PERSISTENT MARKOV PARTITIONS FOR RATIONAL MAPS

MARY REES

Abstract. A construction is given of Markov partitions for some rational maps, which persist over regions of parameter space, not confined to single hyperbolic components. The set on which the Markov partition exists, and its boundary, are analysed.

The first result of this paper is a construction of Markov partitions for some rational maps, including non-hyperbolic rational maps (Theorem 1.1). Of course, results of this type have been around for many decades. We comment on this below. There is considerable freedom in the construction. In particular, the construction can be made so that the partition varies isotopically to a partition for all maps in a sufficiently small neighbourhood of the original one (Lemma 2.1). So the partition is not specific, like the Yoccoz puzzle, and also less specific than other partitions which have been developed to exploit the ideas on analysis of dynamical planes and parameter space which were pioneered using the Yoccoz puzzle. We then investigate the boundary of the set of rational maps for which the partition exists in section 2, in particular in Theorem 2.2. We also explore the set in which the partition does exist, in section 3, in particular in Theorem 3.2. We show how parameter space is partitioned, using a partition which is related to the Markov partitions of dynamical planes – in much the usual manner – and show that all the sets in the partition are nonempty. We are able to apply some of the results of [14] in our setting, in particular in the analysis of dynamical planes. The main tool used in the results about the partitioning the subset of parameter space admitting a fixed Markov partition is the λ-lemma [11].

It is natural to start our study with hyperbolic rational maps. For some integer N which depends on f, the iterate f^N of a hyperbolic map f is expanding on the Julia set $J = J(f)$ with respect to the spherical metric. The full expanding property does not hold for a parabolic rational map on its Julia set, but a minor adjustment of it does. Given any closed subset of the Julia set disjoint from the parabolic orbits, the map f^N is still expanding with respect to the spherical metric, for a suitable N.

Definition A *Markov partition for f* is a set $\mathcal{P} = \{P_1, \cdots P_n\}$ such that:

- $\overline{\text{int}(P_i)} = P_i$;
- P_i and P_j have disjoint interiors if $i \neq j$;
- $\cup_{i=1}^n P_i = \overline{\mathbb{C}}$;
- each P_i is a union of connected components of $f^{-1}(P_j)$ for varying j.

1. Construction of partitions

We shall use the following definition of Markov partition for a rational map $f : \overline{\mathbb{C}}$.

Our first theorem applies to a familiar "easy" class of rational maps. In particular, we assume that every critical orbit is attracted to an attractive or parabolic periodic orbit. The most important property of the Markov partitions yielded by this theorem, however, is that the set of rational maps for which they exist is open – and if this open set contains a rational map with at least one parabolic periodic point, the open set is not contained in a single hyperbolic component.

Theorem 1.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map such that every critical point is in the Fatou set, and such that the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Let F_0 be the union of the periodic Fatou components. Let Z be a finite forward invariant set which includes all parabolic points. Let $G_0 \subset \overline{\mathbb{C}}$ be a connected piecewise C^1 graph such that the following hold.*

- *All components of $\overline{\mathbb{C}} \setminus G_0$ are topological discs, as are the closures of these components.*
- *$G_0 \cap (F_0 \cup Z) = \emptyset$, any component of $\overline{\mathbb{C}} \setminus G_0$ contains at most component of $F_0 \cup Z$, and G_0 has at most one component of intersection with any Fatou component.*
- *G_0 is trivalent, that is, exactly three edges meet at each vertex.*
- *The closures of any two components of $\overline{\mathbb{C}} \setminus G_0$ intersect in at most a single component, which, if it exists, must be either an edge together with the endpoints of this edge, or a single vertex, by the previous conditions.*

Then there exists $G' \subset \overline{\mathbb{C}} \setminus (F_0 \cup Z)$ isotopic to G_0 in $\overline{\mathbb{C}} \setminus (F_0 \cup Z)$ and such that $G' \subset f^{-N}(G')$ for some N. Given any neighbourhood U of G_0, G' can be chosen with $G' \subset U$, for sufficiently large N.

Moreover, there is a connected graph $G \subset \bigcup_{i \geq 0} f^{-i}(G')$ with finitely many vertices and edges, with $G \subset f^{-1}(G)$, and such that:

1. *any point of G is connected to some point of $f^{-i}(G')$, for some $0 \leq i < N$, by a path which crosses at most two components of $\overline{\mathbb{C}} \setminus f^{-i}(G')$;*

(2) any component of $\overline{\mathbb{C}} \setminus G$ contains at most one periodic Fatou component.

Hence $\mathcal{P} = \{\overline{U} : U$ is a component of $\overline{\mathbb{C}} \setminus G\}$ is a Markov partition for f, such that each set in the partition contains at most one periodic Fatou component. The boundary of the closure of any component of $\overline{\mathbb{C}} \setminus G$ is a quasi-circle.

For quite some time, I thought that there was no general result of this type in the literature, that is, no general result giving the existence of such a graph and related Markov partition for a map f with expanding properties. To some extent, this is true. One would expect to have a result of this type for smooth expanding maps of compact Riemannian manifolds, for which the derivative has norm greater than one with respect to the Riemannian metric. I shall call such maps *expanding local diffeomorphisms.* Of course, an expanding map of a compact metric space is never invertible. Also, a rational map is never expanding on the whole Riemann sphere, unless one allows the metric to have singularities — because of the critical points of the map. A hyperbolic rational map is an expanding local diffeomorphism on a neighbourhood of the Julia set, but such a neighbourhood is not forward invariant. The invertible analogue of expanding local diffeomorphisms is Axiom A diffeomorphisms. There is, of course, an extensive literature on these, dating from the 1960's and '70's. The existence of Markov partitions for Axiom A diffeomorphisms was proved by Rufus Bowen [3], who developed the whole theory of describing invertible hyperbolic systems in terms of their symbolic dynamics in a remarkable series of papers. Bowen's results are in all dimensions. The construction of the sets in these Markov partitions is quite general, and the sets are not shown to have nice properties. In fact results appear to be in the opposite direction: [4], for example, showing that boundaries of Markov partitions of Anosov toral diffeomorphisms of the three-torus are never smooth – a relatively mild, but interesting pathology, which, in itself, has generated an extensive literature.

The existence of Markov partitions for expanding maps of compact metric spaces appears as Theorem 4.5.2 in the recent book by Przytycki and Urbanski [12]. But there is no statement, there, about topological properties of the sets in the partition. I only learnt relatively recently (from Feliks Pzrytycki, among others) about the work of F.T. Farrell and L.E. Jones on expanding local diffeomorphisms, in particular about their result in dimension two [7]. Their result is a version of the statement in Theorem 1.1 – more general in some respects – about an invariant graph G for f^N for a suitably large N. In the Farrell-Jones set-up, f is an expanding local diffeomorphism of a compact two-manifold. Unaware of their result, the first version of this paper included my own proof of the theorem above – which of course has different hypotheses from the Farrell-Jones result. Other such results have also been obtained relatively recently in other contexts, for example by Bonk and Meyer in [2], where Theorem 1.2 states that the n'th iterate $F = f^n$ of an expanding Thurston map f admits an invariant Jordan curve, if n is sufficiently large, and consequently, by Corollary 1.5, F admits cellular Markov partitions of a certain type. My proof made an assumption of conformality which is not in the Farrell-Jones result. I also claimed a proof for f, rather than f^N. There is no such result in [7]. If $G' \subset \overline{\mathbb{C}}$ is a graph satisfying $G' \subset f^{-N}(G')$, then the set $G^0 = \bigcup_{i=0}^{N-1} f^{-i}(G')$ satisfies $G^0 \subset f^{-1}(G^0)$. But G^0 might not be a graph with finitely many edges and vertices. In the first version of this paper a proof was given that G^0 was, nevertheless, such a graph. However, on seeing the result, Mario Bonk and others warned that the method of proof did not appear to take account of counter-examples in similar contexts, and was likely to be flawed - as indeed it was.

The statement now proved is not that G^0 itself is a graph with finitely many vertices and edges — although it might well be — but that there is such a graph $G \subset \bigcup_{i \geq 0} f^{-i}(G')$ with $G \subset f^{-1}(G)$. I have now changed to proof of the existence of the graph for f^N, for sufficiently large N, to one closer to the Farrell-Jones result. Perhaps not surprisingly, the first part of the proof – both the statement and proof of Lemma 1.3 below – are very close to those of Farrell-Jones, even though they were produced independently. But the proof (in 1.5) that the limit graph is homeomorphic to the graphs in the sequence has been changed to essentially that of Farrell-Jones, avoiding the use of conformality. This is partly because their proof is shorter and more straightforward, and partly because variants of these ideas are needed in any case, in the proof, in 1.6 to 1.10, for f rather than f^N.

As a corollary of Theorem 1.1, we have the following.

Corollary 1.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with connected Julia set J, such that the forward orbit of each critical point is attracted to an attractive or parabolic periodic orbit, and such that the closure of any Fatou component is a closed topological disc, and all of these are disjoint. Then there exists a graph $G \subset \overline{\mathbb{G}}$ such that the following hold.*

(1) $G \subset f^{-1}G$.
(2) G does not intersect the closure of any Fatou component.
(3) All components of $\overline{\mathbb{C}} \setminus G$ are topological discs.
(4) Any component of $\overline{\mathbb{C}} \setminus G$ contains at most one periodic Fatou component of f.
(5) The boundary of any component of $\overline{\mathbb{C}} \setminus G$ is a quasi-circle.

In particular, the set of closures of components of $\overline{\mathbb{C}} \setminus G$ *is a Markov partition for* f*.*

Proof. We can choose the graph G_0 of Theorem 1.1 to satisfy the conditions of 1.1 and also property 2 above. Then choose the neighbourhood U of G_0 of 1.1 sufficiently small and N sufficiently large that any path from U to the closure of a periodic Fatou component must cross at least 3 components of $\overline{\mathbb{C}} \setminus f^{-N}(U)$. Then any such path must also cross at least 3 components of $\overline{\mathbb{C}} \setminus f^{-N}(G')$. So by condition 2 of Theorem 1.1, G is disjoint from the closures of periodic Fatou components. □

The first step in the proof of 1.1 is a lemma about the existence of subgraphs – which, as already stated, parallels methods in Farrell-Jones, section 1 of [7].

Lemma 1.3. *Let* f*,* F_0*,* Z *and* G_0 *be as in 1.1. Let* $F(G_0)$ *denote the union of* G_0 *and all sets* $\overline{F}$ *such that* F *is a Fatou component intersected by* G_0*. Then the following holds for* δ *sufficiently small given* δ_1*. Let* Γ *be another graph which also has these properties, and such that every component of* $\overline{\mathbb{C}} \setminus \Gamma$ *within* $2\delta_1$ *of* $F(G_0)$ *is either within* δ *of a Fatou component intersected by* G_0*, or has diameter* $< \delta$*. Then there is a subgraph* G_1 *of* Γ *which is in the* δ_1*-neighbourhood of* $F(G_0)$*, such that* G_1 *can be isotoped to* G_0 *in this neighbourhood.*

Remark Many of the vertices of G_1 are likely to be bivalent rather than trivalent, but these are the only types which occur.

Proof. Perturbing all intersections of G_0 with Fatou components to the boundaries of those components, and, if this creates non-transverse self-intersections, a bit beyond, we can assume that G_0 is contained in the Julia set of f. The hypotheses on Γ then ensure that there is a point of Γ within δ of each point of G_0. Write $\delta_0 = \delta_1/3$ and suppose $\delta < \delta_1/18$. So now we aim to find $G_1 \subset \Gamma$ within a $3\delta_0$-neighbourhood of G_0 itself, which can be isotoped to G_0 in this neighbourhood. We identify a vertex $v_1 = v_1(v)$ of Γ within δ of each vertex v of G_0. These are to be the vertices of G_1. Let G_v be the connected component of $G_0 \cap B_{\delta_0}(v)$ which contains v. We shall find three arcs in $\Gamma \cap B_{\delta_0}(v_1)$ starting from v_1, disjoint apart from the starting point at v_1, with exactly one ending within δ of each of the endpoints of G_v. Suppose that we can find such sets for each vertex v of Γ. We denote by $G_{v,1}$ this union of $v_1(v)$ and the three attached arcs in Γ. Then $G_{v,1}$ can be isotoped to G_v within a $3\delta_0$-neighbourhood of G_v (because $B_{\delta_0}(v_1)$ has diameter $2\delta_0$, and $B_{\delta_0}(v_1) \subset B_{\delta_0+\delta}(v)$). We can assume without loss of generality that the $2\delta_1$-neighbourhoods of vertices of G_0 are disjoint, and let δ_2 be such that the $2\delta_2$-neighbourhoods of the components of $G_0 \setminus \bigcup_v G_v$ are disjoint. Now we assume that $\delta < \delta_2$. Then for each edge of G_0 between a pair of vertices v and v_2, there is a unique pair of endpoints from $G_{v,1}$ and $G_{v_2,1}$ within δ of this edge, and we can find an arc between them in G, staying within δ of the edge in G_0. The resulting path between $v_1(v)$ and $v_1(v_2)$ might not be an arc, but if it is not an arc, then any self-intersections only occur within 2δ of $\partial B_{\delta_0}(v) \cup \partial B_{\delta_0}(v_2)$, and the path can then be replaced by an arc in Γ within δ of this path. The construction ensures that all the arcs between distinct pairs of vertices are disjoint, apart from intersections at the vertices. The union of these arcs, joined at the vertices $v_1(v)$, is then the required graph G_1.

So it remains to construct the sets $G_{v,1}$. For the moment, all edges and vertices referred to are edges and vertices of G. By a 2-*cell* (*of* G) we mean the closure of a component of $\overline{\mathbb{C}} \setminus \Gamma$. We fix a vertex $v_1 = v_1(v)$ of Γ, given a vertex v of G_0. We denote by $S_r(v_1)$ the collection of points which can be reached by crossing at most r 2-cells from v_1. Then each $S_r(v_1)$ is a connected topological surface with boundary. $S_1(v_1)$ is a closed topological disc, but in general $S_r(v_1)$ might have several boundary components. However, so long as $S_r(v_1) \subset B_{\delta_0}(v_1)$, there is a particular boundary component $\partial_1 S_r(v_1)$ which separates $S_r(v_1)$ from $\partial B_{\delta_0}(v_1)$. Now we claim we can draw three arcs from v_1 in $\Gamma \cap B_{\delta_0}(v_1)$ which successively cross $\partial_1 S_r(v_1)$ for increasing r, so long as $S_r(v_1) \subset B_{\delta_0}(v_1)$.

We prove this by induction on r. It is true for $r = 1$ because there are three edges from v_1 to $\partial S_1(v_1)$. Suppose inductively it is true for $r - 1$, and suppose that $S_r(v_1) \subset B_{\delta_0}(v_1)$. By definition, $\partial_1 S_r(v_1)$ and $\partial_1 S_{r-1}(v_1)$ are disjoint. All we need are three disjoint paths through edges joining these two closed curves. To do this, we consider the 2-cells in $S_r(v_1) \setminus \mathrm{int}(S_{r-1}(v_1))$ which have boundary intersecting $\partial_1 S_r(v_1)$. Any such 2-cell must also have boundary intersecting $\partial_1 S_{r-1}(v_1)$. There are at least three vertices on $\partial S_r(v_1)$, because otherwise we have two 2-cells with disconnected intersection between boundaries. There must also be at least three vertices in the boundaries of these 2-cells on $\partial_1 S_{r-1}(v_1)$, for the same reason. If there are not three disjoint paths through these 2-cell boundaries from $\partial_1 S_r(v_1)$ to $\partial_1 S_{r-1}(v_1)$, then there must be two 2-cells which intersect in at least one vertex on each of $\partial_1 S_r(v_1)$ and $\partial_1 S_{r-1}(v_1)$, and do not intersect along the edges in between. But then, once again, there are two 2-cells with disconnected intersection.

We can continue this process for $S_r(v_1)$ so long as $S_r(v_1) \subset B_{\delta_0}(v_1)$. So now consider the largest r such that this holds. The diameter of $\partial_1 S_r(v_1)$ is $\geq \delta_0 - \delta > \delta_0/2$. Since we are assuming that $\delta < \delta_1/18$, the same is true for $\partial_1 S_{r-1}(v_1)$ and $\partial_1 S_{r-2}(v_1)$. We can then modify the three arcs to $\partial_1 S_r(v_1)$, by cutting one off at $\partial_1 S_{r-2}(v_1)$, one at $\partial_1 S_{r-1}(v_1)$, and then extending the three arcs round $\partial_1 S_i(v_1)$ for each of $r-2 \leq i \leq r$, to the nearest points on each of these boundaries to the three endpoints of G_v. For δ sufficiently small given δ_2, the endpoints of these arcs are distance $> 4\delta$ apart. The arcs can then be extended by disjoint arcs outside $S_i(v_1)$ (for $r-2 \leq i \leq r$ in the respective cases) in Γ, to within δ of each of the endpoints of G_v, as required.

□

We will prove Theorem 1.1 using Lemma 1.4.

Lemma 1.4. *Let f, Z, G_0 be as in 1.1 to 1.3. As in 1.3, let $F(G_0)$ be the union of G_0 and the closures of any components of the Fatou set of f which are intersected by G_0. Let U be a neighbourhood of $F(G_0)$ with C^1 boundary such that the distance from any point of $F(G_0)$ to ∂U is at least a proportion bounded from 0 of the distance of that point from $\partial F(G_0)$. Let $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Then for all sufficiently large N, depending on G_0, ε and λ, there are a graph G_1 and a piecewise C^1 homeomorphism h of $\overline{\mathbb{C}}$ such that:*

- *$G_1 \subset f^{-N}(G_0)$ and G_1 is contained in the ε - neighbourhood of $F(G_0)$;*
- *h is isotopic to the identity, is the identity outside U_1, where $f^N(U_1) = U$ and $\overline{U_1} \subset U$, and $h(G_0) = G_1$;*
- *$g = f^N \circ h$ is expanding on G_0, and f^N is expanding on U_1 (using the spherical derivative), both with expansion constant $\geq \lambda^{-1}$, and $g(U_1) = U$ for a set U_1 containing G_0 with $\overline{U_1} \subset U$.*

Proof. If N is sufficiently large given δ then every component of $f^{-N}(\overline{\mathbb{C}} \setminus G_0)$ either has spherical diameter $< \delta$, or is within the δ-neighbourhood of some Fatou component. This is simply because, if B_1 is any closed set, and S is any univalent local inverse of f^n defined on an open set B_2 containing B_1 then the diameter of SB_1 tends to zero uniformly with n, independent of S. This is true whenever f has no Siegel discs or Herman rings, so holds under our assumptions. In fact, our assumptions ensure that we can take B_2 to be any open set which is disjoint from the closures of the critical forward orbits. In particular, we can take B_2 to be a sufficiently small neighbourhood of the closure B_1 of any component of $\overline{\mathbb{C}} \setminus G_0$ which does not contain a periodic Fatou component. We can also take B_1 to be any closed simply -connected set in $\overline{W_1 \setminus W_2}$ for any component W_1 of $\overline{\mathbb{C}} \setminus G_0$ and periodic Fatou component W_2 with $W_2 \subset W_1$, and in the complement of a neighbourhood of the set of parabolic points. It follows from the fact that G_0 satisfies the properties of 1.3, that $f^{-N}(G_0)$ satisfies the properties of Γ of 1.3 if δ is sufficiently small given δ_1. So, for $\delta_1 < \varepsilon/2$, we choose

$$G_1 \subset f^{-N}(G_0) \cap B_{\delta_1}(G_0)$$

as in 1.3. In particular, G_1 is isotopic to G_0, and the isotopy can be performed within a δ_1-neighbourhood of $F(G_0)$. We assume that δ_1 is sufficiently small that this neighbourhood is tubular and contained in U.

It is clear that, if we choose G_0 to be piecewise C^1, then we can find a piecewise C^1 h isotopic to the identity mapping G_0 to G_1. It remains to show that we can ensure the required expanding properties of $f^N \circ h$. For this, it suffices to bound the derivative of h on G_0 from 0, independently of N, because the minimum of the derivative of f^N on $U_{1,N}$ tends to ∞ with N, where $U_{1,N}$ is the component of $f^{-N}(U)$ which contains G_1. Of course we have to map vertices of G_0 to the nearby vertices of G_1, and edges of G_0 to the corresponding edges of G_1. We have an upper bound on the length of edges of G_0, and of course a lower bound on the corresponding edges of G_1 which is independent of N. We choose N sufficiently large, and write $U_{1,N} = U_1$. We have $\overline{U_1} \subset U$, provided that N is sufficiently large. So we choose h to be the identity outside U_1, a homeomorphism inside U_1 to have constant derivative restricted to each edge of G_0. The lower bound on the derivative of h on G_0 then comes from the upper bound on the length of edges of G_0.

□

1.5. **Proof of Theorem 1.1 for some N.** Let G_0 and G_1 be the graphs as in Lemma 1.4, and let h and U_1 be as in Lemma 1.4, so that f^N is expanding on U_1 and $f^N \circ h = g$ is expanding on G_0, and h is the identity outside U_1. Inductively, define U_{n+1} to be the component of $f^{-N}(U_n)$ contained in U_n, for all $n \geq 1$. Define $h = h_1$, and, inductively, for $n \geq 2$, define $h_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be h_{n-1} outside U_n and $f^N \circ h_n = h_{n-1} \circ f^N$ on U_n. It follows that $f^N \circ h_n = h_{n-1} \circ f^N$ on $\overline{\mathbb{C}}$, for all $n \geq 2$, while $f^N \circ h_1 = g$ on $\overline{\mathbb{C}}$. Also, define $G_n = h_n(G_{n-1})$ and $\varphi_n = h_n \circ \cdots h_1$ for all $n \geq 1$. Then inductively we see that

$$f^N \circ \varphi_n = \varphi_{n-1} \circ g \text{ on } \overline{\mathbb{C}},$$

for all $n \geq 1$, where we define φ_0 to be the identity. Hence

$$f^{Nk} \circ \varphi_n = \varphi_{n-k} \circ g^{Nk} \text{ on } \overline{\mathbb{C}}$$

for all $0 \leq k \leq n$. If follows that, as $\lambda^{-1} > 1$ is the expansion constant of f^N on U_1,

$$d(\varphi_{n+1}(w), \varphi_n(w)) \leq \lambda^n d(w, \varphi_1(w))$$

for all $w \in U$. It follows that φ_n converges uniformly on U to a continuous $\varphi : U \to U$. The set $G' = \varphi(G_0)$ is then the required graph with $G' \subset f^{-N}(G')$, provided that φ is a homeomorphism on G_0.

To show that $\varphi : G_0 \to \varphi(G_0)$ is a homeomorphism, it suffices to show that it is injective. Let $\mathcal{P}_0$ be the set of closures of components of $\overline{\mathbb{C}} \setminus G_0$. Then $\mathcal{P}_0$ is a partition of $\overline{\mathbb{C}}$. We denote by $f^{-n}(\mathcal{P}_0)$ and $g^{-n}(\mathcal{P}_0)$ the partitions of $\overline{\mathbb{C}}$ for which the sets are the components of sets $f^{-n}(P)$ and $g^{-n}(P)$ respectively, for $P \in \mathcal{P}_0$. Then $h = \varphi_1$ maps sets in the partition $g^{-1}(\mathcal{P}_0)$ to sets in the partition $f^{-N}(\mathcal{P}_0)$, and φ_n maps sets in the partition $g^{-n}(\mathcal{P}_0)$ to sets in the partition $f^{-nN}(\mathcal{P}_0)$, for all $n \geq 1$. We now consider the intersection of the partition $g^{-n}(\mathcal{P}_0)$ with G_0. Let x, $y \in G_0$ with $x \neq y$. We want to show that $\varphi(x) \neq \varphi(y)$. For sufficiently large n, any path between x and y must cross at least five sets in the partition $g^{-n}(\mathcal{P}_0)$. So any path between $\varphi_n(x)$ and $\varphi_n(y)$ must cross at least five sets in the partition $f^{-nN}(\mathcal{P}_0)$. But $\varphi_m(x)$ and $\varphi_{m+1}(x)$ are in the same element of the partition $f^{-mN}(\mathcal{P}_0)$ for all m. We claim that $\varphi_m(x)$ is in the same, or adjacent, partition element of $f^{-n}(\mathcal{P}_0)$, as $\varphi_n(x)$, for all $m \geq n$, and similarly for y, which is enough to show that $\varphi(x) \neq \varphi(y)$. We see this as follows. If N is sufficiently large given N_1, we can ensure that any path between two sets in the partition $\mathcal{P}_0$ which are not adjacent must cross at least N_1 sets in the partition $f^{-N}(\mathcal{P}_0)$. It follows that, for any i, any path between two sets in the partition $f^{-i}(\mathcal{P}_0)$ which are not adjacent must cross at least N_1 sets in the partition $f^{-i-N}(\mathcal{P}_0)$. So if $m \geq n+1$ and $\varphi_{n+1}(x)$ and $\varphi_m(x)$ are in the same or adjacent sets in the partition $f^{-(n+1)N}(\mathcal{P}_0)$, then, since $\varphi_n(x)$ and $\varphi_{n+1}(x)$ are in the same set of the partition $f^{-nN}(\mathcal{P}_0)$, we see that $\varphi_n(x)$ and $\varphi_m(x)$ are in the same, or adjacent, sets in the partition $f^{-nN}(\mathcal{P}_0)$, provided $N_1 \geq 5$. We now assume this, and hence, by induction, $\varphi_n(x)$ and $\varphi_m(x)$ are in the same, or adjacent. set in the partition $f^{-nN}(\mathcal{P}_0)$, for all $m > n$.

1.6. **Start of Proof of Theorem 1.1.** The set $G^0 = \cup_{i=0}^{N-1} f^{-i}(G')$ satisfies $G^0 \subset f^{-1}(G^0)$, but G^0 might not be a finite graph, if $f^{-i}(G')$ and $f^{-j}(G')$ have infinitely many points of intersection for some $0 \leq i < j \leq N-1$. We aim to find a finite connected graph $G \subset \cup_{i\geq 0} f^{-i}(G')$ which satisfies $G \subset f^{-1}(G)$ and the other conditions of 1.1. We shall construct G as $\lim_{n\to\infty} \Gamma_n$, where $\Gamma_n \subset f^{-n}(G^0)$. For this, we need a number of lemmas.

Lemma 1.7. *There is a graph G_1' with $G_1' \subset G' \subset f^{-n}(G_1')$ for some $n \geq 1$, and an integer N_1 dividing N such that $G_1' = f^{N_1}(G_1')$ and there are no arcs in $G_1' \cap f^{-i}(G_1')$ for $0 < i < N_1$.*

Proof. Let

$$G_1' = \bigcap_{k\geq 0} f^{kN}(G').$$

Then $G_1' = f^{k_1N}(G')$ for some $k_1 \geq 0$. Then G_1' is a non-empty subgraph satisfying $f^N(G_1') = G_1'$. We claim that G_1' is connected. For if it is not, then each component of G_1' is contained in a separate component of

$$\bigcup_{i\geq 0} f^{-iN}(G_1') \cap G' = G',$$

and G' itself is disconnected. The set of arcs in $G_1' \cap f^{-i}(G_1')$ is forward invariant under f^N, and by the expanding property of f^N on G_1', must be an open and closed subset of G_1', which is therefore all of G_1'. So if there is an arc in $G_1' \cap f^{-i}(G_1')$, we have $f^i(G_1') = G_1'$, and then we also have $f^{N_1}(G_1') = G_1'$ where N_1 is the gcd of i and N. We then also have

$$f^{N_1}(f^{-iN_1}(G_1')) \subset f^{-iN_1}(G_1')$$

for all $i \geq 0$, and in particular this holds for $i = k_1$, and $f^{-k_1N_1}(G_1')$ is a graph which contains G'. □

So from now on we assume that there are no arcs in $G' \cap f^{-i}(G')$ for $0 < i < N$.

If G^0 is not a finite graph, the set of accumulation points X of

$$\bigcup_{0\leq i<j<N} (f^{-i}(G') \cap f^{-j}(G'))$$

is a closed set without interior. We distinguish between two types of points in X. If x is of the first, and simpler type, for each $0 \leq i < N$ and sufficiently small arc e ending at x in an edge of $f^{-i}(G')$ which ends at x, and for each $0 \leq j < N$, at most one edge of $f^{-j}(G')$ intersects e. The point x must be of this first type unless it is a vertex of $f^{-j}(G')$ for some $0 \leq j < N$. But if it is such a vertex, then there is a second possibility: there might be at least two arcs e and e' of $f^{-j}(G')$ which intersect only at their common endpoint, which is x, and which both intersect the same edge e'' of $f^{-i}(G')$ for some $0 \leq i < N$ with $i \neq j$, and the intersection points of e and e' with e'' accumulate on x.

The set X is forward invariant under f, and is the union of the finitely many sets X_A for sets $A \subset \{i : 0 \leq i < N\}$ with $\#(A) \geq 2$, where X_A is defined to be the set of accumulation points of

$$\bigcap_{i\in A} f^{-i}(G')$$

with $A \subset \{i : 0 \leq i < N\}$. The complement of X in $f^{-\ell}(G')$, for any $0 \leq \ell < N$, is a countable union of intervals, each being a finite intersection of the complementary intervals of sets $\bigcap_{i\in A} f^{-i}(G')$.

Lemma 1.8. *The endpoints of any complementary interval of X in any edge of $f^{-\ell}(G')$, for any $0 \leq \ell < N$, must be eventually periodic.*

Proof. It suffices to show that, for each set $A \subset \{j : 0 \leq j < N\}$ with $\#(A) \geq 2$, and with $0 \in A$, the endpoints of each complementary interval of X_A in G' are eventually periodic. We only need to consider endpoints which are mapped under f^r to $f^{-i}(G')$ if and only if $i - r \in A \bmod N$, that is, endpoints which are mapped under f^{Nj} to X_A for all $j \geq 0$, and not to X_B for any set B properly containing A. To see this, it suffices to show that, for some $\varepsilon > 0$, the forward orbit of each complementary interval of X_A in G' either contains an interval of diameter $\geq \varepsilon$, or has eventually periodic endpoints. In the former case, there are only finitely many of these, and hence they must be eventually periodic. We see this as follows. The image under f^N of a complementary interval of diameter $< \varepsilon$ is strictly larger, and the image of an interval of diameter $\geq \varepsilon$ is also of diameter $\geq \varepsilon$, unless the interval maps forward to contain points of X_A. But if this happens, the complementary interval must contain points of $f^{-N}(X_A) \setminus X_A$, so must contain points of $f^{-N-i}(G') \setminus f^{-i}(G')$ for some $i \in A$. Now for $\varepsilon > 0$ sufficiently small, this means that there must be a vertex of $f^{-i}(G')$ on the arc of $f^{-N-i}(G')$ between the point of X_A and the point of $f^{-N}(X_A)$. Since there are only finitely many vertices, this can only happen for finitely many intervals. So apart from finitely many intervals, each interval eventually maps forward to have length $\geq \varepsilon$, and then has finite forward orbit. □

Now we will construct a new Markov partition of G^0. Of course, the edges of the graphs $f^{-i}(G')$ give a Markov partition of G^0 for f. But the edges might intersect in infinitely many points.

Lemma 1.9. *Given $\varepsilon > 0$, there exists a Markov partition $\mathcal{R}(G^0)$ of G^0, for f, into sets P of diameter $< \varepsilon$ such that $\partial P \cap G^0$ consists of finitely many points, each one of which is either a vertex of $f^{-i}(G')$ for some $0 \leq i < N$ or in X.*

Proof. For $A \subset \{i : 0 \leq i < N\}$, we define $A + 1 = \{i + 1 \bmod N : i \in A$. If $\#(A)$ is maximal, then $f^n(X_A) \subset X_{A+n}$ for all $n \geq 0$. We define

$$X' = \bigcup_{A\subset\{i:0\leq i<N\},\#(A)\geq 1} \{x \in X_A : \text{ if } f^n(x) \in X_B, \text{ then } B \subset A{+}n, \text{ for all } n \geq 0\}.$$

Then $f(X') \subset X'$, and since the possibility $\#(A) = 1$ is included, for each $x \in G^0$, there is $n = n(x)$ such that $f^n(x) \in X'$.

Now we can choose a finite cover $\mathcal{U}_0$ of X' by discs in $\overline{\mathbb{C}}$ of spherical diameter $< \varepsilon/2$. We do this as follows. If $\#(A)$ is maximal subject to $X_A \neq \emptyset$, then $X_A \subset X'$ and, of course, X_A is closed. So we start by taking a finite cover $\mathcal{U}_{1,0}$ of the union of the non-empty X_A for maximal $\#(A)$, such that each set Uin the cover does intersect such X_A and $U \cap G^0 \subset X_A$. Then inductively we take a cover $\mathcal{U}_{i+1,0}$ of

$$\bigcup_B X_B \cap (X' \setminus \cup_{1\le j\le i, U\in\mathcal{U}_j}),$$

where the union in B is over those B for which $\#(B)$ is maximal subject to

$$X_B \setminus \bigcup_{1\le j\le i, U\in\mathcal{U}_{j,0}} \neq \emptyset,$$

for $1 \le i < r$, until we obtain

$$X' \subset \bigcup_{1\le i\le r, U\in\mathcal{U}_{i,0}} U$$

Then we define

$$\mathcal{U}_0 = \bigcup_{i=1}^{r} \mathcal{U}_{i,0}.$$

Then by compactness of G^0, there is $N_0 \ge 0$ such that

$$G^0 \subset \bigcup_{0\le n\le N_0, U\in\mathcal{U}_0} f^{-n}(U).$$

We can assume that ε is small enough that f is a local homeomorphism from $U\cap G^0$ to $f(U)\cap G^0$, for all $U \in \mathcal{U}_0$. Of course $f|U$ is a local homeomorphism, assuming, as we may do, that ε sufficiently small given the bound from the critical points. But if f is not a local homeomorphism to G^0 then some other edge of G^0 must map forward and accumulate on $f(x)$, contradicting maximality.

Now we need to perturb the boundaries of the discs in $\mathcal{U}_0$ so that the intersection of G^0 with each boundary is finite, and in fact consisting of eventually periodic points in X. To do this we take a finite cover of the boundary by open discs in $\overline{\mathbb{C}}$ such that each one intersects at most one edge of $f^{-i}(G')$ for each $0 \le i < N$. We can also assume that for each set V in this cover there is $A(V)$ as before, that is, $X_A(V) \cap V \neq \emptyset$ and

$$X \cup V \subset \bigcup_{B\subset A(V)} X_B.$$

We extend to add to U each arc of an edge between points of $X_B \cap V$, or else remove them all. The arcs are short enough that for each such arc I, we have

$$I \cap X \subset \bigcup_{B\subset A(V)} X_B.$$

The boundaries of the sets in the cover will intersect just finitely many complementary arcs of X_A in each graph $f^{-i}(G')$ (for $0 \le i < N$) and the endpoints of these arcs are eventually periodic. Write $\mathcal{U}_1$ for the collection of sets obtained by modifying the sets of $\mathcal{U}_0$.. We still have that $f : U \cap$

$G^0 \to f(U) \cap G^0$ is a local homeomorphism, for each $U \in \mathcal{U}_1$, assuming ε is sufficiently small. Now let $\mathcal{U}_2$ be the sets of

$$\bigvee_{i=0}^{N_1} f^i(\mathcal{U}_1).$$

For suitable N_1 the sets of $\mathcal{U}_2$, intersected with G^0, form a Markov partition of $\bigcup_{U \in \mathcal{U}_1} U$. Then $f^{-N_0}(\mathcal{U}_2)$ covers G^0. The sets $U \cap G^0$, for $U \in f^{-N_0}(\mathcal{U}_2)$ form the required partition $\mathcal{R}(G^0)$ of G^0.

□

1.10. **The iterative construction of G.** We can extend the partition $\mathcal{R}(G^0)$ of G^0 to a partition $\mathcal{R} = \mathcal{R}(\overline{\mathbb{C}})$ of $\mathbb{C}$ by adding to $P \in \mathcal{R}(G^0)$ any components of $\overline{\mathbb{C}} \setminus G^0$ which are bounded by P, and also adding to $\mathcal{R}$ the closures of any components of the complement of $\cup\{P : P \in \mathcal{R}(G^0)$. Then $\mathcal{R}$ is a Markov partition of $\overline{\mathbb{C}}$ for f. Now, using the partition $\mathcal{R}(G^0)$ of G^0, and the larger partition $\mathcal{R}$ of $\overline{\mathbb{C}}$, we construct the graph G by an iterative process, $G = \lim_{n\to\infty} \Gamma_n$, with $\Gamma_0 \subset G^0$, and $\Gamma_{n+1} \subset f^{-1}(\Gamma_n)$. As might be expected, this is similar to the iterative process used to construct G' from the graphs G_n in 1.5. We start by choosing $\Gamma_0 \subset G^0$ to be a union of finitely many arcs, intersecting only in endpoints, such that in each set P there is a union of arcs between each pair of points in $\partial P \cap G^0$. To do this, we can put circular order on the finitely many points of $G^0 \cap \partial P$, join two adjacent points by an arc in ∂P, then join the next point in the order to these by an arc in $P \cap G^0$ which intersects the first arc only in its endpoint, and then similarly add other arcs to the other points in $\partial P \cap G^0$, proceeding in circular order. Then, since $\mathcal{R}(G^0)$ s a Markov partition, there is a homeomorphism k_1 of $\overline{\mathbb{C}}$ which maps Γ_0 into $f^{-1}(\Gamma_0)$ which is isotopic to the identity via an isotopy fixing the boundary points of the sets P, for $P \in \mathcal{R}(G^0)$. We then define $\Gamma_1 = k_1(\Gamma_0)$. Then we can define sequences k_n of homeomorphisms of $\overline{\mathbb{C}}$ and graphs Γ_n by the properties:

- $f \circ k_{n+1} = k_n \circ f$ for $n \geq 1$ and k_n and k_{n+1} are isotopic via an isotopy which is constant on vertices of $f^{-i}(G')$ (for $0 \leq i < N$) and on boundary points of P, for $P \in \mathcal{P}(G^0)$. For $n \geq 2$, this means that the isotopy between k_n and k_{n+1} is a lift under f of the isotopy between k_{n-1} and k_n.
- $\Gamma_{n+1} = k_{n+1}(\Gamma_n)$ for all $n \geq 0$.

Then we define $\psi_n = k_n \circ \cdots \circ k_1$ for all $n \geq 1$, and ψ_0 to be the identity. Then $f \circ \psi_n = \psi_{n-1} \circ f \circ k_1$ for $n \geq 1$, and $f^s \circ \psi_n = \psi_{n-s} \circ (f \circ k_1)^s$ for all $0 \leq s \leq n$. We note that $(f \circ k_1)(\Gamma_0) = \Gamma_0$. Since f^N is expanding, we see that ψ_n converges to a continuous map ψ which satisfies $f \circ \psi = \psi \circ (f \circ k_1)$.

Then $\psi(\Gamma_0)$ is our required graph G, provided that ψ is injective on Γ_0. Let $\mathcal{Q}(\Gamma_0) = \{P \cap \Gamma_0 : P \in \mathcal{R}\}$. Then $\mathcal{Q}(\Gamma_0)$ is a partition of Γ_0, which is Markov with respect to $f \circ k_1$. We define

$$\mathcal{R}_n(G^0) = f^{-n}(\mathcal{R}(G^0))$$
$$= \{P' : P' \text{ is a component of } f^{-n}(P), \text{ some } P \in \mathcal{R}(G^0)\},$$
$$\mathcal{R}_n = f^{-n}(\mathcal{R}) = \{P' : P' \text{ is a component of } f^{-n}(P), \text{ some } P \in \mathcal{R}\},$$

and

$$\mathcal{Q}_n(\Gamma_0)) = (f \circ k_1)^{-n}(\mathcal{Q}(\Gamma_0))$$
$$= \{Q' : Q' \text{ is a component of } (f \circ k_1)^{-n}(Q), \text{ some } Q \in \mathcal{Q}(\Gamma_0)\}.$$

Then ψ_n maps sets of $\mathcal{Q}_n(\Gamma_0)$ into sets of $\mathcal{R}_n(G^0)$. We can choose k_1 so that the maximum diameter of sets in $\mathcal{Q}_n(\Gamma_0)$ tends to 0 as $n \to \infty$. To do this, note that Γ_0 is a finite union of topological arcs. These arcs have infinite length with respect to the spherical metric, but can be parametrised by intervals. The map $f \circ k_1$ has a Markov matrix A with respect to the partition $\mathcal{Q}(\Gamma_0)$. The largest modulus eigenvalue of A is real and > 1. For A^n has integer entries for all $n \geq 1$, and the eigenvector of the largest modulus eigenvalue has positive entries. If the largest modulus eigenvalue has modulus < 1 then $A^n \to 0$ as $n \to \infty$, and if the largest modulus eigenvalue has modulus 1, then there is an eigenvector with positive entries with modulus 1. Both of these contradict transitivity of f on $f^{n_1}(G^0)$ for some $n_1 > 0$. So we have an eigenvector of A with positive entries and with real eigenvalue $\lambda_1 > 1$. Then we can parametrise the arcs of Γ_0 by intervals of the lengths given by the eigenvector, and define k_1 so that $f \circ k_1$ is linear with respect to these lengths, and use this length as a metric on Γ_0. This metric is compatible with the topology of Γ_0, and $f \circ k_1$ is expanding with respect to this metric, with expansion constant λ_1. The proof of injectivity of ψ is now similar to the proof of injectivity in 1.5. If $x \neq y$, then for all sufficiently large n, x and y are in sets of $\mathcal{Q}_n(\Gamma_0)$ which do not intersect, and the images of these sets under ψ_n, which are in disjoint sets of $\mathcal{R}_n$, are such that any path between these two sets must cross at least five sets of $\mathcal{R}_n$. Then $\psi_{n+m}(x)$ is in the same set in $\mathcal{R}_n$ for all $m \geq 0$, and similarly for $\psi_{n+m}(y)$, and the image under $\psi_{n+m} \circ \psi_n^{-1}$ of any path joining $\psi_n(x)$ and $\psi_n(y)$ crosses the same sets of $\mathcal{R}_n$. Hence any path joining $\psi_{n+m}(x)$ and $\psi_{n+m}(y)$ must cross at least five sets of $\mathcal{R}_n$ for all $m \geq 0$ and the same is true for any path joining $\psi(x)$ and $\psi(y)$, and therefore $\psi(x) \neq \psi(y)$.

For property 2 of 1.1, note that, for each $x \in \Gamma_0$, x and $\psi(x)$ are in the same set of $\mathcal{R}$. But for some $0 \leq i < N$, a set of $\mathcal{R}$ either intersects at most one edge of $f^{-i}(G')$ or at most one vertex of $f^{-i}(G')$. Hence there is a path from x to $\psi(x)$ which crosses at most two components of $\overline{\mathbb{C}} \setminus f^{-i}(G')$.

1.11. **Proof that boundaries are quasi-circles.** Now G is a graph with finitely many edges and vertices, disjoint from $Z \cup F_0$, and $G \subset f^{-1}(G)$. It remains to show that every simple closed loop γ in G is a quasi-circle. We use the characterisation of quasi-circles in terms of the bounded turning property of [1]. For a suitable n, let U_n be the neighbourhood of G consisting of the closures of all components of $f^{-n}(U \setminus G)$ which either intersect G or are separated from G by at most two other components. For a suitable r_0, the ratios of the minimum diameters of paths between two components of $f^{-n}(U \setminus G)$ which are not adjacent, through at most r_0 components, is bounded, where the bound depends only on f and r_0 and the minimum distance between vertices of G. This ratio is independent of n, and $G \subset f^{-n}(G)$ for all n. So for any two points w_1 and $w_2 \in G$ we can choose n so that they are separated by between r_1 and r_0 sets in $f^{-n}(U \setminus G)$, for a suitable n, and the bounded turning property follows.

2. Boundary of existence of Markov partition

The main motivation for constructing Markov partitions as in Section 1 is that Markov partitions with such properties exist on an open subset of a suitable parameter space. One can then use such partitions to analyse dynamical planes of maps in a subset of parameter space, and this subset of parameter space itself, and try to follow at least part of the programme introduced by Yoccoz for quadratic polynomials, and generalised by others, including Roesch [14] to other families of rational maps.

We have the following lemma.

Lemma 2.1. *Let f be a rational map. Let $G \subset \overline{\mathbb{C}}$ be a graph, and U a connected closed neighbourhood of G such that the following hold.*

- *$G \subset f^{-1}(G)$.*
- *U is disjoint from the set of critical values of f.*
- *U contains the component of $f^{-1}(U)$ containing G, and, for some $N > 0$, int(U) contains the component of $f^{-N}(U)$ containing G.*

Then for all g sufficiently close to f in the uniform topology, the properties above hold with g replacing f and a graph $G(g)$ isotopic to the graph $G = G(f)$ above, and varying continuously with g.

In particular, these properties hold for nearby g, if f is a rational map such that the forward orbit of every critical point is attracted to an attractive or parabolic periodic orbit, the closures of any two periodic Fatou components are disjoint, and G is a graph with the properties above, and which is also disjoint from the closure of any periodic Fatou component.

Proof. First we note that the hypotheses do hold for f and G as in the final sentence. For if we take any sufficiently large n given ε, every component of

$\overline{\mathbb{C}} \setminus f^{-n}(G)$ is either within ε of a single Fatou component or has diameter $< \varepsilon$. Under the given hypotheses on f, only finitely many Fatou components of f have diameter $< \varepsilon$. So for sufficiently large n, if we take U to be the union of the closures of all components of $\overline{\mathbb{C}} \setminus f^{-n}(G)$ which intersect G, then U contains the component of $f^{-1}(U)$ containing G. Moreover, if we take N sufficiently large given f and U, the maximum diameter of any component of $f^{-N}(W)$, for any component W of $\overline{\mathbb{C}} \setminus f^{-n}(G)$, is strictly less than the minimum distance of G from ∂U. For this N, the component U_1 of $f^{-N}(U)$ which contains G is contained in $\mathrm{int}(U)$. We can also assume, by taking N sufficiently large, that $f^N : U_1 \to U$ is expanding in the spherical metric, with expansion constant suitably large for what follows. (Proof of expansion is a standard argument, but the proof of a slightly more precise statement is given in Lemma 2.3 below.)

For future purpose, we write G_0 for G. Then for g sufficiently close to f there is a component $U_1(g)$ of $g^{-N}(U)$ which varies isotopically for g near f, with $U_1(f) = U_1$, $U_1(g) \subset U$ and $g^N : U_1(g) \to U$ is sufficiently strongly expanding for the methods of Section 1 to work. It follows that there is a graph $G_1(g) \subset g^{-N}(G_1(g)) \cap U_1(g)$ isotopic to G_0 with $G_1(f) = G_0$. Then as in Section 1 we construct $G_n(g)$ inductively varying isotopically for g near f, with:

- $G_n(f) = G_0$ for all n;
- $G_{n+1}(g) \subset g^{-N}(G_n(g))$ for all n;
- $G_0 = G_0(g)$ for all g near f.

For g sufficiently near f, the expansion constant of g^N on $U_1(g)$ is sufficiently strong that 1.4 holds, and hence we obtain $G(g) = \lim_{n\to\infty} G_n(g)$ isotopic to G_0 with $G(g) \subset \mathrm{int}(U) \cap g^{-N}(G(g))$. We claim that we also have $G(g) \subset g^{-1}(G(g))$. For suppose not so. Then we have an isotopy of $G(g)$ into $g^{-1}(G(g))$, which extends continuously from the inclusion of $G_0 = G_0(f)$ in $f^{-1}(G_0)$. This isotopy lifts to an isotopy of $g^{-N}(G(g))$ into $g^{-N-1}(G(g))$, for which the maximum distance is strictly less. But this gives a contradiction because this lifted isotopy includes the original one. So $G(g) \subset g^{-1}(G(g))$, as required. □

So we see that there are natural conditions under which an isotopically varying graph $G(g)$ exists, with $G(g) \subset g^{-1}(G(g))$, for an open connected set of g which are not all hyperbolic. In fact these open connected sets will intersect infinitely many hyperbolic components. We also have an isotopically varying Markov partition $\mathcal{P}(g)$ given by

$$\mathcal{P}(g) = \{\overline{U} : U \text{ is a component of } \overline{\mathbb{C}} \setminus G(g)\}.$$

We now proceed to investigate the boundary of the set of g in which $G(g)$ and $\mathcal{P}(g)$ exist.

Theorem 2.2. *Let V be a connected component of an affine variety over $\mathbb{C}$ of rational maps V in which the set $Y(f)$ of critical values varies isotopically. Let V_1 be a maximal connected subset of V such that, for $g \in V_1$, there exist a finite connected graph $G(g)$, a closed neighbourhood $U(g)$ of $G(g)$, and an integer $n(g) > 0$ with the following properties.*

- *$G(g)$ varies isotopically with g for $g \in V_1$.*
- *$G(g) \subset g^{-1}(G(g))$.*
- *$\partial U(g) \subset g^{-n(g)}(G(g)) \setminus G(g)$.*
- *$U(g)$ contains the component of $g^{-1}(U(g))$ which contains $G(g)$.*
- *$Y(g) \cap U(g) = \emptyset$.*
- *For any component W of $\overline{\mathbb{C}} \setminus G(g)$, all components of $g^{-1}(W)$ are discs.*

Then if $V_2 \subset V_1$ is a set such that $\overline{V_2} \setminus V_1 \neq \emptyset$, where the closure denotes closure in V, the integer $n(g)$ is unbounded for $g \in V_2$.

Definition We shall say that $Y(g)$ is *combinatorially bounded from $G(g)$ for $g \in V_2$* if $n(g)$ as above is bounded for $g \in V_2$, that is, for some N, there is a closed neighbourhood $U(g)$ of $G(g)$ with boundary in $g^{-N}(G(g)) \setminus G(g)$ which is disjoint from $Y(g)$, for all $g \in V_2$, and such that $U(g)$ contains the component of $g^{-1}(U(g))$ which contains $G(g)$.
Remarks 1. Because the critical value set $Y(g)$ varies isotopically for $g \in V_1$, the set of critical points also varies isotopically.
2. For $g \in V_1$, the hypotheses of 2.1 are satisfied by the set $U = U(g)$ as in Theorem 2.2, with $N = n(g)$ because $g^{-n(g)}(\partial U(g)) \subset g^{-2n(g)}(G(g)) \setminus g^{-n(g)}(G(g))$ is disjoint from $\partial U(g)$.

We now establish some basic properties of the dynamics of g in a neighbourhood of $G(g)$, for $g \in V_1$.

Lemma 2.3. *Let $g \in V_1$, for V_1 as in 2.2 and let $U(g)$ be as in 2.2. Then for sufficiently large N, $g^N : G(g) \to G(g)$ is expanding with respect to the spherical metric. If $U_1(g)$ denotes the component of $g^{-N}((U(g))$ which contains $G(g)$, and the modulus of any component of int$(U(g)) \setminus U_1(g)$ adjacent to $\partial U(g)$ is bounded below, then the expansion constant of g^N is bounded from 1.*

Proof. We write $U(g) = U$ and $U_1(g) = U_1$. Since there are no critical values of g in U, $g^N : U_1 \to U$ is a local isometry on the interior, with respect to the Poincaré metrics on $\text{int}(U_1)$ and $\text{int}(U)$. But the Poincaré metric d_1 on $\text{int}(U_1)$ is strictly larger than the restriction d to $\text{int}(U_1)$ of the Poincaré metric d_1 on $\text{int}(U)$. If modulus$(A) \geq c > 0$ for any component A

of $\text{int}(U)\setminus U_1$ adjacent to ∂U, then $d_1 \geq \mu(c)d$ for $\mu(c) > 1$. So the derivative of g^N on U_1 with respect to the Poincaré metric on $\text{int}(U)$ is strictly $> \mu(c)$ in modulus. $\square$

2.4. **Real-analytic coordinates on $G(g)$.** A key idea in the proof of 2.2 is to use real-analytic coordinates on the graph $G(g)$ for $g \in V_1$, provided by the normalisations of the sets in the complement of the graph. Let $P_i(g)$ be the closures of the components of $\overline{\mathbb{C}} \setminus G(g)$ for $1 \leq i \leq k$, so that

$$G(g) = \bigcup_{i=1}^{k} \partial P_i(g).$$

We have uniformising maps $\varphi_{i,g} : P_i(g) \to \{z : |z| \leq 1\}$ for each $1 \leq i \leq k$, which are holomorphic between interiors, and unique up to post-composition with Möbius transformations. Then we have a collection of maps $\varphi_{j,g} \circ g \circ \varphi_{i,g}^{-1}$, defined on subsets of the closed unit disc, and mapping onto the closed unit disc. Each of these maps is holomorphic on the intersection of its domain with the open unit disc , and extends by the Schwarz reflection principle to a holomorphic map on the reflection $z \mapsto \overline{z}^{-1}$ of this domain in the unit circle. In particular, each such map is real analytic on the intersection of its domain with the unit circle.

Now $g : g^{-1}(P_i(g)) \to P_i(g)$ is a branched covering, and, by assumption, each component of $g^{-1}(P_i(g))$ is conformally a disc, and the closure of each component is a closed topological disc. Let $I(i)$ denote the (finite) set of components of $g^{-1}(P_i(g))$. Let $\psi_{i,g} : g^{-1}(P_i(g)) \to \{z : |z| \leq 1\} \times I(i)$ be a uniformising map, once again, holomorphic on the interior and unique up to post-composition with a Möbius transformation on each component. Then $\varphi_{i,g} \circ g \circ \psi_{i,g}^{-1}$ is a disc-preserving Blaschke product on each of a finite union of discs, mapping each one to the same disc whose degree is the degree of $g|P_i(g)$ — with no other restriction, unless we normalise the maps $\varphi_{i,g}$ and $\psi_{i,g}$ in some way, which we might want to do. Each map $\varphi_{i,g} \circ g \circ \varphi_{j,g}^{-1}$, where defined, is of the form $(\varphi_{i,g} \circ g \circ \psi_{i,g}^{-1}) \circ \psi_{i,g} \circ \varphi_{j,g}^{-1}$. Now we establish an expansion property of these maps.

Lemma 2.5. *Let $X(g)$ denote the vertex set of $G(g)$. Suppose that N is such that for any i and j and component Q of $g^{-N}(P_j(g))$ with $Q \subset P_i(g)$, at least one component of $\partial P_i(g) \setminus \partial Q$ contains at least two vertices of $G(g)$, and the moduli of*

$$\left(\bigcup_{i\in I} P_i(g), g^{-N}(X(g)) \cap \partial(\bigcup_{i\in I} P_i(g))\right)$$

are bounded for any finite set I such that $\bigcup_{i\in I} P_i(g)$ is a topological disc. Then the expansion constants of the maps $\varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1}$ with respect to *the Euclidean metric, where defined, are bounded from* 1 *for some bounded* $\ell \geq 1$.

Remark If D denotes the closed unit disc and $A \subset \partial D$ is a finite set, then we say that the moduli of (D, A) are bounded if A contains less than four points, or if the cross-ratio of any subset of of A consisting of four points is bounded above and below. If Q is a closed topological disc and $B \subset \partial Q$ is finite, then we say that the moduli of (Q, B) are bounded if the moduli of $(\varphi(Q), \varphi(B))$ are bounded, where $\varphi : Q \to D$ is a homeomorphism which is holomorphic on the interior of Q.

Proof. For the maps $\varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1}$, it suffices to bound below the derivative of $\varphi_{i,g} \circ g^N \circ \varphi_{j,g}^{-1}$, with respect to a suitable metric d_p which we can show to be boundedly Lipschitz equivalent to the Euclidean metric d_e. Then the derivative of $\varphi_{i,g} \circ g^{N\ell} \circ \varphi_{j,g}^{-1}$ with respect to to d_p is $\geq \mu^\ell$, and if d_p/d_e is bounded between $C^{\pm 1}$ for some $C \geq 1$, we see that the derivative with respect to d_e is $\geq C^{-1}\mu^\ell$, giving expansion for all ℓ such that $C^{-1}\mu^\ell > 1$. So it remains to define d_p so that these properties are satisfied. This is the restriction of a Poincaré metric on a suitable surface, one for each component e of $\partial Q \cap \partial P_i(g)$, or union of two such components round a vertex of $g^{-N}(G(g))$ in ∂Q, where Q is the closure of a component of $\overline{\mathbb{C}} \setminus g^{-N}(G(g))$ with $Q \subset P_i(g)$ and $e \subset \partial Q$. For each such component, we consider a union Q' of closures of components of $\overline{\mathbb{C}} \setminus g^{-N}(G(g))$ contained in $P_i(g)$, such that Q' is a topological disc and such that the connected component e' of $\partial Q' \cap \partial P_i(g)$ which contains e has e in its interior. We can assume without loss of generality, replacing $G(g)$ by $g^{-M}(G(g))$ if necessary, that the image of Q' under g^N is also a closed topological disc – obviously of the form $\cup_{j \in J} P_j(g)$ — and that g^N is a homeomorphism on e'. So there is a map of Q' to $\{z : |z| \leq 1, \mathrm{Im}(z) \geq 0\}$ which maps e' to the interval $[-1, 1]$, and which is conformal on the interior. We then take the restriction of the Poincaré metric on the unit disc to $(-1, 1)$. This is the metric d_p on $\mathrm{int}(e') \supset e$. The image of e under g^N is an edge of $G(g)$ in $\partial P_j(g)$, or a union of two edges round a vertex in $\partial P_j(g)$, for some $j \in J$. We take the corresponding metric d_p on each edge of $g^{-N}(G(g))$ in $\partial P_j(g)$. Take any edge e_1 of $g^{-N}(G(g))$ or union of two edges of $g^{-N}(G(g))$ which are subsets of edges of $G(g)$, adjacent to a vertex of $G(g)$ in $P_j(g)$, with $e_1 \subset e$. Let Q_1 be the component of $\overline{\mathbb{C}} \setminus g^{-N}(G(g))$, and $e_1 \subset \partial Q_1$ and $Q_1 \subset g^N(Q)$. Let Q_1' be the union of closures of components of $\overline{\mathbb{C}} \setminus g^{-N}(G)$ with $Q_1 \subset Q_1'$ which is used to define the metric d_p on e_1. Then $Q_1' \subset g^N(Q')$, and by the hypotheses, if we double $g^N(Q')$ across $g^N(e')$ by Schwarz reflection, and then normalise, the image of the double of Q_1' within this is contained in

$\{z : |z| \leq r\}$, for some r bounded by 1. It follows that g^N is expanding with respect to the metric d_p, with expansion constant bounded from 1. □

Now each edge of $G(g)$ is in the image of two maps $\varphi_{i_1,g}$ and $\varphi_{i_2,g}$, where the edge is a connected component of $\partial P_{i_1}(g) \cap \partial P_{i_2}(g)$. Since $G(g) \subset g^{-1}(G(g))$, it is also the case that each edge is contained in a union of components of sets $g^{-1}(P_{j_1}(g) \cap P_{j_2}(g))$, where these sets are disjoint apart from some common endpoints. It follows that from g, we obtain two real-analytic maps $h_{1,g}$ and $h_{2,g}$, each mapping a finite union of intervals to itself, mapping endpoints to endpoints, except for being two-valued at finitely many interior points in the intervals, but at these points, the right and left-derivatives exist and coincide, so that the derivative is single valued at such points, and extends continuously in the neighbourhood of any such point. These two maps are quasi-symmetrically conjugate, because the maps $\varphi_{i,g}$ are quasiconformal. The quasi-symmetry is unique, and the pair $(\overline{\mathbb{C}}, g^{-1}(G))$ can be reconstructed from it, up to Möbius transformation of $\overline{\mathbb{C}}$. Now we can make this idea more precise. Lemma 2.5 shows that the hypotheses are satisfied.

Lemma 2.6. *Let $I_{i,r}$ be finite intervals for $1 \leq i \leq k$ and $r = 1, 2$. Let $h_1 : \bigcup_{i=1}^k I_{i,1} \to \bigcup_{i=1}^k I_{i,1}$ and $h_2 : \bigcup_{i=1}^k I_{i,2} \to \bigcup_{i=1}^k I_{i,2}$ be two C^2 maps which are multivalued just at points which are mapped to endpoints of intervals, but with well-defined continuous derivatives at such points, such that $h_r(I_{i,r})$ is a union of intervals $I_{j,r}$ for each of $r = 1, 2$, and $I_{j,1} \subset h_1(I_{i,1})$ if and only if $I_{j,2} \subset h_2(I_{i,2})$, and $I_{i,r} \cap h_1^{-1}(I_{j,r})$ has at most one component, for both $r = 1, 2$. Suppose also that there is N such that h_1^n and h_2^n are expanding with respect to the Euclidean metric for all $n \geq N$. Then h_1 and h_2 are quasi-symmetrically conjugate, with the norm of the quasi-symmetric conjugacy bounded in terms of N and of the bound of the expansion constants of h_1^N and h_2^N from 1.*

Proof. This is standard. We simply choose $\varphi_0 : \bigcup_{i=1}^k I_{i,1} \to \bigcup_{i=1}^k I_{i,2}$ to be an affine transformation (for example) restricted to $I_{i,1}$, mapping $I_{i,1}$ to $I_{i,2}$, for each $1 \leq i \leq k$. Then φ_n is defined inductively by the properties $h_2 \circ \varphi_{n+1} = \varphi_n \circ h_1$ and $\varphi_{n+1}(I_{i,1}) = I_{i,2}$ for each $1 \leq i \leq k$. Then $\varphi_0 \circ h_1^n = h_2^n \circ \varphi_n$ for all n, and we deduce from this that $|\varphi_n(x) - \varphi_{n+1}(x)| \leq C_2\lambda^n$ for all x and n, for some constant C_2 depending on C_1, where $|h_2^n(x) - h_2^n(y)| \geq C_1\lambda^{-n}$ for all n and all x and y such that $h_2^m(x)$ and $h_2^m(y)$ are in the same set $I_{i_m,2}$, for all $0 \leq m \leq n$. Then φ_n converges uniformly to φ, with $\varphi \circ h_1 = h_2 \circ \varphi$. Similarly, using the expanding properties of h_1, we deduce that φ_n^{-1} converges uniformly to φ^{-1}.

To prove quasi-symmetry of φ, we use the standard result that $(h_r^n)'$ varies by a bounded proportion on any interval J such that $h_r^n(J)$ is a union of at most two subintervals of $\bigcup_{i=1}^k I_{i,r}$. This uses continuity of the derivative across the finitely many discontinuities of h_r. So then given any $x \neq y \in \bigcup_{i=1}^k I_{i,1}$ such that $|x-y|$ is sufficiently small, we choose the greatest n such that $|h_1^n(x) - h_1^n(y)| \leq c$, for a suitable constant $c > 0$ such that any interval of $\bigcup_{i=1}^k I_{i,1}$ which has length $\leq c$ is mapped to a union of at most two intervals of $\bigcup_{i=1}^k I_{i,1}$. Then $|h_1^{n+p}(x) - h_1^{n+p}(y)|$ is bounded above and below for any bounded p, and $(h_1^{n+p})'$ varies by a bounded proportion on the interval $[x, y]$. So does the derivative S', on the smallest interval containing $h_1^{n+p}(x)$, $h_1^{n+p}(y)$, where S is the branch of $h_2^{-(n+p)}$ such that $\varphi_{n+p} = S \circ \varphi_0 \circ h_1^{n+p}$. We can choose p so that each of the points $h_1^n(x)$, $h_1^n(y)$, $h_1^n((x+y)/2)$ is separated by at least two points from $\bigcup_{i=1}^k h_1^{-p}(\partial I_{i,1})$ — but only boundedly many, by the bound on p. Now $\varphi_m = \varphi_{n+p}$ on $\bigcup_{i=1}^k h_1^{-(n+p)}(\partial I_{i,1})$ for all $m \geq n+p$, and hence $\varphi = \varphi_{n+p}$ on $\bigcup_{i=1}^k h_1^{-(n+p)}(\partial I_{i,1})$. If z_1, z_2 and z_3 are any three distinct points of $\bigcup_{i=1}^k h_1^{-(n+p)}(\partial I_{i,1})$ which are either between x and y, or the nearest point on one side, then $|\varphi_{n+p}(z_1) - \varphi_{n+p}(z_2)|/|\varphi_{n+p}(z_1) - \varphi_{n+p}(z_3)|$ is bounded and bounded from 0, that is, $|\varphi(z_1) - \varphi(z_2)|/|\varphi(z_1) - \varphi(z_3)|$ is bounded and bounded from 0. But then since $|\varphi(x) - \varphi((x+y)/2)|$ is bounded between some such $|\varphi(z_1) - \varphi(z_2)|$ and $|\varphi(z_1) - \varphi(z_3)|$, and similarly for $|\varphi(y) - \varphi((x+y)/2)|$, we have upper and lower bounds on $|\varphi(x) - \varphi((x+y)/2)|/|\varphi(y) - \varphi((x+y)/2)|$, and quasi-symmetry follows. □

We deduce the following.

Lemma 2.7. *Let V_1 be as in Theoreom 2.2. For $f \in V_1$, let $P_i(f)$, $\varphi_{i,f}$ and $\psi_{i,f}$ be as previously defined. Let $\{g_n : n \geq 0\}$ be any sequence in V_1 such that $Y(g_n)$ is combinatorially bounded from $G(g_n)$ for $n \geq 0$, and let $g_n \to g$. Let $X(g_n)$ denote the vertex set of $G(g_n)$. Then $g \in V_1$ if the moduli of*

$$\left(\bigcup_{i\in I} P_i(g_n), g^{-\ell}(X(g_n)) \cap \partial\left(\bigcup_{i\in I} P_i(g_n)\right)\right)$$

are bounded as $n \to \infty$ for any fixed ℓ, and any finite set I such that $\bigcup_{i\in I} P_i(g_n)$ is a topological disc, and, using this to normalise the maps φ_{i,g_n} and ψ_{i,g_n}, the disc-preserving Blaschke products $\varphi_{i,g_n} \circ g_n \circ \psi_{i,g_n}^{-1}$ are also bounded.

Proof. The bounds on moduli and Blaschke products ensure that the real analytic maps h_{1,g_n} and h_{2,g_n} have derivatives which are bounded above and below. Also, they extend to Blaschke products on neighbourhoods of intervals of the unit circle. By the hypotheses, there is a closed neighbourhood $U(g_n)$ of the graph $G(g_n)$, disjoint from $Y(g_n)$, such that $U(g_n)$ has boundary in $g_n^{-r}(G(g_n))$ for some r independent of n. Moreover, $U(g_n)$ contains the component of $g_n^{-1}(U(g_n))$ containing $G(g_n)$, and there is N independent of n, such that $\text{int}(U(g_n))$ contains the component $U_1(g_n)$ of $g_n^{-N}(U(g_n))$ containing $G(g_n)$. We have seen from 2.5 and 2.6 that the maps h_{1,g_n} and h_{2,g_n} are boundedly quasi-symmetrically conjugate, that is, there is a quasi-symmetric homeomorphism φ_n whose domain is the domain and image of h_{1,g_n} and whose image is the domain and image of h_{2,g_n}, that is, a finite union of intervals in each case, such that $\varphi_n \circ h_{1,g_n} = h_{2,g_n} \circ \varphi_n$.

Then φ_n can be used to define a Beltrami differential μ_n on $\overline{\mathbb{C}}$, which is uniformly bounded independently of n, as follows. This sphere is, topologically, a finite union of discs, with the boundary of each disc written as a finite union of arcs, and with each arc identified with one other, from a different disc, by φ_n in one direction and φ_n^{-1} in the other. It is convenient to identify this sphere with the Riemann sphere $\overline{\mathbb{C}}$, in such a way that each of the discs has piecewise smooth boundary, and the maps identifying the copies of the closed unit disc with the image discs in $\overline{\mathbb{C}}$ are piecewise smooth. The union of the images of copies of the unit circle form a graph $\Gamma \subset \overline{\mathbb{C}}$. We then define a quasi-conformal homeomorphism ψ_n from the union of copies of the closed unit disc to $\overline{\mathbb{C}}$ such that, whenever I_1 and I_2 are arcs on the boundaries of discs D_1 and D_2, identified by $\varphi_n : I_1 \to I_2$, we have ψ_n on I_2 is defined by $\psi_n \circ \varphi_n^{-1}$, using $\varphi_n^{-1} : I_2 \to I_1$ and $\psi_n : I_1 \to \overline{\mathbb{C}}$. The q-c norm of ψ_n can clearly be bounded in terms of the q-s norm of φ_n, and the identification we choose of the copies of the closed unit disc with their images in $\overline{\mathbb{C}}$. This means that the q-c norm of φ_n can be bounded independently of n. We then define $\mu_n = (\varphi_n)_* 0$ on the image of each copy of the open unit disc, where 0 simply denotes the Beltrami differential which is 0 everywhere on the open unit disc. Then μ_n is defined a.e. on $\overline{\mathbb{C}}$, and is uniformly bounded, in n, in the L_∞ norm.

So there is a quasi-conformal map $\chi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, with q-c norm which is uniformly bounded in n, such that $\mu_n = \chi_n^* 0$, where, here, 0 denotes the Beltrami differential which is 0 everywhere on $\overline{\mathbb{C}}$. By construction, there is a conformal map of $\overline{\mathbb{C}}$ which maps $\chi_n(\Gamma)$ to $G(g_n)$. So we can assume without loss of generality that $\chi_n(\Gamma) = G(g_n)$. By taking limits, we can assume that χ_n has a limit χ in the uniform topology, which is a quasi-conformal homeomorphism. So $\chi_n(\Gamma)$ has a limit $\chi(\Gamma)$, which is also a graph, and since $G(g_n) \subset g_n^{-1}(G(g_n))$, we have $\chi(\Gamma) \subset g^{-1}(\chi(\Gamma))$. The sequence of sets $U(g_n)$ also has a limit $U(g)$ with boundary in $g^{-r}(\chi(\Gamma))$, such that $U(g)$ is a closed neighbourhood of $g(\chi(\Gamma))$, contains the component of $g^{-1}(U(g))$ which contains $\chi(\Gamma)$, and such that $\text{int}(U(g)$ contains the component of $g^{-N}(U(g))$ containing $\chi(\Gamma)$. So we have $g \in V_1$, with $\chi(\Gamma) = G(g)$, as required. □

Since $G(g)$ varies isotopically in V_1, the set $X(g)$ of vertices of $G(g)$ also varies isotopically in V_1. But $X(g)$ is a finite forward invariant set for all $g \in V_1$. Hence $X(g)$ varies locally isotopically for g in the dense open subset V_0 of V such that the multiplier of any periodic points in $X(g)$ is not 1, and there are no critical points in $X(g)$. We have $V_1 \subset V_0$.
Definition A path α with endpoints in $X(g)$ *has homotopy length* $\leq M$ if it can be isotoped, by an isotopy which is the identity on $X(g)$, to be arbitrarily uniformly close to a path in $G(g)$ which crosses $\leq M$ edges of $G(g)$.

Lemma 2.8. *Let V and V_1 be as in 2.2. Let V_0 be as above. Fix $g_0 \in V_1$. Let W_0 be a path-connected compact subset of V containing g_0, and let $M_0 > 0$ be given. There is $M_1 = M_1(M_0, W_0)$ with the following property. Let $g \in V_1$ be joined in V to g_0 by a path in V_0. If e is an edge of $G(g)$ and $e' \subset e$ is a connected set which shares its first endpoint with e, and α is any extension of e' by spherical length $\leq M_0$ to a path with both endpoints in $X(g)$, then α has homotopy length $\leq M_1$.*

Proof. Let g_t be a path between g_0 and $g = g_1$. Since $V \setminus V_0$ has codimension two, we can assume without loss of generality, enlarging W_0 if necessary, that $g_t \in V_0 \cap W_0$ for all t, so that $X(g_t)$ varies isotopically. We can choose the path g_t so that its length is bounded in terms of W_0, using any suitable Riemannian metric on V, for example, that coming from the embedding of V in $\mathbb{C}^m$ (since V is an affine variety).

Now given $N > 1$, there is k such that $g^k(e'')$ is a union of at least N edges for each edge e'' of $G(g)$. This is true for all $g \in V_1$, because the dynamics of the map $g : G(g) \to G(g)$ is independent of g. We take $N = 2$. For this k (or, indeed, any strictly positive integer), $\bigcup_{\ell \geq 0} g^{-\ell k}(X(g))$ is dense in $G(g)$, because, for any edge e of $G(g)$, the maximum diameter of any component of $g^{-n}(e)$ tends to 0 as $n \to \infty$. So it suffices to prove the lemma for $e' \subset e$ sharing first endpoint with e and with the second endpoint in $g^{-\ell k}(X(g))$ for some $\ell \geq 0$, but we cannot obtain any bound on ℓ. So fix such an e'. For each $i \leq \ell$, let $e_{ik} = e_{ik}(g) \subset e$ such that $g^{ik}(e_{ik})$ is an edge of $G(g)$, hence with endpoints in $X(g)$, such that the second endpoint of e' is in e_{ik}, and is not the first endpoint of e_{ik}.

Any point of $\overline{\mathbb{C}}$ is spherical distance $\leq \pi$ from a point of $X(g)$ (assuming the sphere has radius 1). Any path of bounded (spherical) distance between points of $X(g)$ is homotopically bounded, because of the bounded distance between $X(g_0)$ and $X(g)$. We suppose for contradiction that, for any path α_0 of length $\leq M_0$ from the second endpoint of e' to a point of $X(g)$, the path $e' * \alpha_0$ has homotopy length $\geq M_1$. Then $g^k(e' \cup \alpha_0)$ has homotopy length $\geq 2M_1$. Now let α_k be a path of length $\leq M_0$ connecting the second endpoint of $g^k(e')$ to $X(g)$. Now we have a bound on the homotopy length of $g^k(e' \setminus e_k)$ depending only on k, because this is a union of a number of edges of $G(g)$, where the number is bounded in terms of k. We also have a bound in terms of k and M_0 (and on g_0, but g_0 is fixed throughout) on the spherical length of $\overline{\alpha_k} * g^k(\alpha_0)$, where $\overline{\alpha_k}$ denotes the reverse of α_k. This is because the bound on the path between g_0 and g gives a bound on the spherical derivative of g^k in terms of M_0 and k. If φ is the homeomorphism of $\overline{\mathbb{C}}$ given by the isotopy from the identity mapping $X(g)$ to $X(g_0)$, then φ is bounded in terms of M_0. So we have a bound on the spherical length of $\varphi(\overline{\alpha_k} * g^k(\alpha_0))$. This is a path between points of $X(g_0)$. So we have a bound on the homotopy length of this path in terms of M_0 and k (and g_0, but this is fixed throughout). But the homotopy length is the same as the homotopy length of $\overline{\alpha_k} * g^k(\alpha_0)$. So both $g^k(e' \setminus e_k)$ and $\overline{\alpha_k} * g^k(\alpha_0)$ have homotopy length $\leq M_0'$ where M_0' is bounded in terms of M_0 and k. So then $g^k(e' \cap e_k) * \alpha_k$ has homotopy length $\geq 2M_1 - 2M_0' > M_1$ assuming that M_1 is sufficiently large given M_0' and k, that is, sufficiently large given M_0. Similarly, for each i, $g^k((e' \cap e_{(i-1)k}) \setminus e_{ik})$ and $\overline{\alpha_{ik}} * g^k(\alpha_{(i-1)k})$ have homotopy length $\leq M_0'$, and hence we prove by induction that $g^{ik}(e_{ik} \cap e')$ has homotopy length $> M_1$ for all $i \geq 0$. For $i = \ell$ we obtain the required contradiction, because $g^{\ell k}(e' \cap e_{\ell k})$ is a single edge. □

Corollary 2.9. *Let V, V_1, g_0, M_0, W_0 and g be as in 2.8. There is $M_2 > 0$, depending on M_0, W_0 and g_0 with the following property. If e' is any path in an edge of $G(g)$ then e' is homotopic, via a homotopy fixing endpoints and $X(g)$, to a path of (spherical) length $\leq M_2$.*

Proof. It suffices to prove this for paths with one endpoint at $X(g)$, because $e' = \overline{e_1'} * e_2'$ for two such paths in the same edge as e'. So now assume that e' shares an endpoint with e. Then by 2.8, we can extend e' by spherical length $\leq M_0$ to a path α with both endpoints in $X(g)$ so that α is homotopic, via a homotopy fixing $X(g)$, to an arbitrarily small neighbourhood of a path crossing $\leq M_1$ edges of $G(g)$. Because the movement of $X(g_0)$ to $X(g)$ is bounded, this means that α is homotopic, via a homotopy fixing $X(g)$, to a path of spherical length $\leq M_2'$. Then since e' can be obtained from α by adding length M_0, we obtain the required bound on γ with $M_2 = M_2' + M_0$. □

Lemma 2.10. *Let V, V_1, g_0, M_0, W_0 and g be as in 2.8. There is $\varepsilon > 0$ depending on M_0 and g_0 such that for each i, there is some point in $P_i(g)$ which is distance $\geq \varepsilon$ from $\partial P_i(g)$.*

Proof. It suffices, for some $x \in P_i(g)$ and for some fixed n, to find a lower bound on the length of $g^n\alpha$, where α is any path from x to $\partial P_i(g)$. By

2.9, we can extend $g^n\alpha$ by a path γ in some $\partial P_j(g)\cap g^n(\partial P_i(g))$ to a point of $X(g)$, such that γ is homotopic, via a homotopy fixing endpoints and $X(g)$, to a path of length $\leq M_2$, and such that any extension of γ at the other end by a path of length $\leq M_0$ to a point of $X(g)$ has homotopy length $\leq M_1$, by 2.8. Both M_1 and M_2 are independent of n. But we can choose $x\in g^{-n}(X(g))$, for some n, so that if α' is any path from x to $\partial P_i(g)\cap X(g)$ then the homotopy length of $g^n\alpha'$ is $>M_3$, where M_3 is sufficiently long to force spherical length $>2M_2$. We do this using the bound on the isotopy distance between $X(g)$ and $X(g_0)$, and the dynamics of g_0 on the graph $G(g_0)$. Then the spherical length of $g^n\alpha$ is $>M_2$, which gives us a strictly positive lower bound on the spherical length of α: in terms if n, which means, ultimately, in terms of M_0. □

In a similar way, we can prove the following.

Lemma 2.11. *Let V, V_1, g_0, M_0 , W_0 and g be as in 2.8. Let A be any embedded annulus which is a union of $N_1\geq 1$ components of sets $g^{-r}(P_i(g))$ (for varying i) surrounding a union of $N_2\geq 1$ components of sets $g^{-r}(P_j(g))$ (for varying j). Then the modulus of A is bounded and bounded from* 0*, where the bounds depend on N_1, N_2, M_0, g_0 and r.*

Proof. It suffices to prove this with $r=0$, since the result remains true under branched covers, just depending on r and the degree of g_0. The upper bound is clear, from the bound on the diameter of the sets $P_i(g)$ from 2.8 and on the lower bound on the interior of sets $P_j(g)$ in 2.10. Actually a lower bound on the diameter of the sets $P_j(g)$ is enough, and this is easily obtained. So now we need to bound the modulus below. For this, we need to bound below the length (in the spherical metric) of any path γ between the two boundary components of A. As in 2.10, it suffices to bound below the length of $g^n(\gamma)$, for some fixed n, and it suffices to show that this length tends to ∞ with n. As in 2.10, it suffices to prove this for paths with endpoints in $X(g)$, in distinct components of ∂A, and this length tends to ∞ because of the bounded homotopy distance of points in $X(g)$ from $X(g_0)$, and the homotopy length tends to ∞. □

Then using this, we can prove the following.

Lemma 2.12. *Let V, V_1, g_0, M_0, W_0 and g be as in 2.8. The moduli of $\left(\bigcup_{i\in I}P_i(g),g^{-N}(X(g))\cap\partial\left(\bigcup_{i\in I}P_i(g)\right)\right)$ are bounded whenever $\bigcup_{i\in I}P_i(g)$ is a topological disc.*

Proof. . Write $Q=\bigcup_{i\in I}P_i(g)$, for any fixed I such that Q is a topological disc. If (x_1,x_2,x_3,x_4) is an ordered quadruple of four points of $\partial Q\cap g^N(X(g))$, with $x_1$and x_2 not separated in ∂Q by the set $\{x_3,x_4\}$, then we define the modulus of (x_1, x_2, x_3, x_4) to be the modulus of the rectangle $\varphi(Q)$ where φ is conformal on the interior and the vertices are the points $\varphi(x_i)$. In turn, we define modulus to be the modulus of the annulus formed by identifying the edge of the rectangle joining $\varphi(x_1)$ and $\varphi(x_2)$ to the edge joining $\varphi(x_3)$ and $\varphi(x_4)$. So it suffices to bound below the modulus of each such quadruple (x_1, x_2, x_3, x_4). But then it suffices to do it in the case when x_1 and x_2 come from adjacent points of $g^{-N}(X(g))$ on ∂Q, and similarly for x_3 and x_4, because modulus$(A_1) \leq$ modulus(A_2) if $A_1 \subset A_2$ and the inclusion is injective on π_1. But if we have two disjoint edges on ∂Q, we can make an annulus which includes Q and encloses a union of partition elements $P_j(g)$. The partition elements $P_j(g)$ are those with edges on one path in ∂Q between the edges associated with (x_1, x_2) and (x_3, x_4). So the lower bound on the modulus of (x_1, x_2, x_3, x_4) comes from the lower bound of this annulus, which was obtained in 2.11. □

2.13. **Proof of Theorem 2.2.** We recall that we are making the assumption that $Y(g_n)$ is combinatorially bounded from $G(g_n)$. We need to check that the assumptions of Lemma 2.7 are satisfied, since Theorem 2.2 will then immediately follow. Lemma 2.12 gives the bounds on the moduli of $\left(\bigcup_{i\in I} P_i(g_n), g_n^{-N}(X(g_n)) \cap \partial\left(\bigcup_{i\in I} P_i(g_n)\right)\right)$. By 2.11, the set $Y(g_n)$ is bounded from $G(g_n)$ by a union of annuli of moduli bounded from 0. Together with the bound on the moduli of $(P_i(g_n), g_n^{-N}(X(g_n)) \cap \partial P_i(g_n))$, which is just used for normalisation, this gives the required bound on the Blaschke products $\varphi_{i,g_n} \circ g_n \circ \psi_{i,g_n}^{-1}$ of 2.7, and the proof is completed.

3. Parametrisation of existence set of Markov partition

In Section 2, the parameter space V was a connected component of an affine variety over $\mathbb{C}$. In this section, we put more restrictions on V. In particular, the restrictions include that V is of complex dimension one. This means that we are looking at a familiar scenario, in which it is reasonable to suppose that parameter space can be described by movement of a single critical value. It is certainly possible that the ideas generalise to higher dimensions. But there are still new features to consider, even for V of complex dimension one.

We consider the case when V is a parameter space of quadratic rational maps g with numbered critical points for which one critical point $c_1(g)$ is periodic of some fixed period and the other, $c_2(g)$, is free to vary. The family of such maps, quotiented by Möbius conjugation, is of complex dimension one, and is well known to have no finite singular points. (See, for example, Theorem 2.5 of [8].) So V, or a natural quotient of it, is a Riemann surface, with some punctures at ∞, where the degree of the map degenerates. So we assume from now on that V is a Riemann surface. We write $v_1(g) = g(c_1(g))$ and $v_2(g) = g(c_2(g))$ for the critical values. Fix a critically finite $g_0 \in V$ for which a graph $G(g_0)$ exists with $G(g_0) \subset g_0^{-1}(G(g_0))$ and $v_2(g_0) \notin G(g_0)$. There are simple conditions on $G(g_0)$ under which the results of Section 2 hold. It is enough to assume that $G(g_0)$ does not intersect the boundary of any periodic Fatou component, separates critical values and separates periodic Fatou components. In particular, this ensures that the diameters of the components of $\overline{\mathbb{C}} \setminus f^{-n}(G_0)$, with closures intersecting $G(g_0)$, tend to 0 as $n \to \infty$. Write

$$\mathcal{P} = \mathcal{P}(g_0) = \{\overline{U} : U \text{ is a component of } \overline{\mathbb{C}} \setminus G(g_0)\}.$$

We write $V(G(g_0), g_0)$ for the largest connected set of $g \in V$ containing g_0 for which there exists a graph $G(g)$ varying isotopically from $G(g_0)$ with:

- $G(g) \subset g^{-1}(G(g))$;
- a neighbourhood $U(g)$ of $G(g)$ with boundary in $g^{-r}(G(g))$ for some r and not containing $v_2(g)$;
- $U(g)$ contains the component of $g^{-1}(G(g))$ which contains $G(g)$;
- $G(g)$ separates the critical values $v_1(g)$ and $v_2(g)$.

Thus, $V(G(g_0), g_0)$ is the set V_1 defined directly after 2.1, if we replace f and $G(f)$ by g_0 and $G(g_0)$, and assume suitable conditions, as above, on $G(g_0)$. We write $V(G(g_0))$ for the union of sets $V(G(g_1), g_1$ for which there is a homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\varphi(G(g_0)) = G(g_1), \quad \varphi(g_0^i(v_1(g))) = g_1^i(v_1(g)) \text{ for } i \geq 0, \quad \varphi(v_2(g_0)) = v_2(g_1),$$

and

$$\varphi \circ g_0 = g_1 \circ \varphi \text{ on } G(g_0).$$

Thus, $G(g)$ exists for all $g \in V(G(g_0))$, and varies isotopically on each component of $V(G(g_0))$, so that there is a homeomorphism from $G(g_0)$ to $G(g)$ with properties as above. This is slightly ambiguous notation, because the definition of $V(G(g_0))$ uses the isomorphism class of the dynamical system $(G(g_0), g_0)$, not just the homeomorphism class of the graph $G(g_0)$, but this seems the best option available.

For $g \in V(G(g_0))$, we write

$$\mathcal{P}(g) = \{\overline{P} : P \text{ is a component of } \overline{\mathbb{C}} \setminus G(g)\}.$$

Where it is convenient to do so, we shall write $G_0(g)$ for $G(g)$. In Section 2 we found a partial characterisation of the boundary of this set. Now we want to try and obtain a parametrisation of the set $V(G(g_0), g_0)$. For any $g \in V(G(g_0))$, and integer $n \geq 0$, we define

$$G_n(g) = g^{-n}(G(g)),$$

$$\mathcal{P}_n(g) = \bigvee_{i=0}^{n} g^{-i}(\mathcal{P}(g)) = \{\overline{U} : U \text{ is a component of } \overline{\mathbb{C}} \setminus G_n(g)\}.$$

3.1. **The possible graphs.** Let $g_0 \in V$ and $G(g_0)$ be as above. Following a common strategy, we want to use the dynamical plane of g_0 to investigate the variation of dynamics in $V(G(g_0), g_0)$. Let $G(g)$ be the graph which varies isotopically from $G(g_0)$ for $g \in V(G(g_0), g_0)$. Then $G_1(g) = g^{-1}(G(g))$ also varies isotopically with g. This is not true for $n > 1$. But nevertheless, it is possible to determine inductively all the possible graphs $G_n(g)$ up to isotopy, for $g \in V(G(g_0))$. The different possibilities for $G_n(g)$, up to isotopy, are determined from the different possibilities for $G_{n-1}(g)$ up to isotopy, together with the position, up to homeomorphism fixing $G_{n-1}(g)$, of $v_2(g)$ in $G_{n-1}(g)$ or its complement. Inductively, this means that the different possibilities for $G_n(g)$ (and $\mathcal{P}_n(g)$), up to isotopy, are determined by $(Q_i(g) : 0 \leq i \leq n-1)$, where:

- $Q_0 = Q_0(g)$ is the set in $\mathcal{P}(g)$ with $v_2(g) \in \text{int}(Q_0)$;
- $Q_{i+1}(g) \subset Q_i(g)$ for $0 \leq i \leq n-1$;
- $Q_i(g) \in \mathcal{P}_i(g)$ or $Q_i(g)$ is an edge of $G_i(g)$ or a vertex of $G_i(g)$;
- $v_2(g) \in Q_i(g)$ for $i \leq n-1$ and $v_2(g) \in \text{int}(Q_i(g)$ if $Q_i \in \mathcal{P}_i(g)$, and $v_2(g)$ is not an endpoint of $Q_i(g)$ if $Q_i(g)$ is an edge of $G_i(g)$.

Inductively, this means that the different possibilities for $Q_n(g)$ are determined by $Q_i(g)$, for $0 \leq i \leq n-1$, and hence so is the graph $G_n(g)$, up to homeomorphism of $\overline{\mathbb{C}}$, and the dynamical system $(G_n(g), g)$, up to isomorphism. So the different possibilities for any sequence $(Q_i : 0 \leq i \leq n-1)$ as above, or even any infinite sequence $(Q_i : i \geq 0)$ with these properties, are determined by $g_0 : G_1(g_0) \to G(g_0)$, up to homeomorphism of $\overline{\mathbb{C}}$ which is the identity on ∂Q_0. We will write $\mathcal{Q}$ for the set of sequences, either finite or infinite, up to equivalence, where two sequences $(Q_i : i \geq 0)$ and $(Q_i' : i \geq 0)$ are regarded as equivalent if there is a homeomorphism φ of $\overline{\mathbb{C}}$ which maps Q_i to Q_i' for all $i \geq 0$. We will write $\mathcal{Q}_\infty$ for the set of infinite sequences in $\mathcal{Q}$, and $\mathcal{Q}_n$ for the set of finite sequences $(Q_0, \cdots Q_n)$ in $\mathcal{Q}$. For $\underline{Q} = (Q_0, \cdots Q_{n-1}) \in \mathcal{Q}$, we write $V(\underline{Q})$ for the set of $g \in V(G(g_0))$ such that $(Q_i(g) : 0 \leq i \leq n-1)$ is equivalent to $(Q_0, \cdots Q_{n-1})$. We write $G(Q_0, \cdots Q_{n-1})$ and $\mathcal{P}(Q_0, \cdots Q_{n-1})$ for the graph $G_n(g)$ and $\mathcal{P}_n(g)$, up to isotopy, for any g such that $(Q_0(g), \cdots Q_{n-1}(g))$ is equivalent to $\underline{Q}$. This means that all the dynamical systems $(G_n(g), g)$, for $g \in V(\underline{Q})$, are isomorphic. If $g_1 \in V(\underline{Q})$, we write $V(\underline{Q}; g_1)$ for the component of $V(\underline{Q})$ containing g_1. In particular, all the graphs $G_n(g)$ for $g \in V(\underline{Q})$ are homeomorphic. For $g_1 \in V(\underline{Q})$ and $g \in V(\underline{Q}; g_1)$, the graph $G(g)$ varies isotopically. This isotopy is, of course, an ambient isotopy, because any isotopy of a graph in a two-dimensional manifold is an ambient isotopy.

This isotopy is actually a bit more general, which will be important later. Let $(Q_0, \cdots Q_n) \in \mathcal{Q}$, so that $Q_i \in \mathcal{P}(Q_0, \cdots Q_{i-1})$ for $1 \leq i \leq n$. Let $Q'_{n-1} \subset Q_{n-1}$ be an edge or point of $G(Q_0, \cdots Q_{n-1})$. Then $g^{-1}(G(Q_0, \cdots Q_{n-1}) \setminus Q'_{n-1})$ varies isotopically for $g \in V(Q_0, \cdots Q_{n-1}; g_0) \cup V(Q_0, \cdots Q_{n-2}, Q'_{n-1}; g_0)$. This means that if $g \in V(Q_0, \cdots Q_{n-1}; g_0)$ and $h \in V(q_0, \cdots Q_{n-2}, Q'_{n-1}; g_0)$, then

$$G_n(g) \setminus g^{-}(Q'_{n-1}(g)), \quad G_n(h) \setminus h^{-1}(Q'_{n-1}(h))$$

are isotopic, where $Q'_{n-1}(g)$ and $Q'_{n-1}(h)$ are the images of Q'_{n-1} under the isotopic homeomorphisms of $G(Q_1, \cdots Q_{n-1})$ to $G_{n-1}(g)$ and $G_{n-1}(h)$.

For $g \in V(G(g_0))$, we also define

$$\mathcal{P}_\infty(g) = \bigcap_{n=0}^{\infty} \{Q_n : Q_n \subset Q_{n-1},\ Q_n \in \mathcal{P}_n(g) \text{ for all } n \geq 0\}.$$

Then $\mathcal{P}_\infty(g)$ is a collection of closed sets whose union is the whole sphere. If $v_2(g)$ is *not persistently recurrent* then all the sets in $\mathcal{P}_\infty(g)$ are either points or Fatou components for g. This follows from [14].

For any $\underline{Q} = (Q_i : i \geq 0) \in \mathcal{Q}_\infty$, we also define

$$V(\underline{Q}) = \bigcap_{n=1}^{\infty} (V(Q_0, \cdots Q_n) \cup V(Q_0, \cdots Q_{n-1}, \partial Q_n)),$$

where $V(Q_0, \cdots Q_{n-1}, \partial Q_n)$ is the union of all those $V(Q_0, \cdots Q_{n-1}, Q'; g_0)$ such that $Q' \subset Q$ and Q' is an edge or vertex of $G(Q_0, \cdots Q_{n-1}) \setminus G(g_0)$. For each n, we have

$$V(G(g_0)) = \bigcup_{\underline{Q} \in \mathcal{Q}_n} V(\underline{Q})$$

and

$$V(G(g_0)) = \bigcup_{\underline{Q} \in \mathcal{Q}_\infty} V(\underline{Q}).$$

We now have the notation in place to state the main theorem of this section. A branched covering f of $\overline{\mathbb{C}}$ is said to be *critically finite* if the postcritical set $Z(f) = \{f^n(c) : c \text{ critical}, n > 0\}$ is finite.

Theorem 3.2. *Let V be the Riemann surface consisting of a connected component of the set of quadratic rational maps f with numbered critical values $v_1(f)$ and $v_2(f)$, such that $v_1(f)$ is of some fixed period, quotiented by Möbius conjugation (all as previously stated). Let $g_0 \in V$ be such that there exists a finite connected graph $G(g_0) \subset \overline{\mathbb{C}}$ with the following properties.*

- $G(g_0) \subset g^{-1}(G(g_0))$
- *$G(g_0)$ separates the critical values.*

- *$G(g_0)$ does not intersect the boundary of any periodic Fatou component intersecting the forward orbit of $v_1(g_0)$.*
- *Any component of $\overline{\mathbb{C}}\setminus G(g_0)$ contains at most one Fatou component intersecting the forward orbit of $v_1(g_0)$.*
- *$v_2(g_0)\in g_0^{-r}(G(g_0))\setminus G(g_0)$ for some $r\geq 1$.*

Let $\mathcal{Q}$ be defined using $G(g_0)$. Let $\underline{Q}\in\mathcal{Q}$.

- *If $g_1\in V(G(g_0))$ and $v_2(g_1)\in g_1^{-s}(G(g_1))\setminus G(g_1)$ for some $s\geq 1$, then $V(G(g_0),g_0)=V(G(g_1),g_1)$.*
- *$V(\underline{Q})\cap V(G(g_0),g_0)$ is nonempty, connected and its complement in $V(G(g_0),g_0)$ is connected.*
- *If there is some n such that*

  $$Q_i\subset G(Q_0,\cdots Q_{n-1})int(Q_0(g))\text{ for all }i\geq n,$$

  or if there is n such that

  $$\bigcap_{i\geq 0}Q_i(g)\subset int(Q_n(g))\text{ for all }g\in V(Q_0,\cdots Q_n),$$

  $$g^m\left(\bigcap_{i\geq 0}Q_i(g)\right)\cap int(Q_n(g))=\emptyset\text{ for all }m>0,$$

  then $V(\underline{Q})\cap V(G(g_0),g_0)$ is a single point.
- *If $\underline{Q}=(Q_0,\cdots Q_n)\in\mathcal{Q}_n$ and if $Q_i\in\mathcal{P}(Q_0,\cdots Q_{i-1}))$ for each $1\leq i\leq n$, then $V(\underline{Q})$ is open, and*

  $$\overline{V(\underline{Q})}\subset V(\underline{Q})\cup V(Q_0,\cdots Q_{n-1},\partial Q_n),$$

  where the closure is taken in $V(G(g_0))$.

Remark 1. As already explained at the start of this section, the properties specified for g_0 and $G(g_0)$ ensure that $V(G(g_0),g_0)$ satisfies the conditions for a set V_1 as in section 2, in particular in 2.1.
2. The theorem does not state that $V(G(g_0))$ is connected, but appears to come close to this. All maps g_1 as in the statement of the theorem are in the same component of $V(G(g_0))$.

For the rest of this section, we keep the hypotheses of Theorem 3.2, and we use the notation that we have established. The following proposition shows that the possibilities for $\underline{Q}$ can be analysed by simply looking at those $\underline{Q}=(Q_i)\in\mathcal{Q}$ for which all the Q_i are topological discs.

Proposition 3.3. *For any $(Q_0,\cdots Q_n)\in\mathcal{Q}_n$, there is $(Q_0,Q_1'\cdots Q_n')\in\mathcal{Q}_n$ such that Q_i' is a topological disc for all $0\leq i\leq n$, and $Q_i\subset Q_i'$ for $0<i\leq n$, and there are isotopic subgraphs $G'(Q_0,\cdots Q_{n-1})$ and $G'(Q_0,Q_1'\cdots Q_{n-1}')$ of $G(Q_0,\cdots Q_{n-1})$ and $G(Q_0,Q_1',\cdots Q_{n-1}')$ such that $Q_i'\subset G'(Q_0,Q_1',\cdots Q_{n-1})$*

for all $1 \le i \le n-1$ *with* $Q_i' \ne Q_i$, *and the isotopy between* $G(Q_0, Q_1, \cdots Q_i)$ *and* $G(Q_0, Q_1', \cdots Q_i')$ *extends to the isotopy between* $G(Q_0, Q_1, \cdots Q_{n-1})$ *and* $G(Q_0, Q_1', \cdots Q_{n-1}')$ *for all* $0 \le i < n-1$.

This is not difficult. The main step is the following.

Lemma 3.4. *If* e *is any edge of* $G_n(g) \setminus G(g)$, *for any* $g \in V(G(g_0))$ *and any integer* $n \ge 1$, *then* $e \cap g^{-m}(e) = \emptyset$ *for any* $m \ge 1$.

Proof. It suffices to prove this for $n = 1$, because any edge e of $G_n(g) \setminus G(g)$ is a contained in $g^{1-n}(e')$ for some edge e' of $G_1(g) \setminus G(g)$. So now we assume that e is an edge of $G_1(g) \setminus G(g)$. Now $G_1(g) = g^{-1}(G(g))$. So

$$g^{-m}(G_1(g) \setminus G(g)) = g^{-(m+1)}(G(g)) \setminus g^{-m}(G(g)).$$

So

$$g^{-m}(G_1(g) \setminus G(g)) \cap g^{-m}(G(g)) = \emptyset$$

for all $m \ge 0$. But $G(g) \subset g^{-1}(G(g)) = G_1(g)$, and hence $G(g) \subset g^{-m}(G(g))$ for all $m \ge 0$ and $G_1(g) \subset g^{-m}(G(g))$ for all $m \ge 1$. So

$$g^{-m}(G_1(g) \setminus G(g)) \cap G_1(g) = \emptyset$$

for all $m \ge 1$, as required.

□

Proof of the proposition. We prove this by induction on n. If $n = 1$ then there is nothing to prove, because $G(g)$ is isotopic to $G(g_0)$. So we assume it is true for $n - 1 \ge 1$, and we need to prove that it is also true for n. If Q_n is a topological disc, there is nothing to prove. Otherwise, there is a least $1 \le i \le n$ such that Q_i is not a topological disc. Then Q_i is an edge or point of $G(Q_0, \cdots Q_{i-1})$. Let $Q_i(g)$ be the corresponding isotopically varying edge or point of $G(Q_0, \cdots Q_{i-1})$ for $g \in V(Q_0, \cdots Q_{i-1})$. Fix such a g. Write $e = Q_i(g)$ if $Q_i(g)$ is an edge of $G_i(g)$. Otherwise, let e be an edge of $G_i(g)$ in $\partial Q_{i-1}(g)$ which contains the point $Q_i(g)$. Let Q_i' be any closed topological disc such that $(Q_0, \cdots Q_{i-1}, Q_i') \in \mathcal{Q}_i$ with $Q_i \subset Q_i'$. It has already been noted in 3.1 that if $g \in V(Q_0, \cdots Q_{i-1}, Q_i)$ and $h \in V(Q_0, \cdots Q_{i-1}, Q_i')$ then $g^{-1}(G_i(g) \setminus Q_i(g))$ and $h^{-1}(G_i(h) \setminus Q_i(h))$ are isotopic. Then by 3.4, $e \cap g^{-m}(e) = \emptyset$ for all $g \in V(Q_0, \cdots Q_i) \cup V(Q_0, \cdots Q_{i-1}, Q_i')$ and all $m > 0$. So $Q_\ell \cap g^{i-\ell}(e) = \emptyset$ for all $i < \ell \le n$ and for all such g. For $i \le \ell \le n$ we choose a topological disc Q_ℓ' so that $(Q_0, \cdots Q_{i-1}, Q_i' \cdots Q_\ell') \in \mathcal{Q}_\ell$ and $Q_\ell \subset Q_\ell'$. Once Q_i' has been chosen, the choice of Q_ℓ' for $\ell > i$ is unique. So then by induction on ℓ, we have that if $g \in V(Q_0, \cdots Q_\ell)$ and $h \in V(Q_0, \cdots Q_{i-1}, Q_i' \cdots Q_\ell')$, then $G_{\ell+1}(g) \setminus g^{-1}(Q_\ell(g))$ and $G_{\ell+1}(h) \setminus h^{-1}(Q_\ell(h))$ are isotopic. This gives the required result if we define $G'(Q_0, \cdots Q_{\ell+1})$ to be the subgraph of $G(Q_0, \cdots Q_{\ell+1})$ which is isotopic to $G_{\ell+1}(g) \setminus g^{-1}(Q_\ell(g))$, and similarly for $G'(Q_0, \cdots Q_{i-1}, Q_i', \cdots Q_{\ell+1}')$. The claimed extension properties hold, by construction.

The following lemma uses Thurston's theorem for critically finite branched coverings, and the set-up for this. See [13] or [6] for more details. Two critically finite branched coverings f_0 and f_1 are said to be *Thurston equivalent* if there is a homotopy f_t ($t \in [0,1]$ through critically finite branched coverings, such that the postcritical set $Z(f_t)$ varies isotopically for $t \in [0,1]$. Thurston's theorem gives a necessary and sufficient condition for a critically finite branched covering f of $\overline{\mathbb{C}}$ to be Thurston equivalent to a critically finite rational map. The rational map is then unique up to conjugation by a Möbius transformation. The condition is in terms of non-existence of loop sets in $\overline{\mathbb{C}} \setminus Z(f)$ with certain properties. In the case of degree two branched coverings, the criterion reduces to the non-existence of a *Levy cycle*, as is explained in the proof below.

Lemma 3.5. *Let $(Q_0, \cdots Q_n) \in \mathcal{Q}_n$ where Q_n is a closed topological disc (that is, the closure of a component of $\overline{\mathbb{C}} \setminus G(Q_0, \cdots Q_{n-1})$ if $n \geq 1$, or $Q_0 = Q_0(g_0)$ if $n = 0$) and that $\underline{Q} = (Q_i : 0 \leq i < N) \in \mathcal{Q}$ for $N > n+1$, possibly $N = \infty$, with $Q_i \subset Q_n \cap G(Q_0, \cdots Q_n) \cap int(Q_{n-1})$ for $i > n$ and such that $\bigcap_{i \geq 0} Q_i$ represents an eventually periodic point. Suppose that $V(Q_0, \cdots Q_n) \neq \emptyset$. Then $V(\underline{Q}) = \{g_1\}$ for some $g_1 \in V$.*

Remark. Note that there is no statement, as yet, that $g_1 \in V(G(g_0), g_0)$. That will come later.

Proof. Let $g \in V(Q_1, \cdots Q_n)$. Then $G(Q_0, \cdots Q_n)$ is isotopic to $G_{n+1}(g)$, and the isotopy carries $\cap_{i \geq 0} Q_i$ to a point z_0 in $G_{n+1}(g)$, which, like $v_2(g)$, is in $\text{int}(Q_{n-1}(g)) \cap Q_n(g)$. We can construct a path $\beta : [0,1] \to Q_n(g) \cap \text{int}(Q_{n-1}(g))$ with $\beta(0) = v_2(g)$ and $\beta(1) = \cap_{0 \leq i < N} Q_i(g) = z_0$. We can also choose β so that $\beta([0,1)) \subset \text{int}(Q_n(g))$. The hypotheses ensure that either $z_0 \in G_{n+1}(g) \setminus G_n(g)$ or $z_0 \in G_n(g) \setminus G_{n-1}(g)$. Either way, the endpoint-fixing homotopy class of β is uniquely determined in $\overline{\mathbb{C}} \setminus \{g^i(z_0) : i > 0\}$. This means that the Thurston-equivalence class of the post-critically finite branched covering $\sigma_\beta \circ g$ is well defined, where σ_β is a homeomorphism which is the identity outside an arbitrarily small neighbourhood of β and maps $\beta(0)$ to $\beta(1) = z_0$.

Then we claim that $\sigma_\beta \circ g$ is Thurston equivalent to a rational map. Since this is a branched covering of degree two, it suffices to prove the non-existence of a Levy cycle. By definition, a Levy cycle is an isotopy class of a collection of distinct and disjoint simple closed loops, where the isotopy is in the complement of the postcritical set. In the present case, it is convenient to consider isotopy in the complement of a potentially larger forward invariant set X consisting of the union of the forward orbits of z_0,

$c_1(g)$ and the vertices of $G_0(g)$. Thurston's Theorem adapts naturally to this setting. A Levy cycle for $\sigma_\beta \circ g$ is then the isotopy class in $\overline{\mathbb{C}} \setminus X$ of a finite set $\{\gamma_i : 1 \leq i \leq r\}$ of distinct and disjoint simple closed loops, such that there is a component γ_i' of $(\sigma_\beta \circ g)^{-1}(\gamma_{i+1})$ (writing $\gamma_1 = \gamma_{r+1}$, so that this also makes sense if $i = r$), such that γ_i and γ_i' are isotopic in $\overline{\mathbb{C}} \setminus X$, for $1 \leq i \leq r$. We consider the case when $z_0 \in \partial Q_n(g) \cap \text{int}(Q_{n-1}(g) \subset G_n(g) \setminus G_{n-1}(g)$. The other case, when $z_0 \in G_{n+1}(g) \cap \text{int}(Q_n(g) \subset G_{n+1}(g) \setminus G_n(g)$ can be dealt with similarly. The γ_i can also be chosen to have only transversal intersections with $G_{n-1}(g)$. We have $z_0 \notin G_{n-1}(g)$. So $(\sigma_\beta \circ g)^{-1}(G_{n-1}(g)) = g^{-1}(G_{n-1}(g)) = G_n(g)$. Now $(\sigma_\beta \circ g)^{-1}(\gamma_{i+1})$ has two components γ_i' and γ_i'', each of them mapped homeomorphically to γ_{i+1} by $\sigma_\beta \circ g$. Each transverse intersection between γ_i and $G_{n-1}(g)$ in $\overline{\mathbb{C}} \setminus X$ lifts to two transverse intersections between $\gamma_i' \cup \gamma_i''$ and $G_n(g) \supset G_{n-1}(g)$ in $\overline{\mathbb{C}} \setminus (\sigma_\beta \circ g)^{-1}(X)$, one of these intersections with γ_i' and one with γ_i''. Because of the isotopy between γ_i and γ_i', the intersection on γ_i' must be in $G_{n-1}(g)$ and must be essential in $\overline{\mathbb{C}} \setminus X$. So this means that each arc on γ_{i+1} between essential intersections in $G_{n-1}(g)$ lifts to an arc on γ_i' between essential intersections in $G_{n-1}(g)$, and this arc can be isotoped in the complement of X to an arc on γ_i between essential intersections in $G_{n-1}(g)$. Since $g^{-1}(G_{n-j}(g) \setminus G_{n-j-1}(g)) = G_{n-j+1}(g) \setminus G_{n-j}(g)$, it follows by induction on $j \geq 1$ that all intersections between γ_i and $G_{n-1}(g)$ are in $G_0(g)$. So every arc of intersection of γ_i with $G_{n-1}(g)$ must be with $G_0(g)$, and in a single set of $\mathcal{P}_{n-1}(g)$ adjacent to a vertex of $G_0(g) = G(g)$. If n is large enough, this is clearly impossible, because successive arcs are too far apart. But we can assume n is large enough to make this impossible, by replacing γ_i by γ_i^m if necessary, where $\gamma_i^0 = \gamma_i$ and $\gamma_i^1 = \gamma_i'$ and γ_i^{m+1} is isotopic to γ_i^m, obtained by lifting, under $\sigma_\beta \circ g$, the isotopy between γ_{i+1}^m and γ_{i+1}^{m-1}, writing $\gamma_1^m = \gamma_{r+1}^m$. It follows that all intersections between γ_i^m and $G_0(g)$ are in a single set of $\mathcal{P}_{n+m-1}(g)$, adjacent to a vertex of $G_0(g)$. If m is large enough, this is, once again, impossible.

So Thurston's Theorem for critically finite branched coverings implies that $\sigma_\beta \circ g$ is Thurston equivalent to a unique rational map g_1. From the definitions, we have $g_1 \in V(\underline{Q})$. By the uniqueness statement in Thurston's Theorem, we have $V(\underline{Q}) = \{g_1\}$. For if $g_2 \in V(\underline{Q}$ and $v_1(g_1) \in G_{m+1}(g) \setminus G_m(g_1)$ for $m = n$ or $n - 1$ then there is a homeomorphism φ of $\overline{\mathbb{C}}$ which maps $G_m(g_1)$ to $G_m(g_2)$ which conjugates dynamics of g_1 and g_2 on these graphs, and maps $v_2(g_1)$ to $v_2(g_2)$ and $g_1^i(v_1(g_1)$ to $g_2^i(v_1(g_2))$ for all $i \geq 0$. So $\varphi \circ g_1 \circ \varphi^{-1}$ and g_2 are homotopic through branched coverings which are constant on $G_m(g_2)$, and on the postcritical sets. $\square$

The following lemma, like the preceding one, gives a condition under which $V(\underline{Q})$ is nonempty. It has some overlap with the preceding one, but is of a rather different type. it uses the λ-Lemma of Mane, Sullivan and Sad [11] rather than Thurston's Theorem, and is a result about connected sets of maps rather than critically finite maps. 3.6 has no uniqueness statement. The two lemmas complement each other in the proof of 3.2.

Lemma 3.6. *Let $g_1 \in V(G(g_0))$. Let $Q_{n-1} \in \mathcal{P}_{n-1}(G_1)$ and let $v_2(g_1) \in int(Q_{n-1}) \cap G_n(g_1)$ for some $n \geq 1$. Then $V(\underline{Q}, g_1) \neq \emptyset$ for all $(\underline{Q}) = (Q_i')$ with $Q_i' = Q_i$ for $i \leq n-1$ such that $\cap_i Q_i$ is in the same component of $G_n \cap int(Q_0) \cap Q_{n-1}$ as $v_2(g_1)$.*

Proof. From the hypotheses on g_1, the graph $G_n(g)$ varies isotopically for $g \in V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1)$, and the dynamics of maps in $V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1)$ are conjugate in the following sense. There is a homeomorphism

$$\varphi_{g,h} : G_n(h) \to G_n(g), \qquad (g,h) \in (V(Q_0, \cdots Q_{n-1} \cup \partial Q_{n-1}; g_1) \cap V)^2,$$

such that the map $(g,h) \mapsto \varphi_{g,h}$ is continuous, using the uniform topology on the image and $\varphi_{g,h} \circ h = g \circ \varphi_{g,h}$ on $G_i(h)$, and $\varphi_{h,h}$ is the identity. Each preperiodic point in $G_n(g)$ varies holomorphically for $g \in V(Q_0, \cdots Q_{n-1}; g_1)$, that is, $\varphi_{g,h}(z)$ varies holomorphically with g for each preperiodic point $z \in G_n(g_1)$. But preperiodic points are dense in $G_n(g_1)$. (For example, the backward orbits of vertices of $G_n(g_1)$ are dense in $G_n(g_1)$, by the expansion properties of g_1 on $G_n(g_1)$ established in 2.3.) It follows by the λ-Lemma [11] that $(z,g) \mapsto \varphi_{g,h}(z)$ is continuous in (z,g), and holomorphic in $g \in V(Q_0, \cdots Q_{n-1}; g_1)$ for each $z \in G_n(g_1)$. (In fact it is also possible to prove this by standard hyperbolicity arguments.) Now we assume without loss of generality, conjugating by a Möbius transformation if necessary, that $Q_{n-1}(g) \subset \mathbb{C}$ for $g \in V(Q_0, \cdots Q_{n-1}; g_1)$, in particular, $\{v_2(g)\} \cup (G_n(g) \cap Q_{n-1}(g)) \subset \mathbb{C}$. We consider the maps

$$\psi(z,g) = \varphi_{g,g_1}(z) - v_2(g)$$

for $z \in G_n(g_1) \cap Q_{n-1}(g_1)$. The map $(z,g) \mapsto \psi(z,g)$ is, once again, continuous in (z,g) and holomorphic in $g \in V(Q_0, \cdots Q_{n-1}; g_1)$. Now write $z_0 = v_2(g_1)$, so that $z_0 \in G_n(g_1) \setminus G_{n-1}(g_1)$. The map $g \mapsto \psi(z_0, g)$ is holomorphic in g and the inverse image of a disc round 0 is a topological disc containing z_0 in its interior. By continuity, the same is true for z sufficiently near z_0. Hence for all z sufficiently near z_0, the map $g \mapsto \psi(z,g)$ has a zero. This argument shows that the set of $z \in Q_{n-1}(g_1) \cap \text{int}(Q_0) \cap G_n(g_1)$ for which $g \mapsto \psi(z,g) : V(Q_0, \cdots Q_{n-1}) \to \mathbb{C}$ has a zero in $V(Q_0, \cdots Q_{n-1})$ is open, because z_0 can be replaced by any other point z in $Q_{n-1}(g_0) \cap \text{int}(Q_0) \cap G_n(g_0)$. But the set is also closed in $\text{int}(Q_0(g_1)) \cap Q_{n-1}(g_1) \cap G_n(g_1)$. For suppose

$\psi(z_k, g_k) = 0$ and $z_k \to z$. Then either some subsequence of g_k has a limit g, in which case $\psi(z, g) = 0$ for any such g, and the proof is finished, or $g_k \to \infty$ in V.

We now have to deal with the situation that $g_k \to \infty$ in V. In this case, we can assume that all z_k are in a single edge of $G_n(g_1)$. We will now show that this implies the existence of a Levy cycle for the unique map $h_1 \in G(Q_0, \cdots Q_{n-1}, Q'_n)$, where Q'_n is a vertex of $G_n(g_1) \setminus G_0(g_1)$. This contradicts the result of 3.5, and hence $g_k \to \infty$ is impossible. We use certain facts about the ends of V. These appear in Stimson's thesis [16] and in various other papers, for example [8]. Choosing suitable representatives of g_k up to Möbius conjugation,chosen, in particular, so that $c_1(g_k) = 1$ for all k, g_k converges to a periodic Möbius transformation $g(z) = e^{2\pi ir/q}z$ for some integer $q \geq 2$ and some $r \geq 1$ which is coprime to q, and the set $\{g_k^i(v_1(g_k)) : i \geq 0\} \cup \{v_2(g_k)\} = Z_1(g_k)$ converges $Z_1(g) = \{e^{2\pi ij/q} : 0 \leq j \leq q-1\}$. Let $\overline{V}$ be the compactification of V obtained by adding the Möbius transformations at infinity and consider a fixed $g \in \overline{V} \setminus V$. The parametrisation can be chosen so that the other critical point $c_2(g_k) = 1 + \rho_k$ where $\lim_{k\to\infty} \rho_k = 0$. Passing to a subsequence if necessary, we may assume that g_k is in a single branch of V near g. Then $(g_k^q(1 + z\rho_k) - 1)/\rho_k$ has a limit as $k \to \infty$ for z bounded and bounded from $\frac{1}{2}$, which is the quadratic map

$$h : z \mapsto qa + z + \frac{1}{4(z - \frac{1}{2})}$$

for a constant $a \neq 0$.

Because of the nature of h, it follows that all the eventually periodic points of g_k whose forward orbits have size $\leq N$ lie in the $C|\rho_k|$-neighbourhood of $Z(g)$, if k is sufficiently large given N, for a suitable constant C. We will call this neighbourhood U_1. So if N is a bound on the number of vertices of $G_n(g_k)$ — which is, of course, the same for all k — then all vertices of $G_n(g_k)$ lie in U_1, for all sufficiently large k. If the edge e of $G_n(g_k)$ between one vertex and $v_2(g_k)$ is contained in a single component of U_1, then the boundary of U_1 provides a Levy cycle for h_1, where Q'_n is taken to be this vertex, and this gives the required contradiction. Now $e \subset G_n(g_k) \setminus G(g_k)$, and we claim that $e \subset U_1$, up to isotopy preserving the set X which is the union of the vertex set of $G_n(g_k)$ and the set $\{g_k^i(v_1(g_k)) : i \geq 0\}$. We consider only essential intersections between $G_n(g_k) \setminus G(g_k)$ and ∂U_1 under isotopies preserving X. If γ is an arc of essential intersection then it must be in the inverse image under g_k of an arc which contains one or more arcs of essential intersection. Since the number of such arcs is finite, each arc must be in the inverse image of exactly one other, and the inverse image of each arc contains exactly one other. But then each edge must be contained in a periodic edge of $G_n(g_k) \setminus G_0(g_k)$. But there are none. So there are no essential intersections with ∂U_1. In particular, $e \subset U_1$ up to isotopy preserving X, as required. □

Corollary 3.7. *For all $(Q_0, \cdots Q_n) \in \mathcal{Q}_n$, if $V(Q_0, \cdots Q_n) \neq \emptyset$, then it is connected.*

Proof. By 3.6, for any nonempty component $V(Q_0, \cdots Q_n; g_1)$ of $V(Q_0, \cdots Q_n)$,

$$V(Q_0, \cdots Q_n; g_1) \cap V(\underline{Q}) \neq \emptyset$$

for any $\underline{Q} \in \mathcal{Q}$ such that $\underline{Q}$ extends $(Q_0, \cdots Q_n)$.

In particular, if $V(Q_0, \cdots Q_n, g_2)$ is another component of $V(Q_0, \cdots Q_n)$, then there is $\underline{Q}$ with $\bigcap_{i \geq 0} Q_i$ representing an eventually periodic point such that $V(\underline{Q})$ which intersects both components. But this is impossible, because $V(\underline{Q})$ contains a single critically finite map. So $V(Q_0, \cdots Q_n)$ is connected. □

Lemma 3.8. $V(\underline{Q}) \cap V(G(g_0), g_0) \neq \emptyset$ *for all* $\underline{Q} \in \mathcal{Q}$.

Proof. By 3.6, $V(\underline{Q}) \cap V(G(g_0), g_0) \neq \emptyset$ for all $\underline{Q}$ with $\cap_{i \geq 0} Q_i \subset \partial Q_n$ for any $(Q_0, \cdots Q_n) \in \mathcal{Q}_n$ and such that $V(Q_0, \cdots Q_{n-1}, \partial Q_n) \cap V(G(g_0), g_0) \neq \emptyset$ with $Q_n \subset \mathrm{int}(Q_0)$, because then $\partial Q_n \cap \mathrm{int}(Q_0)$ is connected. This means that if $V(\underline{Q}) \cap V(G(g_0), g_0) \neq \emptyset$, then we have $V(\underline{Q'}) \cap V(G(g_0), g_0) \neq \emptyset$ for any $\underline{Q'}$ which can be connected to $\underline{Q}$ by sets $\partial Q^i_{n_i}$, for varying n_i and $\underline{Q^i} = (Q^i_0 \cdots Q^i_{n_i})$ with $Q^i_{n_i} \subset \mathrm{int}(Q_0)$. But any $\underline{Q}$ and $\underline{Q'}$ can be connected in this way. □

Lemma 3.9. $V(\underline{Q})$ *is singleton, and contained in* $V(G(g_0), g_0)$, *if there is* $n \geq 1$ *such that either* $\bigcap_{i=0} Q_i(g) \subset G_n(g) \cap$ *int*$(Q_0(g))$ *or* $\bigcap_{i=0}^{\infty} Q_i(g) = \underline{Q}(g) \subset$ *int*$(Q_n(g))$ *and such that* $g^k(\underline{Q}(g)) \cap$ *int*$(Q_n(g)) = \emptyset$ *for all* $k > 0$, *and for at least one* $g \in V(\underline{Q})$.

Proof. In both cases, the set $\underline{Q}(g) = \bigcap_{i=0}^{\infty} Q_i(g)$ is well-defined for all $g \in V(Q_0, \cdots Q_n)$. It is a point, which follows from the result of [14] about non-persistently-recurrent points, but in any case the construction of a nested sequence of annuli of moduli bounded from 0 is straightforward. Moreover $z(g) = \underline{Q}(g)$ is the limit of a sequence $z_\ell(g)$ of eventually periodic points in $G_\ell(g)$with the same property of being defined for all $g \in V(Q_0, \cdots Q_n)$. Fix $g_0 \in V(Q_0, \cdots Q_n)$ and write $z(g_0) = \underline{Q}(g_0)$ and $z_\ell(g_0)$ for the sequence of eventually preperiodic points under g_0 with $\lim_{\ell \to \infty} z_\ell(g_0) = z(g_0)$. Then since $g \mapsto \psi(z_\ell(g_0), g)$ is holomorphic in g and has a single zero g_ℓ, the same is true for the limiting holomorphic function $g \mapsto \psi(z(g_0), g)$. The single zero is the unique point in $V(\underline{Q})$. By 3.8, $V(\underline{Q})$ is contained in $V(G(g_0), g_0)$. □

Now the following lemma completes the proof of Theorem 3.2.

Lemma 3.10. *The complement of* $\overline{V(\underline{Q};g_0)}$ *has exactly one component in* $V(G(g_0),g_0)$ *for all* $\underline{Q}\in\mathcal{Q}$*, for* $\underline{Q}\neq Q_0$.

Proof. If $\underline{Q}=(Q_i:i\geq 0)\in\mathcal{Q}_\infty$ and the complement of $V(\underline{Q})$ has more than one component in $V(G(g_0),g_0)$, then the same is true for the complement of $\overline{V(Q_0,\cdots Q_n;g_0)}$, for some n. So it suffices to show that the complement of $\overline{V(Q_0,\cdots Q_n)}$ has at most one component in for each $(Q_0,\cdots Q_n)\in\mathcal{Q}_n$. So suppose this is not true. Then $\partial V(Q_0,\cdots Q_n;g_0)\cap V(G(g_0),g_0)$ is disconnected. But

$$\partial V(Q_0,\cdots Q_n;g_0)\cap V(G(g_0),g_0)\subset V(Q_0,\cdots Q_{n-1},\partial Q_n\setminus\partial Q_0).$$

Moreover, if we fix $h\in V(Q_0,\cdots Q_n)$ there is a continuous surjective map

$$\Phi:V(Q_0,\cdots Q_{n-1},\partial Q_n\setminus\partial Q_0)\to\partial Q_n(h)\setminus\partial Q_0(h),$$

defined by

$$\Phi(g)=\varphi_{g,h}^{-1}(v_2(g)),$$

where $\varphi_{g,h}$ is as in the proof of 3.6. By 3.6 to 3.8, $\Phi^{-1}(\Phi(g))$ is connected for each g. In fact if $v_2(g)$ is critically finite, then this already follows from 3.5. Also, $\Phi(\partial V(Q_0,\cdots Q_n))\supset\partial Q_n(g)\cap\text{int}(Q_0(g))$ by the proof of 3.6. So if $\partial V(Q_0,\cdots Q_n)\cap V(G(g_0),g_0)$ can be written as a disjoint union of two nonempty closed sets X_1 and X_2 in $V(G(g_0),g_0)$, we have $X_j=\Phi^{-1}(\Phi(X_j))$. Since X_j is closed and bounded (and hence compact), we see that $\Phi(X_j)$ is also closed (and bounded and compact) and the sets $\Phi(X_1)$ and $\Phi(X_2)$ are also disjoint. So then $\partial Q_n(g)\cap\text{int}(Q_0(g))$ is disconnected, giving a contradiction.

□

References

[1] Ahlfors, L., Quasiconformal reflections, Acta Math., 109 (1963), 291-301.
[2] Bonk, M. and Meyer, D., Expanding Thurston Maps, ArXiv:1009.3467,to appear in the AMS Mathematical Surveys and Monographs Series.
[3] Bowen, Rufus, Markov partitions for Axiom A diffeomorphisms, Amer. J. Math. 92 (1970) 725-747.
[4] Bowen, Rufus, Markov partitions are not smooth, Proc. AMS 71 (1978) 130-132.
[5] Carleson, L. and Gamelin, T.D., Complex Dynamics, Universitext: Tracts in Mathematics, Springer-Verlag, 1993.
[6] Douady, A. and Hubbard, J.H., A proof of Thurston's topological characterization of rational functions. Acta Math., 171 (1993), 263-297.
[7] Farrell, F.T. and Jones, L.E., Markov Cell Structures for Expanding Maps in Dimension Two, Transactions AMS 259 (1979), 315-327.
[8] Kiwi, J. and Rees, M., Counting Hyperbolic Components, J. London Math. Soc., 88 (3) (2013) 669-698.

[9] Lehto, O., Univalent Functions and Teichmüller spaces, Springer-Verlag, 1987
[10] Lehto, O. and Virtanen, K.I., Quasiconformal mappings in the plane, Die Grundlehren der mathematischen Wissenschaften 126 (2nd edition) Springer-Verlag.
[11] Mané, R., Sad., P., and Sullivan, D., On the dynamics of rational maps, Ann. Sci. Ecole Norm. Sup., (4) 16 (1983), 193-217.
[12] Przytycki, F. and Urbanski, M., Conformal Fractals: Ergodic Theory Methods LMS Lecture Note Series 371, CUP 2010.
[13] Rees, M., Views of Parameter Space, Topographer and Resident. Astérisque, 288 (2003).
[14] Roesch, P., Puzzles de Yoccoz pur les applications à l'allure rationelle, l'Enseignement Mathé matique 45 (1999) 133-168.
[15] Shub, M., Endomorphisms of compact differentiable manifolds, Amer J. Math. 91 (1969) 175-199.
[16] Stimson, J., Degree two rational maps with a periodic critical point. Thesis, University of Liverpool, July 1993.