technical problems to complement a previous study, we would like to stress its practical relevance in insurance and finance. When the insurance business goes insolvent, the insurer will of course become more conservative with investments. Moreover, stock market crashes will certainly increase the prudence of not only banking but also insurance regulators. These require the insurer to observantly adjust between the insurance and financial markets, leading to negatively dependent insurance and financial risks. Note that under the FGM framework $\theta = -1$ exhibits an extremely negative, thus the least risky, scenario in which hypothetically the insurer complies with the most conservative self-adjustment mechanism. The ruin probability for $\theta = -1$ should be smaller, at least asymptotically, than that for $-1 < \theta \leq 1$, as is confirmed by our main results in Section 4.2. Thus, the present study devoted to the least risky scenario of the FGM framework offers some new insight into the insolvency of the insurance business in the presence of dependent insurance and financial risks.

The rest of this chapter consists of four sections. Section 4.2 presents main results and corollaries, and Sections 4.3, 4.4 prove the main results and corollaries, respectively.

4.2 Main Results

For simplicity, we say that a real function $a(\cdot)$ defined on $\mathbb{R}_+$ is an auxiliary function if it satisfies $0 \le a(x) < x/2$, $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$.

We recall here some facts, which will be used tacitly for a few times in this thesis. If $F \in \mathcal{L}$ and $0 < \hat{y} < \infty$, then by Theorem 2.2(iii) of Cline and Samorodnitsky (1994) we have $F \otimes G \in \mathcal{L}$. Moreover, if $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$, then by Theorem 2.1 of Cline and Samorodnitsky (1994) as recalled in Lemma 4.3.4 below, $F \otimes G \in \mathcal{S}$.

Recall that the dependence structure of (X, Y) is described by the joint distribution (2.4.8) with $\theta = -1$; that is

$$\Pi(x, y) = F(x)G(y)\left(1 - \overline{F}(x)\overline{G}(y)\right) \tag{4.2.1}$$

with F on $\mathbb{R}$ and G on $\mathbb{R}_+$. Introduce independent random variables X^*, Y^*, Y_1^*, Y_2^*, Y_3^*, $\ldots$, with the first identically distributed as X and others identically distributed as Y. Also recall $X_\wedge^*$ and $Y_\wedge^*$ introduced at the end of Section 2.4. Denote by H the distribution of the product XY and, as in (2.1.1), denote by $H^* = F \otimes G$ the distribution of the product X^*Y^*. As before, $\hat{y}$ denotes the essential upper bound of Y.

In the first result below, the condition $0 < \hat{y} \le 1$ indicates that there are risk-free investments only:

Theorem 4.2.1 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1) with $F \in \mathcal{S}$ and $0 < \hat{y} \le 1$. Then it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right), \tag{4.2.2}$$

where, and throughout the thesis, the usual convention $\prod_{j=2}^{1} Y_j^* = 1$ is in force.

In the second result below, the condition $1 \le \hat{y} \le \infty$ means the presence of risky investments:

Theorem 4.2.2 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1) with $F \in \mathcal{L}$, $1 \le \hat{y} \le \infty$ and $H \in \mathcal{S}$. Then the relation*

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* Y^* \prod_{j=2}^{i} Y_j^* > x\right) \tag{4.2.3}$$

holds for each $n \in \mathbb{N}$ under either of the following groups of conditions:

(i) there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\big(\overline{H}(x)\big)$ and $\overline{H}(x-a(x)) \sim \overline{H}(x)$;

(ii) $M_*(F) > 0$, and there is an auxiliary function $a(\cdot)$ such that $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$.

Lemma 4.3.5 below gives an asymptotic expression (4.3.11) for $\overline{H}(x)$ in terms of the tails of products of independent random variables. This expression can help us easily verify the conditions $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$ and $\overline{H}\left(x - a(x)\right) \sim \overline{H}(x)$ in Theorem 4.2.2 in a given situation. We remark that the condition $M_*(F) > 0$ in Theorem 4.2.2(ii) is really mild and does not exclude any distribution of practical interest. Under its help, however, we are able to get rid of the troublesome condition $\overline{H}\left(x - a(x)\right) \sim \overline{H}(x)$.

Recall Theorem 3.1 of Chen (2011), which, for $\theta \in (-1, 1]$, shows the asymptotic formula

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr\left(XY \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.2.4}$$

The three formulas (4.2.2)–(4.2.4) look very different from each other but they are intrinsically consistent. Actually, according to relation (4.3.11) for $\theta = -1$, (4.2.3) corresponds to (4.2.4) with the distribution of XY replaced by another distribution with tail asymptotically equivalent to $\Pr\left(X^* Y_\wedge^* > x\right) + \Pr\left(X_\wedge^* Y^* > x\right)$. Subsequently, (4.2.2) is also consistent with (4.2.4) since, as Lemma 4.3.2 below shows, we have $\Pr\left(X_\wedge^* Y^* > x\right) = o\left(\Pr\left(X^* Y_\wedge^* > x\right)\right)$ when $0 < \hat{y} < \infty$.

Lemma 4.2 of Chen (2011) shows that, for $\theta \in (-1, 1]$,

$$\Pr\left(XY > x\right) \sim (1 - \theta) \Pr\left(X^* Y^* > x\right) + \theta \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right). \tag{4.2.5}$$

Notice that, as anticipated, the right-hand side of (4.2.5) increases in θ. By comparing (4.3.11) to (4.2.5), one can gain a quantitative understanding on how significantly the asymptotic ruin probability decreases when the parameter θ switches from its normal range $(-1, 1]$ to its extremum -1.

It is naturally asked if one of the two sums on the right-hand side of (4.2.3) is negligible. The answer is diverse. Listed below are some important special cases, showing that sometimes the second sum on the right-hand side of (4.2.3) is negligible and, hence, relation (4.2.3) reduces to relation (4.2.2), but sometimes not.

Corollary 4.2.1 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1) with $F \in \mathcal{S}$ and $0 < \hat{y} < \infty$. Then relation (4.2.2) holds for each $n \in \mathbb{N}$.*

Corollary 4.2.1 extends Theorem 4.2.1 by relaxing the restriction on Y from $0 < \hat{y} \leq 1$ to $0 < \hat{y} < \infty$.

Corollary 4.2.2 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1). Then relation (4.2.2) holds for each $n \in \mathbb{N}$ under either of the following groups of conditions:*

(i) $F \in \mathcal{C}$ and $\mathrm{E}[Y^p] < \infty$ for some $p > M^*(F)$;

(ii) $F \in \mathcal{L} \cap \mathcal{D}$ with $M_*(F) > 0$, and $\mathrm{E}[Y^p] < \infty$ for some $p > M^*(F)$.

In Corollaries 4.2.1 and 4.2.2(i), if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$, then applying Breiman's theorem (see Cline and Samorodnitsky (1994), who attributed it to Breiman (1965)) to relation (4.2.2), we obtain

$$\psi(x; n) \sim \mathrm{E}\left[(Y_\wedge^*)^\alpha\right] \frac{1 - (\mathrm{E}\left[Y^\alpha\right])^n}{1 - \mathrm{E}\left[Y^\alpha\right]} \overline{F}(x), \tag{4.2.6}$$

where the ratio $\frac{1-(\mathrm{E}[Y^\alpha])^n}{1-\mathrm{E}[Y^\alpha]}$ is understood as n in the case $\alpha = 0$. Relation (4.2.6) confirms that relation (3.2) of Chen (2011) still holds for $\theta = -1$. Comparing both, one again gains a quantitative understanding on how the asymptotic ruin probability decreases as the parameter θ decreases to its negative extremum.

In the next two corollaries we look at a critical situation with the same heavy-tailed insurance and financial risks. The first one below addresses the regular variation case:

Corollary 4.2.3 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1). If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, $\overline{F}(x) \sim c\overline{G}(x)$ for some $c > 0$, and $\mathrm{E}[Y^\alpha] = \infty$, then it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim \left(c\mathrm{E}\left[(Y_\wedge^*)^\alpha\right] + \mathrm{E}\left[\left(X_\wedge^{*+}\right)^\alpha\right]\right) \Pr\left(\prod_{j=1}^{n} Y_j^* > x\right), \tag{4.2.7}$$

where $X_\wedge^{*+}$ denotes the positive part of $X_\wedge^*$.

The second one below addresses the rapid variation case:

Corollary 4.2.4 *Let the random pair (X, Y) follow a bivariate FGM distribution (4.2.1). If $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$ and $\overline{F}(x) \sim c\overline{G}(x)$ for some $c > 0$, then it holds for each $n \in \mathbb{N}$ that*

$$\psi(x; n) \sim (1 + c) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{n} Y_j^* > x\right). \tag{4.2.8}$$

4.3 Proofs of Theorems 4.2.1–4.2.2

4.3.1 General Derivations

The following general derivations will be used in the proofs of both Theorems 4.2.1 and 4.2.2.

Notice the decomposition

$$\Pi = F^2 G + F G^2 - F^2 G^2 \tag{4.3.1}$$

and the obvious facts that $X_\vee^*$ is distributed by F^2 and $Y_\vee^*$ by G^2. It follows that

$$\begin{aligned}
& \Pr(XY > x) \\
= \; & \Pr(X_\vee^* Y^* > x) + \Pr(X^* Y_\vee^* > x) - \Pr(X_\vee^* Y_\vee^* > x) \\
= \; & 2\Pr(X^* Y^* > x) - \Pr(X_\wedge^* Y^* > x) \\
& +2\Pr(X^* Y^* > x) - \Pr(X^* Y_\wedge^* > x) \\
& -4\Pr(X^* Y^* > x) + 2\Pr(X^* Y_\wedge^* > x) + 2\Pr(X_\wedge^* Y^* > x) - \Pr(X_\wedge^* Y_\wedge^* > x) \\
= \; & \Pr(X^* Y_\wedge^* > x) + \Pr(X_\wedge^* Y^* > x) - \Pr(X_\wedge^* Y_\wedge^* > x). \qquad (4.3.2)
\end{aligned}$$

Define

$$T_n = \sum_{i=1}^n X_i \prod_{j=i}^n Y_j, \qquad n \in \mathbb{N}. \tag{4.3.3}$$

Note that T_n is identically distributed as S_n in (3.1.2) due to the i.i.d. assumption on the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$, and that it fulfills the recursive formula

$$T_{n+1} = (T_n + X_{n+1}) Y_{n+1}, \qquad n \in \mathbb{N}. \tag{4.3.4}$$

Similarly to the derivation of (4.3.2), starting from (4.3.4) and applying the decomposition in (4.3.1) we have

$$\begin{aligned}
& \Pr(T_{n+1} > x) \\
= \; & \Pr((T_n + X_\vee^*) Y^* > x) + \Pr((T_n + X^*) Y_\vee^* > x) - \Pr((T_n + X_\vee^*) Y_\vee^* > x) \\
= \; & \Pr((T_n + X^*) Y_\wedge^* > x) + \Pr((T_n + X_\wedge^*) Y^* > x) - \Pr((T_n + X_\wedge^*) Y_\wedge^* > x) \\
= \; & I_1(x) + I_2(x) - I_3(x). \qquad (4.3.5)
\end{aligned}$$

The following lemma is well known and can be found in Embrechts and Goldie (1980), Cline (1986, Corollary 1) and Tang and Tsitsiashvili (2003, Lemma 3.2):

Lemma 4.3.1 *Let F_1 and F_2 be two distributions on $\mathbb{R}$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$ and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.*

4.3.2 Proof of Theorem 4.2.1

In the proof of Theorem 4.2.1 we need the following two lemmas:

Lemma 4.3.2 *Let* (X, Y) *follow a bivariate FGM distribution (4.2.1). If* $F \in \mathcal{S}$ *and* $0 < \hat{y} < \infty$*, then*

$$\begin{aligned} \Pr\left(X_{\wedge}^{*} Y^{*} > x\right) &= o\left(\Pr\left(X^{*} Y_{\wedge}^{*} > x\right)\right), & (4.3.6)\\ \Pr\left(XY > x\right) &\sim \Pr\left(X^{*} Y_{\wedge}^{*} > x\right). & (4.3.7)\end{aligned}$$

Proof. Without loss of generality, assume $\hat{y} = 1$. We have

$$\frac{\Pr\left(X_{\wedge}^{*} Y^{*} > x\right)}{\Pr\left(X^{*} Y_{\wedge}^{*} > x\right)} \leq \frac{\overline{F}^{2}(x)}{\Pr\left(X^{*} > 2x, Y_{\wedge}^{*} > 1/2\right)} = \frac{1}{\overline{G}^{2}(1/2)} \frac{\overline{F}^{2}(x)}{\overline{F}(2x)}.$$

By Lemma 5.1 of Cai and Tang (2004), the right-hand side above converges to 0 as $x \to \infty$. This proves relation (4.3.6). Looking at (4.3.2), relation (4.3.6) implies that the second term on the right-hand side of (4.3.2) and, hence, the third term there also, is negligible. Then relation (4.3.2) gives relation (4.3.7). ■

The following lemma will enable us to conduct an induction procedure in the proof of Theorem 4.2.1:

Lemma 4.3.3 *In addition to the conditions in Lemma 4.3.2, assume* $0 < \hat{y} \leq 1$. *Then* $XY + X^{*}$ *follows a subexponential distribution with tail satisfying*

$$\Pr\left(XY + X^{*} > x\right) \sim \Pr\left(XY > x\right) + \Pr\left(X > x\right).$$

Proof. As recalled in Chapter 2, the conditions $F \in \mathcal{L}$ and $0 < \hat{y} \leq 1$ imply that $\Pr\left(X^{*} Y_{\wedge}^{*} > x\right)$ is a long tail, and so is $\Pr\left(XY > x\right)$ due to relation (4.3.7). The condition $0 < \hat{y} \leq 1$ implies that $\Pr\left(XY > x\right) \leq \Pr\left(X > x\right)$. Thus, the desired results follow from Lemma 4.3.1. ■

Proof of Theorem 4.2.1. As analyzed by Chen (2011), it suffices to prove the relation

$$\Pr\left(T_{n} > x\right) \sim \sum_{i=1}^{n} \Pr\left(X^{*} Y_{\wedge}^{*} \prod_{j=2}^{i} Y_{j}^{*} > x\right). \tag{4.3.8}$$

Note that the first term on the right-hand side of (4.3.8) is a subexponential tail and the other terms are long tails and are not greater than the first. Thus, as in Lemma 4.3.1, the right-hand side of (4.3.8) indeed gives a subexponential tail for T_{n}.

We employ the method of induction to complete the proof of (4.3.8). Lemma 4.3.2 shows that relation (4.3.8) holds for $n = 1$. Now assume that relation (4.3.8) holds for some n and we are going to prove it for $n+1$. Our proof is based on the recursive formula (4.3.4).

First deal with $I_1(x)$ in (4.3.5). Since $F \in \mathcal{S}$ and $0 < \hat{y} \leq 1$, we have

$$\Pr\left(T_n > x\right) \leq \Pr\left(\sum_{i=1}^{n} X_i > x\right) \sim n\overline{F}(x).$$

By Lemma 4.3.1,

$$\begin{aligned} I_1(x) &= \int_0^1 \Pr\left(T_n + X^* > \frac{x}{y}\right) \Pr\left(Y^*_\wedge \in \mathrm{d}y\right) \\ &\sim \int_0^1 \left(\Pr\left(T_n > \frac{x}{y}\right) + \Pr\left(X^* > \frac{x}{y}\right)\right) \Pr\left(Y^*_\wedge \in \mathrm{d}y\right) \\ &= \Pr\left(T_n Y^*_\wedge > x\right) + \Pr\left(X^* Y^*_\wedge > x\right). \end{aligned}$$

Now we turn to $I_2(x)$ in (4.3.5). Note that $X^*_\wedge Y^*$, T_n and $T_n Y^*$ are all long tailed, one can choose some auxiliary function $a(\cdot)$ such that the following relations hold simultaneously:

$$\begin{cases} \Pr\left(X^*_\wedge Y^* > x \pm a(x)\right) \sim \Pr\left(X^*_\wedge Y^* > x\right), \\ \Pr\left(T_n > x \pm a(x)\right) \sim \Pr\left(T_n > x\right), \\ \Pr\left(T_n Y^* > x \pm a(x)\right) \sim \Pr\left(T_n Y^* > x\right). \end{cases} \tag{4.3.9}$$

Also note that, by relation (4.3.6) and relation (4.3.8) for n,

$$\Pr\left(X^*_\wedge Y^* > x\right) = o(1) \Pr\left(X^* Y^*_\wedge > x\right) = o(1) \Pr\left(T_n > x\right). \tag{4.3.10}$$

We derive

$$\begin{aligned} I_2(x) &\leq \Pr\left(\left(T_n + X^*_\wedge\right) Y^* > x, T_n Y^* \leq a(x)\right) \\ &\quad + \Pr\left(\left(T_n + X^*_\wedge\right) Y^* > x, X^*_\wedge Y^* \leq a(x)\right) \\ &\quad + \Pr\left(\left(T_n + X^*_\wedge\right) Y^* > x, T_n Y^* > a(x), X^*_\wedge Y^* > a(x)\right) \\ &\leq \Pr\left(X^*_\wedge Y^* > x - a(x)\right) + \Pr\left(T_n Y^* > x - a(x)\right) \\ &\quad + \Pr\left(\left(T_n + X^*_\wedge\right) Y^* > x, X^*_\wedge Y^* > a(x)\right) \\ &\leq o(1) \Pr\left(T_n > x\right) + (1 + o(1)) \Pr\left(T_n Y^* > x\right) \\ &\quad + \Pr\left(T_n + X^*_\wedge Y^* > x, X^*_\wedge Y^* > a(x)\right), \end{aligned}$$

where in the last step we used (4.3.9)–(4.3.10). For the last term on the right-hand side above, we have

$$\begin{aligned} &\Pr\left(T_n + X^*_\wedge Y^* > x, X^*_\wedge Y^* > a(x)\right) \\ \leq\; &\Pr\left(T_n + X^*_\wedge Y^* > x\right) - \Pr\left(T_n + X^*_\wedge Y^* > x, -a(x) \leq X^*_\wedge Y^* \leq a(x)\right) \\ \leq\; &(1 + o(1)) \left(\Pr\left(T_n > x\right) + \Pr\left(X^*_\wedge Y^* > x\right)\right) - (1 + o(1)) \Pr\left(T_n > x + a(x)\right) \\ =\; &o(1) \Pr\left(T_n > x\right), \end{aligned}$$

where in the second step we used Lemma 4.3.1 and in the last step we used (4.3.9)–(4.3.10). It follows that

$$I_2(x) \leq (1+o(1))\Pr(T_nY^* > x) + o(1)\Pr(T_n > x).$$

On the other hand, by (4.3.9),

$$\begin{aligned} I_2(x) &\geq \Pr\left((T_n - a(x))Y^* > x, -a(x) \leq X_\wedge^* \leq a(x)\right) \\ &\geq \Pr(T_nY^* > x + a(x))\Pr(-a(x) \leq X_\wedge^* \leq a(x)) \\ &\geq (1+o(1))\Pr(T_nY^* > x). \end{aligned}$$

For $I_3(x)$ in (4.3.5), by going along the same lines of the derivation for $I_2(x)$ and changing every Y^* to $Y_\wedge^*$, we obtain

$$(1+o(1))\Pr(T_nY_\wedge^* > x) \leq I_3(x) \leq (1+o(1))\Pr(T_nY_\wedge^* > x) + o(1)\Pr(T_n > x).$$

Plugging all these estimates for $I_1(x)$, $I_2(x)$ and $I_3(x)$ into (4.3.5), we obtain

$$\begin{aligned} &\Pr(T_{n+1} > x) \\ \leq\ & (1+o(1))\Pr(T_nY_\wedge^* > x) + (1+o(1))\Pr(X^*Y_\wedge^* > x) \\ & + (1+o(1))\Pr(T_nY^* > x) + o(1)\Pr(T_n > x) \\ & - (1+o(1))\Pr(T_nY_\wedge^* > x) \\ =\ & o(1)\left(\Pr(T_nY_\wedge^* > x) + \Pr(T_n > x)\right) + (1+o(1))\left(\Pr(X^*Y_\wedge^* > x) + \Pr(T_nY^* > x)\right) \\ =\ & o(1)\Pr(T_n > x) + (1+o(1))\left(\Pr(X^*Y_\wedge^* > x) + \sum_{i=1}^{n}\Pr\left(X^*Y_\wedge^*Y^*\prod_{j=2}^{i}Y_j^* > x\right)\right) \\ =\ & o(1)\sum_{i=1}^{n}\Pr\left(X^*Y_\wedge^*\prod_{j=2}^{i}Y_j^* > x\right) + (1+o(1))\sum_{i=1}^{n+1}\Pr\left(X^*Y_\wedge^*\prod_{j=2}^{i}Y_j^* > x\right) \\ \sim\ & \sum_{i=1}^{n+1}\Pr\left(X^*Y_\wedge^*\prod_{j=2}^{i}Y_j^* > x\right), \end{aligned}$$

where in the third and fourth steps we used (4.3.8) for n. The lower asymptotic bound can be derived similarly. Hence, (4.3.8) holds for $n+1$. ■

4.3.3 Proof of Theorem 4.2.2

The following lemma is a restatement of Theorem 2.1 of Cline and Samorodnitsky (1994), which is crucial for establishing our Theorem 4.2.2(i):

Lemma 4.3.4 *Let F be a distribution on $\mathbb{R}$ and G be a distribution on $\mathbb{R}_+$. We have $H^* = F \otimes G \in \mathcal{S}$ if $F \in \mathcal{S}$ and there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H^*}(x)\right)$ and $\overline{F}(x - a(x)) \sim \overline{F}(x)$.*

The lemma below dismisses the dependence structure of X and Y:

Lemma 4.3.5 *Let (X, Y) follow a bivariate FGM distribution (4.2.1) with $\hat{y} = \infty$. If there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H^*}(x)\right)$, then*

$$\Pr(XY > x) \sim \Pr(X^*Y^*_\wedge > x) + \Pr(X^*_\wedge Y^* > x). \tag{4.3.11}$$

Proof. By (4.3.2), we need only prove that the last term on its right-hand side is negligible, namely,

$$\Pr(X^*_\wedge Y^*_\wedge > x) = o(1)\left(\Pr(X^*Y^*_\wedge > x) + \Pr(X^*_\wedge Y^* > x)\right). \tag{4.3.12}$$

For this purpose, we do the split

$$\Pr(X^*_\wedge Y^*_\wedge > x) = \Pr(X^*_\wedge Y^*_\wedge > x, (Y^*_\wedge \le a(x)) \cup (Y^*_\wedge > a(x))) = J_1(x) + J_2(x).$$

By conditioning on $Y^*_\wedge$, we have

$$J_1(x) \le \overline{F}\left(\frac{x}{a(x)}\right) \int_0^{a(x)} \overline{F}\left(\frac{x}{y}\right) \Pr(Y^*_\wedge \in \mathrm{d}y) = o(1) \Pr(X^*Y^*_\wedge > x).$$

It is easy to see that

$$J_2(x) = o(1) \Pr(X^*_\wedge Y^* > x). \tag{4.3.13}$$

Actually, on the one hand, it is clear that

$$J_2(x) \le \Pr(Y^*_\wedge > a(x)) = \overline{G}^2(a(x));$$

while on the other hand, by Jensen's inequality we have

$$\Pr(X^*_\wedge Y^* > x) = \int_0^\infty \overline{F}^2\left(\frac{x}{y}\right) \Pr(Y^* \in \mathrm{d}y) \ge \overline{H^*}^2(x).$$

Relation (4.3.13) follows since $\overline{G}(a(x)) = o\left(\overline{H^*}(x)\right)$. Thus, relation (4.3.12) holds. ■

With (X, Y) following a bivariate FGM distribution (4.2.1), we see that the condition $\overline{G}(a(x)) = o\left(\overline{H^*}(x)\right)$ is slightly more general than the condition $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$ since $\overline{H}(x) \lesssim 2\overline{H^*}(x)$ by (4.3.11).

The following lemma is a counterpart of Lemma 4.3.3:

Lemma 4.3.6 *In addition to the conditions in Lemma 4.3.5, assume $F \in \mathcal{L}$ and $H \in \mathcal{S}$. Then $XY + X^*$ follows a subexponential distribution with tail satisfying*

$$\Pr(XY + X^* > x) \sim \Pr(XY > x) + \Pr(X > x).$$

Proof. By Lemma 4.3.5,

$$\Pr(XY > x) \gtrsim \Pr(X^* Y^*_\wedge > x) \geq \Pr(X^* > x)\Pr(Y^*_\wedge \geq 1).$$

Thus, the result follows from Lemma 4.3.1. ■

Define

$$V_n = \sum_{i=1}^{n} X_i Y_i \prod_{j=i+1}^{n} Y_j^*, \qquad n \in \mathbb{N}.$$

Lemma 4.3.7 *Let (X, Y) follow a bivariate FGM distribution (4.2.1) with $\hat{y} = \infty$ and $H \in \mathcal{S}$.*

(i) If there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$ and $\overline{H}(x - a(x)) \sim \overline{H}(x)$, then each V_n follows a subexponential distribution with tail satisfying

$$\Pr(V_n > x) \sim \sum_{i=1}^{n} \Pr\left(X^* Y^*_\wedge \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X^*_\wedge \prod_{j=1}^{i} Y_j^* > x\right);$$

(ii) If $M_*(F) > 0$, then the restriction $\overline{H}(x - a(x)) \sim \overline{H}(x)$ on the auxiliary function $a(\cdot)$ is unnecessary.

Proof. For simplicity, write $Z_i = X_i Y_i$ for $i = 1, \ldots, n$. Notice that the sequence $\{V_n, n \in \mathbb{N}\}$ fulfills the recursive equation

$$V_{n+1} = V_n Y^*_{n+1} + Z_{n+1}.$$

Applying Lemmas 4.3.4 and 4.3.1, we can conduct a standard induction procedure to prove that, for each $n \in \mathbb{N}$, the sum V_n follows a subexponential distribution with tail satisfying

$$\Pr(V_n > x) \sim \sum_{i=1}^{n} \Pr\left(Z_i \prod_{j=i+1}^{n} Y_j^* > x\right) = \sum_{i=1}^{n} \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.3.14}$$

For case (i), we refer the reader to the proofs of Theorem 3.1 of Tang (2006b), Theorem 3.1 of Chen (2011) and, in particular, Theorem 1.2 of Zhou et al. (2012) for similar discussions. For case (ii), see Theorem 4.1 of Tang (2006a).

We can also conduct a standard induction procedure to prove that, for each $i = 2, \ldots, n$ and every $a > 0$,

$$\Pr\left(\prod_{j=2}^{i} Y_j^* > ax\right) = o(1)\Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right).$$

Consequently, it is easy to see that, for each $i = 2, \ldots, n$, there is some auxiliary function $a_i(\cdot)$ such that

$$\Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right) = o(1) \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right). \tag{4.3.15}$$

For each $i = 2, \ldots, n$, we split each probability on the right-hand side of (4.3.14) as

$$\Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right) = \Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) + O(1) \Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right).$$

By conditioning on $\prod_{j=2}^{i} Y_j^*$ and applying Lemma 4.3.5, the first term on the right-hand side above is asymptotically equivalent to

$$\begin{aligned}
& \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) + \Pr\left(X_\wedge^* Y^* \prod_{j=2}^{i} Y_j^* > x, \prod_{j=2}^{i} Y_j^* \le a_i(x)\right) \\
= \; & \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) + O(1) \Pr\left(\prod_{j=2}^{i} Y_j^* > a_i(x)\right).
\end{aligned}$$

By (4.3.15), it follows that

$$\Pr\left(Z_i \prod_{j=2}^{i} Y_j^* > x\right) \sim \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right).$$

Substituting this into (4.3.14) leads to the desired result. ∎

Proof of Theorem 4.2.2. Recall T_n introduced in (4.3.3) and the recursive formula (4.3.4). Same as before, it suffices to prove the relation

$$\Pr(T_n > x) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right). \tag{4.3.16}$$

Lemma 4.3.7 shows that the right-hand side of (4.3.16) indeed gives a subexponential tail for T_n.

Similarly to the proof of Theorem 4.2.1, we employ the method of induction to prove (4.3.16). Lemma 4.3.5 shows that relation (4.3.16) holds for $n = 1$. Now assume that relation (4.3.16) holds for some n and we are going to prove it for $n + 1$.

For this purpose, we still start from the decomposition in (4.3.5). For $I_1(x)$, since T_n is subexponential, X^* is long tailed, and $\Pr(X^* > x) = O(\Pr(T_n > x))$, by conditioning on $Y_{\wedge}^{*}$ and applying Lemma 4.3.1 we have

$$\begin{aligned} I_1(x) &= \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \Pr\left(T_n + X^* > \frac{x}{y}\right) \Pr\left(Y_{\wedge}^{*} \in \mathrm{d}y\right) \\ &= (1+o(1)) \int_0^{a(x)} \left(\Pr\left(T_n > \frac{x}{y}\right) + \Pr\left(X^* > \frac{x}{y}\right)\right) \Pr\left(Y_{\wedge}^{*} \in \mathrm{d}y\right) \\ &\quad + O(1)\overline{G}^2(a(x)) \\ &= (1+o(1)) \left(\Pr\left(T_n Y_{\wedge}^{*} > x\right) + \Pr\left(X^* Y_{\wedge}^{*} > x\right)\right) + O(1)\overline{G}^2(a(x)). \end{aligned}$$

In the same way, we have

$$I_2(x) = (1+o(1)) \left(\Pr\left(T_n Y^* > x\right) + \Pr\left(X_{\wedge}^{*} Y^* > x\right)\right) + O(1)\overline{G}(a(x))$$

and

$$I_3(x) = (1+o(1)) \left(\Pr\left(T_n Y_{\wedge}^{*} > x\right) + \Pr\left(X_{\wedge}^{*} Y_{\wedge}^{*} > x\right)\right) + O(1)\overline{G}^2(a(x)).$$

Plugging these estimates into (4.3.5) and using the condition $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$ and relations (4.3.11)–(4.3.12), we obtain

$$\Pr\left(T_{n+1} > x\right) \sim \Pr\left(X^* Y_{\wedge}^{*} > x\right) + \Pr\left(X_{\wedge}^{*} Y^* > x\right) + \Pr\left(T_n Y^* > x\right).$$

For the last term above, by conditioning on Y^* and applying relation (4.3.16) for n it is easy to show that

$$\Pr\left(T_n Y^* > x\right) \sim \sum_{i=1}^{n} \Pr\left(X^* Y_{\wedge}^{*} \prod_{j=2}^{i+1} Y_j^* > x\right) + \sum_{i=1}^{n} \Pr\left(X_{\wedge}^{*} \prod_{j=1}^{i+1} Y_j^* > x\right).$$

Thus, relation (4.3.16) holds for $n+1$. ■

4.4 Proofs of Corollaries 4.2.1–4.2.4

4.4.1 Proof of Corollary 4.2.1

When $0 < \hat{y} \le 1$, the result comes directly from Theorem 4.2.1. When $1 \le \hat{y} < \infty$, by the condition $F \in \mathcal{S}$ and Lemma 4.3.2, we have $\overline{H}(x) \sim \Pr\left(X^* Y_{\wedge}^{*} > x\right)$ and, hence, $H \in \mathcal{S}$. Furthermore, $H \in \mathcal{S} \subset \mathcal{L}$ implies the existence of an auxiliary function $a(\cdot)$ satisfying $\overline{H}\left(x - a(x)\right) \sim \overline{H}(x)$. Since $\hat{y} < \infty$, the condition $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$ in Theorem 4.2.2(i) holds trivially for every such function $a(\cdot)$. Thus, all conditions of Theorem 4.2.2(i) are fulfilled.

For each $i = 1, \ldots, n$, by conditioning on $\prod_{j=2}^{i} Y_j^*$ and using relation (4.3.6),

$$\begin{aligned}
\Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) &= \int_0^{\hat{y}^{i-1}} \Pr\left(X_\wedge^* Y_1^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&= o(1) \int_0^{\hat{y}^{i-1}} \Pr\left(X^* Y_\wedge^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right).
\end{aligned}$$

Substituting this into relation (4.2.3) leads to relation (4.2.2).

4.4.2 Proof of Corollary 4.2.2

(i) By Theorem 3.3(iv) of Cline and Samorodnitsky (1994), the conditions $F \in \mathcal{C}$ and $\mathrm{E}[Y^p] < \infty$ for some $p > M^*(F)$ imply that $\overline{H^*}(x) \asymp \overline{F}(x)$. By Lemma 3.5 of Tang and Tsitsiashvili (2003), the relation $x^{-q} = o\left(\overline{F}(x)\right)$ holds for every $q > M^*(F)$. Define an auxiliary function $a(x) = x^r$ for some $r \in (M^*(F)/p, 1)$. We have

$$\overline{G}\left(x^r\right) \leq x^{-rp}\mathrm{E}[Y^p] = o\left(\overline{H^*}(x)\right).$$

Thus, Lemma 4.3.5 is applicable and gives relation (4.3.11). For the two terms on the right-hand side of (4.3.11), we have, respectively, $\Pr\left(X^* Y_\wedge^* > x\right) \asymp \overline{F}(x)$ and

$$\begin{aligned}
\Pr\left(X_\wedge^* Y^* > x\right) &\leq \int_0^{x^r} \overline{F}^2\left(\frac{x}{y}\right) G(\mathrm{d}y) + \overline{G}\left(x^r\right) \\
&\leq \overline{F}\left(\frac{x}{x^r}\right) \overline{H^*}(x) + o\left(\overline{F}(x)\right) \\
&= o\left(\overline{F}(x)\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* > x\right). \qquad (4.4.1)
\end{aligned}$$

It follows from (4.3.11) that

$$\overline{H}(x) \sim \Pr\left(X^* Y_\wedge^* > x\right) \asymp \overline{F}(x). \qquad (4.4.2)$$

Thus, $a(x) = x^r$ satisfies the conditions $\overline{G}\left(a(x)\right) = o\left(\overline{H}(x)\right)$ and $\overline{H}\left(x - a(x)\right) \sim \overline{H}(x)$ in Theorem 4.2.2(i) too. By the first relation in (4.4.2) and Theorem 3.4(ii) of Cline and Samorodnitsky (1994), it is easy to see that $H \in \mathcal{C} \subset \mathcal{S}$. Thus, all conditions of Theorem 4.2.2(i) are satisfied and we have relation (4.2.3). For each

$i = 2, \ldots, n$, similarly to (4.4.1), by conditioning on $\prod_{j=2}^{i} Y_j^*$ we obtain

$$\begin{aligned}
\Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) &\leq \int_0^{x^r} \Pr\left(X_\wedge^* Y^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&\quad + \Pr\left(\prod_{j=2}^{i} Y_j^* > x^r\right) \\
&= o(1) \int_0^{x^r} \Pr\left(X^* Y_\wedge^* > \frac{x}{y}\right) \Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) + o\left(\overline{F}(x)\right) \\
&= o(1) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right). \qquad (4.4.3)
\end{aligned}$$

Thus, the second sum on the right-hand side of relation (4.2.3) is negligible and we finally obtain relation (4.2.2).

(ii) With $a(x) = x^r$ for some $r \in (M^*(F)/p, 1)$, the verifications of the conditions of Lemma 4.3.5 and Theorem 4.2.2(ii) are similar to those in the proof of Corollary 4.2.2(i), and the proofs of relations (4.4.1)–(4.4.3) are also the same. A major difference is that we need to apply Theorems 2.2(iii) and 3.3(ii) of Cline and Samorodnitsky (1994) to the first relation in (4.4.2) to verify $H \in \mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$.

4.4.3 Proof of Corollary 4.2.3

As in the proof of Corollary 2.1 of Chen and Xie (2005), by Fatou's lemma we have

$$\liminf_{x \to \infty} \frac{\overline{H^*}(x)}{\overline{G}(x)} \geq \int_0^\infty \liminf_{x \to \infty} \frac{\overline{G}(x/y)}{\overline{G}(x)} F(\mathrm{d}y) = \mathrm{E}[X_+^\alpha] = \infty. \qquad (4.4.4)$$

It follows that $\overline{G}(x) = o(\overline{H^*}(x))$. By Lemma 3.2 of Chen and Xie (2005), there is an auxiliary function $a(\cdot)$ such that $\overline{G}(a(x)) = o\left(\overline{H^*}(x)\right)$ holds. By Lemma 4.3.5,

$$\begin{aligned}
\overline{H}(x) &\sim \Pr(X^* Y_\wedge^* > x) + \Pr(X_\wedge^* Y^* > x) \\
&\sim \mathrm{E}\left[(Y_\wedge^*)^\alpha\right] \overline{F}(x) + \mathrm{E}\left[\left(X_\wedge^{*+}\right)^\alpha\right] \overline{G}(x) \\
&\sim \left(c\mathrm{E}\left[(Y_\wedge^*)^\alpha\right] + \mathrm{E}\left[\left(X_\wedge^{*+}\right)^\alpha\right]\right) \overline{G}(x),
\end{aligned}$$

where the second step is due to Breiman's theorem. Hence, $H \in \mathcal{R}_{-\alpha}$ and the same auxiliary function $a(\cdot)$ satisfies $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$. Thus, relation (4.2.3) holds by Theorem 4.2.2(ii).

Next we simplify (4.2.3) to (4.2.7). For each $i = 2, \ldots, n$, by the Corollary of Embrechts and Goldie (1980), $\prod_{j=2}^{i} Y_j^*$ follows a distribution belonging to the class $\mathcal{R}_{-\alpha}$. Based on the same reasoning as above we see that there is some auxiliary function $\tilde{a}_i(\cdot)$ such that

$$\Pr\left(\prod_{j=2}^{i} Y_j^* > \tilde{a}_i(x)\right) = o(1)\Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right).$$

We have

$$\begin{aligned}
\Pr\left(X^* Y_\wedge^* \prod_{j=2}^{i} Y_j^* > x\right) &\sim \int_0^{\tilde{a}_i(x)} \Pr\left(X^* Y_\wedge^* > \frac{x}{y}\right)\Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&\sim c\mathrm{E}\left[(Y_\wedge^*)^\alpha\right]\int_0^{\tilde{a}_i(x)} \Pr\left(Y^* > \frac{x}{y}\right)\Pr\left(\prod_{j=2}^{i} Y_j^* \in \mathrm{d}y\right) \\
&\sim c\mathrm{E}\left[(Y_\wedge^*)^\alpha\right]\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right),
\end{aligned}$$

where the second step is due to Breiman's theorem. Similarly, for each $i = 2, \ldots, n$,

$$\Pr\left(X_\wedge^* \prod_{j=1}^{i} Y_j^* > x\right) \sim \mathrm{E}\left[\left(X_\wedge^{*+}\right)^\alpha\right]\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right).$$

Substituting these asymptotic results into (4.2.3) gives

$$\psi(x;n) \sim \left(c\mathrm{E}\left[(Y_\wedge^*)^\alpha\right] + \mathrm{E}\left[\left(X_\wedge^{*+}\right)^\alpha\right]\right)\sum_{i=1}^{n}\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right).$$

Similarly to (4.4.4), $\Pr\left(\prod_{j=1}^{i} Y_j^* > x\right) = o(1)\Pr\left(\prod_{j=1}^{n} Y_j^* > x\right)$ for every $i = 1, \ldots, n-1$. Then relation (4.2.7) follows.

4.4.4 Proof of Corollary 4.2.4

The following lemma will be needed in the proof of Corollary 4.2.4:

Lemma 4.4.1 *For two distributions F on $\mathbb{R}$ and G on $\mathbb{R}_+$, if $F \in \mathcal{R}_{-\infty}$ and $G \in \mathcal{R}_{-\infty}$, then $H^* = F \otimes G \in \mathcal{R}_{-\infty}$.*

Proof. By (2.1.1), it holds for every $a > 0$ that

$$\frac{\overline{G}(ax)}{\overline{H^*}(x)} \le \frac{\overline{G}(ax)}{\overline{F}(2/a)\overline{G}(ax/2)} \to 0.$$

Thus, there is an auxiliary function $a(\cdot)$ such that $\overline{G}\left(a(x)\right) = o\left(\overline{H^*}(x)\right)$. By (2.1.1) again, for every $z > 1$,

$$\frac{\overline{H^*}(xz)}{\overline{H^*}(x)} \le \frac{\int_0^{a(x)} \overline{F}\left(xz/y\right) G(\mathrm{d}y)}{\int_0^{a(x)} \overline{F}\left(x/y\right) G(\mathrm{d}y)} + \frac{O\left(\overline{G}(ax)\right)}{\overline{H^*}(x)} \le \sup_{0<y\le a(x)} \frac{\overline{F}\left(xz/y\right)}{\overline{F}\left(x/y\right)} + o(1) \to 0.$$

Hence, $H^* \in \mathcal{R}_{-\infty}$. ■

Proof of Corollary 4.2.4. Since $\overline{F}(x) \sim c\overline{G}(x)$ and $\Pr(X_\wedge^* > x) \sim c^2 \Pr(Y_\wedge^* > x)$, by Lemma A.5 of Tang and Tsitsiashvili (2004) we have

$$\Pr(X_\wedge^* Y^* > x) \sim c^2 \Pr(Y_\wedge^* Y^* > x) \sim c \Pr(X^* Y_\wedge^* > x). \tag{4.4.5}$$

From the proof of Lemma 4.4.1, there is an auxiliary function $\tilde{a}(\cdot)$ such that $\overline{G}(\tilde{a}(x)) = o\left(\overline{H^*}(x)\right)$. Then by Lemma 4.3.5, relation (4.3.11) holds. It follows from relations (4.3.11) and (4.4.5) that

$$\Pr(XY > x) \sim \Pr(X^* Y_\wedge^* > x) + \Pr(X_\wedge^* Y^* > x) \sim (1+c) \Pr(X^* Y_\wedge^* > x).$$

By Corollary 2.1 of Tang (2006a), $\Pr(X^* Y_\wedge^* > x)$ is a subexponential tail and, hence, $H \in \mathcal{S}$. Again from the proof of Lemma 4.4.1, there is an auxiliary function $a(\cdot)$ such that

$$\overline{G}(a(x)) \sim \frac{1}{c}\overline{F}(a(x)) = o(1) \Pr(X^* Y_\wedge^* > x) = o\left(\overline{H}(x)\right).$$

Thus, relation (4.2.3) holds by Theorem 4.2.2(ii).

Next we simplify (4.2.3) to (4.2.8). For each $i = 1, \ldots, n-1$, since $X_\wedge^* Y^* \prod_{j=2}^i Y_j^*$ is rapidly-varying tailed by Lemma 4.4.1, we have

$$\begin{aligned}
\Pr\left(X_\wedge^* Y^* \prod_{j=2}^i Y_j^* > x\right) &= o(1) \Pr\left(X_\wedge^* Y^* \prod_{j=2}^i Y_j^* > \frac{x}{2^{n-i}}\right) \\
&= o(1) \Pr\left(X_\wedge^* Y^* \prod_{j=2}^i Y_j^* > \frac{x}{2^{n-i}}, Y_{i+1}^* > 2, \ldots, Y_n^* > 2\right) \\
&= o(1) \Pr\left(X_\wedge^* Y^* \prod_{j=2}^n Y_j^* > x\right).
\end{aligned}$$

Similarly, it holds for each $i = 1, \ldots, n-1$ that

$$\Pr\left(X^* Y_\wedge^* \prod_{j=2}^i Y_j^* > x\right) = o(1) \Pr\left(X^* Y_\wedge^* \prod_{j=2}^n Y_j^* > x\right).$$

It follows from relation (4.2.3) that

$$\psi(x; n) \sim \Pr\left(X^* Y_\wedge^* \prod_{j=2}^n Y_j^* > x\right) + \Pr\left(X_\wedge^* Y^* \prod_{j=2}^n Y_j^* > x\right).$$

Since by Lemma 4.4.1 the product $\prod_{j=2}^n Y_j^*$ is rapidly-varying tailed, applying Lemma A.5 of Tang and Tsitsiashvili (2004) and relation (4.4.5) we obtain

$$\Pr\left(X_\wedge^* Y^* \prod_{j=2}^n Y_j^* > x\right) \sim c \Pr\left(X^* Y_\wedge^* \prod_{j=2}^n Y_j^* > x\right).$$

Relation (4.2.8) follows. ■

Part III

Ruin Probabilities in a continuous-time dependent risk model

Chapter 5

Infinite-time Absolute Ruin in Dependent Renewal Risk Models with Constant Force of Interest [3]

5.1 Introduction

Consider the following renewal risk model with constant premium rate and constant force of interest. In this model, the claim sizes $X_i,\ i \in \mathbb{N}$, arriving at successive renewal epochs with inter-arrival times $\vartheta_i,\ i \in \mathbb{N}$, form a sequence of i.i.d. copies of a generic random pair (X, ϑ) with common distribution F on $[0, \infty)$ and G_ϑ on $(0, \infty)$. The arrival times of the successive claims, $\tau_i = \sum_{j=1}^{i} \vartheta_j, i \in \mathbb{N}$, with $\tau_0 = 0$, construct a renewal counting process $N(t) = \sup\{j \in \mathbb{N} : \tau_j \leq t\}$. Let $x \geq 0$ be the initial reserve of an insurance company and let $c > 0$ and $r > 0$ be the constant premium rate and the constant force of interest rate, respectively. Thus, the future value of a capital x after time t becomes $x\mathrm{e}^{rt}$. Denote by $W_r(t)$, the total reserve up to time t, and let it satisfy

$$W_r(t) = x\mathrm{e}^{rt} + c\int_0^t \mathrm{e}^{r(t-y)}\mathrm{d}y - \sum_{i=1}^{N(t)} X_i \mathrm{e}^{r(t-\tau_i)}, \qquad t \geq 0.$$

Hence, the classical ruin probability by time $T \leq \infty$ is defined by

$$\varphi(x, T) = \Pr\left(\inf_{0\leq t\leq T} W_r(t) < 0 \,\middle|\, W_r(0) = x\right).$$

In the actuarial literature, the finite-time or infinite-time ruin probability was defined as the probability of the surplus falling below zero, which has been widely investigated by many researchers, see, e.g. Asmussen (1998), Klüppelberg and Stadtmüller (1998), Konstantinides et al. (2002) and Tang (2005). Tang

[3] This chapter is based on Liu and Yang (2015).

(2007) established a tail asymptotic formula which holds uniformly in time. However, as commented by Embrechts et al. (1994) the threshold 0 above does not make sense in reality since an insurance company will not be allowed to proceed with its business when its wealth stays at a too low level. By introducing a default threshold ζ, the default probability by time $T \leq \infty$ becomes $\psi(x,\zeta,T) = \Pr\left(\inf_{0\leq t\leq T} W_r(t) < \zeta \middle| W_r(0) = x\right)$, which is referred to as the absolute ruin probability when $\zeta = -c/r$. The insurer will not be able to repay it debts when the surplus process hits such boundary. Some investigations have been made in absolute ruin probability; see, e.g. Embrechts et al. (1994), Cai (2007), Konstantinides et al. (2010). Inspired by the literature, we define the probability of infinite-time absolute ruin as

$$\psi(x,\infty) = \Pr\left(\inf_{t\geq 0} W_r(t) < -\frac{c}{r}\middle| W_r(0) = x\right), \qquad x \geq 0, \tag{5.1.1}$$

which can also be rewritten as

$$\psi(x,\infty) = \Pr\left(\sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j > x + \frac{c}{r}\right), \qquad x \geq 0, \tag{5.1.2}$$

where $Y_i = \mathrm{e}^{-r\vartheta_i}, i \geq 1$, can be considered as discount factor according to the constant force of interest r.

As shown by (5.1.2), the probability of infinite-time absolute ruin in (5.1.1) has been transferred to the tail probability of all discounted future claims exceeding the initial surplus plus the total discounted premium $x+c/r$. The standard renewal risk model was first proposed by Andersen (1957), assuming that $\{X_i, i \in \mathbb{N}\}$ and $\{\vartheta_i, i \in \mathbb{N}\}$ were mutually independent, which is for mathematical convenience but far away from reality. Various dependence structures were introduced to the risk model by many researchers. Chen and Yuen (2007) obtained a formula for the ruin probability in a renewal risk model with constant interest force in which the claims were pairwise negatively dependent and extended regularly varying tailed. Yang and Wang (2010) obtained two weak asymptotic equivalent formulae for the renewal risk model with a constant premium under the structure of pairwise negatively quadrant dependent (NQD). Recently, Yang et al. (2013) derived asymptotic results for the infinite-time absolute ruin probabilities in some time-dependent renewal risk models.

Motivated by the researchers above, we allow the generic random pair (X, ϑ) or (X, Y) to follow a more general dependence structure in Section 2. This structure captures the impact of dependence between each pair of the claim size and inter-arrival time prior to the claims. For example, the waiting time for a claim is dependent on the claim size. Alternatively, the discount factor can be affected indirectly by the changing of the waiting time. We firstly derive two asymptotic formulas for the infinite-time absolute ruin probabilities under the condition that the distribution of the claim-size belongs to the intersection of the class of all convolution-equivalent distributions and the class of rapidly-varying-tailed distributions. Then we extend the obtained results to a larger distribution class. Finally, we consider an extension for the Farlie-Gumbel-Morgenstern case with parameter in the whole interval $[-1, 1]$. All estimates achieved for the infinite-time absolute ruin probabilities capture the impact of dependence structure of (X, ϑ) or (X, Y).

The rest of the present this chapter is organized as follows. In the next section, we provide the main results. In Section 5.3, we prove them.

5.2 Main results

Now we turn to the dependence structure between the claim size and inter-arrival time by a comprehensive treatment of copulas. Let (X, ϑ) be a random vector with continuous marginal distribution $F(x)$ and $G_\vartheta(t)$, then the dependence structure between X and ϑ is characterized in terms of a unique bivariate copula by Sklar's theorem, with its joint distribution $V(x,t) = C\left(F(x), G_\vartheta(t)\right)$. The reader is referred to Nelsen (1998) or Joe (1997).

Clearly, such a bivariate function with respect to the copula $C(u, v)$ can also be described as

$$\overline{V}(x,t) = \overline{C}\left(\overline{F}(x), \overline{G_\vartheta}(t)\right) = \Pr(X > x, \vartheta > t).$$

Assume that the copula function $C(u, v)$ is absolutely continuous, denote by $C_1(u,v) := \frac{\partial}{\partial u}C(u,v)$, $C_2(u,v) := \frac{\partial}{\partial v}C(u,v)$, and $C_{12}(u,v) := \frac{\partial^2}{\partial u \partial v}C(u,v)$, then

$$\begin{aligned}\overline{C}_2(u,v) &:= \frac{\partial}{\partial v}\overline{C}(u,v) = 1 - C_2(1-u, 1-v),\\ \overline{C}_{12}(u,v) &:= \frac{\partial^2}{\partial u \partial v}\overline{C}(u,v) = C_{12}(1-u, 1-v).\end{aligned}$$

The following assumption we made can be attributed to Asimit and Badescu (2010), which has also been used to illustrate the impact of the dependence on the tail behaviour of the product of two random variables, recently. Similar assumption related to Assumption 5.2.1 can also be found in Yang and Konstantinides (2014).

Assumption 5.2.1 *The relation*

$$\overline{C}_2(u,v) \sim u\overline{C}_{12}(0+, v), \qquad u \downarrow 0,$$

holding uniformly on $(0, 1]$.

We remark that Assumption 5.2.1 is equivalent to

$$1 - C_2(u,v) \sim (1-u)C_{12}(1-,v), \qquad u \uparrow 1,$$

holds uniformly on $[0,1)$. There are three commonly-used copula functions satisfying Assumption 5.2.1 given in Asimit and Badescu (2010).

1. The Ali-Mikhail-Haq copula is of the form

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \qquad \theta \in (-1,1),$$

with $\overline{C}_{12}(0+,v) = 1+\theta(1-2v)$.

2. The FGM copula is of the form

$$C(u,v) = uv + \theta uv(1-u)(1-v), \qquad \theta \in (-1,1),$$

with $\overline{C}_{12}(0+,v) = 1+\theta(1-2v)$.

3. The Frank copula is of the form

$$C(u,v) = -\frac{1}{\theta}\log\left(1+\frac{(\mathrm{e}^{-\theta u -1})(\mathrm{e}^{-\theta v-1})}{\mathrm{e}^{-\theta}-1}\right), \qquad \theta \neq 0,$$

with $\overline{C}_{12}(0+,v) = \theta \mathrm{e}^{\theta(1-v)}/(\mathrm{e}^{\theta}-1)$.

Recall the renewal risk model with constant premium rate and constant force of interest in Section 5.1. In the sequel, denote by F on $[0,\infty)$, G_ϑ on $(0,\infty)$ and G on $(0,1)$ the distributions of the claim size X, the inter-arrival time ϑ and the discount factor Y, respectively.

In the following result, we allow the generic random pair (X,ϑ) to follow a dependence structure with its joint distribution $V(x,t) = C(F(x), G_\vartheta(t))$, introduced in Section 2, and establish an asymptotic relation for $\psi(x,\infty)$.

Theorem 5.2.1 *In the renewal risk model with constant force of interest rate $r>0$, assume that (X_i,ϑ_i), $i\in\mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X,ϑ) satisfying Assumption 5.2.1 with $F \in \mathcal{S}(\gamma)\cap\mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, then $\mathrm{E}\mathrm{e}^{\gamma S_\infty}<\infty$, where $S_\infty = \sum_{i=1}^{\infty} X_i \mathrm{e}^{-r\tau_i}$, and*

$$\psi(x,\infty) \sim \mathrm{e}^{-\gamma c/r}\mathrm{E}\mathrm{e}^{\gamma S_\infty}\int_0^\infty \overline{F}(x\mathrm{e}^{rt})G_{\vartheta_c}(\mathrm{d}t), \tag{5.2.1}$$

where $G_{\vartheta_c}(\mathrm{d}t) = C_{12}\left[1-, G_\vartheta(t)\right]G_\vartheta(\mathrm{d}t)$.

As mentioned in Section 5.1, the discount factor can also be influenced by the claim size. We assume that the generic pair (X,Y) follows a dependence structure via its joint distribution $H(x,y) = C(F(x),G(y))$. We will show in the next subsection an example to explain our second result and extend it to case that the generic pair (X,Y) follows a bivariate FGM distribution.

Theorem 5.2.2 *In the renewal risk model with constant force of interest rate $r > 0$, assume that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) under Assumption 5.2.1 with $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, then $\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty$, where $S_\infty = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j$, $\quad Y_j = \mathrm{e}^{-r\vartheta_i}$, and*

$$\psi(x, \infty) \sim \mathrm{E}\mathrm{e}^{\gamma S_\infty} \Pr(XY_c > x + c/r), \tag{5.2.2}$$

where Y_c is distributed by $G_c(\mathrm{d}y) = C_1[1-, G(\mathrm{d}y)] = C_{12}[1-, G(y)]G(\mathrm{d}y)$.

Next, we generalize the above two results under the condition that the distribution of claim-size belongs to a larger intersection class of O-subexponential distributions, rapidly-varying-tailed distributions and the distributions in the class $\mathcal{L}(\gamma)$.

Theorem 5.2.3 *In the renewal risk model with constant force of interest rate $r > 0$, assume that (X, ϑ) or (X, Y) is dependent according to Assumption 5.2.1 with $F \in \mathcal{L}(\gamma) \cap \mathcal{OS} \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, then relation (5.2.1) or (5.2.2) holds with $\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty$.*

5.2.1 An extension: Farlie-Gumbel-Morgenstern Copula

It is easy to see that the FGM copula is satisfied by the Assumption 5.2.1 when $\theta \in (-1, 1)$. In this subsection, we only take $\theta = 1$ and $\theta = -1$ into consideration.

As done by Chen (2011), we assume that $(X_i, Y_i), i \in \mathbb{N}$, form a sequence of i.i.d. random pairs with generic random pair (X, Y). However, the components of (X, Y) are dependent and follow a joint bivariate FGM distribution (2.4.8). The general derivations will be mainly used in following proof. We observe the decomposition

$$\Pi = (1 + \theta)\, FG - \theta F^2 G - \theta F G^2 + \theta F^2 G^2,$$

or, equivalently,

$$\Pi = (1 + \theta)\, FG - \theta(1 - \overline{F}^2)G - \theta F(1 - \overline{G}^2) + \theta(1 - \overline{F}^2)(1 - \overline{G}^2). \tag{5.2.3}$$

Moreover, for a random variable X, introduce two independent random variables $X^*_\vee = X^*_1 \vee X^*_2$ and $X^*_\wedge = X^*_1 \wedge X^*_2$, which are independent of all other source of randomness, where X^*_1 and X^*_2 are two i.i.d. copies of X. Actually, if X is distributed by F, then $X^*_\vee$ is distributed by F^2 and the tail of $X^*_\wedge$ is $\overline{F}^2$, see Yang et al. (2011). We allow the generic pair (X, Y) to follow a bivariate FGM distribution.

Theorem 5.2.4 *In the renewal risk model with constant force of interest rate $r > 0$, assume that (X_i, Y_i), $i \in \mathbb{N}$, are a sequence of i.i.d. random pairs with generic random pair (X, Y) following a common bivariate FGM distribution function (2.4.8) with $\theta \in [-1, 1]$. If $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$, then $\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty$, where $S_\infty = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j$, and*

$$\psi(x, \infty) \sim \mathrm{E}\mathrm{e}^{\gamma S_\infty} \Pr(XY_\theta > x + c/r),$$

where Y_θ is distributed by G_θ with $G_\theta(y) = (1-\theta)G(y) + \theta G^2(y)$.

5.3 Proofs of main results

Before we prove the first result, we introduce the following lemma, which is a restatement of a result in Rogozin (1999).

Lemma 5.3.1 *Let F, F_1 and F_2 be three distributions on $[0, \infty)$. Assume that $F \in \mathcal{S}(\gamma)$, $\gamma \geq 0$, and the limits $k_i = \lim \overline{F_i}(x)/\overline{F}(x)$ exist and are finite, $i = 1, 2$. Then, it holds that*

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = k_1 \int_{0-}^{\infty} \mathrm{e}^{\gamma t} F_2(\mathrm{d}t) + k_2 \int_{0-}^{\infty} \mathrm{e}^{\gamma t} F_1(\mathrm{d}t).$$

Motivated by Grey (1994) and Konstantinides et al. (2010), we complete the proof of Theorem 5.2.1 below. The proof of Theorem 5.2.2 is similar to that of Theorem 5.2.1, and we omit it.

5.3.1 Proof of Theorem 5.2.1

Introduce a random variable Z with distribution F, independent of $\{(X, Y), (X_i, Y_i), i \in \mathbb{N}\}$. Then for sufficiently large $x_0 > 0$ and all $x \geq x_0$ we have that

$$\begin{aligned}
&\Pr\left((Z + X)\, Y > x\right) = \int_0^\infty \Pr\left(X + Z > x\mathrm{e}^{rt} | \vartheta = t\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & \int_0^\infty \int_0^{x\mathrm{e}^{rt} - x_0} \Pr\left(X > x\mathrm{e}^{rt} - u | \vartheta = t\right) F\left(\mathrm{d}u\right) G_\vartheta(\mathrm{d}t) + \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt} - x_0\right) G_\vartheta(\mathrm{d}t) \\
=\ & \int_0^\infty \int_0^{x\mathrm{e}^{rt} - x_0} \overline{C}_2\left[\overline{F}\left(x\mathrm{e}^{rt} - u\right), \overline{G}_\vartheta(t)\right] F\left(\mathrm{d}u\right) G_\vartheta(\mathrm{d}t) + \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt} - x_0\right) G_\vartheta(\mathrm{d}t).
\end{aligned} \tag{5.3.1}$$

The Assumption 5.2.1 implies that for any $\varepsilon > 0$, there exists a positive number $x_1 \leq x_0$ such that for all $x \geq x_1$ and all $t > 0$,

$$(1-\varepsilon)\overline{C}_{12}[0+, \overline{G_\vartheta}(t)]\overline{F}(x) \leq \overline{C}_2[\overline{F}(x), \overline{G_\vartheta}(t)] \leq (1+\varepsilon)\overline{C}_{12}[0+, \overline{G_\vartheta}(t)]\overline{F}(x). \tag{5.3.2}$$

Thus, by (5.3.1) and (5.3.2) we have that for all $x \geq x_0 \geq x_1$,

$$
\begin{aligned}
& \Pr\left((Z+X)\,Y > x\right) \\
\leq\ & (1+\varepsilon) \int_0^\infty \int_0^{x\mathrm{e}^{rt}-x_0} \overline{C}_{12}\left[0+, \overline{G}_\vartheta(t)\right] \overline{F}\left(x\mathrm{e}^{rt}-u\right) F\left(\mathrm{d}u\right) G_\vartheta(\mathrm{d}t) \\
& + \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt}-x_0\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & (1+\varepsilon) \int_0^\infty \left(\overline{F^{*2}}\left(x\mathrm{e}^{rt}\right) - \overline{F}\left(x\mathrm{e}^{rt}\right)\right) G_{\vartheta_c}(\mathrm{d}t) + \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt}-x_0\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & (1+\varepsilon)^2 \int_0^\infty \left(2E\mathrm{e}^{\gamma X}-1\right) \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t) + (1+\varepsilon)\mathrm{e}^{\gamma x_0} \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt}\right) G_\vartheta(\mathrm{d}t) \\
=\ & o(\overline{F}(x)).
\end{aligned}
\tag{5.3.3}
$$

Here, in the last step we used the dominated convergence theorem and $F \in \mathcal{R}_{-\infty}$, which implies that $\overline{F}\left(x\mathrm{e}^{rt}\right) = o(\overline{F}(x))$. Therefore, by (5.3.3) there is some sufficiently large $x_2 > 0$ such that for all $x > x_2$,

$$
\Pr\left((X+Z)\,Y > x\right) \leq \overline{F}(x), \tag{5.3.4}
$$

Construct a new conditional random variable $Z_0 = (Z|Z > x_2)$, whose distribution F_{Z_0} still belongs to the class $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. It is easy to see that

$$
(Z_0+X)\,Y \overset{\mathrm{d}}{\leq} Z_0 \tag{5.3.5}
$$

holds for all $x > 0$, where the symbol $\overset{\mathrm{d}}{\leq}$ denotes 'stochastically not larger than'. Actually, $\Pr(Z_0 > x) = 1$ for all $x \leq x_2$, while it still holds for all $x \leq x_2$ by relation (5.3.4). Consequently, it holds equivalently for all $x > 0$,

$$
\Pr((Z_0+X)\,Y > x) \leq \overline{F_{Z_0}}(x).
$$

Therefore, we have that $(Z_0+X_1)\,Y_1 \overset{\mathrm{d}}{\leq} Z_0,\ (Z_0+X_2)\,Y_2 \overset{\mathrm{d}}{\leq} Z_0$ and $((Z_0+X_2)\,Y_2+X_1)\,Y_1 \overset{\mathrm{d}}{\leq} Z_0$. Write

$$
S_n = \sum_{i=1}^n X_i \prod_{j=1}^i Y_j, \quad n \geq 1.
$$

Hence, $S_1 = X_1Y_1 \overset{\mathrm{d}}{\leq} Z_0$ and $S_2 = X_1Y_1 + X_2Y_2Y_1 \overset{\mathrm{d}}{\leq} Z_0$, which leads to $S_n \overset{\mathrm{d}}{\leq} Z_0$ for all $n \geq 1$. Moreover, as $n \to \infty$, we get that $S_\infty \overset{\mathrm{d}}{\leq} Z_0$, which implies that $\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty$. Let $\tilde{S}_\infty$, independent of $\{(X_n, Y_n), n \geq 1\}$, be a copy of S_∞. Then, for every $n \geq 1$,

$$
S_\infty \overset{\mathrm{d}}{=} S_n + \tilde{S}_\infty \prod_{j=1}^n Y_j,
$$

where $\overset{\mathrm{d}}{=}$ stands for the equality in distribution. Then,

$$S_\infty \overset{\mathrm{d}}{\leq} S_n + Z_0 \prod_{j=1}^{n} Y_j. \tag{5.3.6}$$

Clearly, for $n \geq 2$,

$$\begin{aligned}
\Pr\left(S_\infty > x\right) &\leq \Pr\left(S_n + Z_0 \prod_{j=1}^{n} Y_j > x\right) \\
&= \int_0^\infty \Pr\left(X_1 + \sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} \right. \\
&\qquad \left. + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt} \right| \vartheta_1 = t \Big) G_\vartheta(\mathrm{d}t).
\end{aligned} \tag{5.3.7}$$

Note that for some large x_3, by Assumption 5.2.1 and (5.3.2) we have that for all $x \geq x_3$,

$$\begin{aligned}
&\Pr\left(X_1 + \sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt} \right| \vartheta_1 = t \Big) \\
\leq & \int_{0-}^{x\mathrm{e}^{rt}-x_3} \Pr\left(X_1 > x\mathrm{e}^{rt} - v \middle| \vartheta_1 = t\right) \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} \in \mathrm{d}v\right) \\
& + \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt} - x_3\right) \\
= & \int_{0-}^{x\mathrm{e}^{rt}-x_3} \left(\overline{C}_2\left[\overline{F}\left(x\mathrm{e}^{rt} - v\right), \overline{G}_\vartheta(t)\right]\right) \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} \in \mathrm{d}v\right) \\
& + \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt} - x_3\right) \\
\leq & (1+\varepsilon) \Pr\left(X_1 + \sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt}\right) C_{12}\left[1-, G_\vartheta(t)\right] \\
& + \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\mathrm{e}^{rt} - x_3\right).
\end{aligned} \tag{5.3.8}$$

For all $n \geq 3$ and some large $x_4 \geq x_3$, again by Assumption 5.2.1, (5.3.2) and $F \in \mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$, we have that for all $x \geq x_4$,

$$\begin{aligned}
& \Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\right) \\
\leq\ & \int_0^\infty \Pr\left(\sum_{i=2}^{n} X_i + Z_0 > x\mathrm{e}^{rt} \middle| \vartheta_2 = t\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & \int_0^\infty \int_{0-}^{x\mathrm{e}^{rt}-x_4} \Pr\left(X_2 > x\mathrm{e}^{rt} - v \middle| \vartheta_2 = t\right) \Pr\left(\sum_{i=3}^{n} X_i + Z_0 \in \mathrm{d}v\right) G_\vartheta(\mathrm{d}t) \\
& + \int_0^\infty \Pr\left(\sum_{i=3}^{n} X_i + Z_0 > x\mathrm{e}^{rt} - x_4\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & (1+\varepsilon)^2 \int_0^\infty \left((n-1)\left(\mathrm{E}\mathrm{e}^{\gamma X}\right)^{n-2} \mathrm{E}\mathrm{e}^{\gamma Z_0} + \frac{\left(\mathrm{E}\mathrm{e}^{\gamma X}\right)^{n-1}}{\overline{F}(x_2)}\right) \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t) \\
& + (1+\varepsilon) \int_0^\infty \left((n-1)\left(\mathrm{E}\mathrm{e}^{\gamma X}\right)^{n-3} \mathrm{E}\mathrm{e}^{\gamma Z_0} + \frac{\left(\mathrm{E}\mathrm{e}^{\gamma X}\right)^{n-2}}{\overline{F}(x_2)}\right) \mathrm{e}^{\gamma x_4}\overline{F}(x\mathrm{e}^{rt}) G_\vartheta(\mathrm{d}t) \\
=\ & o(\overline{F}(x)). \qquad (5.3.9)
\end{aligned}$$

Plugging (5.3.8) and (5.3.9) into (5.3.7), by Lemma 5.3.1, leads to

$$\Pr\left(S_\infty > x\right) \lesssim (1+\varepsilon)\mathrm{E}\mathrm{e}^{\gamma\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j}\right)} \int_0^\infty \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t). \qquad (5.3.10)$$

It is easy to see that $\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j}$ converges to S_∞ in distribution as $n \to \infty$. Therefore, the asymptotic upper bound can be obtained by the dominated convergence theorem and the arbitrariness of $\varepsilon > 0$,

$$\limsup_{x\to\infty} \frac{\Pr(S_\infty > x)}{\int_0^\infty \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t)} \leq \mathrm{E}\mathrm{e}^{\gamma S_\infty}. \qquad (5.3.11)$$

Similarly, for the lower bound, by Assumption 5.2.1 and (5.3.2), we have that for some sufficiently large x_5 and all $x \geq x_5$,

$$
\begin{aligned}
& \Pr\left(S_\infty > x\right) \geq \Pr\left(S_n > x\right) \\
= & \int_0^\infty \Pr\left(X_1 + \sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} > x\mathrm{e}^{rt} \middle| \vartheta_1 = t\right) G_\vartheta(\mathrm{d}t) \\
\geq & \int_0^\infty \int_{0-}^{x\mathrm{e}^{rt}-x_5} \overline{C}_2\left[\overline{F}(x\mathrm{e}^{rt}-u), \overline{G}_\vartheta(t)\right] \Pr\left(\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} \in \mathrm{d}u\right) G_\vartheta(\mathrm{d}t) \\
\geq & (1-\varepsilon)\int_0^\infty \int_{0-}^{x\mathrm{e}^{rt}-x_5} \overline{C}_{12}\left[0+, \overline{G}_\vartheta(t)\right] \overline{F}(x\mathrm{e}^{rt}-u) \\
& \Pr\left(\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} \in \mathrm{d}u\right) G_\vartheta(\mathrm{d}t) \\
= & (1-\varepsilon)\int_0^\infty \left(\Pr\left(X_1 + \sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} > x\mathrm{e}^{rt}\right)\right. \\
& - \Pr\left(\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} > x\mathrm{e}^{rt}\right) \\
& \left. - \int_{x\mathrm{e}^{rt}-x_5}^{x\mathrm{e}^{rt}} \Pr(X_1 > x\mathrm{e}^{rt}-u) \Pr\left(\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} \in \mathrm{d}u\right)\right) G_{\vartheta_c}(\mathrm{d}t) \\
\geq & (1-\varepsilon)\int_0^\infty \left(\Pr\left(X_1 + \sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} > x\mathrm{e}^{rt}\right)\right. \\
& \left. - 2\Pr\left(\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} > x\mathrm{e}^{rt} - x_3\right)\right) G_{\vartheta_c}(\mathrm{d}t) \\
\sim & (1-\varepsilon)\mathrm{E}\mathrm{e}^{\gamma\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j}} \int_0^\infty \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t), \qquad (5.3.12)
\end{aligned}
$$

where the last step holds due to Lemma 5.3.1 and (5.3.9).

Note that $\mathrm{E}\mathrm{e}^{\gamma\sum_{k=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j}} \to \mathrm{E}\mathrm{e}^{\gamma S_\infty}$, $n \to \infty$. Then, the lower bound is derived

$$
\liminf_{x\to\infty} \frac{\Pr(S_\infty > x)}{\int_0^\infty \overline{F}(x\mathrm{e}^{rt}) G_{\vartheta_c}(\mathrm{d}t)} \geq \mathrm{E}\mathrm{e}^{\gamma S_\infty}. \qquad (5.3.13)
$$

By Lemma 4.4 of Konstantinides et al. (2010), the distribution of the product $X\mathrm{e}^{-r\vartheta_c}$ still belongs to the class $\mathcal{S}(\gamma)$. Therefore, the desired relation (5.2.1) follows from (5.1.2), (5.3.11) and (5.3.13).

5.3.2 Proof of Theorem 5.2.3

It has been pointed out by Klüppelberg and Villasenor (1991) that the inclusion $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma) \cap \mathcal{OS}$ is strict. Furthermore, it is well known that all distributions in the class $\mathcal{L}(\gamma)$ with $\gamma > 0$ belong to the class $\mathcal{R}_{-\infty}$. Consequently, the intersection $\mathcal{L}(\gamma) \cap \mathcal{OS} \cap \mathcal{R}_{-\infty}$ is larger than $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$, $\gamma \geq 0$.

The following lemma comes from Cheng et al. (2012).

Lemma 5.3.2 *Let F_1 and F_2 be two distributions on $[0,\infty)$ with $F_1 \in \mathcal{L}(\gamma) \cap \mathcal{OS}$, $\gamma \geq 0$. Let $F = F_1 * F_2$, if $\overline{F_2}(x) = o\left(\overline{F_1}(x)\right)$, then for each $n \geq 1$,*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) \int_0^\infty \mathrm{e}^{\gamma u} F_2(\mathrm{d}u).$$

Now we turn to the proof of Theorem 5.2.3, which is a modification of that of Theorem 5.2.1. We only give the sketch, and all of the random variables and constants are the same as those in the proof of Theorem 5.2.1. As in (5.3.1) and (5.3.3), for any $\varepsilon > 0$, by $F \in \mathcal{OS} \cap \mathcal{L}(\gamma)$ we have that

$$\begin{aligned}
\Pr\left((Z+X)Y > x\right) &= \int_0^\infty \Pr\left(X + Z > x\mathrm{e}^{rt} | \vartheta = t\right) G_\vartheta(\mathrm{d}t) \\
&\leq (1+\varepsilon) \int_0^\infty \left(\overline{F^{*2}}\left(x\mathrm{e}^{rt}\right) - \overline{F}\left(x\mathrm{e}^{rt}\right)\right) G_{\vartheta_c}(\mathrm{d}t) \\
&\quad + \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt} - x_0\right) G_\vartheta(\mathrm{d}t) \\
&\leq (1+\varepsilon) \left(\int_0^\infty O\left(\overline{F}\left(x\mathrm{e}^{rt}\right)\right) G_{\vartheta_c}(\mathrm{d}t) \right. \\
&\quad \left. + \mathrm{e}^{\gamma x_0} \int_0^\infty \overline{F}\left(x\mathrm{e}^{rt}\right) G_\vartheta(\mathrm{d}t) \right) \\
&= o(\overline{F}(x)),
\end{aligned}$$

which implies that for some large x_2, (5.3.4) holds for all $x \geq x_2$. Then, as above, construct $Z_0 = (Z | Z > x_2)$, whose distribution F_{Z_0} still belongs to the class $\mathcal{L}(\gamma) \cap \mathcal{OS} \cap \mathcal{R}_{-\infty}$. Following the steps of proof of Theorem 5.2.1, similarly to (5.3.9), for $n \geq 2$, we can obtain from Assumption 5.2.1 and (5.3.2) that for all $x \geq x_4$,

$$\begin{aligned}
&\Pr\left(\sum_{i=2}^n X_i \mathrm{e}^{-r\sum_{j=2}^i \vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^n \vartheta_j} > x\right) \\
\leq\ & \int_0^\infty \Pr\left(\sum_{i=2}^n X_i + Z_0 > x\mathrm{e}^{rt} \,\middle|\, \vartheta_2 = t\right) G_\vartheta(\mathrm{d}t) \\
\leq\ & (1+\varepsilon) \int_0^\infty \Pr\left(\sum_{i=2}^n X_i + Z_0 > x\mathrm{e}^{rt}\right) G_{\vartheta_c}(\mathrm{d}t) \\
& + \int_0^\infty \Pr\left(\sum_{i=3}^n X_i + Z_0 > x\mathrm{e}^{rt} - x_4\right) G_\vartheta(\mathrm{d}t).
\end{aligned} \tag{5.3.14}$$

Note that for each $i = 2, \ldots, n$,

$$X_i \overset{\mathrm{d}}{\leq} Z_0,$$

which implies that for all $t \geq 0$,

$$\Pr\left(\sum_{i=2}^{n} X_i + Z_0 > x\mathrm{e}^{rt}\right) \leq \overline{F_{Z_0}^{n*}}(x). \tag{5.3.15}$$

By $F_{Z_0} \in \mathcal{OS}$ and Proposition 2.4 of Shimura and Watanabe (2005), we know that for all $n \geq 1$ and $x \geq 0$,

$$\frac{\overline{F_{Z_0}^{n*}}(x)}{\overline{F_{Z_0}}(x)} \leq CK^n, \tag{5.3.16}$$

where C and K are two positive constants. Then, by (5.3.14) and the dominated convergence theorem, in order to prove

$$\Pr\left(\sum_{i=2}^{n} X_i \mathrm{e}^{-r\sum_{j=2}^{i}\vartheta_j} + Z_0 \mathrm{e}^{-r\sum_{j=2}^{n}\vartheta_j} > x\right) = o(\overline{F}(x)), \tag{5.3.17}$$

it suffices to show that for any fixed $t > 0$,

$$\Pr\left(\sum_{i=2}^{n} X_i + Z_0 > x\mathrm{e}^{rt}\right) = o(\overline{F}(x)). \tag{5.3.18}$$

Indeed, relation (5.3.18) holds because of (5.3.15), (5.3.16) and $F \in \mathcal{OS} \cap \mathcal{R}_{-\infty}$.

Therefore, along the lines of the proof of Theorem 5.2.1, the desired relations (5.3.11) and (5.3.13) follow from (5.3.7), (5.3.8), (5.3.17), (5.3.10), (5.3.12) and by applying Lemma 5.3.2.

5.3.3 Proof of Theorem 5.2.4

Recall that the FGM copula is satisfied by Assumption 5.2.1 for $\theta \in (-1, 1)$ as described in Section 5.2. Therefore, applying Theorem 5.2.2, it is obvious to see that Theorem 5.2.4 holds when $\theta \in (-1, 1)$. Thus, in this subsection, we only prove the case $\theta = -1$. One can easily prove the case $\theta = 1$, by proceeding along the same lines below.

Now we begin the proof of Theorem 5.2.4. Let X' be a random variable, independent of $\{(X, Y)(X_i, Y_i),\ i \in \mathbb{N}\}$, with distribution F. Applying the decomposition in (5.2.3),

$$\begin{aligned}
\Pr\left((X + X')\,Y > x\right) &= (1+\theta)\Pr\left((X^* + X')\,Y^* > x\right) \\
&\quad -\theta\Pr\left((X^*_{\wedge} + X')\,Y^* > x\right) - \theta\Pr\left((X^* + X')\,Y^*_{\wedge} > x\right) \\
&\quad +\theta\Pr\left((X^*_{\wedge} + X')\,Y^*_{\wedge} > x\right) \\
&= (1+\theta)I_1(x) - \theta I_2(x) - \theta I_3(x) + \theta I_4(x).
\end{aligned}$$

When $\theta = -1$, it reduces to

$$\Pr\left((X + X')\, Y > x\right) = I_2(x) + I_3(x) - I_4(x). \tag{5.3.19}$$

We derive from $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ and Lemma 5.3.1 that

$$\begin{aligned} I_2 &= \int_0^1 \Pr\left(X_\wedge^* + X' > xy^{-1}\right) G(\mathrm{d}y) \\ &\sim \mathrm{E}e^{\gamma X_\wedge^*} \int_0^1 \overline{F}(xy^{-1}) G(\mathrm{d}y) \\ &= o(\overline{F}(x)). \end{aligned}$$

Here, we note that $\mathrm{E}e^{\gamma X_\wedge^*} < \infty$ because of $\overline{F_{X_\wedge^*}}(x) = (\overline{F}(x))^2 = o(\overline{F}(x))$ and $\mathrm{E}e^{\gamma X} < \infty$. Similarly, $I_3 = o(\overline{F}(x))$ and $I_4 = o(\overline{F}(x))$. Thus, there exists some large $x_6 > 0$ such that, for all $x > x_6$,

$$\Pr\left((X' + X)Y > x\right) \le \overline{F}(x).$$

Let $R_0 = (X'|X' > x_6)$ be a new conditional random variable, whose distribution still belongs to the intersection $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. As (5.3.5), we claim that

$$Y(X + R_0) \overset{\mathrm{d}}{\le} R_0,$$

which implies that $S_n \overset{\mathrm{d}}{\le} R_0$ for all $n \ge 1$. Letting $n \to \infty$ yields

$$S_\infty \overset{\mathrm{d}}{\le} R_0,$$

implying $\mathrm{E}e^{\gamma S_\infty} < \infty$.

By $F \in \mathcal{S}(\gamma)$ and Lemma 3.4.2, we have that

$$\frac{\Pr(X_\wedge^* Y^* > x)}{\Pr(X^* Y_\wedge^* > x)} \le \frac{\Pr(X_\wedge^* > x)}{\Pr(X^* Y_\wedge^* > x)} \le \frac{(\overline{F}(x))^2}{(\overline{G}(1/2))^2 \overline{F}(2x)} \to 0, \tag{5.3.20}$$

and, clearly,

$$\Pr(X_\wedge^* Y_\wedge^* > x) = \int_0^1 \left(\overline{F}(xy^{-1})\right)^2 \Pr(Y_\wedge^* \in \mathrm{d}y) = o(\Pr(X^* Y_\wedge^* > x)). \tag{5.3.21}$$

Thus, it follows from (5.2.3), (5.3.20) and (5.3.21) that

$$\Pr(XY > x) \sim \Pr(X^* Y_\wedge^* > x). \tag{5.3.22}$$

Let Z' be a random variable, independent of $\{(X, Y)(X_i, Y_i), i \in \mathbb{N}\}$, with $\Pr(Z' > x) \le \overline{F}(x)$ for all $x > 0$, and

$$\Pr(Z' > x) \sim c \Pr(X^* Y_\wedge^* > x), \quad \text{with} \quad c > \mathrm{E}e^{\gamma Z'},$$

which implies that $\Pr(Z' > x) = o(\overline{F}(x))$ by $F \in \mathcal{R}_{-\infty}$. Furthermore, it is easy to see that $Z' \overset{\mathrm{d}}{\leq} R_0$ and $\mathrm{E}e^{\gamma Z'} \leq \mathrm{E}e^{\gamma R_0} < \infty$. Therefore, the positive c always exits. For example, we can choose $c > \mathrm{E}e^{\gamma R_0}$. Since $Y_\wedge^*$ has a finite upper endpoint 1, by $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$, Theorem 1.1 of Tang (2006) and Lemma 2.2 of Tang and Tsitsiashvili (2004) show that the distribution of Z' also belongs to $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. Hence, applying the decomposition of (5.2.3) with $\theta = -1$,

$$\begin{aligned}\Pr\left((X + Z')Y > x\right) &= \Pr\left((X_\wedge^* + Z')Y^* > x\right) + \Pr\left((X^* + Z')Y_\wedge^* > x\right) \\ &\quad - \Pr\left((X_\wedge^* + Z')Y_\wedge^* > x\right) \\ &= J_2(x) + J_3(x) - J_4(x). \end{aligned} \tag{5.3.23}$$

By $\Pr(Z' > x) = o(\overline{F}(x))$ and Lemma 5.3.1,

$$\begin{aligned} J_3(x) &= \int_0^1 \Pr\left(X^* + Z' > x\right) \Pr(Y_\wedge^* \in \mathrm{d}y) \\ &= \mathrm{E}e^{\gamma Z'} \Pr(X^* Y_\wedge^* > x). \end{aligned} \tag{5.3.24}$$

By (5.3.20), Lemma 5.3.1 and $F_{Z'} \in \mathcal{R}_{-\infty}$,

$$\begin{aligned} J_2(x) &= \int_0^1 \Pr\left(X_\wedge^* + Z' > xy^{-1}\right) \Pr(Y^* \in \mathrm{d}y) \\ &\sim \mathrm{E}e^{\gamma X_\wedge^*} \Pr(Z' Y^* > x) \\ &= o\left(\Pr(X^* Y_\wedge^* > x)\right). \end{aligned} \tag{5.3.25}$$

Similarly,

$$J_4(x) = o(\Pr(X^* Y_\wedge^* > x)). \tag{5.3.26}$$

From relations (5.3.23)–(5.3.26), we derive that

$$\Pr\left((X + Z')Y > x\right) \sim \mathrm{E}e^{\gamma Z'} \Pr(X^* Y_\wedge^* > x).$$

By $c > \mathrm{E}e^{\gamma Z'}$, the inequality

$$\Pr\left((X + Z')Y > x\right) \leq \Pr(Z' > x)$$

holds for all $x \geq x_7$, where x_7 is a sufficiently large constant. Hence, we can construct a new conditional random variable $Q_0 = (Z' | Z' > x_7)$ whose distribution also belongs to the intersection $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. By similar arguments as (5.3.5), it follows that

$$(X + Q_0)Y \overset{\mathrm{d}}{\leq} Q_0, \tag{5.3.27}$$

which indicates that $S_n \overset{\mathrm{d}}{\leq} Q_0$ for all $n \geq 1$. Letting $n \to \infty$ yields $S_\infty \overset{\mathrm{d}}{\leq} Q_0$. By similar arguments as (5.3.6), we have that

$$S_\infty \overset{\mathrm{d}}{=} S_n + \tilde{S}_\infty \prod_{j=1}^n Y_j \overset{\mathrm{d}}{\leq} S_n + Q_0 \prod_{j=1}^n Y_j,$$

where $\tilde{S}_\infty$ is an independent copy of S_∞. Therefore, for every $n \geq 1$,

$$\Pr(S_\infty > x) \leq \Pr\left(S_n + Q_0 \prod_{j=1}^{n} Y_j > x\right). \tag{5.3.28}$$

For each $n \geq 1$, introduce

$$U_n = \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j + Q_0 \prod_{j=1}^{n} Y_j \overset{\mathrm{d}}{=} S_n + Q_0 \prod_{j=1}^{n} Y_j, \quad U_0 = Q_0,$$

which satisfies the recursive equation

$$U_n = (U_{n-1} + X_n)\, Y_n.$$

Clearly,

$$\begin{aligned}
\Pr(U_n > x) &= \Pr((U_{n-1} + X_n)\, Y_n > x) \\
&= (1+\theta)\Pr((U_{n-1} + X^*)\, Y^* > x) - \theta \Pr((U_{n-1} + X^*_\wedge) Y^* > x) \\
&\quad -\theta \Pr((U_{n-1} + X^*) Y^*_\wedge > x) + \theta \Pr((U_{n-1} + X^*_\wedge) Y^*_\wedge > x) \\
&= (1+\theta) K_1(x) - \theta K_2(x) - \theta K_3(x) + \theta K_4(x).
\end{aligned} \tag{5.3.29}$$

In the following, we only consider the case $\theta = -1$. The proof of the result for $\theta = 1$ is similar, and the case $\theta \in (-1, 1)$ has been investigated by Theorem 5.2.2. We mainly prove that for each $n \geq 1$,

$$\Pr(U_n > x) \sim \mathrm{E}e^{\gamma U_{n-1}} \Pr(X^* Y^*_\wedge > x) \tag{5.3.30}$$

holds by induction. Using relation (5.3.1) for $n = 1$ and $\theta = -1$, we can write

$$\begin{aligned}
\Pr(U_1 > x) &= \Pr((X_1 + U_0) Y_1 > x) \\
&= \Pr((U_0 + X^*_\wedge) Y^* > x) + \Pr((U_0 + X^*) Y^*_\wedge > x) \\
&\quad - \Pr((U_0 + X^*) Y^*_\wedge > x) \\
&= K_2(x) + K_3(x) - K_4(x).
\end{aligned} \tag{5.3.31}$$

For $K_3(x)$, by $\Pr(U_0 > x) = \Pr(Q_0 > x) = o(\overline{F}(x))$ and Lemma 5.3.1 we obtain that

$$\begin{aligned}
K_3(x) &= \int_0^1 \Pr(U_0 + X^* > xy^{-1}) \Pr(Y^*_\wedge \in \mathrm{d}y) \\
&\sim \mathrm{E}e^{\gamma U_0} \Pr(X^* Y^*_\wedge > x).
\end{aligned} \tag{5.3.32}$$

Similarly to (5.3.25), by (5.3.20) and Lemma 5.3.1 we have that

$$\begin{aligned}
K_2(x) &= \int_0^1 \Pr(U_0 + X^*_\wedge > xy^{-1}) \Pr(Y^* \in \mathrm{d}y) \\
&\sim \mathrm{E}e^{\gamma X^*_\wedge} \Pr(U_0 Y^* > x) \\
&= o\left((X^* Y^*_\wedge > x)\right).
\end{aligned} \tag{5.3.33}$$

Similarly,

$$K_4(x) = o\left((X^*Y_\wedge^* > x)\right). \tag{5.3.34}$$

Thus, it follows from (5.3.31)–(5.3.34) that

$$\Pr(U_1 > x) \sim \mathrm{E}e^{\gamma U_0} \Pr(X^*Y_\wedge^* > x),$$

which means that (5.3.30) holds for $n = 1$. Now we assume that relation (5.3.30) holds for $n-1$ and then we prove that it holds for $n \geq 2$. Since $F \in \mathcal{R}_{-\infty}$, by our induction assumption, it is easy to show that

$$\Pr\left(U_{n-1} > x\right) = o(\overline{F}(x)). \tag{5.3.35}$$

Moreover, by $F \in \mathcal{S}(\gamma)$, Theorem 1.1 of Tang (2006) gives that the product $X^*Y_\wedge^*$ follows a distribution in the class $\mathcal{S}(\gamma)$, and $Y_\wedge^*$ has an upper endpoint 1. This, combined with the induction assumption, yields $F_{U_{n-1}} \in \mathcal{S}(\gamma)$. Thus, by Lemma 5.3.1 we have that

$$\Pr(U_{n-1} + X^* > x) \sim \mathrm{E}e^{\gamma U_{n-1}} \overline{F}(x),$$

which implies that

$$K_3(x) = \int_0^1 \Pr(U_{n-1} + X^* > xy^{-1}) \Pr(Y_\wedge^* \in \mathrm{d}y) \sim \mathrm{E}e^{\gamma U_{n-1}} \Pr(X^*Y_\wedge^* > x). \tag{5.3.36}$$

As for $K_2(x)$, similarly to (5.3.20), by $F \in \mathcal{S}(\gamma)$ and Lemma 3.4.2, we have that

$$\frac{\Pr\left(X_\wedge^* > x\right)}{\Pr\left(U_{n-1} > x\right)} \leq \frac{(\overline{F}(x))^2}{\Pr(XY > x)} \sim \frac{(\overline{F}(x))^2}{\Pr(X^*Y_\wedge^* > x)} \to 0.$$

Hence, by $F_{U_{n-1}} \in \mathcal{S}(\gamma)$, Lemma 5.3.1 and the induction assumption, we have that

$$\Pr(U_{n-1} + X_\wedge^* > x) \sim \mathrm{E}e^{\gamma X_\wedge^*} \Pr(U_{n-1} > x) \sim \mathrm{E}e^{\gamma X_\wedge^*} \mathrm{E}e^{\gamma U_{n-2}} \Pr(X^*Y_\wedge^* > x),$$

which implies that

$$\begin{aligned}
K_2(x) &= \int_0^1 \Pr(U_{n-1} + X_\wedge^* > xy^{-1}) G(\mathrm{d}y) \\
&\sim \mathrm{E}e^{\gamma X_\wedge^*} \mathrm{E}e^{\gamma U_{n-2}} \int_0^1 \Pr(X^*Y_\wedge^* > xy^{-1}) G(\mathrm{d}y) \\
&= \mathrm{E}e^{\gamma X_\wedge^*} \mathrm{E}e^{\gamma U_{n-2}} \Pr(X^*Y_\wedge^* Y^* > x) \\
&= \mathrm{E}e^{\gamma X_\wedge^*} \mathrm{E}e^{\gamma U_{n-2}} \int_0^1 \Pr(X^*Y^* > xy^{-1}) \Pr(Y_\wedge^* \in \mathrm{d}y) \\
&= o(1) \int_0^1 \overline{F}(xy^{-1}) \Pr(Y_\wedge^* \in \mathrm{d}y) \\
&= o(\Pr(X^*Y_\wedge^* > x)),
\end{aligned} \tag{5.3.37}$$

where the fifth step holds because of the fact

$$\Pr(X^*Y^* > x) = \int_0^1 \overline{F}(xy^{-1})G(\mathrm{d}y) = o(\overline{F}(x)),$$

due to $F \in \mathcal{R}_{-\infty}$ and the dominated convergence theorem. Similarly,

$$K_4(x) = o(\Pr(X^*Y_\wedge^* > x)). \tag{5.3.38}$$

Plugging (5.3.36)–(5.3.38) into (5.3.29) leads to (5.3.30) holding for n.

It follows from (5.3.28) and (5.3.30) that

$$\begin{aligned} \Pr(S_\infty > x) &\leq \Pr(U_n > x) \\ &\sim \mathrm{E}e^{\gamma U_{n-1}} \Pr(X^*Y_\wedge^* > x). \end{aligned}$$

Note that $U_{n-1} \stackrel{\mathrm{d}}{=} S_{n-1} + Q_0 \prod_{j=1}^{n-1} Y_j$ converges to S_∞ in distribution as $n \to \infty$. Therefore, the asymptotic upper bound is established by

$$\limsup_{x\to\infty} \frac{\Pr(S_\infty > x)}{\Pr(X^*Y_\wedge^* > x)} \leq \mathrm{E}e^{\gamma S_\infty}.$$

As for the asymptotic lower bound, similarly to (5.3.30), we have that

$$\begin{aligned} \Pr(S_\infty > x) &\geq \Pr(S_n > x) \\ &\sim \mathrm{E}e^{\gamma T_{n-1}} \Pr(X^*Y_\wedge^* > x), \end{aligned}$$

where $T_n = \sum_{i=1}^n X_i \prod_{j=1}^i Y_j \stackrel{\mathrm{d}}{=} S_n,\ n \geq 1$, satisfies the recursive equation

$$T_n = (T_{n-1} + X_n)Y_n.$$

Clearly, the sum T_{n-1} converges to S_∞ in distribution as $n \to \infty$ as well. Therefore, we derive the lower bound

$$\liminf_{x\to\infty} \frac{\Pr(S_\infty > x)}{\Pr(X^*Y_\wedge^* > x)} \geq \mathrm{E}e^{\gamma S_\infty}.$$

Appendix A

Below we provide the Matlab code used for numerical studies in Chapter 3.

Here is the Matlab code for Montle Carlo simulation of the finite time ruin with light-tailed case :

```
 % this is a script to simulate the finite ruin probability using Montle
% Carlo method.
clc;clear all;
%% Parameters
% Number of MC simulations, the N in the paper
NumberOfSims =10^7;

% Discrete time horizon, the n in the paper
TimeHorizon_n = 4;
% Risk free rate, the r_i in the paper
RiskFree_r = 0.0125;
% Proportion invested in the risky assets, the pi_i in the paper
Risky_pi = 0.8; p_hat = 0.05;
% Initial capital, the x in the paper
for Capital_x =3600:4000
% From the copula we choose, the following parameters stands
copula_theta = 0.8;
%% Generate two independent random uniform V1 and V2
% V_1
V1_i = unifrnd(0,1,[TimeHorizon_n,NumberOfSims]);
% V_2
V2_i = unifrnd(0,1,[TimeHorizon_n,NumberOfSims]);
temp_array_a = 1 + copula_theta .* (1 - 2 .* V1_i );
temp_array_b = sqrt ( temp_array_a .^2 - 4 .* (temp_array_a -1) .* V2_i);
% output U1_i, U2_i;
U1_i = V1_i ;
```

```
U2_i = 2.* V2_i ./ (temp_array_a + temp_array_b);

%% Random number generation
% X_i: for the case r=0 we choose a light-tailed IG distribution
% for inverse gaussian distribution we need: mu and lambda
mu = 1; lambda = 1;
% generate a random variate from a normal distribution with mu=0 and sig=1
W_i = normrnd(0,1,[TimeHorizon_n,NumberOfSims]);
P_i = W_i .^2;
IG_i = mu + mu^2 .* P_i/(2*lambda)-mu/(2*lambda) * sqrt(4*mu*lambda
      .*P_i+mu^2 *P_i.^2);
% genrate all needed random numbers in a matrix
if U1_i > mu *(mu + IG_i).^(-1);
    X_i = IG_i;
else
X_i = mu^2 * IG_i .^(-1);
end

% Y_i: we use a uniform distribution range from 0 to y_hat
% details see: doc unifrnd
y_hat = 1 / ((1-Risky_pi) * (1 + RiskFree_r));
if U2_i > (1-p_hat);
    Y_i = y_hat;
else Y_i = y_hat .*  U2_i .* (1-p_hat);
end
% Cumprod_Y_i: the cum-product of Y_i as will be used in each scenario
Cumprod_Y_i = cumprod(Y_i);
% Cumprod_Y_i: the cum-product of Y_i as will be used in each scenario
Cumprod_Y_i = cumprod(Y_i);


%% Monte Carlo Simulation part
% For each simulation, we first calculate all the outcomes at each discrete
% time point, then do the summation, and find the maximum at last.

% outcomes_each: individual outcome at each discrete time point in a matrix
outcomes_each = X_i .* Cumprod_Y_i;
% outcomes_all: all the outcomes with summation at all possible time point
outcomes_all = cumsum(outcomes_each);
```

```
% find the maximum of each simulation path, put in an array
outcoees_max = max(outcomes_all);

% if the maximum > initial capital take as 1, else as 0.
ruin_counts = sum(outcoees_max > Capital_x);

%% Out put the ruin probability
ruin_pro =ruin_counts / NumberOfSims;
z(Capital_x - 3599) = Capital_x;
y1(Capital_x - 3599) = ruin_pro;
fprintf('The ruin probability within finite time is: %.12f.\n',ruin_pro);
% plot(Capital_x,ruin_pro,'+'); hold on;
end;
plot(z, smooth(y1,0.25,'rloess'),'k'); hold on;
```

Here is the Matlab code for Montle Carlo simulation of the finite time ruin on the asymptotic solutions with light-tailed case :

```
% this is a script to simulate the finite ruin probability using CMC on the
% asymptotic solutions.
clc;clear all;
%% Parameters
% Number of MC simulations, the N in the paper
NumberOfSims = 10^7;

% Discrete time horizon, the n in the paper
TimeHorizon_n = 4;
% Risk free rate, the r_i in the paper
RiskFree_r = 0.0125;
% Proportion invested in the risky assets, the pi_i in the paper
Risky_pi = 0.8;
% Initial capital, the x in the paper
for Capital_x =3600:4000
% From the copula we choose, the following parameters stands
copula_theta = 0.8; p_hat = 0.05;
% y_hat as in the paper
y_hat = 1 / ((1-Risky_pi) * (1 + RiskFree_r));
% for inverse gaussian distribution we need: mu and lambda
mu = 1; lambda = 1;
% gamma as a parameter
```

```
gamma = lambda/(2*mu^2);
%% We estimate the asymptotic solutions in three parts:
%  Ruin = constant * the C part * the probability part
%  Since the independence of each part, we start with the C part

%% CMC on the C part
% simulation trials on the expectation estimation
% (to save time we set NumberOfSims and est_E_sims the same)
est_E_sims = NumberOfSims;
% Random number generations


%% Random number generation
% X_i: for the case r=0 we choose a light-tailed IG distribution
% for inverse gaussian distribution we need: mu and lambda
mu = 1; lambda = 1;
% generate a random variate from a normal distribution with mu=0 and sig=1
W_i = normrnd(0,1,[est_E_sims,1]);
P_i = W_i .^2;
IG_i = mu + mu^2 .* P_i/(2*lambda)-mu/(2*lambda) * sqrt(4*mu*lambda
      .*P_i+mu^2 *P_i.^2);
% genrate all needed random numbers in a matrix
V1_i= unifrnd(0,1,[est_E_sims,1]);
% genrate all needed random numbers in a matrix
if V1_i > mu *(mu + IG_i).^(-1);
    X_i = IG_i;
else
X_i = mu^2 * IG_i .^(-1);
end
% make a distribution for inverse guassian
pd = makedist('inversegaussian', 'mu',mu,'lambda', lambda);
% for inverse gaussian distribution
syms x



% calulation of what is inside of the expectation for all times
% a temporary row vector
for i=1:TimeHorizon_n-1
```

```
fun_i = @(x) (1 + copula_theta * (1 - p_hat) * (1-2 .* cdf(pd,x))).* pdf(pd,x);
inside_exp_all_i =  integral( fun_i,0,Inf);
end;
% % calulation all the expectations
% workout C_gamma the C part as in the paper
C_gamma = cumprod(inside_exp_all_i); C_gamma = C_gamma(end);


%% CMC on the probability part
% Random number generations
% to save time, we use the random X_i's generated before

% Y_i: we use a uniform distribution range from 0 to y_hat
% details see: doc unifrnd

% V_2
V2_i = unifrnd(0,1,[TimeHorizon_n,NumberOfSims]);
% genrate all needed random numbers in a matrix
if V2_i > (1-p_hat);
    Y_i = y_hat;
else
Y_i = y_hat .*  V2_i .* (1-p_hat);
end

% Cumprod_Y_i: the cum-product of Y_i as will be used in each scenario
Cumprod_Y_i = cumprod(Y_i); Cumprod_Y_i = Cumprod_Y_i(end,:);
% if the X_i .* Cumprod_Y_i > Capital_x) capital take as 1, else as 0.
pro_part = sum(X_i .* Cumprod_Y_i' > Capital_x) / NumberOfSims;

%% Out put the ruin probability
ruin_pro =   (1 + copula_theta * (1 - p_hat)) * C_gamma * pro_part;
z(Capital_x - 3599) = Capital_x;
y2(Capital_x -3599) = ruin_pro;
fprintf('The ruin probability within finite time is: %.12f.\n',ruin_pro);
% plot(Capital_x,ruin_pro,'+'); hold on;
end;
plot(z, smooth(y2,0.25,'rloess'),'--k'); hold on;
```

Here is the Matlab code for simulation of the finite time ruin using EVT with light-tailed case :

```
% this is a script to simulate the finite ruin probability using EVT on the
% asymptotic solutions.
clc;clear all;
%% Parameters

% Number of MC simulations, the N in the paper
NumberOfSims = 10^7;

% Discrete time horizon, the n in the paper
TimeHorizon_n = 4;
% Risk free rate, the r_i in the paper
RiskFree_r = 0.0125;
% Proportion invested in the risky assets, the pi_i in the paper
Risky_pi = 0.8;
% Initial capital, the x in the paper
for Capital_x =3200:3700
% From the copula we choose, the following parameters stands
copula_theta = 0.8; p_hat = 0.05;
% y_hat as in the paper
y_hat = 1 / ((1-Risky_pi) * (1 + RiskFree_r));
% for inverse gaussian distribution we need: mu and lambda
mu = 1; lambda = 1;
% gamma as a parameter
gamma = lambda/(2*mu^2);
%% CMC on the probability part
% Random number generations
% to save time, we use the random X_i's generated before

% Y_i: we use a uniform distribution range from 0 to y_hat
% details see: doc unifrnd

% V_2
V2_i = unifrnd(0,1,[1,NumberOfSims]);
% genrate all needed random numbers in a matrix
if V2_i > (1-p_hat);
    Y_i = y_hat;
else
Y_i = y_hat .*  V2_i * (1-p_hat);
end
```

```
%% We estimate the asymptotic solutions in three parts:
%  Ruin = constant * the C part * the probability part
%  Since the independence of each part, we start with the C part
% make a distribution for inverse guassian
pd = makedist('inversegaussian', 'mu',mu,'lambda', lambda);
% for inverse gaussian distribution
syms x



% calulation of what is inside of the expectation for all times
% a temporary row vector
for i=1:TimeHorizon_n-1
fun_i = @(x) (1 + copula_theta * (1 - p_hat) * (1-2 .* cdf(pd,x))).* pdf(pd,x);
inside_exp_all_i =  integral( fun_i,0,Inf);
end;
% % calulation all the expectations
% workout C_gamma the C part as in the paper
C_gamma = cumprod(inside_exp_all_i); C_gamma = C_gamma(end);
% B_hat
Fun_B = @(x)(1-cdf(pd,x));
B_hat= integral(Fun_B,Capital_x/(y_hat^(TimeHorizon_n)),Inf)
      /(1-cdf(pd,Capital_x/(y_hat^(TimeHorizon_n))));
pro_part1=1-cdf(pd,Capital_x/((y_hat)^(TimeHorizon_n)));
pro_counts=sum(Y_i>y_hat -(y_hat^(TimeHorizon_n+1)/Capital_x)* B_hat);
pro_part2=pro_counts/NumberOfSims;
%% Out put the ruin probability
ruin_pro = C_gamma*(1 + copula_theta * (1 - p_hat))
          * pro_part1* (pro_part2^TimeHorizon_n);
z(Capital_x - 3199) = Capital_x;
y3(Capital_x - 3199) = ruin_pro;
fprintf('The ruin probability within finite time is: %.12f.\n',ruin_pro);
end;
plot(z, smooth(y3,0.25,'rloess'),'-.k'); hold on;
```

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