On Incentive Issues in Practical Auction Design

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Abstract

Algorithmic mechanism design studies the allocation of resources to selfish agents, who might behave strategically to maximize their own utilities. This thesis studies these incentive issues arsing from four different settings, that are motivated by real-life applications. We model the settings and problems by appropriately extending or generalizing classical economic models. After that we systematically analyze the auction design problems by using methods from both economic theory and computer science.

The first problem is the auction design problem for selling online rich media advertisement. In this market, multiple advertisers compete for a set of slots that are arranged in a line, such as a banner on a website. Each buyer desires a particular number of consecutive slots and has a private per-click valuation while each slot is associated with a quality factor. Our goal is to maximize the auctioneer's expected revenue given buyers' consecutive demand. This is motivated by modeling buyers who may require these to display a large size ad. Three major pricing mechanisms, the Bayesian pricing model, the maximum revenue market equilibrium model and an envy-free solution model are studied in this setting.

The second setting is for fund-raising scenarios, where a revenue target is usually specified. We are interested in designing truthful auctions that maximize the probability to achieve this revenue target, rather than in maximizing the expected revenue. We study this topic from the perspective of Bayesian auction design in digital good auctions. We present an algorithm to find the optimal truthful auction for two buyers with independent valuations and show the problem is NP-hard when the number of buyers is arbitrary or the distributions are correlated. We also investigate simple auctions in this setting and provide approximately optimal solutions.

Third, we study double auction market design where the trading broker wants to maximize its total revenue by buying low from the sellers and selling high to the buyers in a Bayesian setting. For single-parameter setting, we develop a maximum mechanism for the market maker to maximize its own revenue. For the more general case where each seller's product may be different, we consider a number of various settings in terms of constraints on supplies and demands. For each of them, we develop a polynomial time computable truthful mechanism for the market maker to achieve a revenue at least

a constant factor times the revenue of any other truthful mechanism.

Finally, we study the inefficiency of mixed equilibria of all-pay auctions in three different environments – combinatorial, multi-unit and single-item auctions. First, we consider item-bidding combinatorial auctions where m all-pay auctions run in parallel, one for each good. For fractionally subadditive valuations, we strengthen the upper bound by proving some structural properties of mixed Nash equilibria. Next, we design an all-pay mechanism with a randomized allocation rule for the multi-unit auction, which admits a unique, approximately efficient, pure Nash equilibrium. Finally, we analyze single-item all-pay auctions motivated by their connection to crowdsourcing contests and show tight bounds on the PoA of social welfare, revenue and maximum bid.

Contents

List of Figures

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Chapter 1

Introduction

With the emergence of the Internet Economy and Electronic Commerce, one of the challenging problems arising in computer science is to design and analyze algorithms taking input not only from the objective reality but also from selfish agents, who may *manipulate* their private information to maximize their own *utility*. *Algorithmic mechanism design* is such a field that studies these incentive issues arsing from real-life applications by using methodologies from both economic theory and theoretical computer science. Prominent successful applications include sponsored search auction[37, 77] that is used by search engines to sell advertisement slots and FCC spectrum auctions[23] that are recruited by the US government to sell the licenses for electromagnetic spectrum.

Despite this remarkable progress, a number of applications still require theoretical model and systematic study. This thesis is dedicated for this purpose, which concentrates on studying the incentive issues in practical auction design. We analyze four different practical problems arising from applications and provide different extensions and generalizations for existing economic models. In order to tackle these problems, we employ classical concepts in economics and computational techniques from theoretical computer science. For all these problems, we either give a solution which is computationally efficient or provide hardness results by reductions from classical hard problems in computer science.

The rest of this chapter is organized as follows. In Section 1.1, we provide some basic knowledge for auction design by presenting elementary concepts and illustrative examples. The goal of this part is to help readers get familiar with the primary notions and terminologies in algorithmic mechanism design. After that, we give a brief overview for each problem that is going to be studied in this thesis. Each section corresponds to a problem studied in a chapter of the thesis. We will discuss how we consider these problems from the viewpoint of theoretical computer science. Moreover, we will lightly motivate the problems and briefly discuss the interesting questions arising there.

1.1 Background

1.1.1 Auctions

Auctions are widely used for the exchange of a very large variety of commodities. The most prevalent auction is the *English Auction*, also called open ascending price auction [78], where the potential buyers iteratively increase the price starting from the auctioneer's *reserve price* until there is only one interested buyer. Another interesting type of auction is the *Dutch auction*, which is also called the open descending price auction [58]. In contrast to the English auction, in the Dutch auction, the auctioneer starts by announcing a high enough price such that no buyers are interested. Then the price gradually decreases until it is affordable for a buyer.

One might note that in both English and Dutch auctions, multiple rounds of interactions are needed between the auctioneer and buyers. In particular, buyers need to rise the price iteratively in the English auction while in the Dutch auction, the auctioneer is required to announce descending prices continuously. One direct question here is whether these multi-round auctions can be simulated by *one-shot* auctions or *sealed-bid* auctions. Actually, the affirmative answer to this question is given by the *Revelation Principle* [65], which states that when designing mechanisms, we only need to focus on the sealed-bid auctions where bidders are only allowed to submit their bids once (in sealed envelopes) to represent their willingness to pay, which is without loss of generality.

However, even restricted in sealed-bid auctions, there are still a number of auctions, including the first price auction and the second price auction (Vickrey auction) [78]. Given the buyers' sealed bids, both auctions sell the item to the buyer with the highest bid but ask the winner to pay his bid or the second highest bid respectively. Vickrey [78] shows that the English auction is equivalent to the second-price auction. We give an example here to illustrate the intuition behind it. Consider an auctioneer who wishes to sell an item to two potential buyers where the first buyer would like to pay 10 pounds for the item. In auction theory, this value 10 is called buyer 1's *valuation*. Suppose the second buyer has valuation 8 and the English auction starts with a reserve price 5. During the auction, buyer 1 first rises the price to 6, then the price increases to 7 by the other buyer and finally stops with 8. This is because the *utility* of the second player, which is defined as the difference between his valuation and the price, is 0 when the price is 8. The outcome is that the first buyer is the winner and pays 8 for the item. Now consider what happens in a second price auction, the buyers bid 10 and 8 respectively and then buyer 1 wins the item by paying 8. That is exactly the same as the outcome in the English auction.

1.1.2 Nash Equilibrium and Truthfulness

As we mentioned before, in the auction, buyers have their own objectives which are represented by their own utility functions. In order to maximize their utility, buyers will behave strategically, like making incremental price increases in the English auction, manipulating their bids in sealed-bid auctions. To model and analyze these strategic plays is an ultimate goal in game theory. The most prominent concept is *Nash equilibrium* named after John Nash [66]. In a Nash equilibrium, every player/buyer plays a best strategy which maximizes his utility given other players' strategies. In other words, every player in the Nash equilibrium makes a *best response* to other players' strategies. In sealed-bid auctions, the differences among the strategies are represented by the different submitted bids. So in a Nash equilibrium of a sealed-bid auction, no bidder can improve his utility by changing his own bid unilaterally.

Truthfulness is the most important concept in auction theory and a main criterion when designing auctions [78]. In particular, buyers have incentive to deviate from truthfully reporting if this deviation can improve their own utilities. Roughly speaking, an auction is truthful if it is a Nash equilibrium when buyers use their true valuations as their bids. It has some advantages to use truthful auctions instead of non-truthful ones. First, buyers don't need to do any computation or optimization when participating the auction since bidding true value is a *dominant strategy* in this game. Second, the auctioneer can easily infer buyers' valuations from their bids. This information is really important to get buyers' preferences and optimize subsequent sales. Finally, for theoretical interest, truthful auctions are easy to analyze since no manipulation exists in the auction. A notable example of truthful auction is the second price auction. We give the primary idea of the proof. Given buyers' valuations and truthful bids, it is clear that losers can only win the item by bidding higher than their valuations. This will result in negative utilities for them. On the other hand, the winner cannot decrease his payment because the price he pays depends on the second highest buyer's bid, not on his own bid.

It is also worth mentioning that even in truthful auctions, the truthfully bidding might not be the only Nash equilibrium. For instance, consider the example we described above, i.e. a second price auction with two buyers whose valuations are 10 and 8 respectively. Clearly, bidding 10 and 8 is a truthful Nash equilibrium here. But bidding 0 and 11 is also a Nash equilibrium in this auction. It seems like that the second buyer makes a bid that is higher than his valuation and given this bluffing strategy, the first buyer cannot do anything better than bidding 0. Actually, such strategic bidding needs the complete information of all buyers' valuations, which is almost impossible in common scenarios.

1.1.3 Auction Design

Before considering sealed-bid auction design, we need to represent these auctions in a mathematical expression. In particular, every sealed-bid auction can be characterized by a pair of functions, i.e. *allocation function* and *payment function*. An allocation function is a mapping from buyers' bidding vector to an allocation vector which indicates the allocation of the items among buyers. In single-item case, this is just the index of the winning buyer. Similarly, a payment function is a mapping from buyers' bids to a payment vector which encodes the buyers' final payments. For example, in first-price auction, the allocation function is to give the item to the highest bidder and the payment function is to charge the winner his bid and nothing for the losers.

Given these representations, we can see a pair of allocation and payment functions specifies an auction. In order to design auction, we need to specify which auctions are good or useful in applications. There are two major objectives in auction design – *social welfare* and *revenue*. The social welfare is the sum of all buyers' valuations of the final allocation. In single-item case, the social welfare is exactly the winner's valuation. So in order to maximize the social welfare, we need to sell the item to the buyer with the highest valuation (not the highest bid since bidders may manipulate their bids). The revenue is the sum of all bidder's payments, which is a common objective for the auctioneer.

Therefore, an auction design problem can be viewed as an algorithm design problem in the following sense. Given the auction setting (like single-item or multi-item), the goal is to design an algorithm where for any bidders' valuations, the objective (like social welfare or revenue) is optimized or approximately optimized. For single-item case, it has been shown that the second-price auction is an social-optimal truthful auction that always sells the item to the bidder with the highest valuation. But for revenue maximization, no such universally optimal auction exists. This is because given buyers' valuations, the optimal way to extract revenue is to sell the item with a fixed price equal to the highest valuation. Clearly, this revenue in this auction is optimal since no bidder would like to pay more than his valuation. But this type of auctions might not be optimal for other buyer profiles.

1.1.4 Bayesian Auction Design: An Algorithmic View

In the auction design problems considered above, we assume the auctioneer has no knowledge about buyers' valuations and aims to design an auction which is optimal for any valuation profiles. But in many real-life scenarios, the auctioneer might be able to learn buyers' valuations from their consumption habits or previous trade records. Such information is called *prior knowledge* that is obtained prior to the execution of an auction. Normally, this prior knowledge is represented by a distribution of buyers' valuations. For instance, the auctioneer may assume the buyers' valuations are drawn

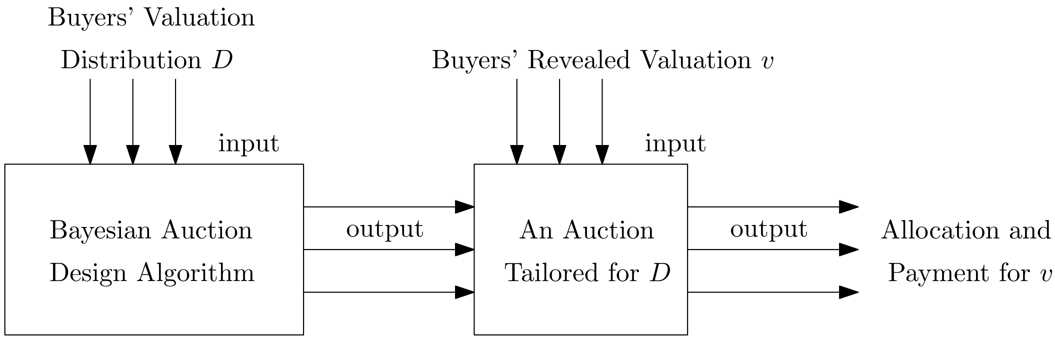


Figure 1.1: Illustration for Bayesian Auction Design

from a *uniform* distribution with a carefully selected support or a *normal* distribution with properly estimated mean and variance.

On the other hand, from buyers' perspective, Nash equilibrium is a slightly strong concept which requires them to have the full information of other players. An appropriate relaxation is to use the prior knowledge to be the common knowledge among buyers. That is, each buyer knows the distributions of other buyers' valuations instead of the exact valuations. This setting is called *Bayesian setting* or *incomplete information* setting, which is broadly studied in economics [46]. In contrast, we call the setting without prior knowledge *prior-free* setting. The corresponding concept of Nash equilibrium in Bayesian settings is called *Bayesian Nash Equilibrium.* Similarly, we can define *Bayesian truthfulness*, for which we defer the definitions to Chapter 2.

In the Bayesian setting, the auction design problem takes buyers' valuation distributions as an additional input to that in prior-free setting. So our goal in Bayesian auction design is to design an algorithm that take distributions as input and outputs an auction tailored for the given distribution (illustrated in Figure 1.1). As mentioned above, an auction is represented by a pair of functions, which might not be able to be represented in polynomial size of the number of items and buyers. So how to represent the outputted auction in a succinct way is already a challenging problem. Despite these difficulties, Myerson developed an auction which maximizes the auctioneer's expected revenue for any product distribution in his celebrated work [65]. A simple example is that the optimal way to sell one item to a buyer whose value for the item is uniformly distributed in $[0, 1]$ is to price the item at 0.5. The expected revenue is 0.25. Another example is that for independent, identical and regularly distributed buyers, the second-price auction with a distribution related reserve price is the optimal auction which maximizes the expected revenue. We will discuss more details about Myerson's auction in Chapter 2.

1.2 Rich Media Advertisement

Auctions for selling online advertisement have been extensively studied in the literature, especially for sponsored search auctions, which account for a major part of search engines revenue. In the traditional model of sponsored search auction[37, 77], advertisement positions in the search results are allocated to advertisers. Each position is characterized by a quality number, representing the number of clicks it could provide. Each bidder is associated with a valuation that represents his monetary valuation per click and his valuation for an advertisement slot is the product of his per-click value and the slot's quality number.

Besides sponsored search auction, other format for online ads like banner advertisement and rich media advertisement also raise a large amount of revenue for Internet companies. From the standpoint of advertiser, the significant difference between rich media advertisement and sponsored search is that, in rich media advertisement market, each buyer desires a particular number of *consecutive* slots which is needed to display his specific advertisement. Note that this demand constraint is called *sharp* in the sense that the buyer will buy the whole bundle of slots or nothing. A question risen here is how to extend and generalize the traditional sponsored search auction model to a model suitable for rich media advertisement market. Furthermore, it is also a challenging task to decide the allocations and the prices in order to maximize the total revenue of the market maker.

From the viewpoint of theoretical computer science, the buyers' consecutive and sharp demands impose new combinatorial constraints upon the traditional models. As a consequence, providing complete solutions to the revenue optimization problems in this setting requires some amount of care and effort. In addition, it is also desirable to see a rigorous analysis of the potential impact of these new combinatorial constraints on auction design and a systematic study of the associated mechanism design issues.

1.3 Fund Raising

Suppose a start-up company would like to raise a specific amount of capital investment from a number of potential investors. One of the most difficult problems they are facing is how to decide a suitable amount of money for each investment proposal associated with different types of investors. In terms of auction theory, this problem can be viewed as an auctioneer (the start-up company) wants to sell identical items (their business plan) to a number of buyers (investors). In addition, the objective of the auctioneer is to maximize the probability to achieve some given revenue target. Actually, in Internet crowd-sourcing platforms that support fund-raising for business start-ups (Kickstarter, Indiegogo, RocketHub etc.), it is typical to aim for some amount of money. Similarly, a common feature of charitable fund-raising is the identification of a target revenue to be raised.

By contrast, in most work that considers revenue maximization in auctions, the objective is expected revenue, as opposed to the probability to achieve some target. Despite remarkable progress on the front of maximizing expected revenue, it is still unclear how to design auctions that attempt to maximize the probability to achieve some given revenue target. While a fund-raising effort is not the same thing as an auction, to some extent it can be modeled as one: an approach to a donor (or investor) corresponds to an attempt to sell an item to a would-be buyer. Besides fund raising, it is also desirable to raise a particular amount of money in other auction scenarios. For example, in a bankruptcy situation, the administrator may wish to sell a collection of items to repaying the debt. And while the FCC spectrum auction wants to raise as much money as possible, it is also required to cover its costs.

In the perspective of theoretical analysis, it is more difficult to maximize the probability to achieve some given revenue target than to maximize the expected revenue. Maximizing the probability can be viewed as maximizing an utility function of the total revenue. This utility function takes value 1 if the revenue is at least the target revenue and 0 otherwise. Compared with maximization of expected revenue, this utility function is no longer linear. Moreover, it is also not separable among buyers. Due to the lack of linearity and buyer-separability, the auctioneer must consider the optimization problem in a global way instead of considering each buyer separately.

1.4 Trading Broker

With the explosion of Internet economy, numerous industry e-marketplaces aim at conducting trading transactions between potential consumers and commodity retailers or service providers. Important practical applications include online trading platforms (e.g. eBay, Amazon), advertisement exchange services (e.g., Google, Yahoo). In fact, these online business-to-business marketplaces can be regarded as trading brokers in bilateral exchange environments.

In the economic theory, this trading broker problem is modeled as double auctions where the market maker would like to maximize its total revenue by buying low from the sellers and selling high to the buyers. It would be interesting to study the following question in this setting. How does a mechanism decide the optimal prices or payments for the participating sellers or buyers? How can a mechanism explicitly take into account the truthfulness and incentive issues arising from both sellers and buyers?

Unlike the classical auction where only buyers are strategically playing, in double auctions, both sellers and buyers are selfish and aim at maximizing their own utilities. Recall that in the one-side auction setting, an important technique to make auction truthful is to use other buyers' bids to decide a buyer's payment. However, in double auction, we also need to use the seller's cost to set buyers' payments. A simple example is that the broker should not make a trade if the buyer's valuation is lower than the seller's reported cost. This is a clearly a new ingredient added into the model. So designing truthful auctions for both sides does require much more carefully analysis. It would be more challenging to extend recent remarkable progress on multi-item revenue maximizing auctions to the settings with double auctions. More aggressively, one may expect a general framework for reducing the problems in two-side market (double auctions) to the corresponding ones in one-side market (traditional auctions).

1.5 Crowdsourcing Contest

The term "crowdsourcing" is used to refer to the methods of soliciting solutions to tasks via open calls to large-scale communities. Many crowdsourcing sites are developed to complete tasks like graphical design of logos, labeling of an image data-set, answering of an individuals question and programming for a specific problem. A common feature among these crowdsourcing competitions is the monetary or non-monetary rewards, which are provided to the best submission or a set of selected good submissions.

Crowdsourcing competitions can be modeled as *all-pay* auctions where all players, even the losers, pay their bids. This type of auction is widely used to model the competitions that agents make irreversible investments without knowing the outcome. In a crowdsourcing contest, the payments correspond to the effort that players make in order to win the reward and the private value is the rate at which the contestant works.

Clearly, this all-pay auctions are not truthful since all players are required to pay their bids even if they get nothing at the end of the auction. So in order to study the incentive issues arising from all-pay auctions, we should examine the Nash equilibria in the games induced from the auctions. It would be interesting to study the economic efficiency of the Nash equilibrium in all-pay auctions. That is to compare the social welfare extracted from all-pay auctions and other auctions (e.g., first-price auctions, second price auctions). Actually, this can be examined via adopting the famous notion called *Price of Anarchy* (PoA)[56]. PoA is defined as the ratio between the social welfare in the worst Nash equilibrium and the optimal social welfare without strategic behaviors. It is also challenging to extend the analysis of all-pay auction from single-item setting to multi-item setting. Besides social welfare, one can also investigate other objectives like revenue and the quality of the best submission.

Chapter 2

Preliminaries

In this chapter, we introduce basic definitions and essential concepts needed in the rest of this thesis. We also present a description of the common notations used in the thesis. Noting that the settings studied in next chapters are different from each other, we only introduce the most general notations and concepts here and leave the specific ones to the preliminaries of each chapter separately.

2.1 Auction Design

An auction setting consists of n bidders/buyers and m items to be sold. Each bidder has a private interest of the items, represented by a valuation function v_i. We use b_i to denote a pure strategy (a bid) of buyer i and it might be a single value or a vector, depending on the auction. We denote by $\mathbf{b}_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ the strategies of all players except for i. Any *mixed strategy* B_i of player i is a probability distribution over b_i.

An auction/mechanism can be viewed as a pair of rules: allocation rule x and payment rule p. Given a buyers' bidding profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, the allocation of buyer i is $x_i(\mathbf{b})$ and his payment is $p_i(\mathbf{b})$. Throughout this thesis, we assume buyers have quasi-linear utilities, i.e. $u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - p_i(\mathbf{b})$. We also assume the buyers are fully rational in the auction, i.e. every buyer aims at maximizing his own utility. In different settings, buyers' allocations are subject to various constraints. For example, in the unit-demand setting, each buyer can be assigned at most one item. We say an auction setting is single-parameter if each buyer's valuation can be represented by a single parameter, otherwise we say it is multi-parameter setting.

In most of the remaining chapters, we assume that all buyers' values are distributed independently according to publicly known bounded distributions following the work of [65]. The distribution of each buyer i is represented by a Cumulative Distribution Function (CDF) F_i and a Probability Density Function (PDF) f_i. The auction design in this Bayesian model will take these distributions of buyers' valuations as input and output an auction represented by allocation and payment rules.

Given a specific auction and bidding profile, we consider two important benchmarks in this auction, i.e. social welfare and revenue. More precisely, the *social welfare* is the sum of all buyers' valuations, $SW(\mathbf{x}) = \sum_{i\in[n]} v_i(x_i(\mathbf{b}))$. and the *revenue* equals the sum of the payments, $REV(\mathbf{p}) = \sum_i p_i(\mathbf{b})$ where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$.

For simplicity, if the allocation rule $\mathbf{X}$ is clear from the context, we use $SW(\mathbf{b})$, $v_i(\mathbf{b})$ and $u_i(\mathbf{b})$ instead of $SW(\mathbf{X}(\mathbf{b}))$, $v_i(X_i(\mathbf{b}))$ and $u_i(\mathbf{X}(\mathbf{b}))$, to express social welfare, valuation and utility of player i for the allocation $\mathbf{X}(\mathbf{b})$.

We use the standard notation $[s]$ to denote the set of integers from 1 to s, i.e. $[s] = \{1, 2, \ldots, s\}$. We sometimes use $\sum_i$ instead of $\sum_{i\in[n]}$ to denote the summation over all buyers and $\sum_j$ instead of $\sum_{j\in[m]}$ for items, and the terms $\mathrm{E}_{\mathbf{v}}$ and $\mathrm{E}_{v_{-i}}$ denote, respectively, an expectation over buyer valuation vectors $\mathbf{v}$ sampled from the prior, and an expectation over valuation vectors v_{-i}, representing buyers other than i. The vector $\mathbf{v}$ of all the buyers' values is sometimes written as (v_i, v_{-i}), v_i being the i-th entry of $\mathbf{v}$ and v_{-i} the joint bids of all buyers other than i.

2.2 Incentive Compatibility

One of the most important criteria in auction design is incentive compatibility. That is, truthful bidding is a dominant strategy for any buyer participating the auction. Formally,

Definition 2.1. A mechanism M is called *Incentive Compatible* (IC) iff the following inequalities hold for all i, v_i, v_i'.

$$u_i(v_i, v_{-i}) \geq u_i(v_i', v_{-i}) \tag{2.1}$$

Similarly, we can define a weak notion called Bayesian Incentive Compatibility.

Definition 2.2. A mechanism M is called *Bayesian Incentive Compatible* (BIC) iff the following inequalities hold for all i, v_i, v_i'.

$$\mathrm{E}_{v_{-i}}[u_i(v_i, v_{-i})] \geq \mathrm{E}_{v_{-i}}[u_i(v_i', v_{-i})] \tag{2.2}$$

Clearly, any Incentive compatible mechanism is also Bayesian Incentive Compatible.

2.3 Myerson's Optimal Auction

In the single-parameter settings, we review the seminal work by Myerson [65] in the following lemmas.

Lemma 2.3 (From [65])**.** *A mechanism $M = (\mathbf{x}, \mathbf{p})$ is Bayesian Incentive Compatible if and only if, letting $\underline{v}_i$ be a lower bound on values taken by v_i:*
a) $\mathrm{E}_{v_{-i}}[x_i(v_i, v_{-i})]$ is monotone non-decreasing as a function of v_i for any agent i,
b) $\mathrm{E}_{v_{-i}}[p_i(v_i, v_{-i})] = v_i \mathrm{E}_{v_{-i}}[x_i(v_i, v_{-i})] - \int_{\underline{v}_i}^{v_i} \mathrm{E}_{v_{-i}}[x_i(z, v_{-i})]dz$.

Lemma 2.4 (From [65])**.** *For any BIC mechanism* $M = (\mathbf{X}, \mathbf{p})$*, the expected revenue* $\mathrm{E}_{\mathbf{v}}[\sum_i p_i(\mathbf{v})]$ *is equal to the virtual surplus* $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i) x_i(\mathbf{v})]$ *where* $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ *denote the virtual value of buyer* i *when his valuation is* v_i*.*

2.4 Nash Equilibrium

For auctions that are not incentive compatible, we analyze strategic behavior and incentive issues by studying the Nash equilibrium in the induced game.

Definition 2.5. (*Nash equilibria*) A bidding profile $\mathbf{b}$ forms a pure Nash equilibrium if for all bids b'_i and every player i, $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$. Similarly, a mixed bidding profile $\mathbf{B} = \times_i B_i$ is a mixed Nash equilibrium if for all bids b'_i and every player i, $\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathrm{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$. Clearly, any pure Nash equilibrium is also a mixed Nash equilibrium.

Definition 2.6 (Price of anarchy)**.** Let $\mathcal{I}([n], [m], \mathbf{v})$ be the set of all instances, i.e. $\mathcal{I}([n], [m], \mathbf{v})$ includes the instances for every set of bidders and items and any possible valuations that the bidders might have for the items. The mixed price of anarchy, PoA, of a mechanism is defined as

$$\mathrm{PoA} = \max_{I \in \mathcal{I}} \max_{\mathbf{B} \in \mathcal{E}(I)} \frac{SW(\mathbf{O})}{\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{b})]},$$

where $\mathcal{E}(I)$ is the class of mixed Nash equilibria for the instance $I \in \mathcal{I}$ and $\mathbf{O}$ is the optimal allocation that maximizes the social welfare. The pure PoA is defined as above but restricted in the class of pure Nash equilibria.

2.5 Fairness and Envy-freeness

Besides incentive issues, we also consider the fairness of the auction. In this thesis, we study a widely used fairness concept called envy-freeness.

Definition 2.7. (Envy-freeness) An auction $(\mathbf{x}, \mathbf{p})$ is envy-free if no bidders can improve his utility by swapping his allocation and payments with another player. Formally, for all buyer i and i', $u_i(\mathbf{b}) \geq v_i(x_{i'}(\mathbf{b})) - \mathbf{p}_{i'}(\mathbf{b})$.

Another similar concept for fairness in auction design is called competitive equilibrium or market equilibrium.

Definition 2.8. (Competitive Equilibrium) We say an outcome of the market $(\mathbf{x}, \mathbf{p})$ is a *competitive equilibrium* if it is envy-free and all unallocated items are priced at zero.

Chapter 3

Rich Media Advertising: Auction Design with Consecutive Demand

In this chapter, we consider the optimal pricing problem for a model of the rich media advertisement market, that has other related applications. In this market, there are multiple buyers (advertisers), and items (slots) that are arranged in a line such as a banner on a website. Each buyer desires a particular number of *consecutive* slots and has a per-unit-quality value v_i (dependent on the ad only) while each slot j has a quality q_j (representing j's click-through rate, in the context of position auctions). The valuation that buyer i has for item j is the product $v_i q_j$. Our interest in buyers who demand multiple consecutive slots is motivated by modeling buyers who may require these to display a large size ad; thus we extend (and generalize) the traditional position auction model. We want to decide the allocations and the prices in order to maximize the total revenue of the market maker.

We study three major pricing mechanisms, the Bayesian pricing model, the maximum revenue market equilibrium model and an envy-free solution model. Under the Bayesian model, we design a polynomial-time computable truthful mechanism that optimizes the revenue. For the market equilibrium paradigm, we find a polynomial-time algorithm to obtain the maximum revenue market equilibrium solution. In the envy-free setting, an optimal solution is presented for the case where the buyers have the same demand for the number of consecutive slots.

This chapter is based on a joint work [30] with Paul Goldberg, Xiaotie Deng, Yang Sun and Jinshan Zhang, which appears in SAGT 2013.

3.1 Overview

Ever since the pioneering studies on pricing protocols for sponsored search advertisement, especially with the generalized second price auction (GSP), by Edelman, Ostrovsky, and Schwarz [37], as well as Varian [77], market making mechanisms have attracted much attention from the research community in understanding their effectiveness for the revenue maximization task facing platforms providing Internet advertisement services. In the traditional advertisement setting, advertisers negotiate ad presentations and prices with website publishers directly. An automated pricing mechanism simplifies this process by creating a bidding game for the buyers of advertisement space over an IT platform. It creates a complete competition environment for the price discovery process. Accompanying the explosion of the online advertisement business, there is a need for a complete picture of the pricing methods that may be used by advertisers and ad space providers.

In addition to search advertisements, display advertisements have been widely used in commercial webpages. They have a rich format of displays such as text ads and rich media ads. Compared with sponsored search, there is a lack of systematic studies of the associated mechanism design issues. The market maker faces a combinatorial problem of whether to assign a large space to one large rich media ad or multiple small text ads, as well as how to decide on the prices charged for them. We present a study of the allocation and pricing mechanisms for displaying slots in this environment where some buyers would like to have one slot and others may want several consecutive slots in a display panel. In addition to webpage ads, another motivation for our study is TV advertising where inventories of a commercial break are usually divided into slots of a few seconds each, and slots have various qualities measuring their expected number of viewers and the corresponding attractiveness.

We discuss three types of mechanisms and consider the revenue maximization problem under these mechanisms, and compare their effectiveness in revenue maximization under a dynamic setting where buyers may change their bids to improve their utilities. Our results make an important step towards understanding the advantages and disadvantages of their uses in practice. Assume the ad supplier divides the ad space into small enough slots (pieces) such that each advertiser is interested in a position with a fixed number of *consecutive* pieces. In modeling values to the advertisers, we modify the position auction model from the sponsored search market [37, 77] where each ad slot is measured by the Click Through Rates (CTR), with users' interest expressed by a click on an ad. Since display advertising is usually sold on a per impression (CPM) basis instead of a per click basis (CTR), the quality factor of an ad slot stands for the expected impression it brings per unit of time. Unlike in the traditional position auctions, people may have varying demands (need different spaces to display their ads) in a rich media ad auction for the market maker to decide on slot allocations and their prices.

3.1.1 Main Results

We have a set of *buyers* (advertisers) and a set of *items* to be sold (the ad slots on a web page). We address the challenge of computing prices that satisfy certain desirable properties. Next we describe the elements of the model in more detail.

Items. Our model considers the geometric organization of ad slots, which commonly has the slots arranged in some sequence (typically, from top to bottom in the right-hand side of a web page). The slots are of variable quality. In the study of sponsored search auctions, a standard assumption is that the quality (corresponding to click-through rate) is higher at the beginning of the sequence and then monotonically decreases. Here we consider a generalization where the quality may go down and up, subject to a limit on the total number of local maxima (which we call *peaks*), corresponding to focal points on the web page. As we will show later, without this limit the revenue maximization problem is NP-hard.

Buyers. A buyer (advertiser) may want to purchase multiple slots, so as to display a larger ad. Such slots should be *consecutive* in the sequence. Each buyer i has a fixed *demand* d_i, the number of slots she needs for her ad. Two important aspects of this are

- *sharp* multi-unit demand, that is, buyer i should be allocated d_i items, or none at all; there is no point in allocating any fewer
- *consecutiveness* of the allocated items, in the pre-existing sequence of items.

These constraints give rise to a new and interesting combinatorial pricing problem.

Valuations. We assume that each buyer i has a parameter v_i representing the value she assigns to a slot of unit quality. Valuations for multiple slots are additive, so that a buyer with demand d_i would value a block of d_i slots to be their total quality, multiplied by v_i. This is an extension of the valuation model considered by Edelman et al. [37] and Varian [77] in their seminal work for keywords advertising where the buyers are unit-demand.

Pricing mechanisms. Given the valuations and demands from the buyers, the market maker decides on a price vector for all slots and an allocation of slots to buyers, as an output of the market. The question is one of which output the market maker should choose to achieve certain objectives. We consider three approaches:

- *Truthful mechanism* whereby the buyers report their demands (publicly known) and values (private) to the market maker; then prices are set so as to ensure that the buyers have no incentive to report incorrect valuations. We give a revenue-maximizing approach (i.e., maximizing the total price paid), within this framework.
- *Competitive equilibrium* whereby we prescribe certain constraints on the prices so as to guarantee certain well-known notions of fairness and envy-freeness.

- *Envy-free solution* whereby we prescribe certain constraints on the prices and allocations so as to achieve envy-freeness, as explained below.

The mechanisms we exhibit are computationally efficient. We also perform experiments to compare the revenues obtained from these three mechanisms.

3.1.2 Literature Review

The theoretical study of position auctions (of single slots) under the generalized second price auction was initiated in [37, 77]. There has been a series of studies of position auctions in deterministic settings [59]. Our consideration of position auctions in the Bayesian setting fits in the general one-dimensional auction design framework. Our study considers continuous distributions on buyers' values. For discrete distributions, Cai et al. [11] presents an optimal mechanism for budget constrained buyers without demand constraints in multi-parameter settings and subsequently they also give a general reduction from revenue to welfare maximization in [9]; for buyers with both budget constraints and demand constraints, 2-approximate mechanisms [1] and 4-approximate mechanisms [6] exist in the literature.

There are extensive studies on multi-unit demand in economics, see for example [12, 38]. Chen et al. [17] first considered sharp multi-unit demand, where a buyer with demand d should be allocated d items or none at all, but with no further combinatorial constraint, such as the consecutiveness constraint that we consider here. The sharp demand setting is in contrast with a "relaxed" multi-unit demand (in which one can buy a subset of at most d items), where it is well known that the set of competitive equilibrium prices is non-empty and forms a distributive lattice [45, 73]. This immediately implies the existence of an equilibrium with maximum possible prices, that consequently maximizes the revenue. Demange, Gale, and Sotomayor [29] proposed a combinatorial dynamics which always converges to a revenue maximizing (or minimizing) equilibrium for unit demand; their algorithm can be easily generalized to relaxed multi-unit demand. A strongly related work to our consecutive settings is the work of Rothkopf et al. [70], where the authors presented a dynamic programming approach to compute the maximum social welfare of consecutive settings when all the qualities are the same. Hence, our dynamic programming approach for general qualities in Bayesian settings is a non-trivial generalization of their setting.

Organization This chapter is organized as follows. In Section 3.2 we describe the details of our rich media ads model and the related solution concepts. In Section 3.3, we study the problem under the Bayesian model and provide a Bayesian Incentive Compatible auction with optimal expected revenue for the special case of the single peak in quality values of advertisement positions. Then in Section 3.4, we extend the optimal auction to the case with limited peaks/valleys and show that it is NP-hard to maximize revenue without this limit. Next, in Section 3.5, we turn to the full information setting and propose an algorithm to compute the competitive equilibrium with maximum revenue. In Section 3.6, NP-hardness of envy-freeness for consecutive multi-unit demand buyers is shown. We also design a polynomial time solution for the special case where all advertisers demand the same number of ad slots.

3.2 Preliminaries

In our model, a rich media advertisement instance consists of n advertisers and m advertising slots. Each slot (or item) $j \in \{1, \ldots, m\}$ is associated with a (publicly known) number q_j which can be viewed as the quality or the desirability of the slot. Each advertiser (or buyer) i wants to display her own ad that occupies d_i consecutive slots on the webpage. In addition, each buyer has a private number v_i representing her valuation and thus, the i-th buyer's value for item j is $v_{ij} = v_i q_j$.

In this chapter, we will say that slot j is assigned to a set of buyers B, to denote that j is assigned to some buyer in B. We call the set of all slots assigned to B the allocation to B. In addition, a buyer will be called a winner if he succeeds in displaying his ad and a loser otherwise.

We represent a feasible assignment by a vector $\mathbf{X} = (X_1, \ldots, X_n)$, where X_i is the set of items allocated to buyer i, i.e. $j \in X_i$ denotes item j is assigned to buyer i. Thus we have $\{X_i\}$ is pairwise disjoint, which can be viewed as a partition of items. Given a fixed assignment $\mathbf{X}$, we use t_i to denote the total quality that buyer i is assigned, precisely, $t_i = \sum_{j \in X_i} q_j$. In general, when $\mathbf{X}$ is a function of buyers' bids $\mathbf{v}$, we define t_i to be a function of $\mathbf{v}$ such that $t_i(\mathbf{v}) = \sum_{j \in X_i(\mathbf{v})} q_j$.

When we say that slot qualities have a single peak, we mean that there exists a peak slot k such that for any slot $j \leq k$ on the above side of k, $q_j \geq q_{j-1}$ and for any slot $j \geq k$ on the below side of k, $q_j \geq q_{j+1}$.

3.2.1 Bayesian Mechanism Design

In Section 3.3 and 3.4, we assume that all buyers' values are distributed independently according to publicly known bounded distributions following the work of [65]. The distribution of each buyer i is represented by a Cumulative Distribution Function (CDF) F_i and a Probability Density Function (PDF) f_i. In addition, we assume that the concave and convex closure and integration of those functions can be computed efficiently.

An auction $M = (\mathbf{X}, \mathbf{p})$ consists of an allocation function $\mathbf{X}$ and a payment function $\mathbf{p}$. More precisely, $\mathbf{X}$ specifies the allocation of items to buyers and $\mathbf{p} = (p_i)_i$ specifies the buyers' payments, where both $\mathbf{X}$ and $\mathbf{p}$ are functions of the reported valuations $\mathbf{v}$. Our objective is to maximize the expected revenue of the mechanism $Rev(M) = \mathrm{E}_{\mathbf{v}}\left[\sum_i p_i(\mathbf{v})\right]$ by using Bayesian incentive compatible mechanisms. For completeness, we restated the definition of BIC in Chapter 2.

Definition 3.1. A mechanism M is called *Bayesian Incentive Compatible* (BIC) iff the following inequalities hold for all i, v_i, v_i'.

$$\mathrm{E}_{v_{-i}}[v_i t_i(\mathbf{v}) - p_i(\mathbf{v})] \geq \mathrm{E}_{v_{-i}}\left[v_i t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i})\right] \tag{3.1}$$

Furthermore, we say M is *Incentive Compatible* if M satisfies a stronger condition that $v_i t_i(\mathbf{v}) - p_i(\mathbf{v}) \geq v_i t_i(v_i'; v_{-i}) - p_i(v_i'; v_{-i})$, for all $\mathbf{v}, i, v_i'$,

To put it in words, in a BIC mechanism, no player can improve her *expected* utility (expectation taken over other players' bids) by misreporting her value. An IC mechanism satisfies the stronger requirement that no matter what the other players declare, no player has incentives to deviate.

3.2.2 Competitive Equilibrium and Envy-free Solution

In Section 3.5, we study the revenue maximizing competitive equilibrium and envy-free solution in the full information setting instead of the Bayesian setting. An outcome of the market is a pair $(\mathbf{X}, \mathbf{p})$, where $\mathbf{X}$ specifies an allocation of items to buyers and $\mathbf{p}$ specifies prices paid. Given an outcome $(\mathbf{X}, \mathbf{p})$, recall $v_{ij} = v_i q_j$, let $u_i(\mathbf{X}, \mathbf{p})$ denote the *utility* of i.

Definition 3.2. A tuple $(\mathbf{X}, \mathbf{p})$ is a *consecutive envy-free pricing* solution if every buyer is consecutive envy-free, where a buyer i is consecutive envy-free if the following conditions are satisfied:

- if $X_i \neq \emptyset$, then (i) X_i is d_i consecutive items. $u_i(\mathbf{X}, \mathbf{p}) = \sum_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other subset of consecutive items T with $|T| = d_i$, $u_i(\mathbf{X}, \mathbf{p}) = \sum_{j \in X_i} (v_{ij} - p_j) \geq \sum_{j \in T} (v_{ij} - p_j)$;
- if $X_i = \emptyset$ (i.e., i wins nothing), then, for any subset of consecutive items T with $|T| = d_i$, $\sum_{j \in T} (v_{ij} - p_j) \leq 0$.

In the literature, there have been two other notions of envy-free allocation, namely, sharp item envy-freeness [17] and bundle envy-freeness [39]. Sharp item envy-free requires that no buyer should envy any bundle of items whose size equals her demand, while bundle envy-free is the (weaker) stipulation that no-one should envy the bundle bought by any other buyer. Note that Definition 3.2 is about item envy-freeness; a weaker notion of consecutive bundle envy-freeness would require that no buyer should envy a bundle allocated entirely to some other buyer having the same demand.

Example 3.1 (Three types of envy-freeness)**.** *Suppose there are two buyers i_1 and i_2 with per-unit-quality $v_{i_1} = 10$, $v_{i_2} = 8$ and demands $d_{i_1} = 1$, $d_{i_2} = 2$. Three items j_1,*

j_2, j_3 have quality $q_{j_1} = q_{j_3} = 1$ and $q_{j_2} = 3$. The optimal solution of the three types of envy-freeness are as follows:

- *The optimal consecutive envy-free solution, $X_{i_1} = \{j_3\}$, $X_{i_2} = \{j_1, j_2\}$ and $p_{j_1} = p_{j_3} = 6$ and $p_{j_2} = 26$ with total revenue 38;*
- *Optimal sharp item envy-free solution, $X_{i_1} = \{j_2\}$, $X_{i_2} = \{j_1, j_3\}$ and $p_{j_1} = p_{j_3} = 8$ and $p_{j_2} = 28$ with total revenue 44;*
- *Optimal (relaxed) bundle envy-free solution, $X_{i_1} = \{j_2\}$, $X_{i_2} = \{j_1, j_3\}$ and $p_{j_1} = p_{j_3} = 8$ and $p_{j_2} = 30$ with total revenue 46;*

Definition 3.3. (Competitive Equilibrium) We say an outcome of the market $(\mathbf{X}, \mathbf{p})$ is a *competitive equilibrium* if it satisfies two conditions.

- $(\mathbf{X}, \mathbf{p})$ must be consecutive envy-free (Definition 3.2).
- The unsold items must be priced at zero.

We are interested in the revenue maximizing competitive equilibrium and envy-free solutions. It is well known that a competitive equilibrium always exists for unit demand buyers (even for general v_{ij} valuations) [73]. For our consecutive multi-unit demand model, however, a competitive equilibrium may not always exist as the following example shows.

Example 3.2 (Competitive equilibrium may not exist)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 10$ and $v_{i_2} = 9$, respectively. Let their demands be $d_{i_1} = 1$ and $d_{i_2} = 2$, respectively. There are two items j_1, j_2, both with unit quality $q_{j_1} = q_{j_2} = 1$. If i_1 wins an item, without loss of generality, say j_1, then j_2 is unsold and $p_{j_2} = 0$; by envy-freeness of i_1, we have $p_{j_1} = 0$ as well. Thus, i_2 envies the bundle $\{j_1, j_2\}$. On the other hand, if i_2 wins both items, then $p_{j_1} + p_{j_2} \leq v_{i_2 j_1} + v_{i_2 j_2} = 18$, implying that $p_{j_1} \leq 9$ or $p_{j_2} \leq 9$. Therefore, i_1 is not envy-free. Hence, there is no competitive equilibrium in the given instance.*

In the unit demand case, it is well-known that the set of equilibrium prices forms a distributive lattice; hence, there exist extremes which correspond to the maximum and the minimum equilibrium price vectors. In our consecutive demand model, however, even if a competitive equilibrium exists, maximum equilibrium prices may not exist.

Example 3.3 (Maximum equilibrium need not exist)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 1$, $v_{i_2} = 10$ and demands $d_{i_1} = 1, d_{i_2} = 2$, and two items j_1, j_2 with unit quality $q_{j_1} = q_{j_2} = 1$. It can be seen that allocating the two items to i_2 at prices $(19, 1)$ or $(1, 19)$ are both revenue maximizing equilibria; but there is no equilibrium price vector that is at least both $(19, 1)$ and $(1, 19)$.*

Because of the consecutive multi-unit demand, it is possible that some items are 'over-priced'; this is a significant difference between consecutive multi-unit and unit demand models. Formally, in a solution $(\mathbf{X}, \mathbf{p})$, we say an item j is *over-priced* if there is a buyer i such that $j \in X_i$ and $p_j > v_i q_j$. That is, the price charged for item j is larger than its contribution to the utility of its winner.

Example 3.4 (Over-priced items)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 20, v_{i_2} = 10$ and demands $d_{i_1} = 1$ and $d_{i_2} = 2$, and three items j_1, j_2, j_3 with qualities $q_{j_1} = 3, q_{j_2} = 2, q_{j_3} = 1$. We can see that the allocations $X_{i_1} = \{j_1\}, X_{i_2} = \{j_2, j_3\}$ and prices $(45, 25, 5)$ constitute a revenue maximizing envy-free solution with total revenue 75, where item j_2 is over-priced. If no items are over-priced, the maximum possible prices are $(40, 20, 10)$ with total revenue 70.*

3.3 Optimal Auction for the Single Peak Case

The goal of this section is to present our optimal auction for the single peak case that serves as an elementary component in the general case later. En route, several principal techniques are examined exhaustively to the extent that they can be applied directly in the next section. With these techniques, we show that the optimal Bayesian Incentive Compatible auction can be represented by a simple Incentive Compatible one. Furthermore, this optimal auction can be implemented efficiently. Given some mechanism, let $T_i(v_i) = \mathrm{E}_{v_{-i}}[t_i(\mathbf{v})]$ and $P_i(v_i) = \mathrm{E}_{v_{-i}}[p_i(\mathbf{v})]$. We also use $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ to denote the virtual value of buyer i when his valuation is v_i. Similar to Myerson [65], we suppose $P_i(\underline{v}_i) = \underline{v}_i T_i(\underline{v}_i)$ otherwise the revenue can be improved by increasing $P_i(\underline{v}_i)$. From Myerson [65], we obtain the following three lemmas.

Lemma 3.4 (From [65])**.** *A mechanism $M = (\mathbf{X}, \mathbf{p})$ is Bayesian Incentive Compatible if and only if, letting $\underline{v}_i$ be a lower bound on values taken by v_i:*
a) $T_i(v_i)$ is monotone non-decreasing for any agent i,
b) $P_i(v_i) = v_i T_i(v_i) - \int_{\underline{v}_i}^{v_i} T_i(z)dz$.

Lemma 3.5 (From [65])**.** *For any BIC mechanism $M = (\mathbf{X}, \mathbf{p})$, the expected revenue $\mathrm{E}_{\mathbf{v}}[\sum_i P_i(v_i)]$ is equal to the virtual surplus $\mathrm{E}_{\mathbf{v}}[\sum_i \phi_i(v_i) t_i(\mathbf{v})]$.*

We assume $\phi_i(\cdot)$ is monotone increasing, i.e. the distribution is regular. If not, Myerson's ironing technique can be applied to make $\phi_i(\cdot)$ monotone — it is here that we invoke our assumption that we can efficiently compute the convex closure of a continuous function and integration. Lemma 3.6 follows directly from Lemmas 3.4 and 3.5.

Lemma 3.6. *Suppose that $\mathbf{X}$ is the allocation function that maximizes $\mathrm{E}_{\mathbf{v}}[\phi_i(v_i) t_i(\mathbf{v})]$ subject to the constraints that $T_i(v_i)$ is monotone non-decreasing for any fixed profile v_{-i}*

of the other bidders, and any agent i is assigned either d_i consecutive slots or nothing. Suppose also that

$$p_i(\mathbf{v}) = v_i t_i(\mathbf{v}) - \int_{\underline{v}_i}^{v_i} t_i(v_{-i}, s_i) ds_i \tag{3.2}$$

Then $(\mathbf{X}, \mathbf{p})$ represents an optimal mechanism for the consecutive multiple-slot ad auction problem.

We will use dynamic programming to maximize the virtual surplus in Lemma 3.5. First we show the following useful lemma which states that all the allocated slots are consecutive (see Figure 3.1).

Lemma 3.7. *Suppose that ad slot qualities have the single-peak property. There exists an optimal allocation $\mathbf{X}$ that maximizes $\sum_i \phi_i(v_i)t_i(\mathbf{v})$, having the following property. For any unassigned slot j, it must be that either $\forall j' > j$, slot j' is unassigned or $\forall j' < j$, slot j' is unassigned.*

Proof. Let $\mathbf{X}$ be an allocation maximising the sum of the virtual values. If $\mathbf{X}$ does not satisfy the property in the statement of the lemma, we show how to modify $\mathbf{X}$ so as to satisfy the property while preserving optimality. Let j be a slot that violates the property, in that there are allocated slots to either side of j. Letting k be a maximum-quality slot, assume $j \geq k$ (where the case $j \leq k$ is similar). Modify $\mathbf{X}$ by taking all buyers allocated slots to the right of j, and moving their allocation one position to the left. The sum of virtual valuations cannot decrease, since any buyer's allocation is to slots having at least as high quality. Also, the modification reduces the number of slots violating the condition, so if applied repeatedly, we end up with an optimal allocation having the claimed property. □

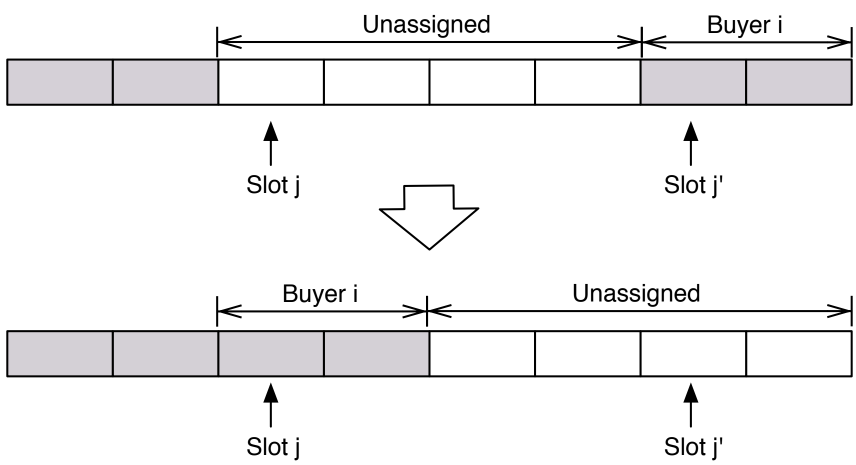


Figure 3.1: Illustration of the Proof of Lemma 3.7
By reassigning i the slots from j, we make the set of assigned slots consecutive.

By renumbering the bidders in the order of the virtual values, we can assume all the buyers are sorted in a non-increasing order according to their virtual values. Next, we prove that the consecutiveness of allocated slots holds for any buyers sets $[s] \subseteq [n]$. That is, Lemma 3.8 says that there exists an optimal allocation that always allocates the first s buyers consecutive slots, for $s \in [n]$.

Lemma 3.8. *Suppose that ad slot qualities have the single-peak property. There exists an optimal allocation* $\mathbf{X}$ *having the following property. For any slot* j *not allocated to buyers in* $[s]$*, we either have* $\forall j' > j$*, slot* j' *is not allocated to any member of* $[s]$*, or* $\forall j' < j$*, slot* j' *is not allocated to any member of* $[s]$*.*

Proof. The idea is to pick an arbitrary optimal allocation $\mathbf{X}$ and modify it to the desired one. Suppose $\mathbf{X}$ does not satisfy the property on a subset $[s]$. By Lemma 3.7, there is no unassigned slots in the middle of allocations to set $[s]$. Then there must be a slot assigned to a buyer i not in the set $[s]$ that separates the allocations to $[s]$. We use W_i to denote the allocated slots of buyer i. Suppose slot k is the peak. There are two cases to be considered:

Case 1. $k \notin W_i$. Let j and j' be the leftmost and rightmost slot in W_i respectively. We consider two cases $q_j \geq q_{j'}$ and $q_j < q_{j'}$. We only prove for the first case and the proof for the other case is symmetric. If $q_j \geq q_{j'}$, we find the leftmost slot $j_1 > j'$ assigned to $[s]$ and the rightmost slot $j_2 < j_1$ not assigned to $[s]$. In addition, let $i_1 \in [s]$ be the buyer that j_1 is assigned to and $i_2 > s$ be the buyer that j_2 is assigned to. In single peak case, it is easy to check $q_j \geq q_{j'}$ implies that all the slots assigned to i_2 have higher quality than i_1's. Thus swapping the positions of i_1 and i_2 will always increase the virtual surplus, $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ as illustrated in Figure 3.2. By continuing to do this, we can eliminate all slots not allocated to $[s]$ in the middle of allocation to $[s]$ and attain the desired optimal solution.

Case 2. $k \in W_i$. Suppose $W_i = \{j_1^i, j_2^i, \cdots, j_{u_i}^i\}$ with $j_1^i < j_2^i < \cdots < j_{u_i}^i$ and there exists $1 \leq e \leq u_i$ such that $k = j_e^i$. Let a and b be the left and right neighbour buyers of i winning slots next to W_i. As we know $a, b \in [s]$, hence, $\phi_a(v_a) \geq \phi_i(v_i)$ and $\phi_b(v_b) \geq \phi_i(v_i)$. Let $W_a = \{j_1^a, j_2^a, \cdots, j_{u_a}^a\}$ and $W_b = \{j_1^b, j_2^b, \cdots, j_{u_b}^b\}$ denote the allocated slots of buyer a and b respectively, where $j_1^a < j_2^a < \cdots < j_{u_a}^a$ and $j_1^b < j_2^b < \cdots < j_{u_b}^b$. As $k \in W_i$, then $q_{j_1^i} \geq q_{j_{u_a}^a}$ and $q_{j_{u_i}^i} \geq q_{j_1^b}$ (note that $j_{u_a}^a$ and j_1^b are the indices of slots with the largest qualities in W_a and W_b respectively). We will show that either swapping winning slots of i with a or with b will increase the virtual surplus. To prove this, there four cases needed to be considered: (1). $u_i \geq u_a$ and $u_i \geq u_b$; (2). $u_i \geq u_a$ and $u_i < u_b$; (3). $u_i < u_a$ and $u_i \geq u_b$; (4). $u_i < u_a$ and $u_i < u_b$. We only prove the case (1) since the other cases can be proved similarly. Now, suppose $u_i \geq u_a$ and $u_i \geq u_b$, then we must have either (i). $\sum_{k=1}^{u_b} q_{j_k^i} \geq \sum_{k=1}^{u_b} q_{j_k^b}$ or (ii). $\sum_{k=1}^{u_a} q_{j_{u_i-k+1}^i} \geq \sum_{k=1}^{u_a} q_{j_k^a}$. Suppose (i) is not true, that is $\sum_{k=1}^{u_b} q_{j_k^i} < \sum_{k=1}^{u_b} q_{j_k^b}$, if $u_b \le e$, then we have $q_{j_1^i} \le q_{j_{u_b}^i}$, as a result

$$u_b q_{j_1^i} \le \sum_{k=1}^{u_b} q_{j_k^i} < \sum_{k=1}^{u_b} q_{j_k^b} \le u_b q_{j_1^b} \le u_b q_{j_{u_i}^i},$$

thus, $q_{j_1^i} < q_{j_{u_i}^i}$; otherwise $u_b > e$, then it must also hold that $q_{j_1^i} \le q_{j_{u_i}^i}$ (otherwise, for any $1 \le \ell \le u_b$, $q_{j_\ell^i} \ge q_{j_{u_i}^i} \ge q_{j_1^b}$ implying that $\sum_{k=1}^{u_b} q_{j_k^i} \ge u_b q_{j_1^b} \ge \sum_{k=1}^{u_b} q_{j_k^b}$, a contradiction). In both cases, it is obtained that $q_{j_1^i} \le q_{j_{u_i}^i}$, therefore,

$$\sum_{k=1}^{u_a} q_{j_{u_i-k+1}^i} \ge u_a q_{j_1^i} \ge \sum_{k=1}^{u_a} q_{j_k^a}$$

implying (ii) is true. Thus, if (i) is true, by simple calculations, swapping winning slots of i with b will increase the virtual value (since $\phi_b(v_b) \ge \phi_i(v_i)$), otherwise swapping winning slots of i with a will increase the virtual surplus (since $\phi_a(v_a) \ge \phi_i(v_i)$). Then keep doing it by the method of Case 1 until eliminating all slots not allocated to $[s]$ in the middle of allocation to $[s]$ and attaining the desired optimal solution. □

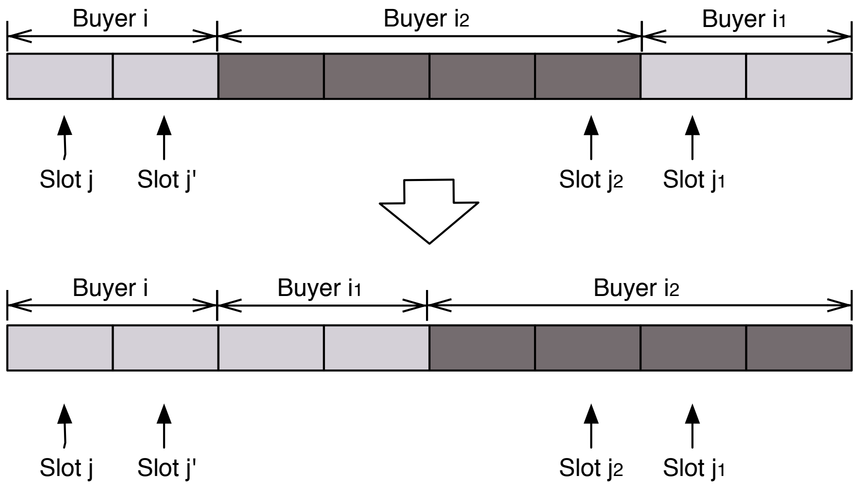


Figure 3.2: Illustration of the proof of Lemma 3.8
Slots with light color are assigned to $[s]$. By swapping the positions of i_1 and i_2, we make the allocations to $[s]$ consecutive.

Since there exists an optimal solution that assigns slots to $[s]$ consecutively (Lemma 3.8), we can express the slots allocated to $[s]$ as an interval $[\ell, r]$. Let $g[s, \ell, r]$ denote the maximized value of our objective function $\sum_i \phi_i(v_i)t_i(\mathbf{v})$ when we only consider the first s buyers and the allocation of $[s]$ is exactly the interval $[\ell, r]$. Then we have the following recurrence.

$$g[s,\ell,r] = \max \begin{cases} g[s-1,\ell,r] \\ g[s-1,\ell,r-d_s] + \phi_s(v_s)\sum_{j=r-d_s+1}^{r} q_j \\ g[s-1,\ell+d_s,r] + \phi_s(v_s)\sum_{j=\ell}^{\ell+d_s-1} q_j \end{cases} \tag{3.3}$$

Our summary statement is as follows.

Theorem 3.9. *The mechanism that applies the allocation rule according to Dynamic Programming (3.3) and payment rule according to Equation (3.2) is an optimal mechanism for the banner advertisement problem with single peak qualities.*

Proof. To complete the proof, it suffices to prove that $T_i(v_i)$ is monotone non-decreasing. More specifically, we prove a stronger fact, that $t_i(v_i, v_{-i})$ is non-decreasing as v_i increases. Given other buyers' bids v_{-i}, the monotonicity of t_i is equivalent to $t_i(v_i, v_{-i}) \leq t_i(v_i', v_{-i})$ if $v_i' > v_i$. Assuming that $v_i' > v_i$, the regularity of ϕ_i implies that $\phi_i(v_i) \leq \phi_i(v_i')$. If $\phi_i(v_i) = \phi_i(v_i')$, then $t_i(v_i, v_{-i}) = t_i(v_i', v_{-i})$ and we are done.

Consider the case that $\phi_i(v_i) < \phi_i(v_i')$. Let Q and Q' denote the total quantities obtained by all the other buyers except buyer i in the mechanism when buyer i bids v_i and v_i' respectively.

$$\phi_i(v_i')t_i(v_i', v_{-i}) + Q' \geq \phi_i(v_i')t_i(v_i, v_{-i}) + Q$$
$$\phi_i(v_i)t_i(v_i, v_{-i}) + Q \geq \phi_i(v_i)t_i(v_i', v_{-i}) + Q'.$$

The above inequalities are due to the optimality of allocations when i bids v_i and v_i' respectively. It follows that

$$\phi_i(v_i')(t_i(v_i, v_{-i}) - t_i(v_i', v_{-i})) \leq Q' - Q$$
$$\phi_i(v_i)(t_i(v_i, v_{-i}) - t_i(v_i', v_{-i})) \geq Q' - Q$$

By the fact that $\phi_i(v_i) < \phi_i(v_i')$, it must be $t_i(v_i, v_{-i}) \leq t_i(v_i', v_{-i})$. □

3.4 Multiple Peaks Case

Suppose now that there are only h peaks (local maxima) in the qualities. Thus, there are at most $h-1$ valleys (local minima). Since h is a constant, we can enumerate all the buyers occupying the valleys. After this enumeration, we can divide the sequence of slots into at most h consecutive pieces, each of which is single-peaked. Theorem 3.10 shows, using similar properties as those in Lemma 3.7 and 3.8, how we can obtain a larger size (polynomial-time) dynamic program similar to dynamic program (3.3) to solve the problem.

Theorem 3.10. *There is a polynomial algorithm to compute revenue maximization problem in Bayesian settings where the qualities of slots have a constant number of peaks.*

Proof. Our proof is based on the single peak algorithm. Assume there are h peaks, thus there must be $h-1$ valleys. Suppose these valleys are indexed $j_1, j_2, \cdots, j_{h-1}$. In an optimal allocation, for any j_k, $k = 1, 2, \cdots, h-1$, j_k must be allocated to a buyer or unassigned to any buyer. If j_k is assigned to a buyer, say, buyer i, since i would buy d_i consecutive slots, j_k may appear in the ℓth position of these d_i consecutive slots. Hence, by this brute force, each j_k will at most have $\sum_i d_i + 1 \leq mn + 1$ possible positions to be allocated. In all, all the valleys have $(mn+1)^h$ possible allocated positions. For each of this allocation, the slots are broken into h single peak slots. We can obtain similar properties to those in Lemma 3.7 and 3.8. Without loss of generality, suppose the rest of the buyers are still the set $[n]$, with non-increasing virtual value. Since the optimal solution always assigns to $[s]$ consecutively, we can express the allocations to $[s]$ as intervals denoted by $[\ell_i, r_i]$, $i = 1, 2, \cdots, h$, where $[\ell_i, r_i]$ lies in the i-th single peak slot. Let $g[s, \ell_1, r_1, \cdots, \ell_h, r_h]$ denote the maximized value of our objective function $\sum_i \phi_i(v_i) t_i(\mathbf{v})$ when we only consider the first s buyers and the allocations of $[s]$ are exactly intervals $[\ell_i, r_i]$, $i = 1, 2, \cdots, h$. Then we have the following dynamic program: $g[s, \ell_1, r_1, \cdots, \ell_h, r_h]$ can be evaluated as

$$\max_{i \in [d]} \begin{cases} g[s-1, \ell_1, r_1, \cdots, \ell_h, r_h] \\ g[s-1, \ell_1, r_1, \cdots, \ell_i, r_i - d_s, \cdots, \ell_h, r_h] + \phi_s(v_s) \sum_{j=r_i-d_s+1}^{r_i} q_j \\ g[s-1, \ell_1, r_1, \cdots, \ell_i + d_s, r_i, \cdots, \ell_h, r_h] + \phi_s(v_s) \sum_{j=\ell_i}^{\ell_i+d_s-1} q_j \end{cases}$$

Note that the dynamic programming runs in polynomial time provided that the number of peaks is constant. □

Now we consider the case without the constant peak assumption and prove the following hardness result.

Theorem 3.11. *The revenue maximization problem for allocating consecutive ad slots (with unrestricted qualities) to buyers with fixed demands, is NP-hard.*

Proof. We prove the NP-hardness by reducing from the 3-Partition problem, which is to decide whether a given multi-set of integers can be partitioned into triplets that all have the same sum. More precisely, given a multi-set S of $3n$ positive integers, can S be partitioned into n triplets $S_1, \ldots, S_n$ such that the sum of the numbers in each subset is equal? We use the fact that 3-Partition remains NP-complete in a strong sense [42], meaning that it remains NP-complete even when the integers in S are bounded above by a polynomial in n.

Given an instance of 3-Partition $(a_1, a_2, \ldots, a_{3n})$, we construct an instance of the advertising problem with $3n$ advertisers and $m = n + \sum_i a_i$ slots. Note that m is polynomial in n due to the fact that all a_i are bounded by a polynomial in n. In the advertising instance, the valuation v_i for each advertiser i is 1 and his demand d_i is defined as a_i. Moreover, for any advertiser, his valuation distribution is that $v_i = 1$ with probability 1. Then everyone's virtual value is exactly 1. Maximizing revenue is equivalent to maximizing the simplified function $\sum_i \sum_{j \in X_i} q_j$.

Let $B = \sum_i a_i / n$. We define the quality of slot j to be 0 if j is a multiple of $B+1$, otherwise $q_j = 1$. That can be illustrated as follows.

$$\underbrace{1\,1 \cdots 1}_{B}\; 0\; \underbrace{1\,1 \cdots 1}_{B}\; 0 \ldots \underbrace{1\,1 \cdots 1}_{B}\; 0$$

It is not hard to see that the optimal revenue is $\sum_i a_i$ iff there is a solution to this 3-PARTITION instance. □

3.5 Competitive Equilibrium

In this section, we study the revenue maximizing competitive equilibrium in the full information setting. To simplify the following discussions, we sort all buyers in non-increasing order of their values, i.e., $v_1 \geq v_2 \geq \cdots \geq v_n$. (By contrast, in Section 3.3, we sorted them in non-increasing order of virtual values, which are not relevant to competitive equilibrium.)

We say that an allocation $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ is *efficient* if $\mathbf{Y}$ maximizes the total social welfare, that is, $\sum_i \sum_{j \in Y_i} v_{ij}$ is maximized over all the possible allocations. We call $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ an *equilibrium price* if there exists an allocation $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{p})$ is a competitive equilibrium. The following lemma is implicitly stated in [45]; for completeness, we give a proof below.

Lemma 3.12. *If allocation $\mathbf{Y}$ is efficient, then for any equilibrium price $\mathbf{p}$, $(\mathbf{Y}, \mathbf{p})$ is a competitive equilibrium.*

Proof. Since $\mathbf{p}$ is an equilibrium price, there exists an allocation $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{p})$ is a competitive equilibrium. As a result, by envy-freeness, $u_i(\mathbf{X}, \mathbf{p}) \geq u_i(\mathbf{Y}, \mathbf{p})$ for any $i \in [n]$. Let $T = [m] \setminus \cup_i Y_i$, then we have

$$\begin{aligned}
& \sum_i \sum_{j \in Y_i} v_{ij} - \sum_{j=1}^{m} p_j \geq \sum_i \sum_{j \in X_i} v_{ij} - \sum_{j=1}^{m} p_j = \sum_i \sum_{j \in X_i} v_{ij} - \sum_i \sum_{j \in X_i} p_j \\
= \; & \sum_i u_i(\mathbf{X}, \mathbf{p}) \geq \sum_i u_i(\mathbf{Y}, \mathbf{p}) = \sum_i \sum_{j \in Y_i} v_{ij} - \sum_i \sum_{j \in Y_i} p_j \\
= \; & \sum_i \sum_{j \in Y_i} v_{ij} - \sum_{j=1}^{m} p_j + \sum_{j \in T} p_j \qquad (3.4)
\end{aligned}$$

where the first inequality is due to $\mathbf{Y}$ being efficient and first equality due to $u_i(\mathbf{X}, \mathbf{p})$ being competitive equilibrium (unallocated item priced at 0). Therefore, $\sum_{j \in T} p_j = 0$ and the above inequalities are all equalities. $\forall i : u_i(\mathbf{X}, \mathbf{p}) = u_i(\mathbf{Y}, \mathbf{p})$. Further, because the price is the same, we have

- For every loser i and every set Z of consecutive items with $|Z| = d_i$, we have $u_i(Z) \leq 0$.
- For every winner i and every set Z of consecutive items with $|Z| = d_i$, we have

$$u_i(Y_i) = u_i(X_i) \geq u_i(Z).$$

Therefore, $(\mathbf{Y}, \mathbf{p})$ is a competitive equilibrium. □

By Lemma 3.12, to find a revenue maximizing competitive equilibrium, we can first find an efficient allocation and then use linear program to settle the prices. We develop the following dynamic program to find an efficient allocation. We first only consider there is one peak in the quality order of items. The case with constant peaks is similar to the above approaches, for general peak case, as shown in above Theorem 3.11, finding one competitive equilibrium is NP-hard if the competitive equilibrium exists, and determining existence of competitive equilibrium is also NP-hard. This is because that considering the instance in the proof of Theorem 3.11, it is not difficult to see the constructed instance has an equilibrium if and only if 3 PARTITION has a solution.

Recall that all the values are sorted in non-increasing order, $v_1 \geq v_2 \geq \cdots \geq v_n$. $g[s, \ell, r]$ denotes the maximized value of social welfare when we only consider first s buyers and the allocation of s is exactly the interval $[\ell, r]$. Then we have the following recurrence.

$$g[s, l, r] = \max \begin{cases} g[s-1, \ell, r] \\ g[s-1, \ell, r-d_s] + v_s \sum_{j=r-d_s+1}^{r} q_j \\ g[s-1, \ell+d_s, r] + v_s \sum_{j=\ell}^{\ell+d_s-1} q_j \end{cases} \tag{3.5}$$

By tracking procedure 3.5, an efficient allocation denoted by $\mathbf{X}^* = (X_1^*, X_2^*, \cdots, X_n^*)$ can be found. The price $\mathbf{p}^*$ such that $(\mathbf{X}^*, \mathbf{p}^*)$ is a revenue maximization competitive equilibrium can be determined from the following linear program. Let T_i be any consecutive number of d_i slots, for all $i \in [n]$.

$$\begin{aligned} \max \quad & \sum_{i \in [n]} \sum_{j \in X_i^*} p_j \\ s.t. \quad & p_j \geq 0 && \forall\, j \in [m] \\ & p_j = 0 && \forall\, j \notin \cup_{i \in [n]} X_i^* \\ & \sum_{j \in X_i^*} (v_i q_j - p_j) \geq \sum_{j' \in T_i} (v_i q_{j'} - p_{j'}) && \forall\, i \in [n] \\ & \sum_{j \in X_i^*} (v_i q_j - p_j) \geq 0 && \forall i \in [n] \end{aligned}$$

Clearly there is only a polynomial number of constraints. The constraints in the first line represent that all the prices are non negative (no positive transfers). The constraint in the second line means unallocated items must be priced at zero (market clearance condition). And the constraint in the third line contains two aspects of information. First, for any loser k with $X_k = \emptyset$, the utility that k gets from any consecutive number of d_k is no more than zero, which makes all the losers envy-free. The second aspect is that any winner i with $X_i \neq \emptyset$ must receive a bundle with d_i consecutive slots maximizing its utility over all d_i consecutive slots, which together with the constraint in the fourth line (winner's utilities are non negative) guarantees that all winners are envy-free.

Theorem 3.13. *Under the condition of a constant number of peaks in the qualities of slots, there is a polynomial time algorithm to decide whether there exists a competitive equilibrium or not and to compute a revenue maximizing revenue market equilibrium if one does exist. If the number of peaks in the qualities of the slots is unbounded, both the problems are NP-complete.*

Proof. Clearly the above linear program and procedure (3.5) run in polynomial time. If the linear program output a price $\mathbf{p}^*$, then by its constraint conditions, $(\mathbf{X}^*, \mathbf{p}^*)$ must be a competitive equilibrium. On the other hand, if there exist a competitive equilibrium $(\mathbf{X}, \mathbf{p})$ then by Lemma 3.12, $(\mathbf{X}^*, \mathbf{p})$ is a competitive equilibrium, providing a feasible solution of above linear program. By the objective of the linear program, it must be a revenue maximizing one.

The NP-hardness follows from Theorem 3.11. □

3.6 Consecutive Envy-freeness

We first prove a negative result on computing the revenue maximization problem in general demand case. We show it is NP-hard even if all the qualities are the same.

Theorem 3.14. *The revenue maximization problem for allocating consecutive ad slots to envy-free buyers is NP-hard, even if all slot qualities are the same.*

Proof. Using similar ideas in the proof of Theorem 3.11, we prove the NP-hardness by reducing from the 3-PARTITION problem. Given an instance of 3-PARTITION $(a_1, a_2, \ldots, a_{3n})$. Let $B = \sum_i a_i/n$. We construct an instance for advertising problem with $3n+1$ advertisers and $m = B + 1 + n + \sum_i a_i$ slots. It should be mentioned that m is polynomial of n due to the fact that all a_i are bounded by a polynomial of n. In the advertising instance, the valuation v_i for each advertiser i is 1 and his demand d_i is defined as a_i and there is another buyer with valuation 2 for each slot and with demand $B+1$. The quality of each slot j is 1. It is not hard to see that the optimal revenue is $nB+2(B+1)$ if and only if there is a solution to this 3-PARTITION instance, the optimal solution is illustrated as follows.

$$\underbrace{1\,1\cdots1}_{B+1}\ \underbrace{1}_{\text{unassigned}}\ \underbrace{1\,1\cdots1}_{B}\ \underbrace{1}_{\text{unassigned}}\ \underbrace{1\,1\cdots1}_{B}\ \underbrace{1}_{\text{unassigned}}\ \cdots\underbrace{1\,1\cdots1}_{B}$$

□

Although the hardness in Theorem 3.14 indicates that finding the optimal revenue for general demand in polynomial time is impossible, it does not however rule out the important special case where the demand is uniform, that is, $d_i = d$. If in addition we have slots that are arranged in non-increasing order of their qualities, that is, $q_1 \geq q_2 \geq \cdots \geq q_m$, then we have the following computational positive result.

Theorem 3.15. *There is a polynomial-time algorithm to compute the consecutive envy-free solution in the case where all the buyers have the same demand and slots are arranged in non-increasing order of quality.*

The proof of Theorem 3.15 is based on bundle envy-free solutions; in fact we will prove the bundle envy-free solution is also a consecutive envy-free solution by defining price of items properly. Thus, we first need to give the result on bundle envy-free solutions. Suppose d is the uniform demand for all the buyers. Let T_i be the slot set allocated to buyer i, $i = 1, 2, \cdots, n$. Let P_i be the total payment of buyer i and p_j be the price of slot j. Let t_i denote the total qualities obtained by buyer i, $t_i = \sum_{j \in T_i} q_j$, and $\alpha_i = iv_i - (i-1)v_{i-1}$, $\forall i \in [n]$.

Theorem 3.16. *The revenue maximization problem of bundle envy-freeness is equivalent to solving the following optimization problem.*

$$\begin{aligned} \textit{Maximize:} \quad & \sum_{i=1}^{n} \alpha_i t_i \\ s.t. \quad & t_1 \geq t_2 \geq \cdots \geq t_n \\ & T_i \subset [m], \quad T_i \cap T_k = \emptyset \quad \forall i, k \in [n] \end{aligned} \tag{3.6}$$

Proof. Recall P_i denotes the payment of buyer i, and we next prove that the linear program (3.6) actually gives an optimal solution of bundle envy-free pricing. By the definition of bundle envy-free, where buyer i would not envy buyer $i+1$ and vice versa, we have

$$v_i t_i - P_i \geq v_i t_{i+1} - P_{i+1} \tag{3.7}$$

$$v_{i+1} t_{i+1} - P_{i+1} \geq v_{i+1} t_i - P_i \tag{3.8}$$

By summing up the above two inequalities, we have $(v_i - v_{i+1})(t_i - t_{i+1}) \geq 0$. Hence, if $v_i > v_{i+1}$, then $t_i \geq t_{i+1}$. From (3.7), we get $P_i \leq v_i(t_i - t_{i+1}) + P_{i+1}$. The maximum payment of buyer i is

$$P_i = v_i(t_i - t_{i+1}) + P_{i+1}, \tag{3.9}$$

Together with $t_i \geq t_{i+1}$, the above equation implies (3.7) and (3.8). Furthermore, the maximum payment of n is $P_n = t_n v_n$. (3.9) together with $t_i \geq t_{i+1}$ and $P_n = t_n v_n$ would make everyone bundle envy-free, the arguments are as follows.

All the buyers must be bundle envy free. By (3.9), we have $P_i - P_{i+1} = v_i(t_i - t_{i+1})$, hence $P_i = \sum_{k=i}^{n-1} v_k(t_k - t_{k+1}) + P_n$. Noticing that if $t_i = 0$, then $P_i = 0$, which means i is a loser. For any buyer $j < i$, we have

$$P_j - P_i = \sum_{k=j}^{i-1} v_k(t_k - t_{k+1}) \leq \sum_{k=j}^{i-1} v_j(t_k - t_{k+1}) = v_j(t_j - t_i)$$

By rearranging the terms, we have $v_j t_i - P_i \le v_j t_j - P_j$, that means buyer j would not envy buyer i. Similarly,

$$P_j - P_i = \sum_{k=j}^{i-1} v_k(t_k - t_{k+1}) \ge \sum_{k=j}^{i-1} v_i(t_k - t_{k+1}) = v_i(t_j - t_i)$$

We have $v_i t_i - P_i \ge v_i t_j - P_j$ that implies i would not envy buyer j.

Now we are ready to calculate $\sum_{i=1}^{n} P_i$ based on (3.9) and $t_{n+1} = 0$.

$$\begin{aligned}\sum_{i=1}^{n} P_i &= \sum_{i=1}^{n}\left[\sum_{k=i}^{n-1} v_k(t_k - t_{k+1}) + P_n\right] = \sum_{i=1}^{n}\sum_{k=i}^{n} v_k(t_k - t_{k+1}) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{k} v_k(t_k - t_{k+1}) = \sum_{k=1}^{n} k v_k(t_k - t_{k+1}) \\ &= \sum_{k=1}^{n} k v_k t_k - \sum_{k=1}^{n}(k-1)v_{k-1}t_k = \sum_{i=1}^{n}\alpha_i t_i\end{aligned}$$

We know the revenue maximizing problem of bundle envy-freeness can be formalized as (3.6). □

Since consecutive envy-free solutions are a subset of (sharp) bundle envy-free solutions, hence the optimal value of the optimization problem (3.6) gives an upper bound of optimal objective value of consecutive envy-free solutions. Note that the optimization problem (3.6) can be solved by dynamic programming. Let $g[s, j]$ denote the optimal objective value of the following LP with some set in $[j]$ allocated to all the buyers in $[s]$:

$$\begin{aligned}\text{Maximize:} \quad & \sum_{i=1}^{s}\alpha_i t_i \\ \text{s.t.} \quad & t_1 \ge t_2 \ge \cdots \ge t_s \\ & T_i \subset [j], \quad T_i \cap T_k = \emptyset \quad \forall i, k \in [s]\end{aligned}$$

We can derive the following equations since the monotonicity of the envy-free allocation guarantees that the players with higher value should get better positions.

$$g[s, j] = \max\begin{cases} g[s, j-1] \\ g[s-1, j-d] + \alpha_s \sum_{u=j-d+1}^{j} q_u \end{cases}$$

Next, we show how to modify the (sharp) bundle envy-free solution to consecutive envy-free solutions by properly defining the slot price of T_i, for all $i \in [n]$. Suppose the optimal winner set of bundle envy-free solution is $[L]$. Assume the optimal allocation and price of bundle envy-free solution are $T_i = \{j_1^i, j_2^i, \cdots, j_d^i\}$ with $j_1^i \ge j_2^i \ge \cdots \ge j_d^i$ and P_i respectively, for all $i \in [L]$.

Proof of Theorem 3.15. Define the price of T_i iteratively: $p_{j_k^L} = v_L q_{j_k^L}$, for all $k \in [d]$; $p_{j_k^i} = v_i(q_{j_k^i} - q_{j_k^{i+1}}) + p_{j_k^{i+1}}$ for $k \in [d]$ and $i \in [n]$. Now we observe that the price defined by the above procedure is still a bundle envy-free solution. This is because by induction, we have $P_i = \sum_{k=1}^{d} p_{j_k^i}$. Hence, we need only to check the prices defined as above and allocations T_i constitute a consecutive envy-free solution. In fact, we prove a strong version: suppose T_i's are consecutive from top to bottom in a line S, we will show each buyer i would not envy any consecutive sub line of S comprising d slots. For any i, we consider two cases.

Case 1. Buyer i would not envy the slots below his slots.
for any consecutive line T below i with size d, suppose T comprises of slots won by buyer k (denoted such slot set by U_k) and $k+1$ (denoted such slot set by U_{k+1} and let $\ell = |U_{k+1}|$) where $k \geq i$. Recall that $t_i = \sum_{j \in T_i} q_j$, then

$$\begin{aligned}
&\sum_{j \in T_i} p_j - \sum_{j \in T} p_j = v_i(t_i - t_{i+1}) + P_{i+1} - \sum_{j \in U_k \cup U_{k+1}} p_j \\
= \;& v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + P_k - \sum_{j \in U_k \cup U_{k+1}} p_j \\
= \;& v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + \sum_{j \in T_k \setminus U_k} p_j - \sum_{j \in U_{k+1}} p_j \\
= \;& v_i(t_i - t_{i+1}) + v_{i+1}(t_{i+1} - t_{i+2}) + \cdots + \sum_{u=1}^{\ell} v_k(q_{j_u^k} - q_{j_u^{k+1}}) \\
\leq \;& v_i(t_i - t_{i+1}) + v_i(t_{i+1} - t_{i+2}) + \cdots + \sum_{u=1}^{\ell} v_i(q_{j_u^k} - q_{j_u^{k+1}}) \\
= \;& v_i t_i - v_i \sum_{j \in T} q_j.
\end{aligned}$$

By rewriting $\sum_{j \in T_i} p_j - \sum_{j \in T} p_j \leq v_i t_i - v_i \sum_{j \in T} q_j$ as $v_i t_i - \sum_{j \in T_i} p_j \geq v_i \sum_{j \in T} q_j - \sum_{j \in T} p_j$, we get the desired result.

Case 2. Buyer i would not envy the slots above his slots. For any consecutive line T above i with size d, suppose T comprises of slots won by buyer k (denoted such slot set by U_k) and $k-1$ (denoted such slot set by U_{k-1} and let $\ell = |U_{k-1}|$) where $k \leq i$. Recall that $t_i = \sum_{j \in T_i} q_j$, then

$$\begin{aligned}
&\sum_{j \in T} p_j - \sum_{j \in T_i} p_j = \sum_{j \in U_{k-1} \cup U_k} p_j - \sum_{j \in T_i} p_j \\
= \;& \sum_{u=d-\ell+1}^{d} v_{k-1}(q_{j_u^{k-1}} - q_{j_u^k}) + \sum_{j \in T_k} p_j - \sum_{j \in T_i} p_j \\
= \;& \sum_{u=d-\ell+1}^{d} v_{k-1}(q_{j_u^{k-1}} - q_{j_u^k}) + v_k(t_k - t_{k+1}) + \cdots + v_{i-1}(t_{i-1} - t_i) \\
\geq \;& \sum_{u=d-\ell+1}^{d} v_i(q_{j_u^{k-1}} - q_{j_u^k}) + v_i(t_k - t_{k+1}) + \cdots + v_i(t_{i-1} - t_i) \\
= \;& v_i \sum_{j \in T} q_j - v_i t_i.
\end{aligned}$$

By rewriting $\sum_{j\in T} p_j - \sum_{j\in T_i} p_j \geq v_i \sum_{j\in T} q_j - v_i t_i$ as $v_i t_i - \sum_{j\in T_i} p_j \geq v_i \sum_{j\in T} q_j - \sum_{j\in T} p_j$ we get the desired result. □

3.7 Conclusion

The rich media pricing models for consecutive demand buyers in the context of Bayesian truthfulness, competitive equilibrium and envy-free solution paradigm are investigated in this chapter. As a result, an optimal Bayesian incentive compatible mechanism is proposed for various settings such as single peak and multiple peaks. In addition, to incorporate fairness e.g. envy-freeness, we also present a polynomial-time algorithm to decide whether or not there exists a competitive equilibrium and to compute a revenue maximized market equilibrium if one does exist. For envy-free settings, though the revenue maximization of general demand case is shown to be NP-hard, we still provide optimal solution of common demand case. Besides, our simulation shows a reasonable relationship of revenues among these schemes and a generalized GSP for rich media ads.

Even though our main motivation arises from the rich media advert pricing problem, our models have other potential applications. For example TV ads can also be modeled under our consecutive demand adverts where inventories of a commercial break are usually divided into slots of fixed sizes, and slots have various qualities measuring their expected number of viewers and corresponding attractiveness. With an extra effort to explore the periodicity of TV ads, we can extend our multiple peak model to one involved with cyclic multiple peaks.

Besides single consecutive demand where each buyer only has one demand choice, the buyer may have more options to display his ads, for example select a large picture or a small one to display them. Our dynamic programming algorithm (3.3) can also be applied to this case (the transition function in each step selects maximum value from $2k + 1$ possible values, where k is the number of choices of the buyer).

Another reasonable extension of our model would be to add budget constraints for buyers, i.e., each buyer cannot afford the payment more than his budget. By relaxing the requirement of Bayesian incentive compatible (BIC) to one of approximate BIC, this extension can be obtained by the recent milestone work of Cai et al. [9]. It remains an open problem how to do it under the exact BIC requirement. It would also be interesting to handle it under the market equilibrium paradigm for our model.

Chapter 4

Fund Raising: Auction Design with a Revenue Target

In many fund-raising situations, a revenue target is specified. This suggests that the fund-raiser is interested in maximizing the probability to achieve this revenue target, rather than in maximizing the expected revenue. We study this topic from the perspective of Bayesian mechanism design, in a setting where a seller has a certain good that he can supply at no cost, and there are buyers whose joint valuation for the good comes from some given prior distribution. We present an algorithm to find an optimal truthful auction for two buyers with independent valuations via a direct characterization of the optimal auction. In contrast, we show the problem is NP-hard when the number of buyers is arbitrary or the distributions are correlated. Both negative results can be modified to show NP-hardness of designing auctions for risk-averse sellers.

Furthermore, we investigate the design of *simple* auctions for multiple buyers, again in the context of a revenue target. For *Sequential Posted Price Auctions*, we provide a FPTAS to compute the optimal posted prices for a given sequence of buyers. For *Monopoly Price Auctions*, we apply the results of [25] on sparse covers of distributions to obtain a PTAS in a setting where the seller has a constraint on discriminatory pricing, consisting of a fixed set of prices he may use.

This chapter is based on a joint work [44] with Paul Goldberg, which appears in SAGT 2015.

4.1 Overview

There is a considerable literature on the algorithmic challenge of designing auctions that maximise the expected revenue obtained from a set of buyers. In this chapter we consider a related objective where instead of maximising the expected revenue, the auctioneer has been given some revenue target T, and wishes to maximise the probability of raising at least T. This objective gives rise to new and interesting algorithmic challenges, and has some plausible real-world motivations, discussed below.

We work in the classical Bayesian setting of a collection of buyers whose valuations (prices they are willing to pay) for items being sold, are assumed to be drawn from some known prior distribution D. We are interested in designing mechanisms that are incentive compatible and individually rational. D in combination with a mechanism M results in a distribution over the revenue R obtained. A standard objective is to choose M to maximise the expected value of R. A more general setting assumes a non-decreasing "utility of money" function u, and aims to maximize the expectation of $u(R)$. In this revenue-target setting, u is a shifted Heaviside function, equal to 0 for $R < T$ and 1 for $R \geq T$. Certain concave functions u have been used to model risk aversion, however the functions u considered here are not concave.

In this chapter we focus on the "digital goods" setting, where the seller can supply unlimited copies of some good, at no cost. We also assume that the buyers have unit demand, so that a buyer's type is represented by a probability distribution over his valuation for a copy of the item. This special case is a simplified model of the fund-raising situations mentioned below. In the context of digital goods and unit demands, maximisation of the *expected* revenue can be decomposed into revenue-maximisation from each buyer independently. In contrast, when we switch to a revenue target, we find that the deal offered to a buyer should depend on the outcomes of the deals offered to other buyers.

This revenue-target setting is motivated by various real-world scenarios. Charitable fund-raising typically identify a target revenue to be raised. Similarly, in Internet crowd-sourcing platforms that support fund-raising for business start-ups (Kickstarter, Indiegogo, RocketHub etc.), it is typical to aim for some amount of money, and if that target is not reached, the would-be investors get their money back. (Our model doesn't properly capture this situation; we mention it to emphasise the importance of revenue targets in practice.) While a fund-raising effort is not the same thing as an auction, to some extent it can be modelled as one: an approach to a donor (or investor) corresponds to an attempt to sell an item to a would-be buyer. In cases where goods are sold at auction, it may be more desirable to raise a particular amount of money than to maximise the expected revenue. For example, in a bankruptcy situation, the administrator may wish to sell a collection of items so as to prioritise repaying the top-tier creditors. And while the FCC spectrum auction wants to raise as much money as possible, it is also required to cover its costs.

4.1.1 Main Results

We consider the problem parametrised by the number of buyers n, and the support size m of their value distributions. With multiple buyers, it is #P-complete to compute the exact success probability (probability to achieve revenue target T) for a given auction (Proposition 4.2). Given this obstacle, in Section 4.3 we consider a basic case of two buyers having independent valuations. We exhibit a polynomial-time algorithm to exactly compute the optimal truthful auction that maximises the probability to achieve T, given as input any discrete prior distributions. We do this via a structural characterisation of auctions that optimise the probability of achieving a given revenue target. This characterisation totally differs from the one maximising expected revenue and allows us to restrict to auctions with a geometric property that makes the problem tractable.

We show contrasting hardness results for correlated valuations or n buyers with independent distributions. Specifically, it is shown to be NP-complete to compute the optimal auction for three buyers having correlated valuations and NP-hard for n buyers with independent distributions. Note that, in the latter case, a truthful auction may not necessarily be succinctly representable. We overcome the obstacle via proving the hardness for a class of succinct auctions and showing there exists a truthful auction with good performance if and only if there exists a good succinct auction in the constructed instance.

Our main algorithmic results are in Section 4.6, for two prevalent auctions following the trend of designing *simple* auctions. The first one is the *Sequential Posted Price Auction* introduced by Chawla et al. [14] to approximate the expected revenue in multi-dimensional Bayesian mechanism design. In this auction, the seller offers a take-it-or-leave-it price to each buyer sequentially. Given a sequence of buyers, we are able to provide a *fully polynomial-time approximation scheme* (FPTAS) to compute an approximately optimal sequential posted price auction that maximizes the success probability with an additive error. Second, we consider the *Monopoly Price Auction* where the seller offers take-it-or-leave-it prices to buyers simultaneously. This type of auctions was studied in [47] for selling goods with limited supply. We apply results of [25] on sparse covers of Poisson binomial distributions to obtain a PTAS when the seller has a limitation on discriminatory pricing, i.e., is only allowed to use few distinct prices.

4.1.2 Literature Review

There has been a long line of research on maximizing expected revenue in Bayesian mechanism design starting from the seminal work by Myerson [65]. Recently, Cai et al. [9] developed a general framework reducing revenue maximization to social welfare maximization. They also applied the framework to optimize certain non-linear functions [10]. However, the mechanisms they derived are randomized and Bayesian truthful, not deterministic truthful mechanisms studied in this chapter.

Another line of research studied auction design for risk-averse sellers that can be regarded as maximizing a concave function of the revenue (cf. [71]). Sundararajan and Yan [75] studied the auction design problem for a risk averse seller and gave robust mechanisms (without knowledge of the concave function) which achieve constant approximations when buyers' distributions are independent. The approximation ratio has been improved to $e/(e-1)$ by Bhalgat et al. [5] by using the knowledge of concave functions. Our work complements their results by providing some corresponding intractability results.

We mention several negative results on revenue maximization in deterministic mechanism design. Diakonikolas et al. [33] showed that it is NP-hard to maximize revenue given a welfare constraint. Chen et al. [19] proved that it is NP-hard to maximize revenue in a multi-dimensional setting with a single unit-demand buyer when the valuations of items are independently distributed. For correlated buyers, Papadimitriou and Pierrakos [67] proved that it is NP-hard to approximate the optimal expected revenue for a single-item auction. However, in digital goods setting, the revenue maximizing auction can be constructed easily by computing the optimal price for each bidder separately based on their distributions conditioned on others' bids.

The study of digital goods auctions was initiated by Goldberg et al. [43]. Recently, Chen et al. [18] derived the optimal competitive auction with the benchmark defined to measure worst-case over all buyer profiles. In contrast, our benchmark measure is the average cases based on the prior distribution.

Threshold probability maximization is a classical objective in stochastic optimization and has been studied for several combinatorial optimization problems (cf. [60] and references therein). However no incentive issues were considered before when optimizing this objective. The technique we apply to approximate the optimal monopoly price auction is based on [25]. These results have been shown helpful in computing Nash Equilibria [26] and learning sums of random variables [24]. But to our knowledge, our result is their first application in auction design.

4.2 Preliminaries

Auction Setting We study an auction environment where a seller want to sell n copies of an item to n bidders. Each bidder/buyer i is interested in a single copy of the item and values it at a privately known value v_i. A valuation profile $\mathbf{v}$ is the vector of all bidders' valuations, i.e. $\mathbf{v} = (v_1, \ldots, v_n)$. We consider a deterministic single-round sealed-bid auction where each bidder submits a bid b_i to express how much he is willing to pay for the item. After soliciting submitted bids $\mathbf{b} = (b_1, \ldots, b_n)$, the seller must decide whether each bidder i wins an item and how much he needs to pay. Bidder i's utility is the difference between his value v_i and his payment if he wins a item; otherwise he pays 0 and gets utility 0 to guarantee the individual rationality, that is no bidders will get a negative utility in the auction.

We assume every bidder in the auction is rational and aims to maximize his own utility by choosing the best bidding strategy. An auction is said to be truthful if for each bidder i, bidding his true valuation (i.e. $b_i = v_i$) is a dominant strategy no matter what the other bidders bid. It is known that truthful auctions can be characterized by *bid-independent* auctions where for each bidder i, the auction computes a threshold price p_i that does not depend on b_i but may depend on the bids of the other bidders $\mathbf{b}_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. In other words, there exists a pricing function for bidder i such that $p_i = f_i(\mathbf{b}_{-i})$ and i wins the item iff $b_i \geq p_i$ and his payment is p_i if he wins. So it suffices to consider bid-independent auctions when designing truthful auctions.

Thus any truthful or bid-independent auction A can be represented by n pricing functions $(f_1, \ldots, f_n)$ where f_i is the pricing function for bidder i which maps other bidders' valuations $\mathbf{v}_{-i}$ to the threshold price p_i. For convenience, we use $x_i(\mathbf{v})$ to denote the allocation rule of the auction, i.e. $x_i(\mathbf{v}) = 1$ if bidder i wins an item when the valuation profile is $\mathbf{v}$; otherwise $x_i(\mathbf{v}) = 0$. Hence, the revenue of A on profile $\mathbf{v}$ is $R^A(\mathbf{v}) = \sum_{i \in [n]} x_i(\mathbf{v}) f_i(\mathbf{v}_{-i})$ where $[n]$ denotes the set $\{1, \ldots, n\}$. We also use $R_i^A(\mathbf{v})$ to denote the revenue of the auction A from bidder i, i.e., $R_i^A(\mathbf{v}) = x_i(\mathbf{v}) f_i(\mathbf{v}_{-i})$. We will omit A from the notation if the auction is clear from the context.

Auction Design with a Revenue Target We assume the seller has prior knowledge of the bidders' valuations, which is represented by a distribution on the valuation profile $\mathbf{v}$. In particular, we use D to denote the distribution on the valuation profile and V to denote the support of D. We denote the probability that the valuation profile is $\mathbf{v}$ by $\Pr[\mathbf{v}]$ for all $\mathbf{v} \in V$. Obviously, the distribution D can be represented in the size of V (denoted by $|V|$ or $|D|$) by explicitly describing $\Pr[\mathbf{v}]$ for all $\mathbf{v} \in V$. We also use $V_i = \{v_i^1, \ldots, v_i^{m_i}\}$ to denote the set of all possible value of v_i in D, where m_i is $|V_i|$ and $v_i^1 < v_i^2 < \cdots < v_i^{m_i}$. For convenience, we define $v_i^0 = 0$ and assume $0 \in V_i$.

We say the bidders' valuations are independently distributed if D is a product distribution, i.e. $D = \times_{i \in [n]} D_i$ where D_i is the distribution on buyer i's valuations; otherwise they are correlated. For convenience, we say the bidders are independent (or correlated) according to whether their valuations are independently distributed. For independent bidders, D can be represented using space $O(n \cdot m)$ where $m = \max_i m_i$.

We consider a seller who is endowed with a revenue target T and his utility is 1 if the revenue raised in the auction is at least T; otherwise his utility is 0. Given an instance $\mathcal{I} = (D, T)$ with the profile distribution D and revenue target T, the seller's utility in an auction A is $\Pr_{\mathbf{v} \sim D}[R_A(\mathbf{v}) \geq T]$. We also call this value the *performance* of auction A on instance $\mathcal{I}$. So an auction is an optimal truthful auction for an instance $\mathcal{I}$ if no truthful auction can outperform A on the instance $\mathcal{I}$. Similarly, we say A is c-additive approximately optimal if no truthful auction can perform better than the performance of A plus a parameter c. It is without loss of generality to assume the range of pricing function for bidder i is V_i as shown in the following proposition.

Proposition 4.1. *For any distribution profile D and truthful auction A, there exists another truthful auction A' such that the range of pricing functions for bidder i in A' is V_i for all $i \in [n]$ and $R^{A'}(\mathbf{v}) \geq R^A(\mathbf{v})$ for all profiles $\mathbf{v}$.*

Proof. Let $f_1, \ldots, f_n$ be the pricing functions used in A. Recall that $V_i = \{v_i^1, \ldots, v_i^{m_i}\}$ is the set of all possible values of v_i in D. It is w.l.o.g. to assume that $f_i(\mathbf{v}_{-i}) \leq v_i^{m_i}$; otherwise we can set $f_i(\mathbf{v}_{-i}) = v_i^{m_i}$ without decreasing the revenue. Given pricing functions $f_1, \ldots, f_n$, we construct $f'_1, \ldots, f'_n$ such that $f'_i(\mathbf{v}_{-i}) = v_i^j$ if $f'_i(\mathbf{v}_{-i}) \in (v_i^{j-1}, v_i^j]$. It is easy to see that for any profile $\mathbf{v}$, if bidder i wins an item in A, he also wins an item in A', i.e., $x'_i(\mathbf{v}) \geq x_i(\mathbf{v})$. By the construction of f'_i, we have $f'_i(\mathbf{v}_{-i}) \geq f_i(\mathbf{v}_{-i})$ for all $\mathbf{v}$. Thus, $R^{A'}(\mathbf{v}) = \sum_i x'_i(\mathbf{v}) f'_i(\mathbf{v}_{-i}) \geq \sum_i x_i(\mathbf{v}) f_i(\mathbf{v}_{-i}) = R^A(\mathbf{v})$ for any profile $\mathbf{v}$. □

Simple Auctions We consider two types of simple auctions called monopoly price auctions and sequential posted price auctions. A *monopoly price auction* is a truthful auction with pricing functions $(f_1, \ldots, f_n)$ where each function f_i depends only on the prior distribution D and not on the other bids $\mathbf{b}_{-i}$. We say an auction is a *sequential posted price auction* with respect to an order σ if f_i may depend on D together with the bids of buyers who precede i in σ, i.e. $(b_1, \ldots, b_{i-1})$ if buyers are indexed according to σ. The following proposition shows the hardness of evaluating the performance of a given monopoly price auction.

Proposition 4.2. *Given a monopoly price auction for independent bidders, it is #P-complete to compute the probability of achieving a revenue target.*

Proof. The proposition follows from the following theorem proved in [54].

Theorem 4.3 (Theorem 2.1 in [54])**.** *Given Bernoulli trials $X_1, \ldots, X_n$, where X_i takes the value s_i with probability q_i and the value 0 with $1 - q_i$, it is #P-complete to compute $\Pr[\sum_i X_i > 1]$.*

To see it, given the monopoly prices $p_1, \ldots, p_n$, the revenue from each bidder is a Bernoulli trial taking value p_i when $v_i \geq p_i$ and the value 0 otherwise. □

4.3 Optimal auction for two independent bidders

In this section, we describe our optimal truthful auction for two independent bidders. We first give a high-level idea for the proof. First of all, we show that it is w.l.o.g. to assume the bidders' valuations are in $\{0, 1, \ldots, m\}$ and the revenue target is m. Then we show that the mechanism which can be described as two pricing function f_1, f_2 must have non-increasing f_1 and f_2. After that, we show instead of searching all possible f_1 and f_2, one can only compute the optimal f_1 and the values of f_2 follows by a direct calculation. Finally, we give a geometric characterization of the optimal f_1, which allows us to find the optimal solution by an elaborate dynamic program.

Recall that any truthful auction for two bidders can be represented by two pricing functions f_1 and f_2. By Proposition 4.1, we only need to consider $f_1 : V_2 \to V_1$ which maps bidder 2's valuations to bidder 1's threshold prices and $f_2 : V_1 \to V_2$. First of all, we show that the general problem reduces to a restricted version where bidders' distributions have support $\{0, \ldots, m\} \times \{0, \ldots, m\}$ and the target revenue is m, for some positive integer m. The intuition is mapping values of one agent to indices and mapping values of the second agent to intervals of $T - v_1$.

Lemma 4.4. *Given any instance $\mathcal{I} = (D, T)$ with an independent profile distribution $D = D_1 \times D_2$ (D_i having support V_i) and a target revenue T, there exists an integer $m \leq \min\{|V_1|, |V_2|\} + 1$ and another instance $\mathcal{I}' = (D', T')$ such that*

(a) $D' = D_1' \times D_2'$ has the support $\{0, \ldots, m\} \times \{0, \ldots, m\}$ and $T' = m$

(b) Given an instance $\mathcal{I}$, the instance $\mathcal{I}'$ can be found in time linear in m

(c) Given any optimal truthful auction for $\mathcal{I}'$, it is possible to construct an optimal truthful auction for $\mathcal{I}$ in time linear in m.

Proof. Given an instance $\mathcal{I} = (D, T)$, it is w.l.o.g. to assume $|V_1| \leq |V_2|$. We define m to be the index such that $m - 1$ is the maximal index $i \in [|V_1|]$ such that $v_1^i < T$. It follows that $m \leq |V_1| + 1$. For convenience, we add the valuation T into V_1 if for all $v_1 \in V_1$, $v_1 < T$ and the same modification for V_2 as well. Recall that $v_1^0 = v_2^0 = 0$. We construct the following two mappings $g_1 : V_1 \to [0, \ldots, m]$ and $g_2 : V_2 \to [0, \ldots, m]$ as follows: $g_1(v_1^i) = \min\{i, m\}$ and $g_2(v_2^j) = k$ if $v_2^j \in [T - v_1^{m-k}, T - v_1^{m-k-1})$. It is easy to see g_1 and g_2 are monotone non-decreasing. Then we define the inverse function for g_1 and g_2 as follows. Define $g_1^{-1}(i) = v_1^i$. Define $g_2^{-1}(j)$ to be the smallest $v_2 \in V_2$ such that $v_2 \geq T - v_1^{m-j}$. Note that g_2^{-1} is well-defined since there always exists $v_2 \in V_2$ such that $v_2 \geq T$ after the modification described above. It is easy to check the following properties hold: $v_1 \geq g_1^{-1}(g_1(v_1))$ for all $v_1 \in V_1$, $v_2 \geq g_2^{-1}(g_2(v_2))$ for all $v_2 \in V_2$, $i = g_1(g_1^{-1}(i))$ for all $i = 0, \ldots, m$ and $j = g_2(g_2^{-1}(j))$ for all $j \in \{0, \ldots, m\}$ such that there exists $v_2 \in V_2$, $g_2(v_2) = j$.

Now we consider the instance $\mathcal{I}' = (D', m)$ where D' is the modified distribution by replacing v_1^i and v_2^j by $g_1(v_1^i)$ and $g_2(v_2^j)$ respectively in D. Note that we may map different valuations in V_1 to the same $v_1' \in \{0, \ldots, m\}$. In this case, we set the probability of v_1' in D_1' to be the sum of the probabilities of all valuations in V_1 such that $g_1(v_1) = v_1'$. Similar modifications will be applied to D_2' as well. Given D_2', we have $j = g_2(g_2^{-1}(j))$ for all j with positive probability in D_2' since there exists $v_2 \in V_2$, $g_2(v_2) = j$ for such j.

It is clear that $\mathcal{I}'$ satisfies (a) and (b) in the lemma. In order to prove (c), we first show that there exists a mapping from any truthful auction A' for $\mathcal{I}'$ to a truthful auction A for $\mathcal{I}$ such that A and A' have the same performance. More specifically, given any truthful auction $A' = (f_1', f_2')$ for $\mathcal{I}'$, let $A = (f_1, f_2)$ be a truthful auction for $\mathcal{I}$ such that $f_1(v_2) = g_1^{-1}(f_1'(g_2(v_2)))$ for all $v_2 \in V_2$ and $f_2(v_1) = g_2^{-1}(f_2'(g_1(v_1)))$ for all $v_1 \in V_1$. We will show that $\Pr_{\mathbf{v}\sim D}[R_A(\mathbf{v}) \geq T] = \Pr_{\mathbf{v}'\sim D'}[R_{A'}(\mathbf{v}') \geq m]$. It is sufficient to show that for any profile $\mathbf{v} \in V$ with positive probability in D, $R_A(\mathbf{v}) \geq T$ if and only if $R_{A'}(\mathbf{v}') \geq m$ where $\mathbf{v}' = (g_1(v_1), g_2(v_2)) \in V'$. Suppose $R_A(\mathbf{v}) \geq T$, by definition we have $x_1(\mathbf{v})f_1(v_2) + x_2(\mathbf{v})f_2(v_1) \geq T$. Recall that $x_1(\mathbf{v})$ is the allocation rule of A for bidder 1. We can show $x_1'(\mathbf{v}') = x_1(\mathbf{v})$ by the following deduction.

$$\begin{aligned} &x_1(\mathbf{v}) = 1 \Leftrightarrow v_1 \geq f_2(v_2) \Leftrightarrow g_1(v_1) \geq g_1(f_2(v_2)) \\ &\Leftrightarrow g_1(v_1) \geq g_1(g_1^{-1}(f_1'(g_2(v_2^j)))) \Leftrightarrow g_1(v_1) \geq f_1'(g_2(v_2^j)) \Leftrightarrow x_1'(\mathbf{v}') = 1 \end{aligned}$$

The second equivalence follows by the monotonicity of g_1 and the fourth one is due to the fact that $i = g_i^{-1}(g_1(i))$. By similar arguments, we also have $x_2'(\mathbf{v}) = x_2(\mathbf{v})$. In order to show $R_{A'}(\mathbf{v}') \geq m$, consider the following cases: Case (1): $x_1(\mathbf{v})f_1(v_2) \geq T$, then we have $g_1(f_1(v_2)) = m$. So $R_{A'}(\mathbf{v}') \geq m$ follows from $x_1'(\mathbf{v}') = x_1(\mathbf{v})$. Case (2): $x_2(\mathbf{v})f_2(v_1) \geq T$, the analysis for this case is similar to previous case by changing from bidder 1 to bidder 2. Case (3): $x_1(\mathbf{v})f_1(v_2) < T$ and $x_2(\mathbf{v})f_2(v_1) < T$, then it must be the case that $x_1(\mathbf{v}) = x_2(\mathbf{v}) = 1$ and $f_1(v_2) + f_2(v_1) \geq T$ since $R_A(\mathbf{v}) \geq T$. Thus it is sufficient to show that $g_1(f_1(v_2)) + g_2(f_2(v_1)) \geq m$. Let i be the index such that $v_1^i = f_1(v_2)$. Since $f_1(v_2) < T$ and $f_2(v_1) \geq T - f_1(v_2)$, we have $g_1(f_1(v_2)) = i$ and $g_2(f_2(v_1)) \geq m - i$. Combine both cases, we get $R_A(\mathbf{v}) \geq T$ implies $R_{A'}(\mathbf{v}') \geq m$. The other part of the proof follows by similar arguments.

Given an instance $\mathcal{I}$ and g_1, g_2 defined as above, we say a truthful auction with pricing functions (f_1, f_2) for $\mathcal{I}$ is regular if $g_1(v_1^i) = g_1(v_1^k)$ implies $f_2(v_1^i) = f_2(v_1^k)$ for all $i, k \in \{0, \ldots, |V_1|\}$ and $g_2(v_2^j) = g_2(v_2^\ell)$ implies $f_1(v_1^i) = f_1(v_1^\ell)$ for all $j, \ell \in \{0, \ldots, |V_2|\}$. To put it in words, if two valuations have the same image in g_1 or g_2, then the pricing functions take the same value on them.

Next we show that there always exists a regular optimal truthful auction. Given an optimal auction $A = (f_1, f_2)$, it is w.l.o.g. to assume that $f_1(v_2) \leq T$ and $f_2(v_1) \leq T$ for all $v_1 \in V_1$ and $v_2 \in V_2$ otherwise we can decrease the value of f_1 or f_2 to T without decreasing the probability to get the target revenue T. Suppose that $g_1(v_1^i) = g_1(v_1^k)$ and $f_2(v_1^i) \neq f_2(v_1^k)$ for some i, k. W.l.o.g. we assume $\Pr[v_1 = v_1^i] > 0$ and $\Pr[v_1 = v_1^k] > 0$ otherwise we can made them equal without decreasing the probability. Since $g_1(v_1^i) = g_1(v_1^k)$, we have $v_1^i \geq T$ and $v_1^k \geq T$ by the definition of g_1. Now we modify A to a new auction B by only changing the value of $f_2(v_1^i)$ to $f_2(v_1^k)$. It suffices to show that the performance should be the same. Recall that $R_1^A(\mathbf{v})$ and $R_2^A(\mathbf{v})$ denote the revenue of A from bidder 1 and bidder 2 respectively when the profile is $\mathbf{v}$. Since $f_1(v_2) \leq T$, we have $R_1^A(v_1^i, v_2) = R_1^A(v_1^k, v_2)$ for any $v_2 \in V_2$. Moreover, $R_1^A(\mathbf{v}) = R_1^B(\mathbf{v})$ since the pricing function for bidder 1 are the same in A and B. Also note that $R_2^A(v_1^k, v_2) = R_2^B(v_1^i, v_2)$ since the pricing function of bidder 2 in B has value $f_2(v_1^k)$ on v_1^i. Suppose to the contrary that the probability to get revenue T decreases in B. Since we only change the value of $f_2(v_1^i)$, it is $\Pr[R^B(\mathbf{v}) \geq T | v_1 = v_1^i] < \Pr[R^A(\mathbf{v}) \geq T | v_1 = v_1^i]$. Since the bidders are independent, we get

$$\sum_{j \leq |V_2|} \Pr[v_2 = v_2^j] \cdot \mathbf{1}_{R_1^A(v_1^i, v_2^j) + R_2^A(v_1^i, v_2^j) \geq T} > \sum_{j \leq |V_2|} \Pr[v_2 = v_2^j] \cdot \mathbf{1}_{R_1^B(v_1^i, v_2^j) + R_2^B(v_1^i, v_2^j) \geq T}$$

where $\mathbf{1}_{\text{cond}}$ is the indicator function of the condition cond, i.e. it is 1 if cond is true and 0 otherwise. Since $R_1^B(v_1^i, v_2) = R_1^A(v_1^i, v_2) = R_1^A(v_1^k, v_2)$ and $R_2^A(v_1^k, v_2) = R_2^B(v_1^i, v_2)$,

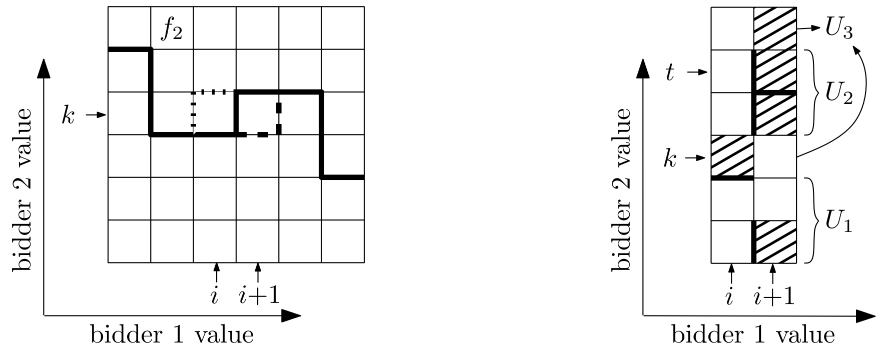
$$\sum_{j \leq |V_2|} \Pr[v_2 = v_2^j] \cdot \mathbf{1}_{R_1^A(v_1^i, v_2^j) + R_2^A(v_1^i, v_2^j) \geq T} > \sum_{j \leq |V_2|} \Pr[v_2 = v_2^j] \cdot \mathbf{1}_{R_1^A(v_1^k, v_2^j) + R_2^A(v_1^k, v_2^j) \geq T}$$

which contradicts with the optimality of A since we can set $f_2(v_1^k) = f_2(v_1^i)$ in A and increase the probability of getting T. So we can modify A to satisfy that $g_1(v_1^i) = g_1(v_1^k)$ implies $f_2(v_1^i) = f_2(v_1^k)$. For the other part that $g_2(v_2^j) = g_2(v_2^\ell)$ implies $f_1(v_1^i) = f_1(v_1^\ell)$, we can use similar arguments. The only difference is that if $g_2(v_2^j) = g_2(v_2^\ell)$, we cannot get $R_2(v_1^i, v_2^j) = R_2(v_1^i, v_2^\ell)$. But since $R_1(\mathbf{v})$ will only take values in V_1, we have $R_1(v_1^i, v_2^j) + R_2(v_1^i, v_2^j) \geq T \Leftrightarrow R_1(v_1^i, v_2^\ell) + R_2(v_1^i, v_2^\ell) \geq T$ by the definition of g_2. Roughly speaking, v_2^j and v_2^ℓ have no difference, if we only care whether the revenue is at least T or not. Then existence of regular optimal truthful auctions follows by using above arguments for f_2 on f_1.

After that, we will show there exists a mapping from any regular truthful auction A for $\mathcal{I}$ to a truthful auction A' for $\mathcal{I}'$ such that A and A' have the same performance. Formally, for any regular truthful auction $A = (f_1, f_2)$ for $\mathcal{I}$, let $A' = (f_1', f_2')$ be a truthful auction for $\mathcal{I}'$ such that $f_1'(j) = g_1(f_1(g_2^{-1}(j))$ for all $j = 0, \dots, m$ and $f_2'(i) = g_2(f_2(g_1^{-1}(i))$ for all $i = 0, \dots, m$. We will show that $\Pr_{\mathbf{v} \sim D}[R_A(\mathbf{v}) \geq T] = \Pr_{\mathbf{v}' \sim D'}[R_{A'}(\mathbf{v}') \geq m]$. Note that unlike the previous mapping, we require the regularity of A in this mapping. The proof of this mapping follows from similar arguments in the proof of the previous mapping by using regular condition for the valuations that have the same image in g_1 or g_2.

Finally, we prove (c) by using above two mappings. Given an optimal truthful auction A' for $\mathcal{I}'$, we use the first mapping to get a truthful auction A. Suppose to the contrary that A is not optimal for $\mathcal{I}$, then there exits a regular optimal truthful auction B which has better performance than A. We use the second mapping on B to get a truthful auction B' for $\mathcal{I}'$. By the property of these two mappings, we have the performance of B and B' are the same and better than A and A' which contradicts with the optimality of A'. □

For the rest of this section, we assume $V_1 = V_2 = \{0, \dots, m\}$ and $T = m$. We also use q_1^i and q_2^j to denote probabilities $\Pr[v_1 = i]$ and $\Pr[v_2 = j]$ respectively and $R(i, j)$ to be the revenue from the profile (i, j). Regarding pricing functions, we can assume $f_1(0) = m$ and $f_2(0) = m$, since otherwise we can increase $f_1(0)$ or $f_2(0)$ to m without loss of the objective. In the following lemmas, we show that there exists an optimal auction with several nice properties. The first one is monotonicity of f_1 and f_2. Intuitively, the lemma says once one bidder's valuation increases, the seller will get more revenue from this bidder and set a lower price for the other bidder as a consequence.



(a) The dotted line shows the modified form of f_2 by setting $f_2(i) = k+1$ if $R(i,k) < m$. Otherwise, if $R(i,k) \geq m$, the dashed line shows the modification by setting $f_2(i+1) = k$.

(b) The illustration of partitions U_i and profiles with revenue at least m in A (marked by shaded squares). The vertical and horizontal bold lines are f_1 and f_2 respectively.

Figure 4.1: Illustrations of the proof of Lemma 4.5.

We use a grid to illustrate all the profiles in D. The square (i,j) is the profile $v_1 = i$ and $v_2 = j$. The bold lines in the grid denote the value of pricing functions f_1 or f_2. If square (i,j)'s left (or bottom) boundary is bold, that means $f_1(j) = i$ (or $f_2(i) = j$).

Lemma 4.5. *There exists an optimal truthful auction for two independent bidders such that the pricing functions are monotonically non-increasing.*

Proof. Let $A = (f_1, f_2)$ be an optimal auction. Suppose that f_1 and f_2 are not non-increasing. We exhibit a procedure to modify f_2 to be non-increasing without changing f_1 or the performance of A. A similar procedure can then be used to modify f_1, so the lemma follows by applying the procedure to f_1 and f_2. Before presenting the proof, we introduce a new notation $S(i,j)$ defined as $\{j' \in \{0,\ldots,m\} | R_1(i,j') + \mathbf{1}_{j' \geq j} \cdot j \geq T\}$. Intuitively, the set $S(i,j)$ includes all the valuations of bidder 2 such that the revenue is at least T when $v_1 = i$ and $f_2(i)$ is re-set to j. Recall that q_2^j is the probability of $v_2 = v_2^j$. We claim that $\sum_{j \in S(i,f_2(i))} q_2^j \geq \sum_{j \in S(i,j')} q_2^j$ for any $i, j' \in \{0,\ldots,m\}$. Otherwise, the performance of A can be increased by changing $f_2(i)$ to j'.

Suppose f_2 is increasing for some i, i.e., $f_2(i) < f_2(i+1)$. For simplicity of presentation, we use k and t to be the values of $f_2(i)$ and $f_2(i+1)$. So when we use $f_2(i)$ or $f_2(i+1)$ below, we are referring the modification of f_2, not the values of them. By Proposition 4.1 and the hypothesis that f_2 is increasing at i, we know $k < t \leq m$.

We describe our procedure as illustrated in Figure 4.1(a). If $R(i,k) < m$, equivalently $k + R_1(i,k) < m$, we increase $f_2(i)$ to $k+1$. Obviously, this modification will only decrease the revenue for profile (i,k). Since $R(i,k) < m$ before the adjustment, this modification dose not decrease the objective, i.e. the probability to get the target revenue m.

If $R(i,k) \geq m$, equivalently $k + R_1(i,k) \geq m$, we decrease $f_2(i+1)$ from t to k. We will show that the objective will not decrease after this adjustment. By the independence of D and the definition of S, it is sufficient to prove $\sum_{j\in S(i+1,k)} q_2^j \geq \sum_{j\in S(i+1,t)} q_2^j$. In order to prove this, we partition the set $\{0,\ldots,m\}$ into three parts associated with the value k: $U_1 = \{j | j < k\}$, $U_2 = \{j | j \geq k \text{ and } f_1(j) = i+1\}$ and $U_3 = \{j | j \geq k \text{ and } f_1(j) \neq i+1\}$. See Figure 4.1(b) for an illustration.

If $j \in U_1$, we get $j < k < t$ since f_2 is increasing on i. Thus, the revenue from bidder 2 is 0 when $v_1 = i$ or $v_1 = i+1$, i.e. $R_2(i,j) = R_2(i+1,j) = 0$. Then we have $R_1(i,j) + R_2(i,j) \geq m$ iff $R_1(i,j) \geq m$ that will not be effected by changing $f_2(i)$. So we can see $S(i,k) \cap U_1 = S(i,t) \cap U_1$ and similarly $S(i+1,k) \cap U_1 = S(i+1,t) \cap U_1$.

If $j \in U_2$, we get $R_1(i,j) = 0$ and $R_1(i+1,j) = i+1$ by $f_1(j) = i+1$. Then we have $R_1(i+1,j) + \mathbf{1}_{j\geq k} \cdot k = i+1+k > R_1(i,k) + k \geq m$ since the revenue from bidder 1 cannot be higher than his valuation. Thus, $S(i+1,k) \cap U_2 = U_2 \supseteq S(i+1,t) \cap U_2$. On the other hand, $R_1(i,j) + R_2(i,j) = R_2(i,j) = k < m$. So $S(i,k) \cap U_2 = \emptyset$.

If $j \in U_3$, we have $R_1(i,j) = R_1(i+1,j)$ by $f_1(j) \neq i+1$. So $S(i+1,k) \cap U_3 = S(i,k) \cap U_3$ and $S(i+1,t) \cap U_3 = S(i,t) \cap U_3$. So by optimality of f_2, we have

$$\begin{aligned}\sum_{j\in S(i+1,k)\cap U_3} q_2^j = \sum_{j\in S(i,k)\cap U_3} q_2^j = \sum_{j\in S(i,k)} q_2^j - \sum_{j\in S(i,k)\cap U_1} q_2^j \\ \geq \sum_{j\in S(i,t)} q_2^j - \sum_{j\in S(i,t)\cap U_1} q_2^j \geq \sum_{j\in S(i,t)\cap U_3} q_2^j = \sum_{j\in S(i+1,t)\cap U_3} q_2^j\end{aligned}$$

The second equality is due to $S(i,k) \cap U_2 = \emptyset$ and the first inequality follows from the optimality of f_2 and $S(i,k) \cap U_1 = S(i,t) \cap U_1$.

Combining all together, we have shown that $\sum_{j\in S(i+1,k)} q_2^j \geq \sum_{j\in S(i+1,t)} q_2^j$ by considering each partition separately. Thus, setting $f_2(i+1) = k$ when $k + R_1(i,k) \geq m$ will not decrease the objective function. Therefore, f_2 can be made non-increasing by repeating this procedure. □

By Lemma 4.4 we assume the valuations of both bidders are in $\{0,\ldots,m\}$ and the target revenue is m. So for any profile $\mathbf{v}$ such that $v_1 < m$ and $v_2 < m$, the seller must sell items to both bidders to achieve the target revenue. Based on this observation, we are able to show another property of f_1 and f_2.

Lemma 4.6. *There exists an optimal truthful auction $A = (f_1, f_2)$ for two independent* *bidders such that f_1 is non-increasing and for any $i \in \{0, \dots, m\}$,*

$$f_2(i) = \begin{cases} m & \text{if } \forall j \in \{0,\dots,m\}, i < f_1(j) \\ j & \text{if } \exists j \in \{0,\dots,m\}, f_1(j) \le i < f_1(j-1) \\ f_2(m-1) & \text{if } \forall j \in \{0,\dots,m\}, i \ge f_1(j), \text{ i.e. } i = m \text{ since } f_1(0) = m \end{cases}$$

Proof. Given an optimal auction $A = (f_1, f_2)$ that satisfies Lemma 4.5. We describe how to modify A to satisfy the conditions in this lemma. If $i < f_1(m)$, for any j, we have $i < f_1(m) \le f_1(j)$ by the monotonicity of f_1. Thus $R_1(i, j) = 0$ then setting $f_2(i) = m$ is the only choice to raise revenue m. For i such that $f_1(j) \le i < f_1(j-1)$ for some j, we consider the following two cases if $f_2(i) \ne j$.

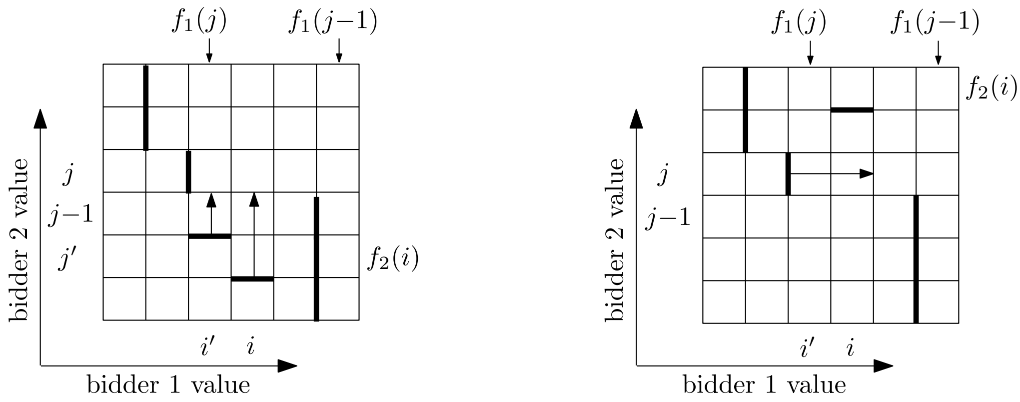
Case (1): $f_2(i) < j$. Then for all $i' \le i$ such that $f_2(i') < j$, we change $f_2(i')$ to j as shown in Figure 4.2(a). It is clear that f_2 is still monotone. By doing this, we only decrease the revenue for profiles (i', j') with i' described above and j' such that $f_2(i) \le j' \le j-1$. We claim that the seller cannot raise a revenue m in A for these profiles. This is because $f_1(j') \ge f_1(j-1) > i \ge i'$, equivalently $R_1(i', j') = 0$ and $f_2(i') \le f_2(i) < j \le m$ equivalently $R_2(i', j') < m$ by the monotonicity of f_2. That is, the seller didn't sell the item to bidder 1 in A and raised a revenue at most $f_2(i') < m$ from bidder 2.

Case (2): $f_2(i) > j$. Then we change $f_1(j)$ to $i+1$ as shown in Figure 4.2(b). It is easy to check it is still monotone since $i + 1 \le f_1(j-1)$. By doing this, we only decrease the revenue for the profile (i', j) with i' such that $f_1(j) \le i' \le i$. Similarly, we claim that the seller cannot raise the target revenue in A for these profiles since $f_2(i') \ge f_2(i) > j$ and $f_1(j) < f_1(j-1) \le m$.

If $i \ge f_1(j)$ for all j, it must be $i = m$ since $f_1(0) = m$. So setting $f_2(m) = f_2(m-1)$ is well defined. Obviously, the modified f_2 is monotone. It is also optimal because for $i = m-1$, by above result, we have $f_1(f_2(m-1)-1) > m-1$. So for all $f_2(m) \le j < f_2(m-1)$, we have $f_1(j) \ge f_1(f_2(m-1)-1) > m-1$, i.e. $f_1(j) = m$. Hence, increasing $f_2(m)$ to $f_2(m-1)$ will not effect the seller to get the revenue m from these profiles.

Combining all these together, we define the procedure and the lemma follows by repeating it. □

Intuitively, the optimal auction described in the above lemma divides all profiles into four areas. In area one, the auction allocates nothing and in area two it sells both items. In area three (or four), the auction only sells a single copy with a price m to bidder 1 (or bidder 2). In addition, as shown in Figure 4.3, the values of f_2 in this auction only depend on f_1. Thus, in order to design the optimal auction, we only need to find the optimal f_1, then a suitable f_2 follows by Lemma 4.6. Before characterizing the optimal f_1, we introduce some new notations. Given a non-increasing function f_1, let $J \subseteq [m]$ be the set of indices such that $f_2(j) < f_2(j-1)$. We denote the set J by $\{j_1, j_2, \dots, j_{|J|}\}$ with an increasing order, i.e. $j_\ell < j_{\ell+1}$. Let $i_\ell = f_1(j_\ell)$ as illustrated



(a) Modification when $f_2(i) < j$. (b) Modification when $f_2(i) > j$.

Figure 4.2: Illustration of the proof of Lemma 4.6

in Fig 4.4. We also define $i_0 = j_{|J|+1} = m + 1$ for simplicity. Then for all $\ell = 1, \ldots, |J|$ and $j_\ell \le j < j_{\ell+1}$, $f_1(j) = i_\ell$ by the definition of j_ℓ. In addition, for all $\ell = 1, \ldots, |J|$ and $i_\ell \le i < i_{\ell-1}$, $f_2(i) = j_\ell$ by Lemma 4.6. This is because j_ℓ is the j such that $f_1(j) \le i < f_1(j-1)$. Then we can prove the following lemma.

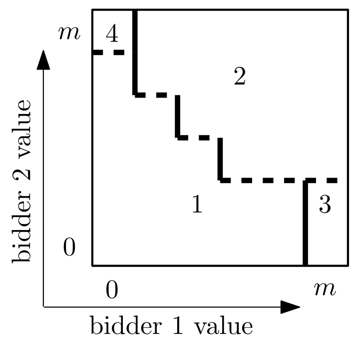


Figure 4.3: Illustration of the computation of f_2 for a given f_1 based on Lemma 4.6.

Again, the vertical bold lines are f_1 and the horizontal dashed lines are the resulting f_2. We also mark the four areas mentioned in the text.

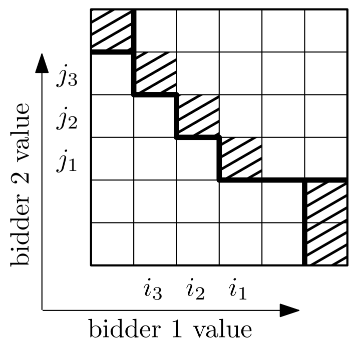


Figure 4.4: Illustration of the definition of the set J.

The values j_ℓ and i_ℓ when the pricing functions f_1 and f_2 are given as vertical and horizontal bold lines respectively. The shawed squares illustrate the profiles with revenue at least m.

Lemma 4.7. *There exists an optimal auction $A = (f_1, f_2)$ such that $i_\ell + j_\ell = m$ for all $\ell = 1, \ldots, |J|$ where i_ℓ and j_ℓ are defined by f_1 as above.*

Proof. Given an optimal auction $A = (f_1, f_2)$ satisfying Lemma 4.5 and Lemma 4.6, we define a procedure to change A to satisfy this lemma as well. If $i_\ell + j_\ell < m$ for some $\ell \in [|J|]$, it is easy to see that $R(i_\ell, j_\ell) = i_\ell + j_\ell < m$. So increasing $f_1(j_\ell)$ by 1 will not decrease the objective. Obviously, after this modification, f_1 is still monotone since $f_1(j_\ell - 1) > f_1(j_\ell)$ by the definition of j_ℓ. After repeating this modification, we get an auction such that $i_\ell + j_\ell \geq m$ for all $\ell \in [|J|]$.

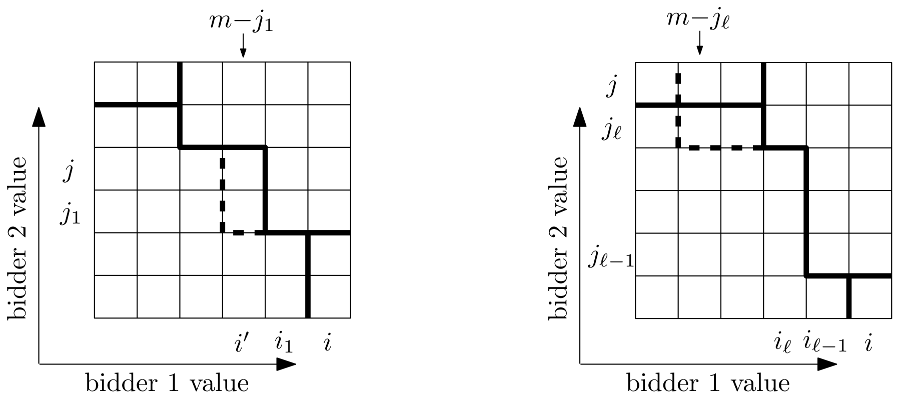
Now we describe the procedure by induction on ℓ. For the basic case $\ell = 1$, we make $i_1 + j_1 = m$ by setting $f_1(j) = m - j_1$ for all $j \geq j_1$ such that $f_1(j) > m - j_1$ and changing f_2 according to Lemma 4.6 as shown in Figure 4.5(a). It is obvious that the resulting f_1 is still non-increasing. Then it suffices to show this modification will not decease the objective. We prove this by two steps. In the first step, we show that the changes of f_1 (without changing f_2) will not decrease the objective. By the calculation above this lemma, $f_2(i) \geq f_2(m) = j_1$ for all i. If A get the revenue m for any profile (i, j) with $j \geq j_1$ and $f_1(j) > m - j_1$, we must have $R_2(i, j) \geq j_1$ since $R_1(i, j) \leq f_1(j) \leq f_1(j_1) < f_1(j_1 - 1) \leq m$. So by setting $f_1(j)$ to $m - j_1$, the seller could also get the revenue m for these profiles. This completes the first step. In the second step, we show that after changing f_1, the changes of f_2 according to Lemma 4.6 will not decrease the objective. Note that the changes we made on f_2 are setting $f_2(i') = j_1$ for all $m - j_1 \leq i' < i_1$. For all such i', the revenue from bidder 1 did not change for the case when $v_1 = i_1$ and $v_1 = i'$, i.e. $R_1(i_1, j') = R_1(i', j')$ for all $j' \in \{0, \ldots, m\}$. So setting $f_2(i') = j_1$ is also an optimal choice by the independence of the distribution and the optimality of A (the arguments are identical to the ones for proving the existence of regular auctions in the proof of Lemma 4.4).

Inductive step: similar with the modification in base, we make $i_\ell + j_\ell = m$ by setting $f_1(j) = m - j_\ell$ for all $j \geq j_\ell$ such that $f_1(j) > m - j_\ell$ and changing f_2 according to Lemma 4.6 as shown in Figure 4.5(b). It is obvious that the resulting f_1 is still non-increasing and this modification will not change the value of $\{j_1, \ldots, j_{\ell-1}\}$ and $\{i_1, \ldots, i_{\ell-1}\}$. So it suffices to show this modification will not decease the objective. First, we show that for all $i \geq i_{\ell-1}$ and $j \geq j_\ell$, the seller cannot get the target revenue from (i, j). By the calculation before this lemma and the induction hypothesis, we have $R_2(i, j) \leq f_2(i) \leq j_{\ell-1} = m - i_{\ell-1}$. On the other hand $R_1(i, j) \leq f_1(j) \leq f_1(j_\ell) = i_\ell < i_{\ell-1}$. So the revenue $R(i, j)$ is $R_1(i, j) + R_2(i, j) < m$. Then the inductive step follows from similar arguments in the base case by ignoring all the profiles (i, j) with $i \geq i_{\ell-1}$ and $j \geq j_\ell$.

Therefore, we can get the desired auction by applying this procedure from $\ell = 1$ to $|J|$ and complete the proof of the Lemma. □

By the above lemma, we can characterize the optimal auction by only using the set J, i.e. the values of $\{j_1, \ldots, j_{|J|}\}$. Given the set J, we can compute f_1 and f_2 by Lemma 4.7 and 4.6 respectively. Based on this characterization, we are able to show the main theorem in this section.

Theorem 4.8. *Given a distribution $D = D_1 \times D_2$ for two independent bidders and a*



(a) Modification for the base case.

(b) Modification for the inductive step.

Figure 4.5: Illustration of the proof of Lemma 4.7

target revenue for the seller, an optimal truthful auction can be found in time $O(m^3)$ *where* $m = \min\{|D_1|, |D_2|\}$.

Proof. By Lemma 4.4, it suffices to show the result for the case of $V_1 = V_2 = \{0, \ldots, m\}$ and $T = m$. Then, by Lemma 4.7, we only need to consider the auctions satisfying the conditions in Lemma 4.5, 4.6 and 4.7. Actually, any such auction can be represented by a set $J \subseteq \{0, \ldots, m\}$ denoted by $\{j_1, \ldots, j_{|J|}\}$ such that $0 < j_1 <, \ldots, < j_{|J|} < m$. Then we can set $i_\ell = m - j_\ell$ by Lemma 4.7 and compute the non-decreasing pricing function f_1 according to $\{i_\ell\}$ and $\{j_\ell\}$, and the other function f_2 by Lemma 4.6. Note that if $J = \emptyset$, that is $f_1(j) = f_2(i) = m$ for all $i, j \in \{0, \ldots, m\}$.

Given an auction defined by a set J, for any profile (i, j) such that $R(i, j) \geq m$, it should be in one of the three cases: (1) $R_1(i, j) = m$ (2) $R_2(i, j) = m$ (3) $R_1(i, j) < m$ and $R_2(i, j) < m$. For the case (1), it must be $j = m$ and $f_2(i) = m$. By Lemma 4.7, we have $i < f_1(j_1) = i_1$. For the case (2), it must be $i = m$ and $f_1(j) = m$. By the definition of j_ℓ, we have $j > j_{|J|}$. For the case (3), there must exists $\ell \in [|J|]$ such that $f_1(j) = i_\ell$ and $f_2(i) = j_\ell$. Then we have $i_\ell \leq i < i_{\ell-1}$ and $j_\ell \leq j < j_{\ell+1}$. Therefore given J, we can mark all the profiles where the raised revenue is at least m as shown in Figure 4.4.

The above analysis allow us to express the probability of getting the target revenue as a function of $\{j_1, \ldots, j_{|J|}\}$ if $J \neq \emptyset$. For convenience, let $q(i^-, i^+, j^-, j^+)$ be the probability that $i^- \leq v_1 < i^+$ and $j^- \leq v_2 < j^+$ and $i_\ell = m - j_\ell$. We also use $J_{<\ell}$ (or $J_{\geq \ell}$) to denote the subset of J where the elements are less than j_ℓ (or at least j_ℓ). Then the probability $\Pr_{\mathbf{v} \sim D}[R(\mathbf{v}) \geq T]$ can be expressed as

$$q(m, m+1, 0, j_1) + q(0, i_{|J|}, m, m+1) + \sum_{\ell \in [|J|]} q(i_\ell, i_{\ell-1}, j_\ell, j_{\ell+1}) \qquad (*)$$

So to design the optimal auction is equivalent to maximizing above formula. Note that any term in Formula $(*)$ that contains j_ℓ as a variable, only contain $j_{\ell-1}$ and $j_{\ell+1}$ as other variables. This allows us to show the optimal substructure of this problem. That is, if J maximizes formula $(*)$, $J_{\geq \ell}$ also maximize the part of formula $(*)$ that only contains $J_{\geq \ell}$ as variables given the value of $j_{\ell-1}$. This is because otherwise we can improve the value of formula $(*)$ by setting $J_{\geq \ell}$ optimally without changing $J_{<\ell}$.

Based on this observation, we are able to develop a dynamic programming for this problem. For any $i \in [m]$ and $k \in [i-1]$ we define $W[i,k]$ to be the optimal probability you can get when $v_1 < i$, $v_2 > m-i$, $f_2(i) = m-i$ and $f_2(i-1) = m-k$. Intuitively, we solve the optimal auction for all profiles by breaking it down into the sub-problems for profiles with $v_1 < i$ and $v_2 > m-i$ and the values i and k satisfies $f_2(i) = m-i$ and $f_2(i-1) = m-k$. That is $j_{\ell-1} = m-i$ and $j_\ell = m-k$ for some ℓ. If we only consider the profiles with $v_1 < i$, $v_2 > m-i$, the probability is the part of Formula $(*)$ that only contains $J_{\geq \ell}$ as variables. So we can compute $W[i,k]$ recursively as follows based on the optimal substructure of Formula $(*)$.

$$W[i,k] = \max \begin{cases} q(0,k,m,m+1) + q(k,i,m-k,m+1) & \\ W[k,t] + q(k,i,m-k,m-t) & 1 \leq t < k \end{cases}$$

The first line in the maximum is to consider the case that $k = m - j_{|J|}$, i.e. k is the last element in J and the second line is for the other case that $j_\ell = m-k$ and $j_{\ell+1} = m-t$. Then the final solution is

$$\max \begin{cases} q(m,m+1,0,m+1) + q(0,m,m,m+1) & \\ q(m,m+1,0,m-i) + q(0,i,m,m+1) + q(i,m+1,m-i,m+1) & 1 \leq i < m \\ W[i,k] + q(m,m+1,0,i) + q(i,m+1,m-i,m-k) & 1 \leq k < i < m \end{cases}$$

The first line is to consider the case that $J = \emptyset$, that is $f_1(j) = f_2(i) = m$ for all $i, j \in \{0, \ldots, m\}$. The second line is to consider the case that J is a singleton, i.e. $|J| = 1$. The last line is for the other case, i.e. $|J| \geq 2$. The correctness of the dynamic programming follows from the optimal substructure of Formula $(*)$. Since the number of states is at most m^2 and the time of computing each state is $O(m)$, the running time is $O(m^3)$ □

4.4 NP-Completeness for three correlated bidders

In this section, we show that in the case of three correlated bidders, it is NP-complete to decide whether there exists a truthful auction such that the probability of getting revenue T is at least Q for the given distribution D and target revenue T. Specifically, the input of the problem is the distribution D expressed explicitly as its probability mass function in the size of D, the target revenue T and a probability Q. We state our main theorem in this section.

Theorem 4.9. *It is NP-complete to compute an optimal auction for three correlated bidders.*

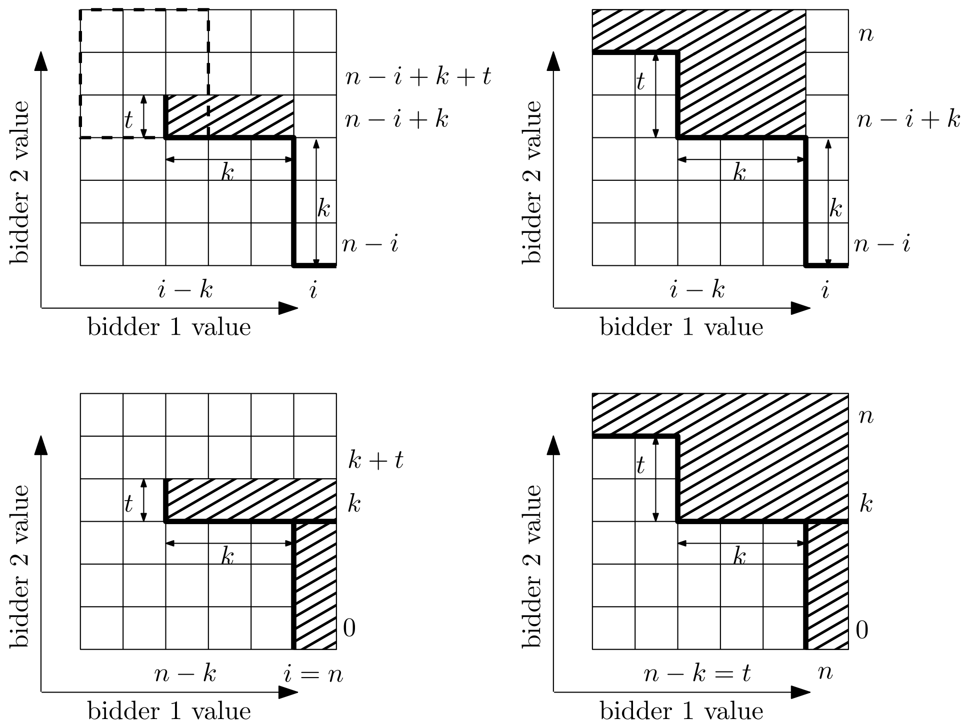


Figure 4.6: Illustration of the dynamic programming.

Proof. Membership in NP follows from noting that any truthful auction A can be expressed by three pricing functions f_1, f_2, f_3, that use space polynomial in the input size. So we can guess the values of $f_1(v_2, v_3)$, $f_2(v_1, v_3)$ and $f_3(v_1, v_3)$ for all $(v_1, v_2, v_3) \in V$ and then compute the probability of getting revenue T by considering the profiles one by one.

We show NP-hardness by reducing from the problem Vertex Cover. Given a graph with G with n vertices labeled $\{1, 2, \ldots, n\}$ and m edges, we construct an instance with three bidders where the support of the valuation distributions of bidder 1 and bidder 2 are $[n]$ and bidder 3's valuation are taken from $[n] \cup \{1.5\}$[1]. We also set the target revenue T to be $n + 2$. Then NP-hardness will be shown in two steps. First we show NP-hardness for finding an optimal auction for bidders 1 and 2, where bidder 3 has his valuation fixed at $v_3 = n$, but contributes revenue according to a pricing function $h(v_1, v_2)$, where h is part of the input. Next we construct a distribution for $v_3 < n$ that endogenizes h in the sense that any truthful auction with good performance must use h as the pricing function for bidder 3. First of all, we define the particular function h as follows.

$$h(v_1, v_2) = \begin{cases} 0 & \text{if } v_1 + v_2 \le n \\ 1 & \text{if } v_1 + v_2 = n + 1 \\ v_1 + v_2 - n - 1 & \text{if } v_1 + v_2 > n + 1 \end{cases}$$

Now we consider the profiles with $v_3 = n$. Since $h(v_1, v_2) < n$, the revenue from bidder

[1] Technically the value 1.5 used here can be replaced by any value between 1 and 2.

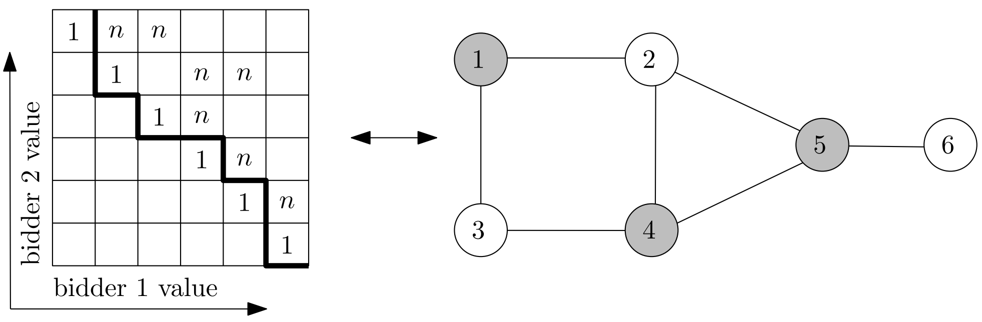


Figure 4.7: An example for the construction of the distribution for $v_3 = n$ when $n = 6$. The numbers 1 and n in the (v_1, v_2)-pair boxes are proportional to their probabilities. The corresponding auction is shown on the left if the vertex cover of the graph is $\{1, 4, 5\}$ shown as gray nodes on the right.

3 is exactly given by h, i.e. $R_3(v_1, v_2, n) = h(v_1, v_2)$ for all v_1, v_2. So the revenue needed from bidder 1 and 2 to achieve the target revenue $n+2$ is $n+2-h(v_1, v_2)$. Recall that $\Pr[\mathbf{v}]$ denotes the probability when the profile is $\mathbf{v}$. For simplicity of presentation, we work with probabilities higher than 1 that can be normalized by dividing by the sum of all probabilities later. Given the graph G, we define a correlated distribution as follows (illustrated in Figure 4.7).

$$\Pr[(v_1, v_2, n)] = \begin{cases} 0 & \text{if } v_1 + v_2 \leq n \\ 1 & \text{if } v_1 + v_2 = n + 1 \\ n & \text{if } v_1 + v_2 > n + 1 \text{ and } (v_1, n + 1 - v_2) \in E \\ 0 & \text{if } v_1 + v_2 > n + 1 \text{ and } (v_1, n + 1 - v_2) \notin E \end{cases}$$

We will show that G has a vertex cover with size at most $k < n$ iff there exists an auction whose probability to achieve the target revenue is at least $m \cdot n + n - k$ where m is the number of edges in G. For necessity, suppose S is a vertex cover for G and $|S| \leq k$. We construct an auction such that $f_1(v_2, n) = n + 2 - v_2$ if $n + 1 - v_2 \in S$ otherwise $f_1(v_2, n) = n+1-v_2$ and $f_2(v_1, n) = n+2-v_1$ if $v_1 \in S$ otherwise $f_2(v_1, n) = n+1-v_1$. Moreover we set the pricing function f_3 for bidder 3 to be the particular h defined above. Then for any (v_1, v_2) such that $v_1 + v_2 > n + 1$ and $\Pr[(v_1, v_2, n)] > 0$, we have $(v_1, n + 1 - v_2) \in E$ by the construction of $\Pr[(v_1, v_2, n)]$. Since S is a vertex cover for G, we have $v_1 \in S$ or $n + 1 - v_2 \in S$. So the revenue from the profile (v_1, v_2, n) is $f_1(v_1, n)+f_2(v_2, n)+f_3(v_1, v_2)$ that is at least $n+1-v_2+n+1-v_1+v_1+v_2-n-1+1 = n + 2$. Thus the probability that these profiles contribute to the event of raising the target revenue is $m \cdot n$. On the other hand, for any (v_1, v_2) such that $v_1 + v_2 = n + 1$ and $v_1 \notin S$, the revenue of the profile (v_1, v_2, n) is $n + 1 - v_1 + n + 1 - v_2 + 1 = n + 2$. Therefore the total probability is at least $m \cdot n + n - k$.

For the sufficiency, suppose A is an auction whose probability is at least $m \cdot n+n-k$. Note that we assume A use h as the pricing function for bidder 3. Let S be the set of vertices v_1 such that A doesn't get the target revenue from the profile $(v_1, n + 1 - v_1)$.

Since the probability achieved by A is at least $m{\cdot}n{+}n{-}k$ and $k < n$, it is easy to see that $|S| \leq k$ and A achieves the target revenue for all profile (v_1, v_2) with $\Pr[v_1, v_2, n] = n$. For any edge $(v_1, n+1-v_2) \in E$, it is w.l.o.g. to assume $v_1 > n+1-v_2$, i.e., $v_1 + v_2 > n+1$. Then we have $\Pr[v_1, v_2, n] = n$ by the construction. So the revenue obtained from this profile by A is at least $n+2$, i.e. $f_1(v_2, n) + f_2(v_1, n) + f_3(v_1, v_2) \geq n+2$. So we have $f_1(v_2, n) > n+1-v_2$ or $f_2(v_1, n) > n+1-v_1$. That is $v_1 \in S$ or $n+1-v_2 \in S$. Therefore S is a vertex cover of G.

Next, we show how to construct the distribution for $v_3 < n$ such that any auction with good performance must use h be the pricing function for bidder 3. We divide all profiles into $n+1$ layers according to the value of v_3. In layer 0, $v_3 = 1$; in layer 1, $v_3 = 1.5$; and for all $k = 2, \ldots, n$, $v_3 = k$. The general idea is to use layer $k \in \{0, \ldots, n-1\}$ to make f_3 equal to h on the profiles (v_1, v_2) such that $v_1 + v_2 = n + 1 + k$. Finally, we use the last layer n (when $v_3 = n$) to encode the VERTEX COVER instance as described above.

We present the inductive construction of the layers from 1 to $n-1$. For layer 0, only the profiles on the diagonal have positive probabilities, i.e. $\Pr[(v_1, v_2, 1)] = n$ iff $v_1 + v_2 = n+1$. So in order to get all the probabilities in this layer, the auction must set the pricing function for bidder 3 as $f_3(v_1, v_2) = 1$ for all (v_1, v_2) such that $v_1 + v_2 = n+1$.

For layer 1, we first set $\Pr[(v_1, n+2-v_1, 1.5)] = n$ for all $v_1 \in [n]$ and $\Pr[(v_1, n+1-v_1, 1.5)] = n$ for all odd v_1. We will show in order to get all the positive probabilities from layer 0 and layer 1, the auction must set $f_3(v_1, v_2) = 1$ for all $v_1 + v_2 = n+2$. Note that for any (v_1, v_2) such that $v_1 + v_2 = n+2$, either v_1 is odd or $n+1-v_2 = v_1 - 1$ is odd. That is either $\Pr[(v_1, v_2 - 1, 1.5)] = n$ or $\Pr[(v_1 - 1, v_2, 1.5)] = n$ by the construction. Since $f_3(v_1, v_2 - 1) = f_3(v_1 - 1, v_2) = 1$ by the construction of layer 0, we have either $f_2(v_1, n) = v_2 - 1$ or $f_1(v_2, n) = v_1 - 1$ to guarantee the auction get all probabilities in layer 1. That is $f_1(v_2, n) + f_2(v_1, n) = v_1 + v_2 - 1 = n+1$. So we need to set $f_3(v_1, v_2) \geq 1$ to get the target revenue. Since $v_3 = 1.5$ in this layer, the value of $f_3(v_1, v_2)$ cannot be higher than 1.5. By Proposition 4.1, it is w.l.o.g. to set it 1 because the target revenue and the valuations of the other bidders are integers.

Finally, for layer $k \geq 2$, we set $\Pr[(v_1, v_2, 1.5)] = n$ for all $v_1 + v_2 = n+1+k$. Then We construct a graph G' with n vertices labeled by $[n]$ and a set of edges denoted by E such that $(v_1, v_2) \in E$ iff $v_1 + n + 1 - v_2 = n + 1 + k$ as shown in Figure 4.8. Since the graph consists of k chains, there exists an exact cover S' for this graph. That is for each $(v_1, v_2) \in E$, one and only one of $\{v_1, v_2\}$ is in S'. Actually this can be proven constructively by setting $S' = \{1, 2, \ldots, k, 2k+1, 2k+2, \ldots, 3k, \ldots\}$ as labeled in Figure 4.8. Given this exact cover S', we define the distribution for layer k as follows: $\Pr[(v_1, n+1-v_1, k)] = n$ for all $v_1 \in S'$ and $\Pr[(v_1, n+2-v_1, k)] = n$ if $n+1-v_2 \notin S'$ or $v_1 \notin S'$ as shown in Figure 4.8. Intuitively, we construct the distribution such

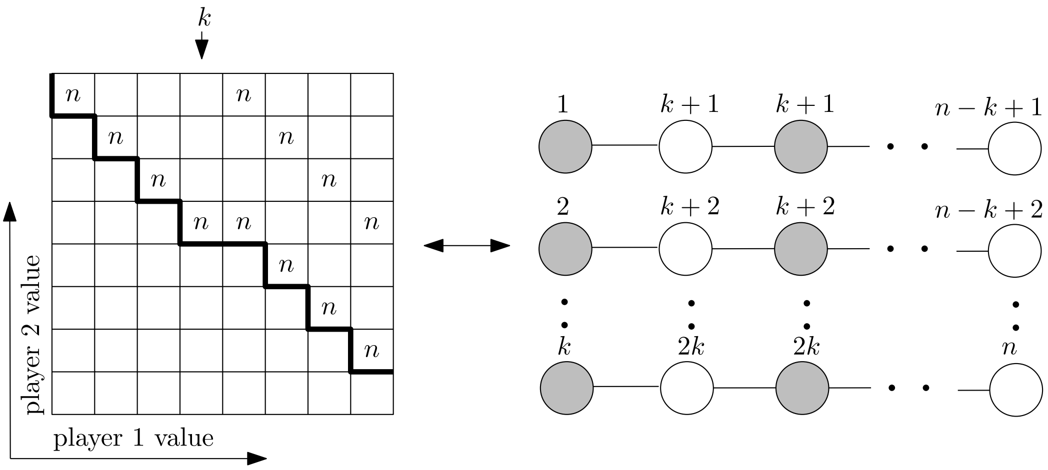


Figure 4.8: An illustration for the construction of layer k.
The gray vertices form the exact cover S when n is a multiple of k and n/k is even. The left graph shows the construction and optimal auction when $n = 8$ and $k = 4$.

that for any (v_1, v_2) with $v_1 + v_2 = n + 1 + k$, either $\Pr[(v_1, n + 1 - v_1, k)] = n$ and $\Pr[(n{+}2{-}v_2, v_2, k)] = n$ or $\Pr[(v_1, n{+}2{-}v_1, k)] = n$ and $\Pr[(n{+}1{-}v_2, v_2, k)] = n$. These profiles will restrict the values of $f_1(v_2, k)$ and $f_2(v_1, k)$. Furthermore, we require for any $(v_1, n{+}2{-}v_1)$ with positive probability, not both $(v_1, n{+}1{-}v_1)$ and $(v_1{-}1, n{+}2{-}v_1)$ have positive probabilities to guarantee that $f_2(v_1, k) + f_1(n + 2 - v_1, k) \geq n + 1$.

Now we prove that in order to get all the probabilities in first n layers from layer 0 to layer $n - 1$, the auction must set h to be the pricing function for bidder 3. Let $A = (f_1, f_2, f_3)$ be the auction that gets the target revenue for all profiles with positive probabilities. It suffices to show $f_3(v_1, v_2) = h(v_1, v_2)$ for all (v_1, v_2) such that $v_1 + v_2 \geq n + 1$ since no profile with $v_1 + v_2 < n + 1$ has positive probability. Recall that we have shown $f_3(v_1, v_2) = h(v_1, v_2)$ for all $v_1 + v_2 = n + 1 + k$ when $k = 0$ and $k = 1$. We will prove this for any $k \in \{2, \ldots, n\}$ base on the results for $k = 0$ and $k = 1$. If the auction get the revenue $n + 2$ for the profile $(v_1, n + 1 - v_1, k)$, it must be $f_1(n + 1 - v_1, k) = v_1$ and $f_2(v_1, k) = n + 1 - v_1$ since $f_3(v_1, v_2) = 1$. Similarly, since $f_3(v_1, n{+}2{-}v_1) = 1$, if the auction get the revenue $n{+}2$ for the profile $(v_1, n{+}2{-}v_1, k)$, it must be $f_1(n{+}2{-}v_1, k) \leq v_1$ and $f_2(v_1, k) \leq n{+}2{-}v_1$. By the construction of layer k and the property of exact cover, for any (v_1, v_2) such that $v_1 + v_2 = n + 1 + k$, either $\Pr[(v_1, n + 1 - v_1, k)] = n$ and $\Pr[(n + 2 - v_2, v_2, k)] = n$ or $\Pr[(v_1, n + 2 - v_1, k)] = n$ and $\Pr[(n + 1 - v_2, v_2, k)] = n$. So the revenue from bidder 1 and bidder 2 is at most $n + 1 - v_1 + n + 2 - v_2 = n + 2 - k$. To achieve the target revenue $n + 2$, the price for bidder 3 must be k. Therefore, we show that $f_3(v_1, v_2) = h(v_1, v_2)$ for all (v_1, v_2) such that $v_1 + v_2 \geq n + 1$.

Combining the construction for $v_3 = n$ and $v_3 < n$, we can easily show that the graph G has a vertex cover with size at most k iff there exists a truthful auction can get the probability $m \cdot n + n - k + s \cdot n$ where s is the total number of profiles with positive probabilities when $v_3 < n$. □

We are able to show a similar NP-completeness result for risk-averse sellers

Theorem 4.10. *It is NP-complete to compute the optimal utility auction to maximize a concave function of the revenue for three correlated bidders.*

Proof. Its membership in NP is obvious by similar arguments in the proof of Theorem 4.9. We prove NP-hardness by reducing from the following 3D-POINT-COVER problem.

Problem 1 (Problem 6.3 in [49]). Given a set S of n points $(x_i, y_i, z_i)(i = 1, ..., n)$ and an integer k, recognize whether there exist k lines parallel to the axes whose union contains S.

Given an instance of 3D-POINT-COVER problem in $[2..n]^3$, we construct the following valuation distribution and concave utility function. $u(x) = x$ if $x \leq 4$ and $u(x) = 4$ otherwise. $\Pr[(1, 1, 1)] = n^9$, $\Pr[(x, 1, 1)] = \Pr[(1, y, 1)] = \Pr[(1, 1, z)] = n^7$ and $\Pr[(1, y, z)] = \Pr[(x, 1, z)] = \Pr[(x, y, 1)] = 1$. In addition, $\Pr[(x, y, z)] = n^3$ if $(x, y, z) \in S$ and 0 otherwise. It is easy to see that the optimal auction will set $f_1(1, 1) = f_2(1, 1) = f_3(1, 1) = 1$ to get utility 3 from $(1, 1, 1)$ and $f_1(1, z) = f_1(y, 1) = f_2(x, 1) = f_2(1, z) = f_3(x, 1) = f_3(1, y) = 1$ to get utility 3 from $(x, 1, 1)$, $(1, y, 1)$ and $(1, 1, z)$ for any $x, y, z \geq 2$.

We will show that the optimal utility for this instance is at least $3n^9 + 3n^7(n - 1) + 4n^3 \cdot |S| + 9(n - 1)^2 - k$ iff there exists a k cover for the corresponding 3D-POINT-COVER instance. For sufficiency, suppose there exists a k cover K for the instance. We construct the following auction where $f_1(x, y) = 2$ if the line of (x, y) is in the cover, and $f_1(x, y) = 1$ otherwise. It is easy to check for any (x, y, z) with positive probability, at least one of (x, y), (y, z), (x, z) is 2, so the revenue is at least 4. So the total utility is $3n^9 + 3n^7(n - 1) + 4n^3 \cdot |S| + 9(n - 1)^2 - k$. For necessity, if there exists an auction has utility $3n^9 + 3n^7(n - 1) + 4n^3 \cdot |S| + 9(n - 1)^2 - k$, let K be the set of all $f_1(y, z) > 1$. It is easy to check this is a point cover for the instance. □

4.5 NP-hardness for n independent bidders

In this section, we show that in the case of independent bidders, it is NP-hard to decide whether there exists a truthful auction such that the probability of getting the revenue T is at least Q for the given distribution D. More specifically, the input is the independent distribution $D = \times_i D_i$ with each D_i expressed explicitly, a target revenue T and a probability Q. In contract with Section 4.4, a truthful auction for n bidders cannot necessarily be represented succinctly in space polynomial in the input size even if each bidder has only two possible valuations. The following is our main theorem in this section.

Theorem 4.11. *It is NP-hard to compute the optimal auction for independent bidders even when each bidder has only two possible valuations, i.e.* $|V_i| = 2$.

Proof. We prove this theorem in two steps. First, we show that it is NP-hard to compute the optimal monopoly pricing auction. Then we show that in the hard instance for optimal monopoly pricing, the existence of a truthful auction with good performance is equivalent to the existence of a monopoly pricing auction with good performance.

Recall that in the monopoly pricing auction, the seller specifies a threshold price for each bidder independent of any other buyers' bids. So this type of auction can be represented by a pricing vector $(p_1, \ldots, p_n)$ where p_i denotes the monopoly price for bidder i. We reduce from the SUBSET-SUM problem. That is given a set of positive integers denoted by $a_1, \ldots, a_n$, ask if there is a subset S such that $\sum_{i \in S} a_i = K$ for some positive integer K.

Given an instance of SUBSET-SUM, we construct the following instance for optimal auction design. Let $H = \sum_i a_i + 1$ and $W = \max_i\{a_i\}$. It is w.l.o.g. to assume $K \leq nW$. For each integer a_i in the SUBSET-SUM instance, we construct a bidder i whose valuation is H with probability $q_i = a_i \cdot \varepsilon$ and $H + a_i$ with probability $1 - q_i$ where $\varepsilon = 2^{-n}/W^2$. We also set the target revenue T to be $nH + K$. Since $H > \sum_i a_i$, the revenue reaches the target only if every bidder wins an item and pays at least H. By Proposition 4.1, the value of p_i should be either H or $H + a_i$. We will show that there exists a subset with sum K if and only if there exists a truthful auction with performance at least $1 - K\varepsilon$ in the constructed instance.

For necessity, given a set S such that $\sum_{i \in S} a_i = K$. We set $p_i = H + a_i$ for all $i \in S$. Then the performance of the auction is $\sum_{i \in S}(1 - q_i) \geq 1 - \varepsilon \cdot \sum_{i \in S} a_i \geq 1 - K\varepsilon$. For sufficiency, given a monopoly pricing auction A with performance at least $1 - K\varepsilon$. Let S be the set of bidders such that $p_i = H + a_i$. Since $\sum_i p_i \geq T$, we have $\sum_{i \in S} a_i \geq K$. On the other hand, the probability of getting revenue $\sum_i p_i$ is

$$\prod_{i \in S}(1 - q_i) \leq 1 - \varepsilon \cdot \sum_{i \in S} a_i + \sum_{i \neq j} q_i q_j \leq 1 - \varepsilon \cdot \sum_{i \in S} a_i + O(n^2 \cdot (\varepsilon \cdot W)^2) \leq 1 - \varepsilon \cdot \sum_{i \in S} a_i + o(\varepsilon).$$

Thus, we have $1 - \varepsilon \cdot \sum_{i \in S} a_i + o(\varepsilon) \geq 1 - K\varepsilon$ since the performance of A is at least $1 - K\varepsilon$. Hence, $\sum_{i \in S} a_i \geq K$ since K is an integer.

To prove the hardness for any truthful auctions, it suffices to show in the above instance, there exists a truthful auction with performance at least $1 - K\varepsilon$ if and only if there must exist a monopoly pricing auction whose performance is at least $1 - K\varepsilon$. The sufficiency is trivial since monopoly pricing auctions are truthful. For the necessity, given a truthful auction A with performance at least $1 - K\varepsilon$, since the profile that all bidders have high values ($v_i = H + a_i$) appears with a relatively high probability i.e. $\prod_{i\in[n]}(1-q_i)$, the revenue of A should reach the target on this profile. Let S be the set of bidders who pays strictly higher than H on this profile. Then we have $\sum_{i\in S} a_i \geq K$. Since A is truthful, we know every bidder $i \in S$ cannot wins an items in the profile that $v_i = H$ and $v_j = H + a_j$ for all $j \neq i$. So the performance of A is at most $1 - \sum_{i\in S} q_i \prod_{j\neq i}(1-q_j) = 1 - \sum_{i\in S} q_i + o(\varepsilon)$. Since the performance of A is at least $1 - K\varepsilon$, we have $1 - K\varepsilon \leq 1 - \sum_{i\in S} q_i + o(\varepsilon)$. By the assumption that K is an integer, we have $\sum_{\in S} q_i \leq K\varepsilon$. We construct a monopoly pricing auction by setting $p_i = H + a_i$ iff $i \in S$. So its performance is $\prod_{i\in S}(1-q_i) \geq 1 - \sum_{i\in S} q_i \geq 1 - K\varepsilon$ since $\sum_{i\in S} a_i \geq K$. □

We are able to show a similar NP-hardness result for risk-averse sellers.

Theorem 4.12. *It is NP-hard to compute the optimal utility auction to maximize a concave function of the revenue for independent bidders.*

Proof. We prove the theorem by reducing from SUBSET-SUM. That is given a set of positive integers denoted by $a_1, \ldots, a_n$, we ask if there is a subset S such that $\sum_{i\in S} a_i = K$ for some positive integer K. Let $H = \sum_i a_i + 1$, $W = \max_i\{a_i\}$ and $T = n \cdot H + K$. Consider the following concave function, $f(x) = x$ if $x \leq T$, otherwise $f(x) = T$. We use the similar distribution in the proof of Theorem 4.11. That is every bidder has valuation H with probability q_i and $H + a_i$ with probability $1 - q_i$. We set $q_i = \varepsilon \cdot a_i/(K + a_i + 1)$ where $\varepsilon = 2^{-n}/(W^2 \cdot T)$.

Now we show that there exists a subset with sum K if and only if there exists a truthful auction with performance at least $T - \varepsilon \cdot (K + 1/2)$. For the sufficiency, suppose there exists a truthful auction A with performance at least $T - \varepsilon \cdot W$. Let $\mathbf{h}$ denote the profile when all bidders have high values and $\mathbf{h}^i$ be the profile when all bidders except i have high values. Since the profile $\mathbf{h}$ has a relatively high probability, i.e. $1 - O(\varepsilon)$, the revenue of A should be at least T on this profile. Otherwise the performance of A is at most $T - 1 + n \cdot \varepsilon \cdot T \cdot K < T - 1 + 2^{-2n} < T - \varepsilon \cdot (K + 1/2)$ since $K \leq n \cdot W$. Let S be the set of bidders who pays strictly higher than H on the profile $\mathbf{h}$. We have $\sum_{i\in S} a_i \geq K$. Then for each profile $\mathbf{h}^i$ with $i \in S$, the revenue is at most $(n-1)H + \sum_{j\neq i} a_j = nH - a_i - 1 = T - K - a_i - 1$ since $H = \sum_i a_i + 1$ and $T = nH + K$. So the performance of A is at most $T - \sum_{i\in S} \Pr[\mathbf{h}^i] \cdot (K + a_i + 1) \leq T - \sum_{i\in S} q_i \cdot (K + a_i + 1) + o(\varepsilon)$. Since the performance of A is at least $T - \varepsilon \cdot (K + 1/2)$, we have $\sum_{i\in S} a_i \leq K$ by the definition of q_i and the fact that $\{a_i\}_{i=1}^n$ and K are integers. Therefore, $\sum_{i\in S} a_i = K$.

For the necessity, given a set S such that $\sum_{i\in S} a_i = K$. We construct a truthful auction $A = (f_1, \ldots, f_n)$ with performance at least $T - \varepsilon \cdot (K + 1/2)$ where f_i is the pricing function for bidder i. Recall that $\mathbf{h}$ denotes the profile when all bidders have high values and $\mathbf{h}^i$ is the profile when all bidders except i have high values. In order to define the pricing function f_i, we also use $\mathbf{h}_{-i}$ to denote the vector of valuations except bidder i in profile $\mathbf{h}$. Similarly we also define $\mathbf{h}^i_{-j}$ to be the vector of valuations except bidder j in profile $\mathbf{h}^i$. Then we set $f_i(\mathbf{h}^j_{-i}) = H + a_i$ for all $j \neq i$ and $f_i(\mathbf{h}_{-i}) = H + a_i$ if $i \in S$, otherwise $f_i(\mathbf{h}_{-i}) = H$. For the profile $\mathbf{h}$, the revenue is $\sum_i f_i(\mathbf{h}_{-i}) = nH + \sum_{i\in S} a_i = T$. For the profile $\mathbf{h}_i$ with $i \notin S$, the revenue is $\sum_{j\neq i} f_j(\mathbf{h^i}_{-j}) + f_i(\mathbf{h}_{-i}) = nH + \sum_{j\neq i} a_i \geq nH + \sum_{j\in S} a_j = T$ since $\mathbf{h}^i_{-i} = \mathbf{h}_{-i}$. For the profile $\mathbf{h}_i$ with $i \in S$, the revenue is $\sum_{j\neq i} f_j(\mathbf{h^i}_{-j}) = (n-1)H + \sum_{j\neq i} a_j = T - K - a_i - 1$ since $f_i(\mathbf{h}^i_{-i}) = f_i(\mathbf{h}_{-i}) = H + a_i > v_i$. Combining all these together, the performance of A is at least

$$
\begin{aligned}
& (\Pr[\mathbf{h}] + \sum_{i\notin S} \Pr[\mathbf{h}^i]) \cdot T + \sum_{i\in S} \Pr[\mathbf{h}^i] \cdot (T - (K + a_i + 1)) \\
= & \left(\prod_{i\in[n]} (1-q_i) + \sum_{i\notin S} q_i \prod_{j\neq i} (1-q_j) \right) \cdot T + \sum_{i\in S} q_i \prod_{j\neq i} (1-q_j) \cdot (T - (K + a_i + 1)) \\
\geq & \left(1 - \sum_{i\in[n]} q_i + \sum_{i\notin S} q_i (1 - \sum_{j\neq i} q_j) \right) \cdot T + \sum_{i\in S} q_i (1 - \sum_{j\neq i} q_j) \cdot (T - (K + a_i + 1)) \\
\geq & \; T - \sum_{i\in[n]} \sum_{j\neq i} q_i \cdot q_j \cdot T - \sum_{i\in S} q_i \cdot (K + a_i + 1) \\
= & \; T - \sum_{i\in S} a_i - o(\varepsilon) \geq T - (K + 1/2) \cdot \varepsilon
\end{aligned}
$$

The first inequality comes from the fact that $\prod_{i\in[n]}(1-q_i) \geq 1 - \sum_{i\in[n]} q_i$ and the last equality is due to $q_i = \varepsilon \cdot a_i/(K + a_i + 1)$ and $\varepsilon = 2^{-n}/(W^2 \cdot T)$. Therefore, we prove the correctness of the reduction and the theorem follows. □

4.6 Optimal simple auctions for n independent bidders

In this section, we study the following simple auctions for sellers with a target revenue when the bidders are independent. In Section 4.6.1, we present an additive FPTAS for computing approximately optimal *sequential posted price* auctions with respect to a fixed order σ. Then in Section 4.6.2 we show an additive PTAS for optimal *monopoly price auctions*, in a setting where the seller is restricted to using a constant number of distinct prices.

4.6.1 Optimal Sequential Posted Price Auction

We first present a pseudo-polynomial time algorithm to compute optimal sequential posted prices via dynamic programming. Then we show that this algorithm can be modified to be a FPTAS with respect to additive error. We order the bidders with respect to the fixed order σ.

Recall that in a sequential posted price mechanism, the seller offers take-it-or-leave-it prices to the buyers sequentially with respect to a given order σ and the computation of the price for buyer i is based on the results of all buyers preceding i, together with the valuation distributions. Note that the optimal sequential posted price for any sequence of buyers, performs at least as well as the optimal monopoly price auction. In contrast with the objective of expected revenue maximization, our objective of a target revenue means that the price offered to bidder i may depend on the revenue gained from the first $i-1$ bidders. This allows us to solve the problem by the following dynamic program. Let $Q[i,r]$ be the maximal probability to achieve revenue r by selling items to buyers from i to n. By Proposition 4.1, it is sufficient to consider the case that $p_i \in V_i$ where V_i is the support of buyer i's valuation distribution. It is easy to see $Q[i,r]=1$ if $r \leq 0$ and $Q[i,r]=0$ if $i>n$ and $r>0$. For the other cases when $i \leq n$ and $r>0$ we have

$$Q[i,r] = \max_{p_i \in V_i}\{Q[i+1, r-p_i] \cdot \Pr[v_i \geq p_i] + Q[i+1,r] \cdot (1-\Pr[v_i \geq p_i])\}.$$

Thus the maximal probability to achieve target revenue T from all buyers is $Q[1,T]$. Note that solving the above dynamic program gives a pseudo-polynomial time algorithm for the problem. Actually, we can get an additive FPTAS by rounding the dynamic program properly.

Theorem 4.13. *There exists an additive FPTAS for computing approximately optimal sequential posted price auctions with respect to a fixed order of the buyers. In particular, given $\epsilon \in (0,1)$, an instance $\mathcal{I}=(D,T)$ with n independent buyers and a buyer sequence σ, an ϵ-additive approximately optimal sequential posted price auction with respect to σ can be computed in time $O(m^2 n^2 \log n \cdot 1/\epsilon \log(1/\epsilon))$ where m is the maximal support size, i.e. $\max_{i\in[n]}\{|D_i|\}$.*

Proof. Recall that in the dynamic program, we use $Q[i,r]$ to denote the maximum probability to achieve revenue r by selling items to buyers from i to n. Note that $Q[i,r]$ is a monotone function of r, i.e. $Q[i,r] \geq Q[i,r']$ if $r \leq r'$. So we are able to store $Q[i,r]$ in another data structure. Let $R[i,q]$ be the maximum revenue the seller can get by selling items to buyers from i to n with probability at least q. The relationship between $R[i,q]$ and $Q[i,r]$ is

$$R[i,q] = \max_{r\geq 0}\{r|Q[i,r] \geq q\} \text{ and } Q[i,r] = \max_{q\in[0,1]}\{q|R[i,q] \geq r\}.$$

Given a parameter $\epsilon > 0$, let $\delta = \epsilon/n$ and $K = \lfloor 1/\delta \rfloor$. We set $R[n+1, k\delta]$ to be equal to 0 for all $k \in [K]$. Our FPTAS is based on the computation of $R[i,k\delta]$ for all $i \in [n]$ and $k \in [K]$. Let $Q^A[i,r]$ be the value of $Q[i,r]$ output by our algorithm by making queries to the computed $R[i,k\delta]$. Given the values of $R[i,k\delta]$, we can query the value of $Q^A[i,r] = \max_{k\in[K]}\{k\delta|R[i,k\delta] \geq r\}$ by using binary search in time $\log K$. If $R[i,k\delta] < r$ for all $k \in [K]$, we set $Q^A[i,r]=0$. We say $R[i,k\delta]$ determines the value of $Q^A[i,r]$ if $Q^A[i,r]=k\delta$ and $R[i,k\delta] \geq r$. We compute $R[i,q]$ based on $R[i+1,q]$ as follows. For any revenue r such that there exists $k \in [K]$ and $p_i \in V_i$ such that

$R[i+1,k\delta] = r$ or $R[i+1,k\delta] = r - p_i$, we compute $Q[i,r]$ by using the transition equation in the dynamic program and making queries to $R[i+1,k\delta]$. Note that this $Q[i,r]$ is different from $Q^A[i,r]$ since we have not constructed $R[i,k\delta]$ yet. We use $Q^B[i,r]$ to denote it. By the transition equation in the dynamic program, we have

$$Q^B[i,r] = \max_{p_i \in V_i}\{Q^A[i+1,r-p_i] \cdot \Pr[v_i \geq p_i] + Q^A[i+1,r] \cdot (1 - \Pr[v_i \geq p_i])\}$$

We also save two pointers from $Q^B[i,r]$ to the entries of $R[i+1,k\delta]$ which determine the values of the corresponding $Q^A[i+1,r-p_i]$ and $Q^A[i+1,r]$. After that, we use all values of $Q^B[i,r]$ to construct $R[i,k\delta]$ by setting $R[i,k\delta] = \max_r\{r|Q^B[i,r] \geq k\delta\}$. We also save a pointer from $R[i,k\delta]$ to $Q^B[i,r]$ if $R[i,k\delta] = r$. Then we can construct the sequential posted price auction by tracking back from the entry of $R[1,k\delta]$ which determines the value of $Q^A[1,T]$. Since we are rounding down, it is clear that the performance of this auction is at least $Q^A[1,T]$. To prove the approximation guarantee of the algorithm, it suffices to show that $Q^A[1,T] \geq Q^O[1,T] - n \cdot \delta$ where $Q^O[i,r]$ is the original value of $Q[i,r]$ in the instance without rounding.

We first show that $Q^A[i,r] \geq Q^B[i,r] - \delta$ for all $i \in [n]$ and revenue r such that there exists $k \in [K]$ and $p_i \in V_i$ such that $R[i+1,k\delta] = r$ or $R[i+1,k\delta] = r - p_i$. Let $k' \in [K]$ such that $k'\delta \leq Q^B[i,r] < (k'+1)\delta$. Note that $R[i,k'\delta] \geq r$ by the construction of $R[i,k\delta]$. So by the process of binary search, we have $Q^A[i,r] \geq k'\delta$. Thus, $Q^A[i,r] \geq Q^B[i,r] - \delta$.

Then we are able to show that $Q^A[i,r] \geq Q^O[i,r] - (n+1-i) \cdot \delta$ for all $i \in [n]$ and $r \geq 0$ by induction on i. Base case $i = n$: For any $r \geq 0$, let j be the index such that $v_n^j \leq r < v_n^{j+1}$. By defining $v_n^0 = 0$ and $v_n^{|V_n|+1} = +\infty$, there always exists such j. It is easy to see $Q^O[n,r] = \Pr[v_n \geq r] = \Pr[v_n \geq v_n^{j+1}]$ and

$$Q^A[n,r] = Q^A[n,v_n^{j+1}] \geq Q^B[n,v_n^{j+1}] - \delta = \Pr[v_n \geq v_n^{j+1}] - \delta = Q^O[n,r] - \delta.$$

The second equality is due to the computation of $Q^B[n,v^{j+1}]$. That completes the proof of the base case. For the inductive step from $i+1$ to i, given any $r > 0$, let $p_i \in V_i$ be the price for bidder i used in the optimal sequential posted price auction, i.e.,

$$Q^O[i,r] = Q^O[i+1,r-p_i] \cdot \Pr[v_i \geq p_i] + Q^O[i+1,r] \cdot (1 - \Pr[v_i \geq p_i]).$$

Let r_1 be the value $R[i+1,k\delta]$ which determines the value of $Q^A[i+1,r]$ and r_2 be the value $R[i+1,k'\delta]$ which determines the value of $Q^A[i+1,r-p_i]$. We consider two cases. Case 1: $r_1 \le r_2 + p_i$ then we have

$$\begin{aligned}
&Q^O[i,r] = Q^O[i+1, r-p_i] \cdot \Pr[v_i \ge p_i] + Q^O[i+1,r] \cdot (1 - \Pr[v_i \ge p_i])\} \\
&\le (Q^A[i+1, r-p_i] + (n-i)\delta) \cdot \Pr[v_i \ge p_i] + (Q^A[i+1,r] + (n-i)\delta) \cdot (1 - \Pr[v_i \ge p_i])\} \\
&= Q^A[i+1, r_2] \cdot \Pr[v_i \ge p_i] + Q^A[i+1, r_1] \cdot (1 - \Pr[v_i \ge p_i])\} + (n-i)\delta \\
&\le Q^A[i+1, r_1 - p_i] \cdot \Pr[v_i \ge p_i] + Q^A[i+1, r_1] \cdot (1 - \Pr[v_i \ge p_i])\} + (n-i)\delta \\
&\le Q^B[i, r_1] + (n-i)\delta \le Q^A[i,r] + (n+1-i)\delta
\end{aligned}$$

The first inequality is by the induction hypothesis and the equality on the third line is by the definitions of r_1 and r_2 and the process of binary search. The second last inequality comes from the computation of $Q^B[i, r_1]$ and the last inequality are from $Q^B[i, r_1] \le Q^A[i, r_1] + \delta$ and the monotonicity of Q^A. Case 2: $r_1 > r_2 + p_i$, the argument are similar by replacing r_1 by $r_2 + p_i$. Therefore we prove the approximation guarantee of the algorithm.

Finally, we consider the running time of the algorithm. For each layer i, the total number of values of revenue r we computed $Q^B[i,r]$ for is at most $m \cdot K$. For each $Q^B[i,r]$, we need time $O(m \log K)$ to enumerate $p_i \in V_i$ and make queries to $R[i+1, k\delta]$. By summing over all layers, the total running time is $O(m^2 n \cdot K \log K) = O(m^2 n^2 \log n \cdot 1/\epsilon \log(1/\epsilon))$. □

Remark 1. It should be mentioned that our hardness results in previous sections do not imply any hardness results here. So our FPTAS might not be tight.

4.6.2 Optimal Monopoly Price Auction

In this section, we present a PTAS for computing the optimal monopoly price auction when the seller is restricted to a given constant-sized set of distinct prices, and for each buyer has to select one of those prices for that buyer. Recall that in a monopoly price auction, the seller offers those take-it-or-leave-it prices to the buyers simultaneously, and the prices are only based on the valuation distributions. Our PTAS uses results of [25] on Poisson Binomial Distributions. First of all, we review the definitions and results. For any two random variables X and Y supported on a finite set A, their total variation distance is defined as

$$d_{\mathrm{TV}}(X,Y) = \frac{1}{2} \sum_{a \in A} |\Pr[X = a] - \Pr[Y = a]|.$$

We use the following result in the proof of Theorems 4.17 and 4.18.

Lemma 4.14 (Lemma 2 in [25]). *Let $X_1, \ldots, X_n$ be mutually independent random variables, and let $Y_1, \ldots, Y_n$ be mutually independent random variables. Then*

$$d_{\mathrm{TV}}(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) \le \sum_{i=1}^n d_{\mathrm{TV}}(X_i, Y_i).$$

A distribution is said to be a *Poisson Binomial Distribution* (PBD) of order n if it is a discrete probability distribution consisting of the sum of n independent indicator random variables. The distribution is parameterized by a vector $(r_i)_{i=1}^n \in [0,1]^n$ of probabilities and is denoted by $\mathrm{PBD}(r_1, \ldots, r_n)$. Let $\mathcal{S}_n$ be the set of all PBDs of order n. We review a construction of an efficient and proper ϵ-cover for $\mathcal{S}_n$.

Theorem 4.15 (Theorem 1 in [25]). *For all $n, \epsilon > 0$, there exists a set $\mathcal{S}_{n,\epsilon} \subset \mathcal{S}_n$ such that*

1. *$\mathcal{S}_{n,\epsilon}$ is an ϵ-cover of $\mathcal{S}_n$ in total variation distance; that is, for all $D \in \mathcal{S}_n$, there exists some $D' \in \mathcal{S}_{n,\epsilon}$ such that $d_{\mathrm{TV}}(D, D') \leq \epsilon$,*
2. *$|S_{n,\epsilon}| \leq n^2 + n \cdot (\frac{1}{\epsilon})^{O(\log^2 1/\epsilon)}$,*
3. *$S_{n,\epsilon}$ can be computed in time $O(n^2 \log n) + O(n \log n) \cdot (\frac{1}{\epsilon})^{O(\log^2 1/\epsilon)}$.*

Moreover, all distributions $\mathrm{PBD}(r_1, \ldots, r_n) \in \mathcal{S}_{n,\epsilon}$ in the cover satisfy at least one of the following properties, for some positive integer $t = t(\epsilon) = O(1/\epsilon)$.

- *(t-sparse form) there is some $\ell \leq t^3$ such that, for all $i \leq \ell$, $r_i \in \{\frac{1}{t^2}, \frac{2}{t^2}, \ldots, \frac{t^2-1}{t^2}\}$ and for all $i > \ell$, $r_i \in \{0, 1\}$; or*
- *((n,t)-Binomial form) there is some $\ell \in [n]$ and $q \in \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ such that, for all $i \leq \ell$, $r_i = q$ and for all $i > \ell$, $r_i = 0$; moreover ℓ and q satisfy $\ell q \geq t^2$ and $\ell q(1-q) \geq t^2 - t - 1$.*

In words, every PBD can be approximated by either a sparse PBD or a binomial distribution. Moreover, the following theorem tells us that if the first $O(\log 1/\epsilon)$ moments of two PBDs are the same, then the total variation distance between them is at most ϵ.

Theorem 4.16 (Theorem 3 in [25]). *Let $\mathcal{P} := (p_i)_{i=1}^n \in [0,1/2]^n$ and $\mathcal{Q} := (q_i)_{i=1}^n \in [0,1/2]^n$ be two collections of probability values. Let also $\mathcal{X} := (X_i)_{i=1}^n$ and $\mathcal{Y} := (Y_i)_{i=1}^n$ be two collections of mutually independent indicators with $\mathrm{E}[X_i] = p_i$ and $\mathrm{E}[Y_i] = q_i$, for all $i \in [n]$. If for some $d \in [n]$ the following condition is satisfied: $\sum_{i=1}^n p_i^\ell = \sum_{i=1}^n q_i^\ell$ for all $\ell = 1, \ldots, d$, then*

$$d_{\mathrm{TV}}(\sum_i X_i, \sum_i Y_i) \leq 13(d+1)^{1/4} 2^{-(d+1)/2}.$$

It is easy to see that Theorem 4.16 holds if we replace $[0, 1/2]$ with $[1/2, 1]$. Moreover, by setting $d = O(\log 1/\epsilon)$, this bound becomes at most ϵ. Theorem 4.15 shows that there exists an efficient cover for the set of all PBDs. However, we cannot directly apply this theorem to our problem, since (given prices and prior distributions of a problem instance) the set of associated PBDs (call it S) is a proper subset of $\mathcal{S}_n$, and we need to find a cover that consists of a subset of S. Theorem 4.17 is intended to overcome this obstacle. Given n finite sets $W_1, \dots, W_n$ where $W_i \subset [0,1]$ for all $i \in [n]$, let $W = \times_{i=1}^n W_i$, and let $\mathcal{S}_n(W)$ denote the set of all PBDs such that the probability of the indicator i is in W_i for all $i \in [n]$. That is $\mathcal{S}_n(W) = \{\text{PBD}(r_1, \dots, r_n) | (r_i)_{i=1}^n \in W\}$.

Theorem 4.17. *For all $n, \epsilon > 0$ and any n finite subsets of the unit interval $W_1, \dots, W_n$ let $W = \times_{i=1}^n W_i$. Then there exists a set $\mathcal{S}_{n,\epsilon}(W) \subset \mathcal{S}_n(W)$ such that*

1. *$\mathcal{S}_{n,\epsilon}(W)$ is an ϵ-cover of $\mathcal{S}_n(W)$ in total variation distance; that is, for all $D \in \mathcal{S}_n(W)$, there exists some $D' \in \mathcal{S}_{n,\epsilon}(W)$ such that $d_{\text{TV}}(D, D') \le \epsilon$,*
2. *$\mathcal{S}_{n,\epsilon}(W)$ can be computed in time $(\frac{n}{\epsilon})^{O(\log^2 1/\epsilon)}$ and has size at most $(\frac{n}{\epsilon})^{O(\log^2 1/\epsilon)}$.*

Proof. For any $i \in [n]$ and $r_i \in W_i$, we round r_i down to a multiple of ϵ/n denoted by r_i'. By Lemma 4.14, the total variation distance between the distributions before and after this rounding is at most ϵ. Let d satisfy $13(d+1)^{1/4}2^{-(d+1)/2} < \epsilon$ (the expression is from Theorem 4.16) so that $d = O(\log 1/\epsilon)$. For any $\text{PBD}(r_1, \dots, r_n)$, we define its *moment profile* $(\mu, \nu) = (\mu_1, \dots, \mu_d, \nu_1, \dots, \nu_d)$ to be the $2d$-dimensional vector such that $\mu_\ell = \sum_{i: r_i' \in [0,1/2]} (r_i')^\ell$ and $\nu_\ell = \sum_{i: r_i' \in (1/2,1]} (r_i')^\ell$. Since r_i's are multiples of ϵ/n, $\mu_\ell, \nu_\ell \in \{0, (\epsilon/n)^\ell, 2(\epsilon/n)^\ell, \dots, n\}$ for all $\ell \in [d]$. By Theorem 4.16 and the triangle inequality, if the moment profiles of two PBDs are the same, the total distance between them is at most 3ϵ. So instead of considering all the PBD in $\mathcal{S}_n(W)$, we only need to examine PBDs with different moment profiles. Note that the number of possible moment profiles is at most $(n/\epsilon)^{O(d^2)} = (n/\epsilon)^{O(\log^2 1/\epsilon)}$. Thus we establish the existence of a cover for $\mathcal{S}_n(W)$ with size $(n/\epsilon)^{O(\log^2 1/\epsilon)}$. In order to get a cover that's a subset of $\mathcal{S}_n(W)$, we need to check, for any moment profile (μ, ν) if there exists a probability vector $(r_i)_{i=1}^n \in W$ such that $\text{PBD}(r_1', \dots, r_n')$ has such moment profile. This can be computed by the following dynamic program (a similar dynamic program was used in the proof of Claim 1 in [25]).

Let $A(i, \mu, \nu)$ be the indicator such that $A(i, \mu, \nu) = 1$ iff there exists $r_1 \in W_1, \dots, r_i \in W_i$ such that $\sum_{j \le i: r_i' \in [0,1/2]} (r_i')^\ell = \mu_\ell$ and $\sum_{j \le i: r_i' \in (1/2,1]} (r_i')^\ell = \nu_\ell$ for all $\ell \in [d]$ where $\mathbf{r}'$ is the ϵ/n rounding of $\mathbf{r}$ defined at the beginning of the proof. We initialize all entries to value 0 except $A(0, \mathbf{0}, \mathbf{0}) = 1$. Then the transition equation is $A(i, \mu, \nu) = 1$ iff there exists $r_i \in W_i$ such that $r_i' \in [0, 1/2]$ and $A(i-1, (\mu_\ell - (r_i')^\ell)_{\ell=1}^d, \nu) = 1$ or there exists $r_i \in W_i$ such that $r_i' \in (1/2, 1]$ and $A(i-1, \mu, (\nu_\ell - (r_i')^\ell)_{\ell=1}^d) = 1$. We also save a pointer to the entry $A[i, \mu, \nu]$ from the corresponding entry for buyer $i-1$. It is easy to see that the overall running time to compute A is $\max_i\{|W_i'|\} \cdot n \cdot (n/\epsilon)^{O(\log^2 1/\epsilon)} \le (n/\epsilon)^{O(\log^2 1/\epsilon)}$ where $|W_i'|$ is the number of possible r_i' that is at most n/ϵ. Note that given a moment profile (μ, ν) such $A(n, \mu, \nu) = 1$, we can compute the corresponding probability vector $\mathbf{r}$ by tracing the pointers from this cell of A back to level 1. Therefore, we can define the cover $\mathcal{S}_{n,\epsilon}$ to be the set of all PBDs with such probability vectors. □

Given the above theorem, we are able to obtain an additive PTAS for computing approximately optimal monopoly price auctions, given a fixed set of allowed prices.

Theorem 4.18. *There exists an additive PTAS for computing approximately optimal monopoly price auctions when the seller is restricted to a fixed number of distinct prices. In particular, given* $\epsilon \in (0,1)$*, an instance with* n *independent bidders and* k *distinct prices the seller may use, an* ϵ*-additive approximately optimal monopoly price auction can be computed in time* $(\frac{nk}{\epsilon})^{O(k \log^2 1/\epsilon)}$.

Proof. We use $a_1, \dots, a_k$ to denote the k distinct prices the seller may use. Given a monopoly price auction with price vector $(p_1, p_2, \dots, p_n)$, we use an indicator random variable H_{ij} to indicate that the seller gets revenue a_j from buyer i, that is $H_{ij} = 1$ iff $p_i = a_j$ and $v_i \geq a_j$. Let $H_j = \sum_{i \in [n]} H_{ij}$ and $H = \sum_{j \in [k]} a_j H_j$. Note that H is the random variable for the total revenue raised in this auction. Since the H_{ij} are indicator random variables, the H_j are Poisson Binomial random variables due to the independence among bidders. So H can be viewed as a weighted sum of k Poisson Binomial random variables. Let r_{ij} denote the probability of getting revenue exactly a_j from buyer i. Then the distribution of H_j is $\text{PBD}(r_{1j}, \cdots, r_{nj})$. The distribution of H can be represented by the vector $\mathbf{r} = (r_{ij})_{i \in [n], j \in [k]}$. Let W_i be the set of all possible $(r_{i1}, \dots, r_{ik})$ such that $r_{ij} = \Pr[v_i \geq a_j]$ if the seller use price a_j for bidder i and $r_{ij} = 0$ otherwise . It is clear that the set $W = \times_{i \in [n]} W_i$ is the set of all probability vector $\mathbf{r}$ corresponding to a feasible pricing vector $\mathbf{p}$.

Note that for any two random variables X, Y and any value T,

$$|\Pr[X \geq T] - \Pr[Y \geq T]| \leq d_{\text{TV}}(X, Y).$$

So if there exists an ϵ-cover for the set of all possible distribution of H parameterized by $\mathbf{r} \in W$, we can explore the pricing rules in the cover instead of all possible pricing rules to find a sequence of monopoly prices which approximately maximize $\Pr[H \geq T]$. In order to get such a cover, we need to modify the dynamic program used in the proof of Theorem 4.17 to be k-dimensional. The moment profile $(\mu^1, \dots, \mu^k, \nu^1, \dots, \nu^k)$ is defined as $\mu^j = (\mu^j_1, \dots, \mu^j_d), \nu^j = (\nu^j_1, \dots, \nu^j_d)$ and $\mu^j_\ell, \nu^j_\ell \in \{0, (\frac{\epsilon}{nk})^\ell, 2(\frac{\epsilon}{nk})^\ell, \dots, n\}$ for all $\ell \in [d]$ and $j \in [k]$. By a similar argument to Theorem 4.17 and Lemma 4.14, all the possible moment profiles is already an ϵ-cover. Define $A[i, \mu^1, \dots, \mu^k, \nu^1, \dots, \nu^k]$ to be the indicator such that it is equal to 1 iff there exists $\mathbf{r}_1 \in W_1, \dots, \mathbf{r}_i \in W_i$ such that for all $j \in [k]$ and $\ell \in [d]$,

$$\sum_{i' \leq i : r'_{i'j} \in [0,1/2]} (r'_{i'j})^\ell = \mu^j_\ell \quad \text{and} \quad \sum_{i' \leq i : r'_{i'j} \in (1/2,1]} (r'_{i'j})^\ell = \nu^j_\ell$$

where $\mathbf{r}'$ is a $\frac{\epsilon}{nk}$-rounding of $\mathbf{r}$ such that r'_{ij} is a multiple of $\frac{\epsilon}{nk}$ and $r_{ij} - \frac{\epsilon}{nk} < r'_{ij} \leq r_{ij}$ for all $i \in [n]$ and $j \in [k]$.

Similarly to the proof of Theorem 4.17, A can be computed by the following dynamic program. Inductively, to compute layer $i+1$, we consider all the non-zero entries of layer i and for every such non-zero entry and every possible prices a_j, we find which entry of layer $i+1$ we would transition to if we choose $p_i = a_j$, i.e. $r_{ij} = \Pr[v_i \geq a_j]$ and $r_{ij'} = 0$ for all $j' \neq j$. It is easy to see the overall running time to compute A is $(\frac{nk}{\epsilon})^{O(k \log^2 1/\epsilon)}$. In addition, we can find the corresponding monopoly prices for any distribution in this cover by tracing the pointers in the computation of A. Therefore, we can enumerate all possible pricing rules in this cover with size at most $(\frac{nk}{\epsilon})^{O(k \log^2 1/\epsilon)}$ to find the optimal pricing which maximizes $\Pr[H \geq T]$.

The final step is to compute $\Pr[H \geq T]$ given a price vector $\mathbf{p}$. By Theorem 4.15, we know any PBD can be approximated by a sparse PBD or a binomial distribution. For the given price vector, we can get the corresponding H_j for all $j \in [k]$. We use Theorem 4.15 to compute H'_j from H_j such that H'_j is either a k/ϵ-sparse PBD or a binomial distribution and $d_{\mathrm{TV}}(H'_j, H_j) \leq \epsilon/k$ for all $j \in [k]$. Then we compute $\Pr[H'_j = T_j]$ for any value $T_j \in [0, \ldots, n]$ and $j \in [k]$. This computation can be done efficiently since H'_j is either a k/ϵ-sparse PBD or a binomial distribution. By Lemma 4.14, we have $d_{\mathrm{TV}}(H', H) \leq \epsilon$ where $H' = \sum_j a_j H'_j$. Finally we compute $\Pr[H' \geq T] = \sum_{(T_j)_j : \sum_j a_j T_j \geq T} \prod_j \Pr[H'_j = T_j]$ by enumerating all possible $T_1, \ldots, T_k$. Since the distance between H and H' is at most ϵ, we have $\Pr[H \geq T] \geq \Pr[H' \geq T] - \epsilon$. Combine all these together, we get the additive PTAS with running time $(\frac{nk}{\epsilon})^{O(k \log^2 1/\epsilon)}$. □

Remark 2. It should be mentioned that our hardness results in previous sections do not imply our PTAS is tight.

4.7 Illustrative Examples

Claim 4.19. Monopoly price auctions cannot guarantee a c-multiplicative approximation of the optimal truthful auction with any $c < n$ even for two independent bidders.

As shown in Example 4.1, the monopoly price auction can perform extremely badly with respect to multiplicative error even in the case of two independent bidders.

Example 4.1. *The two bidders have the same distribution, that is $v_i = k$ with probability ϵ^k for $k = 1, \ldots, n$. It is not hard to see that the seller can get the revenue $n+1$ with probability at least $n \cdot \epsilon^{n+1}$ by setting $f_2(i) = n+1-i$ and $f_1(j) = n+1-j$ for all $i, j = 1, \ldots, n$. However, no monopoly price auction can do better than $\epsilon^{n+1} + o(\epsilon^{n+1})$ when ϵ is small enough. So the multiplicative approximation ratio is at least n.*

Claim 4.20. For independent bidders, monopoly price auctions cannot guarantee a c-additive approximation of the optimal truthful auction with any $c < 1/e$ where $e \approx 2.718$ is the Euler's number.

Example 4.2. *For each bidder i, we construct the distribution such that $v_i = 2$ with probability $1/n$ and $v_i = 1$ with probability $1 - 1/n$. It is easy to check the seller can get revenue $n + 1$ with probability $(1 - 1/n)^{n-1} \approx 1/e$ by setting $f_i(1, \ldots, 1) = 2$ and $f_i(1, \ldots, 1, 2, 1, \ldots, 1) = 1$. However no monopoly price auction can do better than $2/n$. To see this, suppose the auction sets the threshold price to be 2 for k bidders. So the revenue is the $n - k + R$ where R is the random variable denoting the revenue from the bidders with $p_i = 2$. Since $\mathrm{E}[R] = 2k/n$, $\Pr[R \geq k + 1] \leq \frac{2k/n}{k+1} \leq 2/n$ by Markov inequality. So the probability of getting revenue $n + 1$ is at most $2/n$.*

Claim 4.21. Lemma 4.5 does not hold for two correlated bidders.

As shown in Example 4.3, there exists an instance with two correlated bidders, where no optimal truthful auction consists of non-increasing pricing functions.

Example 4.3. *Bidder 1's valuations are $1, 2, \ldots, m$ and bidder 2's valuations are 1 and 2. The probability of $v_1 = i$ and $v_2 = j$ is $1/m$ if $i + j$ is odd, otherwise it is 0. It is not hard to see the optimal auction can get the revenue 3 with probability 1 by setting setting $f_1(1) = 2$, $f_1(2) = 1$ and $f_2(i) = 1$ if i is odd otherwise $f_2(i) = 2$. However, no truthful auction with non-increasing pricing functions can do better than $1/2$.*

4.8 Conclusion

In this chapter, we study auction design with a revenue target in the Bayesian setting and provide computational positive and negative results in terms of the number of buyers and whether valuations are independently distributed. We see several promising directions for future work. For independent buyers, a direct open problem is to generalize our characterization to three or more buyers. That may be achievable via an induction on the number of buyers, characterizing the optimal auction for three buyers by using the case with two buyers as a substructure. Another direction is to approximate the optimal auction via designing simple auctions. We find several examples to show the lower bounds but the upper bound is still open. Finally, we point out an interesting problem of computing optimal monopoly prices without the limitation on distinct prices. This problem seems to be hard; we just know that it is contained in $NP^{\#P}$.

Chapter 5

Trading Broker: Double Auction Design

We study double auction market design where the market maker wants to maximize its total revenue by buying low from the sellers and selling high to the buyers. We consider a Bayesian setting where buyers and sellers have independent probability distributions on the values of products on the market.

For the simplest setting, each seller has the same kind of indivisible good with a bounded (integer) amount that can be sold to a buyer, who may demand a bounded number of copies. We develop an optimal mechanism for the market maker to maximize its own revenue.

For the more general case where each seller's product may be different, we consider a number of varieties in terms of constraints on supplies and demands. For each of them, we develop a polynomial time computable truthful mechanism for the market maker to achieve a revenue at least a constant α times the revenue of any other truthful mechanism.

This chapter is based on a joint work [31] with Paul Goldberg, Xiaotie Deng, and Jinshan Zhang, which appears in ISAAC 2012.

5.1 Overview

We consider a double auction market maker who collects valuations from buyers and sellers about a certain product to decide on the prices each seller gets and each buyer pays. The buyers may want to buy many units and the sellers may have many units to part with. The buyers and sellers may have different valuations of the product, and the probability distributions of the valuations are public knowledge but each valuation, sampled from its distribution, is known only to its own buyer or seller. For simplicity, we assume that the probability distributions are independent. For the sellers and buyers, they know their own private values exactly. The market maker purchases the products from the sellers and sells them to the buyers. Our goal is to design a market mechanism

Table 5.1: Results

	Dimension	Demand	Supply	Distribution	Results
Sec. 5.3	Single	Arbitrary	Arbitrary	Continuous (Discrete)	Optimal
Sec. 5.4	Multi	Arbitrary	Arbitrary	Continuous	1/4-Approx
Sec. 5.4	Multi	Arbitrary	Arbitrary	Discrete	1/4-Approx
Sec. 5.5	Multi	Unlimited	Arbitrary	Discrete	Optimal
Sec. 5.5	Multi	Arbitrary	Unlimited	Discrete	Optimal

For the demand column, "Arbitrary" refers to the case where buyers can buy at most d_i items where d_i can be an arbitrary number and "Unlimited" means $d_i = +\infty$. The supply column is similar.

that maximizes the revenue of the market maker. In other words, the market maker is to buy the same amount of products from the sellers as the amount sold to the buyers with the objective of maximizing the difference of its collected payment from the buyers and the total amount paid to the sellers. When in addition we assume public knowledge of distributions of buyers' private values from the previous sales, we call it a revenue maximizing Bayesian double auction market maker.

There have been many double auction institutions, each of which may be suitable for one type of market environment [41]. Ours is motivated by the growing use of discriminative pricing models over the Internet such as one that is studied in [32] for the prior-free market environment. A possible realistic setting for applications of our model could be Google's ad exchange where Google could play a market maker for advertisers and webpage owners [63]. One may also use it for a market model of Groupon. Our use of the Bayesian model is justified by the repeated uses of a commercial system by registered users. It allows the market maker to gain Bayesian information of the users' valuations of the products being sold. Therefore, the Bayesian model adequately describes the knowledge of the market maker, buyers and sellers for the optimal mechanism design.

5.1.1 Main Results

We provide optimal or constant approximate mechanisms for various settings for double auction design. The parameters considered in our discussion are related to important market design issues. Those include one or multi dimensional problems (meaning, one product or multiple different types of products). The buyers can have demand constraints or not. The sellers can be supply constrained or not. Players' values may be drawn from a continuous or from a discrete distribution. The results are summarized in Table 5.1.1. In the Bayesian Mechanism Design problems, there are two computational processes involved. The first one is to design an optimal or approximate mechanism which can be viewed as a function mapping bidders' profiles to allocation and payment outcomes. Since the function maps potentially exponentially many profiles to outcomes, a succinct representation of the function is an important part in the Bayesian mechanism design. The second process is the implementation of the mechanism, i.e., given a bid profile, we run the mechanism to compute the outcome allocation and payment scheme. Our results imply that all mechanisms described in the table can be represented in polynomial size and can be found and implemented in polynomial time.

5.1.2 Literature Review

Auction design plays an important role in economics in general and especially in electronic commerce [55]. Of particular interest, is the problem of maximizing the auctioneer's revenue, referred as the optimal auction design problem. A number of research works have focused on this issue. Myerson, in his seminal paper [65], characterized the optimal auction for the single-item setting in the Bayesian model. Recently, efforts have been made on extending Myerson's results to border settings [35, 67, 69].

Unlike Myerson's optimal auction result, finding the optimal solution is not easy for multi-dimensional settings. Recent research interest has turned toward approximate mechanisms [1, 15]. Cai et al. [9] presented a characterization of a rather general multi-dimensional setting and proposed an efficient mechanism for the special case where no bidders are demand constrained. Using similar ideas, Alaei et al. [2] present a general framework for reducing multi-agent service problems to single-agent ones.

The double auction design problem becomes more complicated since the market maker acts as the middle man to bring buyers and sellers together. A guide to the literature in micro-economics on this topic can be found in [41]. The profit maximization problem for the single buyer/single seller setting has been studied by Myerson and Satterthwaite [64]. Our optimal double auction is a direct extension of their work and, to our best knowledge, fills a clear gap in the economic theory of double auctions. Deshmukh et al. [32], studied the revenue maximization problem for double auctions where the auctioneer has no prior knowledge about bids. Their prior-free model is essentially different from ours. More auction mechanism design problems were studied by many researchers in recent years, but as far as we know, not in the context of optimal double auction design in the Bayesian setting. The most related one is by Jain and Wilkens [50], where they studied the market intermediation problem in a setting with a single unit-demand buyer and a group of sellers. They gave several constant approximate mechanisms with various buyer behaviour assumptions. While our setting assumes the existence of a monopoly platform, Rochet and Tirole [68] and Armstrong [3] introduced several different models for the two-sided market and studied the platform competition problem.

5.2 Preliminaries

Throughout the chapter we focus on Bayesian incentive compatible mechanisms only. Informally, a mechanism is Bayesian incentive compatible if it is optimal (in the expected utility) for each buyer and each seller to bid its true value of the items. We will formally define this concept later. As a consequence, we should consider their bids to be their true valuations and restrict our discussion to (direct revelation) mechanisms that result in less or equal utility if one deviates to report a false value.

Therefore, we will use the notation v_{ij} to represent the ith buyer's (true) bid for one of the jth seller's items and w_j for the jth seller's (true) bid. We will drop the "(true)" subsequently as deviations of bids from the true valuations will be clearly stated. The ith buyer's bid can be denoted by a vector v_i and bids of all buyers can be denoted by v or sometimes $(v_i; v_{-i})$ where v_{-i} is the joint profile of all other bidders. Similarly, we use w and $(w_j; w_{-j})$ for the sellers' bid. [1]

In our model, all players' bids are assumed to be distributed independently according to publicly known distributions, V for buyers, W for sellers. Note that we also assume that V and W should be bounded, i.e. $v_{ij} \in [\underline{v}_{ij}, \overline{v}_{ij}]$ and $w_j \in [\underline{w}_j, \overline{w}_j]$.

Before introducing the formal notations of double auction, we recall some preliminaries for Myerson's auction [65] as mentioned in Chapter 2. In Myerson's auction, there is one item to sell and there are n buyers, whose valuations are drawn independently from some distributions, each of whom are risk-neutral expected utility maximizers, bidding for the item. The auctioneer of the auction is the seller who would like to find a mechanism consisting of an allocation rule and payment rule given the bids of buyers such that the mechanism is Bayesian incentive compatible and individual rationality and the expected revenue of the seller is maximized. Myerson characterized the Bayesian incentive compatible mechanism as monotonicity of allocation rule and payment rule and based on this, he converted the revenue maximization problem into social welfare maximization problem and resolved the problem completely. This is also called one dimensional one side Bayesian mechanism design since each buyer's valuation is single dimensional.

The idea of double auction design is similar to Myerson's auction while maintaining some differences. First, the auctioneer is not the item holder but intermediary agent who buys items from sellers and sells the items to buyers while simultaneously maximizing his own revenue. We will still adopt Bayesian settings and assume all the participators are selfish, e.g. risk-neutral expected utility maximizers. Second, we need to clarify the utility of sellers and not only to make sure that buyers are Bayesian incentive compatible and individual rational but also to guarantee these two properties for sellers. Third, the mechanism we considered includes the single dimension case

[1] We use semi-colon to separate the profile of a special player with others and use comma to separate the buyers' profiles with sellers'.

where sellers hold the same items, similar to Myerson's result, the optimal mechanism for single dimension is resolved. We extended to the analysis of single dimension to multi dimension by employing recent elegant techniques developed for one side multi dimensional mechanism design. More precise notations and definitions will be presented bellow.

The outcome of a mechanism M consists of four random variables (x, p, y, q) where x and p are the allocation function and payment functions for buyers, y and q for sellers. That is, buyer i receives item j with probability $x_{ij}(v, w)$ and pays $p_i(v, w)$; seller j sells her item with probability $y_j(v, w)$ and gets a payment $q_j(v, w)$. Thus, the expected revenue of the mechanism is $R(M) = \mathrm{E}_{v,w}[\sum_i p_i(v, w) - \sum_j q_j(v, w)]$ where $\mathrm{E}_{v,w}$ is short for $\mathrm{E}_{v\sim V,w\sim W}$.

In general, a buyer may buy more than one item from the mechanism. We assume buyers' valuation functions are additive, i.e. $v_i(S) = \sum_{j\in S} v_{ij}$. For each buyer i, let d_i denote the demand constraint for buyer i, i.e. buyer i cannot buy more than d_i items. Similarly, let k_j be the supply constraint for seller j, i.e. seller j cannot sell more than k_j items. By the Birkhoff-von Neumann theorem [52, 35, 27], it suffices to satisfy $\sum_j x_{ij} \le d_i$ and $y_j = \sum_i x_{ij} \le k_j$.

Let $U_i(v, w) = \sum_j x_{ij}(v, w)v_{ij} - p_i(v, w)$ be the expected utility of buyer i when the profile of all players is (v, w), which is identical to usual definition of utility for buyers [65]. Similarly, the expected utility of sellers is the expected selling price of his item minus the multiplication of his true valuation for the item and the probability that item is sold, and we use $T_j(v, w) = q_j(v, w) - y_j(v, w)w_j$ to be the expected utility of seller j when the profile of all players is (v, w). We proceed to formally define the concepts of Bayesian Incentive Compatibility of mechanisms and ex-interim Individual Rationality of the buyers and sellers:

Definition 5.1. A double auction mechanism M is said to be *Bayesian Incentive Compatible (BIC)* iff the following inequalities hold for all i, j, v, w.

$$\begin{aligned} \mathrm{E}_{v_{-i},w}[U_i(v, w)] &\ge \mathrm{E}_{v_{-i},w}[U_i((v'_i; v_{-i}), w)] \\ \mathrm{E}_{v,w_{-j}}[T_j(v, w)] &\ge \mathrm{E}_{v,w_{-j}}[T_j(v, (w'_j; w_{-j}))] \end{aligned} \tag{5.1}$$

We note that, if $U_i(v, w) \ge U_i((v'_i; v_{-i}), w)$ and $T_j(v, w) \ge T_j(v, (w'_j; w_{-j}))$ for all v, w, v'_i, w'_j, we say M is *Incentive Compatible.*

To illustrate incentive compatibility, two well-known auctions are sufficient, one is first price auction, that is, the bidder with highest bid wins and pays his bids, the other one is second price auction, that is, the bidder with highest bid wins and pays the second highest bids. The first price auction is not incentive compatible, for example, the second highest bids is \$8 and his true value is \$10, if he bids \$10, he wins, but pays \$10 getting utility 0, however, if he lies to bid \$9, he still wins and pay \$9, getting utility \$1 > 0, therefor, he has incentive to lie. The truthfulness of second price auction is well known [78].

Besides Bayesian incentive compatibility, another important concept is individual rationality, which requires each participant's (expected) utility is non negative, ensuring his participation into the game since no participation guarantees his utility is zero.

Definition 5.2. A double auction mechanism M is said to be *ex-interim Individual Rational (IR)* iff the following inequalities hold for all i, j, v, w.

$$\begin{aligned} \mathrm{E}_{v_{-i},w}[U_i(v,w)] &\geq 0 \\ \mathrm{E}_{v,w_{-j}}[T_j(v,w)] &\geq 0 \end{aligned} \tag{5.2}$$

Similarly, we note that, if $U_i(v, w) \geq 0$ and $T_j(v, w) \geq 0$ for all v, w, we say M is *ex-post Individual Rational.*

We say a mechanism is feasible if each buyer and seller are Bayesian incentive compatible and ex-interim individual rational, and simultaneously demand and supply constraints are satisfied.

Not all the mechanisms are individual rational, for example, fixed price auction, there is one item to be sold and the price is set to be \$10, however, some buyer may value the item \$8 and the item will be allocated to him and charge him \$10 if he participates, hence, he will not participate in this auction.

Finally, we present the formal definition of approximate mechanism.

Definition 5.3 (α-approximate Mechanism [69])**.** Given a set $\mathbb{M}$ of any mechanisms, we say mechanism $M \in \mathbb{M}$ is an α-approximate mechanism in $\mathbb{M}$ iff for each mechanism $M' \in \mathbb{M}$, for any set of buyers and sellers $\alpha \cdot R(M') \leq R(M)$. A mechanism is *optimal* in $\mathbb{M}$ if it is an 1-approximate mechanism in $\mathbb{M}$.

5.3 Optimal Single-Dimensional Double Auction

In this section, we consider the single-dimensional double auction design problem where all sellers sell identical items, that is for all $j, j' \in [m]$, $v_{ij} = v_{ij'}$. Moreover, as shown in Table 5.1.1, in this section we assume the bidders' bids are drawn from continuous distributions[2]. Let f_i, F_i be the probability density function (PDF) and cumulative distribution function (CDF) for buyer i's value, g_j, G_j be the PDF and CDF for seller j's value.

Our mechanism can be viewed as a generalization of the classical Myerson's Optimal Auction [65]. We show that a similar optimal double auction can be found in this single-dimensional setting. In addition, in Section 5.4 this optimal mechanism will be used to construct a constant approximate mechanism for a multi-dimensional setting.

[2]The case for discrete distributions is the same as continuous distribution [79]

Recall that Myerson's virtual value function is defined as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ for each buyer. In the double auction, we define the virtual value functions for buyers and sellers as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $r_j(w_j) = w_j + \frac{G_j(w_j)}{g_j(w_j)}$. If $c_i(v_i)$ is not an increasing function of v_i or r_j is not decreasing, by Myerson's ironing technique, we can use the ironed virtual value function $\bar{c}_i$ and $\bar{r}_j$. W.l.o.g, we assume the buyers are sorted in decreasing order with respect to $\bar{c}_i(v_i)$ and all sellers are in increasing order with respect to $\bar{r}_j(w_j)$. Let $D = \max_{i,j}\{\min\{\sum_{s=1}^{i} d_s, \sum_{t=1}^{j} k_j\} | \bar{c}_i(v_i) > \bar{r}_j(w_j)\}$. Thus, we can define the optimal auction in the spirit of maximizing virtual surplus.

$$x_i(v,w) = \begin{cases} d_i & \text{if } \sum_{s\le i} d_s \le D \\ D - \sum_{s<i} d_s & \text{if } \sum_{s<i} d_s < D < \sum_{s\le i} d_s \\ 0 & \text{otherwise} \end{cases}$$

$$y_j(v,w) = \begin{cases} k_j & \text{if } \sum_{t\le j} k_t \le D \\ D - \sum_{s<j} k_s & \text{if } \sum_{t<j} k_t < D < \sum_{t\le j} k_t \\ 0 & \text{otherwise} \end{cases}$$

$$p_i(v,w) = x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i((s;v_{-i}),w)ds$$

$$q_j(v,w) = y_j(v,w)w_j + \int_{w_j}^{\overline{w}_j} y_j(v,(t;w_{-j}))dt$$

Theorem 5.4. *The above mechanism is an optimal (revenue) mechanism for the single-dimensional double auction setting. Under the assumption that the integration and convex hull of f, g can be computed in polynomial time, the mechanism can be found and implemented. Moreover, the mechanism is deterministic, incentive compatible and ex-post Individual Rational.*

Proof. Let $\hat{U}_i(x,p,v_i)$ be the expected utility for buyer i when his bid is v_i and the mechanism uses allocation function x and payment function p. Similarly, we use $\hat{T}_j(y,q,w)$ to denote seller j's expected utility.

$$\hat{U}_i(x,p,v_i) = \mathrm{E}_{v_{-i},w}[v_i x_i(v,w) - p_i(v,w)]$$
$$\hat{T}_j(y,q,w_j) = \mathrm{E}_{v,w_{-j}}[q_j(v,w) - w_j y_j(v,w)]$$

Similarly, the expected utility for the auctioneer is

$$R(x,p,y,q) = \mathrm{E}_{v,w}[\sum_i p_i(v,w) - \sum_j q_j(v,w)] \tag{5.3}$$

We call a mechanism (x,p,y,q) feasible if and only if it satisfies the following constraints.

$$\begin{gathered} x_i(v,w) \le d_i, \quad y_j(v,w) \le k_j \\ \sum_i x_i(v,w) \le \sum_j y_j(v,w) \\ x_i(v,w), y_j(v,w) \ge 0 \end{gathered} \tag{5.4}$$

$$\hat{U}_i(p,x,v_i) \ge 0, \quad \hat{T}_j(q,y,w_j) \ge 0 \tag{5.5}$$

$$
\begin{aligned}
\hat{U}_i(p,x,v_i) &\geq \hat{U}_i(p,x,v_i') \\
\hat{T}_j(q,y,w_j) &\geq \hat{T}_j(q,y,w_j')
\end{aligned} \tag{5.6}
$$

As is well known, the Incentive Compatibility is equivalent to the monotonicity. Given a mechanism (p,x,q,y), we define $H_i(x,v_i) = \mathrm{E}_{v_{-i},w}[x_i(v,w)]$ and $L_i(y,w_j) = \mathrm{E}_{v,w_{-j}}[y_j(v,w)]$. Then we have the following lemma.

Lemma 5.5. *A mechanism (x,p,y,q) is feasible if and only if the following conditions hold:*

$$
\begin{aligned}
\text{if} \quad v_i' \leq v_i \quad \text{then} \quad H_i(x,v_i') \leq H_i(x,v_i) \\
\text{if} \quad w_j' \leq w_j \quad \text{then} \quad L_j(y,w_j') \geq L_j(y,w_j)
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
\hat{U}_i(p,x,v_i) &= \hat{U}_i(p,x,\underline{v}_i) + \int_{\underline{v}_i}^{v_i} H_i(x,v_i')dv_i' \\
\hat{T}_j(q,y,w_j) &= \hat{T}_j(q,y,\overline{w}_j) + \int_{w_j}^{\overline{w}_j} L_j(y,w_j')dw_j'
\end{aligned} \tag{5.8}
$$

$$
\hat{U}_i(p,x,\underline{v}_i) \geq 0, \hat{T}_j(q,y,\overline{w}_j) \geq 0 \tag{5.9}
$$

and inequalities (5.4)

Proof. The IC constraint (5.6) is equivalent to

$$
\begin{aligned}
\hat{U}_i(p,x,v_i) &\geq \mathrm{E}_{v_{-i},w}[v_i x_i((v_i';v_{-i}),w) - p_i((v_i';v_{-i}),w)] \\
&= \mathrm{E}_{v_{-i},w}[(v_i - v_i' + v_i')x_i((v_i';v_{-i}),w) - p_i((v_i';v_{-i}),w)] \\
&= \hat{U}_i(p,x,v_i') + (v_i - v_i')H_i(x,v_i') \\
\hat{T}_j(q,y,w_j) &\geq \hat{T}_j(q,y,w_j') + (w_j' - w_j)L_j(y,w_j')
\end{aligned} \tag{5.10}
$$

Using (5.10) twice, we have (5.7). By integrating H_i and L_j, we have (5.8). The proof of the necessary part is similar. □

Now we can characterize the optimal double auction in the following lemma.

Lemma 5.6. *Suppose that (x,y) maximizes*

$$
\mathrm{E}_{v,w}[\sum_i c_i(v_i)x_i(v,w) - \sum_j r_j(w_j)y_j(v,w)]
$$

subject to the constraints (5.4) and (5.7). Suppose also that

$$
\begin{aligned}
p_i(v,w) &= x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i((v_i';v_{-i}),w)dv_i' \\
q_j(v,w) &= y_j(v,w)w_j + \int_{w_j}^{\overline{w}_j} y_j(v,(w_j';w_{-j}))dw_j'
\end{aligned} \tag{5.11}
$$

Then (x,p,y,q) represents an optimal auction.

Proof. Recalling (5.3), we may write the auctioneer's objective function as

$$
\begin{aligned}
R(p,x,q,y) &= \mathrm{E}_{v,w}[\sum_i p_i(v,w) - \sum_j q_j(v,w)] \\
&= \sum_i \mathrm{E}_{v,w}[x_i(v,w)v_i] + \sum_i \mathrm{E}_{v,w}[p_i(v,w) - x_i(v,w)v_i] \\
&- \sum_j \mathrm{E}_{v,w}[y_j(v,w)w_j] - \sum_j \mathrm{E}_{v,w}[q_j(v,w) - y_j(v,w)w_j]
\end{aligned}
\tag{5.12}
$$

But, using Lemma 5.5, we have

$$
\begin{aligned}
&\mathrm{E}_{v,w}[p_i(v,w) - x_i(v,w)v_i] = -\mathrm{E}_{v_i}[\hat{U}_i(p,x,v_i)] \\
&= -\mathrm{E}_{v_i}[\hat{U}_i(p,x,\underline{v}_i) + \int_{\underline{v}_i}^{v_i} H_i(x,v_i')dv_i'] \\
&= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\overline{v}_i}\int_{\underline{v}_i}^{v_i} H_i(x,v_i')f_i(v_i)dv_i'dv_i \\
&= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\overline{v}_i}\int_{v_i'}^{\overline{v}_i} H_i(x,v_i')f_i(v_i)dv_idv_i' \\
&= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\overline{v}_i}\int_{v_i'}^{\overline{v}_i} f_i(v_i)dv_iH_i(x,v_i')dv_i' \\
&= -\hat{U}_i(p,x,\underline{v}_i) - \mathrm{E}_{v_{-i},w}[\int_{v_i}^{\overline{v}_i} f_i(v_i')dv_i'x_i(v,w)]
\end{aligned}
\tag{5.13}
$$

Similarly,

$$
\begin{aligned}
&-\mathrm{E}_{v,w}[q_j(v,w) - y_j(v,w)w_j] \\
&= -\hat{T}_j(q,y,\overline{w}_j) - \mathrm{E}_{v,w_{-j}}[\int_{\underline{w}_j}^{w_j} g_j(w_j')dw_j'y_i(v,w)]
\end{aligned}
$$

Substituting (5.13) into (5.12) gives us,

$$
\begin{aligned}
R(p,x,q,y) = &-\sum_i \hat{U}_i(p,x,\underline{v}_i) - \sum_j \hat{T}_j(p,x,\overline{w}_j) \\
&+ \mathrm{E}_{v,w}[\sum_i c_i(v_i)x_i(v,w)] - \mathrm{E}_{v,w}[\sum_j r_j(w_j)y_j(v,w)]
\end{aligned}
\tag{5.14}
$$

So the auctioneer's problem is to maximize (5.14) subject to the constraints (5.4), (5.7), (5.8) and (5.9). In this formulation, p, q appear only in the first two terms and in the constraints (5.8) and (5.9). These two constraints may be rewritten as

$$
\mathrm{E}_{v_{-i},w}[x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i((v_i';v_{-i}),w)dv_i' - p_i(v,w)] = \hat{U}_i(p,x,\underline{v}_i) \geq 0
$$

$$
\mathrm{E}_{v,w_{-j}}[q_j(v,w) - y_j(v,w)w_j - \int_{w_j}^{\overline{w}_j} y_j(v,(w_j';w_{-j}))dw_j'] = \hat{T}_j(q,y,\overline{w}_j) \geq 0
$$

If the seller chooses p, q according to (5.11), then he satisfies both (5.8) and (5.9), and he gets the best possible value for (5.14). So we can drop p, q from the problem entirely. This completes the proof of the lemma. □

By the above lemma, we can reduce the optimal double auction design problem to a combinatorial optimization problem. Subject to constraint (5.4), our greedy mechanism always maximizes $\sum_i c_i(v_i)x_i(v,w) - \sum_j r_j(w_j)x_j(v,w)$ for all v, w. If c_i is not increasing and r_j is not decreasing, we just use the standard Myerson's technique to refine c_i and r_j to $\bar{c}_i$ and $\bar{r}_j$.

Therefore, we complete the proof of Theorem 5.4. □

Finally, we give an example for $n = m = 1$, $v_1 \in [0, 100]$, $f_1(v_1) = 1/100$, $w_1 \in [0, 100]$ and $g_1(w_1) = 1/100$. We also suppose that $d_i = k_j = 1$. It is easy to see, $c_1(v_1) = 2v_1 - 100$ and $r_1(w_1) = 2w_1$. So our optimal auction is to make a trade iff $v_1 - w_1 \geq 50$. Then we charge the buyer $50 + w_1$ and pay the seller $v_1 - 50$. So our revenue is $100 - (v_1 - w_1)$. This is interesting because $(v_1 - w_1)$ is the social welfare in this game. It follows (perhaps counter-intuitively) that the revenue decreases when the social welfare increases.

5.4 Approximate Multi-Dimensional Double Auction

In this section, we provide a general framework for approximately reducing the double auction design problem for multiple buyers and sellers to the subproblem for a single pair of buyer and seller. As an application, we apply the framework to construct a $\frac{1}{4}$-approximate mechanism for the multi-dimensional setting. Our approach is inspired by the work of Alaei [1] which provides a general framework for the one sided auction.

We first give a high-level idea of our approach. It is not hard to see, following the previous section, we can also write the multi-dimensional double auction design problem as a linear program (with exponential size). Our first step is to relax the feasibility constraint (demand and supply constraints) from the ex-post ones to ex-ante ones. Clearly, this relaxation will not decrease the expected revenue. Then we can use the solution of the magician problem described in [1] to modify an ex-ante feasible allocation to a ex-post feasible allocation with a constant fraction loss. In order to solve the relaxed optimization problem, we need to define Primary Mechanisms which is only for a single buyer and a single seller. As we will show, this single-buyer and single-seller problem can be solved efficiently.

Recall that all bids are drawn from publicly known distributions and our goal is to maximize the expected revenue for the auctioneer. It should be emphasized that, in this section, we assume the buyers' values for different items are independent, i.e. v_{ij} and $v_{ij'}$ are independent. To use Alaei's general framework, we also assume each buyer can at most buy one copy of items from one seller. This is w.l.o.g. because we can remove this assumption by constructing k_j duplicate sellers (each with one copy item to sell) for each seller.

First of all, we introduce the concept of Primary Mechanism which can be viewed as a mechanism between one buyer and one seller.

Definition 5.7 (Primary Mechanism/Primary Benchmark)**.**
A *primary mechanism* denoted by M_{ij} for buyer i and seller j is a single buyer and single seller mechanism which allows specifying an upper bound on the ex-ante expected probability $\bar{k}_{ij}$ of allocating the item j to the buyer i. A *primary benchmark* denoted by $\bar{R}_{ij}$ is a concave function such that the optimal revenue of any primary mechanism M_{ij} subject to $\bar{k}_{ij}$ is upper bounded by $\bar{R}_{ij}(\bar{k}_{ij})$.

Intuitively, for any allocation rule, define the ex-ante probability of assigning the jth seller's items to the i-th buyer as $\bar{k}_{ij} = \mathrm{E}_{v_i,w_j}[x_{ij}(v_i, w_j)]$. Then we can divide the supply constraints $\sum_i x_{ij}(v, w) \leq k_j$ and demand constraints $\sum_j x_{ij}(v, w) \leq d_i$ to the ex-ante probability constraints, $\sum_i \bar{k}_{ij} \leq k_j$ and $\sum_j \bar{k}_{ij} \leq d_i$. Then we compute the optimal ex-ante probability by convex programming. Obviously, the optimal solution of the relaxed problem must be an upper bound for any original solution. Unfortunately, the solution obtained by convex programming may not be a feasible solution of the original problem. To solve this problem, Alaei introduced the following rounding process to round the relaxed solution to a feasible one.

Lemma 5.8 (γ-Conservative Magician (Theorem 2 in [1]))**.** *In the Magician problem, a magician is presented with a series of boxes one by one. He has k magic wands that can be used to open the boxes. On each box is written a probability q_i. If a wand is used on a box, it opens, but with probability at most q_i the wand breaks. Given $\sum_i q_i \leq k$ and any $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$, a γ-conservative magician guarantees that each box is opened with an ex-ante expected probability at least γ.*

Using the above lemma, we describe our mechanism for multi-dimensional double auction problem. Recall that in the classical auction setting, all items are sold by the auctioneer. However, in the double auction setting, items are sold by different sellers and more efforts should be taken to handle the truthfulness issue of sellers. We extend Alaei's rounding mechanism from one-dimension (considering buyers one by one) to two-dimension (considering each pair of buyer and seller sequentially) as follows.

Mechanism (Modified γ-Pre-Rounding Mechanism)

(I) Solve the following convex program and let $\bar{k}_{ij}$ denote an optimal assignment for it.

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{i\in[n],j\in[m]} \bar{R}_{ij}(x_{ij}) & & \text{(CP)}\\
\text{Subject to:} \quad & \sum_{j\in[m]} x_{ij} \le d_i & \text{for all } i \in [n] &\\
& \sum_{i\in[n]} x_{ij} \le k_j & \text{for all } j \in [m] &\\
& x_{ij} \in [0,1] & \text{for all } i \in [n],\, j \in [m] &
\end{aligned}$$

(II) For each buyer i, create an instance of γ-conservative magician with d_i wands (this will be referred to as the buyer i's magician). For each item j create an instance of γ-conservative magician with k_j wands (this will be referred to as the seller j's magician).

(III) For each pair of buyer and seller (i, j):

(a) Write $\bar{k}_{ij}$ on a box and present it to the buyer i's magician and the seller j's magician.

(b) If both of them open the box, run $M_{ij}(\bar{k}_{ij})$ on buyer i and seller j otherwise consider next pair.

(c) If the mechanism buys an item from seller j and sells it to buyer i, then break the wands of buyer i's magician and seller j's magician.

Theorem 5.9 (Modified γ-Pre-Rounding Mechanism)**.** *Suppose for each buyer and seller pair (i, j), we have an α-approximate primary mechanism M_{ij} and a corresponding primary benchmark $\bar{R}_{ij}$ [3]. Then for any $\gamma \in [0, 1-\frac{1}{\sqrt{k^*+3}}]$ where $k^* = \min_{i,j}\{d_i, k_j\}$, the Modified γ-Pre-Rounding Mechanism is a $\gamma^2 \cdot \alpha$-approximation mechanism.*

Proof. The proof is similar to the proof of Theorem 7 in [1]. First, we prove that the expected revenue of any mechanism is upper bounded by $\sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$. For any mechanism $M = (x, p, y, q)$, let $k_{ij} = E_{v,w} x_{ij}(v, w)$. Due to the feasibility of M, k_{ij} must be a feasible solution of the convex programming (CP). So we have,

$$R(M) = \sum_i \sum_j R_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$$

Then it suffices to show that for each pair (i, j), our mechanism can gain the revenue $\bar{R}_{ij}(\bar{k}_{ij})$ with probability at least $\gamma^2 \cdot \alpha$, i.e. each box will be opened with probability at least γ^2 (this is because the γ-conservative magician for the buyer is independent to that for the seller and each of them chooses to open the box with ex-ante probability γ, the box will be opened iff both magicians choose to open the box). This can be deduced from Lemma 5.8 easily. □

[3]Since we require the valuations of the buyer for different items are independent, $\bar{R}_{ij}$ has a budget balanced cross monotonic cost sharing scheme defined in Definition 6 of [1]

Then we consider the multi-dimensional double auction design problem and present a constant approximate mechanism. For each buyer and seller pair i, j, we use the mechanism in Section 5.3 for one-dimensional cases to be the primary mechanism M_{ij} and the expected revenue of M_{ij} to be the primary benchmark $\bar{R}_{ij}$.

Theorem 5.10. *Assume that all bidders' bids are drawn from continuous distributions. A $\frac{1}{4}$ approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

Proof. Now we use the similar approach in Section 5.3 to prove that the optimal allocation rule must be the solution of the following optimization problem.

$$\begin{aligned} \text{Maximize: } & E_{v_i,w_j}[x_{ij}(v_i,w_j)(\bar{c}_i(v_i)-\bar{r}_j(w_j)] \\ \text{Subject to: } & E_{v_i,w_j}[x_{ij}(v_i,w_j)] \le \bar{k}_{ij} \\ & x_{ij}(v_i,w_j) \in [0,1] \end{aligned}$$

The above problem can be solved by the our previous algorithm for the one dimension case where we allocate one item to buyer i if $E_{v_i,w_j}[x_{ij}(v_i,w_j)] \le \bar{k}_{ij}$ and $c_i(v_i) \ge r_j(w_j)$. Then by the pricing rule described in Section 5.3, we can compute the optimal revenue as follows.

$$\bar{R}_{ij}(\bar{k}_{ij}) = \int_{\underline{v}_i}^{\overline{v}_i} \int_{\underline{w}_j}^{\min\{\bar{r}_j^{-1}(\bar{c}_i(v_i)),G_j^{-1}(\bar{k}_{ij})\}} \left(\bar{c}_i^{-1}(\bar{r}_j(w_j)) - \bar{r}_j^{-1}(\bar{c}_i(v_i))\right) dw_j dv_i$$

In the above formula, we use $\bar{c}_i^{-1}$ and $\bar{r}_j^{-1}$ to denote the inverse function of $\bar{c}_i$ and $\bar{r}_j$ respectively. However, $\bar{c}_i^{-1}$ and $\bar{r}_j^{-1}$ are non-decreasing, so $\bar{c}_i^{-1}(y) = \arg\min_x\{\bar{c}_i(x) = y\}$ is well-defined, so as $\bar{r}_j^{-1}$.

Note M_{ij} is optimal for buyer i and seller j and $\bar{R}_{ij}$ is the expected revenue of M_{ij}, i.e. $R(M_{ij})$. Let $M'_{ij}(\lambda, x, y)$ be the randomized mechanism which runs $M_{ij}(x)$ with probability λ and $M_{ij}(y)$ with probability $1-\lambda$. Then for all $x, y, \lambda \in [0,1]$, we have

$$\begin{aligned} & \lambda \cdot \bar{R}_{ij}(x) + (1-\lambda)\cdot \bar{R}_{ij}(y) \\ = & \lambda \cdot R(M_{ij}(x)) + (1-\lambda)\cdot R(M_{ij}(y)) \\ = & R(M'_{ij}(\lambda,x,y)) \\ \le & R(M(\lambda x + (1-\lambda)y)) \\ = & \bar{R}_{ij}(\lambda x + (1-\lambda)y) \end{aligned}$$

Therefore, $\bar{R}_{ij}(x)$ is a concave function. Hence, we obtain an 1-approximate primary mechanism M_{ij} and a corresponding primary benchmark $\bar{R}_{ij}$. By Theorem 5.9, we have a γ^2-approximation mechanism, where $\gamma = 1 - \frac{1}{\sqrt{k^*+3}} \ge \frac{1}{2}$ since $k^* = \min_{i,j}\{d_i, k_j\} \ge 1$. □

For the discrete distribution case, the optimal mechanism for single buyer and single seller can be computed by linear programming. So we have the similar result.

Theorem 5.11. *Assume that all bidders' bids are drawn from discrete distributions. A* $\frac{1}{4}$ *approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

5.5 Optimal Mechanism for Discrete Distributions

In this section, we consider the multi-dimensional double auction when all the bidders' value distributions are discrete. Unlike Section 5.4, we consider two special cases of the problem. One is the case where all buyers have unlimited demand, i.e., $d_i = +\infty$ for all buyer i and the other one is the case where all sellers have unlimited supply, i.e. $k_j = +\infty$ for all seller j. In this section, we focus on the former. The mechanism and the proof of the latter are similar.

Recall that, in the multi-dimensional setting, the auctioneer collects each buyer's bid, denoted by a vector $v_i = (v_{i1}, \ldots, v_{im})$ drawn from a public known distribution V_i and seller's bid denoted by w_j drawn from W_j. Throughout this section, V_i and W_j are discrete distributions and we use f_i and g_j to denote their probability mass function, i.e. $f_i(t) = \Pr[v_i = t]$ and $g_j(t) = \Pr[w_j = t]$. It should be emphasized that, unlike Section 5.4, we do not need to assume that the buyer's bids for each item should be independent, i.e. v_{ij} and $v_{ij'}$ can be correlated in this section. We also add a dummy buyer 0 with only one type v_0 for buyers and seller 0 with w_0 for sellers.

Our approach is motivated by the results of Cai et al. [11] and Alaei et al. [2] which require a reduced form of x, y, p, q denoted by $\bar{x}$, $\bar{y}$, $\bar{p}$ and $\bar{q}$ respectively, defined as follows:

$$\begin{array}{ll} \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)] & \bar{y}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[y_j(v, w)] \\ \bar{p}_i(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[p_i(v, w)] & \bar{q}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[q_j(v, w)] \end{array} \tag{5.15}$$

Now we are ready to convert an optimization problem of x, p, y, q to a problem of $\bar{x}, \bar{p}, \bar{y}, \bar{q}$ which can be represented by a linear program with polynomial size in T, n and m where T is the maximum among all $|V_i|$ and $|W_j|$.

Then BIC constraints (5.1) and IR constraints (5.2) can be rewritten as

$$\begin{array}{lcl} \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i, w_j)v_{ij} - \bar{p}_i(v_i, w_j)] & \geq & \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i', w_j)v_{ij} - \bar{p}_i(v_i', w_j)] \\ \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j)w_j)] & \geq & \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j') - \bar{y}_j(v_i, w_j')w_j] \\ \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i, w_j)v_{ij} - \bar{p}_i(v_i, w_j)] & \geq & 0 \\ \mathrm{E}_{v_i}[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j)w_j)] & \geq & 0 \end{array} \tag{5.16}$$

Finally, the mechanism should satisfy the supply constraints, i.e., for each item j and profiles $v, w, y_j(v, w) = \sum_i x_{ij}(v, w) \leq k_j$. Note that there is no demand constraint on buyers. Without loss of generality, we assume that $k_j = 1$ for all j. Otherwise, we can normalize x by setting $x'_{ij}(v, w) = x_{ij}(v, w)/k_j$ and refine v, w by setting $v'_{ij} = k_j v_{ij}$ and $w'_j = k_j w_j$ such that $k'_j = 1$ for all item j.

For the single-item setting of classical auction, i.e. $m = 1$ and seller's value for his item is always 0, Alaei et al. [2] prove a sufficient and necessary condition for the supply constraint.

Lemma 5.12 (Theorem 2 in [2])**.** *A reduced allocation rule $\bar{x}$ is feasible if and only if it can be implemented by the Stochastic Sequential Allocation (SSA) algorithm for some choice of stochastic transition table. In other words, there exists an ex-post implementation x of $\bar{x}$ such that $\bar{x}_i(v_i) = \mathrm{E}_{v_{-i}}[x_i(v)]$ and $\sum_i x_i(v) \leq 1$ for all v iff there exists (s, z) such that*

$$\begin{array}{lll} s_0(v_0, 0) = 1 & & \\ s_i(v_i, i) = \sum_{k=0}^{i-1} \sum_{v_k \in V_k} z_{ki}(v_k, v_i) & \forall i, v_i \in V_i & \\ s_k(v_k, i) = s_k(v_k, i-1) - \sum_{v_i \in V_i} z_{ki}(v_k, v_i) & \forall i, k < i, v_k \in V_k & \\ z_{ki}(v_k, v_i) \leq s_k(v_k, i-1) f_i(v_i) & \forall i, k < i, v_i \in V_i, v_k \in V_k & \\ \bar{x}_i(v_i) f_i(v_i) = s_i(v_i, n) & \forall i, v_i \in V_i & \end{array} \tag{5.17}$$

Moreover, given any feasible reduced allocation rule $\bar{x}$, the ex-post of x can be found efficiently.

We generalize Lemma 5.12 to a multi-dimensional double auction setting.

Lemma 5.13. *Given a reduced form $\bar{x}$, there exists an ex-post implementation x such that $x_{ij}(v, w) \geq 0, \sum_i x_{ij}(v, w) \leq 1$ and $\bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)]$ iff there exists (s, z) such that, for each seller j and $w_j \in W_j$*

$$\begin{array}{ll} s_0^{(j)}(v_0, w_j, 0) = 1 & \\ s_i^{(j)}(v_i, w_j, i) = \sum_{k=0}^{i-1} \sum_{v_k \in V_k} z_{ki}^{(j)}(v_k, v_i, w_j) & \forall i, v_i \in V_i \\ s_k^{(j)}(v_k, w_j, i) = s_k^{(j)}(v_k, w_j, i-1) - \sum_{v_i \in V_i} z_{ki}^{(j)}(v_k, v_i, w_j) & \forall i, k < i, v_k \in V_k \\ z_{ki}^{(j)}(v_k, v_i, w_j) \leq s_k^{(j)}(v_k, w_j, i-1) f_i(v_i) & \forall i, k < i, v_i \in V_i, v_k \in V_k \\ \bar{x}_{ij}(v_i, w_j) f_i(v_i) = s_i^{(j)}(v_i, w_j, n) & \forall i, v_i \in V_i \end{array} \tag{5.18}$$

Moreover, given any feasible reduced allocation rule $\bar{x}$, the ex-post of $\bar{x}$ can be found efficiently.

Proof. First, we prove that given a reduced form $\bar{x}$, there exists an ex-post implementation x such that

$$\begin{array}{l} x_{ij}(v, w) \geq 0 \\ \sum_i x_{ij}(v, w) \leq 1 \\ \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)] \end{array}$$

if and only if there exists an ex-interim implementation $\hat{x}$ such that

$$\begin{array}{l} \hat{x}_{ij}(v, w_j) \geq 0 \\ \sum_i \hat{x}_{ij}(v, w_j) \leq 1 \\ \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}}[\hat{x}_{ij}(v, w_j)] \end{array}$$

The necessary part is obvious by just setting $\hat{x}_{ij}(v, w_j) = \mathrm{E}_{w_{-j}}[x_{ij}(v, w)]$. And the sufficiency can be checked by letting $x_{ij}(v, w) = \hat{x}_{ij}(v, w_j)$.

Now the lemma can be proved by straightforwardly applying Lemma 5.12 for all j and w_j. □

Finally, we convert the problem of multi-dimensional double auction design problem to a linear program with reduced form which can be solved in polynomial time in m, n, T.

Theorem 5.14. *Assume all bidders' bids are drawn from discrete distributions and all bidders are without demand constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

Proof. By Lemma 5.13, it suffices to prove the recuded form defined in (5.15) can be computed in polynomial time. Actually, it can be computed by solving the following linear program.

$$\begin{array}{lll} \text{Max:} & \sum_{i,j} \mathrm{E}_{v_i,w_j}[p_i(v_i,w_j) - q_j(v_i,w_j)] & \\ \text{s.t.} & \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i,w_j)v_{ij} - \bar{p}_i(v_i,w_j)] & \geq \ \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v'_i,w_j)v_{ij} - \bar{p}_i(v'_i,w_j)] \\ & \mathrm{E}_{v_i}[\bar{q}_j(v_i,w_j) - \bar{y}_j(v_i,w_j)w_j)] & \geq \ \mathrm{E}_{v_i}[\bar{q}_j(v_i,w'_j) - \bar{y}_j(v_i,w'_j)w_j] \\ & \mathrm{E}_{w_j}[\sum_j \bar{x}_{ij}(v_i,w_j)v_{ij} - \bar{p}_i(v_i,w_j)] & \geq \ 0 \\ & \mathrm{E}_{v_i}[\bar{q}_j(v_i,w_j) - \bar{y}_j(v_i,w_j)w_j)] & \geq \ 0 \\ & \sum_i x_{ij}(v_i,w_j) & \leq \ k_j \end{array} \tag{5.19}$$

Then by lemma 5.13, we can find the ex-post allocations in polynomial time. □

Theorem 5.15. *Assume that all bidders' bids are drawn from discrete distributions and all sellers are without supply constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

The proof of the above theorem is similar to Theorem 5.14.

5.6 Conclusion

In this chapter, we present several optimal or approximately-optimal auctions for a double auction market. Double auction platforms have started to gain importance in electronic commerce. One possible example is the ad exchange market proposed to bring advertisers and web publishers together [63]. There are other potentials in setting up electronic platforms for sellers and buyers of other types of resources such as in the context of cloud computing.

Our results on the one hand show the power of recent significant progress in one-sided markets, and on the other hand raise new challenges in the development of mathematical and algorithmic tools for market design.

Chapter 6

Crowdsourcing Contests: The Efficiency of All-Pay Auctions

In this chapter, we study the inefficiency of mixed equilibria of all-pay auctions in three different environments – combinatorial, multi-unit and single-item auctions. First, we consider item-bidding combinatorial auctions where m all-pay auctions run in parallel, one for each good. For fractionally subadditive valuations, we strengthen the upper bound from 2 [76] to 1.82 by proving some structural properties that characterize the mixed Nash equilibria of the game. Next, we design an all-pay mechanism with a randomized allocation rule for the multi-unit auction. We show that, for bidders with submodular valuations, the mechanism admits a unique, 75% efficient, pure Nash equilibrium. The efficiency of this mechanism outperforms all the known bounds on the price of anarchy of mechanisms used for multi-unit auctions. Finally, we analyze single-item all-pay auctions motivated by their connection to contests and show tight bounds on the PoA of social welfare, revenue and maximum bid.

This chapter is based on a joint work [20] with Georgios Chirstodoulou and Alkmini Sgouritsa, which appears in ESA 2015.

6.1 Overview

It is a common economic phenomenon in competitions that agents make irreversible investments without knowing the outcome. *All-pay* auctions are widely used in economics to capture such situations, where all players, even the losers, pay their bids. For example, a lobbyist can make a monetary contribution in order to influence decisions made by the government. Usually the group invested the most increases their winning chances, but all groups have to pay regardless of the outcome. In addition, all-pay auctions have been shown useful to model rent seeking, political campaigns and R&D races. There is a well-known connection between all-pay auctions and *contests* [74]. In particular, the all-pay auction can be viewed as a single-prize contest, where the payments correspond to the effort that players make in order to win the competition.

In this chapter, we study the efficiency of mixed Nash equilibria in all-pay auctions with complete information, from a worst-case analysis perspective, using the *price of anarchy* [56] as a measure. As social objective, we consider the *social welfare*, i.e. the sum of the bidders' valuations. We study the equilibria induced from all-pay mechanisms in three fundamental resource allocation scenarios; combinatorial auctions, multi-unit auctions and single-item auctions.

In a combinatorial auction a set of items are allocated to a group of selfish individuals. Each player has different preferences for different subsets of the items and this is expressed via a *valuation set* function. A multi-unit auction, can be considered as an important special case, where there are multiple copies of a single good. Hence the valuations of the players are not set functions, but depend only on the number of copies received. Multi-unit auctions have been extensively studied since the seminal work by Vickrey [78]. As already mentioned, all-pay auctions have received a lot of attention for the case of a single item, as they model all-pay contests and procurements via contests.

6.1.1 Main Results

Combinatorial Auctions. Our first result is on the PoA of simultaneous all-pay auctions with item-bidding that was previously studied by Syrgkanis and Tardos [76]. For simultaneous all-pay auctions with fractionally subadditive valuations, it was previously shown that the price of anarchy was at most 2 [76] and at least $e/(e-1) \approx 1.58$ [22]. We narrow further this gap, improving the upper bound to 1.82. In order to obtain the bound, we come up with several structural theorems that characterize mixed Nash equilibria in simultaneous all-pay auctions.

Multi-unit Auctions. Our next result shows a novel use of all-pay mechanisms to the multi-unit setting. We propose an all-pay mechanism with a randomized allocation rule inspired by Kelly's seminal proportional-share allocation mechanism [53]. We show that this mechanism admits a *unique*, 75% efficient *pure* Nash equilibrium and no other mixed Nash equilibria exist, when bidders' valuations are submodular. As a consequence, the price of anarchy of our mechanism outperforms all current price of anarchy bounds of prevalent multi-unit auctions including uniform price auction [61] and discriminatory auction [28], where the bound is $e/(e-1) \approx 1.58$.

Single-item Auctions. Finally, we study the efficiency of a single-prize contest that can be modeled as single-item all-pay auction. We show a tight bound on the PoA for mixed equilibria which is approximately 1.185. Following previous study on the procurement via contest, apart from the social welfare, we also study two other standard objectives, *revenue and maximum bid.* We evaluate the performance of all-pay auctions in the prior-free setting, i.e. no distribution over bidders' valuation is assumed. We show that both the revenue and the maximum bid of any mixed Nash equilibrium are at least as high as $v_2/2$, where v_2 is the second highest valuation. In contrast, the revenue and the maximum bid in some mixed Nash equilibrium may be less than $v_2/2$ when using reward structure other than allocating the entire reward to the highest bidder. This result coincides with the optimal crowdsourcing contest developed in [16] for the setting with prior distributions. We also show that in conventional procurements (modeled by first-price auctions), v_2 is exactly the revenue and maximum bid in the worst equilibrium. So procurement via all-pay contests is a 2-approximation to the conventional procurement in the context of worst-case equilibria.

6.1.2 Literature Review

The inefficiency of Nash equilibria in auctions has been a well-known fact (see e.g. [57]). Existence of efficient equilibria of simultaneous sealed bid auctions in full information settings was first studied by Bikhchandani [8]. Christodoulou, Kovács and Schapira [21] initiated the study of the (Bayesian) PoA of simultaneous auctions with item-bidding. Several variants have been studied since then [7, 48, 40], as well as multi-unit auctions [61, 28].

Syrgkanis and Tardos [76] proposed a general smoothness framework for several types of mechanisms and applied it to settings with fractionally subadditive bidders obtaining several upper bounds (e.g., first price auction, all-pay auction, and multi-unit auction). Christodoulou et al. [22] constructed tight lower bounds for first-price auctions and showed a tight PoA bound of 2 for all-pay auctions with subadditive valuations. Roughgarden [72] presented an elegant methodology to provide PoA lower bounds via a reduction from the hardness of the underlying optimization problems.

All-pay auctions and contests have been studied extensively in economic theory. Baye, Kovenock and de Vries [4], fully characterized the Nash equilibria in single-item all-pay auction with complete information. The connection between all-pay auctions and crowdsourcing contests was proposed in [34]. Chawla et al. [16] studied the design of optimal crowdsourcing contest to optimize the maximum bid in all-pay auctions when agents' value are drawn from a specific distribution independently.

6.2 Preliminaries

In a *combinatorial auction*, n *players* compete on m *items* with unit supply. Every player (or *bidder*) $i \in [n]$ has a valuation function $v_i : \{0,1\}^m \to \mathbb{R}^+$ which is monotone and normalized, that is, $\forall S \subseteq T \subseteq [m]$, $v_i(S) \leq v_i(T)$, and $v_i(\emptyset) = 0$. The outcome of the auction is represented by a tuple of $(\mathbf{X}, \mathbf{p})$ where $\mathbf{X} = (X_1, \ldots, X_n)$ specifies the allocation of items (X_i is the set of items allocated to player i) and $\mathbf{p} = (p_1, \ldots, p_n)$ specifies the buyers payments (p_i is the payment of player i for the allocation $\mathbf{X}$). In the *simultaneous item-bidding* auction, every player $i \in [n]$ submits a non-negative bid b_{ij} for each item $j \in [m]$. The items are then allocated by independent auctions, i.e.

the allocation and payment rule for item j only depend on the players' bids on item j. In a simultaneous *all-pay* auction the allocation and payment for each player is determined as follows: each item $j \in [m]$ is allocated to the bidder i^* with the highest bid for that item, i.e. $i^* = \arg\max_i b_{ij}$, and each bidder i is charged an amount equal to $p_i = \sum_{j \in [m]} b_{ij}$.

Definition 6.1 (Valuations)**.** Let $v : 2^{[m]} \to \mathbb{R}$ be a valuation function. Then v is called *a) additive*, if $v(S) = \sum_{j \in S} v(j)$; *b) submodular*[1], if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$; *c) fractionally subadditive* or *XOS*, if v is determined by a finite set of additive valuations f_k such that $v(S) = \max_k f_k(S)$.

The classes of the above valuations are in increasing order of inclusion.

Multi-unit Auction. In a multi-unit auction, m copies of an item are sold to n bidders. Here, bidder i 's valuation is a function that depends on the number of copies he gets. That is $v_i : \{0, 1, \ldots, m\} \to \mathbb{R}^+$ and it is non-decreasing and normalized, with $v_i(0) = 0$. We say a valuation v_i is *submodular*, if it has non-increasing marginal values, i.e. $v_i(s + 1) - v_i(s) \geq v_i(t + 1) - v_i(t)$ for all $s \leq t$.

Nash equilibrium and price of anarchy We use b_i to denote a pure strategy of player i and it might be a single value or a vector, depending on the auction. So, for the case of m simultaneous auctions, $b_i = (b_{i1}, \ldots, b_{im})$. We denote by $\mathbf{b}_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ the strategies of all players except for i. Any *mixed strategy* B_i of player i is a probability distribution over b_i.

For any profile of strategies, $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{X}(\mathbf{b})$ denotes the allocation under the strategy profile $\mathbf{b}$. The valuation of player i for the allocation $\mathbf{X}(\mathbf{b})$ is denoted by $v_i(\mathbf{X}(\mathbf{b}))$. The *utility* u_i of player i is defined as the difference between her valuation and payment: $u_i(\mathbf{X}(\mathbf{b})) = v_i(\mathbf{X}(\mathbf{b})) - p_i(\mathbf{b})$.

Definition 6.2. (*Nash equilibria*) A bidding profile $\mathbf{b}$ forms a pure Nash equilibrium if for all bids b'_i and every player i, $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$. Similarly, a mixed bidding profile $\mathbf{B} = \times_i B_i$ is a mixed Nash equilibrium if for all bids b'_i and every player i, $\mathrm{E}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \geq \mathrm{E}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$. Clearly, any pure Nash equilibrium is also a mixed Nash equilibrium.

Our global objective is to maximize the sum of the valuations of the players for their received allocations, i.e., to maximize the *social welfare* $SW(\mathbf{X}) = \sum_{i \in [n]} v_i(X_i)$. For an *optimal allocation* $\mathbf{O}(\mathbf{v}) = \mathbf{O} = (O_1, \ldots, O_n)$, $SW(\mathbf{O}) = \max_{\mathbf{X}} SW(\mathbf{X})$. In Section 6.5, we also study two other objectives: the *revenue*, which equals the sum of the payments, $\sum_i p_i$, and the *maximum payment*, $\max_i b_i$. We also refer to the maximum payment as the *maximum bid.*

[1] As an equivalent definition, submodular valuations are exactly the valuations with *decreasing marginal bids*, meaning that $v(\{j\} \cup T) - v(T) \leq v(\{j\} \cup S) - v(S)$ holds for any item j and any $S \subseteq T$.

For simplicity, if the allocation rule $\mathbf{X}$ is clear from the context, we use $SW(\mathbf{b})$, $v_i(\mathbf{b})$ and $u_i(\mathbf{b})$ instead of $SW(\mathbf{X}(\mathbf{b}))$, $v_i(X_i(\mathbf{b}))$ and $u_i(\mathbf{X}(\mathbf{b}))$, to express social welfare, valuation and utility of player i for the allocation $\mathbf{X}(\mathbf{b})$.

Let $\mathbf{B} = (B_1, \ldots, B_n)$ be a profile of mixed strategies. Given the profile $\mathbf{B}$, we fix the notation for the following *cumulative distribution functions (CDF):* G_{ij} is the CDF of the bid of player i for item j; F_j is the CDF of the highest bid for item j and F_{ij} is the CDF of the highest bid for item j if we exclude the bid of player i. Observe that $F_j = \Pi_k G_{kj}$, and $F_{ij} = \Pi_{k \neq i} G_{kj}$. We also use $\varphi_{ij}(x)$ to denote the probability that player i gets item j by bidding x. Then, $\varphi_{ij}(x) \leq F_{ij}(x)$. When we refer to a single item, we may drop the index j. Whenever it is clear from the context, we will use shorter notation for expectations, e.g. we use $\mathrm{E}[u_i(b)]$ instead of $\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$. For simplicity, we also use $u(\mathbf{B})$ to denote $\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[u(\mathbf{b})]$.

6.3 Combinatorial Auctions

In this section we prove an upper bound of 1.82 for the mixed price of anarchy of simultaneous all-pay auctions when bidders' valuations are fractionally subadditive. This result improves over the previously known bound of 2 due to [76]. We first state our main theorem and present the key ingredients. Then we prove these ingredients in the following subsections.

6.3.1 Proof Outline

Here we present a (very short) sketch of the proof highlights of the upper bound.

Theorem 6.3. *The mixed PoA for simultaneous all-pay auctions with fractionally subadditive bidders is at most* 1.82.

Proof Sketch. We first illustrate the main ideas by focusing on a single item APA. W.l.o.g. we assume bidder 1 has the highest valuation v_1 among all bidders. First we came up with the following two lower bounds on the social welfare in equilibrium,

$$SW(\mathbf{B}) \geq A + \int_0^{v_1 - A} 1 - F(x)dx, \qquad SW(\mathbf{B}) \geq \int_0^{v_1 - A} \sqrt{F(x)}dx$$

where $F(x)$ is the CDF of $\max_i\{b_i\}$ and $A = \max_x\{F_1(x) \cdot v_1 - v_1\}$. Note that $F_1(x)$ is the CDF of $\max_{i \neq 1}\{b_i\}$. The first inequality is derived from the existing upper bound 2. The proof of the second inequality is based on the structure of mixed equilibria in APAs. By definition, we have $F_i(x) \cdot v_i - x \geq F_i(y) \cdot v_i - y$ if bidder i bids x in the Nash. By taking limits when $y \to x$, we have that $1/v_i$ equals to the derivative of F_i at x. So $SW(\mathbf{B})$ can be rewritten as $\sum_i \int_x^{v_1} F_i(x) g_i(x) \frac{1}{F_i'(x)} dx \geq \int_x^{v_1 - A} \sum_i \frac{g_i(x)}{\sum_{k \neq i} \frac{g_k(x)}{G_k(x)}}$ by using $F_i(x) = \prod_{k \neq i} G_k(x)$. Then we can adapt the following proposition to get the second lower bound for $SW(\mathbf{B})$.

Proposition 6.4. *For any integer $l \geq 2$, any positive real $G_i \leq 1$ and positive real g_i, for $1 \leq i \leq n$,*

$$\sum_{i=1}^{l} \frac{g_i}{\sum_{k \neq i} \frac{g_k}{G_k}} \geq \sqrt{\prod_{i=1}^{l} G_i}$$

The bound 1.82 can be derived by an optimal convex combination of these two lower bounds for $SW(\mathbf{B})$.In order to generalize the proof from a single to multiple items, we introduce a notion, that we call expected marginal valuation denoted by $v_{ij}(x)$ for which we show that $F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(y) \cdot v_{ij}(x) - y$. This allows us to treat each item separately and get the improved upper bound for simultaneous APAs.

6.3.2 Full Proof

Proof. Given a valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, let $\mathbf{O} = (O_1, \ldots, O_n)$ be a fixed optimal solution that maximizes the social welfare. Since v_i is a fractionally subadditive valuation, let $f_i^{O_i}$ be a maximizing additive function w.r.t O_i. Let $j \in O_i$ be one of the items that i receives. We denote by o_j item j's contribution to the optimal social welfare, that is, $o_j = f_i^{O_i}(j)$. The optimal social welfare is thus $SW(\mathbf{O}) = \sum_j o_j$. In order to bound the price of anarchy, we consider only items with $o_j > 0$, as it is without loss of generality to omit items with $o_j = 0$.

For a fixed mixed Nash equilibrium $\mathbf{B}$, recall that by F_j and F_{ij} we denote the CDFs of the maximum bid on item j among all bidders, with and without the bid of bidder i, respectively. Observe that $F_j(x) \leq F_{ij}(x)$. For any item $j \in O_i$, let $A_j = \max_{x \geq 0} \{F_{ij}(x) o_j - x\}$.

As a key part of the proof we use the following two inequalities that bound from below the social welfare in any mixed Nash equilibrium $\mathbf{B}$.

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} (A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx) \quad (6.1)$$

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx \quad (6.2)$$

Inequality (6.1), suffices to provide a weaker upper bound of 2 (see [22]). The proof of Inequality (6.2) is much more involved, and requires deeper understanding of the properties of equilibria of the induced game. We postpone their proofs to Section 6.3.3 (Lemma 6.5) and Section 6.3.4 (Lemma 6.6) respectively.

By combining (6.1) and (6.2) we get

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \cdot \sum_j \left(A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx + \lambda \cdot \int_0^{o_j - A_j} \sqrt{F_j(x)} dx \right), \quad (6.3)$$

for any $\lambda \geq 0$. It suffices to bound from below the right-hand side of (6.3) with respect to the optimal social welfare. For any cumulative distribution function F, and any positive real number v, let

$$R(F, v) \stackrel{\text{def}}{=} A + \int_0^{v-A} (1 - F(x))dx + \lambda \cdot \int_0^{v-A} \sqrt{F(x)}dx,$$

where $A = \max_{x \geq 0}\{F(x) \cdot v - x\}$. Then inequality (6.3) can be rewritten as $SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j)$. Finally, we show a lower bound on $R(F, v)$ that holds for any CDF F and any positive real v.

$$R(F, v) \geq \frac{3 + 4\lambda - \lambda^4}{6} \cdot v. \tag{6.4}$$

The proof of inequality 6.4 is given in Section 6.3.5 (Lemma 6.27). Finally, we obtain that for any $\lambda > 0$,

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j) \geq \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot \sum_j o_j = \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot SW(\mathbf{O})$$

We conclude that the price of anarchy is at most $\frac{6\lambda+6}{3+4\lambda-\lambda^4} \simeq 1.82$ by taking $\lambda = 0.56$. □

6.3.3 Proof of Inequality (6.1)

This section is devoted to the proof of the following lower bound.

Lemma 6.5. *$SW(\mathbf{B}) \geq \sum_{j \in [m]}(A_j + \int_0^{o_j - A_j} (1 - F_j(x))dx)$.*

Proof. Recall that $A_j = \max_{x \geq 0} \{F_{ij}(x)o_j - x\}$. We can bound bidder i's utility in the Nash equilibrium $\mathbf{B}$ by $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$. To see this, consider the deviation for bidder i, where he bids only for items in O_i, namely, for each item j, he bids the value x_j that maximizes the expression $F_{ij}(x_j)o_j - x_j$. Since for any obtained subset $T \subseteq O_i$, he has value $v_i(T) \geq \sum_{j \in T} o_j$, and the bids x_j must be paid in any case, the expected utility with these bids is at least $\sum_{j \in O_i} \max_{x \geq 0} (F_{ij}(x)o_j - x) = \sum_{j \in O_i} A_j$. With $\mathbf{B}$ being an equilibrium, we infer that $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$.

By summing up over all bidders, we have

$$\begin{aligned} SW(\mathbf{B}) &= \sum_{i \in [n]} u_i(\mathbf{B}) + \sum_{i \in [n]} \sum_{j \in [m]} \mathrm{E}[b_{ij}] \geq \sum_{j \in [m]} A_j + \sum_{j \in [m]} \sum_{i \in [n]} \mathrm{E}[b_{ij}] \\ &\geq \sum_{j \in [m]} (A_j + \mathrm{E}[\max_{i \in [n]}\{b_{ij}\}]) \geq \sum_{j \in [m]} \left(A_j + \int_0^{o_j - A_j} (1 - F_j(x))dx \right). \end{aligned}$$

The first equality holds because $SW(\mathbf{B}) = \sum_i \mathrm{E}_{\mathbf{b}}[v_i(\mathbf{b})] = \sum_i \mathrm{E}_{\mathbf{b}}[u_i(\mathbf{b}) + \sum_{j \in [m]} b_{ij}]$. The second inequality follows because $\sum_i b_{ij} \geq \max_i b_{ij}$ and the last one is implied by the definition of the expected value of any positive random variable. □

6.3.4 Proof of Inequality (6.2)

In this section, we prove the following lemma for any mixed Nash equilibrium $\mathbf{B}$.

Lemma 6.6. $SW(\mathbf{B}) \geq \sum_{j\in[m]} \int_0^{o_j - A_j} \sqrt{F_j(x)}dx$.

First we show a useful lemma that holds for fractionally subadditive valuations.

Lemma 6.7. *For any fractionally subadditive valuation function v,*

$$v(S) \geq \sum_{j\in[m]} \left(v(S) - v(S \setminus \{j\})\right).$$

Proof. Let f be a maximizing additive function of S for the fractionally subadditive valuation v; then by definition $v(S) = f(S)$ and for every item j it holds that $v(S \setminus \{j\}) \geq f(S \setminus \{j\})$. Then,

$$\sum_{j\in[m]} \left(v(S) - v(S \setminus \{j\})\right) \leq \sum_{j\in[m]} \left(f(S) - f(S \setminus \{j\})\right) = \sum_{j\in S} f(j) = v(S).$$

□

We will use the following technical proposition.

Proposition 6.8. *For any integer $n \geq 2$, any positive reals $G_i \leq 1$ and positive reals g_i, for $1 \leq i \leq n$,*

$$\sum_{i=1}^{n} \frac{g_i}{\sum_{k\neq i} \frac{g_k}{G_k}} \geq \sqrt{\prod_{i=1}^{n} G_i}.$$

In order to prove the proposition, we will minimize the left hand side of the inequality over all G_i and g_i, such that

$$0 < G_i \leq 1 \qquad g_i > 0 \quad (i \in [n]) \quad \text{where } \prod_{t=1}^{n} G_t \quad \text{is a constant.} \tag{6.5}$$

We introduce the following notation:

$$H = \sum_{i=1}^{n} \frac{g_i}{\sum_{t=1,t\neq i}^{n} \frac{g_t}{G_t}} \qquad \text{and} \qquad \forall i, \qquad H_i = \frac{g_i}{\sum_{t=1,t\neq i}^{n} \frac{g_t}{G_t}}.$$

Note that $H = \sum_{i=1}^{n} H_i$. Our goal is to minimize H over all possible variables G_i and g_i under the constraints (6.5), and eventually show $H \geq \sqrt{\prod_{i=1}^{n} G_i}$. We also use the notation $\mathbf{G} = (G_i)_i$, $\mathbf{g} = (g_i)_i$, $H = H(\mathbf{G}, \mathbf{g})$ and $H_i = H_i(\mathbf{G}, \mathbf{g})$, $\forall i$.

Lemma 6.9. *For every $\mathbf{G}$ and $\mathbf{g}$ that minimize $H(\cdot, \cdot)$ under constraints (6.5):*

1. *If $G_i < 1$ and $G_j < 1$, then $H_i = H_j$,*

2. *If $G_i = G_j = 1$ then $g_i = g_j$.*

We prove Lemma 6.9, by proving Lemmas 6.10 and 6.11.

Lemma 6.10. *Under constraints* (6.5)*, if* $\mathbf{G}$ *and* $\mathbf{g}$ *minimize* $H(\cdot,\cdot)$*, then for every* $G_i < 1$ *and* $G_j < 1$*,* $H_i(\mathbf{G},\mathbf{g}) = H_j(\mathbf{G},\mathbf{g})$*.*

Proof. For the sake of contradiction, suppose that there exist $G_i < 1$ and $G_j < 1$ such that (w.l.o.g.) $H_i(\mathbf{G},\mathbf{g}) > H_j(\mathbf{G},\mathbf{g})$. Let

$$r = \min\left\{\left(\frac{H_i(\mathbf{G},\mathbf{g})}{H_j(\mathbf{G},\mathbf{g})}\right)^{\frac{1}{2}}, \frac{1}{G_j}\right\}.$$

Notice that $r > 1$.

Claim: We claim that $H(\mathbf{G},\mathbf{g}) > H(\mathbf{G}',\mathbf{g}')$, where $\mathbf{G}' = (\frac{G_i}{r}, rG_j, \mathbf{G}_{-ij})$ and $\mathbf{g}' = (\frac{g_i}{r}, rg_j, \mathbf{g}_{-ij})$.
As usual $\mathbf{G}_{-ij}$ stands for $\mathbf{G}$ vector after eliminating G_i and G_j (accordingly for $\mathbf{g}_{-ij}$). Therefore $\mathbf{G}'$ and $\mathbf{g}'$ are the same as $\mathbf{G}$ and $\mathbf{g}$ by replacing G_i, G_j, g_i, g_j by $\frac{G_i}{r}$, rG_j, $\frac{g_i}{r}$, rg_j, respectively.

Proof of the claim: Notice that

$$\frac{g'_i}{G'_i} = \frac{g_i/r}{G_i/r} = \frac{g_i}{G_i}, \qquad \frac{g'_j}{G'_j} = \frac{rg_j}{rG_j} = \frac{g_j}{G_j} \qquad \text{and} \qquad \forall s \neq i,j, \quad G'_s = G_s \quad \text{and} \quad g'_s = g_s.$$

Therefore, $\forall s \neq i,j$, $H_s(\mathbf{G},\mathbf{g}) = H_s(\mathbf{G}',\mathbf{g}')$. So, we only need to show that $H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) > H_i(\mathbf{G}',\mathbf{g}') + H_j(\mathbf{G}',\mathbf{g}')$.

$$\begin{aligned}
H_i(\mathbf{G}',\mathbf{g}') + H_j(\mathbf{G}',\mathbf{g}') &= \frac{g'_i(x)}{\sum_{t=1,t\neq i}^{n} \frac{g'_t(x)}{G'_t(x)}} + \frac{g'_j(x)}{\sum_{t=1,t\neq j}^{n} \frac{g'_t(x)}{G'_t(x)}} \\
&= \frac{g_i(x)/r}{\sum_{t=1,t\neq i}^{n} \frac{g_t(x)}{G_t(x)}} + \frac{rg_j(x)}{\sum_{t=1,t\neq j}^{n} \frac{g_t(x)}{G_t(x)}} \\
&= \frac{H_i(\mathbf{G},\mathbf{g})}{r} + rH_j(\mathbf{G},\mathbf{g}) \\
&= \left(\frac{1}{r} - 1\right) H_i(\mathbf{G},\mathbf{g}) + (r-1)H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&\leq \left(\frac{1}{r} - 1\right) r^2 H_j(\mathbf{G},\mathbf{g}) + (r-1)H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&= -(r-1)^2 H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&< H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}).
\end{aligned}$$

In the above inequalities we used that $r > 1$ and $r^2 \leq \frac{H_i(\mathbf{G},\mathbf{g})}{H_j(\mathbf{G},\mathbf{g})}$. The claim contradicts the assumption that $H(\mathbf{G},\mathbf{g})$ is the minimum, so the lemma holds.

□

Lemma 6.11. *Under constraints* (6.5)*, if* $\mathbf{G}$ *and* $\mathbf{g}$ *minimize* $H(\cdot,\cdot)$*, then for every* $G_i = G_j = 1$*,* $g_i = g_j$*.*

Proof. For the sake of contradiction, suppose that there exist $G_i = G_j = 1$ such that $g_i \neq g_j$. We will prove that for $\mathbf{g'} = (\frac{g_i+g_j}{2}, \frac{g_i+g_j}{2}, g_{-ij})$ (i.e. for every $k \neq i, j$, $g'_k = g_k$, and $g'_i = g'_j = \frac{g_i+g_j}{2}$), $H(\mathbf{G}, \mathbf{g}) > H(\mathbf{G}, \mathbf{g'})$.

Notice that for every $k \neq i, j$, $H_k(\mathbf{G}, \mathbf{g'}) = H_k(\mathbf{G}, \mathbf{g})$, since $g_i + g_j = g'_i + g'_j$ and $G_i = G_j = 1$. Hence it is sufficient to show that $H_i(\mathbf{G}, \mathbf{g}) + H_j(\mathbf{G}, \mathbf{g}) \geq H_i(\mathbf{G}, \mathbf{g'}) + H_j(\mathbf{G}, \mathbf{g'})$. Let $A_{ij} = \sum_{t \neq j, t \neq i} \frac{g_t}{G_t}$.

$$
\begin{aligned}
& H_i(\mathbf{G}, \mathbf{g}) + H_j(\mathbf{G}, \mathbf{g}) - H_i(\mathbf{G}, \mathbf{g'}) - H_j(\mathbf{G}, \mathbf{g'}) \\
= \; & \frac{g_i}{g_j + A_{ij}} + \frac{g_j}{g_i + A_{ij}} - \frac{g_i}{\frac{g_i+g_j}{2} + A_{ij}} - \frac{g_j}{\frac{g_i+g_j}{2} + A_{ij}} \\
= \; & \frac{g_i}{g_j + A_{ij}} + \frac{g_j}{g_i + A_{ij}} - \frac{2g_i + 2g_j}{g_i + g_j + 2A_{ij}} \\
= \; & g_i \frac{(g_i + A_{ij})((g_i + g_j + 2A_{ij}) - 2(g_j + A_{ij}))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} \\
+ \; & g_j \frac{(g_j + A_{ij})((g_i + g_j + 2A_{ij}) - 2(g_i + A_{ij}))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} \\
= \; & \frac{g_i(g_i + A_{ij})(g_i - g_j) + g_j(g_j + A_{ij})(g_j - g_i)}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} \\
= \; & \frac{(g_i - g_j)(g_i^2 - g_j^2 + A_{ij}(g_i - g_j))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} \\
= \; & \frac{(g_i - g_j)^2(g_i + g_j + A_{ij})}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} > 0,
\end{aligned}
$$

which contradicts the assumption that $\mathbf{G}$ and $\mathbf{g}$ minimize $H(\cdot, \cdot)$. □

Lemma 6.12. *If $H_i = H_j$, then:*

1. $g_i = g_j \Leftrightarrow G_i = G_j$,
2. $(g_i = rg_j > 0$ *and* $r \geq 1) \Rightarrow G_i \geq r^2 G_j$.

Proof. Let $A_{ij} = \sum_{t \neq j, t \neq i} \frac{g_t}{G_t}$; then $H_i = \frac{g_i}{\frac{g_j}{G_j} + A_{ij}}$. By assumption:

$$
\begin{aligned}
\frac{g_i}{\frac{g_j}{G_j} + A_{ij}} &= \frac{g_j}{\frac{g_i}{G_i} + A_{ij}} \\
\frac{g_i^2}{G_i} + g_i A_{ij} &= \frac{g_j^2}{G_j} + g_j A_{ij} \\
(g_i - g_j) A_{ij} &= \frac{g_j^2}{G_j} - \frac{g_i^2}{G_i}.
\end{aligned}
$$

If $g_i = g_j$ then $\frac{1}{G_j} - \frac{1}{G_i} = 0$, so $G_i = G_j$.
If $G_i = G_j$ then $(g_i - g_j)(g_i + g_j + A_{ij} G_i) = 0$. Under constraints (6.5), $A_{ij} G_i > 0$ and $g_i, g_j > 0$, so $g_i - g_j = 0$ which results in $g_i = g_j$.
If $g_i = rg_j$, with $r \geq 1$ then $(g_i - g_j) A_{ij} \geq 0$ and so $\frac{1}{G_j} - \frac{r^2}{G_i} \geq 0$, which implies $G_i \geq r^2 G_j$. □

Lemma 6.13. *For n, k integers, $n \geq 2$, $1 \leq k \leq n$, $0 < a \leq 1$ and $g > 0$:*

$$L = \frac{kg}{(k-1)\frac{g}{a} + n - k} + \frac{n-k}{k\frac{g}{a} + n - k - 1} \geq a.$$

Proof. We distinguish between two cases, 1) $k > \frac{1}{1-\sqrt{a}}$ and 2) $k \leq \frac{1}{1-\sqrt{a}}$.
Case 1 ($k > \frac{1}{1-\sqrt{a}}$): For $k = n$, $L = \frac{k}{k-1}a \geq a$. We next show that $\frac{dL}{dg} \leq 0$, for $n \geq 2$, $1 \leq k < n$, $0 < a \leq 1$ and $g > 0$.

$$\begin{aligned}
\frac{dL}{dg} = \frac{(n-k)k}{\left(\frac{(k-1)g}{a} + n - k\right)^2} - \frac{(n-k)k}{\left(\frac{kg}{a} + n - k - 1\right)^2 a} &\leq 0 \\
\left(\frac{(k-1)g}{a} + n - k\right)^2 - \left(\frac{kg}{a} + n - k - 1\right)^2 a &\geq 0 \\
\left(\frac{(k-1)g}{a} + n - k - \left(\frac{kg}{a} + n - k - 1\right) a^{\frac{1}{2}}\right) &\geq 0 \\
\left(\frac{g}{a}\left(k - 1 - ka^{\frac{1}{2}}\right) + (n-k)\left(1 - a^{\frac{1}{2}}\right) + a^{\frac{1}{2}}\right) &\geq 0 \\
k - 1 - ka^{\frac{1}{2}} &\geq 0
\end{aligned}$$

which is true by the case assumption. Therefore, L is non-increasing and so it is minimized for $g = \infty$. Hence, $L \geq \frac{k}{k-1}a \geq a$.
Case 2 ($k \leq \frac{1}{1-\sqrt{a}}$): L is minimized ($dL/dg(g^*) = 0$) for $g^* = \frac{a\left(\sqrt{a} + (n-k)\left(1-\sqrt{a}\right)\right)}{k\sqrt{a} - k + 1}$, therefore:

$$L \geq \frac{k^2\left(1-\sqrt{a}\right)^2 + k\left(a - n\left(1-\sqrt{a}\right)^2 - 1\right) + n)}{(n-1)},$$

which is minimizes for $k = \frac{n}{2} + \frac{\left(1+\sqrt{a}\right)}{2\left(1-\sqrt{a}\right)}$. However, for $n \geq 2$, $\frac{n}{2} + \frac{\left(1+\sqrt{a}\right)}{2\left(1-\sqrt{a}\right)} \geq \frac{1}{1-\sqrt{a}}$. Notice, though, that for $k \leq \frac{1}{1-\sqrt{a}}$, L is decreasing, so it is minimized for $k = \frac{1}{1-\sqrt{a}}$. Therefore, $L \geq \sqrt{a} \geq a$.

□

Proof. (Proposition 6.8)
Let $\mathbf{G}$ and $\mathbf{g}$ minimize $H(\cdot, \cdot)$ and also let $S = \{i | G_i < 1\}$ and $F = \prod_{t=1}^{n} G_t$. Moreover, given Lemma 6.9, for $g_i = \hat{g}$ for every $i \notin S$ and $j = \arg\min_{i \in S} g_i$, $H(\mathbf{G}, \mathbf{g})$ can be written as:

$$H(\mathbf{G}, \mathbf{g}) = |S| \frac{g_j}{\sum_{t \in S, t \neq j} \frac{g_t}{G_t} + (n - |S|)\hat{g}} + (n - |S|) \frac{\hat{g}}{\sum_{t \in S} \frac{g_t}{G_t} + (n - |S| - 1)\hat{g}}.$$

Let $g_i = r_i g_j$, for every $i \in S$. Since $j = \arg\min_{i \in S} g_i$, then for every $i \in S$, $r_i \geq 1$. By using Lemma 6.12:

$$
\begin{aligned}
H(\mathbf{G},\mathbf{g}) &= |S|\frac{g_j}{\sum_{t\in S,t\neq j}\frac{r_t g_j}{G_t^{\frac{1}{2}}G_t^{\frac{1}{2}}}+(n-|S|)\hat{g}}+(n-|S|)\frac{\hat{g}}{\sum_{t\in S}\frac{r_t g_j}{G_t^{\frac{1}{2}}G_t^{\frac{1}{2}}}+(n-|S|-1)\hat{g}} \\
&\geq |S|\frac{g_j}{\sum_{t\in S,t\neq j}\frac{g_j}{F^{\frac{1}{2}}}+(n-|S|)\hat{g}}+(n-|S|)\frac{\hat{g}}{\sum_{t\in S}\frac{g_j}{F^{\frac{1}{2}}}+(n-|S|-1)\hat{g}} \\
&= |S|\frac{g_j}{(|S|-1)\frac{g_j}{F^{\frac{1}{2}}}+(n-|S|)\hat{g}}+(n-|S|)\frac{\hat{g}}{|S|\frac{g_j}{F^{\frac{1}{2}}}+(n-|S|-1)\hat{g}}.
\end{aligned}
$$

Let $g=\frac{g_j}{\hat{g}}$, then:

$$
H(\mathbf{G},\mathbf{g})\geq\frac{|S|g}{(|S|-1)\frac{g}{F^{\frac{1}{2}}}+n-|S|}+\frac{n-|S|}{|S|\frac{g}{F^{\frac{1}{2}}}+n-|S|-1}.
$$

If $|S|=0$, $H(\mathbf{G},\mathbf{g})\geq\frac{n}{n-1}\geq 1\geq\sqrt{F}$. else, due to Lemma 6.13, $H(\mathbf{G},\mathbf{g})\geq\sqrt{F}$.

□

We are now ready to proceed with the proof of Lemma 6.6. We first state a proof sketch here to illustrate the main ideas.

Sketch of Lemma 6.6. Recall that G_{ij} is the CDF of the bid of player i for item j. For simplicity, we assume $G_{ij}(x)$ is non-decreasing, continuous and differentiable, with $g_{ij}(x)$ being the PDF of player i's bid for item j. The general case is considered later. First, we define the *expected marginal valuation* of item j w.r.t player i.

$$
v_{ij}(x)\stackrel{\text{def}}{=}\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b})\cup\{j\})-v_i(X_i(\mathbf{b})\setminus\{j\})|b_{ij}=x]
$$

Given the above definition and a careful characterization of mixed Nash equilibria, we are able to show $F_{ij}(x)\cdot v_{ij}(x)=\mathrm{E}[v_i(X_i(\mathbf{b}))-v_i(X_i(\mathbf{b})\setminus\{j\})|b_{ij}=x]$ and $\frac{1}{v_{ij}(x)}=\frac{dF_{ij}(x)}{dx}$ for any x in the support of G_{ij}. Let $g_{ij}(x)$ be the derivative of $G_{ij}(x)$. Using Lemma 6.7, we have

$$
\begin{aligned}
SW(\mathbf{B}) &= \sum_i \mathrm{E}[v_i(X_i(\mathbf{b}))]\geq\sum_i\sum_j\mathrm{E}[v_i(X_i(\mathbf{b}))-v_i(X_i(\mathbf{b})\setminus\{j\})] \\
&\geq\sum_i\sum_j\int_0^{o_j-A_j}\mathrm{E}[v_i(X_i(\mathbf{b}))-v_i(X_i(\mathbf{b})\setminus\{j\})|b_{ij}=x]\cdot g_{ij}(x)dx \\
&\geq\sum_i\sum_j\int_0^{o_j-A_j}F_{ij}(x)\cdot v_{ij}(x)\cdot g_{ij}(x)dx,
\end{aligned}
$$

where the second inequality follows by the law of total probability. By using the facts that $F_{ij}(x)=\prod_{k\neq i}G_{kj}(x)$ and $\frac{1}{v_{ij}(x)}=\frac{dF_{ij}(x)}{dx}$, for any $x>0$ such that $g_{ij}(x)>0$ and $F_j(x)>0$, we obtain

$$
F_{ij}(x)\cdot v_{ij}(x)\cdot g_{ij}(x)=\frac{F_{ij}(x)\cdot g_{ij}(x)}{\frac{dF_{ij}}{dx}(x)}=\frac{\prod_{k\neq i}G_{kj}(x)\cdot g_{ij}(x)}{\sum_{k\neq i}\left(g_{kj}\cdot\prod_{s\neq k\wedge s\neq i}G_{sj}\right)}=\frac{g_{ij}(x)}{\sum_{k\neq i}\frac{g_{kj}(x)}{G_{kj}(x)}}.
$$

For every $x > 0$, we use Proposition 6.8 only over the set S of players with $g_{ij}(x) > 0$. After summing over all bidders we get,

$$\sum_{i\in[n]} F_{ij}(x)\cdot v_{ij}(x)\cdot g_{ij}(x) \geq \sum_{i\in S} \frac{g_{ij}(x)}{\sum_{k\neq i, k\in S}\frac{g_{kj}}{G_{kj}}} \geq \sqrt{\prod_{i\in S} G_{ij}(x)} \geq \sqrt{F_j(x)}.$$

Note that the above inequality holds even for $x > 0$, such that $F_j(x) = 0$. Finally, by merging the above inequalities, we conclude that

$$SW(\mathbf{B}) \geq \sum_{j\in[m]} \int_0^{o_j - A_j} \sqrt{F_j(x)}dx.$$

□

Recall that o_j is the contribution of item j to the optimum social welfare. If player i is the one receiving item j in the optimum allocation, then $A_j = \max_{x\geq 0}\{F_{ij}(x)\cdot o_j - x\}$. The proof of Lemma 6.6 needs a careful technical preparation that we divided into a couple of lemmas.

First of all, we define the expected marginal valuation of item j for player i. For given mixed strategy B_i, the distribution of bids on items in $[m]\setminus\{j\}$ depends on the bid b_{ij}, so one can consider the given conditional expectation:

Definition 6.14. Given a mixed bidding profile $\mathbf{B} = (B_1, B_2, \ldots, B_n)$, the expected marginal valuation $v_{ij}(x)$ of item j for player i when $b_{ij} = x$ is defined as

$$v_{ij}(x) \stackrel{\text{def}}{=} \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b})\cup\{j\}) - v_i(X_i(\mathbf{b})\setminus\{j\})|b_{ij} = x].$$

For a given $\mathbf{B}$, let $\varphi_{ij}(x)$ denote the probability that bidder i gets item j when she bids x on item j. It is clear that φ_{ij} is non-decreasing and $\varphi_{ij}(x) \leq F_{ij}(x)$ (they are equal when no ties occur).

Lemma 6.15. *For a given $\mathbf{B}$, for any bidder i, item j and bids $x \geq 0$ and $y \geq 0$,*

$$\varphi_{ij}(y)\cdot v_{ij}(x) = \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x],$$

where $\mathbf{b}'$ is the modified bid of $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b}$ except that $b'_{ij} = y$.

Proof.

$$\begin{aligned}
& \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x] \\
= \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\Pr(j\in X_i(\mathbf{b}')|b_{ij} = x) \\
+ \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\notin X_i(\mathbf{b}')]\Pr(j\notin X_i(\mathbf{b}')|b_{ij} = x) \\
= \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\Pr(j\in X_i(\mathbf{b}')|b_{ij} = x) \\
= \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\cdot\varphi_{ij}(y) \\
= \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')\cup\{j\}) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\cdot\varphi_{ij}(y) \\
= \ & \mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')\cup\{j\}) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x]\cdot\varphi_{ij}(y) \\
= \ & \varphi_{ij}(y)\cdot v_{ij}(x).
\end{aligned}$$

The second equality is due to $\mathrm{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}'))) - v_i(X_i(\mathbf{b}') \setminus \{j\})|b_{ij} = x, j \notin X_i(\mathbf{b}')] = 0$; the third one holds because $b'_{ij} = y$, and that other players' bids have distribution $\times_{k\neq i} B_k$. The fourth one is obvious, since $X_i(\mathbf{b}') = X_i(\mathbf{b}') \cup \{j\}$ given that $j \in X_i(\mathbf{b}')$. The last two equalities follow from the fact that $v_i(X_i(\mathbf{b}') \cup \{j\}) - v_i(X_i(\mathbf{b}') \setminus \{j\})$ is independent of the condition $j \in X_i(\mathbf{b}')$ and of the player i's bid on item j. □

Definition 6.16. Given a Nash equilibrium $\mathbf{B}$, we say a bid x is *good for bidder i and item j* (or $b_{ij} = x$ is *good*) if $\mathrm{E}[u_i(\mathbf{b})] = \mathrm{E}[u_i(\mathbf{b})|b_{ij} = x]$, otherwise we say $b_{ij} = x$ is *bad*.

Lemma 6.17. *Given a Nash equilibrium $\mathbf{B}$, for any bidder i and item j,* $\Pr[b_{ij}$ *is bad*$] = 0$.

Proof. The lemma follows from the definition of Nash equilibrium; otherwise we can replace the bad bids with good bids and improve the bidder's utility. □

Lemma 6.18. *Given a Nash equilibrium $\mathbf{B}$, for any bidder i, item j, good bid x and any bid $y \geq 0$,*

$$\varphi_{ij}(x) \cdot v_{ij}(x) - x \geq \varphi_{ij}(y) \cdot v_{ij}(x) - y.$$

Moreover, for a good bid $x > 0$, $\varphi_{ij}(x) > 0$ holds.

Proof. Let $\mathbf{b}'$ be the modified bid of $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b}$ except that $b'_{ij} = y$.

$$\mathrm{E}[u_i(\mathbf{b})] = \mathrm{E}[u_i(\mathbf{b})|b_{ij} = x] \geq \mathrm{E}[u_i(\mathbf{b}')|b_{ij} = x].$$

Now we consider the difference between the above two terms:

$$\begin{aligned}
0 &\leq \mathrm{E}[u_i(\mathbf{b})|b_{ij} = x] - \mathrm{E}[u_i(\mathbf{b}')|b_{ij} = x] \\
&= \mathrm{E}[v_i(X_i(\mathbf{b})) - b_{ij}|b_{ij} = x] - \mathrm{E}[v_i(X_i(\mathbf{b}')) - b'_{ij}|b_{ij} = x] \\
&= \mathrm{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})|b_{ij} = x] - \mathrm{E}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}') \setminus \{j\}|b_{ij} = x] + y - x \\
&= (\varphi_{ij}(x) \cdot v_{ij}(x) - x) - (\varphi_{ij}(y) \cdot v_{ij}(x) - y).
\end{aligned}$$

The second equality holds since $X_i(\mathbf{b}) \setminus \{j\} = X_i(\mathbf{b}') \setminus \{j\}$; the third equality holds by Lemma 6.15.

Finally, $\varphi_{ij}(x) > 0$ for positive good bids follows by taking $y = 0$, since with $\varphi_{ij}(x) = 0$ the left hand side of the inequality would be negative. □

Next, by using the above lemma, we are able to show several structural results for Nash equilibria.

Definition 6.19. Given a mixed strategy profile $\mathbf{B}$, we say that a positive bid $x > 0$ is in bidder i's *support on item j*, if for all $\varepsilon > 0$, $G_{ij}(x) - G_{ij}(x - \varepsilon) > 0$.

Lemma 6.20. *Given a mixed strategy profile $\mathbf{B}$, if a positive bid x is in bidder i's support on item j, then for every $\varepsilon > 0$, there exists $x - \varepsilon < x' \leq x$ such that x' is good.*

Proof. Suppose on the contrary that there is an $\varepsilon > 0$ such that for all x', such that $x - \varepsilon < x' \leq x$, x' is bad. Then $\Pr[b_{ij} \text{ is bad}] \geq G_{ij}(x) - G_{ij}(x - \varepsilon) > 0$ (given that x is in the support), which contradicts Lemma 6.17. □

Lemma 6.21. *Given a Nash equilibrium* $\mathbf{B}$*, if* $x > 0$ *is in bidder* i*'s support on item* j*, then there must exist another bidder* $k \neq i$ *such that* x *is also in the bidder* k*'s support on item* j*, i.e. for all* $\varepsilon > 0$*,* $G_{kj}(x) - G_{kj}(x - \varepsilon) > 0$*.*

Proof. Assume on the contrary that for each player $k \neq i$, there exists $\varepsilon_k > 0$ such that $G_{kj}(x) - G_{kj}(x - \varepsilon_k) = 0$. Clearly, for $\varepsilon = \min\{\varepsilon_k | k \neq i\}$ it holds that $G_{kj}(x) - G_{kj}(x - \varepsilon) = 0$ for all bidders $k \neq i$. That is $\varphi_{ij}(x) = \varphi_{ij}(x - \varepsilon)$. By Lemma 6.20, there exists $x - \varepsilon < x' \leq x$ such that x' is good for player i. Since φ_{ij} is a non-decreasing function and $\varphi_{ij}(x) = \varphi_{ij}(x - \varepsilon)$, we have $\varphi_{ij}(x') = \varphi_{ij}(x - \varepsilon)$. By Lemma 6.18, $\varphi_{ij}(x') \cdot v_{ij}(x') - x' \geq \varphi_{ij}(x - \varepsilon) \cdot v_{ij}(x') - x + \varepsilon$ which contradicts the fact that $\varphi_{ij}(x') = \varphi_{ij}(x - \varepsilon)$ and $x' > x - \varepsilon$. □

Lemma 6.22. *Given a Nash equilibrium* $\mathbf{B}$*, for bidder* i *and item* j*, there are no* $x > 0$ *such that* $\Pr[b_{ij} = x] > 0$*, i.e. there are no mass points in the bidding strategy, except for possibly* 0*.*

Proof. Assume on the contrary that there exists a bid $x > 0$ such that $\Pr[b_{ij} = x] > 0$ for some bidder i and item j. By Lemma 6.17, x is good for bidder i and item j, and $\varphi_{ij}(x) > 0$ by Lemma 6.18.

According to Lemma 6.21, there must exist a bidder k such that x is in her support on item j. We can pick a sufficiently small ε such that $\varepsilon < (x - \varepsilon) \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]$. This can be done since $(x - \varepsilon)$ increases when ε decreases. Due to Lemma 6.20 there exists $x - \varepsilon < x' \leq x$ such that x' is good for bidder k and item j. Now we consider the following two cases for x'.

Case 1: $v_{kj}(x') \leq x'$. Then $\varphi_{kj}(x') \cdot v_{kj}(x') - x' \leq \varphi_{kj}(x') \cdot x' - x' \leq (1 - \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]) \cdot x' - x' < 0$, contradicting Lemma 6.18. The first inequality holds by the case assumption. The second holds because player k cannot get item j with bid x' whenever player i gets it by bidding x. The last inequality holds because both $\varphi_{ij}(x) > 0$ and $\Pr[b_{ij} = x] > 0$.

Case 2: $v_{kj}(x') > x'$. Then there exists a sufficiently small ε' such that $\varepsilon' \leq (x - \varepsilon) \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] - \varepsilon$. So $\varepsilon + \varepsilon' \leq x' \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]$. Then,

$$
\begin{aligned}
&\varphi_{kj}(x + \varepsilon') \cdot v_{kj}(x') - x - \varepsilon' \\
\geq &(\varphi_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]) \cdot v_{kj}(x') - x - \varepsilon' \\
> &\varphi_{kj}(x') \cdot v_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] \cdot x' - x' - (x - x') - \varepsilon' \\
> &\varphi_{kj}(x') \cdot v_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] \cdot x' - x' - \varepsilon - \varepsilon' \\
\geq &\varphi_{kj}(x') \cdot v_{kj}(x') - x',
\end{aligned}
$$

which contradicts Lemma 6.18. Here the first inequality holds because the probability that player k gets the item with bid $x + \varepsilon'$ is at least the probablity that he gets it by bidding x' plus the probability that i bids x and gets the item (these two events for $\mathbf{b}_{-k}$ are disjoint). The second inequality holds by case assumption, and the rest hold by our assumptions on ε and ε'. □

Lemma 6.23. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i *and item* j*,* $\varphi_{ij}(x) = F_{ij}(x)$ *for all* $x > 0$*.*

Proof. The lemma follows immediately from Lemma 6.22. The probablity that some player $k \neq i$ bids exactly x is zero. Thus $F_{ij}(x)$ equals the probability that the highest bid of players other than i is strictly smaller than x, and $1 - F_{ij}(x)$ is the probability that it is strictly higher. Therefore $\varphi_{ij}(x) = F_{ij}(x)$. □

Lemma 6.24. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i*, item* j *and good bids* $x_1 > x_2 > 0$*,* $v_{ij}(x_1) \geq v_{ij}(x_2)$*.*

Proof. By Lemma 6.18, we have $(\varphi_{ij}(x_1) - \varphi_{ij}(x_2)) \cdot v_{ij}(x_1) \geq x_1 - x_2$ and $(\varphi_{ij}(x_2) - \varphi_{ij}(x_1)) \cdot v_{ij}(x_2) \geq x_2 - x_1$. Combining these two inequalities, we have

$$\frac{1}{v_{ij}(x_1)} \leq \frac{\varphi_{ij}(x_1) - \varphi_{ij}(x_2)}{x_1 - x_2} \leq \frac{1}{v_{ij}(x_2)}.$$

□

Lemma 6.25. *Given a Nash equilibrium* $\mathbf{B}$ *and item* j*, let* $T = \sup\{x | x$ *is in some bidder's support on item* $j\}$*. For any bid* $x < T$*,* x *is in some bidder's support on item* j*.*

Proof. Assume on the contrary that there exist a bid $x < T$ such that x is not in any bidder's support. Then there exists $\delta > 0$ such that $G_{ij}(x) = G_{ij}(x - \delta)$ for all bidder i. Let $y = \sup\{z | \forall i, G_{ij}(x) = G_{ij}(z)\}$. By Lemma 6.22, G_{ij} is continuous. So we have $G_{ij}(y) = G_{ij}(x) = G_{ij}(x - \delta)$ for any bidder i. That is $F_{ij}(y) = F_{ij}(x - \delta)$ for any bidder i.

By the definition of supremum, there exists a bidder k such that for any $\varepsilon > 0$, $G_{kj}(y + \varepsilon) > G_{kj}(x) = G_{kj}(y)$. By Lemma 6.17, there exists a good bid $y^+ \in (y, y + \varepsilon]$ for bidder k and item j. We pick a sufficiently small ε such that $(F_{kj}(y^+) - F_{kj}(y)) \cdot v_{kj}(y^+) < \delta$. This can be done since F_{kj} is continuous by Lemma 6.22 and v_{kj} is non-decreasing by Lemma 6.24.

$$\begin{aligned}
&F_{kj}(x - \delta) \cdot v_{ij}(y^+) - x + \delta \\
=&F_{ij}(y) \cdot v_{ij}(y^+) - x + \delta \\
>&F_{ij}(y) \cdot v_{ij}(y^+) - y^+ + \delta \\
>&F_{ij}(y^+) \cdot v_{ij}(y^+) - y^+,
\end{aligned}$$

which contradicts Lemma 6.18 and Lemma 6.23. □

Lemma 6.26. *Given a Nash equilibrium* $\mathbf{B}$*, if* $x > 0$ *is a good bid for bidder* i *and item* j*, and* F_{ij} *is differentiable in* x*, then*

$$\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx}.$$

Proof. Notice that $v_{ij}(x) \neq 0$ by Lemma 6.18. By Lemma 6.18 and 6.23, we have $F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(y) \cdot v_{ij}(x) - y$ for all $y \geq 0$. So for any $\varepsilon > 0$,

$$F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(x+\varepsilon) \cdot v_{ij}(x) - x - \varepsilon$$

$$F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(x-\varepsilon) \cdot v_{ij}(x) - x + \varepsilon.$$

That is,

$$\frac{F_{ij}(x+\varepsilon) - F_{ij}(x)}{\varepsilon} \leq \frac{1}{v_{ij}(x)},$$

$$\frac{F_{ij}(x) - F_{ij}(x-\varepsilon)}{\varepsilon} \geq \frac{1}{v_{ij}(x)}.$$

The lemma follows by taking the limit when ε goes to 0. □

Proof of Lemma 6.6. Since $G_{ij}(x)$ is non-decreasing, continuous (Lemma 6.22) and bounded by 1, $G_{ij}(x)$ is differentiable on almost all points. That is, the set of all non-differentiable points has Lebesgue measure 0. So it will not change the value of integration if we remove these points. Therefore it is without loss of generality to assume $G_{ij}(x)$ is differentiable for all x. Let $g_{ij}(x)$ be the derivative of $G_{ij}(x)$, i.e. probability density function for bidder i's bidding on item j. Using Lemma 6.7, we have

$$\begin{aligned}
SW(\mathbf{B}) &= \sum_i \mathrm{E}[v_i(X_i(\mathbf{b}))] \\
&\geq \sum_i \sum_j \mathrm{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})] \\
&\geq \sum_i \sum_j \int_0^{o_j - A_j} \mathrm{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \cdot g_{ij}(x) dx \\
&\geq \sum_i \sum_j \int_0^{o_j - A_j} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) dx.
\end{aligned}$$

The second inequality follows by the law of total probability, and the third is due to Lemmas 6.15 and 6.23. By Lemma 6.26 and the fact that $F_{ij}(x) = \prod_{k \neq i} G_{kj}(x)$, if x is good, $g_{ij}(x) > 0$ and $G_{ij}(x) > 0$ we have for all j

$$\begin{aligned}
F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) =& \frac{F_{ij}(x) \cdot g_{ij}(x)}{\frac{dF_{ij}}{dx}(x)} \\
= \frac{\prod_{k \neq i} G_{kj}(x) \cdot g_{ij}(x)}{\sum_{k \neq i} \left(g_{kj} \cdot \prod_{s \neq k \wedge s \neq i} G_{sj} \right)} =& \frac{g_{ij}(x)}{\sum_{k \neq i} \frac{g_{kj}(x)}{G_{kj}(x)}}.
\end{aligned}$$

By concentrating on a specific item j, let S_x be the set of bidders so that x is in their support. We next show that $|S_x| \geq 2$ for all $x \in (0, o_j - A_j]$. Recall that $A_j = \max_x \{F_{ij}(x) \cdot o_j - x\}$ for the bidder i who receives j in $\mathbf{O}$. Let $h_{ij} = \min\{x | F_{ij} = 1\}$ (we use minimum instead of infimum, since, by Lemma 6.22, F_{ij} is continuous). By definition h_{ij} should be in some bidder's support. Moreover, $A_j \geq F_{ij}(h_{ij}) \cdot o_j - h_{ij} = o_j - h_{ij}$, resulting in $o_j - A_j \leq h_{ij}$. Therefore, by Lemma 6.25, for all $x \in (0, o_j - A_j]$, x is in some bidder's support and by Lemma 6.21, there are at least 2 bidders such that x is in their supports.

By the definition of derivative, for all $i \notin S_x$, $g_{ij}(x) = 0$. Similarly, we have $g_{ij}(x) > 0$ and $G_{ij}(x) > 0$ for all $i \in S_x$ by definition 6.19. Moreover, for every $i \in S_x$, x is good for bidder i and item j, since x is in their support. So, for any fixed $x \in (0, o_j - A_j]$, $\sum_{i \in [n]} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) = \sum_{i \in S_x} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x)$, and according to Proposition 6.8,

$$\sum_{i \in [n]} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) \geq \sum_{i \in S_x} \frac{g_{ij}(x)}{\sum_{k \neq i, k \in S_x} \frac{g_{kj}}{G_{kj}}} \geq \sqrt{\prod_{i \in S_x} G_{ij}(x)} \geq \sqrt{\prod_{i \in [n]} G_{ij}(x)}.$$

Merging all these inequalities,

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{\prod_{i \in [n]} G_{ij}(x)} dx = \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx.$$

□

6.3.5 Proof of Inequality (6.4)

In this section we prove the following technical lemma.

Lemma 6.27. *For any CDF F and any real $v > 0$, $R(F, v) \geq \frac{3+4\lambda-\lambda^4}{6} v$.*

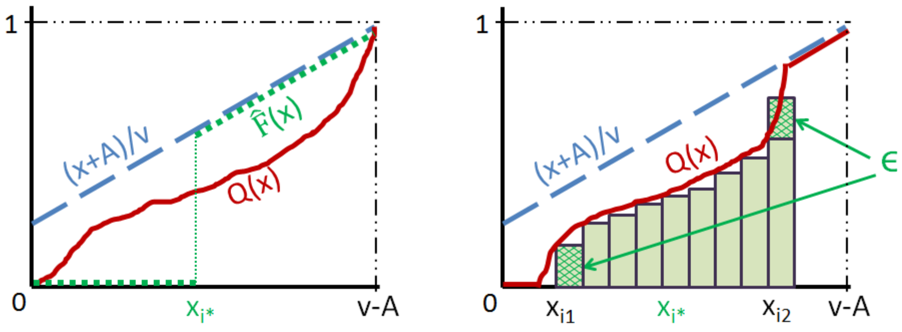
In order to obtain a lower bound for $R(F, v)$ as stated in the lemma, we show first that we can restrict attention to cumulative distribution functions of a simple special form, since these constitute worst cases for $R(F, v)$. In the next lemma, for an arbitrary CDF F we will define a simple piecewise linear function $\hat{F}$ that satisfies the following two properties.

$$\int_0^{v-A} (1 - \hat{F}(x)) dx = \int_0^{v-A} (1 - F(x)) dx \text{ and } \int_0^{v-A} \sqrt{\hat{F}(x)} dx \leq \int_0^{v-A} \sqrt{F(x)} dx.$$

Once we establish this, it will be convenient to lower bound $R(\hat{F}, v)$ for the given type of piecewise linear functions $\hat{F}$.

Lemma 6.28. *For any CDF F and real $v > 0$, there always exists another CDF $\hat{F}$ such that $R(F, v) \geq R(\hat{F}, v)$ that is defined by*

$$\hat{F}(x) = \begin{cases} 0 & \text{if } x \in [0, x_0] \\ \frac{x+A}{v} & \text{if } x \in (x_0, v - A] \end{cases}$$



(a) (b)

Figure 6.1: Figure (a) illustrates $\hat{F}(x) = \lim_{l\to\infty} \hat{F}_l(x)$ and figure (b) shows how Q' is derived from Q.

where $A = \max_{x\geq 0}\{F(x) \cdot v - x\}$.

Proof. First notice that $\max_{x\geq 0}\{\hat{F}(x) \cdot v - x\} = A$. By the definition of Riemann integration, we can represent the integration as the limit of Riemann sums. For any positive integer l, let R_l be the Riemann sum if we partition the interval $[0, v - A]$ into small intervals of size $(v - A)/l$. That is

$$R_l(F, v) = A + \frac{v - A}{l} \cdot \left(\sum_{i=0}^{l-1}(1 - F(x_i)) + \lambda \cdot \sum_{i=0}^{l-1} \sqrt{F(x_i)}\right)$$

where $x_i = \frac{i}{l} \cdot (v - A)$. So we have $R(F, v) = \lim_{l\to\infty} R_l(F, v)$.

For any given l, let i^* be the index such that $\sum_{i>i^*}(x_i + A)/v < \sum_{i=0}^{l-1} F(x_i)$ and $\sum_{i>=i^*}(x_i + A)/v \geq \sum_{i=0}^{l-1} F(x_i)$. We define $\hat{F}_l$ as follows.

$$\hat{F}_l(x) = \begin{cases} 0 & \text{if } x < x_{i^*} \\ \sum_{i=0}^{l-1} F(x_i) - \sum_{i>i^*}(x_i + A)/v & \text{if } x \in [x_{i^*}, x_{i^*+1}) \\ (x + A)/v & \text{if } x \in [x_{i^*+1}, v - A] \end{cases}$$

It is straight-forward to check that $\hat{F}(x) = \lim_{l\to\infty} \hat{F}_l(x)$, as described in the statement of the lemma. We will show that for any l, $R_l(F, v) \geq R_l(\hat{F}_l, v)$. Then the lemma follows by taking the limit, since $R_l(F, v) \to R(F, v)$, and $R_l(\hat{F}, v) \to R(\hat{F}, v)$. Figure 6.1(a) illustrates $\hat{F}(x)$ (when we take the limit of l to infinity).

By the construction of $\hat{F}_l$, it is easy to check that $\sum_{i=0}^{l-1} F(x_i) = \sum_{i=0}^{l-1} \hat{F}_l(x_i)$ and $\max_x\{\hat{F}_l(x) \cdot v - x\} = A$. Then in order to prove $R_l(F, v) \geq R_l(\hat{F}_l, v)$, it is sufficient to prove that $\sum_{i=0}^{l-1} \sqrt{F(x_i)} \geq \sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$. Let $\mathcal{Q}$ be the set of CDF functions such that $\forall Q \in \mathcal{Q}$, $\sum_{i=0}^{l-1} Q(x_i) = \sum_{i=0}^{l-1} F(x_i)$ and $A = \max_{x\geq 0}\{Q(x) \cdot v - x\}$, meaning further that $Q(x) \leq (x + A)/v$, for all $x \geq 0$. We will show that $\hat{F}_l(x)$ has the minimum value for the expression $\sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$ within $\mathcal{Q}$.

Assume on the contrary that some other function $Q \in \mathcal{Q}$ has the minimum value for $\sum_{i=0}^{l-1} \sqrt{Q(x_i)}$ within $\mathcal{Q}$ and $Q(x_j) \neq \hat{F}_l(x_j)$ for some x_j. Let i_1 be the smallest index such that $Q(x_{i_1}) > 0$ and i_2 be the largest index such that $Q(x_{i_2}) < (x_{i_2} + A)/v$. By the monotonicity of Q, we have $i_1 \leq i_2$. Due to the assumption that $Q(x_j) \neq \hat{F}_l(x_j)$ for some x_j and $\sum_{i=0}^{l-1} \sqrt{Q(x_i)} \leq \sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$, we get $i_1 \neq i_2$. So $i_1 < i_2$ and $Q(x_{i_1}) < Q(x_{i_2})$ by the monotonicity of CDF functions. Now consider another CDF function Q' such that $Q'(x_i) = Q(x_i)$ for all $i \neq i_1 \wedge i \neq i_2$, $Q'(x_{i_1}) = Q(x_{i_1}) - \epsilon$ and $Q'(x_{i_2}) = Q(x_{i_2}) + \epsilon$ where $\epsilon = \min\{Q(x_{i_1}), (x_{i_2} + A)/v - Q(x_{i_2})\}$. Figure 6.1(b) shows how we modify Q to Q'. It is easy to check $Q' \in \mathcal{Q}$ and $\sum_{i=0}^{l-1} \sqrt{Q(x_i)} > \sum_{i=0}^{l-1} \sqrt{Q'(x_i)}$ which contradicts the optimality of Q. The inequality holds because of $\sqrt{a} + \sqrt{b} > \sqrt{a-c} + \sqrt{b+c}$ for all $0 < c < a < b$, which can be proved by simple calculations. □

Now we are ready to proceed with the proof of Lemma 6.27.

Proof of Lemma 6.27. By Lemma 6.28, for any fixed $v > 0$, we only need to consider the CDF's that have the following form. For any positive A and x_0 such that $x_0 + A \leq v$,

$$F(x) = \begin{cases} 0 & \text{if } x \in [0, x_0] \\ \frac{x+A}{v} & \text{if } x \in (x_0, v-A] \end{cases}$$

Clearly, $\max_{x \geq 0}\{F(x) \cdot v - x\} = A$. Let $t = \frac{A+x_0}{v}$. Then

$$\begin{aligned} R(F,v) &= A + \int_0^{v-A} 1 - F(x)dx + \lambda \cdot \int_0^{v-A} \sqrt{F(x)}dx \\ &= v - \frac{v}{2} \cdot \left(\frac{x+A}{v}\right)^2 \Bigg|_{x0}^{v-A} + \lambda \cdot \frac{2v}{3} \cdot \left(\frac{x+A}{v}\right)^{\frac{3}{2}} \Bigg|_{x_0}^{v-A} \\ &= v - \frac{v}{2} \cdot (1 - t^2) + \lambda \cdot \frac{2v}{3} \cdot (1 - t^{\frac{3}{2}}) \\ &= v \cdot \left(\frac{1}{2}(1 + t^2) + \frac{2\lambda}{3}(1 - t^{\frac{3}{2}})\right) \end{aligned}$$

By optimizing over t, the above formula is minimized when $t = \lambda^2 \leq 1$. That is,

$$R(F,v) \geq v \cdot \left(\frac{1}{2}(1 + \lambda^4) + \frac{2\lambda}{3}(1 - \lambda^3)\right) = \frac{3 + 4\lambda - \lambda^4}{6} \cdot v$$

□

6.4 Multi-unit Auctions

In this section, we propose a randomized all-pay mechanism for the multi-unit setting, where m identical items are to be allocated to n bidders. Markakis and Telelis [61] and de Keijzer et al. [28] have studied the price of anarchy for several multi-unit auction formats. The current best upper bound obtained was 1.58 for both pure and mixed Nash equilibria.

We propose a *randomized* all-pay mechanism that induces a *unique pure* Nash equilibrium, with an improved price of pnarchy bound of 4/3. We call the mechanism Random proportional-share allocation mechanism (PSAM), as it is a randomized version of

Kelly's celebrated proportional-share allocation mechanism for divisible resources [53]. The mechanism works as follows (illustrated as Mechanism 1).

Each bidder submits a non-negative real b_i to the auctioneer. After soliciting all the bids from the bidders, the auctioneer associates a real number x_i with bidder i that is equal to $x_i = \frac{m \cdot b_i}{\sum_{i \in [n]} b_i}$. Each player pays their bid, $p_i = b_i$. In the degenerate case, where $\sum_i b_i = 0$, then $x_i = 0$ and $p_i = 0$ for all i.

We turn the x_i's to a random allocation as follows. Each bidder i secures $\lfloor x_i \rfloor$ items and gets one more item with probability $x_i - \lfloor x_i \rfloor$. An application of the Birkhoff-von Neumann decomposition theorem guarantees that given an allocation vector $(x_1, x_2, \ldots, x_n)$ with $\sum_i x_i = m$, one can always find a randomized allocation[2] with random variables $X_1, X_2, \ldots, X_n$ such that $\mathrm{E}[X_i] = x_i$ and $\Pr[\lfloor x_i \rfloor \leq X_i \leq \lceil x_i \rceil] = 1$.

We next show that the game induced by the Random PSAM when the bidders have submodular valuations is *isomorphic* to the game induced by Kelly's mechanism for a single divisible resource when bidders have piece-wise linear concave valuations. For convenience, we review the definition of isomorphism between games as appears in Monderer and Shapley [62].

Definition 6.29. [62]. Let Γ_1 and Γ_2 be games in strategic form with the same set of players $[n]$. For $k = 1, 2$, let $(A^i_k)_{i \in [n]}$ be the strategy sets in Γ_k, and let $(u^i_k)_{i \in [n]}$ be the utility functions in Γ_k. We say that Γ_1 and Γ_2 are isomorphic if there exists bijections $\phi^i : a^i_1 \to a^i_2$, $i \in [n]$ such that for every $i \in [n]$ and every $(a^1, a^2, \ldots, a^n) \in \times_{i \in [n]} A^i_1$,

$$u^i_1(a^1, a^2, \ldots, a^n) = u^i_2(\phi^1(a^1), \phi^2(a^2), \ldots, \phi^n(a^n)).$$

Theorem 6.30. *Any game induced by the Random PSAM applied to the multi-unit setting with submodular bidders is* isomorphic *to a game induced from Kelly's mechanism applied to a single divisible resource with piece-wise linear concave functions.*

[2] As an example, assume $x_1 = 2.5, x_2 = 1.6, x_3 = 1.9$. One can define a random allocation such that assignments $(3, 2, 1)$, $(3, 1, 2)$ and $(2, 2, 2)$ occur with probabilities 0.1, 0.4, and 0.5 respectively.

Mechanism 1: Random PSAM

Input: Total number of items m and all bidders' bid $b_1, b_2, \ldots, b_n$
Output: Ex-post allocations $X_1, X_2, \ldots, X_n$ and payments $p_1, p_2, \ldots, p_n$
if $\sum_{i \in [n]} b_i > 0$ **then**
  foreach *bidder* $i = 1, 2, \ldots, n$ **do**
    $x_i \leftarrow \frac{m \cdot b_i}{\sum_{i \in [n]} b_i}$;
    $p_i \leftarrow b_i$;
  Sample $\{X_i\}_{i \in [n]}$ from $\{x_i\}_{i \in [n]}$ by using Birkhoff-von Neumann decomposition theorem such that $\lfloor x_i \rfloor \leq X \leq \lceil x_i \rceil$ and the expectation of sampling X_i is x_i;
else Set $\mathbf{X} = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$;
Return X_i and p_i for all $i \in [n]$;

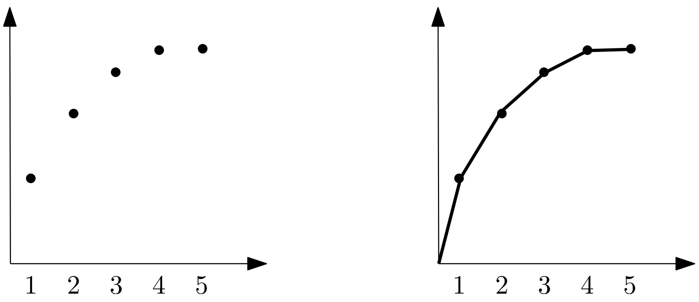


Figure 6.2: Illustration of the concave function.
The left part of the figure depicts some submodular function f, while the right part depicts the modified concave function g. One can verify that g is concave if f is submodular.

Proof. For each bidder i's submodular valuation function $f_i : \{0, 1, \ldots, m\} \to R^+$, we associate a concave function $g_i : [0, 1] \to R^+$ such that,

$$\text{for every } x \in [0, m], \qquad g_i(x/m) = f_i(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (f_i(\lfloor x \rfloor + 1) - f_i(\lfloor x \rfloor)). \quad (6.6)$$

Essentially, g_i is the piecewise linear function that comprises the line segments that connect $f_i(k)$ with $f_i(k + 1)$, for all nonnegative integers k. It is easy to see that g_i is concave if f_i is submodular (see also Figure 6.4 for an illustration).

We use identity functions as the bijections ϕ^i of Definition 6.29. Therefore, it suffices to show that, for any pure strategy profile $\mathbf{b}$, $u_i(\mathbf{b}) = u'_i(\mathbf{b})$, where u_i and u'_i are the bidder i's utility functions in the first and second game, respectively. Let $x_i = \frac{m \cdot b_i}{\sum_i b_i}$, then

$$\begin{aligned} u_i(\mathbf{b}) &= (x_i - \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor + 1) + (1 - x_i + \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor) - b_i \\ &= f_i(\lfloor x_i \rfloor) + (x_i - \lfloor x_i \rfloor)(f_i(\lfloor x_i \rfloor + 1) - f_i(\lfloor x_i \rfloor)) - b_i \\ &= g_i\left(\frac{x_i}{m}\right) - b_i = g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i = u'_i(\mathbf{b}), \end{aligned}$$

where $g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i$ is the utility of player i, under strategy profile $\mathbf{b}$, in Kelly's mechanism. □

Given submodular functions $(f_i)_i$, let $(g_i)_i$ be the associated concave functions as defined in (6.6). We can show the following equivalence between optimal welfares.

Lemma 6.31. *The optimum social welfare in the multi-unit setting, with submodular valuations $\mathbf{f} = (f_1, \ldots, f_n)$, is equal to the optimal social welfare in the divisible resource allocation with concave valuations $\mathbf{g} = (g_1, \ldots g_n)$, where $\mathbf{g}$ is derived from $\mathbf{f}$ as described in* (6.6).

Proof. For any valuation profile $\mathbf{v}$ and (randomized) allocation $\mathcal{A}$, we denote by $SW_{\mathbf{v}}(\mathcal{A})$ the social welfare of allocation $\mathcal{A}$ under the valuations $\mathbf{v}$. For any fractional allocation

$\mathbf{x} = (x_1, \ldots, x_n)$, such that $\sum_i x_i = m$, let $\mathbf{X}(\mathbf{x}) = (X_1(\mathbf{x}), \ldots, X_n(\mathbf{x}))$ be the random allocation as computed by the Random PSAM given the fractional allocation $\mathbf{x}$. Also let $\mathbf{o} = (o_1, \ldots, o_n)$ and $\mathbf{O} = (O_1, \ldots, O_n)$ be the optimal allocations in the divisible resource allocation problem and in the multi-unit auction, respectively.

First we show that $SW_{\mathbf{g}}(\mathbf{o}) \geq SW_{\mathbf{f}}(\mathbf{O})$. Consider the fractional allocation $\mathbf{o}' = (o'_1, \ldots, o'_n)$, where $o'_i = O_i/m$, for every i. Then it is easy to see that for every i, $g_i(o'_i) = f_i(\lfloor O_i \rfloor) + (O_i - \lfloor O_i \rfloor) \cdot (f_i(\lfloor O_i \rfloor + 1) - f_i(\lfloor O_i \rfloor)) = f_i(O_i)$, since O_i is an integer. Therefore, $SW_{\mathbf{g}}(\mathbf{o}) \geq SW_{\mathbf{g}}(\mathbf{o}') = SW_{\mathbf{f}}(\mathbf{O})$, by the optimality of $\mathbf{o}$.

Now we show $SW_{\mathbf{f}}(\mathbf{O}) \geq SW_{\mathbf{g}}(\mathbf{o})$. Note that for any fractional allocation $\mathbf{x}$, such that $\sum_j x_j = m$, $\mathrm{E}_{\mathbf{X}(\mathbf{x})}[f_i(X_i(\mathbf{x}))] = f_i(\lfloor x_i \rfloor) + (x_i - \lfloor x_i \rfloor) \cdot (f_i(\lfloor x_i \rfloor + 1) - f_i(\lfloor x_i \rfloor)) = g_i(x_i/m)$, for every i. By the optimality of $\mathbf{O}$, $SW_{\mathbf{f}}(\mathbf{O}) \geq \mathrm{E}_{\mathbf{X}(m \cdot \mathbf{o})}[SW_{\mathbf{f}}(\mathbf{X}(m \cdot \mathbf{o}))] = SW_{\mathbf{g}}(\mathbf{o})$. □

Theorem 6.30 and Lemma 6.31, allow us to obtain the existence and uniqueness of the pure Nash equilibrium, as well as the price of anarchy bounds of Random PSAM by the corresponing results on Kelly's mechanism for a single divisible resource [51]. Moreover, it can be shown that there are no other mixed equilibria by adopting the arguments of [13] for Kelly's mechanism. The main conclusion of this section is summarized in the following Corollary.

Corollary 6.32. *Random PSAM induces a* unique pure *Nash equilibrium when applied to the multi-unit setting with submodular bidders. Moreover, the price of anarchy of the mechanism is* exactly 4/3.

6.5 Single item auctions

In this section, we study mixed Nash equilibria in a single item all-pay auction. First, in Section 6.5.1 we measure the inefficiency of mixed Nash equilibria, showing tight results for the price of anarchy. Then in Section 6.5.2, we analyze the quality of two other important criteria, the *expected revenue (the sum of bids)* and the quality of the expected *highest submission (the maximum bid)*, which is a standard objective in crowdsourcing contests [16]. For these objectives, we show a lower bound of $v_2/2$, where v_2 is the second highest value among all bidders' valuations. In the following, we drop the word expected while referring to the revenue or to the maximum bid.

We quantify the loss of revenue and the highest submission in the worst-case equilibria. We show that the all-pay auction achieves a 2-approximation comparing to the conventional procurement (modeled as the first price auction), when considering worst-case mixed Nash equilibria; we show in Section 6.5.3 that the revenue and the maximum bid of the conventional procurement equals v_2 in the worst case. We also consider other structures of rewards allocation and conclude that allocating the entire reward to the highest bidder is the only way to guarantee the approximation factor of

2. Roughly speaking, allocating all the reward to the top prize is the optimal way to maximize the maximum bid and revenue among all the prior-free all-pay mechanisms where the designer has no prior information about the participants' skills.

Throughout this section we assume that the players are ordered based on decreasing order of their valuations, i.e. $v_1 \geq v_2 \geq \ldots \geq v_n$.

6.5.1 Social Welfare

Our analysis is based on the characterization of the Nash equilibrium with single item by [4]. En route, we also show the price of anarchy is 8/7 for auctions with two players.

Theorem 6.33. *The mixed price of anarchy of single item all-pay auction is at most* 1.185.

Proof. Based on the results of [4], inefficient Nash equilibria only exist when players' valuations are in the form $v_1 > v_2 = ... = v_k > v_{k+1} \geq ... \geq v_n$ (with $v_2 > 0$), where players $k+1$ through n bid zero with probability 1. W.l.o.g., we assume that $v_1 = 1$ and $v_i = v > 0$, for $2 \leq i \leq k$. Let P_1 be the probability that bidder 1 gets the item in any such mixed Nash equilibrium denoted by $\mathbf{B}$. Then the expected utility of bidder 1 in $\mathbf{b} \sim \mathbf{B}$ can be expressed by $\mathrm{E}[u_1(\mathbf{b})] = P_1 \cdot 1 - \mathrm{E}[b_1]$. Based on the characterization in [4], no player would bid above v in any Nash equilibrium and nobody bids exactly v with positive probability. Therefore, if player 1 deviates to v, she will gets the item with probability 1. By the definition of Nash equilibrium, we have $\mathrm{E}[u_1(\mathbf{b})] \geq \mathrm{E}[u_1(v, \mathbf{b}_{-i})] = 1 - v$, resulting in $P_1 \geq 1 - v + \mathrm{E}[b_1]$.

It has been shown in the proof of Theorem $2C$ in [4], that $\mathrm{E}[b_1]$ is minimized when players 2 through k play symmetric strategies. Following their results, we can extract the following equations (for a specific player i):

$$G_1(x) = \frac{x}{v \prod_{i' \neq 1, i} G_{i'}(x)}, \quad \forall x \in (0, v], \qquad \prod_{i' \neq 1} G_{i'}(x) = 1 - v + x, \quad \forall x \in (0, v]$$

recall that $G_{i'}(x)$ is the CDF according to which player i' bids in $\mathbf{B}$. Since players 2 through k play symmetric strategies, $G_{i'}(x)$ should be identical for $i' \neq 1$. Then, for some $i' \neq 1$,

$$G_1(x) = \frac{x}{v \cdot G_{i'}^{k-2}(x)} = \frac{x}{v \cdot (1 - v + x)^{\frac{k-2}{k-1}}}, \qquad \forall x \in (0, v]$$

Note that $1 - v + x \leq 1$, and so we get $G_1(x) \leq \frac{x}{v(1-v+x)}$ (for two players, $G_1(x) = \frac{x}{v}$) and

$$\mathrm{E}[b_1] \geq \int_0^v \left(1 - \frac{x}{v\,(1 - v + x)}\right) dx = v - 1 - \frac{(1-v)\ln(1-v)}{v}.$$

Now we can derive that $P_1 \geq \frac{1-v}{v} \ln \frac{1}{1-v}$. For two players, $\mathrm{E}[b_1] = \int_0^v (1 - x/v)\, dx = v/2$ and so $P_1 = 1 - v/2$.

The expected social welfare in $\mathbf{B}$ is $\mathrm{E}[SW(b)] \geq P_1 + (1-P_1)v \geq \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v$. The expression, $T(v) = \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v$, is minimized for $v \approx 0.5694$ and therefore, the price of anarchy is at most $T(0.5694) \approx 1.185$. Particularly, for two players, $\mathrm{E}[SW(b)] \geq 1 - v/2 + v^2/2$, which is minimized for $v = 1/2$ and therefore the price of anarchy for two players is at most $8/7$. □

Theorem 6.34. *The mixed price of anarchy of single item all-pay auction is at least* 1.185.

Proof. Consider n players, with valuations $v_1 = 1$ and $v_i = v > 0$, for $2 \leq i \leq n$. Let $\mathbf{B}$ be the Nash equilibrium, where bidders bid according to the following CDFs,

$$G_1(x) = \frac{x}{v\,(1-v+x)^{\frac{n-2}{n-1}}} \quad x \in [0,v], \quad G_i(x) = (1-v+x)^{\frac{1}{n-1}} \quad x \in [0,v], \quad i \neq 1$$

Note that $F_i(x) = \prod_{i' \neq i} G_{i'}(x)$ is the probability of bidder i getting the item when she bids x, for every bidder i.

$$F_1(x) = (1-v+x) \qquad x \in [0,v], \qquad\qquad F_i(x) = \frac{x}{v} \qquad x \in [0,v], \quad i \neq 1.$$

If player 1 bids any value $x \in [0,v]$, her utility is $u_1 = F_1(x) \cdot 1 - x = 1 - v$. Bidding greater than v is dominated by bidding v. If any player $i \neq 1$ bids any value $x \in [0,v]$, her utility is $u_i = F_i(x) \cdot v - x = 0$. Bidding greater than v results in negative utility. Hence, $\mathbf{B}$ is a Nash equilibrium. Let P_1 be the probability that bidder 1 gets the item in $\mathbf{B}$, then

$$\mathrm{E}[SW(b)] = 1 \cdot P_1 + (1-P_1)v = v + (1-v)P_1 = v + (1-v)\int_0^v G_i^{n-1}(x)dG_1(x).$$

When n goes to infinity, $\mathrm{E}[SW(b)]$ converges to $v + (1-v)\int_0^v \frac{1-v}{v(1-v+x)}dx = v + (1-v)\frac{1-v}{v} \ln \frac{1}{1-v} = \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v = T(v)$. If we set $v = 0.5694$, the price of anarchy is at least $T(v) \approx 1.185$.
For $n = 2$, $\mathrm{E}[SW(b)] = v + (1-v)\int_0^v \frac{1-v+x}{v} = v + (1-v)(1-v/2) = 1 - v/2 + v^2/2$, which for $v = 1/2$ results in price of anarchy at least $8/7$. □

6.5.2 Revenue and Maximum Bid

In this section we bound the revenue and the maximum bid of the single-item all-pay auction, for the case of mixed Nash equilibria. Specifically, the revenue and the maximum bid have value of at least $v_2/2$ and this value goes to $v_2/2$ when the number of bidders goes to infinity and v_2/v_1 approaches 0.

Theorem 6.35. *In any mixed Nash equilibrium of the single-item all-pay auction, the revenue and the maximum bid are at least half of the second highest valuation.*

Proof. Let k be any integer greater or equal to 2, such that $v_1 \geq v_2 = \ldots = v_k \geq v_{k+1} \geq \ldots \geq v_n$. Let $F(x) = \prod_i G_i(x)$ be the CDF of the maximum bid h. By the characterization of [4], in any mixed Nash equilibrium, players with valuation less than v_2 do not participate (always bid zero) and there exist two players $1, i$ bidding continuously in the interval $[0, v_2]$. Then, by [4], $F_1 = (v_1 - v_2 + x)/v_1$ and $F_i(x) = x/v_2$, for any $x \in (0, v_2]$. Therefore, we get

$$F(x) = F_i(x)G_i(x) = \frac{x}{v_2}G_i(x).$$

In the proof of Theorem 2C in [4], it is argued that $G_{i_1}(x)$ is maximized (and therefore the expected maximum bid is minimized) when all the k players play symmetrically (except for the first player, in the case that $v_1 > v_2$). So, $F(x)$ is maximized for $G_i = \left(\prod_{i' \neq 1} G_{i'}\right)^{\frac{1}{k-1}} = F_1^{\frac{1}{k-1}} = \left(\frac{v_1 - v_2 + x}{v_1}\right)^{\frac{1}{k-1}}$. Finally we get

$$\begin{aligned} E[h] &= \int_0^\infty (1 - F(x))dx \geq \int_0^{v_2} \left(1 - \frac{x}{v_2}\left(\frac{v_1 - v_2 + x}{v_1}\right)^{\frac{1}{k-1}}\right) dx \\ &\geq v_2 - \int_0^{v_2} \frac{x}{v_2} dx = \frac{1}{2} v_2. \end{aligned}$$

The same lower bound also holds for the expected revenue, which is at least as high as the expected maximum bid. This lower bound is tight for the expected maximum bid, as indicated by our analysis, when k goes to infinity and for the symmetric mixed Nash equilibrium. In the next lemma, we show that this lower bound is also tight for the expected revenue. □

Lemma 6.36. *There exists a mixed Nash equilibrium of the single-item all-pay auction, where the revenue converges to $v_2/2$ when the number of players goes to infinity and v_2/v_1 approaches* 0.

Proof. In [4], the authors provide results for the revenue in all possible equilibria. For the case that $v_1 = v_2$, the expected revenue is always equal to v_2. To show a tight lower bound, we consider the case where $v_1 > v_2$ and there exist k players with valuation v_2 playing symmetrically in the equilibrium, letting k go to infinity. For this case, based on [4], the revenue is equal to[3]

$$\sum_i \mathrm{E}[b_i] = v^2 + (1 - v)\,\mathrm{E}[b_1],$$

where, $\mathrm{E}[b_1] = \int_0^v (1 - G_1(x))\, dx$. From the proof of Theorem 6.35 we can derive that $G_1(x) = F(x)/F_1(x) = \frac{x}{v}(1 - v + x)^{\frac{1}{k-1} - 1} = \frac{x}{v}(1 - v + x)^{-1}$, when k goes to infinity.

[3]For simplicity we assume $v_1 = 1$ and $v_2 = v$.

By substituting we get,

$$\begin{aligned}
\sum_i \mathrm{E}[b_i] &= v^2 + (1-v)\int_0^v \left(1 - \frac{x}{v}(1-v+x)^{-1}\right)dx \\
&= v^2 + (1-v)\left(v - \frac{1}{v}(v + (1-v)\ln(1-v))\right) \\
&= 2v - 1 - \frac{(1-v)^2}{v}\ln(1-v) \\
&= v - (1-v)\left(1 + \frac{1-v}{v}\ln(1-v)\right).
\end{aligned}$$

By taking limits, we finally derive that $\lim_{v \to 0}\left(\frac{\sum_i \mathrm{E}[b_i]}{v}\right) = 1/2$. □

Finally, the next theorem indicates that allocating the entire reward to the highest bidder is the best choice. In particular a prior-free all-pay mechanism is presented by a probability vector $\mathbf{q} = (q_i)_{i \in [n]}$, with $\sum_{i \in [n]} q_i = 1$, where q_i is the probability that the i^{th} highest bidder is allocated the item, for every $i \leq n$.

Theorem 6.37. *For any prior-free all-pay mechanism that assigns the item to the highest bidder with probability strictly less than 1, i.e. $q_1 < 1$, there exists a valuation profile and mixed Nash equilibrium such that the revenue and the maximum bid are strictly less than $v_2/2$.*

Proof. We will assert the statement of the theorem for the valuation profile $(1, v, 0, 0, \ldots, 0)$, where $v \in (0,1)$ is the second highest value. It is safe to assume that $q_2 \in [0, q_1)$ [4]. We show that the following bidding profile is a mixed Nash equilibrium. The first two bidders bid on the interval $[0, v(q_1 - q_2)]$ and the other bidders bid 0. The CDF of bidder 1's bid is $G_1(x) = \frac{x}{v(q_1-q_2)}$ and the CDF of bidder 2's bid is $G_2(x) = x/(q_1 - q_2) + 1 - v$. It can be checked that this is a mixed Nash equilibrium by the following calculations. For every bid $x \in [0, v(q_1 - q_2)]$,

$$u_1(x) = G_2(x) \cdot q_1 + (1 - G_2(x)) \cdot q_2 - x = q_1 - v(q_1 - q_2)$$

$$u_2(x) = G_1(x) \cdot q_1 v + (1 - G_1(x)) \cdot q_2 v - x = q_2 v$$

The expected revenue is

$$\begin{aligned}
&\int_0^{v(q_1-q_2)} (1 - G_1(x))dx + \int_0^{v(q_1-q_2)} (1 - G_2(x))dx \\
=&\int_0^{v(q_1-q_2)} \left(1 - \frac{x}{v(q_1-q_2)}\right)dx + \int_0^{v(q_1-q_2)} \left(1 - \left(\frac{x}{q_1-q_2} + 1 - v\right)\right)dx \\
=&\frac{v(q_1-q_2)}{2} + \frac{v^2(q_1-q_2)}{2}
\end{aligned}$$

When v goes to 0, the revenue go to $v(q_1 - q_2)/2 < v/2$ since $q_1 - q_2 < 1$. Obviously, the same happens with the maximum bid, which is at most the same as the revenue. □

[4] Otherwise, consider the tie-breaking rule that allocates the item equiprobably. Then for $q_2 \geq q_1$, the strategy profile where all players bid zero is strictly dominant.

6.5.3 Conventional Procurement

In this section we give bounds on the expected revenue and maximum bid of the single-item first-price auction. In the following, we just write revenue and maximum bid instead of expected revenue and expected maximum bid, respectively.

Theorem 6.38. *In any mixed Nash equilibrium, the revenue and the maximum bid lie between the two highest valuations. There further exists a tie-breaking rule, such that in the worst-case, these quantities match the second highest valuation (This can also be achieved, under the no-overbidding assumption).*

Lemma 6.39. *In any mixed Nash equilibrium, if the expected utility of any player i with valuation v_i is* 0*, then with probability* 1 *the maximum bid is at least v_i.*

Proof. Consider any mixed Nash equilibrium $\mathbf{b} \sim \mathbf{B}$ and let $h = \max_i\{b_i\}$ be the highest bid; h is a random variable induced by $\mathbf{B}$. For the sake of contradiction, assume that h is *strictly* less than v_i with probability $p > 0$. Then, there exists $\varepsilon > 0$ such that $h < v_i - \varepsilon$ with probability p. Consider now the deviation of player i to pure strategy $s_i = v_i - \varepsilon$. s_i would be the maximum bid with probability p and therefore the utility of player i would be at least $p(v_i - (v_i - \varepsilon)) = p \cdot \varepsilon > 0$. This contradicts the fact that $\mathbf{B}$ is an equilibrium and completes the proof of lemma. □

Lemma 6.40. *In any mixed Nash equilibrium, if v is the highest valuation, any player with valuation strictly less than v has expected utility equal to* 0*.*

Proof. In [22] (Theorem 5.4), they proved that the price of anarchy of mixed Nash equilibria, for the single-item first-price auction, is exactly 1. This means that the player(s) with the highest valuation gets the item with probability 1. Therefore, any player with valuation strictly less than v gets the item with zero probability and hence, her expected utility is 0. □

Consider the players ordered based on their valuations so that $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$. In order to prove Theorem 6.38, we distinguish between two cases: i) $v_1 > v_2$ and ii) $v_1 = v_2$.

Lemma 6.41. *If $v_1 > v_2$, the maximum bid of any mixed Nash equilibrium, is at least v_2 and at most v_1. If we further assume no-overbidding, the maximum bid is exactly v_2.*

Proof. If $v_1 > v_2$, by Lemma 6.40, the expected utility of player 2 equals 0. From Lemma 6.39, the highest bid is at least v_2 with probability 1. Moreover, if there exist players bidding above v_1 with positive probability, then at least one of them (whoever gets the item with positive probability) would have negative utility for that bid and would prefer to deviate to 0; so, the bidding profile couldn't be an equilibrium. Therefore, the maximum bid lies between v_1 and v_2.

If we further assume no-overbidding, nobody, apart from player 1, would bid above v_2. So, the same hold for player 1, who has an incentive to bid arbitrarily close to v_2. □

Corollary 6.42. *If $v_1 > v_2$, there exists a tie breaking rule, under which the maximum bid of the worst-case mixed Nash equilibrium is exactly v_2.*

Proof. Due to Lemma 6.41, it is sufficient to show a tie breaking rule, where there exists a mixed Nash equilibrium with highest bid equal to v_2. Consider the tie-breaking rule where, in a case of a tie with player 1 (the bidder of the highest valuation), the item is always allocated to player 1. Under this tie-breaking rule, the pure strategy profile, where everybody bids v_2 is obviously a pure Nash equilibrium, with v_2 being the maximum bid. □

Lemma 6.43. *If $v_1 = v_2$, the maximum bid of any mixed Nash equilibrium, equals v_2.*

Proof. Consider a set S of $k \geq 2$ players having the same valuation $v_1 = v_2 = \ldots = v_k = v$ and the rest having a valuation strictly less than v. For any mixed Nash equilibrium $\mathbf{b} \sim \mathbf{B}$ and any player i, let G_i and F_i be the CDFs of b_i and $\max_{i' \neq i} b_{i'}$, respectively. We define $l_i = \inf\{x | G_i(x) > 0\}$ to be the infimum value of player's i support in $\mathbf{B}$. We would like to prove that $\max_i l_i = v$. For the sake of contradiction, assume that $\max_i l_i < v$ (Assumption 1).

We next prove that, under Assumption 1, $l_i = l$ for any player $i \in S$ and for some $0 \leq l < v$. We will assume that $l_j < l_i$ for some players $i, j \in S$ (Assumption 2) and we will show that Assumption 2 contradicts Assumption 1. There exists $\varepsilon > 0$ such that $l_j + \varepsilon < l_i$. Moreover, based on the definition of l_j, for any $\varepsilon' > 0$, $G_j(l_j + \varepsilon') > 0$ and so $G_j(l_j + \varepsilon) > 0$. When player's j bid is derived by the interval $[l_j, l_j + \varepsilon]$, she receives the item with zero probability, since $l_i > l_j + \varepsilon$. Therefore, for any bid of her support that is at most $l_j + \varepsilon$, her utility is zero ($G_j(l_j + \varepsilon) > 0$, so there should be such a bid). Since $\mathbf{B}$ is a mixed Nash equilibrium, her total expected utility should also be zero. In that case, Lemma 6.39 contradicts Assumption 1, and therefore Assumption 2 cannot be true (under Assumption 1). Thus, for any player $i \in S$, $l_i = l$ for some $0 \leq l < v$.

Moreover, Lemma 6.40 indicates that no player $i \notin S$ bids above l with positive probability, i.e. $G_i(l) = 1$ for all $i \notin S$. We now show that for any $i \in S$, G_i cannot have a mass point at l, i.e. $G_i(l) = 0$ for all $i \in S$.
Case 1. If $G_i(l) > 0$ for all i, then $p = \prod_i G_i(l) > 0$ is the probability that the highest bid is l, or more precisely, it is the probability that all players in S bid l and a tie occurs. Given that this event occurs, there exists a player $j \in S$ that gets the item with probability p_j strictly less than 1 (this is the conditional probability). Therefore,

player j has an incentive to deviate from l to $l+\varepsilon$, for $\varepsilon < (1-p_j)(v-l)$ (so that $p_j(v-l) < v-(l+\varepsilon)$); this contradicts the fact that $\mathbf{B}$ is an equilibrium.

Case 2. If $G_i(l) > 0$ and $G_j(l) = 0$ for some $i, j \in S$, then l is in the support of player i, but she does never receives the item when she bids l, since player j bids above l with probability 1. Therefore, the expected utility of player i is 0 and due to Lemma 6.39 this cannot happen under Assumption 1.

Overall, we have proved so far that, under Assumption 1 (that now has become $l < v$), $G_i(l) = 0$ for all $i \in S$ and $G_i(l) = 1$ for all $i \notin S$. Since $k \geq 2$, $F_i(l) = \prod_{i' \neq i} G_{i'}(l) = 0$ for all i. Consider any player $i \in S$ and let u_i be her expected utility. Based on the definition of l_i, for any $\varepsilon > 0$, there exists $x(\varepsilon) \in [l, l+\varepsilon]$, such that $x(\varepsilon)$ is in the support of player i. Therefore, $u_i \leq F_i(x(\varepsilon))(v - x(\varepsilon)) \leq F_i(l+\varepsilon)(v-l)$. As F_i is a CDF, it should be right-continuous and so for any $\delta > 0$, there exists some $\varepsilon > 0$, such that $F_i(l+\varepsilon)(v-l) < \delta$ and therefore, $u_i < \delta$. We can contradict Assumption 1, right away by using Lemma 6.39, but we give a bit more explanation. Assume that, in $\mathbf{B}$, the maximum bid h is strictly less than v with probability $p > 0$. Then, there exists some $\varepsilon' > 0$, such that $h < v - \varepsilon'$ with probability p. If we consider any $\delta < p(v-\varepsilon')$, it is straight forward to see that player i has an incentive to deviate to the pure strategy $v - \varepsilon'$. Therefore, we showed that Assumption 1 cannot hold and so the highest bid is at least v with probability 1. Similar to the proof of Lemma 6.41, nobody will bid above v in any mixed Nash equilibrium. □

6.6 Conclusion

All-pay auctions are widely used to model economic agents making irreversible investments in competitions. Specifically, both winners and losers have to pay their bids in (first-price) all-pay auctions. We study the inefficiency of mixed equilibria of all-pay auctions in three different settings — combinatorial auctions, multi-unit auctions and contests.

First, we study item-bidding combinatorial auctions where m all-pay auctions run in parallel, one for each good. We consider fractionally subadditive valuations where the current best upper bound on the price of anarchy is 2 due to [76]. We strengthen this upper bound to 1.82 by proving some structural properties that characterize the mixed Nash equilibria of the game.

Next, we design an all-pay mechanism with a randomized allocation rule for the multi-unit auction. We show that, for bidders with submodular valuations, the mechanism admits a unique, 75% efficient, pure Nash equilibrium. The efficiency of this mechanism outperforms all the known bounds on the price of anarchy of mechanisms used for multi-unit auctions.

Finally, we analyze single-item all-pay auctions motivated by their connection to contests. In a contest, the objective is to design a reward allocation rule to maximize social welfare, sum of bids (revenue) or maximum bid. For the social welfare, we show a tight bound on the price of anarchy of approximately 1.185. For the revenue and maximum bid, we show that they are at least as high as half of the second highest valuation in any mixed Nash equilibrium. In contrast, when using any reward structure other than allocating the entire reward to the highest bidder, the revenue and maximum bid in some mixed Nash equilibrium may be strictly less than half of the second highest valuation.

Chapter 7

Conclusion

In this thesis, open problems and conclusions for each chapter were discussed individually. In this final chapter, we will summarize the entire thesis and discuss relevant broad research issues.

The thesis studies the incentive issues arsing from four different real-life applications. For each of them, we provide an appropriate economic model by extending or generalizing previous results in the literature. We also systematically analyze the auction design and optimization problems in all these models by adopting economic concepts like market equilibrium, truthfulness, envy-freeness. This thesis aims at using methodology from computer science (e.g. approximation algorithms, computational complexity) to investigate the effect of strategic behaviors on algorithm design. More specifically, we generalize the sponsored search auctions to the setting for online rich media advertisement and investigate three major pricing mechanisms there. Second, we use the digital good auctions to model the fund raising problems by altering the auctioneer's objective to maximizing the probability to get a target revenue. We provide both positive and negative results for this setting and show some approximately optimal algorithms for simple auctions. Then we study the double auction setting where a trading broker wants to maximize his total revenue by buying low from the sellers and selling high to the buyers. Our results here extend the recent results for one-side auctions and make more elaborate arguments. Finally, we examine the efficiency of all pay auctions, motivated by its connection to crowdsourcing contests. We improve the previous bound by using structural characterization of the Nash equilibrium in all pay auctions.

Several open problems have been mentioned at the end of each chapter for different settings. But we would like to mention a bit more general and broader related research issues. In the chapter for rich media advertisement, we assume that the advertisement slots are aligned in a line, i.e. single-dimensional displayed. But in many websites, the display of banner advertisement is two-dimensional, i.e. besides the width of the advertisement, we can also consider its height. It would be interesting to generalize our result to this 2D setting. For the chapter for revenue target, we only study the problem for single-parameter buyers, whose valuation can be represented by a single private parameter. It would be a good idea to also look at multi-parameter versions of this problem. For double auction, one major open problem is to give a general framework to reduce the optimization problems in two-sided markets to the ones in the well-studied one-sided market. Actually, a recent work [36] by Dütting et al. has done some work in this direction by investigating the social welfare. However, for the revenue maximization problems, it is still unexplored. Finally, in the chapter for all-pay auctions, the crowdsourcing contests are always modeled as a single-item all-pay auction. It would be more challenging to consider the crowdsourcing contests as multi-item all pay auctions, i.e. participators can attend several contests simultaneously.

Bibliography

[1] Saeed Alaei. Bayesian Combinatorial Auctions: Expanding Single Buyer Mechanisms to Many Buyers. In *FOCS '11: Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*. IEEE Computer Society, October 2011.

[2] Saeed Alaei, Hu Fu, Nima Haghpanah, Jason D. Hartline, and Azarakhsh Malekian. Bayesian optimal auctions via multi- to single-agent reduction. In *EC '12: Proceedings of the 13th ACM Conference on Electronic Commerce*. ACM, June 2012.

[3] Mark Armstrong. Competition in two-sided markets. *The RAND Journal of Economics*, 37(3):668–691, 2006.

[4] Michael R. Baye, Dan Kovenock, and Casper G. de Vries. The All-Pay Auction with Complete Information. *Economic Theory*, 8(2):291–305, August 1996.

[5] Anand Bhalgat, Tanmoy Chakraborty, and Sanjeev Khanna. Mechanism design for a risk averse seller. In *WINE '12: Proceedings of the 8th International Workshop on Internet and Network Economics*, pages 198–211. Springer, 2012.

[6] Sayan Bhattacharya, Gagan Goel, Sreenivas Gollapudi, and Kamesh Munagala. Budget constrained auctions with heterogeneous items. In *STOC '10: Proceedings of the 42nd ACM symposium on Theory of computing*. ACM Request Permissions, June 2010.

[7] Kshipra Bhawalkar and Tim Roughgarden. Welfare guarantees for combinatorial auctions with item bidding. In *SODA '11: Proceedings of the Twenty-Second annual ACM-SIAM symposium on Discrete algorithms*. SIAM, January 2011.

[8] Sushil Bikhchandani. Auctions of Heterogeneous Objects. *Games and Economic Behavior*, January 1999.

[9] Yang Cai, C Daskalakis, and S M Weinberg. Optimal Multi-dimensional Mechanism Design: Reducing Revenue to Welfare Maximization. In *FOCS '12: Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science*, pages 130–139, 2012.

[10] Yang Cai, C. Daskalakis, and S.M. Weinberg. Understanding Incentives: Mechanism Design Becomes Algorithm Design. In *Proceedings of the 2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, FOCS '13, pages 618–627. IEEE, October 2013.

[11] Yang Cai, Constantinos Daskalakis, and S Matthew Weinberg. An algorithmic characterization of multi-dimensional mechanisms. In *STOC '12: Proceedings of the 44th symposium on Theory of Computing*. ACM Request Permissions, May 2012.

[12] Estelle Cantillon and Martin Pesendorfer. Combination Bidding in Multi-Unit Auctions. CEPR Discussion Papers 6083, C.E.P.R. Discussion Papers, February 2007.

[13] Ioannis Caragiannis and Alexandros A. Voudouris. Welfare Guarantees for Proportional Allocations. In Ron Lavi, editor, *Algorithmic Game Theory*, number 8768 in Lecture Notes in Computer Science, pages 206–217. Springer Berlin Heidelberg, 2014.

[14] Shuchi Chawla, Jason D. Hartline, David L Malec, and Balasubramanian Sivan. Multi-parameter mechanism design and sequential posted pricing. In *STOC '10: Proceedings of the 42nd ACM symposium on Theory of computing*. ACM Request Permissions, June 2010.

[15] Shuchi Chawla, Jason D. Hartline, David L Malec, and Balasubramanian Sivan. Prior-Independent Mechanisms for Scheduling. In *STOC '13: Proceedings of the 45th symposium on Theory of Computing*, pages 1–10, April 2013.

[16] Shuchi Chawla, Jason D. Hartline, and Balasubramanian Sivan. Optimal Crowdsourcing Contests. In *Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, pages 856–868, Kyoto, Japan, 2012. SIAM.

[17] Ning Chen, Xiaotie Deng, Paul W. Goldberg, and Jinshan Zhang. On revenue maximization with sharp multi-unit demands. *Journal of Combinatorial Optimization*, pages 1–32, December 2014.

[18] Ning Chen, Nick Gravin, and Pinyan Lu. Optimal Competitive Auctions. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, STOC '14, pages 253–262, New York, NY, USA, 2014. ACM.

[19] Xi Chen, Ilias Diakonikolas, Dimitris Paparas, Xiaorui Sun, and Mihalis Yannakakis. The Complexity of Optimal Multidimensional Pricing. In *SODA '14:*

Proceedings of the Twenty-Fifth annual ACM-SIAM symposium on Discrete algorithms, November 2013.

[20] George Chirstodoulou, Alkmini Sgouritsa, and Bo Tang. On the efficiency of all-pay mechanisms. In *ESA'15*. Springer, 2015.

[21] George Christodoulou, Annamária Kovács, and Michael Schapira. Bayesian Combinatorial Auctions. In *ICALP '08: Proceedings of the 35th international colloquium on Automata, Languages and Programming, Part I*. Springer-Verlag, July 2008.

[22] George Christodoulou, Annamária Kovács, Alkmini Sgouritsa, and Bo Tang. Tight Bounds for the Price of Anarchy of Simultaneous First Price Auctions. *arXiv:1312.2371 [cs]*, December 2013. arXiv: 1312.2371.

[23] Peter Cramton. The fcc spectrum auctions: An early assessment. *Journal of Economics & Management Strategy*, 6(3):431–495, 1997.

[24] Constantinos Daskalakis, Ilias Diakonikolas, and Rocco A. Servedio. Learning Poisson Binomial Distributions. In *Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing*, STOC '12, pages 709–728, New York, NY, USA, 2012. ACM.

[25] Constantinos Daskalakis and Christos Papadimitriou. Sparse covers for sums of indicators. *Probability Theory and Related Fields*, pages 1–27, 2014.

[26] Constantinos Daskalakis and Christos H. Papadimitriou. Approximate Nash equilibria in anonymous games. *Journal of Economic Theory*, 156:207–245, 2015.

[27] Constantinos Daskalakis and Seth Matthew Weinberg. Symmetries and optimal multi-dimensional mechanism design. In *EC '12: Proceedings of the 13th ACM Conference on Electronic Commerce*. ACM Request Permissions, June 2012.

[28] Bart de Keijzer, Evangelos Markakis, Guido Schfer, and Orestis Telelis. On the Inefficiency of Standard Multi-Unit Auctions. In *ESA'13: Proceedings of the 21st European conference on Algorithms*, March 2013.

[29] Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-Item Auctions. *Journal of Political Economy*, 94(4):863–872, August 1986.

[30] Xiaotie Deng, Paul Goldberg, Yang Sun, Bo Tang, and Jinshan Zhang. Pricing ad slots with consecutive multi-unit demand. In Berthold Vöcking, editor, *Algorithmic Game Theory*, volume 8146 of *Lecture Notes in Computer Science*, pages 255–266. Springer Berlin Heidelberg, 2013.

[31] Xiaotie Deng, Paul Goldberg, Bo Tang, and Jinshan Zhang. Revenue maximization in a bayesian double auction market. In Kun-Mao Chao, Tsan-sheng Hsu, and Der-Tsai Lee, editors, *Algorithms and Computation*, volume 7676 of *Lecture Notes in Computer Science*, pages 690–699. Springer Berlin Heidelberg, 2012.

[32] K Deshmukh, A Goldberg, Jason D. Hartline, and A. Karlin. Truthful and competitive double auctions. In *ESA'02: Proceedings of the 10th annual European conference on Algorithms*, pages 127–130. Springer, 2002.

[33] Ilias Diakonikolas, Christos Papadimitriou, George Pierrakos, and Yaron Singer. Efficiency-revenue trade-offs in auctions. In *ICALP'12: Proceedings of the 39th international colloquium conference on Automata, Languages, and Programming.* Springer-Verlag, July 2012.

[34] Dominic DiPalantino and Milan Vojnovic. Crowdsourcing and All-pay Auctions. In *Proceedings of the 10th ACM Conference on Electronic Commerce*, EC '09, pages 119–128, New York, NY, USA, 2009. ACM.

[35] Shahar Dobzinski, Hu Fu, and Robert D Kleinberg. Optimal auctions with correlated bidders are easy. In *STOC '11: Proceedings of the 43rd annual ACM symposium on Theory of computing.* ACM Request Permissions, June 2011.

[36] Paul Dütting, Tim Roughgarden, and Inbal Talgam-Cohen. Modularity and greed in double auctions. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*, EC '14, pages 241–258, New York, NY, USA, 2014. ACM.

[37] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords. *American Economic Review*, 97(1):242–259, March 2007.

[38] Richard Engelbrecht-Wiggans and Charles M. Kahn. Multi-Unit Auctions with Uniform Prices. *Economic Theory*, 12(2):227–258, September 1998.

[39] Michal Feldman, Amos Fiat, Stefano Leonardi, and Piotr Sankowski. Revenue maximizing envy-free multi-unit auctions with budgets. In *EC '12: Proceedings of the 13th ACM Conference on Electronic Commerce.* ACM Request Permissions, June 2012.

[40] Michal Feldman, Hu Fu, Nick Gravin, and Brendan Lucier. Simultaneous Auctions are (almost) Efficient. In *STOC '13: Proceedings of the 45th symposium on Theory of Computing*, September 2012.

[41] Daniel Friedman and John Rust. *The Double Auction Market: Institutions, Theories, and Evidence.* Addison-Wesley Publishing Company, January 1993.

[42] M. Garey and D. Johnson. Complexity Results for Multiprocessor Scheduling under Resource Constraints. *SIAM Journal on Computing*, 4(4):397–411, December 1975.

[43] Andrew V Goldberg, Jason D. Hartline, and Andrew Wright. Competitive auctions and digital goods. In *SODA '01: Proceedings of the twelfth annual ACM-SIAM symposium on Discrete algorithms*. Society for Industrial and Applied Mathematics, January 2001.

[44] Paul Goldberg and Bo Tang. Auction design with a revenue target. In *Algorithmic Game Theory*, Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2015.

[45] Faruk Gul and Ennio Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 1999.

[46] John C Harsanyi. Games with incomplete information played by bayesian players, i–iii: Part i. the basic model&. *Management science*, 50(12_supplement):1804–1817, 2004.

[47] Jason D. Hartline and Tim Roughgarden. Simple versus optimal mechanisms. In *EC '09: Proceedings of the 10th ACM conference on Electronic commerce*. ACM Request Permissions, July 2009.

[48] Avinatan Hassidim, Haim Kaplan, Yishay Mansour, and Noam Nisan. Non-price equilibria in markets of discrete goods. In *EC '11: Proceedings of the 12th ACM conference on Electronic commerce*. ACM, June 2011.

[49] Refael Hassin and Nimrod Megiddo. Approximation algorithms for hitting objects with straight lines. *Discrete Applied Mathematics*, 30(1):29–42, 1991.

[50] Kamal Jain and Christopher A Wilkens. eBay's Market Intermediation Problem. *arXiv*, September 2012.

[51] Ramesh Johari and John N Tsitsiklis. Efficiency Loss in a Network Resource Allocation Game. *Mathematics of Operations Research*, 29(3):407–435, August 2004.

[52] DM Johnson and AL Dulmage. On an algorithm of G birkhoff concerning doubly stochastic matrics. *Canad Math Bull*, 1960.

[53] Frank Kelly. Charging and rate control for elastic traffic. *Eur. Trans. Telecomm.*, 8(1):33–37, January 1997.

[54] Jon Kleinberg, Yuval Rabani, and Éva Tardos. Allocating Bandwidth for Bursty Connections. In *Proceedings of the Twenty-ninth Annual ACM Symposium on Theory of Computing*, STOC '97, pages 664–673, New York, NY, USA, 1997. ACM.

[55] Paul Klemperer. *The economic theory of auctions*. Edward Elgar, January 2000.

[56] Elias Koutsoupias and Christos Papadimitriou. Worst-case equilibria. In *STACS '99: Proceeding of 26th International Symposium on Theoretical Aspects of Computer Science*. Springer-Verlag, March 1999.

[57] Vijay Krishna. *Auction Theory*. Academic Press, 2002.

[58] Vijay Krishna. *Auction Theory*. Academic Press, September 2009.

[59] Sébastien Lahaie. An Analysis of Alternative Slot Auction Designs for Sponsored Search. In *Proceedings of the 7th ACM Conference on Electronic Commerce*, EC '06, pages 218–227, New York, NY, USA, 2006. ACM.

[60] Jian Li and Wen Yuan. Stochastic Combinatorial Optimization via Poisson Approximation. In *Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing*, STOC '13, pages 971–980, New York, NY, USA, 2013. ACM.

[61] Evangelos Markakis and Orestis Telelis. Uniform Price Auctions: Equilibria and Efficiency. In *SAGT*, pages 227–238, 2012.

[62] Dov Monderer and Lloyd S. Shapley. Potential Games. *Games and Economic Behavior*, 14(1):124–143, May 1996.

[63] S. Muthukrishnan. Ad Exchanges: Research Issues. In Stefano Leonardi, editor, *Internet and Network Economics*, number 5929 in Lecture Notes in Computer Science, pages 1–12. Springer Berlin Heidelberg, 2009.

[64] R B Myerson and M A Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, 29(2):265–281, 1983.

[65] Roger B Myerson. Optimal auction design. *Mathematics of Operations Research 1981*, January 1981.

[66] John F Nash et al. Equilibrium points in n-person games. *Proc. Nat. Acad. Sci. USA*, 36(1):48–49, 1950.

[67] Christos H Papadimitriou and George Pierrakos. On optimal single-item auctions. In *STOC '11: Proceedings of the 43rd annual ACM symposium on Theory of computing*. ACM Request Permissions, June 2011.

[68] J C Rochet and J Tirole. Platform competition in twosided markets. *Journal of the European Economic Association*, 1(4):990–1029, 2003.

[69] Amir Ronen. On approximating optimal auctions. In *EC '01: Proceedings of the 3rd ACM conference on Electronic Commerce*. ACM Request Permissions, October 2001.

[70] Michael H. Rothkopf, Aleksandar Pekevc, and Ronald M. Harstad. Computationally Manageable Combinational Auctions. *Management Science*, 44(8):1131–1147, August 1998.

[71] Michael Rothschild and Joseph E. Stiglitz. Increasing risk: I. A definition. *Journal of Economic Theory*, 2(3):225 – 243, 1970.

[72] T. Roughgarden. Barriers to Near-Optimal Equilibria. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 71–80, October 2014.

[73] L. S. Shapley and M. Shubik. The assignment game I: The core. *International Journal of Game Theory*, 1(1):111–130, December 1971.

[74] Ron Siegel. All-Pay Contests. *Econometrica*, 77(1):71–92, January 2009.

[75] Mukund Sundararajan and Qiqi Yan. Robust mechanisms for risk-averse sellers. In *EC '10: Proceedings of the 11th ACM conference on Electronic commerce*. ACM Request Permissions, June 2010.

[76] Vasilis Syrgkanis and Eva Tardos. Composable and Efficient Mechanisms. In *STOC '13: Proceedings of the 45th symposium on Theory of Computing*, November 2012.

[77] Hal R. Varian. Position auctions. *International Journal of Industrial Organization*, 25(6):1163 – 1178, 2007.

[78] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.

[79] Rakesh V. Vohra. *Mechanism Design: A Linear Programming Approach.* Cambridge University Press, May 2011.