Linear Robust H_∞ Stochastic Control Theory on the Insurance Premium-Reserve Processes

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Abstract

This thesis deals with the stability analysis of linear discrete-time premium-reserve (P-R) systems in a stochastic framework. Such systems are characterised by a mixture of the premium pricing process and the medium- and long- term stability in the accumulated reserve (surplus) policy, and they play a key role in the modern actuarial literature. Although the mathematical and practical analysis of P-R systems is well studied and motivated, their stability properties have not been studied thoughtfully and they are restricted in a deterministic framework.

In Engineering, during the last three decades, many useful techniques are developed in linear robust control theory. This thesis is the first attempt to use some useful tools from linear robust control theory in order to analyze the stability of these classical insurance systems.

Analytically, in this thesis, P-R systems are first formulated with structural properties such that time-varying delays, random disturbance and parameter uncertainties. Then as an extension of the previous literature, the results of stabilization and the robust H_∞-control of P-R systems are modelled in stochastic framework. Meanwhile, the risky investment impact on the P-R system stability condition is shown. In this approach, the potential effects from changes in insurer's investment strategy is discussed. Next we develop regime switching P-R systems to describe the abrupt structural changes in the economic fundamentals as well as the periodic switches in the parameters. The results for the regime switching P-R system are illustrated by means of two different approaches: markovian and arbitrary regime switching systems. Finally, we show how robust guaranteed cost control could be implemented to solve an optimal insurance problem.

In each chapter, Linear Matrix Inequality (LMI) sufficient conditions are derived to solve the proposed sub-problems and numerical examples are given to illustrate the applicability of the theoretical findings.

Contents

List of Figures

List of publication and presentation

Publication

1. Yang, L., Pantelous, A.A. and Assa, H. (2015), Robust stability, stabilization and H-infinity control for premium-reserves models in a Markovian regime switching discrete-time framework. *ASTIN Bulletin* (under revision).

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Chapter 1

Introduction

1.1 Research problem and Motivation

The premium pricing process, the medium- and long- term stability in the reserve policy under uncertainty are very challenging issues in the insurance world and particularly for the pricing of General (Non-Life) Insurance products. Additionally, under the Solvency II framework and different other national regulations, the stability and robustness of the model are parameters that have to be also considered very seriously and thoughtfully. Thus, in the insurance market in order for the actuary to be able to price the gross premium (or market premium) accurately, s/he should have a very good feeling about the financial environment where the various uncertainties are appearing in, the constraints that the insurance organization is facing from, and the stochastic nature of many other financial and social variables that can interfere in the model. Ideally, even the delays in reporting the claims and collecting the appropriate information from the client (or/and for the accident or/and event) have to be estimated in order the model to be more pragmatic and eventually realistic. Thus, the modelling of the uncertain parameters turns up to be one the most essential ones that has to be considered properly in the development of any effective premium model and reserve process.

Consequently, the premium-reserve (**P-R**) model has to consider different types of uncertainties as well as to face the impact of the external disturbances. So far, these parameters have not been implemented altogether in P-R modelling. Therefore, in this thesis, some advanced techniques from the linear robust control theory are used in order to investigate the robust stability, stabilization and H_∞ control for the P-R system, bringing in the actuarial science some very fruitful ideas and tools from engineering into a stochastic, discrete-time framework. Here, we are interested not only in examining the stability of the premium process, but also to find a premium such that the stabilization of the process does occur. In more details, the purpose of the robust stabilization is the design of a state feedback controller such as the resulting closed-loop system is robust stochastically stable for all admissible uncertainties. Meanwhile this thesis is to investigate several aspects of the problem of stability and stabilization for this premium pricing and accumulated reserve problem subject to markovian and arbitrary switching regimes.

1.2 Main objectives and contributions

- In Chapter 3, we discuss stability conditions for P-R systems with time-varying delays and outside disturbances in a stochastic framework. This work extends significantly the recent results proposed by Pantelous and Papageorgiou (2013) [51] in a deterministic framework.

- In Chapter 4, we derive robust H_∞ stabilization criteria of the P-R system by considering available risky investments. Here, we discuss the impact of the risky investment on the robust H_∞ stabilization performance.

- In Chapter 5, a linear Markovian regime switching system in discrete-time is used to model the medium- and long- term reserves and the premiums (P-R system) of an insurer. We derive sufficient conditions for stability, the stabilization and the robust H_∞-control of a P-R system and analyse the potential effects from the abrupt structural changes in the economic fundamentals as well as the insurer's strategy over a finite time period.

- In Chapter 6, we derive the result of extended stability and H_∞ controller design for arbitrary regime switching P-R systems.

- In Chapter 7, some preliminary result related to guaranteed cost control approach to determine the optimal performance of the P-R system have been introduced.

1.3 Structure of thesis

Chapter 2 introduces the reader some important relevant concepts from robust control theory. Then, the history of applying control theory is reviewed with emphasis placed on some previous results in actuarial literature, which are essential for the generation of the models in this thesis.

Chapter 3 focuses on extending this classic actuarial problem in stochastic framework. The main goal of the developments in this chapter is to provide a sufficient condition on the design of a state feedback controller such that proposed closed-loop P-R system is robust stochastically stable with parameter uncertainties and random disturbances. First, some assumptions are discussed and most of these assumptions remain same in all the chapters. Then the basic P-R system is introduced and the robust stability and H_∞ stability for the system is derived. Numerical examples are used to show the usefulness of the theorems.

Before introducing the regime-switching models in Chapter 5 and 6, Chapter 4 contains the research of P-R system with predefined risky investment. The results derived in Chapter 3 are extended in Chapter 4 in order to characterize robust H_∞ stability in the discrete-time linear stochastic P-R system with risky investment.

In Chapter 5, the markovian regime switching model is then introduced in great details. The fundamental ideas and the mathematical derivation of the regime-switching P-R system are declared, and the robust LMI stability, stabilization and H_∞ control problem are discussed. We show that in the case of discrete-time markovian switching linear P-R system there is way to analyse stability and generate a feasible controller.

Chapter 6 focuses on the research on the same problem under arbitrary switching framework. Both Chapter 5 and 6 deal with the problem of the stability analysis of switched linear P-R systems. Chapter 5 and 6 contain illustrative examples based on the solutions of sufficient LMI conditions.

Before concluding, Chapter 7 introduces a guaranteed cost control approach to determine the optimal feedback controller when multiple performance targets of the P-R model are required. In short, by a convex optimization problem based on LMI criteria we understand the problem of minimization of a cost function along the stabilization of trajectories of a P-R system.

Chapter 8 is our final chapter. In this chapter we give concluding remarks for the contribution of this thesis and point some feasible further research directions for the topic in this thesis.

1.4 Notation

Throughout this thesis, the matrices are assumed to have compatible dimensions. The superscript T stands for the matrix transposition. $diag\{\cdots\}$ stands for a block-diagonal matrix. For a symmetric matrix $P > 0$ (< 0) means P is positive (negative) definite. I represents identity matrix and 0 denotes zero matrix. $\mathbb{R}^m$ denotes the m dimensional Euclidean space. $\mathbb{N}$ is natural numbers set.

Chapter 2

Literature Review

2.1 Control Theory: A useful tool in Engineering

Let's consider a dynamic system; for example a flying aeroplane. The system is in a certain state which is changing all the time. In the case of an aeroplane the state is described by coordinates of the position of the aeroplane. The changes are caused by internal dynamics of the system and maybe by some outside impulses. The motion of an aeroplane is governed by the laws of kinetics together with disturbances caused e.g. by changes in wind speed or air pressure. The system should be steered towards a certain target. Then, we have an observation which gives some information about the present state of the system. The observation may give complete information about the state, but it may also be disturbed, delayed or otherwise incomplete. Finally, we have some tools or variables which you can control and which affect the state of the system. In an aeroplane that means different piloting measures.

Thus, the challenge which is considered in control theory is to find a rule which best satisfies the criteria chosen. These criteria often suggest that the system should not deviate too much from the target and that the correcting measures cannot be too harsh. In order to do that we must first analyse the statistical properties of the system and build a suitable model for it.

2.1.1 Stability of discrete time delay system

It's easy to notice that time delay exists in various system such as biological, ecological, economic, social, engineering systems etc. One reason is that time-delay parameters are often used to model systems, where procession and transmission of information or material are involved. For example, in economics, the central bank in a country often attempts to influence the economy by adjusting interest rates; the effect of a change in interest rates takes months to be translated into an impact on the economy. It would be much more complicated to manage and control a system with time delays (especially, the system with long delay). It has been shown that time delay is often a source of the generation of oscillation and a source of instability of control systems (Kolmanovskii and Myshkis, 1992 [38]).

Usually, stability conditions for time-delay systems can be classified in two types: delay-dependent and delay-independent stability conditions; the former includes the information on the size of the delay, while the latter does not. Generally speaking, delay independent stability conditions are simpler to apply, while delay-dependent stability conditions are less conservative especially in the case when the time delay is small. Since delay-dependent stability conditions takes into account impact of the upper bound of time delay, which makes the stability condition more relevant. In this thesis, our research is mainly based on delay-dependent stability conditions since the time delay in insurance system is relatively small.

The errors for the system will start to change only after the inherent delay times. Therefore, it is crucial to properly anticipate and understand the existence of delays and not to over-react. Otherwise, the system is very likely to become much more unstable.

At beginning, much attention is being paid to the delay-dependent stability, stabilization, and H_∞ control of linear systems with state delays. Less attention has been paid to discrete-time systems with a time-delay, because a linear discrete-time system with a constant integer time-delay can be transformed into a delay-free system by means of a state-augmentation approach (Mahmoud, 1995 [42]). However this approach is not suitable for systems with either unknown or time-varying delays. Over the past decades, several articles have appeared on this topic. For example, for small time-varying delays, the descriptor model transformation approach was employed to study the delay dependent guaranteed-cost control of uncertain discrete-time delay systems; Chen et al. (2003) [17], Bauer et al.(1993) [4], Kim (2001) [37] and Song and Kim(1998) [64].

In fact, voluminous researches have been done on time-delay systems. A great number of results for delay-dependent stability analysis on time-delay systems have been reported in the literature (see e.g. Chen and Latchman, 1995 [16]; Chu, 1997 [19]; Hui and Hu, 1997 [36]; Dugard and Verriest, 1998 [23]; Su and Chu, 1999 [65]; Hmamed, 2000 [32]; Shi et al., 2000 [61]; Niculescu, 2001 [49]; Boukas and Liu, 2001 [9]; Boukas and Liu, 2002 [10]; Fridman and Shaked, 2002 [26]; Xu et al., 2002 [74]; Xu et al., 2004 [75]; Zhou and Li, 2005 [80]; Chen et al., 2006 [15]; Shu et al., 2006 [63]; Sun et al., 2007 [67]).

Different theorems for delay-dependent stability analysis have been presented in those papers. Meanwhile, many methods have been provided for delay dependent stability for a class of linear discrete-time systems with time-varying delays. Most importantly, appropriate Lyapunov functionals have been constructed to exhibit the delay-dependent systems in those literatures, which provide us fruitful techniques to analyse the problems.

2.1.2 Robustness and robust control

Robust control is a branch of control theory whose approach to controller design explicitly deals with uncertainty. Robust control methods concern a system with admissible uncertain parameters and/or disturbances. Robustness means the systems achieve robust performance and/or stability in the presence of bounded modelling errors. Informally, we call a controller designed to be robust when it would work well for a particular set of parameter uncertainties.

2.1.3 H_∞ control

Definition 2.1. (Infinity norm)(Zhou et al. 1995 [79])

Let V be a vector space over $\mathbb{R}$ and let $\|\cdot\|$ be a norm defined on V. Then V is a normed space. Now let's consider a *Cauchy sequence* x_t in a *Banach space* V_B. Then the corresponding infinity norm is defined as

$$\|x\|_\infty \triangleq \sup\{\mid x_t \mid : t \in \mathbb{N}\}.$$

A vital problem in classical and modern control is how to treat disturbance in control systems. According to the previous numerous literature, probably H_∞ control is the most important example of a robust control technique, which was initially developed by Duncan McFarlane and Keith Glover from Cambridge University. When outside unexpected disturbances enter the system, this method guarantees that the system will not greatly deviate from expected trajectories. H_∞ control is a key approach to deal with robustness (Green and Limebeer, 1995 [29]). The standard H_∞ control problem for delay-free systems was solved in the late 1980s (Zhong, 2006 [78]). Since then, the robust control of time-delay systems has attracted many researchers.

The so-called H_∞ norm (see for instance, Francis and Khargonekar, 1995 [24] or Helton and Merino, 1998 [33]), loosely speaking, focuses on the worst possible case and it tries to minimise the maximum of a (linear) loss function of the state and control variables, for an arbitrary input. In other words, this rule attempts to minimize the loss in system when the circumstances are the worst possible. This actually is the famous min-max decision rule in the game theory.

2.1.4 Linear Matrix Inequality

Definition 2.2. (**Linear Matrix Inequality**) A linear matrix inequality (LMI) is an inequality

$$F(x) < 0, \tag{2.1.1}$$

where F is an affine function.

The LMI (2.1.1) defines a convex constraint on x. That set is

$$\vartheta := \{x \mid F(x) < 0\}, \tag{2.1.2}$$

and the solution of $F(x) < 0$ is convex.

The convex constraint $F(x) < 0$ on x may seem special, but many convex sets can be represented by LMI approach and have more attractive properties than general convex sets.

Lyapunov theory is traditionally applied to the analysis of system stability. Meanwhile, in modern control theory, LMI is a very practical and efficient tool to solve some optimization problems in control theory. Thus, different convex optimization and feasibility problems arising from Lyapunov theory can be transformed to and represented by LMIs under some well-known techniques and lemmas (eg. Schur complement, Moon inequality), see Boyd et al., 1994 [11], Xu and Lam, 2008 [72].

The other main advantage of LMI is that we can possibly get the feasible numerical results for specific problems other than the analytic results. Because nowadays several powerful programming toolboxes for LMI have been developed, the corresponding solution for LMI could be efficiently generated.

Also in Scherer and Weiland (2004) [60], some basic properties of LMIs are discussed which turn out to be very helpful to reduce multiple constraints on an unknown variable to an equivalent constraint involving a single LMI.

According to Boyd et al., 1994 [11], application of LMI in system control is first introduced when Lyapunov published his famous Lyapunov theory in about 1890. He showed the differential equation

$$\frac{d}{dt}x(t) = Ax(t), \tag{2.1.3}$$

is stable if and only if there exists a positive definite matrix P such that

$$A^T P + PA < 0. \tag{2.1.4}$$

In this basic Lyapunov theory, $P > 0$ and $A^T P + PA < 0$ are the basic form of LMI.

In 1940's Lur'e and other researchers apply Lyapunov's methods to real control engineering problems. They solve the resulting LMIs analytically by hand due to lack of consistent theory and computer algorithm. Therefore the result could only apply in basic systems at that time.

In 1960's, Positive-real lemma gives graphical techniques for solving another family of LMIs. Among these graphical techniques, the Root-Locus (see Shinners, 1964 [62]) method is first implemented in classical P-R actuarial problems by Balzer and Benjamin (1982) [3], then appears in Zimbidis and Haberman (2001) [82]. With the restriction and limitation in algorithm of graphical techniques, they present analytical stability condition for the deterministic P-R system at several critical time delay values. However, we can still benefit from the approach by Balzer and Benjamin (1982) [3], Zimbidis and Haberman (2001) [82] due to the fact that many optimization problems in insurance can be formulated (or reformulated) using LMIs.

By 1970, It is shown that a certain algebraic Riccati equation (ARE) can be used to solve the LMI appearing in the Positive-real lemma. In paper Willems 1971 [70] , Willems led to the LMI on quadratic optimal control:

$$\begin{bmatrix} A^T P + PA + Q & PB + C^T \\ B^T P + C & R \end{bmatrix} \geq 0 \tag{2.1.5}$$

From the point of view of the modern control theory, these graphical and Riccati equation methods are all analytic solutions that can be used to solve special forms of LMIs.

In 1980's, Nesterov and Nemirovskii developed interior-point methods. Then several interior point algorithms for LMI problems have been implemented and tested, see Nesterov et al. 1994, Nemirovski 2004 [48], [47]. Along with the development of programming algorithm, these new development make LMIs a much more attractive tool.

Therefore, after 1990's various approaches have been proposed to obtain delay-dependent stability conditions under the LMI approach, and LMI become popular and has played an important role. Another reason which makes LMI conditions appealing is their frequent readiness to solve the corresponding synthesis problems once the stability and/or other performance conditions have been established, especially when state feedback is employed. The recent development on the LMI techniques in deriving delay-dependent stability results for time-delay systems has been shown in the paper Xu and Lam (2008) [72].

In this thesis, LMIs arise as functions of matrix variables rather than scalar valued decision variables. As it is indicated later, LMI setting is popular since it came up, because it can be solved in an efficient, flexible and reliable way. To author's knowledge, this thesis is the first research project to implement modern LMIs techniques to insurance problem for P-R systems. This direction as we will see in the Chapter 8 has the potential to be developed further. In the conclusion chapter, some ideas for further research in insurance and finance are proposed.

2.2 Control Theory in Insurance

Control theory has originally emerged from engineering applications. For instance, the military applications during and since World War II significantly boosted its growth and popularity among engineers and mathematicians. Because many other systems have been observed to have a mathematically corresponding structure, later control theory found many applications in other areas; like communication and networked control systems, transportation, logistics, finance etc.

The development of control theory was initially based on a deterministic framework, but soon it was enlarged to build on stochastic approach. Indeed, stochastic theory is capable of explaining better why some "rule-of-thumb" control rules used in practice had been so successfully compared with the results provided by the deterministic framework. Nowadays, intensive theoretical research is carried out under the stochastic framework, although the deterministic approach has not been forgotten at all, for instance see Pantelous and Papageorgiou (2013) [51].

In spite of its popularity in many other areas, control theory has not belonged to the standard toolbox of actuarial science. In Non-Life insurance world, control theory is a fairly new area of research compared with the long history of actuarial mathematics. Probably the first actuarial publications where the control theory has been involved were the famous papers by De Finetti (1957) [22] and then Borch (1967) [8]. They propose for the classical risk theory problem a control action based on a pre-defined level of the surplus (accumulated) reserve, see Figure 2.1. Both of them suggest a premium refund whenever the surplus exceeds a certain limiting level. Under this arrangement, the premium for the t^{th} year P_t is determined by the following equation:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] + \mathbf{1}_{(R_\Pi - R_t)},$$

where $1 > \theta > 0$ is the loading factor; $\mathbb{E}\left[claims\right]$ is the expected claims of current year, R_t is the reserve value at the end of the t^{th} time period, "$R_\Pi \geq 1$" is the pre-defined limiting (barrier) level of reserve and

$$\mathbf{1}_{(R_\Pi - R_t)} = \begin{cases} R_\Pi - R_t, \text{when} \quad R_\Pi - R_t < 0 \\ 0, \text{ when} \quad R_\Pi - R_t > 0. \end{cases}$$

In De Finetti (1957) [22], a surplus process is modified by the introduction of a constant dividend barrier. In previous research the surplus of the insurer is allowed to grow infinite. However, in that paper, De Finetti tries to improve the classical ruin theory framework by introducing a reflecting barrier for the surplus. When the surplus exceeds the barrier, the excess is immediately distributed as a premium refund and the surplus process restarts from the reflecting barrier, see Figure 2.1. He considers a discrete-time

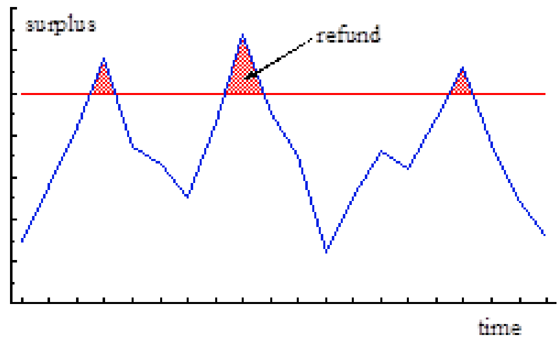

Figure 2.1: De Finetti's approach to control of surplus.

model, in which the periodic gains of a company are +1 (with probability π, $1 > \pi > \frac{1}{2}$) or -1 (with probability $1-\pi$).

The optimisation goal of De Finetti is to maximise the expected total dividend payments of the insurer before the ruin. According to the result in Gerber and Shiu 2004 [28], the ruin probability in his model is

$$(\frac{1-\pi}{\pi})^{R_\Pi - 1}\pi^{d+n}(1-\pi)^m. \tag{2.2.1}$$

d denotes the total dividends. The number of gains of +1 when the surplus at the beginning of the period is less than barrier R_Π is now n, and m is now the number of gains of -1.

In De Finetti model the insurance company has the option to pay out dividends of its surplus to its beneficiaries up to the moment of ruin such that the expected sum of the discounted paid out dividends from time zero until ruin is maximized. He proves that if the surplus process evolves as a random walk, then an optimal way of paying out dividends is according to a barrier strategy.

After De Finetti several control theoretical articles have appeared in actuarial publications. The models in these articles have employed both deterministic and stochastic techniques and have mostly been linear. Actuarial works along this line include Ryder (1977) [58], Cumpston (1978) [20], Bohman (1979) [5], Balzer and Benjamin (1980, 1982) [2, 3], Martin-Löf (1983, 1994) [43, 44], Rantala (1986, 1988) [55, 56, 57], Vandebroek-Dhaene (1990) [68], Zimbidis and Haberman (1993, 2001) [81, 82] , Hipp (1998) [35], Möller (1998) [45] and Cairns (2000) [12]. Most of these focus on studying the properties of a given control rule, though some also explore optimal solutions.

Balzer and Benjamin (1980, 1982) [2, 3], Martin-Löf (1983, 1994) [43, 44] have tried successfully to implement control theory for solving this interesting actuarial problem. They propose a smooth control action for the determination of the premium which is applied periodically and accordingly to the available information of the surplus process.

Thus, according to their research work, the proposed premium equation has finally received the following form:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] - \varepsilon R_{t-1}. \tag{2.2.2}$$

Moreover, Balzer and Benjamin (1980) [2] also discuss the effect of the delay on the stability of the system and the optimal choice for the feedback factor ε when using equation (2.2.2) with the surplus value with 1 year time delay.

Balzer and Benjamin (1982) [3] study further with 4 year time delay. In that paper, a full extension of this kind of investigation is achieved by considering the delay factor as a free parameter τ and by calculating the respective general conditions of stability and optimality for the feedback factor ε. Their result show the system becomes unstable when integer time-delay τ is great than 4. So, the premium equation (2.2.2) becomes,

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] - \varepsilon R_{t-\tau}. \tag{2.2.3}$$

Vandebroek and Dhaene (1990) [68] prove that the premium equation (2.2.3) is the optimal linear *feedback* controller for the premium pricing in the case that we require to minimize the probability of ruin along with a smooth pattern for the development of the premiums and reserves. For solving this problem, they use dynamic programming techniques.

Rantala (1986, 1988) [55, 56] applies elements of control theory for a simultaneous consideration and optimization of the premium and reserve fluctuation. He points out that a suitable control of premiums can lead to a stable and realistic development of the solvency margin.

Bonsdorff (1992) [6] investigates the solvency situation and the financial strength of an insurer within a stochastic model. He points out that the solvency situation and the financial strength of an insurer are affected by nearly all activities and decision-making processes of the insurer such as premium rating, evaluation of the accumulated reserve(surplus) of outstanding claims and investment strategy. It is also affected by external factors such as changes in the underwriting and investment markets, inflation and international economic relations.

He presents the development of the financial situation of an insurer by the basic equation:

$$U(t) = U(t-1) + B(t) + I(t) - X(t) - E(t) \tag{2.2.4}$$

where $U(t)$ is solvency margin or surplus of a company,

$B(t)$ is the premium income in year t,

$I(t)$ is the investment income in year t,

$X(t)$ is the total amount of claims in year t,

$E(t)$ is the operational expense (in a wide sense, including, among other things, dividends).

In Bonsdorff (1992) [6], the equation (2.2.4) provides a year-by-year transition for the financial position. In the equation the premiums are earned premiums. Correspondingly, claims are incurred claims. Investment income consists of cash yield and change in value of assets. All the variables in the equation (2.2.4) are stochastic. The time-delay factor is not considered in his paper.

Similarly with the previous modelling structure, Zimbidis and Haberman (2001) [82] consider a discrete-time equation to describe the development of the accumulated reserve process for an insurance system having constant time-delay and using an equation which evolve from equation (2.2.3) decision function for the determination of the premium strategy.

Their approach says that the development of the accumulated reserve R_t, at the end of each year, assuming also an accumulation factor $1+r$ and $r>0$ which is the respective rate of the investment return of the surplus reserve, is given by

$$R_{t+1} = (1+r)R_t + e(\hat{C}_{t+1} - \varepsilon R_{t-\tau}) - C_{t+1}, \tag{2.2.5}$$

where e is the parameter for the administration expenses and the desired profit margin, which can be expressed as $(1-e)$ of the respective premium.

In their paper, the classical *Root-Locus* (see Shinners, 1964 [62]) method is used for the investigation of the stability of the system and an appropriate feedback factor ε is calculated using a specific premium decision function. Due to the limitation of their method, the analysis of the stability of a P-R process was based on time-invariant parameters and constant delay factors without considering any type of uncertainty.

Recently, Pantelous and Papageorgiou (2013) [51], Pantelous and Yang (2014, 2015) [52, 53] introduce time-varying delays and uncertainties in their P-R systems under different frameworks. In their papers, the stability of the discrete-time P-R systems with norm-bounded parameter uncertainties and time-varying delay are investigated in a deterministic (Pantelous and Papageorgiou, 2013) [51] and stochastic framework (Pantelous and Yang, 2014) [52], respectively. They propose H_∞ criteria to be used for the determination of the premium control rule. Most of these papers focus on studying the properties of a given control rule, though some of them also explore feasible solutions to a specific problem employing different optimality criteria. All papers are based on discrete time approach.

Chapter 3

Robust LMI stability, stabilization and H_∞ control for Premium-Reserve systems with uncertainties

3.1 Introduction

In section 2.2 we have introduced the history of P-R system development in actuarial literature. The stability of P-R system with fixed-time delay is discussed by Zimbidis and Haberman (2001) [82]. Although the result is extended in Pantelous and Papageorgiou (2013) [51], that P-R system is still restricted in deterministic framework. In this chapter, we discuss explicit P-R systems with time-varying delay in stochastic framework. This work extends significantly the recent results proposed by Pantelous and Papageorgiou (2013) [51].

The result in this chapter is mainly based on Pantelous and Yang (2014) [52]. The primary objective is to model the P-R system and provide a useful tool to analyze its stability. This chapter is organized as follows: In order to model P-R system, some key assumptions and preliminary concepts for the model in this section are presented in section 3.2. Then section 3.3 shows how the system is modelled; This P-R process system describes how the accumulated reserve is developing and how the premium is calculated. The parameter uncertainties are assumed to be time-varying, norm-bounded and correlated. The delay is supposed to be time-varying and bounded as well. Time-varying delay enable us to describe a P-R system with uncertain delay. In Chapter 5 and 6, mode-dependant delay which is another type of time delay is introduced. Meanwhile, we concentrate on the P-R system with only risk-free investment option available in this chapter. In section 3.4 and 3.5, the robust stability and stabilization of the system are investigated under the assumption that the system's disturbance is equal to zero, closely following Pantelous and Papageorgiou (2013) [51] ideas. In section 3.6, the system's disturbance is not equal to zero. Our attention is focused on the design of a state feedback controller so that the resulting closed-loop system is robust stochastically stable with a particular disturbance attenuation level $\gamma > 0$ by using results proposed by Xu et al. (2004) [75]. In section 3.5 and 3.6 respectively, two interesting applications for a portfolio of three non-life insurance products illustrate the main findings of this chapter. Moreover, we are assuming that dependency exists among the different policies/products, as well as different uncertainties and a range of time-delays. Section 3.7 concludes this chapter.

3.2 Assumptions

Here, the basic notation and assumptions for our model are described. Similarly as in Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014) [52]. An Insurance company which runs a portfolio of m General (Non-Life) policies (or products lines) is considered, see for instance Booth et al. (1999) [7], Zaks et al. (2006) [77], Pantelous et al. (2009) [50]. The insurer calculates a fair (and as much as possible a competitive premium s/he can) annual gross premium covering the expected claims, the respective administration expenses and the rational profit margin.

Assumption 3.1: We assume that there is a binding agreement between the insurer and the insured indicating that all contracts will remain long term. This assumption is strong but necessary in our model. It prevents withdrawal of the portfolio when the premium needs to be increased due to the feedback controller effect when reserve is negative. The relaxation of this assumption is considered in future research as elements of the game theory will be taken into use. It may be possible to incorporate high penalty for breaking contract in future research.

Assumption 3.2: Let $\underline{P}_t = (P_{1,t}P_{2,t}\cdots P_{m,t})^T$ (where $(\cdot)^T$ is the transpose vector) for $t \in \mathbb{N}$ be the vector of the *premium* paid in insurance lines $1, \ldots, m$ in one time interval. Let $\underline{C_t} = (C_{1,t} C_{2,t} \cdots C_{m,t})^T$ for $t \in \mathbb{N}$ be the vector of the incurred claims which assumed to follow a stochastic process.

In Pantelous and Papageorgiou (2013) [51], the attention for the incurred claims has been restricted into the deterministic case. As an extension in Pantelous and Yang (2014) [52], the incurred claims are assumed to follow a stochastic process.

Assumption 3.3: We denote by $l_2[\Omega, \mathbb{R}^m]$ the space of square-summable $\mathbb{R}^m$-valued vector functions on the probability space $(\Omega, \mathscr{F}, \mathscr{P})$, and we also denote by $l_{e_2}(N; \mathbb{R}^m)$ the space of m-dimensional nonanticipatory square-summable stochastic processes $f(\cdot) = (f(t))_{t \in \mathbb{N}}$ on $\mathbb{N}$ with respect to a filtration $(\mathscr{F}_t)_{t \in \mathbb{N}}$ satisfying:

$$\|f\|_{e_2}^2 = \mathbb{E}\{\sum_{t=0}^{\infty} |f(t)|^2\} = \sum_{t=0}^{\infty} \mathbb{E}\{|f(t)|^2\} < \infty.$$

Here, we assume that $\underline{C_t}$ belongs to this general space, i.e. $l_{e_2}(N; \mathbb{R}^m)$ and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$. Meanwhile, $\underline{P_t}$ and $\underline{C_t}$ are adapted to the filtration $\mathscr{F}_t$.

Assumption 3.4: As described in Zimbidis and Haberman (2001) [82], Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014, 2015) [52, 53], the relationship among the administration expenses, the relative operation costs, the desired profit margin and corresponding premium can be expressed by the equation

$$\textbf{Operation Costs} + \textbf{Profit Margin} = (1 - e) P_t$$

So, the above expression is valid for any year $[t, t+1)$. A typical feature of the operation costs is that they can be estimated. For simplification, we assume that the sum of operation costs and the desired profit is constant percentage of the respective premium. Let us also recall that dividends have already been included in the concept of the desired profits. We won't consider dividends in great detail in this thesis.

Assumption 3.5: In the insurance industry, it is not realistic to assume a pre-determined delay for reporting the full set of information of the total incurred claims to the insurer, and consequently the accurate calculation of the reserves. This assumption has been extensively discussed in Pantelous and Papageorgiou (2013) [51]. Taking into account the ideas in Ackman et al. (1985) [1], it can be easily identify at least five sets of significant factors:

1. The event covered by the insurance policy may not occur at a single instant.

2. Delays may occur before a claimable event is reported to the insurer

3. It may not be possible to determine the magnitude of the claims even if the insured event is already finished.

4. The legal liability of the insurer may not always be clear-cut, and there may be considerable delays before the situation is clarified (possibly involving the court issues).

5. There may be processing delays within the insurer's administration departments.

Therefore, we consider a time-varying delay in this chapter, τ_t, which is upper and lower bounded, i.e. $\tau_{\min} \leq \tau_t \leq \tau_{\max}$ with $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. So, considering a specific time-delay interval, at the end of each year $[t, t+1]$, we have the exact information up to the end of year $t - \tau_t$. The upper and the lower bound for τ_t can be estimated using past experience and statistical data. Moreover, the national and international regulatory policy might be also applied for defining the upper bound of this interval.

Assumption 3.6: The portfolio of m individual insurance policies (or products) can be either *independent* or *dependent*. The different products are dependent when there is interaction among the different reserve accounts. If the individual insurance policies are independent, the matrices J, E and the uncertain parameters ΔE_t and ΔJ_t should be diagonal matrices. However, if dependence exists among the insurance policies we have to use some weighted matrices $J = [(rw)_{ij}]$ and $E = [(ew)_{ij}]$ and uncertain parameter $\Delta J = [\Delta(rw)_{ij}]$ and $\Delta E = [\Delta(ew)_{ij}]$ for $i, j = 1, 2, 3, ...m$ with $\sum_{j=1}^{m} w_{ij} = 1$ for every $i = 1, 2...m$

Assumption 3.7: The state of the insurer is described by one variable only, namely reserve or risk capital. Similarly controller in premiums is the only control variable.

3.3 Model formulation

3.3.1 The Reserve Process

$\underline{R}_t = (R_{1,t}R_{2,t}\cdots R_{m,t})^T$ is the vector of the accumulated reserves, where $R_{i,t}$ is the accumulated reserve of i^{th} product at time t. The accumulated reserve $\underline{R}_t$ is defined by

$$\underline{R}_{t+1} = [J + \Delta J_t]\underline{R}_t(1 + v_t) + e\underline{P}_{t+1} - \underline{C}_{t+1}, \tag{3.3.1}$$

where J is the base investment return matrix and ΔJ_t is the uncertain parameter, $J_t = J + \Delta J_t$. In this chapter, since we assume the accumulated reserve is invested in risk-free assets, ΔJ_t can be assumed to be zero in practice. Now, extending further the existing literature, we assume that v_t is a zero-mean real scalar process on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$ to model different types of financial uncertainties such as inflation, taxation policy etc.

$$\mathbb{E}\{v_t\} = 0, \quad \mathbb{E}\{v_t^2\} = \sigma. \tag{3.3.2}$$

Moreover, we assume that the investment strategy is risk-free, and we are planning to include also risky assets in a future work. Practically speaking, it is true that such short term insurance products (relative to Non-Life insurance policies) are invested predominately in standard bank accounts or/and in short-term "secure" bonds (with duration less than 6 months at the most).

Similar with J, the parameters E and Z are real constant base matrices. ΔE_t and ΔZ_t are the respective parameter uncertainties. For the purpose of the modelling process, J and E respectively could be a risk-free interest rate and a constant-base return to the policyholders. Note that $E + \Delta E_t$ should normally lay in the interval $[0, 1]$, because of restriction in feedback mechanism. Then, Z is a parameter of the control input and e in (3.3.1) is a known real scalar parameter, see Assumption 3.4; Finally, ΔJ_t , ΔE_t and ΔZ_t are unknown matrices representing time-varying parameter uncertainties, and they are assumed to be of the form:

$$[\Delta J_t \quad -e\Delta E_t \quad -e\Delta Z_t] = MF_t[N_1 \quad N_2 \quad N_3], \tag{3.3.3}$$

M, N_1, N_2, N_3 are known real constant matrices and $F_t : \mathbb{N} \to \mathbb{R}^{s\times j}$ is an unknown time-varying matrix function satisfying

$$F_t^T F_t \leq I, \quad \forall t \in \mathbb{N}, \tag{3.3.4}$$

ΔJ_t , ΔE_t and ΔZ_t are said to be admissible if both (3.3.3) and (3.3.4) hold.

'$F_t^T F_t \leq I$' actually represents a convex set. Norm-bounded uncertainty corresponds to a system which matrices range in the polytope of matrices. This means that each parameter is only known to lie in a given fix polytope of matrices described by a specific convex hull. In the norm bounded setting, parameter uncertainty is described with range of parameter values. It means that each parameter ranges between upper and lower bound values. Each parameter uncertainty is correlated through matrix M and matrix function F_t defined by (3.3.3) and (3.3.4). This type of uncertainty modelling is superior than other type of uncertain such us polytopic uncertainty. Because norm-bounded uncertainty has a flexible structure such that it can easily incorporate in system calculation.

3.3.2 The Premium Rating Rule

Pantelous and Papageorgiou (2013) have proposed a feedback mechanism for the premium rating rule to be

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau(t)}, \tag{3.3.5}$$

where $\underline{\hat{C}}$ is the 'claim estimator', which is explained in more details in the next section. E is a known real positive matrix and ΔE_t is a parameter uncertainty, which vary through time.

Compared with Pantelous and Papageorgiou (2013) [51] and Zimbidis and Haberman (2001) [82], see eq.(3.3.5), the stochastic parameter v_t can be implemented as well. Moreover, in order to be able to stabilize the P-R system, the controller U_t is introduced, thus the formula describing the calculation of the premium is now given by

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - \{[E + \Delta E_t]\underline{R}_{t-\tau_t} - [Z + \Delta Z_t]U_t\}(1 + v(t)), \tag{3.3.6}$$

where $\underline{U}_t \in \mathbb{R}^m$ is the control input that has been added in the original system. However, for simplicity, without loss of generality, the state feedback controller is considered to be $\underline{U}_t = K\underline{R}_t$, where the matrix K should be determined. In practice, we can assume Z is an identity matrix and ΔZ equals zero such that the controller is derived directly. As it becomes clearer later in this chapter, the appropriate robust stabilizing controllers for the P-R process are constructed by solving an appropriate LMI (convex optimization) problem.

3.3.3 Claim's Estimator

The claims have been incurred by the end of the accounting year. Since usually a substantial part of the incurred claims is unknown when the balance sheet is compiled, their total value has to be estimated. This estimate is for the claims incurred which is subject to a considerable degree of errors. Meanwhile, the amount of claims in one year would be cleared not until many years in the future, in some insurance lines or cases even in one decade.

The premium P_{t+1} for the $(t+1)$ year is calculated by *claim estimator* $\hat{C}_{t+1}$. As in Zimbidis and Haberman (2001) [82] $\hat{C}_{t+1}$ is determined by the inflation-weighted average of the most recent available claim experience of the f years $[C_{t-\tau_t-f}, C_{t-\tau_t-f+1}, \cdots, C_{t-\tau_t}]$ and a feedback mechanism using the past reserve value of $R_{t-\tau}$.

$$\hat{C}_{t+1} = \frac{1}{Me}[(1+j)^{f+\tau_t}C_{t-\tau_t-f} + (1+j)^{f+\tau_t-1}C_{t-\tau_t-f+1} + \cdots + (1+j)^{\tau_t}C_{t-\tau_t},$$

$$M = \sum_{k=0}^{f}(1+j)^{f+\tau_t-k}.$$

where j is the inflation rate. An inaccurate claims estimation is misleading in many ways and can have fatal consequences. For instance, an underestimation of the claims incurred can result in unprofitable premium level. Underestimation of the claims also leads to a higher probability of insolvency, which can delay corrective action by the management. In this thesis, $\underline{w}_{t+1}$ is one of disturbance to system which is caused by the error between estimated claim value and actual incurred value. It is not persistent in infinite time horizon.

$$\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}(N; \mathbb{R}^m),$$

where e has been explained in Assumption 3.4, $\underline{C}_t = (C_{1,t}C_{2,t}\cdots C_{m,t})^T$ for $t \in \mathbb{N}$ be the vector of the incurred claims which is assumed to follow a stochastic process.

3.3.4 P-R system

In this chapter, the P-R system is developed into a stochastic, discrete-time framework. Theorem in Pantelous and Papageorgiou (2013) [51] is extended and, additionally, the case that the system is affected by external disturbances $\underline{w}_{t+1}$ is also considered. In other words, the actual incurred claims are not exactly the same with the claim estimator, which makes significant difference from Pantelous and Papageorgiou (2013) [51] research work. In their paper, the next period actual claim $\underline{C}_{t+1}$ is assumed to be exactly equal to proportion of respective claim estimator $e\underline{\hat{C}}_{t+1}$. In practice, a premium which is sufficient enough to cover the expected claims and to keep stable the derived reserves is always required. Consequently, the accumulated reserve process is defined by

$$\Theta_1 : \begin{cases} \underline{R}_{t+1} = \{[J + \triangle J_t]\underline{R}_t - e[E + \triangle E_t]\underline{R}_{t-\tau_t}\}(1 + v(t)) + \underline{w}_{t+1}, \\ \underline{R}_t = \underline{\varphi}_t \ for \ t \in [-\tau_{\max}, 0], \end{cases} \tag{3.3.7}$$

where $\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}(N; \mathbb{R}^m)$. We denote the above system as Θ_1. The stochastic disturbance parameter $v(t)$ is defined by eq. (3.3.2).

The P-R system Θ_1 is the basic system without involving any state feedback controller. In Section 3.4, the stability of Θ_1 is investigated and a LMI criterion is developed. Furthermore, as it is visible later on, Θ_1 is extended by introducing a *system feedback controller*. The premium, $\underline{P}_{t+1}$, is given by the eq. (3.3.6), then the accumulated reserve at time $t+1$ follows

$$\underline{R}_{t+1} = [J+\Delta J_t]\underline{R}_t(1+v(t)) - e[E+\Delta E_t]\underline{R}_{t-\tau_t}(1+v(t)) - e[Z+\Delta Z_t]U_t(1+v(t)) + \underline{w}_{t+1}.$$

Also, substituting the control input $\underline{U}_t = K\underline{R}_t$, our new closed loop system becomes

$$\underline{R}_{t+1} = \{[J+\Delta J_t] - e[Z+\Delta Z_t]K\}\underline{R}_t(1+v(t)) - e[E+\Delta E_t]\underline{R}_{t-\tau_t}(1+v(t)) + \underline{w}_{t+1}, \tag{3.3.8}$$

with initial conditions $\underline{R}_t = \underline{\varphi_t} \ for \ t \in [-\tau_{\max}, 0]$. We denote the above system with feedback controller $\underline{U}_t$ as Λ_1.

$$\Lambda_1 : \begin{cases} \underline{R}_{t+1} = \{[J + \Delta J_t] - e[Z + \Delta Z_t]K\}\underline{R}_t(1 + v(t)) - e[E + \Delta E_t]\underline{R}_{t-\tau_t}(1 + v(t)) + \underline{w}_{t+1}, \\ \underline{R}_t = \underline{\varphi}_t \ for \ t \in [-\tau_{\max}, 0], \end{cases}$$

In Section 3.5, a method to construct a state feedback controller $\underline{U}_t$ in order to ensure the robust stability of the system is presented. Both theorems in 3.4 and 3.5 are derived under the assumption that the disturbance input $\underline{w}_{t+1} = e\hat{\underline{C}}_{t+1} - \underline{C}_{t+1} = 0$, which assume that the actual incurred claim cost in next period is equal to the value of claim estimator. However, in section 3.6, we assume that the disturbance input exists, which means that the $\underline{w}_{t+1} = e\hat{\underline{C}}_{t+1} - \underline{C}_{t+1} \neq 0$. A methodology of generating a desired feedback controller is shown in that section.

3.4 Robust stability and Stabilization of the system

In Pantelous and Papageorgiou (2013) [51], the $C_{i,t}$ was the expected incurred claims of the i^{th} product at the beginning of each period, and it has been considered to be norm-bounded scalar. In this section, a similar criterion is derived but in a stochastic framework. Thus, a sufficient condition for the robust stability of the system Θ_1 is given below with Theorem 3.1. However, before we proceed further, the following known lemma and a necessary definition are needed.

Lemma 3.1. *(Wang et al. 1992 [69]) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and F be real matrices of appropriate dimensions such that $\mathcal{D} > 0$ and $F^T F \leq I$. Then, for any scalar $\mu > 0$ such that $\mathcal{D} - \mu\mathcal{B}\mathcal{B}^T > 0$,*

$$(\mathcal{A} + \mathcal{B}\mathcal{F}\mathcal{C})^T\mathcal{D}(\mathcal{A} + \mathcal{B}\mathcal{F}\mathcal{C}) \leq \mathcal{A}^T(\mathcal{D} - \mu\mathcal{B}\mathcal{B}^T)^{-1}\mathcal{A} + \frac{1}{\mu}\mathcal{C}^T\mathcal{C}.$$

Lemma 3.2. *(Schur complement) Let matrix X be*

$$X = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix},$$

then X is negative definite if and only if C and $A - BC^{-1}B^T$ are both negative definite.

$$X < 0 \Longleftrightarrow C < 0, A - B^T C^{-1} B < 0.$$

Definition 3.1. The uncertain stochastic discrete time-delay system Θ_1 is said to be robust stochastically stable if there exists a scalar $c > 0$ such that for all admissible uncertainties

$$\mathbb{E}[\sum_{t=0}^{\infty} |\underline{R}_t|^2] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi_t}|]^2, \tag{3.4.1}$$

when $\underline{w}_{t+1} = 0$, where $\underline{R}_t$ denotes the premium reserve at time t.

Remark 3.1. This definition means that the total value of the accumulated reserve process in the system is bounded by a finite number, i.e. for any "admissible" input the reaction of $\underline{R}_t$ is also bounded in the expected value sense.

To investigate the following robust stability of the system Θ_1 with $\underline{w}_{t+1} = 0$, we introduce the following theorem.

Theorem 3.1. *The uncertain discrete-time stochastic delay system Θ_1 with $\underline{w}_{t+1} = 0$ is robust stochastically stable if there exist matrices $P > 0$, $Q > 0$ and scalar $\mu_1 > 0$, $\mu_2 > 0$ such that the following LMI holds:*

$$\begin{bmatrix} (\tau_{\max} - \tau_{\min} + 1)e^2 Q - P + \mu_1 {N_1}^T N_1 + \sigma\mu_2 {N_1}^T N_1 & \mu_1 {N_1}^T N_2 + \sigma\mu_2 {N_1}^T N_2 & J^T & \sqrt{\sigma} J^T & 0 & 0 \\ \mu_1 {N_2}^T N_1 + \sigma\mu_2 {N_2}^T N_1 & \mu_1 {N_2}^T N_2 + \sigma\mu_2 {N_2}^T N_2 - e^2 Q & -eE^T & -e\sqrt{\sigma} E^T & 0 & 0 \\ J & -eE & -P^{-1} & 0 & M & 0 \\ \sqrt{\sigma} J & -e\sqrt{\sigma} E & 0 & -P^{-1} & 0 & M \\ 0 & 0 & M^T & 0 & -\mu_1 \mathrm{I} & 0 \\ 0 & 0 & 0 & M^T & 0 & -\mu_2 \mathrm{I} \end{bmatrix} < 0. \tag{3.4.2}$$

where the intermediate variables (N_1, N_2, N_3 and σ) are defined in (3.3.2), (3.3.3) and (3.3.4).

Proof 3.1. First, we consider the system Θ_1 with $\underline{w}_{t+1} = 0$, which is

$$\underline{R}_{t+1} = \{[J + \Delta J_t]\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_t}\}(1 + v(t)). \tag{3.4.3}$$

with the initial condition $\underline{R}_t = \underline{\varphi_t} \quad for \quad t = -\tau_{\max}, -\tau_{\max}+1, \cdots, 0$. We denote matrices $\Delta A = [\Delta J_t \quad -e\Delta E_t] = MF_t[N_1 \quad N_2]$, $A = [J \quad -eE]$ and $N = [N_1 \quad N_2]$ in this proof. As in Xu et al. (2004) [75] and Pantelous and Papageorgiou (2013) [51], the following Lyapunov functional candidate for the above closed loop system is considered.

$$V_t(\underline{R}_t) = \underline{R}_t^T P \underline{R}_t + V_t^*(\underline{R}_t) + V_t^{**}(\underline{R}_t), \tag{3.4.4}$$

where

$$V_t^*(\underline{R}_t) = \sum_{i=t-\tau_t}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i, \tag{3.4.5}$$

$$V_t^{**}(\underline{R}_t) = \sum_{j=-\tau_{\min}+2}^{-\tau_{\min}+1} \sum_{i=t+j-1}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i. \tag{3.4.6}$$

Now, we should determine the difference between $\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t]$ and $V_t(\underline{R}_t)$, as this result is used in the final step of our proof.

$$\begin{aligned}
&\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) = \\
&[\underline{R}_{t+1}^T P \underline{R}_{t+1} - \underline{R}_t^T P \underline{R}_t] + [\mathbb{E}[V_{t+1}^*(\underline{R}_{t+1})|\underline{R}_t] - V_t^*(\underline{R}_t)] \\
&+[\mathbb{E}[V_{t+1}^{**}(\underline{R}_{t+1})|\underline{R}_t] - V_t^{**}(\underline{R}_t)].
\end{aligned} \tag{3.4.7}$$

Now, since the time-varying delay $\tau_{\min} \leq \tau_t \leq \tau_{\max}$, denoting $\tilde{Q} = e^2 Q$, we obtain

$$\begin{aligned}
\mathbb{E}[V^*_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V^*_t(\underline{R}_t) &= \sum_{i=t+1-\tau_{t+1}}^{t} \underline{R}_i^T \tilde{Q} \underline{R}_i - \sum_{i=t-\tau_t}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i \\
&= \underline{R}_t^T \tilde{Q} \underline{R}_t + \sum_{i=t+1-\tau_{t+1}}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i \\
&\quad - \sum_{i=t+1-\tau_t}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i - \underline{R}_{t-\tau_t}^T \tilde{Q} \underline{R}_{t-\tau_t} \\
&\leq \underline{R}_t^T \tilde{Q} \underline{R}_t + \sum_{i=t+1-\tau_t}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i + \sum_{i=t+1-\tau_{t+1}}^{t-\tau_{\min}} \underline{R}_i^T \tilde{Q} \underline{R}_i \\
&\quad - \sum_{i=t+1-\tau_t}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i - \underline{R}_{t-\tau_t}^T \tilde{Q} \underline{R}_{t-\tau_t} \\
&\leq \underline{R}_t^T \tilde{Q} \underline{R}_t - \underline{R}_{t-\tau_t}^T \tilde{Q} \underline{R}_{t-\tau_t} + \sum_{i=t+1-\tau_{\max}}^{t-\tau_1} \underline{R}_i^T \tilde{Q} \underline{R}_i.
\end{aligned} \tag{3.4.8}$$

$$\begin{aligned}
\mathbb{E}[V^{**}_{t+1}(R_{t+1})|R_t] - V^{**}_t(R_t) &= \sum_{j=-\tau_{\max}+2}^{-\tau_{\min}+1} \sum_{i=t+j}^{t} \underline{R}_i^T \tilde{Q} \underline{R}_i - \sum_{j=-\tau_{\max}+2}^{-\tau_{\min}+1} \sum_{i=t+j-1}^{t-1} \underline{R}_i^T \tilde{Q} \underline{R}_i \\
&= (\tau_{\max} - \tau_{\min}) \underline{R}_t^T \tilde{Q} \underline{R}_t - \sum_{t-\tau_{\max}+1}^{t-\tau_{\min}} \underline{R}_i^T \tilde{Q} \underline{R}_i.
\end{aligned} \tag{3.4.9}$$

After transferring into a stochastic framework, finally we get

$$\begin{aligned}
&\mathbb{E}[V^*_{t+1}(\underline{R}_t)|\underline{R}_t] - V^*_t(\underline{R}_t) + \mathbb{E}[V^{**}_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V^{**}_t(\underline{R}_t) \\
&\leq (\tau_{\max} - \tau_{\min} + 1) \underline{R}_t^T \tilde{Q} \underline{R}_t - \underline{R}_{t-\tau_t}^T \tilde{Q} \underline{R}_{t-\tau_t}.
\end{aligned} \tag{3.4.10}$$

The Schur complement formula implies

$$P^{-1} - \mu_1^{-1} M M^T > 0 \Rightarrow \frac{1}{\mu_1} M M^T - P^{-1} < 0, \tag{3.4.11}$$

and

$$P^{-1} - \mu_2^{-1} M M^T > 0 \Rightarrow \frac{1}{\mu_2} M M^T - P^{-1} < 0. \tag{3.4.12}$$

Then the LMI becomes

$$\begin{bmatrix} e^2\hat{\tau}Q - P & 0 \\ 0 & -e^2 Q \end{bmatrix} + \mu_1 N^T N + A^T[\frac{1}{\mu_1}MM^T - P^{-1}]A$$

$$+ \mu_2 \sigma N^T N + \sigma A^T[\frac{1}{\mu_2}MM^T - P^{-1}]A < \begin{bmatrix} -\delta I & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.4.13}$$

where δ is a positive scalar and $\hat{\tau} = \tau_{\max} - \tau_{\min} + 1$. Using Lemma 3.1, we can get:

$$\mu_1 N^T N + A^T[P^{-1} - \frac{1}{\mu_1}MM^T]A \geq (A + MFN)^T P(A + MFN). \tag{3.4.14}$$

$$\mu_2 N^T N + A^T[P^{-1} - \frac{1}{\mu_2}MM^T]A \geq (A + MFN)^T P(A + MFN). \tag{3.4.15}$$

In particular, from (3.4.13), (3.4.14) and (3.4.15), it follows that

$$\begin{bmatrix} e^2\hat{\tau}Q - P & 0 \\ 0 & -e^2 Q \end{bmatrix} + (A + MFN)^T P(A + MFN)$$

$$+\sigma(A + MFN)^T P(A + MFN) < \begin{bmatrix} -\delta I & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.4.16}$$

From the Lyapunov functional, we know that

$$V_t(\underline{R}_t) = \underline{R}_t^T P \underline{R}_t + \sum_{i=t-\tau_t}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i + \sum_{j=-\tau_{\max}+2}^{-\tau_{\min}+1} \sum_{i=t+j-1}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i.$$

Because $\tau_{\min} \leq \tau_t \leq \tau_{\max}$ and $\tau_{\max} - \tau_{\min} \geqslant 1$, we get

$$V_t(\underline{R}_t) \leq \underline{R}_t^T P \underline{R}_t + \sum_{i=t-\tau_{\max}}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i + \sum_{j=-\tau_{\max}+2}^{-\tau_{\min}+1} \sum_{i=t-\tau_{\max}}^{t-1} \underline{R}_i^T (e^2 Q)\underline{R}_i.$$

Then, we get $\lambda_{max}(P)|\underline{R}_t|^2 \geq \underline{R}_t^T P \underline{R}_t$ and $\lambda_{\max}(Q)|\underline{R}_t|^2 \geq \underline{R}_t^T Q \underline{R}_t$. $\lambda_{\max}(\quad)$ is the maximum eigenvalue of respective matrix. Thus,

$$V_t(\underline{R}_t) \leq \lambda_{\max}(P)|\underline{R}_t|^2 + \lambda_{\max}(\tilde{Q})\hat{\tau} \sum_{i=t-\tau_{\max}}^{t-1} |\underline{R}_i|^2. \tag{3.4.17}$$

Let $\lambda = \max[\lambda_{\max}(P), \quad \lambda_{\max}(\tilde{Q})]$. Therefore

$$V_t(\underline{R}_t) \leq \lambda |\underline{R}_t|^2 + \lambda \hat{\tau} \sum_{i=t-\tau_{\max}}^{t-1} |\underline{R}_i|^2. \tag{3.4.18}$$

With (3.4.7) and (3.4.10), it yields

$$\begin{aligned}
\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) \leq \; & [(J+\Delta J_t)\underline{R}_t - e(E+\Delta E_t)\underline{R}_{t-\tau_t}]^T P[(J+\Delta J_t)\underline{R}_t \\
& -e(E+\Delta E_t)\underline{R}_{t-\tau_t}] - \underline{R}_t^T P \underline{R}_t \\
& +(\tau_{\max} - \tau_{\min} + 1)\underline{R}_t^T \tilde{Q} \underline{R}_t - \underline{R}_{t-\tau_t}^T \tilde{Q} \underline{R}_{t-\tau_t} \\
= \; & \underline{\eta}^T(t) \Bigg(\begin{bmatrix} e^2 Q\hat{\tau} - P & 0 \\ 0 & -e^2 Q \end{bmatrix} \\
& +(A+MFN)^T P(A+MFN) \\
& +\sigma(A+MFN)^T P(A+MFN) \Bigg) \underline{\eta}(t),
\end{aligned} \tag{3.4.19}$$

where $\underline{\eta}(t) = [\underline{R}_t^T \quad \underline{R}_{t-\tau_t}^T]^T$. Hence, from (3.4.19) and (3.4.16) it is easy to deduce that

$$\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) < -\delta |\underline{R}_t|^2. \tag{3.4.20}$$

Now, summing up both sides of (3.4.20) from time 0 to time N

$$\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_0(\underline{R}_0) < -\delta \sum_{i=0}^{N} |\underline{R}_t|^2. \tag{3.4.21}$$

Then, after taking the expectation on both sides of the above equation, it follows that

$$\mathbb{E}[V_{t+1}(\underline{R}_{t+1})] - \mathbb{E}[V_0(\underline{R}_0)] < -\delta \mathbb{E}[\sum_{t=0}^{N} |\underline{R}_t|^2]. \tag{3.4.22}$$

Thus,

$$\mathbb{E}[\sum_{t=0}^{N} |\underline{R}_t|^2] \leq \frac{1}{\delta} \mathbb{E}[V_0(R_0)]. \tag{3.4.23}$$

Applying (3.4.18) at time $t = 0$ and rearranging, we have

$$V_0(\underline{R}_0) \leq \lambda \hat{\tau} \sum_{i=-\tau_{\max}}^{0} |\underline{R}_i|^2. \tag{3.4.24}$$

Therefore, after using mathematical transformation, the expectation becomes,

$$\mathbb{E}[V_0(\underline{R}_0)] \leq \lambda\hat{\tau}(\tau_{\max}+1) \sup_{-\tau_{\max}\leq t\leq 0} \mathbb{E}[|\underline{\varphi_t}|]^2. \tag{3.4.25}$$

Then, following calculations (3.4.23) and (3.4.25), we get

$$\mathbb{E}[\sum_{t=0}^{N}|\underline{R}_t|^2 \leq c \sup_{-\tau_{\max}\leq t\leq 0} \mathbb{E}[|\underline{\varphi_t}|]^2, \tag{3.4.26}$$

where $c = \frac{1}{\delta}\lambda[\hat{\tau}(\tau_{\max}+1)] > 0$. From (3.4.26), we have

$$\lim_{N\to\infty} \mathbb{E}[\sum_{t=0}^{N}|\underline{R}_t|^2 \leq c \sup_{-\tau_{\max}\leq t\leq 0} \mathbb{E}[|\underline{\varphi_t}|]^2.$$

This indicates that the uncertain discrete-time stochastic delay system Θ_1 with $\underline{w}_{t+1} = 0$ satisfies the Definition 1. Hence the Theorem 3.1 is derived. $\square$

Remark 3.2. Theorem 3.1 provides a sufficient condition for testing the robust stability of an uncertain, stochastic, discrete-time, time-delay P-R system Θ_1 with $\underline{w}_{t+1} = 0$ constructed for a portfolio of different insurance products. Obviously, now the LMI criterion is different compared with what it has been given in Pantelous and Papageorgiou (2013) [51], as P-R process model in this chapter has been extended into a stochastic framework.

Remark 3.3. The idea behind Lyapunov's stability theory is as follows: assume there exists a positive definite function with a unique minimum at the equilibrium. One can think of such a function as a generalised description of the energy of the system. If we perturb the state from its equilibrium, the energy will initially rise. If the energy of the system constantly decreases along the solution of the autonomous system, it will eventually bring the state back to the equilibrium. Such functions are called Lyapunov function.

Lyapunov functions are of great importance for establishing stability of different P-R systems in this thesis include the regime switched systems in Chapter 5 and 6. We shall note some properties and methodology in proof 3.1 are used throughout this thesis.

3.5 Robust Stabilization of the system

So far we gave the sufficient condition for the robust stability of the P-R system Θ_1 with $\underline{w}_{t+1} = 0$. In practice, it is possible that the P-R process can be unstable; however it can be stabilized eventually with the appropriate choice of the premium strategy. Thus, following the ideas by Xu et al. (2004) [75], in this part of the section, we consider a control system in such a way that a feedback controller can be generated in order to stabilize the original P-R process.

Consequently, the P-R system Λ_1 with $\underline{w}_{t+1} = 0$ is considered. The new system has an additional input controller $\underline{U}_t = K\underline{R}_t$. In order to confirm that the new closed-loop system is robust stochastically stable, the previous feedback controller is developed and discussed.

When the feedback controller is determined, the system Λ_1 with $\underline{w}_{t+1} = 0$ is said to be robust stochastically stabilizable with consideration of all admissible noises and model uncertainties. Now, we can derive the following theorem:

Theorem 3.2. *Consider the uncertain discrete stochastic time-delay system Λ_1 with $\underline{w}_{t+1} = 0$. This P-R system is robust stochastically stabilizable if there exist matrices $X > 0$, $Q > 0$, Y and scalar $p_1 > 0$, $p_2 > 0$ such that the following LMI holds,*

$$\left[\begin{array}{cccc}
-X & 0 & XJ^T - Y^T eZ^T & \sqrt{\sigma}XJ^T - \sqrt{\sigma}Y^T eZ^T \\
0 & -Q & -QeE^T & -\sqrt{\sigma}QeE^T \\
JX - eZY & -eEQ & p_1 MM^T - X & 0 \\
\sqrt{\sigma}JX - e\sqrt{\sigma}ZY & -e\sqrt{\sigma}EQ & 0 & p_2 MM^T - X \\
N_1 X + N_3 Y & N_2 Q & 0 & 0 \\
\sqrt{\sigma}N_1 X + \sqrt{\sigma}N_3 Y & \sqrt{\sigma}N_2 Q & 0 & 0 \\
\hat{\tau}X & 0 & 0 & 0
\end{array}\right.$$

$$
\left.\begin{matrix}
XN_1^T + Y^T N_3^T & \sqrt{\sigma}XN_1^T + \sqrt{\sigma}Y^T N_3^T & \hat{\tau}X \\
QN_2^T & \sqrt{\sigma}QN_2^T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-p_1\mathrm{I} & 0 & 0 \\
0 & -p_2\mathrm{I} & 0 \\
0 & 0 & -\hat{\tau}Q
\end{matrix}\right] < 0, \tag{3.5.1}
$$

where $\hat{\tau} = \tau_{\max} - \tau_{\min} + 1$*. Then, a robust stabilizing state feedback controller is given by*

$$
\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t.
$$

Remark 3.4. K is parametrised by YX^{-1} so that we can implement Schur complement to generate LMI 3.5.1. Matrix X and Y is the feasible solution of LMI 3.5.1.

Proof 3.2. Let $\hat{X} = X^{-1}$, $\hat{Q} = Q^{-1}$. Then, pre- and post-multiplying above LMI (3.5.1) by
diag$(X^{-1}, Q^{-1}, I, I, I, I)$, we obtain

$$
\left[\begin{matrix}
-\hat{X} & 0 & (J - eZYX^{-1})^T & (\sqrt{\sigma}J - e\sqrt{\sigma}ZYX^{-1})^T \\
0 & -\hat{Q} & -eE^T & -e\sqrt{\sigma}E^T \\
J - eZYX^{-1} & -eE & pMM^T - \hat{X}^{-1} & 0 \\
\sqrt{\sigma}J - e\sqrt{\sigma}ZYX^{-1} & -e\sqrt{\sigma}E & 0 & p_2MM^T - \hat{X}^{-1} \\
N_1 + N_3YX^{-1} & N_2 & 0 & 0 \\
\sqrt{\sigma}N_1 + \sqrt{\sigma}N_3YX^{-1} & \sqrt{\sigma}N_2 & 0 & 0 \\
\hat{\tau}\mathrm{I} & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
(N_1+N_3YX^{-1})^T & (\sqrt{\sigma}N_1+\sqrt{\sigma}N_3YX^{-1})^T & \hat{\tau}\mathrm{I} \\
N_2^T & \sqrt{\sigma}N_2^T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-p_1\mathrm{I} & 0 & 0 \\
0 & -p_2\mathrm{I} & 0 \\
0 & 0 & -\hat{\tau}\hat{Q}^{-1}
\end{matrix}\right] < \quad 0. \tag{3.5.2}
$$

For convenience, we denote $K = YX^{-1}$, $N_{1K} = N_1 + N_5K$ and $J_K = J - eZK$. Then, the LMI which is given by eq. (3.5.1) is equal to

$$
\begin{bmatrix}
-\hat{X} & 0 & (J_K)^T & (\sqrt{\sigma}J_K)^T & (N_{1K})^T & (\sqrt{\sigma}N_{1K})^T & \hat{\tau}\mathrm{I} \\
0 & -\hat{Q} & -eE^T & -e\sqrt{\sigma}E^T & N_2^T & \sqrt{\sigma}N_2^T & 0 \\
J_K & -eE & p_1MM^T-\hat{X}^{-1} & 0 & 0 & 0 & 0 \\
\sqrt{\sigma}J_K & -e\sqrt{\sigma}E & 0 & p_2MM^T-\hat{X}^{-1} & 0 & 0 & 0 \\
N_{1K} & N_2 & 0 & 0 & -p_1\mathrm{I} & 0 & 0 \\
\sqrt{\sigma}N_{1K} & \sqrt{\sigma}N_2 & 0 & 0 & 0 & -p_2\mathrm{I} & 0 \\
\hat{\tau}\mathrm{I} & 0 & 0 & 0 & 0 & 0 & -\hat{\tau}\hat{Q}^{-1}
\end{bmatrix} < 0. \tag{3.5.3}
$$

Then we pre- and post-multiply inequity (3.5.3) by diag$(I, I, \hat{X}, \hat{X}, I, I)$ and apply the Schur complement formula, we get

$$
\left[\begin{matrix}
\hat{\tau}\hat{Q}-\hat{X}+\hat{p_1}{N_{1K}}^TN_{1K}+\sqrt{\sigma}\hat{p_2}{N_{1K}}^TN_{1K} & \hat{p_1}{N_{1K}}^TN_2+\sqrt{\sigma}\hat{p_2}{N_{1K}}^TN_2 \\
\hat{p_1}{N_2}^TN_{1K}+\sqrt{\sigma}\hat{p_2}{N_2}^TN_{1K} & \hat{p_1}{N_2}^TN_2+\sqrt{\sigma}\hat{p_2}{N_2}^TN_2-\hat{Q} \\
\hat{X}J_K & \hat{X}(-e)E \\
\sqrt{\sigma}\hat{X}J_K & \sqrt{\sigma}\hat{X}(-e)E \\
0 & 0 \\
0 & 0
\end{matrix}\right.
$$

$$\left.\begin{matrix} (J_K)^T\hat{X} & \sqrt{\sigma}(J_K)^T\hat{X} & 0 & 0 \\ -eE^T\hat{X} & -e\sqrt{\sigma}E^T\hat{X} & 0 & 0 \\ -\hat{X} & 0 & \hat{X}M & 0 \\ 0 & -\hat{X} & 0 & \hat{X}M \\ M^T\hat{X} & 0 & -\hat{p}_1\mathrm{I} & 0 \\ 0 & M^T\hat{X} & 0 & -\hat{p}_2\mathrm{I} \end{matrix}\right] < \quad 0. \tag{3.5.4}$$

where $\hat{p} = p^{-1}$. Now, from equation (3.5.4) and LMI (3.4.2), we can apply the controller U_t into the system Λ_1 with $\underline{w}_{t+1} = 0$. Then we get the following closed-loop system:

$$\underline{R}_{t+1} = \{\{[J + \triangle J_t] - e[Z + \triangle Z_t]K\}\underline{R}_t - e[E + \triangle E_t]\underline{R}_{t-\tau_t}\}(1 + v(t)). \tag{3.5.5}$$

Therefore, with Theorem 3.1, the LMI (3.5.4) indicates that the system Λ_1 with $\underline{w}_{t+1} = 0$ is robust stochastically stable. The proof is finished, because the new system is stable with the controller which has to be designed using the LMI (3.5.1). $\square$

Remark 3.5. The results for the robust stability of the pricing process for a portfolio of different insurance products are directly applicable, since the manager can use the existing m-files of the LMI MatLab toolbox to check whether there exist such desirable matrices and scalar such that the required LMI holds. A reasonable controller can be derived by simple calculation $\underline{U}_t = YX^{-1}\underline{R}_t$. However, before using the result, the manager should estimate the specific range of different time-varying parameter uncertainties and the time-delay.

Remark 3.6. The previous theorem provides a sufficient condition for the solvability of the robust stabilization problem for uncertain discrete stochastic time-delay systems. A desired state feedback controller can be obtained by solving the derived LMI (convex optimization problem).

3.5.1 Numerical Application 1.1

In this sub-section, we present a numerical example to illustrate the effectiveness and applicability of the main results of this section and to generate the corresponding input controller.

The life insurance products are not quite suitable in this thesis, because life insurance concerns a long term investment and requires predetermined periodic payments from policyholder. Coverage period for most non-life insurance policies is normally one year.

Let's consider an insurance company which runs a portfolio with 3 general insurance products. We assume that the P-R process is given by an uncertain discrete-time stochastic time-delay system with parameter uncertainties, see (3.3.6) and (3.3.8). Thus, in this example, the input controller $\underline{U}_t = YX^{-1}\underline{R}_t$ is used.

For this portfolio, we have interaction (dependency) among the three products. Our target is to trace the long term movement of the accumulated reserve. Since we introduce the feedback controller, our long-term aim is to stabilize the movement of the accumulated reserve. Practically speaking, this means that we prefer to see accumulated reserve of the insurance company to move around fixed constant level (i.e. not permitting over-shooting). Although it can be kept at lower levels, obviously the company can still make profits as profit margin based on the Assumption 3.4 is allowed.

- First the value of the reserve accounts at $t = 0$ is given by the following matrix,

$$\underline{R}_0 = \begin{bmatrix} R_0(1) \\ R_0(2) \\ R_0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

  i.e. at time $t = 0$, we assume that the reserve account of each product is 0 pounds respectively.

- For the time delay, we assume that the time-varying delay varies between $\tau_{\min} = 1$ and $\tau_{\min} = 3$ (in years). Therefore, we have accurate information upto $-\tau_{\min} = -3$. We set

$$\underline{R}_{-3} = \begin{bmatrix} R_{-3}(1) \\ R_{-3}(2) \\ R_{-3}(3) \end{bmatrix} = \underline{R}_{-2} = \begin{bmatrix} R_{-2}(1) \\ R_{-2}(2) \\ R_{-2}(3) \end{bmatrix} = \underline{R}_{-1} = \begin{bmatrix} R_{-1}(1) \\ R_{-1}(2) \\ R_{-1}(3) \end{bmatrix} = \begin{bmatrix} 27m \\ 34m \\ 16m \end{bmatrix}.$$

- In our model, it is assumed that the insurer can invest the premium surplus into risk-free investments (T-bills) to generate additional income. Since dependencies exist, we have to use weights in the parameter matrix. We assume that the corresponding rate of income is given from the following matrix:

$$J = \begin{bmatrix} 1.030w_{11} & 1.02w_{1,2} & 1.02w_{1,3} \\ 1.05w_{2,1} & 1.040w_{2,2} & 1.02w_{2,3} \\ 1.03w_{3,1} & 1.02w_{3,2} & 1.02w_{3,3} \end{bmatrix}.$$

The value of J can be determined by return rate for each risk-free asset in market.

- We assume the weight ratios $w_{n,m}$ which demonstrates the solvency relation between each product have the following values:

  $w_{1,1} = 0.85$, $w_{1,2} = 0.1$ and $w_{1,3} = 0.05$,

  $w_{2,1} = 0.2$, $w_{2,2} = 0.7$ and $w_{2,3} = 0.1$,

  $w_{3,1} = 0.1$, $w_{3,2} = 0.2$ and $w_{3,3} = 0.7$.

- The parameter E comes from the negative mechanism proposed by Balzer and Benjamin (1980, 1982) [2, 3]. With the indication in Zimbidis and Haberman (2001) [82] and Pantelous and Papageorgiou (2013) [51], the value of E could be the constant base return rate of policyholder rather than issuer.

  For the examples, we assume the value in the parameter matrix E

$$E = \begin{bmatrix} 0.005 * w_{1,1} & 0.006 * w_{1,2} & 0.006 * w_{1,3} \\ 0.004 * w_{2,1} & 0.005 * w_{2,2} & 0.006 * w_{2,3} \\ 0.004 * w_{3,1} & 0.005 * w_{3,2} & 0.006 * w_{3,3} \end{bmatrix},$$

- For parameter e, we let $e = 0.8$, which means $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the administration and operating cost and give company a reasonable profit margin.

- For parameter σ, we let $\sigma = 0.09$, $\sqrt{\sigma} = 0.3$.

- The time-varying unknown parameter uncertainty ΔJ_t , ΔE_t and ΔZ_t have been defined by equation

$$[\Delta J_t \quad -e\Delta E_t \quad -e\Delta Z_t] = MF_t[N_1 \quad N_2 \quad N_3],$$

where

$$M = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.02 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix},\ N_2 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix},\ N_3 = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

In order to get the desirable positive definite matrices X, Q, and Y and parameters p_1 and p_2 such that the LMI criterion is satisfied, we use the already known functions of the Matlab LMI Control toolbox for solving this problem. Then, we can obtain the feasible solution which is given by

$$X = \begin{bmatrix} 1.6024 & -0.7453 & -1.0951 \\ -0.7453 & 1.1437 & -0.1648 \\ -1.0951 & -0.1648 & 1.5895 \end{bmatrix} \times 10^7 \text{ with eigenvalue } \begin{bmatrix} 0.0854 \\ 1.4445 \\ 2.8057 \end{bmatrix} \times 10^7,$$

$$Q = \begin{bmatrix} 2.1645 & -1.1286 & -1.2698 \\ -1.1286 & 2.1266 & -0.1774 \\ -1.2698 & -0.1774 & 1.5113 \end{bmatrix} \times 10^{-7} \text{ with eigenvalue } \begin{bmatrix} 0.1389 \\ 2.0348 \\ 3.6286 \end{bmatrix} \times 10^8,$$

$$Y = \begin{bmatrix} 1.2217 & -0.2364 & -1.0212 \\ -0.0777 & 0.4792 & -0.3010 \\ -0.8879 & -0.2346 & 1.2454 \end{bmatrix} \times 10^7$$

and $p_1 = 7.6283 \times 10^8$, $p_2 = 4.1725 \times 10^8$.

In this case, feasible solution p_1 and p_2 are positive numbers, and matrices X and Q are positive definite. Thus, the pre-defined conditions are satisfied. This means that we can try to generate a proper input controller to stabilize the pricing-reserve system. So the desired feedback controller is given by

$$\underline{U}_t = \begin{bmatrix} 1.3315 & 0.7112 & 0.3486 \\ 0.2918 & 0.6201 & 0.0760 \\ -0.5708 & -0.5287 & 0.3354 \end{bmatrix} \underline{R}_t. \tag{3.5.6}$$

Using Theorem 3.2, by substituting the feedback controller $\underline{U}_t$ into the P-R system

Λ_1 ((3.3.6) and (3.3.8)) with $\underline{w}_t = 0$, the accumulated reserve process can be stabilized. As we can see in the Figure 3.1 and 3.2, the simulation lasts from $t = 0$ to $t = 52$ weeks (i.e. one insurance year). The Figure 3.1 shows the movement of the accumulated reserve for each dependent product. As we can see clearly the accumulated reserve for each product converges to 0 after a certain time-period (no overshoots appear). This indicates that the system always stays in a stable state with the impact of the input controller on the P-R system Λ_1 with $\underline{w}_t = 0$. In Figure 3.2, the movement of P-R system Λ_1 for each dependent product is provided. In this case, by introducing the state feedback controller (3.5.6), we can manipulate the stability of the system Λ_1 with $\underline{w}_t = 0$.

In the next section we would like to consider a more complicated system with disturbance $\underline{w}_t \neq 0$. Thus, the robust stochastic stability of the system is presented.

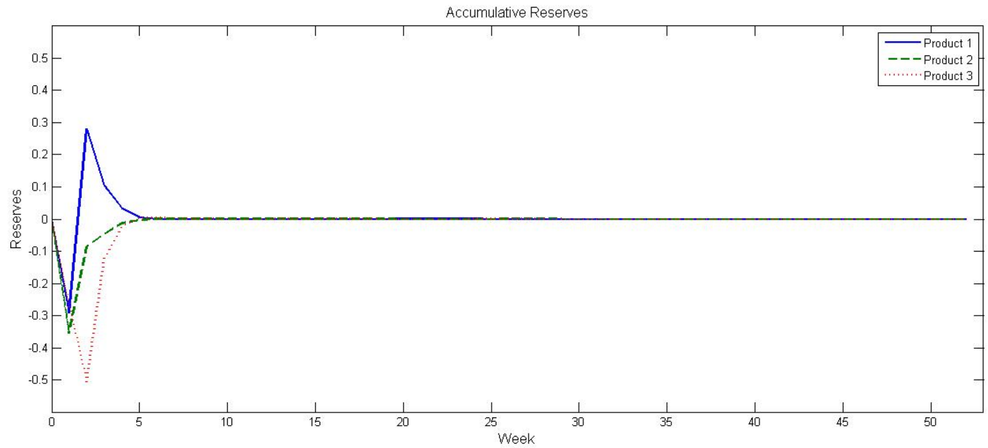

Figure 3.1: The movement of the accumulated reserve for each dependent product.

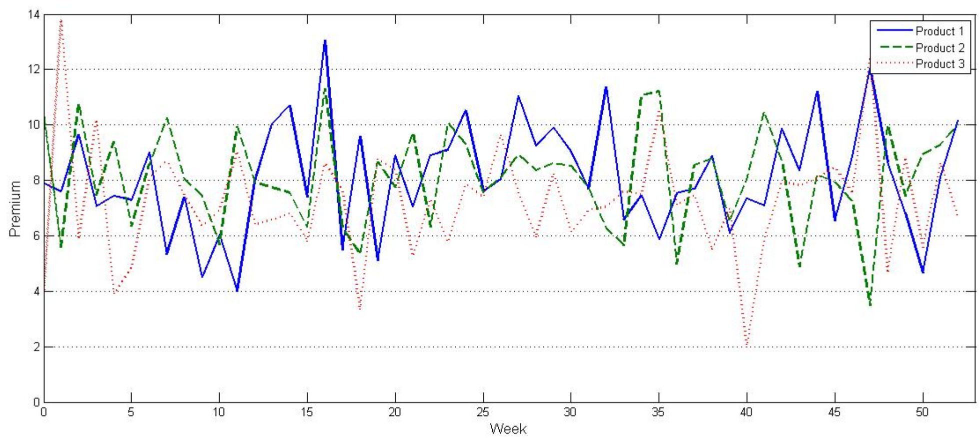

Figure 3.2: The premium process for each dependent product.

3.6 Robust H_∞ control

In section 3.4 and 3.5, the disturbance of the P-R system is assumed to be zero. For the very first time according to the author's knowledge, the disturbance is assumed to be non-zero, i.e. $\underline{w}_t \neq 0$, in system Θ_1 and Λ_1. In order to be able to investigate such kinds of systems, robust H_∞ control is implemented, see Xu et al. (2004) [75]. Here, the state feedback controller $\underline{U}_t = K\underline{R}_t$ is determined such that the resulting closed-loop system is robust stochastically stable with disturbance attenuation level γ which is a given constant performance level. The disturbance attenuation γ is a parameter which measures the accumulated impact of the outside disturbance on the system output. In the insurance industry, we can consider γ as a parameter which measures the influence of the disturbance in the market for the accumulated reserve. Before we present the necessary LMI theorem, we would like to introduce a very useful definition for what it follows.

Definition 3.2. The uncertain stochastic discrete time-delay system system Θ_1 is said to be robust stochastically stable with disturbance attenuation level γ if it is robust stable and (3.6.1) is satisfied,

$$||\underline{L}_t||_{e_2} \leq \gamma ||\underline{w}_t||_{e_2}, \tag{3.6.1}$$

for all nonzero $\underline{w}_t \in l_{e_2}(N);\mathbb{R}^m)$, and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$, where $\gamma > 0$ is a given scalar and $\underline{L}_t = C\underline{R}_t$ is the control output. Details for the control output is discussed in remark 5.1.

From Definition 3.2, one sees that such a P-R system maps finite-energy disturbance $\underline{w}_t$ into the corresponding finite energy output signal $\underline{L}_t$ of the considered P-R system.

3.6.1 Robust H_∞ stability of system Θ_1

First, we consider the system Θ_1. It should be emphasized now that the $\underline{w}_{t+1} \neq 0$.

$$\underline{R}_{t+1} = \{[J + \triangle J_t]\underline{R}_t - e[E + \triangle E_t]\underline{R}_{t-\tau(t)}\}(1 + v(t)) + \underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi_t} \; for \; t \in [-\tau_{\max}, 0], \tag{Θ_1}$$

Theorem 3.3. *Given a constant scalar $\gamma > 0$, the uncertain discrete stochastic time-delay system Θ_1 is robust stochastically stable with disturbance level γ if there exist matrices $P > 0, Q > 0$ and scalar $\mu_1 > 0$, $\mu_2 > 0$ such that the following LMI holds:*

$$\left[\begin{array}{cc}
(\tau_{\max} - \tau_{\min} + 1)e^2Q - P + \mu_1 {N_1}^T N_1 + \sigma\mu_2 {N_1}^T N_1 + C^T C & \mu_1 {N_1}^T N_2 + \sigma\mu_2 {N_1}^T N_2 \\
\mu_1 {N_2}^T N_1 + \sigma\mu_2 {N_2}^T N_1 & \mu_1 {N_2}^T N_2 + \sigma\mu_2 {N_2}^T N_2 - e^2 Q \\
0 & 0 \\
PJ & -ePE \\
\sqrt{\sigma}PJ & -e\sqrt{\sigma}PE \\
0 & 0 \\
0 & 0
\end{array} \right.$$

$$
\left.\begin{array}{ccccc}
0 & J^TP & \sqrt{\sigma}J^TP & 0 & 0 \\
0 & -eE^TP & -e\sqrt{\sigma}E^TP & 0 & 0 \\
\gamma^2\mathrm{I} & \mathrm{I} & 0 & 0 & 0 \\
\mathrm{I} & -P & 0 & PM & 0 \\
0 & 0 & -P & 0 & PM \\
0 & M^TP & 0 & -\mu_1\mathrm{I} & 0 \\
0 & 0 & M^TP & 0 & -\mu_2\mathrm{I}
\end{array}\right] < \quad 0. \tag{3.6.2}
$$

Proof 3.3. From Definition 3.2, two conditions should be satisfied in order the uncertain stochastic system to be robust stochastically stable with disturbance attenuation level γ. One is that the system should be robust stochastically stable. The other condition is given by (3.6.1)

$$
||\underline{L}_t||_{e_2} \leq \gamma||\underline{w}_t||_{e_2},
$$

for all $\underline{w}_t \in l_{e_2}(N);\mathbb{R}^m)$, and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$. First, it is easy to deduce LMI (3.6.2) into the following LMI

$$
\left[\begin{array}{cc}
(\tau_{\max}-\tau_{\min}+1)e^2Q-P+\mu_1{N_1}^TN_1+\sigma\mu_2{N_1}^TN_1 & \mu_1{N_1}^TN_2+\sigma\mu_2{N_1}^TN_2 \\
\mu_1{N_2}^TN_1+\sigma\mu_2{N_2}^TN_1 & \mu_1{N_2}^TN_2+\sigma\mu_2{N_2}^TN_2-e^2Q \\
J & -eE \\
\sqrt{\sigma}J & -e\sqrt{\sigma}E \\
0 & 0 \\
0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
J^T & \sqrt{\sigma}J^T & 0 & 0 \\
-eE^T & -e\sqrt{\sigma}E^T & 0 & 0 \\
-P^{-1} & 0 & M & 0 \\
0 & -P^{-1} & 0 & M \\
M^T & 0 & -\mu_1\mathrm{I} & 0 \\
0 & M^T & 0 & -\mu_2\mathrm{I}
\end{array}\right] < \quad 0.
$$

According to Theorem 3.1, we can conclude that this system is robust stochastically stable. With the next step, our aim is to show that $||\underline{L}_t||_{e2} \leq \gamma||\underline{w}_t||_{e2}$ holds for all nonzero $\underline{w}_t$. To prove this, we define

$$T_N = \mathbb{E}\{\sum_{t=0}^{N}(|\underline{L}_t|^2 - \gamma^2|\underline{w}_t|^2)\}, \tag{3.6.3}$$

where scalar $N > 0$ is an integer. The proof is similar to the derivation of proof in Theorem 3.1, as we can show

$$\begin{aligned}
T_N &= \mathbb{E}\{\sum_{t=0}^{N}(|\underline{L}_t|^2 + \mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) - \gamma^2|\underline{w}_t|^2)\} - \mathbb{E}[V_{N+1}(\underline{R}_{N+1})] \\
&\leq \mathbb{E}\{\sum_{t=0}^{N}(|\underline{L}_t|^2 + \mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) - \gamma^2|\underline{w}_t|^2)\} \\
&\leq \mathbb{E}\{\sum_{t=0}^{N}\underline{\eta}_t^T \underline{\phi_t}\underline{\eta}_t\},
\end{aligned} \tag{3.6.4}$$

where $V_t(R_t)$ is defined by Lyapunov candidate function which has been shown in proof of Theorem 3.1 and

$$\underline{\eta}_t = [\underline{R}_t^T \quad \underline{R}_{t-\tau_t}^T \quad \underline{w}_t^T]^T,$$

$$\underline{\phi_t} = \tilde{\Omega} + [\tilde{A} + \Delta\tilde{A}_t]^T P[\tilde{A} + \Delta\tilde{A}_t] + \sigma[\tilde{A} + \Delta\tilde{A}_t]^T P[\tilde{A} + \Delta\tilde{A}_t]$$

with

$$\tilde{\Omega} = \begin{bmatrix} e^2\hat{\tau}Q - P + C^TC & 0 & 0 \\ 0 & -e^2Q & 0 \\ 0 & 0 & -\gamma^2\mathrm{I} \end{bmatrix}, \quad \tilde{A} = [J \quad -eE \quad I], \quad \Delta\tilde{A} = [\Delta J \quad -e\Delta E \quad 0].$$

The Schur complement formula implies eq. (3.4.11) and (3.4.12), and

$$\tilde{\Omega} + \mu_1 N^T N - \tilde{A}^T[\frac{1}{\mu_1}MM^T - P^{-1}]\tilde{A} + \sigma\mu_2 N^T N - \sigma\tilde{A}^T[\frac{1}{\mu_2}MM^T - P^{-1}]\tilde{A} < 0.$$

With the notation $N = [N_1 \quad N_2 \quad 0]$ and $\Delta\tilde{A} = [\Delta J_t \quad -e\Delta E_t \quad 0] = MF_t[N_1 \quad N_2 \quad 0]$

and by using Lemma 3.1, we can get:

$$\mu_1 N^T N + \tilde{A}^T[P^{-1} - \frac{1}{\mu_1} M M^T]\tilde{A} \geq (\tilde{A} + \Delta\tilde{A})^T P(\tilde{A} + \Delta\tilde{A}).$$

and

$$\mu_2 N^T N + \tilde{A}^T[P^{-1} - \frac{1}{\mu_2} M M^T]\tilde{A} \geq (\tilde{A} + \Delta\tilde{A})^T P(\tilde{A} + \Delta\tilde{A}).$$

From these and $\tilde{\Omega} + \mu_1 N^T N - \tilde{A}^T[\frac{1}{\mu_1} M M^T - P^{-1}]\tilde{A} + \sigma\mu_2 N^T N - \sigma\tilde{A}^T[\frac{1}{\mu_2} M M^T - P^{-1}]\tilde{A} < 0$, it is easy to see that $\underline{\phi_t} < 0$, which implies $T_N < 0$.

Therefore, the inequality $||\underline{L}_t||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}$ holds for all $\underline{w}_t$. This completes our proof, as the robust H_∞ control condition for the P-R system has been proven. $\square$

3.6.2 Feedback controller of system Λ_1

Now, the system Λ_1 is considered taking a state feedback controller such that the resulting closed-loop system to be robust stochastically stabilizable with disturbance attenuation level γ.

$$\underline{R}_{t+1} = \{[J + \Delta J_t] - e[Z + \Delta Z_t]K\}\underline{R}_t(1 + v(t)) - e[E + \Delta E_t]\underline{R}_{t-\tau_t}(1 + v(t)) + \underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi_t} \; for \; t \in [-\tau_{\max}, 0], \tag{Λ_1}$$

Theorem 3.4. *This system Λ_1 is robust stochastically stabilizable with disturbance attenuation γ if there exist matrices $X > 0, Q > 0$ and a scalar $p_1 > 0$, $p_2 > 0$ such that the following matrix inequality holds:*

$$\left[\begin{array}{ccccc}
-X & 0 & 0 & XJ^T - Y^T eZ^T & \sqrt{\sigma}XJ^T - \sqrt{\sigma}Y^T eZ^T \\
0 & -Q & 0 & -QeE^T & -\sqrt{\sigma}QeE^T \\
0 & 0 & -\gamma^2 \mathrm{I} & \mathrm{I}^T & 0 \\
JX - eZY & -eEQ & \mathrm{I} & p_1 M M^T - X & 0 \\
\sqrt{\sigma}JX - e\sqrt{\sigma}ZY & -e\sqrt{\sigma}EQ & 0 & 0 & p_2 M M^T - X \\
N_1 X + N_3 Y & N_2 Q & 0 & 0 & 0 \\
\sqrt{\sigma}N_1 X + \sqrt{\sigma}N_3 Y & \sqrt{\sigma}N_2 Q & 0 & 0 & 0 \\
\hat{\tau}X & 0 & 0 & 0 & 0 \\
CX & 0 & 0 & 0 & 0
\end{array}\right.$$

$$
\begin{array}{cccc}
XN_1^T + Y^TN_3^T & \sqrt{\sigma}XN_1^T + \sqrt{\sigma}Y^TN_3^T & \hat{\tau}X & XC^T \\
QN_2^T & \sqrt{\sigma}QN_2^T & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-p_1\mathrm{I} & 0 & 0 & 0 \\
0 & -p_2\mathrm{I} & 0 & 0 \\
0 & 0 & -\hat{\tau}Q & 0 \\
0 & 0 & 0 & -\mathrm{I}
\end{array}
\Bigg] < 0, \tag{3.6.5}
$$

where $\hat{\tau} = \tau_{\max} - \tau_{\min} + 1$, I *is identity matrix.*

Proof 3.4. The proof of Theorem 3.4 is similar with the proof of Theorem 3.2, so it is omitted.

3.6.3 Numerical Example 1.2

In this sub-section, we extend the example that has been presented previously to show how the robust H_∞ technique can be used in the P-R system process. Thus, the portfolio we simulate is the same with the portfolio assumed in Example 1. However, we should give values to some new parameters involved.

- For parameter γ, mathematical meaning has been shown in Definition 3.2. In practice, we can regards γ as a parameter which measures how much the controller can resist the impact caused by a big disturbance in market. In this example, we set $\gamma = 1.7$ is the given value (but not optimal).
- For the parameter matrix of control output C, we assume $C = [0.2 \quad 0.2 \quad 0.3]$.

In order to get the desirable positive definite matrices X, Q, and Y and parameters p_1 and p_2 such that the LMI criteria is satisfied, we use the already known functions of the Matlab LMI Control toolbox for solving this problem. Then, we can obtain the solution which is given by

$$
X = \begin{bmatrix} 1.5463 & -1.1364 & -0.2748 \\ -1.1364 & 1.8209 & -0.4556 \\ -0.2748 & -0.4556 & 0.4885 \end{bmatrix} \times 10^4 \text{ with eigenvalue } \begin{bmatrix} 0.0006 \\ 1.0157 \\ 2.8395 \end{bmatrix} \times 10^4,
$$

$$Q = \begin{bmatrix} 1.0427 & -0.7083 & -0.4652 \\ -0.7083 & 0.6984 & 0.1850 \\ -0.4652 & 0.1850 & 0.3100 \end{bmatrix} \times 10^6 \text{ with eigenvalue } \begin{bmatrix} 0.0107 \\ 0.2805 \\ 1.7598 \end{bmatrix} \times 10^6,$$

$$Y = \begin{bmatrix} 9.1061 & -5.5834 & -2.3602 \\ -0.4507 & 6.8092 & -4.2375 \\ -3.3289 & 0.0003 & 2.2334 \end{bmatrix} \times 10^3$$

and $p_1 = 8.5682 \times 10^5$ and $p_2 = 8.7581 \times 10^5$.

In this case, since all conditions are satisfied we can generate the the input controller by $\underline{U}_t = YX^{-1}\underline{R}_t$. Thus, the desired state feedback controller is given by

$$\underline{U}_t = \begin{bmatrix} 0.5147 & -0.0442 & -0.2349 \\ 0.6752 & 0.8783 & 0.3315 \\ 0.2399 & 0.3886 & 0.9545 \end{bmatrix} \underline{R}_t.$$

The results are provided for the time-period of $t = 52$ weeks in simulation, and the Figures 3.3, 3.4 and 3.5 derive. Figure 3.3 and 3.5 respectively, show the movement of the accumulated reserve for each dependent product and for whole portfolio. In Figure 3.4, the movement of the charged premium is presented. From those figures, we can clearly see that the controlled reserve for each dependent product fluctuates around 0 after the first week. Obviously, the reason that the reserve is not exactly converge into 0, see also example 1, is related to the fact that new random disturbances affect the system. As we can also observe the state feedback controller $\underline{U}_t$ help to reduce the impact of the disturbance and eventually stabilizes the system quickly. In Figure 3.4, we can see that the premium moves stochastically around a constant level (no drift is observed for any of the available products). Moreover, the premium for each dependent product stays positive for the whole duration of the simulations. Compared this simulation with the example in 3.5.1, we can see the disturbance $\underline{w}_t$ affect significantly the trajectory of accumulated reserves. The accumulated reserves doesn't converge to a fixed level within finite period. However, the state feedback controller $\underline{U}_t$ ensure the fluctuation of the accumulated reserves are bounded with a certain level γ and stable.

In this case, by using the robust H_∞ tool to generate the state feedback controller $\underline{U}_t$, we can manipulate the stability of the system even though the system disturbance $\underline{w}_t \neq 0$.

To calculated the most suitable feasible solution to complex LMI 3.6.5, we use the feasp algorithm in LMI toolbox in Matlab, see Gahinet et al. (1995) [27]). With proper setting, this toolbox will directly give us the feasible solution when it does exists feasible solution.

3.7 Summary

In this chapter we propose a P-R system model for different insurance dependent products. This model considers a negative feedback mechanism for the reserves, it invests the surplus in short-term risk-free assets, and it assumes time-varying, bounded delays for the accumulated reserves into a stochastic, discrete-time framework considering also a set of different norm-bounded parameter uncertainties for the coefficients involved in the model. Thus, the new model extends significantly the one that has been proposed recently by Pantelous and Papageorgiou (2013) [51]. In [51], the P-R system is deterministic and don't consider the impact of disturbance on P-R system.

Moreover, a control parameter is introduced into the system Λ_1 and we present some new ideas to generate an effective state feedback controller for the P-R system stabilizing the unstable nominal system by using a LMI criterion. However, all the derived results are achieved under the assumption that $\underline{w}_t = 0$. Finally, in section 3.6, we assume the disturbance to be non-zero and for the very first time according to the our knowledge, the robust H_∞ control for the reserve process is investigated. With the input controller, the premium is adjusted to reasonable level. Both robust stochastic stability and a pre-specified disturbance attenuation level can be guaranteed for all admissible uncertainties. Corresponding results have been illustrated by introducing two numerical examples.

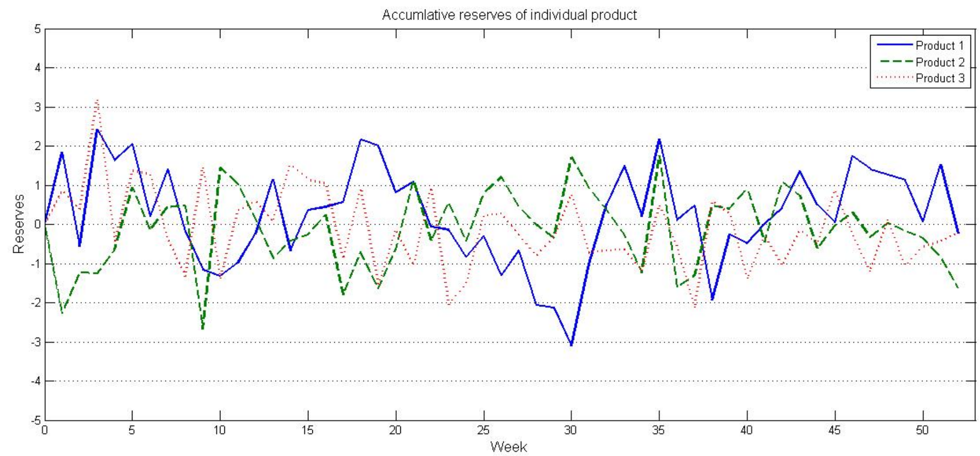

Figure 3.3: The movement of the accumulated reserve for each dependent product.

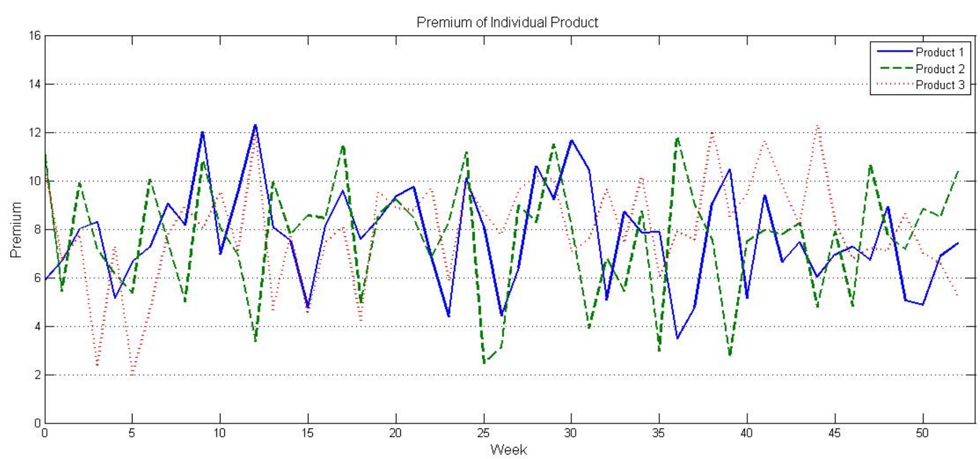

Figure 3.4: The premium process for each dependent product.

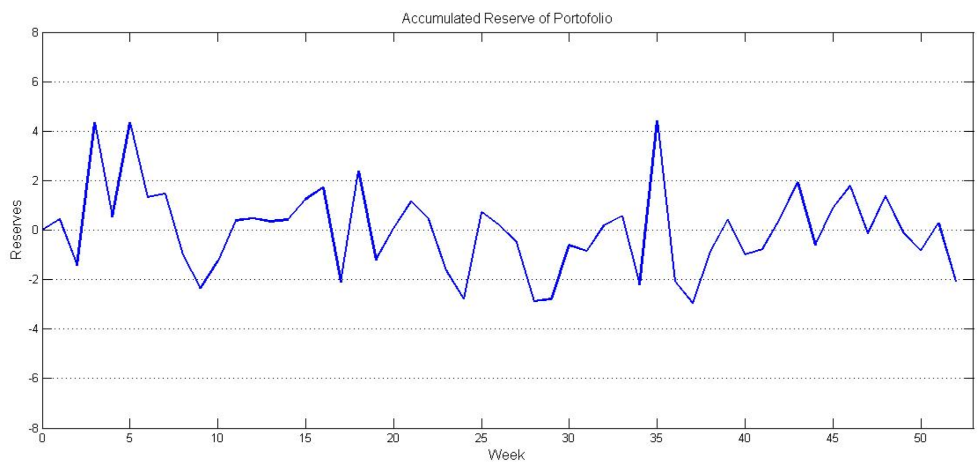

Figure 3.5: The movement of the accumulated reserve for the whole portfolio.

Chapter 4

Predefined Portfolio Strategy

4.1 Introduction

Nowadays, under the Solvency II framework (and different other national regulations), the stability and robustness of the model are parameters that have to be also considered very seriously and thoughtfully. Thus, in the insurance market, in order for the actuary to be able to price the gross (or market) premium accurately, s/he should have a very good feeling about the financial environment where the various uncertainties appear in. Moreover, the constraints that the insurance organization is facing from the market and the stochastic nature of many other financial, social and economic variables and risks that interfere in the model should also be considered. This major extension from the classical literature has been developed and investigated very recently by Pantelous and Yang (2014) [52]. Thus, a P-R system model for different insurance dependent products was constructed into a stochastic, discrete-time framework. Their model considers again a negative feedback mechanism for the reserves, it invests the whole surplus in short-term risk-free assets, and it assumes time-varying, bounded delays for the accumulated reserves considering also a set of different norm-bounded parameter uncertainties for the coefficients involved in the model. Finally, the optimal premium has derived using the ideas of H_∞ control; see the results in Chapter 3 and Pantelous and Yang (2014) [52].

So far, in previous chapter, it has been assumed that all the accumulated reserves are invested in risk-free assets (T-Bills or bank accounts) which, obviously in practice is partially true as (short-term) bonds (or other risky investments) can also be incorporated and they can be used for decreasing the insurance premium*. Additionally, the investment strategy is considered to be pre-defined and unchanged for the whole duration of the process, leaving as a future research plan the optimal asset allocation problem. Therefore, in the present chapter, we investigate the robust H_∞ stabilization performance of the P-R system by considering available, however still pre-defined, risky investment. Obviously, this work extends further the recent results proposed by Pantelous and Yang (2014) [52]. Here, we are interested in investigating how the risky investment might affect the robust H_∞ stabilization performance and in what extend. Again, it appears that we plan to solve different LMI criteria in order to be able to derive a state-feedback controller for the P-R system such that the resulting closed-loop system is robust stochastically stable for all admissible uncertainties.

The result in this chapter is mainly based on Pantelous and Yang (2015) [53]. This chapter is organized as follows: In section 4.2.1, some key assumptions and preliminary concepts for the model are presented. In section 4.2.2, we define the system which considers one risky investment. In section 4.3 we design a state feedback controller such that the resulting closed-loop system is robust stochastically stable with a particular disturbance attenuation level $\gamma > 0$. In this chapter, the available investment is pre-defined, and it contains a single risky and risk-free asset. In section 4.4, an interesting numerical example helps to illustrate the impact of the risky investment in the system. Section 4.5 concludes the chapter.

4.2 Model Description

4.2.1 Assumptions

In this chapter, the basic notations and assumptions for P-R system are same with those appeared in section 3.2 of Chapter 3 (see also Pantelous and Yang, 2014 [52], Pantelous and Yang, 2015 [53]), so unnecessary details are omitted.

4.2.2 Model Formulation

In the present chapter, the P-R system model which considers also a risky asset investment is developed into a stochastic, discrete-time framework. Assume $\underline{R}_t =$

*Matt Wirz article, The Wall Street Journal: "Why Falling Bond Yields Are Raising Your Auto Insurance Premium?": August 13, 2012

$(R_{1,t} R_{2,t} \cdots R_{m,t})^T$ be the vector expression of the accumulated reserves, where $R_{i,t}$ is the accumulated reserves of i^{th} product at time t. Now, the accumulated reserve, $\underline{R}_t$, is given by

$$\underline{R}_{t+1} = m_1[J_1 + \Delta J_{1t}]\underline{R}_t + m_2[J_2 + \Delta J_{2t}]\underline{R}_t + e\underline{P}_{t+1} - \underline{C}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t,$$

Same with that in Chapter 3, the trajectory of premium is formulated in this chapter as follows:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t} - [Z + \Delta Z_t]\underline{U}_t, \tag{4.2.1}$$

$\underline{U}_t \in \mathbb{R}^m$ is the control input. From the above two equations, we can get

$$\begin{aligned}
\underline{R}_{t+1} &= m_1[J_1 + \Delta J_{1t}]\underline{R}_t + m_2[J_2 + \Delta J_{2t}]\underline{R}_t + e\{\underline{\hat{C}}_{n+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t} \\
&\quad -[Z + \Delta Z_t]\underline{U}_t\} - \underline{C}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t \\
&= \{m_1[J_1 + \Delta J_{1t}] + m_2[J_2 + \Delta J_{2t}]\}\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_t} - e[Z + \Delta Z_t]\underline{U}_t \\
&\quad +e\underline{\hat{C}}_{n+1} - \underline{C}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t \\
&= [J' + \Delta J']\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_t} - e[Z + \Delta Z_t]\underline{U}_t + \underline{w}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t,
\end{aligned} \tag{4.2.2}$$

where $\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1}$, $J_{1t} = J_1 + \Delta J_{1t}$, $J_{2t} = J_2 + \Delta J_{2t}$, $J_{3t} = J_3 + \Delta J_{3t}$, $J' = m_1 J_1 + m_2 J_2$, $\Delta J'_t = m_1 \Delta J_{1t} + m_2 \Delta J_{2t}$. J_1 and J_2 are the base investment return matrices for the risk-free and risky asset, accordingly. J_3 is the matrix which represents the volatility of the random process for the risky asset return. J_3 could be estimated by historic data. ΔJ_{1t}, ΔJ_{2t} and ΔJ_{3t} are the corresponding parameter uncertainties. Normally $J_{2t} > J_{1t}$, since insurer demand extra return in order to hold a volatile investment as the classical mean-variance model indicated.

Moreover, $\{v_t\}$ is a zero-mean real scalar process on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. Similar with Chapter 3, it is supposed that

$$\mathbb{E}(v_t) = 0, \quad \mathbb{E}(v_t^2) = 1. \tag{4.2.3}$$

Now, we assume that proportion of the investment of the reserve is split into risk-free and risky asset. Let m_1 be the weight of risk-free investment, and risky investment account for m_2 of total accumulated reserve. Thus, $m_1 + m_2 = 1$. m_1 and m_2 are predefined fixed weight at current stage. Since we consider the risky investment, we assume that the return of risky investment follows stochastic process $\{v_t\}$. $\underline{\hat{C}}$ is the 'claim estimator' described in Chapter 3 and the calculation has been also discussed. E is a known real positive matrix and ΔE_t is a parameter uncertainty, which can be varied with time. Similarly, in this chapter, Z is also a known real constant parameter matrix, and ΔZ_t is the respective parameter uncertainty. In practice, E could be considered as a constant-base return to the policyholders. Normally, we can assume Z is an identity matrix and ΔZ equals zero such that the controller is derived directly. e in eq. (4.2.2) is a known real scalar parameter; $\Delta J^{'}$, ΔE_t and ΔZ_t, ΔJ_{3t} are unknown matrices representing time-varying parameter uncertainties, and are assumed to be form of

$$[\Delta J_t^{'} \quad -e\Delta E_t \quad -e\Delta Z_t \quad m_2\Delta J_{3t}] = MF_t[N_1 \quad N_2 \quad N_3 \quad N_4], \tag{4.2.4}$$

M, N_1, N_2, N_3, N_4 are known real constant matrices and $F_t(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^{s\times j}$ is an unknown time-varying matrix function satisfying

$$F_t^T F_t \leq I, \quad \forall t \in N, \tag{4.2.5}$$

$\Delta J_t^{'}$, ΔE_t, ΔZ_t and ΔJ_{3t} are said to be admissible if both eq.(4.2.4) and eq.(4.2.5) hold.

According to eq.(4.2.2), we define the basic system without controller as

$$\begin{aligned} \underline{R}_{t+1} &= m_1[J_1 + \Delta J_{1t}]\underline{R}_t + m_2[J_2 + \Delta J_{2t}]\underline{R}_t + e\{\underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t}\} \\ &\quad -\underline{C}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t \\ &= \{m_1[J_1 + \Delta J_{1t}] + m_2[J_2 + \Delta J_{2t}]\}\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_t} \\ &\quad +e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t \\ &= [J^{'} + \Delta J^{'}]\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_t} + \underline{w}_{t+1} + m_2[J_3 + \Delta J_{3t}]\underline{R}_t v_t. \end{aligned} \tag{4.2.6}$$

The state feedback controller is calculated by $\underline{U}_t = K\underline{R}_t$, where K is a matrix to be determined. An appropriate robust stabilizing controller $\underline{U}_t$ for a specific P-R system can be constructed by solving the LMI criteria in this chapter. The initial condition is defined by

$$\underline{R_t} = \underline{\varphi_t} \text{ for } t \in [-\tau_{\max}, 0]. \tag{4.2.7}$$

$\underline{\varphi_t}$ is deterministic for a P-R system. We denote system eq. (4.2.6) without controller element and eq. (4.2.7) as Θ_2 and denote system eq. (4.2.2) and eq. (4.2.7) as Λ_2. The stochastic disturbance parameter $v(t)$ is defined by eq. (4.2.3).

(open-loop system)

$$\Theta_2 : \begin{cases} \underline{R_{t+1}} = [J^{'} + \Delta J^{'}]\underline{R_t} - e[E + \Delta E_t]\underline{R_{t-\tau_t}} + \underline{w_{t+1}} + m_2[J_3 + \Delta J_{3t}]\underline{R_t} v_t \\ \underline{R_t} = \underline{\varphi_t} \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}$$

(closed-loop system)

$$\Lambda_2 : \begin{cases} \underline{R_{t+1}} = [J^{'} + \Delta J^{'}]\underline{R_t} - e[E + \Delta E_t]\underline{R_{t-\tau_t}} - e[Z + \Delta Z_t]U_t + \underline{w_{t+1}} + m_2[J_3 + \Delta J_{3t}]\underline{R_t} v_t \\ \underline{R_t} = \underline{\varphi_t} \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}$$

4.3 Robust H_∞ control

In this section, the robust H_∞ control is used to investigate the stability of systems Θ_2 and Λ_2. We extend the previous research by considering the presence of a risky investment. Here, we can determine a state feedback controller in the form of $\underline{U_t} = K\underline{R_t}$ such that the resulting closed-loop system is robust stochastically stable with disturbance attenuation level γ which is a given constant performance level. The disturbance attenuation γ is a parameter which measure the accumulated impact of the outside disturbance on the system output. In the insurance industry, we can consider γ as a parameter which measures the influence of the disturbance in the market for the accumulated reserve, see Pantelous and Yang (2014) [52]. Because insurer always wants to minimize the impact of unexpected events and disturbance, we desire a small γ in P-R system. Before we present the necessary LMI theorem, we would like to introduce a very useful definition. The following definition has same mathematical expression in Definition 3.2, but P-R system consider a risky investment in this chapter.

Definition 4.1. The uncertain stochastic discrete time-delay system system Θ_2 is said to be robust stochastically stable with disturbance attenuation level γ if it is robust stable and the (4.3.1) is satisfied,

$$||\underline{L}_t||_{e_2} \leq \gamma ||\underline{w}_t||_{e_2}, \tag{4.3.1}$$

for all nonzero $\underline{w}_t \in l_{e_2}(N; \mathbb{R}^m)$, and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$, where $\gamma > 0$ is a given scalar and $\underline{L}_t = C\underline{R}_t$ is the control output. Matrix C is a known constant matrix.

4.3.1 Robust H_∞ stability of system Θ_2

First, we consider the system Θ_2. It should be emphasized that the $\underline{w}_{t+1} \neq 0$.

Theorem 4.1. *Given a constant scalar $\gamma > 0$, the uncertain discrete stochastic time-delay system Θ_2 is robust stochastically stable with disturbance level γ if there exist matrices $P > 0, Q > 0$ and scalar $\mu_1 > 0$, $\mu_2 > 0$ such that the following LMI holds:*

$$\begin{bmatrix} e^2\hat{\tau}Q - P + \mu_1 {N_1}^T N_1 + \mu_2 {N_4}^T N_4 + C^T C & \mu_1 {N_1}^T N_2 & 0 & J'^T P & m_2 J_3^T P & 0 & 0 \\ \mu_1 {N_2}^T N_1 & \mu_1 {N_2}^T N_2 - e^2 Q & 0 & -eE^T P & 0 & 0 & 0 \\ 0 & 0 & -\gamma^2 \mathbf{I} & \mathbf{I} & 0 & 0 & 0 \\ PJ' & -ePE & \mathbf{I} & -P & 0 & PM & 0 \\ m_2 P J_3 & 0 & 0 & 0 & -P & 0 & PM \\ 0 & 0 & 0 & M^T P & 0 & -\mu_1 \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & M^T P & 0 & -\mu_2 \mathbf{I} \end{bmatrix} < 0. \tag{4.3.2}$$

Proof 4.1. The proof is similar with that of Theorem 3.3 in Chapter 3, so unnecessary details are omitted. From Definition 3.1, two conditions should be satisfied in order the uncertain stochastic system to be robust stochastically stable with disturbance attenuation level γ. One is that the system should be robust stochastically stable. The other condition is given by (4.3.1) for all $\underline{w}_t \in l_{e_2}(N;\mathbb{R}^m)$, and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$. First, it is easy to transform LMI (4.3.2) into the following LMI

$$\begin{bmatrix} e^2\hat{\tau}Q - P + \mu_1 {N_1}^T N_1 + \mu_2 {N_4}^T N_4 & \mu_1 {N_1}^T N_2 & J'^T & m_2 J_3^T & 0 & 0 \\ \mu_1 {N_2}^T N_1 & \mu_1 {N_2}^T N_2 - e^2 Q & -eE^T & 0 & 0 & 0 \\ J' & -eE & -P^{-1} & 0 & M & 0 \\ m_2 J_3 & 0 & 0 & -P^{-1} & 0 & M \\ 0 & 0 & M^T & 0 & -\mu_1 \mathrm{I} & 0 \\ 0 & 0 & 0 & M^T & 0 & -\mu_2 \mathrm{I} \end{bmatrix} < 0.$$

According to Theorem 3.1 in Chapter 3, we can conclude that this system is robust stochastically stable. With the next step, our aim is to show that $||\underline{L}_t||_{e2} \leq \gamma ||\underline{w}_t||_{e2}$ holds for all nonzero $\underline{w}_t$. To prove this, we can follow the steps in Chapter 4 based on

$$T_N = \mathbb{E}\{\sum_{t=0}^{N}(|\underline{L}_t|^2 - \gamma^2|\underline{w}_t|^2)\}, \tag{4.3.3}$$

where scalar $N > 0$ is an integer. By using results from the proof of Theorem 3.1 in Chapter 3, we can get

$$T_N \leq \mathbb{E}\{\sum_{t=0}^{N} \underline{\eta}_t^T \underline{\phi}_t \underline{\eta}_t\}, \tag{4.3.4}$$

where $V_t(R_t)$ which is defined by Lyapunov candidate function, $\underline{\eta}_t$ and $\underline{\phi_t}$ have been defined in the way as in section 3.6. By using the Schur complement formula and Lemma 3.1, it can be proved that $\underline{\phi_t} < 0$, which implies $T_N < 0$. Therefore, the inequality $||\underline{L}_t||_{e2} \leq \gamma||\underline{w}_t||_{e2}$ holds for all $\underline{w}_t$. This completes our proof, as the robust H_∞ control condition for the P-R system with risky asset has been proven. □

4.3.2 Robust feedback controller of system Λ_2

Now, the system Λ_2 is considered taking into account a state feedback controller such that the resulting closed-loop system to be robust stochastically stabilizable with disturbance attenuation level γ.

Theorem 4.2. *This system Λ_2 is robust stochastically stabilizable with disturbance attenuation γ, if there exist matrices $X > 0, Q > 0$ and a scalar $p_1 > 0$, $p_2 > 0$ such as the following matrix inequality holds :*

$$
\left[\begin{matrix}
-X & 0 & 0 & XJ^{'T} - Y^T e Z^T & m_2 X J_3^T \\
0 & -Q & 0 & -QeE^T & 0 \\
0 & 0 & -\gamma^2 \mathrm{I} & \mathrm{I} & 0 \\
J^{'}X - eZY & -eEQ & \mathrm{I} & p_1 M M^T - X & 0 \\
m_2 J_3 X & 0 & 0 & 0 & p_2 M M^T - X \\
N_1 X + N_3 Y & N_2 Q & 0 & 0 & 0 \\
N_4 X & 0 & 0 & 0 & 0 \\
\hat{\tau} X & 0 & 0 & 0 & 0 \\
CX & 0 & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
XN_1^T + Y^T N_3^T & XN_4^T & \hat{\tau} X & XC^T \\
QN_2^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-p_1 \mathrm{I} & 0 & 0 & 0 \\
0 & -p_2 \mathrm{I} & 0 & 0 \\
0 & 0 & -\hat{\tau} Q & 0 \\
0 & 0 & 0 & -\mathrm{I}
\end{matrix}\right] < 0, \tag{4.3.5}
$$

where $\hat{\tau} = \tau_{\max} - \tau_{\min} + 1$, I *is identity matrix*

Proof 4.2. Since the proof process is similar to Theorem 3.2 in Chapter 3, therefore we omit it here. □

4.4 Numerical Application 2

In this section we conduct the numerical test of the theorems proposed in the previous section. We assume that the investment management team of the General (Non-Life) Insurance company intends to invest the accumulated reserves to a single risk-free and risky asset. Data from the Shanghai Stock Exchange market are used for the risky asset in our application. The weight of risk-free investment is m_1, and the weight of risky investment is m_2. We conduct 11 cases to simulate the effect of the different combinations of m_1 and m_2 into our closed-loop P-R system.

In the same way with Chapter 3, let us consider an insurance company which runs 3 products. We assume that the P-R system is given by an uncertain discrete-time stochastic time-delay system with parameter uncertainties, see Λ_2. Thus, in this example, the input controller $\underline{U}_t = YX^{-1}\underline{R}_t$ is used.

For this portfolio, we have interaction (dependency) among the three products. Our target is to trace the long term movement of the accumulated reserve. Since we introduce the feedback controller, our long-term aim is to stabilize the movement of the accumulated reserve. Practically speaking, it means that we prefer to see the accumulated reserve $\underline{R}_t$ of the insurance company to move around a fixed level (i.e. not permitting over-shooting). Although it can be kept at lower levels, obviously the company can still make profits as profit margin based on the Assumption 3.4 is allowed.

- First the value of the reserve accounts at $t = 0$ is given by the following matrix,

$$\underline{R}_0 = \begin{bmatrix} R_0(1) \\ R_0(2) \\ R_0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

  i.e. at time $t = 0$, we assume that the reserve account of each product is 0 pounds respectively.

- For the time delay, we assume that the time-varying delay varies between $\tau_{\min} = 1$ and $\tau_{\max} = 3$ (in years). Therefore, we have accurate information upto $-\tau_{\max} = -3$. We set

$$\underline{R}_{-3} = \begin{bmatrix} R_{-3}(1) \\ R_{-3}(2) \\ R_{-3}(3) \end{bmatrix} = \underline{R}_{-2} = \begin{bmatrix} R_{-2}(1) \\ R_{-2}(2) \\ R_{-2}(3) \end{bmatrix} = \underline{R}_{-1} = \begin{bmatrix} R_{-1}(1) \\ R_{-1}(2) \\ R_{-1}(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- In our model, it is assumed that the insurer can invest the premium surplus into risk-free investment (T-bills) and risky investment (liquid stock) to generate additional income. Since dependencies exist, we have to use weights in the parameter matrix. We assume that the corresponding rate of income is given from the following matrix:

$$J_1 = \begin{bmatrix} 1.021 * w_{1,1} & 1.021 * w_{1,2} & 1.021 * w_{1,3} \\ 1.021 * w_{2,1} & 1.021 * w_{2,2} & 1.021 * w_{2,3} \\ 1.021 v w_{3,1} & 1.021 * w_{3,2} & 1.021 * w_{3,3} \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 1.039 * w_{1,1} & 1.039 * w_{1,2} & 1.039 * w_{1,3} \\ 1.039 * w_{2,1} & 1.039 * w_{2,2} & 1.039 * w_{2,3} \\ 1.039 * w_{3,1} & 1.039 * w_{3,2} & 1.039 * w_{3,3} \end{bmatrix}.$$

- For parameter J_3, we collect 280 daily data from 1 January 2013 to 7 March 2014 to calculate the historic volatility.

  We let $J_3 = \begin{bmatrix} 0.245 * w_{1,1} & 0.245 * w_{1,2} & 0.245 * w_{1,3} \\ 0.245 * w_{2,1} & 0.245 * w_{2,2} & 0.245 * w_{2,3} \\ 0.245 * w_{3,1} & 0.245 * w_{3,2} & 0.245 * w_{3,3} \end{bmatrix}$.

- The weight ratios $w_{n,m}$ which demonstrates the solvency relation between each product have the following values:

  $w_{1,1} = 0.86$, $w_{1,2} = 0.07$ and $w_{1,3} = 0.07$,

  $w_{2,1} = 0.10$, $w_{2,2} = 0.87$ and $w_{2,3} = 0.03$,

  $w_{3,1} = 0.08$, $w_{3,2} = 0.09$ and $w_{3,3} = 0.83$.

- The parameter E comes from the negative mechanism proposed by Balzer and Benjamin (1980, 1982) [2, 3]. With the indication in Pantelous and Yang (2014) [52], the value of E could be the constant base return rate of policyholder rather than issuer.

  For the examples, we assume the value in the parameter matrix E

$$E = \begin{bmatrix} 0.013 * w_{1,1} & 0.013 * w_{1,2} & 0.013 * w_{1,3} \\ 0.013 * w_{2,1} & 0.015 * w_{2,2} & 0.011 * w_{2,3} \\ 0.013 * w_{3,1} & 0.013 * w_{3,2} & 0.016 * w_{3,3} \end{bmatrix},$$

- For parameter e, we let $e = 0.8$, which means $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the administration and operating cost and give company a reasonable profit margin.

- The time-varying unknown parameter uncertainty $\Delta J_t^{'}$, ΔE_t, ΔZ_t and ΔJ_{3t} have been defined by equation

$$[\Delta J_t^{'} \quad -e\Delta E_t \quad -e\Delta Z_t \quad m_2\Delta J_{3t}] = MF_t[N_1 \quad N_2 \quad N_3 \quad N_4],$$

where

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.008 & 0.002 & 0.002 \\ 0.002 & 0.008 & 0.002 \\ 0.001 & 0.003 & 0.006 \end{bmatrix}, N_2 = \begin{bmatrix} -0.01 & -0.05 & -0.03 \\ -0.05 & -0.01 & -0.05 \\ -0.05 & -0.02 & -0.01 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.3 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, N_4 = \begin{bmatrix} 0.003 & 0.001 & 0.002 \\ 0.002 & 0.008 & 0.002 \\ 0.001 & 0.003 & 0.006 \end{bmatrix}.$$

- Feedback parameter $Z = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$,

- Control output parameter $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For different proportions of m_1, m_2, we can define different investment strategies. Thus, here (see Table 4.1), the following portfolio allocations are considered:

Case	1	2	3	4	5	6	7	8	9	10	11
m_1	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
m_2	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1

Table 4.1: Different portfolio allocations: We start from ($m_1 = 100\%, m_2 = 0\%$) and we end with ($m_1 = 0\%, m_2 = 100\%$)

In order to get the desirable feedback controller for each situation, we use the functions of the Matlab LMI Control toolbox for solving this problem such that the feasible positive definite matrices X, Q, and Y and parameters p_1 and p_2 is calculated. The results for the feedback controllers are as follows:

- When $m_1 = 1, m_2 = 0$,

$$\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t \begin{bmatrix} 0.7955 & 0.2466 & 0.2787 \\ -0.2287 & 0.1545 & -0.3305 \\ 0.1233 & 0.1393 & 0.6138 \end{bmatrix} \underline{R}_t. \tag{4.4.1}$$

- When $m_1 = 0.5, m_2 = 0.5$,

$$\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t \begin{bmatrix} 0.8325 & 0.2876 & 0.3233 \\ -0.2325 & 0.0698 & -0.3317 \\ 0.1873 & 0.2008 & 0.6163 \end{bmatrix} \underline{R}_t. \tag{4.4.2}$$

- When $m_1 = 0, m_2 = 1$,

$$\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t \begin{bmatrix} 0.8285 & 0.2969 & 0.3130 \\ -0.3198 & 0.0570 & -0.3362 \\ 0.1736 & 0.1856 & 0.6172 \end{bmatrix} \underline{R}_t. \tag{4.4.3}$$

As we can see in figures 4.1-4.6, the simulation lasts from $t = 0$ to $t = 52$ time periods. We can assume time period is one year. Figures 4.1 and 4.2 show the movement of the premium and accumulated reserve process for each product respectively, for the case 6, i.e. when equal proportion to risk-free and risky asset ($m_1 = m_2 = 50\%$) exists. In figures 4.3 and 4.4, respectively, a comparison for the premium and the accumulated reserve for the 2nd product for three distinguished cases is provided. Initially, it is assumed that no risky asset is present (Case 1), then equal proportion to risk-free and risky asset is provided (Case 2) and, finally, no risk-free investment is given (Case 3). As it can be seen, there is distinction for both premium and reserve processes, and the investment strategy can affect the decision-making process of the managerial team. With the figures 4.5 and 4.6, the total premium and corresponding accumulated reserve for the same cases as with figure 4.3 and 4.4 are provided. Figure 4.7 demonstrates the disturbance level $\underline{w}_t$ for product 2 in each period. For the first 20 time periods, the disturbance level is much higher than last 32 periods.

Moreover, we can observe from figure 4.7 that in the period when the absolute value of the disturbance $\underline{w}_t$ (which measures the difference between estimated and actual claims occurred) is bigger, the premium for the case 11 (i.e. when only investment in risky asset exist) tends to be lower, for all the products, and it fluctuates more heavily than in any other case. Additionally, in our numerical example, when the disturbance level is lower, the premium is almost equal due to the effect of the feedback controller for each of those cases. The results are really very interesting, and particularly, with consideration of uncertainties, for higher expectation from the risky assets the premium can be lower making the company more competitive, increasing possibly its volume of business.

4.5 Summary

In this chapter, the P-R system for a general insurance product model is modified to incorporate the risky investment so that P-R process could be stabilized using robust H_∞ control. First, we have included the risky-asset in the original system which is defined in Chapter 3. This extends the research in Chapter 3 since the accumulated reserves (surplus) only allows to invest in risk-free investment in Chapter 3. The robust H_∞ stability and the stabilization problems of the new system have been investigated. System controller could be generated by LMI in section 4.3. In section 4.3, a numerical result on the model was conducted by using Matlab and LMI package. It shows the impact of available risky investment on premium and accumulated reserve process. Under some assumptions, the sensitivity of the risky investment weight could be analyzed in the simulation.

In future research work, we may assume the investment plan is not predefined and we design an optimal investment plan for the system under uncertainty. We could also investigate this topic by assuming there exists n risk-free and m risky assets. And weight m_1, m_2 may becomes controllable variables such that we could analyse the optimal mix between weights m_1, m_2. We may introduce monte-carlo simulation to analyse possible consequence of optimal weight. Moreover, it will be interested to extend this topic in order to incorporate elements of the robust guaranteed performance control.

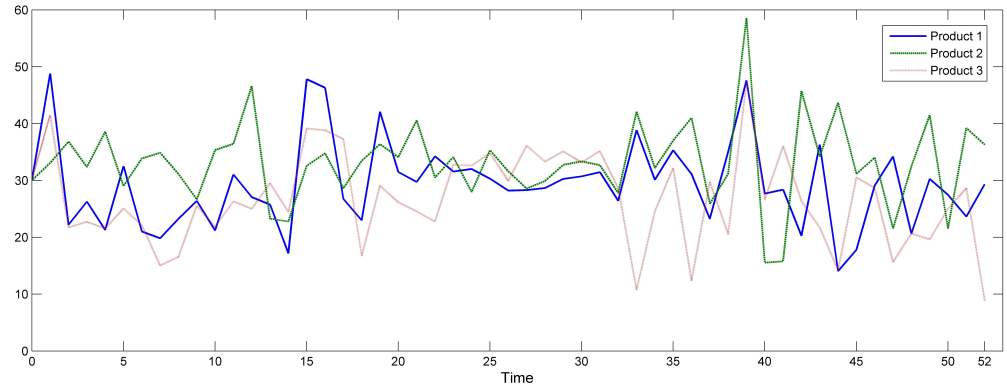

Figure 4.1: The premium for the three products for the case 6: ($m_1 = 50\%, m_2 = 50\%$).

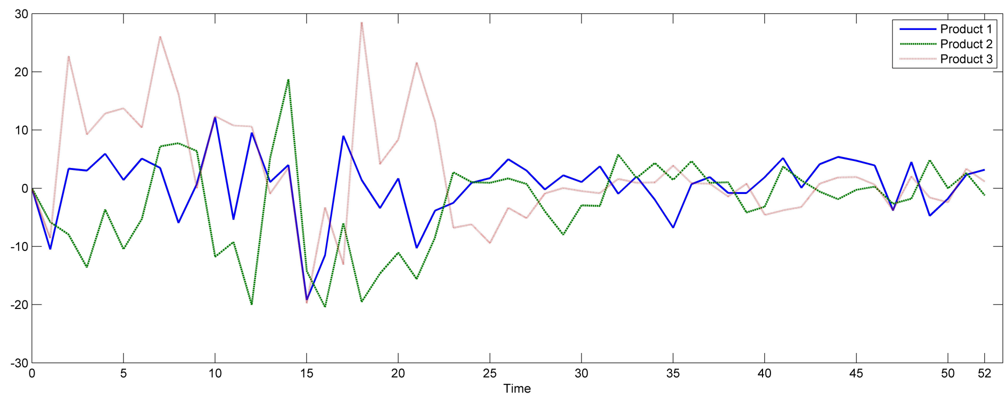

Figure 4.2: The accumulated reserve for the three products for the case 6: ($m_1 = 50\%, m_2 = 50\%$).

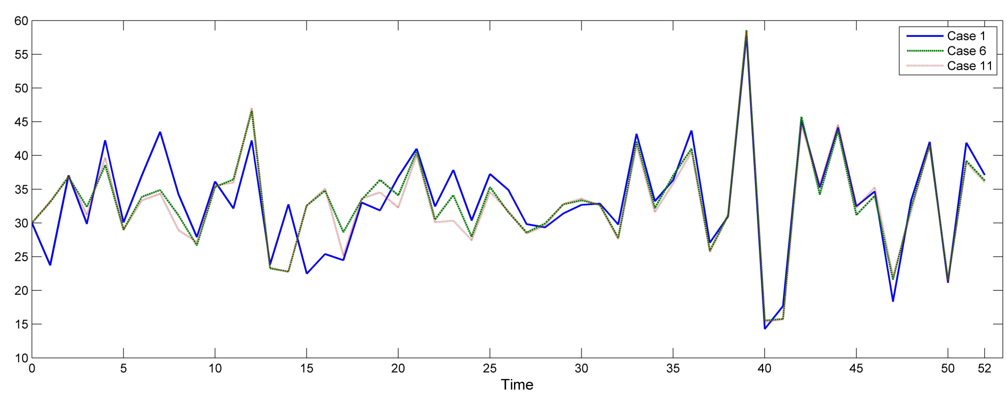

Figure 4.3: The premium for the product 2 for the case 1: ($m_1 = 100\%, m_2 = 0\%$), 6: ($m_1 = 50\%, m_2 = 50\%$) and 11: ($m_1 = 0\%, m_2 = 100\%$).

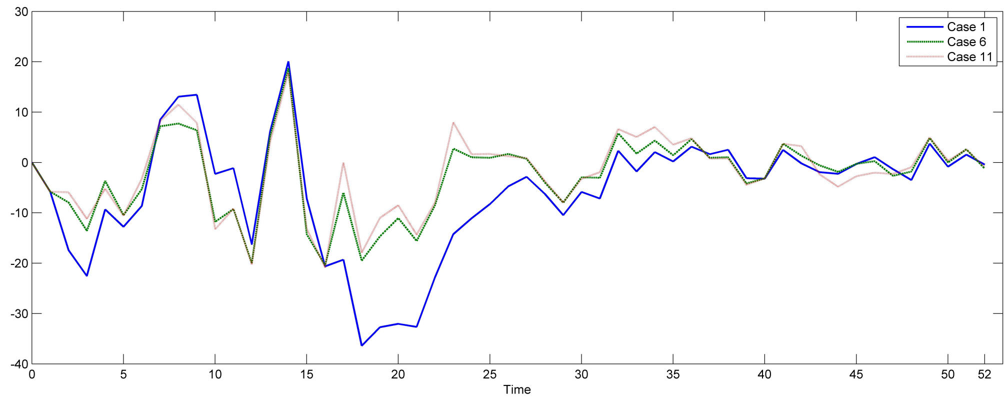

Figure 4.4: The accumulated reserve for the product 2 for the case 1: $(m_1 = 100\%, m_2 = 0\%)$, 6: $(m_1 = 50\%, m_2 = 50\%)$ and 11: $(m_1 = 0\%, m_2 = 100\%)$.

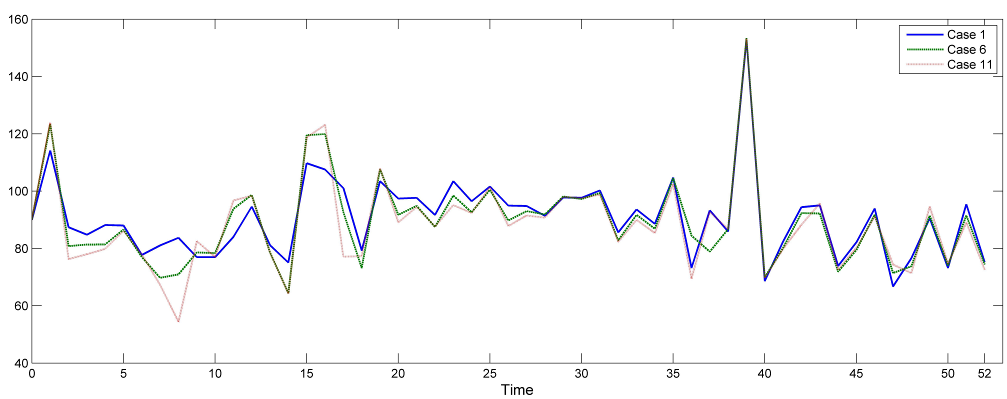

Figure 4.5: The total premium for the case 1: $(m_1 = 100\%, m_2 = 0\%)$, 6: $(m_1 = 50\%, m_2 = 50\%)$ and 11: $(m_1 = 0\%, m_2 = 100\%)$.

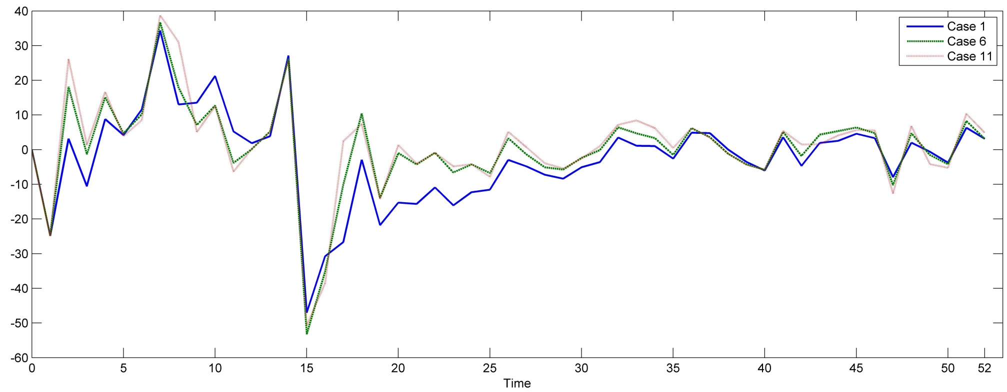

Figure 4.6: The total reserve for the case 1: ($m_1 = 100\%, m_2 = 0\%$), 6: ($m_1 = 50\%, m_2 = 50\%$) and 11: ($m_1 = 0\%, m_2 = 100\%$).

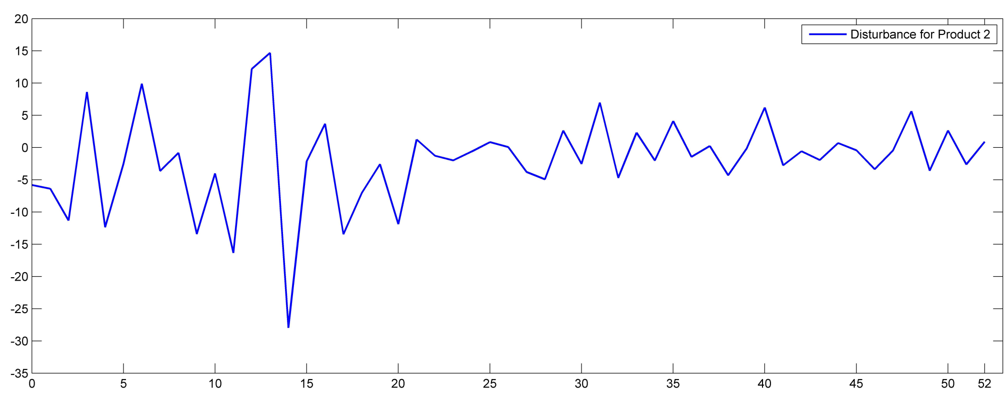

Figure 4.7: Disturbance level w_t for the product 2 for all the cases: from $t = 0$ to $t = 52$.

Chapter 5

Robust stability, stabilization and H_∞ control for markovian regime switching P-R systems

5.1 Introduction

In Chapter 3, we have discussed the robust stability, stabilization and H_∞ control for the premium pricing process, the medium- and long-term stability in the reserve policy under uncertainty and presence of disturbances. During the last two decades, applications of Markovian regime switching models in finance and macroeconomics have received a significant attention among researchers and particularly, market practitioners. However, so far relatively little research has been done in the insurance literature. This chapter is an attempt to consider how a linear Markovian regime switching system in discrete-time could be used to model the medium- and long- term reserves and the premiums (Known as P-R system) of an insurer. Some recently developed techniques from linear robust control theory are applied to explore the stability, the stabilization and the robust H_∞-control of a P-R system and the potential effects from the abrupt structural changes in the economic fundamentals as well as the insurer's strategy over a finite time period.

5.1.1 Regime Switching Systems

In the insurance industry, the interest in time-varying parameter models has been increased in the last decades. The Solvency II framework and the development of some national regulations, have increased the interest in the stability and robustness of the models used to describe insurers' behaviour. One example of that is Pantelous and Papageorgiou (2013) [51], Pantelous and Yang (2014, 2015) [52, 53], which use the recent claim experience and a feedback mechanism based on the surplus value to control the premium level. The main results in those papers are shown in Chapter 3 and 4. But all these models assume only one standard regime for the P-R system. In financial economics, it is indicated that statistical relationships among variables in many macroeconomic/finance models may be inconsistent. Thus, we can model and even possibly predict such shock events in many different ways, since it might contain dramatic changes in the system's behaviour. The abrupt effect on model is mostly associated with events like financial or economic crises or with significant changes in government policies. For the insurance company and its stakeholders, in practice, different strategies should be implemented under different economic environments. Therefore, an "ideal" model of the P-R process should be able to take into account this significant factor. One possible technique, widely used in financial economics, is so called regime switching models. In these models the studied processes are assumed to have several "regimes" with their own regime specific parameters and rules for regime switching.

5.1.2 Markovian Regime Switching Systems

Let us start first with a brief analysis of the basic concepts. First, for dynamical systems, various criteria have been developed to prove their stability; the famous Lyapunov method is the most general one. Thus, the way to establish Lyapunov stability for dynamical systems is by means of Lyapunov functions. Moreover, in feedback control (called also closed loop control), the system output is measured and compared with the desired value; the system continually attempts to reduce the error between the two values. The most important property of the feedback control is that it always compares and adjusts the actual status in order to arrive at the target status. Therefore, the feedback is usually superior to the open loop approach (i.e. feedforward or open loop control, which is based only pre-set values) for practical applications since it is robust against unexpected disturbances and model uncertainty.

In the literature of quantitative finance, regime switching models try to capture the instabilities (or discontinuities) in the different variables involved in the model over long term. Some well-motivated, popular examples are: a) bull and bear regimes alternating in financial markets and their economic impact; b) it is well known that exchange rates tend to alternate protracted periods of depreciation and appreciation; c) monetary policy can change suddenly because of the down and upswings in the economy.

Markovian switching represents the most widely applied and well-known class of regime switching models in both finance and macroeconomics. Many researchers use the Markov properties to describe abrupt changes of different stochastic processes. Guidolin (2011) [31] reviews and summarises the research trends of the application of Markovian switching in finance for the last 20 years. Discrete Markovian jump linear system (DMJLS) may represent a large class of regime switching systems subject to abrupt changes in structures. A discrete Markov chain governs the transition dynamics between the different regimes.

In the present chapter, we assume that time delay and switching signal always exist following closely the ideas by Zimbidis and Haberman (2001) [82], Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014, 2015) [52, 53]. Thus, the model of regime switching systems with time-delay is naturally used to analyse the P-R pricing process. In our framework, the switching dynamics are modelled by a Markovian jump process (see Assumption 5.4), and then the study of the stability and stabilizability is provided for the derived discrete-time Markovian jump P-R system.

The Markovian regime switching system environment used in this chapter increases the flexibility of the parameters and hopefully allows us to model a more representative real market dynamics system. Our objective is to present a new approach to investigate and manipulate the stability of the P-R system. Furthermore, a H_∞ controller for the Markovian jump switched system is designed which guarantees the stability of the switching system.

5.1.3 Structure

This chapter is organized as follows: In section 5.2, the necessary notation and some key assumptions are presented. In section 5.3, the new P-R system is formulated under some particular assumptions and in a Markovian regime switching framework.

In section 5.4 and 5.5, the LMI conditions for robust stabilization and H_∞ control are derived, respectively by using the concepts proposed by Pantelous and Yang (2014) [52] and Boukas and Liu (2001) [9]. Then, a numerical example is exploited to demonstrate the effectiveness of the developed method in section 5.6. Finally, section 5.7 concludes the whole chapter.

5.2 Assumptions

Here, the necessary notation and basic assumptions for our model are described. Some assumptions are almost the same with those in Chapter 3 and 4, so only a brief explanation is provided here.

Assumption 5.1: Same with Assumption 3.1 in Chapter 3.

Assumption 5.2: Same with Assumption 3.2 and Assumption 3.3 in Chapter 3.

Assumption 5.3: Same with Assumption 3.4 in Chapter 3.

Assumption 5.4: Let $\{\sigma_t; t \geq 0\}$ be a discrete-time Markov chain with finite state space $\mathcal{S} = \{1, 2 \cdots N\}$. Denote the state transition matrix by $P = [p_{ij}]_{i,j \in \mathcal{S}}$, i.e., the transition probabilities of $\{\sigma_t, t \geq 0\}$ are given by:

$$Pr[\sigma_{t+1} = j | \sigma_t = i] = p_{ij} \quad \text{for} \quad i, j \in \mathcal{S},$$

with $p_{ij} \geq 0$ for $i, j \in S$, and $\sum_{j=1}^{N} p_{ij} = 1$, for $i \in \mathcal{S}$. The transition probability is time-independent. Thus, the resulting Markov chain is time-homogeneous.

Assumption 5.5: Positive integer τ_i represents the time delay when the system operates in the regime i. Then we denote

$$\tau_{\max} = \max\{\tau_i, i \in \mathcal{S}\},$$

$$\tau_{\min} = \min\{\tau_i, i \in \mathcal{S}\}.$$

We consider a mode-dependent delay, τ_{σ_t}, which is upper and lower bounded, i.e. $\tau_{\min} \leq \tau_{\sigma_t} \leq \tau_{\max}$ with $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. So, considering a specific time-delay interval, at the end of each year $[t, t+1)$, we have the exact information up to the end of the year $t - \tau_t$. The value for τ_i can be estimated using past experience and statistical data. Moreover, the national and international regulatory policy might be also applied for defining the upper bound of this interval.

Assumption 5.6: Same with Assumption 3.6 in Chapter 3.

Assumption 5.7: Same with Assumption 3.7 in Chapter 3.

5.3 Model Formulation

5.3.1 The Premium Rating Rule

The rating of premiums usually depends on available claims experience, on the general and specific market condition and on the strategy and restriction of the company and so on. Therefore, mathematical modelling of this very complicated process is not an easy task. Moreover, it is difficult to find an "ideal" mathematical formula to cover accurately all aspects of the premium setting, thus it is necessary to try to find approximate rules for the anticipated behaviour of the insurer. Zimbidis and Haberman (2001) [82], Pantelous and Papageorgiou (2013) [51] propose a premium rating formula which embeds the feedback mechanism by,

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t},$$

where $\underline{\hat{C}}$ is the *claim estimator*, which is explained in more details in the next section. $\underline{P}_t = (P_{1,t}P_{2,t}\cdots P_{m,t})^T$ for $t \in \mathbb{N}$ be the vector of the *premium* paid in insurance lines $1, \ldots, m$ in one time interval. E is a known real positive matrix, which adjusts the premiums based on the level of the reserve with time lag τ_t and ΔE_t is a parameter uncertainty, which vary through time. Note that $E + \Delta E_t$ should normally lies in the interval $[0, 1]$. τ_t stands for the time delay (see Assumption 5.5). Moreover, in Pantelous and Yang (2014) [52], an additional controller $\underline{U}_t$ in the premium $\underline{P}_t$ is introduced to stabilize the reserve process; how to get $\underline{U}_t$ is explained later. Thus in Pantelous and Yang (2014, 2015) [52, 53], the premium process is formulated as follows:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t} - [Z + \Delta Z_t]\underline{U}_t.$$

Now we assume that the equation above can work in different regimes with regime-specific parameters. Hence the model is developed as a Markovian jump linear system and the premium process is formulated as:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t. \tag{5.3.1}$$

The equation above means that the premium $\underline{P}_t$ at time $t+1$ is $\underline{\hat{C}}_{t+1}$ plus a correction which depends linearly on the past reserve $\underline{R}_{t-\tau_t}$ and the current reserve $\underline{R}_t$ values through $\underline{U}_t$. The dependence can be controlled by varying the values of the involved parameters. Time delay on information is also considered. $\underline{U}_t \in \mathbb{R}^m$ is the control input that has been added in the original system. However, for simplicity, the state feedback controller is considered to depend on the latest value of R: $\underline{U}_t = K_{1i}\underline{R}_t$, where the matrix K_{1i} should be determined by solving an appropriate LMI (convex optimization) problem.

In this model, the insurer can control its financial position. A suitable control of premiums can lead to a stable and realistic evolution of the reserve as well as solvency margin.

5.3.2 The Reserve Process

Let $\underline{R}_t = (R_{1,t}R_{2,t}\cdots R_{m,t})^T$ be the vector expression of the reserves, where $R_{i,t}$ is the reserve of i^{th} insurance line at time t. The reserve, $\underline{R}_t$, evolves according to

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\underline{P}_{t+1} - \underline{C}_{t+1}. \tag{5.3.2}$$

J_{σ_t} is the investment return matrices in time t for the risk-free asset. It is possible to include also risky assets but we leave it for a future work. Practically speaking, it is true that such short term insurance lines (relative to Non-Life insurance policies) are invested predominately in standard bank accounts or/and in short-term *secure* bonds (with duration less than 6 months at the most). Switching signal σ_t is a piecewise constant function of time which takes value i in the finite set $\mathcal{S} = [1 \quad 2\cdots N]$. The Markov chain states represent different system regimes. We assume that the switching signal σ_t is governed by a Markovian jump process (see Assumption 5.4). The premiums are assumed to be the *earned premiums* and claims are *incurred claims*. Investment income consists of cash yield and change in value of assets. All the variables in the basic equation (except e) are stochastic. From the equations (5.3.1) and (5.3.2), we get

$$\begin{aligned}\underline{R}_{t+1} &= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\{\underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t\} - \underline{C}_{t+1} \\ &= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1}.\end{aligned} \tag{5.3.3}$$

The parameters J_i, E_i and Z_i are real constant base matrices. $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are the respective parameter uncertainties. For the purpose of the modelling process, J_i and E_i respectively could be a risk-free interest rate and a constant-base return to the policyholders. Then, Z_i is a parameter of the control input. Normally, Z_i is an identity matrix with proper dimensions. Finally, $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are unknown matrices representing time-varying parameter uncertainties, and they are assumed to be of the form:

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}], \tag{5.3.4}$$

$M_i, N_{1i}, N_{2i}, N_{3i}$ are known real constant matrices and $F_t : \mathbb{N} \to \mathbb{R}^{s\times j}$ is an unknown time-varying matrix function satisfying

$$F_t^T F_t \leq I, \quad \forall t \in \mathbb{N}, \tag{5.3.5}$$

$\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are said to be admissible if they satisfy both (5.3.4) and (5.3.5). Thus we have the following discrete time Markovian jump linear P-R system:

$$\Theta_3 : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1} \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}$$

The system has N system regimes. We denote system Θ_3 without controller element $\underline{U}_t$ and disturbance $\underline{w}_{t+1}$ as Θ_{31}. System Θ_3 without disturbance $\underline{w}_{t+1}$ is denoted as Θ_{32}. System Θ_3 without controller element $\underline{U}_t$ is denoted as Θ_{33}. The observation (see next Remark) is denoted as $\underline{z}_t$, where $\underline{z}_t = C\underline{R}_t$ is the control output.

Remark 5.1. Many H_∞ control problem can be demonstrated by the Figure 5.1, where z is called the controlled output or observation and w is an outside disturbance. Obviously, u is the controller and G is the system/plant. In some systems, it is not possible to directly detect (observe) the accurate status of the state variable y, and we may design and use some observation tools (for instance, we use thermometer to gauge temperature in a heating system). In this situation, we rely on observer z instead of the state variable y to analyse the system process. Intuitively speaking, H_∞ control minimise the maximum impact of w on the observer z (please notice it's not y). In our case, the P-R process is studied in robust H_∞ control framework. The full system Θ_3 should be

$$\Theta_3 : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1} \\ \underline{z}_t = [C_{\sigma_t} + \Delta C_{\sigma_t,t}]\underline{R}_t - e[C^{'}_{\sigma_t} + \Delta C^{'}_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[C^{''}_{\sigma_t} + \Delta C^{''}_{\sigma_t,t}]\underline{U}_t + C^{'''}\underline{w}_{t+1}, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}$$

However, for simplicity, we let $\Delta C_{\sigma_t,t}$, $[C^{'}_{\sigma_t} + \Delta C^{'}_{\sigma_t,t}]$, $[C^{''}_{\sigma_t} + \Delta C^{''}_{\sigma_t,t}]$, $C^{'''}$ to be equal to 0 and $C_{\sigma_t} = C$, then the control output becomes $\underline{z}_t = C\underline{R}_t$ and C is the identity matrix in our numerical example. Practically, it means that we always assume the observation from the system is exactly the accumulated reserve value itself, which doesn't need any other modification. In other words, the current state of accumulated reserve accounts $\underline{R}_t$ can be accurately and directly gauged, although the current value is not the true value due to the time-delay factor. When $\underline{R}_t$ is positive, insurer can pay back part of accumulated reserves as the feedback mechanism indicated. While $\underline{R}_t$ is negative, insurer would like to charge a higher premium to policyholder. Surely, we can give our P-R system another practical meaning by making a more complicated structured observer $\underline{z}_t$.

Remark 5.2. Under the linear control theory framework, the financial position is governed by a linear equation, where the reserve at time t depends linearly on the previous state, on the previous control action and on the disturbance $\underline{w}_{t+1}$.

Both premium and reserve processes have linear relationship with the original claims process. The claims process is a driving force in the system, and the control equation determines how the total energy of the claims process is channelled via the system to the premium and to reserve, respectively. In real world applications it may be a part of an insurance portfolio or line or a company.

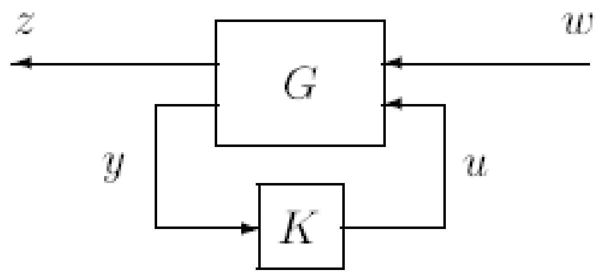

Figure 5.1: Feedback Control System

5.4 Robust Stability

In this section, the robust stability is considered. Before we proceed further we recapitulate the following lemma which is needed later.

Lemma 5.1. *(Xie et al., 1992 [71]) Given appropriately dimensioned matrices Σ_1, Σ_2, Σ_3, with $\Sigma_1^T = \Sigma_1$. Then*

$$\Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T < 0,$$

holds for all F_t, satisfying $F_t^T F_t \leq I$, if and only if for some $\epsilon > 0$,

$$\Sigma_1 + \epsilon \Sigma_2^T \Sigma_2 + \epsilon^{-1} \Sigma_3 \Sigma_3^T < 0.$$

Definition 5.1. The uncertain stochastic discrete time-delay system Θ_1 is said to be robust stochastically stable if there exists a scalar $c > 0$ such that for all admissible uncertainties

$$\mathbb{E}[\sum_{t=0}^{\infty} |\underline{R}_t|^2 | \underline{R}_0, \sigma_0] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2, \tag{5.4.1}$$

when $\underline{w}_{t+1} = 0$, where $\underline{R}_t$ denotes the reserve at time t under initial condition.

5.4.1 Stability of System Θ_{31}

In this subsection, we consider the uncertain discrete time system Θ_3 with state feedback controller $\underline{U}_t = 0$ and disturbance $\underline{w}_{t+1} = 0$. It means that the actual incurred claims are exactly the same with the estimation.

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \tag{Θ_{31}}$$

Theorem 5.1. *For given scalars $\tau_{\max} > \tau_{\min} \geq 0$, the system Θ_{31} is robust stochastically stable, if there exist matrices $X_i > 0$, $L > 0$, $\epsilon_i > 0$, $\forall i \in \mathcal{S}$, such that the following LMI condition holds:*

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0, \tag{5.4.2}$$

where

$$\mathcal{X} = diag\{X_1, \cdots, X_i\}, \quad \Lambda_i = -\mathcal{X} + \epsilon_i H_i^T M_i M_i^T H_i,$$

$$H_i = (\sqrt{p_{i1}} I \cdots \sqrt{p_{iN}} I),$$

$$\frac{1}{\varrho} = 1 + (1 - p_{\min})(\tau_{\max} - \tau_{\min}),$$

and $p_{\min} = \min\{p_{ii}, i \in \mathcal{S}\} for\ i \in \mathcal{S}$.

Proof 5.1. Let matrices $P_i = X_i^{-1}$ and $Q = L^{-1}$. We can construct the Lyapunov functional candidate:

$$V_{\sigma_t}(\underline{R}_t) = V^1(\underline{R}_t) + V^2(\underline{R}_t) + V^3(\underline{R}_t), \tag{5.4.3}$$

where

$$V^1(\underline{R}_t) \triangleq \underline{R}_t^T P_{\sigma_t} \underline{R}_t, \tag{5.4.4}$$

$$V^2(\underline{R}_t) \triangleq \sum_{l=t-\tau_{\sigma_t}}^{t-1} \underline{R}_l^T Q \underline{R}_l, \tag{5.4.5}$$

$$V^3(\underline{R}_t) \triangleq \sum_{k=-\tau_{\max}+1}^{-\tau_{\min}+1} \sum_{l=t+k-1}^{t-1} \underline{R}_l^T \tilde{Q} \underline{R}_l, \tag{5.4.6}$$

and $\tilde{Q} = (1 - p_{\min})Q$. We define $\Delta V_{\sigma_t}(\underline{R}_t) = \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)$. Then, based on the results in Boukas and Liu (2001) [9] and Theorem 1 in Pantelous and Yang (2014) [52], the following equality holds

$$\begin{aligned}\mathbb{E}[V^1(\underline{R}_{t+1})|\underline{R}_t,\sigma_t=i]-V^1(\underline{R}_t) &= \underline{R}_t^T[(J_i+\Delta J_{i,t})^T G_i(J_i+\Delta J_{i,t})-P_i]\underline{R}_t\\ &\quad +2\underline{R}_t^T[J_i+\Delta J_{i,t}-e(E_i+\Delta E_{i,t})]^T G_i[-e(E_i+\Delta E_{i,t})]\underline{R}_t\\ &\quad +\underline{R}_{t-\tau_i}^T[-e(E_i+\Delta E_{i,t})]^T G_i[-e(E_i+\Delta E_{i,t})]\underline{R}_{t-\tau_i},\end{aligned} \tag{5.4.7}$$

where $\mathcal{P}=diag\{P_1,\cdots,P_i\}$ and $G_i=H_i\mathcal{P}H_i^T$. Meanwhile,

$$\begin{aligned}\mathbb{E}[V^2(\underline{R}_{t+1})|\underline{R}_t,\sigma_t=i]-V^2(\underline{R}_t) &= p_{ii}[\sum_{l=t-\tau_i+1}^{t}-\sum_{l=t-\tau_i}^{t-1}]\underline{R}_l^TQ\underline{R}_l\\ &\quad +\sum_{i\neq j}p_{ij}[\sum_{l=t-\tau_j+1}^{t}-\sum_{l=t-\tau_i}^{t-1}]\underline{R}_l^TQ\underline{R}_l\\ &= p_{ii}[\underline{R}_t^TQ\underline{R}_t-\underline{R}_{t-\tau_i}^TQ\underline{R}_{t-\tau_i}]+\sum_{i\neq j}p_{ij}[\sum_{l=t-\tau_j+1}^{t}\\ &\quad -\sum_{l=t-\tau_i+1}^{t-1}]\underline{R}_l^TQ\underline{R}_l-\sum_{j\neq i}p_{ij}\underline{R}_{t-\tau_i}^TQ\underline{R}_{t-\tau_i}\\ &= \underline{R}_t^TQ\underline{R}_t-\underline{R}_{t-\tau_i}^TQ\underline{R}_{t-\tau_i}\\ &\quad +\sum_{i\neq j}p_{ij}[\sum_{l=t-\tau_j+1}^{t-1}-\sum_{l=t-\tau_i+1}^{t-1}]\underline{R}_l^TQ\underline{R}_l.\end{aligned}$$

Note that,

$$\sum_{l=t-\tau_j+1}^{t-1}\underline{R}_l^TQ\underline{R}_l=\sum_{l=t-\tau_{\min}+1}^{t-1}\underline{R}_l^TQ\underline{R}_l+\sum_{l=t-\tau_j+1}^{t-\tau_{\min}}\underline{R}_l^TQ\underline{R}_l.$$

Therefore,

$$\begin{aligned}\mathbb{E}[V^2(\underline{R}_{t+1})|\underline{R}_t,\sigma_t=i]-V^2(\underline{R}_t) &= \underline{R}_t^TQ\underline{R}_t-\underline{R}_{t-\tau_i}^TQ\underline{R}_{t-\tau_i}\\ &\quad +\sum_{i\neq j}p_{ij}[\sum_{l=t-\tau_{\min}+1}^{t-1}+\sum_{l=t-\tau_j+1}^{t-\tau_{\min}}-\sum_{l=t-\tau_i+1}^{t-1}]\underline{R}_l^TQ\underline{R}_l.\end{aligned}$$

Since

$$\sum_{l=t-\tau_{\min}+1}^{t-1}\underline{R}_l^TQ\underline{R}_l\leq\sum_{l=t-\tau_i+1}^{t-1}\underline{R}_l^TQ\underline{R}_l$$

and

$$\sum_{i\neq j} p_{ij} = 1 - p_{ii} \leq 1 - p_{\min},$$

$$\begin{aligned}
\mathbb{E}[V^2(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^2(\underline{R}_t) &\leq \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + \sum_{i\neq j} p_{ij} \sum_{l=t-\tau_j+1}^{t-\tau_{\min}} \underline{R}_l^T Q \underline{R}_l \\
&\leq \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} \\
&\quad +(1 - p_{\min}) \sum_{l=t-\tau_{\max}+1}^{t-\tau_{\min}} \underline{R}_l^T Q \underline{R}_l.
\end{aligned}$$

Also,

$$\mathbb{E}[V^3(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^3(\underline{R}_t) = (\tau_{\max} - \tau_{\min}) \underline{R}_t^T \tilde{Q} \underline{R}_t - \sum_{l=t-\tau_{\max}}^{t-\tau_{\min}} \underline{R}_l^T \tilde{Q} \underline{R}_l. \quad (5.4.8)$$

From (5.4.7), (5.4.8) and (5.4.8), we can show that

$$\begin{aligned}
&\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V_i(\underline{R}_t) \leq \underline{R}_t^T [(J_i + \Delta J_{i,t})^T G_i (J_i + \Delta J_{i,t}) - P_i] \underline{R}_t \\
&+2\underline{R}_t^T [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]^T G_i [-e(E_i + \Delta E_{i,t})] \underline{R}_t \\
&+\underline{R}_{t-\tau_i}^T [-e(E_i + \Delta E_{i,t})]^T G_i [-e(E_i + \Delta E_{i,t})] \underline{R}_{t-\tau_i} \\
&+\underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + (1 - p_{\min})(\tau_{\max} - \tau_{\min}) \underline{R}_t^T Q \underline{R}_t. \quad (5.4.9)
\end{aligned}$$

(5.4.9) is equivalent to

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t) \leq \xi^T(t) \Psi_{\sigma_t} \xi(t), \quad (5.4.10)$$

where

$$\xi(t) = [\underline{R}_t^T \quad \underline{R}_{t-\tau_s}^T]^T,$$

$$\forall i \in \mathcal{S}, \Psi_i = \begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{bmatrix},$$

$$A_{1i} = (J_i + \Delta J_{i,t})^T G_i (J_i + \Delta J_{i,t}) - P_i + \varrho Q,$$

$$A_{2i} = [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]^T G_i [-e(E_i + \Delta E_{i,t})],$$

$$A_{3i} = [-e(E_i + \Delta E_{i,t})]^T G_i [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})],$$

$$A_{4i} = [-e(E_i + \Delta E_{i,t})]^T G_i[-e(E_i + \Delta E_{i,t})] - Q.$$

By Schur complement and $G_i = H_i \mathcal{P} H_i^T$, we can derive a matrix Ω_i from Ψ_i .Therefore,

$$\Omega_i = \Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T, \tag{5.4.11}$$

where

$$\Sigma_1 = \begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i \\ 0 & -Q & -eE_i^T H_i \\ H_i J_i & -eH_i E_i & -\mathcal{P}^{-1} \end{bmatrix} < 0, \tag{5.4.12}$$

$$\Sigma_2 = [0 \quad 0 \quad M_i^T H_i]^T,$$

$$\Sigma_3 = [N_{1i} \quad N_{2i} \quad 0].$$

Similar with the method in Pantelous and Yang (2014) [52] (and references therein), (5.4.1) leads to the following inequality by Schur complement

$$\begin{bmatrix} -X_i + \varrho X_i L^{-1} X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I \end{bmatrix} < 0. \tag{5.4.13}$$

Let $X_i = P_i^{-1}$, $L = Q^{-1}$. Pre and post-multiplying the both sides of (5.4.13) by $diag\{P_i, Q, I, I\}$

$$\begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i & N_{1i}^T \\ 0 & -Q & -eE_i^T H_i & N_{2i}^T \\ H_i^T J_i & -eH_i^T E_i & -\mathcal{P}^{-1} + \epsilon_i H_i^T M_i M_i^T H_i & 0 \\ N_{1i} & N_{2i} & 0 & -\epsilon_i I \end{bmatrix} < 0. \tag{5.4.14}$$

Therefore, if LMI condition (5.4.2) is satisfied, we can show

$$\Sigma_1 + \epsilon_i \Sigma_2 \Sigma_2^T + \epsilon_i^{-1} \Sigma_3^T \Sigma_3 < 0.$$

$$\begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i \\ 0 & -Q & -eE_i^T H_i \\ H_i J_i & -eH_i E_i & -\mathcal{P}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_i H_i^T M_i M_i^T H_i \end{bmatrix} + \begin{bmatrix} N_{1i}^T \\ N_{2i}^T \\ 0 \end{bmatrix} \epsilon_i^{-1} \begin{bmatrix} N_{1i}^T & N_{2i}^T & 0 \end{bmatrix} < 0$$

According to Lemma 5.1, which is the result in Xie et al. (1992) [71], it indicates that:

$$\Omega_i = \Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T < 0.$$

It means those LMI condition (5.4.2) can guarantee that $\Omega_i < 0$. In particular, it follows that

$$\Omega_i < \begin{bmatrix} -\delta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.4.15}$$

$$\Psi_i < \begin{bmatrix} -\delta I & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.4.16}$$

where δ is a positive scalar. Because $\tau_{\min} \leq \tau_i \leq \tau_{\max}$ and $\tau_{\max} - \tau_{\min} \geqslant 1$, we get

$$V_{\sigma_t}(\underline{R}_t) \leq \underline{R}_t^T P \underline{R}_t + \sum_{l=t-\tau_{\max}}^{t-1} \underline{R}_l^T Q \underline{R}_l + \sum_{k=-\tau_{\max}+1}^{-\tau_{\min}+1} \sum_{l=t-\tau_{\max}}^{t-1} \underline{R}_l^T Q \underline{R}_l.$$

Then, we get $\lambda_{max}(P)|\underline{R}_t|^2 \geq \underline{R}_t^T P \underline{R}_t$ and $\lambda_{max}(Q)|\underline{R}_t|^2 \geq \underline{R}_t^T Q \underline{R}_t$. $\lambda_{max}(\quad)$ is the maximum eigenvalue of respective matrix. Thus, following close Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014) [52], we can derive

$$V_{\sigma_t}(\underline{R}_t) \leq \lambda |\underline{R}_t|^2 + \lambda(\tau_{\max} - \tau_{\min} + 1) \sum_{l=t-\tau_{\max}}^{t-1} |\underline{R}_l|^2, \tag{5.4.17}$$

where $\lambda = max[\lambda_{max}(P), \quad \lambda_{max}(Q)]$. Hence, from (5.4.10) and (5.4.16) it is easy to deduce that

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t) < -\delta |\underline{R}_t|^2. \tag{5.4.18}$$

Now, summing up both sides of (5.4.18) over time t

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_0}(\underline{R}_0) < -\delta \sum_{s=0}^{t} |\underline{R}_s|^2. \tag{5.4.19}$$

Then, after taking the expectation on both sides of the above equation, it follows that

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})] - \mathbb{E}[V_{\sigma_0}(\underline{R}_0)] < -\delta \mathbb{E}[\sum_{s=0}^{t} |\underline{R}_s|^2]. \tag{5.4.20}$$

Thus,

$$\mathbb{E}[\sum_{s=0}^{t}|\underline{R}_s|^2] \le \frac{1}{\delta}\mathbb{E}[V_{\sigma_0}(R_0)]. \tag{5.4.21}$$

Applying (5.4.17) at time $t = 0$ and rearranging, we have

$$\begin{aligned} V_{\sigma_0}(\underline{R}_0) &\le \lambda|\underline{R}_0|^2 + \lambda(\tau_{\max} - \tau_{\min} + 1)\sum_{l=-\tau_{\max}}^{-1}|\underline{R}_l|^2 \\ &\le \lambda(\tau_{\max} - \tau_{\min} + 1)\sum_{l=-\tau_{\max}}^{0}|\underline{R}_l|^2. \end{aligned}$$

Therefore, after using mathematical transformation, the expectation becomes,

$$\mathbb{E}[V_{\sigma_0}(\underline{R}_0)] \le \lambda(\tau_{\max} - \tau_{\min} + 1)(\tau_{\max} + 1)\sup_{-\tau_{\max}\le t\le 0}\mathbb{E}[|\underline{\varphi}_t|]^2. \tag{5.4.22}$$

Then, following calculations (5.4.21) and (5.4.22), we get

$$\mathbb{E}[\sum_{s=0}^{t}|\underline{R}_s|^2] \le c\sup_{-\tau_{\max}\le t\le 0}\mathbb{E}[|\underline{\varphi}_t|]^2, \tag{5.4.23}$$

where $c = \frac{1}{\delta}\lambda[(\tau_{\max}-\tau_{\min}+1)(\tau_{\max}+1)] > 0$. The above calculations shows the positive scalar c has relationship with upper and lower bound the time delay, which extend the result in Theorem 1 in Boukas and Liu (2002) [10]. From (5.4.23), we have

$$\lim_{t\to\infty}\mathbb{E}[\sum_{s=0}^{t}|\underline{R}_s|^2] \le c\sup_{-\tau_{\max}\le t\le 0}\mathbb{E}[|\underline{\varphi}_t|]^2.$$

This shows that the system Θ_{31} is robust stochastically stable when LMI condition (5.4.2) is satisfied. □

5.4.2 Stabilization of System Θ_{32}

System Θ_3 with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} = 0$, which we denoted as Θ_{32}, is given by

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t,$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \tag{Θ_{32}}$$

and

$$\underline{U}_t = K_{1i}\underline{R}_t.$$

Theorem 5.2. *Consider the uncertain regime switching system Θ_{32}, this system is robust stochastically stabilizable if there exist matrices $L > 0$, $X_i > 0$, $Y_i > 0$, and $\epsilon_i > 0$, $\forall i \in \mathcal{S}$, such that the following LMI condition holds:*

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i - eY_i Z_i^T H_i & X_i N_{1i}^T + Y_i N_{3i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i - eH_i^T Z_i Y_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i}X_i + N_{3i}Y_i & N_{2i}L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho}L \end{bmatrix} < 0. \tag{5.4.24}$$

In this case, an appropriate robust stabilizing state feedback controller can be chosen as $\underline{U}_t = Y_i X_i^{-1}\underline{R}_t$.

Proof 5.2. From Theorem 5.1, LMI (5.4.24) guarantees that the following system (5.4.25) is robust stochastically stable. (The parameter J_{σ_t}, ΔJ_{σ_t} are replaced by $J_{\sigma_t} + Z_{\sigma_t}K_{\sigma_t}$, $\Delta J_{\sigma_t} + \Delta Z_{\sigma_t}K_{\sigma_t}$.)

$$\underline{R}_{t+1} = [J_{\sigma_t} + Z_{\sigma_t}K_{1\sigma_t} + \Delta J_{\sigma_t} + \Delta Z_{\sigma_t}K_{1\sigma_t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t}]\underline{R}_{t-\tau_t},$$

$$\underline{R}_t = \underline{\varphi_t} \quad \text{for } t \in [-\tau_{\max}, 0]. \tag{5.4.25}$$

Therefore, we have Θ_{32} is robust stochastically stable, since system Θ_{32} and system (5.4.25) describe the same system. The proof is completed. □

Remark 5.3. The theorem above provides a sufficient condition for the solvability of the robust stabilization problem for uncertain regime switching system Θ_{32}. A desired state feedback controller can be obtained by solving the LMI in (5.4.24).

5.5 Robust H_∞ Stability and H_∞ Controller Synthesis

5.5.1 Robust H_∞ Stability

In this sub-section, H_∞ stability is considered. Intuitively H_∞ stability means that the magnitude of movement in output due to the system disturbance is bounded by γ. In our application it means that the worst impact of disturbance in claim process on the reserve level is bounded when the system is robust stochastically stable. Time-delay in this chapter is mode-dependant, which is more conservative than that in Chapter 3 and 4.

Definition 5.2. The uncertain stochastic discrete time-delay system Θ_3 is said to be robust stochastically stable with disturbance attenuation level γ if it is robust stable and the (5.5.1) is satisfied,

$$||\underline{z}_t|\underline{R}_0, \sigma_0||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}, \tag{5.5.1}$$

for all nonzero $\underline{w}_t \in l_{e_2}(N; \mathbb{R}^m)$, and $\underline{w}_t$ is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$, where $\gamma > 0$ is a given scalar and $\underline{z}_t = C\underline{R}_t$ is the control output.

Here we consider the P-R system Θ_{33} which take the impact of outside disturbance $\underline{w}_{t+1}$ into account and without controller. Then the P-R process reduces to

$$\Theta_{33} : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} + \underline{w}_{t+1}, \\ \underline{z}_t = C\underline{R}_t, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}$$

Theorem 5.3. *For given scalars $\tau_{\max} > \tau_{\min} \geq 0$, the system Θ_{33} is robust stochastically stable with disturbance attenuation level $\gamma > 0$, if there exist matrices $L > 0$,*

$X_i > 0$ and $\varepsilon_i > 0$, such that the following LMI condition holds $\forall i \in \mathcal{S}$:

$$\begin{bmatrix} -X_i & 0 & 0 & X_i C^T & X_i J_i^T H_i & X_i N_{1i}^T & X_i \\ 0 & -L & 0 & 0 & -eLE_i^T H_i & LN_{2i}^T & 0 \\ 0 & 0 & -\gamma^2 I & 0 & H_i & 0 & 0 \\ CX_i & 0 & 0 & -I & 0 & 0 & 0 \\ H_i^T J_i X_i & -eH_i^T E_i L & H_i^T & 0 & \Lambda_i & 0 & 0 \\ N_{1i} X_i & N_{2i} L & 0 & 0 & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0. \tag{5.5.2}$$

Proof 5.3. Denote again

$$V_{\sigma_t}(\underline{R}_t) = V^1(\underline{R}_t) + V^2(\underline{R}_t) + V^3(\underline{R}_t), \tag{5.5.3}$$

where $V^1(\underline{R}_t)$, $V^2(\underline{R}_t)$ and $V^3(\underline{R}_t)$ are defined in (5.4.4), (5.4.5), (5.4.6). Following the same procedure as in Theorem 5.1, we can get formulas similar to (5.4.10) and (5.4.11). From (5.5.2), it is easy to deduce the following matrix

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0. \tag{5.5.4}$$

Therefore, Θ_{33} is robust stable. With the next step, our aim is to show that $||\underline{z}_t||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}$ holds for all nonzero $\underline{w}_t$ and $\gamma > 0$. To prove this, we need to define

$$T_{H_\infty} = \mathbb{E}\{\sum_{t=0}^{N}(\underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t)|\underline{R}_0, \sigma_t = 0\}. \tag{5.5.5}$$

With zero initial condition we know $V_{\sigma_0}(\underline{R}_0) = 0$. On the other hand, we have shown in Theorem 5.1 that $\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})] - \mathbb{E}[V_{\sigma_t}(\underline{R}_t)] \leq 0$. Therefore, for any time $\mathbb{T}$ we have that $\mathbb{E}\left(\sum_{t=0}^{\mathbb{T}} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0$ and $V_{\mathbb{T}}(\underline{R}_{\mathbb{T}}) \geq 0$, which after tending $\mathbb{T} \to \infty$ will give us

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0.$$

Using this relation and the definition of T_{H_∞}

$$
\begin{aligned}
T_{H_\infty} &= \mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\
&\quad -\mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t)]\right) \\
&= \mathbb{E}\left(\sum_{t=0}^{\infty}[\xi^T(t)\Psi_{\sigma_t}\xi(t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) - V_{\mathbb{T}+1}(\underline{R}_{\mathbb{T}+1}) + V_{\sigma_0}(\underline{R}_0), \mathbb{T} \to \infty \\
&\leq \mathbb{E}\left(\sum_{t=0}^{\infty}[\xi^T(t)\Psi_{\sigma_t}\xi(t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\
&= \mathbb{E}\sum_{t=0}^{\infty}\eta^T(t)\tilde{\Psi}_{\sigma_t}\eta(t), \qquad (5.5.6)
\end{aligned}
$$

where $\eta(t) = [\underline{R}_t^T \quad \underline{R}_{t-\tau_{\sigma_t}}^T \quad \underline{w}_{t+1}^T]^T$,

$$
\forall i \in \mathcal{S}, \quad \tilde{\Psi}_i = \begin{bmatrix} A_{1i} + C^T C & A_{2i} & 0 \\ A_{3i} & A_{4i} & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix}.
$$

$A_{1i}, A_{2i}, A_{3i}, A_{4i}$ are defined in proof of Theorem 5.1. With the Schur complement, the inequalities conditions in Theorem 5.3 can guarantee that for each $i \in \mathcal{S}$, $\tilde{\Psi}_i < 0$ and therefore we get $T_{H_\infty} < 0$, under zero initial conditions. Then, the system is robust stochastically stable with an H_∞ norm bound γ. □

5.5.2 H_∞ Controller of system Θ_3

Here, we consider the uncertain discrete time system Θ_3 with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} \neq 0$. It means that the actual incurred claims are not the same as the estimator. We use following LMI condition to find a feasible state H_∞ controller to control this process.

Theorem 5.4. *Consider the uncertain regime switching system Θ_3. This system is robust stochastically stabilizable with disturbance attenuation level $\gamma > 0$ if there exist matrices $X_i > 0$, $Y_i > 0$, $L > 0$, and $\epsilon_i > 0$, such that following LMI condition holds:*

$$\left[\begin{array}{cccc} -X_i & 0 & 0 & X_iC^T \\ 0 & -L & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 \\ CX_i & 0 & 0 & -I \\ H_i^T J_i X_i - eH_i^T Z_i Y_i & -eH_i^T E_i L & H_i^T & 0 \\ N_{1i}X_i + N_{3i}X_i & N_{2i}L & 0 & 0 \\ X_i & 0 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{ccc} X_i J_i^T H_i - eY_i Z_i^T H_i & X_i N_{1i}^T + Y_i N_{3i}^T & X_i \\ -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda_i & 0 & 0 \\ 0 & -\epsilon_i I & 0 \\ 0 & 0 & -\frac{1}{\varrho}L \end{array}\right] < 0. \tag{5.5.7}$$

In this case, an appropriate robust stabilizing state feedback controller can be

$$\underline{U_t} = K_{1i}\underline{R_t}, \quad K_{1i} = Y_i X_i^{-1}.$$

Proof 5.4. The proof of Theorem 5.4 is similar with Theorem 5.2, so it is omitted. □

5.5.3 Special case: One dimensional insurance line

So far, the state variable in the model is considered as a multidimensional vector, which means that it can be applied in an insurance company with multiple lines. Just for a better understanding and applicability of the main result of this chapter, here, let us assume that the system Θ_3 contains only one insurance line. Therefore, the parameters and state variables are scalar:

$$R_{t+1} = [j_{\sigma_t} + \Delta j_{\sigma_t,t}]R_t - e[\varepsilon_{\sigma_t} + \Delta\varepsilon_{\sigma_t,t}]R_{t-\tau_t} - e[z_{\sigma_t} + \Delta z_{\sigma_t,t}]U_t + w_{t+1},$$

$$R_t = \varphi_t \text{ for } t \in [-\tau_{\max}, 0]. \tag{5.5.8}$$

Proposition 1. *Consider the above scalar system, This system is robust stochastically stabilizable with disturbance attenuation level* γ *if there exist scalar* $x_i > 0$, $y_i > 0$, $l > 0$, *and* $p_i > 0$, *such that the following condition holds*

$$\begin{bmatrix} -x_i & 0 & 0 & x_i c & x_i j_i h_i - e y_i z_i h_i & x_i n_{1i} + y_i n_{3i} & x_i \\ 0 & -l & 0 & 0 & -e l e_i h_i & l n_{2i} & 0 \\ 0 & 0 & -\gamma^2 I & 0 & h_i & 0 & 0 \\ c x_i & 0 & 0 & -1 & 0 & 0 & 0 \\ h_i j_i x_i - e h_i z_i y_i & -e h_i e_i L & h_i & 0 & \Lambda_i & 0 & 0 \\ n_{1i} x_i + n_{3i} x_i & n_{2i} L & 0 & 0 & 0 & -p_i & 0 \\ x_i & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0, \tag{5.5.9}$$

In this case, an appropriate robust stabilizing state feedback controller can be

$$U_t = K_{1i} R_t, \ \textit{where } K_{1i} = y_i x_i^{-1}.$$

Remark 5.4. When the model is used in studying general financial strength conditions, it is useful at first to define a basic case (nominal system), in which certain specified values are fixed for the parameters of the model. Then sensitivity analysis can be carried out. By varying the size of the portfolio, its composition and other basic parameters, it is possible to study how the business reacts to various external and internal impulses.

Here, j_i, e_i, z_i are known real constant matrices with appropriate dimensions representing the nominal systems for each $i \in \mathcal{S}$.

5.6 Numerical Application 3

In this section, a numerical application for illustrating the applicability of the theoretical results for an insurance company is formulated. We assume that it runs three different insurance lines which are mutually correlated. Then, we use the result from Theorem 5.4 to find out the H_∞ controller such that the total reserve process is stabilized with a particular disturbance attenuation level γ. Let us recall that when the model is applied by a particular insurer, the basic parameters, the parameter uncertainty and disturbance distribution have to be estimated based on real data and realistic assumptions. Here, we assume that the Markovian switching state space is $\mathcal{S} = [1, 2]$, which indicates that there are two different system regimes for the system Θ_3. In the following paragraphs, the necessary parameters are described in details.

- First the value of the reserve accounts at $t = 0$ is given by the following matrix,

$$\underline{R}_0 = \begin{bmatrix} R_0(1) \\ R_0(2) \\ R_0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

  i.e. at time $t = 0$, we assume that the reserve account for each insurance lines is £ 0 pounds, respectively.

- For the time delay, we assume that the mode-dependent delay are $\tau(i = 1) = 3$ for Regime 1 and $\tau(i = 2) = 1$ for Regime 2. Therefore, $\tau_{\min} = 1$ and $\tau_{\max} = 3$:

$$\underline{R}_{-3} = \begin{bmatrix} R_{-3}(1) \\ R_{-3}(2) \\ R_{-3}(3) \end{bmatrix} = \underline{R}_{-2} = \begin{bmatrix} R_{-2}(1) \\ R_{-2}(2) \\ R_{-2}(3) \end{bmatrix} = \underline{R}_{-1} = \begin{bmatrix} R_{-1}(1) \\ R_{-1}(2) \\ R_{-1}(3) \end{bmatrix} = \begin{bmatrix} £270,000 \\ £340,000 \\ £160,000 \end{bmatrix}.$$

- In our model, it is assumed that the insurer can invest the reserve in risk-free investments (T-bills) to generate additional income. Since dependencies among 3 insurance lines exist, we have to use weights in the parameter matrix. We assume that the corresponding rate of income is given from the following matrix:
  For Regime 1

$$J_1 = \begin{bmatrix} 1.021 * w_{1,1} & 1.021 * w_{1,2} & 1.021 * w_{1,3} \\ 1.021 * w_{2,1} & 1.021 * w_{2,2} & 1.021 * w_{2,3} \\ 1.021 * w_{3,1} & 1.021 * w_{3,2} & 1.021 * w_{3,3} \end{bmatrix}.$$

For Regime 2

$$J_2 = \begin{bmatrix} 1.039 * w_{1,1} & 1.039 * w_{1,2} & 1.039 * w_{1,3} \\ 1.039 * w_{2,1} & 1.039 * w_{2,2} & 1.039 * w_{2,3} \\ 1.039 * w_{3,1} & 1.039 * w_{3,2} & 1.039 * w_{3,3} \end{bmatrix}.$$

- The weight ratios w_{nm} which demonstrates the solvency relation between each line have the following values:

  $w_{1,1} = 0.86$, $w_{1,2} = 0.07$ and $w_{1,3} = 0.07$,

  $w_{2,1} = 0.10$, $w_{2,2} = 0.87$ and $w_{2,3} = 0.03$,

  $w_{3,1} = 0.08$, $w_{3,2} = 0.09$ and $w_{3,3} = 0.83$.

- The parameter E comes from the mechanism proposed by Balzer and Benjamin (1980, 1982). The value of E could be the constant base return rate of policyholder rather than issuer.

  For the examples, we assume that the value in the parameter matrix E:

  For Regime 1

$$E_1 = \begin{bmatrix} 0.13 * w_{1,1} & 0.13 * w_{1,2} & 0.13 * w_{1,3} \\ 0.13 * w_{2,1} & 0.13 * w_{2,2} & 0.13 * w_{2,3} \\ 0.13 * w_{3,1} & 0.13 * w_{3,2} & 0.13 * w_{3,3} \end{bmatrix},$$

  For Regime 2

$$E_2 = \begin{bmatrix} 0.18 * w_{1,1} & 0.18 * w_{1,2} & 0.18 * w_{1,3} \\ 0.18 * w_{2,1} & 0.18 * w_{2,2} & 0.18 * w_{2,3} \\ 0.18 * w_{3,1} & 0.18 * w_{3,2} & 0.18 * w_{3,3} \end{bmatrix}.$$

- For the parameter e, we let $e = 0.8$, which means that $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the administration and operating cost and give to the company a reasonable profit margin.

- $\gamma = 3.7$. This is the given value (not optimal) which measures the maximum impact level of the disturbance on the reserves.

- The time-varying unknown parameter uncertainties $\Delta J_{i,n}$, $\Delta E_{i,n}$ and $\Delta Z_{i,n}$, $i \in [1, 2]$ are defined by:

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}],$$

where

$$M_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 0.002 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 0.005 & 0 \\ 0 & 0 & 0.004 \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}, N_{21} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}, N_{31} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

$$N_{12} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}, N_{22} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}, N_{32} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

- We assume that the insurer will change the operating regime influenced by some key economic and market factors which are not constant. In this application, it is assumed that the insurer can switch between 2 regimes. Thus, two different transition probabilities are. Type 2 switching transits more frequently than Type 1.

  Transition probability (Type 1 switching)

$$\Pi_1 = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}.$$

  Transition probability (Type 2 switching)

$$\Pi_2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}.$$

Here, the performance of system under different markovian switching signals is presented. The simulation results are provided for the time-period of $t = 52$ weeks.

By applying the result of the Theorem 5.4, the H_∞ controller is derived, and we get the feedback controller for each regime separately under **Type 1 switching signal** (see Figure 5.2) are as below:

If system is in Regime 1:

$$K_{11} = \begin{bmatrix} 0.9491 & 0.0197 & -0.0047 \\ 0.0867 & 1.1114 & 0.0364 \\ 0.0029 & -0.0794 & 1.0063 \end{bmatrix}.$$

If system is in Regime 2:

$$K_{12} = \begin{bmatrix} 0.9381 & 0.0025 & -0.0189 \\ 0.0792 & 1.1370 & 0.0211 \\ 0.0021 & -0.1045 & 1.0176 \end{bmatrix}.$$

It is clear that under the Type 1 switching signal (Figure 5.2), not too many changes are proposed between the two modes (regimes). Generally speaking, it can be considered as a quite stable case.

Now, when the model is under **Type 2 switching signal** (see Figure 5.3), the controller for each regime is as below.

If system is in Mode 1:

$$K_{11} = \begin{bmatrix} 0.9479 & 0.0216 & -0.0066 \\ 0.0909 & 1.1135 & 0.0382 \\ 0.0012 & -0.0817 & 1.0075 \end{bmatrix}.$$

If system is in Mode 2:

$$K_{12} = \begin{bmatrix} 0.9365 & 0.0013 & -0.0199 \\ 0.0788 & 1.1373 & 0.0204 \\ 0.0021 & -0.1055 & 1.0173 \end{bmatrix}.$$

On contrary under the Type 2 switching signal (see Figure 5.3), the changes between the two modes (regimes) vary frequently. Thus, it can be seen as a quite volatile case.

In Figure 5.4 and 5.5, the movement of the charged premium is presented for the three lines under the Type 1 and 2 signal, respectively. From those figures, we can clearly see that the controlled premium for each dependent line fluctuates around £150,000 (no drift is observed though for any of the available lines and for both signals). Moreover, it should be mentioned that the premium for each dependent line stays positive for the whole duration of the simulations.

Obviously, as we can also observe in the Figure 5.4 and 5.5, the state feedback controller $\underline{U}_t$ helps to reduce the impact of the disturbance and eventually stabilizes the system quickly. Thus, in the Figure 5.6 and 5.7, the movement of the charged reserve is presented for the three lines under the Type 1 and 2 signal, respectively. Finally, it is interested in observing the Figure 5.8, where the total reserve is presented and a comparison is provided for both types of signals. Obviously, the reason that the reserve is not exactly converging to 0, see also Pantelous and Yang (2014) [52], is related to the fact that new random disturbances affect the system. As it is expected, the Type 2 signal gives higher fluctuation compared with the Type 1 signal.

To summarize in this application, by using the robust H_∞ tool to generate the state feedback controller $\underline{U}_t$, we manipulate the stability of the system even though the system disturbance $\underline{w}_t \neq 0$.

5.7 Summary

In this chapter, a Markovian regime switching P-R model for different insurance lines has been proposed in order to describe abrupt changes in structures. This regime switching model considers a negative feedback mechanism for the reserves, invests the surplus in short-term risk-free (T-bills) assets, and also assumes time-varying, bounded delays for the reserves in a stochastic, discrete-time framework. The parameter uncertainties for the coefficients involved in the model are also norm-bounded. Thus, the new model extends significantly the models proposed by Zimbidis and Haberman (2001) [82], Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014, 2015) [52, 53].

Additionally, a control parameter is introduced in the system Θ_3 and some new ideas to generate an effective state feedback controller for the P-R system are presented. The LMI conditions for the robust stabilization and a feasible H_∞ controller are derived through a series of Lemmas and Theorems. Thus, for the very first time, according to our knowledge, a linear robust control theory for Markovian regime switching systems has been implemented in the P-R model. Thus, with the H_∞ controller, the premium is adjusted to reasonable levels for different modes (regimes). Both robust stochastic stability and a pre-specified disturbance attenuation level can be guaranteed for all admissible uncertainties. Corresponding results have been illustrated by introducing a numerical example.

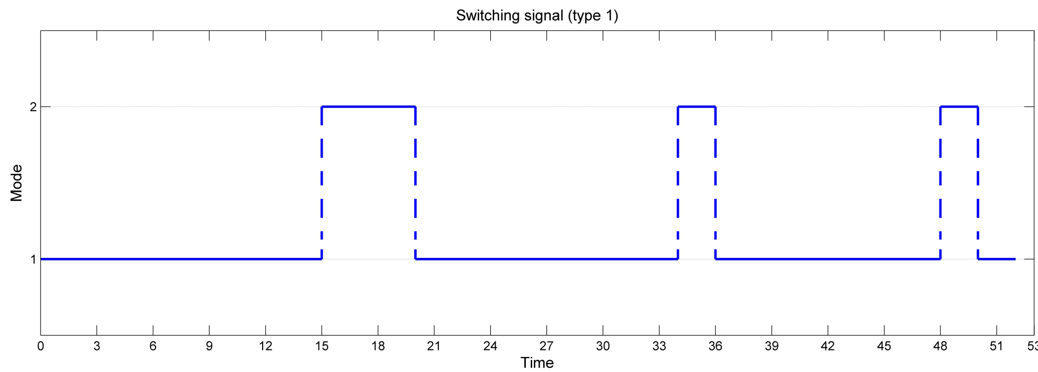

Figure 5.2: Markovian switching signal: Type 1

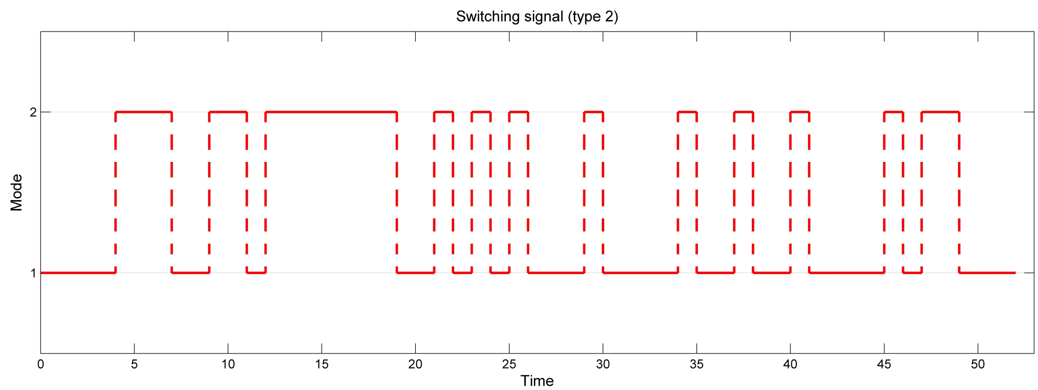

Figure 5.3: Markovian switching signal: Type 2

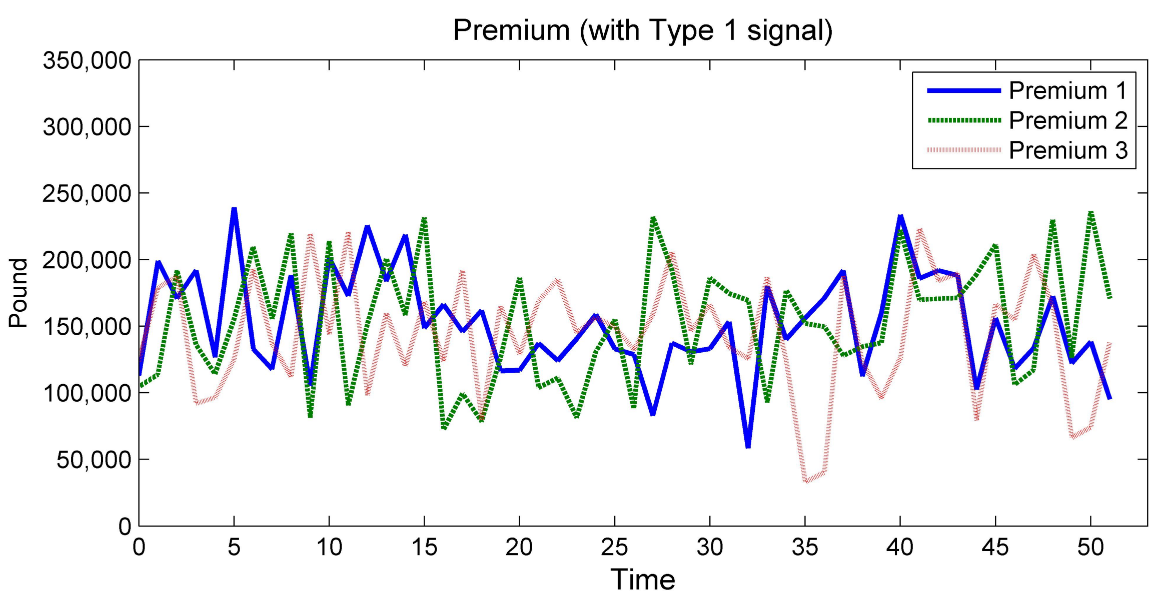

Figure 5.4: The evolution of the three Premiums under the Type 1 signal

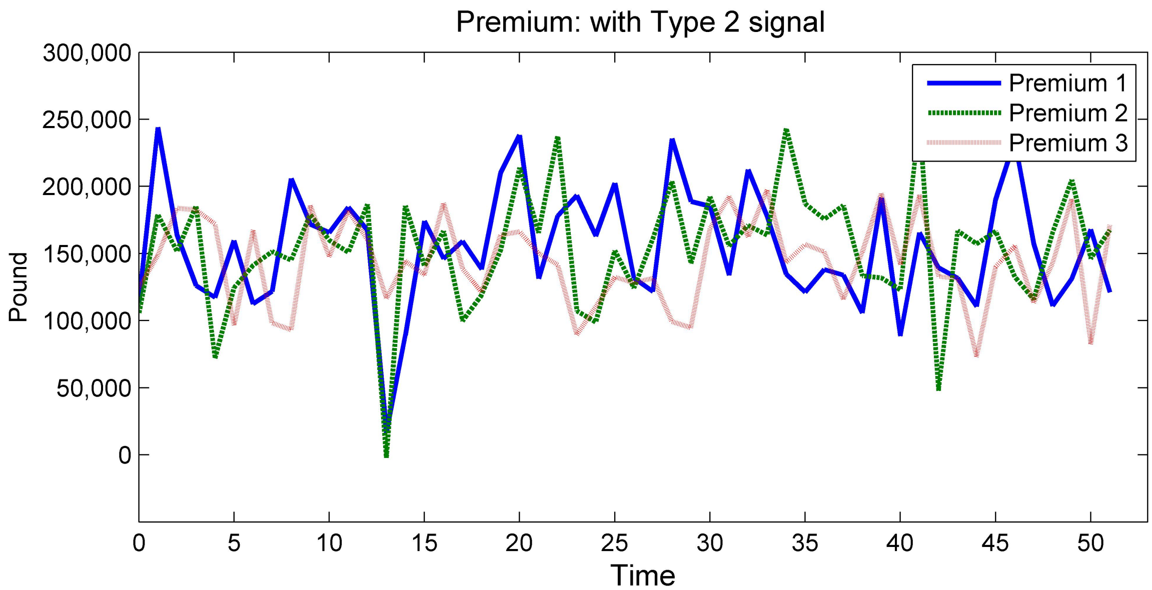

Figure 5.5: The evolution of the three Premiums under the Type 2 signal

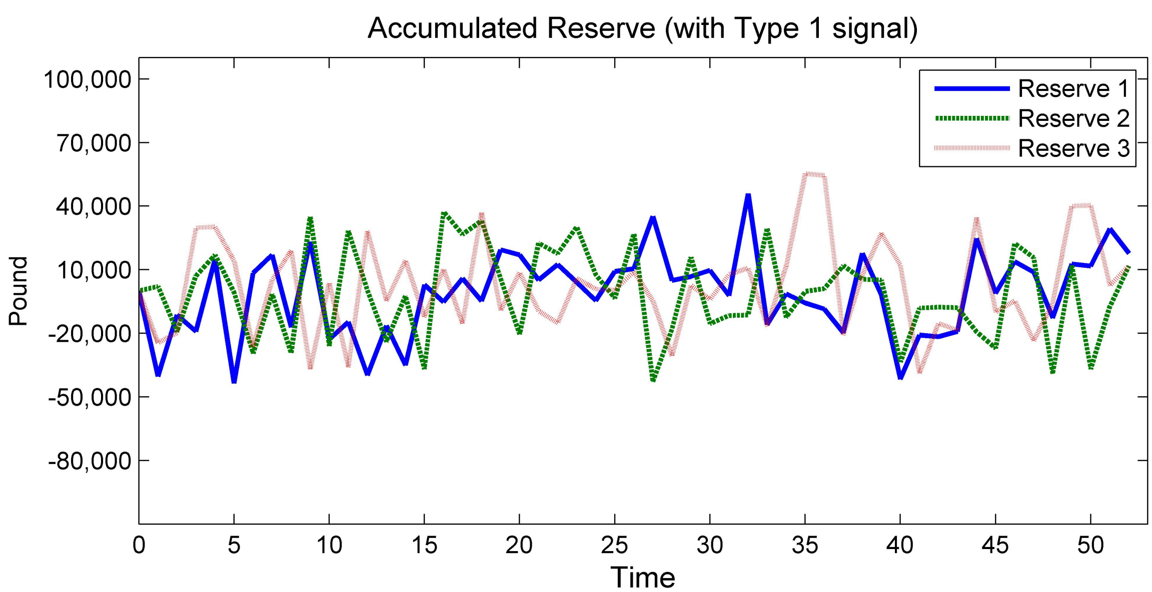

Figure 5.6: The evolution of the accumulated reserves under the Type 1 signal

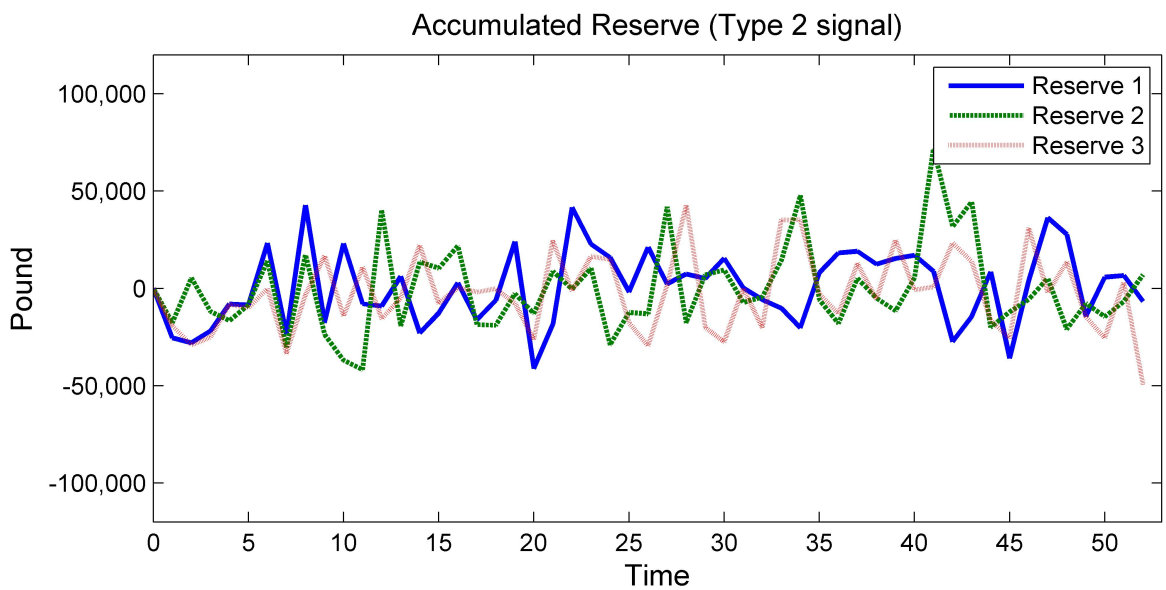

Figure 5.7: The evolution of the accumulated reserves under the Type 2 signal

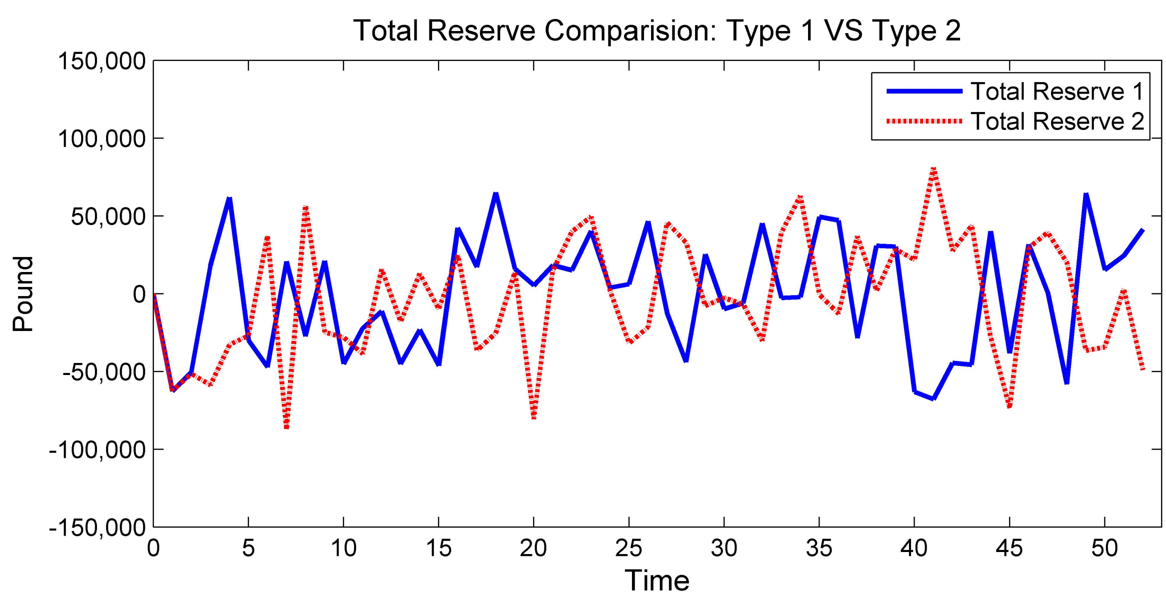

Figure 5.8: The comparison of the total reserve: Type 1 vs Type 2 switching

Chapter 6

Arbitrary Regime Switching System

6.1 Introduction

Regime-switching systems have attracted much attention in the last decade. Regime-switching models have become an powerful modelling tool for applied work, such as in the highway supervisory system, the constrained robotics, the control of aircraft and air traffic control and so on. Particularly we should note applications in measures of economic output, such as real Gross Domestic Product (GDP), which have been used to model and identify the phases of the business cycle, regime shift in inflation and interest rates. Thus many important results related to switched systems have been reported in the literature, see Piger (2011) [59].

Regime-switching models can be classified into two categories: Arbitrary regime switching models and Markov regime switching models. The primary difference between these two approaches is in how the evolution of the state process is modelled. Generally arbitrary regime switching model gives more conservative result than Markov regime switching model, because we know less regime shifts information in arbitrary regime switching model, see Sun et al. (2007) [67].

A model which uses the recent claim experience and a negative feedback mechanism of the known surplus value is proposed in Pantelous and Yang (2014) [52]. That model assumes a time-varying, bounded delay factor, time-varying parameters and different types of norm-bounded uncertainties. In Chapter 5, the impact of the switching regime is analysed by considering a markovian switching signal. However, in contrast to markovian switching regime, it can be also assumed that the switching sequence is not known a priori and look for stability results under arbitrary switching sequences. In this chapter, we would like to reformulate that system and investigate the stability and H_∞ controller of the system which consider an arbitrary switching signal. This would be a complementary research for the Chapter 5.

For P-R system of insurance product, it is worth noting that we investigate the stability under arbitrary switching signal. For this issue, the Lyapunov function method is proposed to study this P-R system.

In this chapter, we investigate the problems of stability analysis and H_∞ controller synthesis for arbitrary regime switching systems. The contribution of this chapter lies in that the extended stability and H_∞ controller design results for regime switching P-R systems with mode-dependent delay are given. First, the correlative assumptions and definitions are proposed. Then the model is transformed into discrete time arbitrary switching systems with mode-dependent time delay, which is quite similar with the system in Chapter 5. Then based on the result in Sun et al. (2007) [67], we apply the descriptor system approach to uncertain discrete-time switched systems with mode-dependent delays by constructing a switched Lyapunov function which is important for the late development. Through some useful lemma and LMI, the new stability criterion is proposed and H_∞ controller is generated, which guarantees the stability of the system in this chapter. Finally, a numerical example is exploited to demonstrate the effectiveness of the developed method.

6.2 Problem formulation

6.2.1 Assumptions

Here, the necessary notation and basic assumptions for our model are described. Some assumptions are almost the same with that in Zimbidis and Haberman (2001) [82], Pantelous and Papageorgiou (2013) [51] and Pantelous and Yang (2014, 2015) [52, 53]. so only a brief explanation for the different assumptions is provided here.

Assumption 6.1: Same with Assumption 3.1 in Chapter 3.

Assumption 6.2: Same with Assumption 3.2 and Assumption 3.3 in Chapter 3.

Assumption 6.3: Same with Assumption 3.4 in Chapter 3.

Assumption 6.4: Let $\{\sigma_t; t \geq 0\}$ be a arbitrary switching signal with state space $\mathcal{S} = \{1, 2 \cdots N\}$. σ_t is a piecewise constant function of time and the transition probability is is unknown or not existed. We assume that the switching signal σ_t is unknown a prior, but its instantaneous value is available in real time.

Assumption 6.5: Positive integer τ_i represents the time delay when the system operates in the regime i. Then we denote

$$\tau_{\max} = \max\{\tau_i, i \in \mathcal{S}\},$$

$$\tau_{\min} = \min\{\tau_i, i \in \mathcal{S}\}.$$

We consider a mode-dependent time-varying delay, τ_t, which is upper and lower bounded, i.e. $\tau_{\min} \leq \tau_t \leq \tau_{\max}$ with $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. So, considering a specific time-delay interval, at the end of each year $[t, t+1]$, we have the exact information up to the end of the year $t - \tau_t$. As indicated in previous chapters' assumption, the value for τ_i can be estimated using past experience and statistical data. Moreover, the national and international regulatory policy might be also applied for defining the upper bound of this interval.

Assumption 6.6: Same with Assumption 3.6 in Chapter 3.

Assumption 6.7: Same with Assumption 3.7 in Chapter 3..

6.2.2 Model Formulation

In the present chapter, the P-R process is described by a arbitrary regime switching system with time-varying delays which extend the model used in Chapter 5. Assume $\underline{R}_t = (R_{1,t}R_{2,t} \cdots R_{m,t})^T$ be the vector expression of the accumulated reserves, where $R_{i,t}$ is the accumulated reserves of i^{th} product at time t. As in Chapter 5, the premium process is formulated as follow:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t. \tag{6.2.1}$$

$\underline{U}_t \in \mathbb{R}^m$ is the control input. Here, we develop the model into the arbitrary switched system. Let $\underline{R}_t = (R_{1,t}R_{2,t} \cdots R_{m,t})^T$ be the vector expression of the reserves, where $R_{i,t}$ is the reserve of i^{th} insurance line at time t. The reserve, $\underline{R}_t$, evolves according to

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\underline{P}_{t+1} - \underline{C}_{t+1}. \tag{6.2.2}$$

Same with those in Chapter 5, J_{σ_t} is the investment return matrices in time t for the risk-free asset. It is possible to include also risky assets but we leave it for a future work. Switching signal σ_t is a piecewise constant function of time which takes value i in the finite set $\mathcal{S} = [1 \quad 2 \cdots N]$. We assume that the switching signal σ_t is governed by a Arbitrary jump process (see Assumption 6.4). The premiums are assumed to be the *earned premiums* and claims are *incurred claims* as well. From the equations (6.2.1) and (6.2.2), we get

$$
\begin{aligned}
\underline{R}_{t+1} &= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\{\underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t\} - \underline{C}_{t+1} \\
&= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1}.
\end{aligned}
$$

The parameters J_i, E_i and Z_i are real constant base matrices. $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are the respective parameter uncertainties. For the purpose of the modelling process, J_i and E_i respectively could be a risk-free interest rate and a constant-base return to the policyholders. Then, Z_i is a parameter of the control input. Finally, $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are unknown matrices representing time-varying parameter uncertainties, and they are assumed to be of the form:

$$
[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}], \tag{6.2.3}
$$

$M_i, N_{1i}, N_{2i}, N_{3i}$ are known real constant matrices and $F_t : \mathbb{N} \to \mathbb{R}^{s\times j}$ is an unknown time-varying matrix function satisfying

$$
F_t^T F_t \leq I, \quad \forall t \in \mathbb{N}, \tag{6.2.4}
$$

$\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are said to be admissible if they satisfy both (6.2.3) and (6.2.4). Thus we have the following discrete time arbitrary regime switching linear P-R system:

$$
\Theta_4 : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1} \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases}
$$

The system has N system regimes. We denote system Θ_4 without controller element $\underline{U}_t$ and disturbance $\underline{w}_{t+1}$ as Θ_{41}. System Θ_4 without disturbance $\underline{w}_{t+1}$ is denoted as Θ_{42}. The observation is denoted as $\underline{z}_t$, where $\underline{z}_t = C\underline{R}_t$ is the control output.

Lemma 6.1. *(Gu et al. 2003 [30]) Assume that* $\tau_t : Z^+ \to 1, 2, \ldots$ *and* $\tau_t < \tau_{\max}$*, where* $\tau_{\max}$ *is a positive integer, then for any positive-definite matrix* $P \in \mathbb{R}^{n\times n}$ *and vector function* $\underline{R}_t$*, we have*

$$\tau_{\max} \sum_{m=t-\tau_{\max}}^{t-1} \underline{Y}_m^T Q \underline{Y}_m > \left\{ \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m^T \} \right\} Q \left\{ \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m \right\}$$

This chapter is concerned with the robust stability analysis and design problems for the arbitrary switched P-R system and it is complementary research of markovian switched system in Chapter 5. Our objective is to present an approach to investigate and manipulate the stability of arbitrary switched P-R system.

6.3 Robust stability and stabilitzation

6.3.1 Robust stability of system Θ_{41}

The uncertain discrete time system Θ_4 with $\underline{U}_t = 0$ and $\underline{w}_{t+1} = 0$.

$$\Theta_{41} : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases} \tag{6.3.1}$$

Theorem 6.1. *The system* Θ_{41} *is robust stochastically stable for any time-varying delay* τ_{σ_t} *satisfying* $\tau_{\max} > \tau_{\min} \geq 0$*, if there exist matrices* P_i*,* $Q > 0$*,* L_i*,* S_i*,* $\epsilon_i > 0$*, such that the following conditions hold for* $\forall (i, j) \in \mathcal{S} \times \mathcal{S}$*:*

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & eL_i^T E_i & L_i^T M_i & (N_{1i} + N_{2i})^T \\ \Lambda_2^T & \Lambda_3 & eS_i^T E_i & S_i^T M_i & 0 \\ eE_i^T L_i & eE_i^T S_i & -Q & 0 & -N_{2i}^T \\ M_i^T L_i & M_i^T S_i & 0 & -\epsilon_i I & 0 \\ N_{1i} + N_{2i} & 0 & -N_{2i} & 0 & -\epsilon_i^{-1} I \end{bmatrix} < 0, \tag{6.3.2}$$

where $\Lambda_1 = L_i^T [J_i - eE_i - I] + [J_i - eE_i - I]^T L_i + P_j - P_i$*,*
$\Lambda_2 = P_j - L_i^T + [J_i - eE_i - I]^T S_i$*,*
$\Lambda_3 = P_j + \tau_{\max}^2 Q - S_i - S_i^T$*.*

Proof 6.1.

$$\begin{aligned}\underline{Y}_t &\triangleq \underline{R}_{t+1} - \underline{R}_t \\ &= [J_i + \Delta J_{i,t} - I]\underline{R}_t - e[E_i + \Delta E_{i,t}]\underline{R}_{t-\tau_{\sigma_t}},\end{aligned} \tag{6.3.3}$$

Since

$$\underline{R}_{t-\tau_{\sigma_t}} = \underline{R}_t - \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m \tag{6.3.4}$$

Then, system Θ_1 can be transformed into

$$\underline{R}_{t+1} = [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]\underline{R}_t + e[E_i + \Delta E_{i,t}] \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m, \tag{6.3.5}$$

$$\underline{Y}_t = \{J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - I\}\underline{R}_t + e[E_i + \Delta E_{i,t}] \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m, \tag{6.3.6}$$

We can construct a switching Lyapunov Function:

$$V_{\sigma_t}(\underline{R}_t) = \underline{R}_t^T P_{\sigma_t} \underline{R}_t + \tau_{\max} \sum_{k=-\tau_{\max}+1}^{0} \sum_{m=t-1+k}^{t-1} \underline{Y}_m^T Q \underline{Y}_m \tag{6.3.7}$$

P_{σ_t}, Q is the feasible solution satisfying 6.3.2. Define $\Delta V_{\sigma_t}(\underline{R}_t) = \mathbb{E}[V_{\sigma_t}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)]$ and the transition regime at time t, $t+1$ are $\sigma_t = i, \sigma_{t+1} = j$.

$$\begin{aligned}\Delta V_{\sigma_t}(\underline{R}_t) &= \mathbb{E}[V_{\sigma_t}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)] \\ &= \underline{R}_{t+1}^T P_j \underline{R}_{t+1} - \underline{R}_t^T P_i \underline{R}_t + \tau_{\max}^2 \underline{Y}_t^T Q \underline{Y}_t - \tau_{\max} \sum_{m=t-\tau_{\max}}^{t-1} \underline{Y}_m^T Q \underline{Y}_m \\ &\leq 2\underline{R}_t^T P_j \underline{Y}_t + \underline{R}_t^T (P_j - P_i)\underline{R}_t + \underline{Y}_t^T (P_j + \tau_{\max}^2 Q)\underline{Y}_t \\ &\quad - \left\{ \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m^T\} \right\} Q \left\{ \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m \right\}.\end{aligned} \tag{6.3.8}$$

From (6.3.6) we know

$$2[\underline{R}_t^T L_i^T + \underline{Y}_t^T X_i^T]\{-\underline{Y}_t + \{J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - I\}\underline{R}_t + e[E_i + \Delta E_{i,t}] \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m\} = 0. \tag{6.3.9}$$

We set $\underline{N}_t = \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m$ and combine (6.3.8) with (6.3.9). Then, the following formula can be derived:

$$\Delta V_{\sigma_t}(\underline{R}_t) \le \xi^T(t)\Psi_{ij}\xi(t), \tag{6.3.10}$$

where

$$\xi(t) = [\underline{R}_t^T \quad \underline{Y}_t^T \quad \underline{N}_t^T]^T,$$

and

$$\Psi_{ij} = \begin{bmatrix} A_1 & A_2 & eL_i^T E_i \\ A_2^T & A_3 & eS_i^T E_i \\ eE_i^T L_i & eE_i^T S_i & -Q \end{bmatrix} [\tag{6.3.11}$$

with

$$A_1 = L_i^T[J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - I] + [J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - I]^T L_i + P_j - P_i,$$

$$A_2 = P_j - L_i^T + [J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - I]^T S_i,$$

$$A_3 = P_j + \tau_{\max}^2 Q - S_i - S_i^T.$$

Then, we can develop (6.3.11) to

$$\begin{aligned} \Psi_{ij} \quad = \quad & \begin{bmatrix} \Lambda_1 & \Lambda_2 & eL_i^T E_i \\ \Lambda_2^T & \Lambda_3 & eS_i^T E_i \\ eE_i^T L_i & eE_i^T S_i & -Q \end{bmatrix} + \begin{bmatrix} L_i^T M_i \\ S_i^T M_i \\ 0 \end{bmatrix} F_t \begin{bmatrix} N_{1i} + N_{2i} & 0 & -N_{2i} \end{bmatrix} \\ & + \begin{bmatrix} (N_{1i} + N_{2i})^T \\ 0 \\ -N_{2i}^T \end{bmatrix} F_t^T \begin{bmatrix} M_i^T L_i & M_i^T S_i & 0 \end{bmatrix}, i, j \in \mathcal{S}. \end{aligned} \tag{6.3.12}$$

Similar with the method in Chapter 3 and Chapter 5, if LMI condition (6.3.2) is satisfied, it can lead to the following inequality by Schur complement

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & eL_i^T E_i \\ \Lambda_2^T & \Lambda_3 & eS_i^T E_i \\ eE_i^T L_i & eE_i^T S_i & -Q \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} L_i^T M_i \\ S_i^T M_i \\ 0 \end{bmatrix} \begin{bmatrix} M_i^T L_i & M_i^T S_i & 0 \end{bmatrix}$$
$$+\epsilon_i \begin{bmatrix} (N_{1i}+N_{2i})^T \\ 0 \\ -N_{2i}^T \end{bmatrix} \begin{bmatrix} N_{1i}+N_{2i} & 0 & -N_{2i} \end{bmatrix} < 0, i, j \in \mathcal{S}. \quad (6.3.13)$$

According to Lemma 5.1, which is the result in Xie et al. (1992) [71], it indicates that

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & eL_i^T E_i \\ \Lambda_2^T & \Lambda_3 & eS_i^T E_i \\ eE_i^T L_i & eE_i^T S_i & -Q \end{bmatrix} + \begin{bmatrix} L_i^T M_i \\ S_i^T M_i \\ 0 \end{bmatrix} F_t \begin{bmatrix} N_{1i}+N_{2i} & 0 & -N_{2i} \end{bmatrix}$$
$$+ \begin{bmatrix} (N_{1i}+N_{2i})^T \\ 0 \\ -N_{2i}^T \end{bmatrix} F_t^T \begin{bmatrix} M_i^T L_i & M_i^T S_i & 0 \end{bmatrix} < 0, i, j \in \mathcal{S}. \quad (6.3.14)$$

It means the LMI condition (6.3.2) can guarantee that $\Psi_{ij} < 0$. Therefore, $\Delta V_{\sigma_t}(\underline{R}_t) < 0$ is always satisfied for all $t \geq 0$. Using the standard Lyapunov stability theory we have that the system Θ_{41} is robust stable when LMI condition 6.3.2 is satisfied. This completes the proof. □

6.3.2 Stabilization of system Θ_{42}

System Θ_4 with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} = 0$, which we denoted as Θ_{42}, is

$$\Theta_{42} : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_{\sigma(t)}} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t, \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \end{cases} \quad (6.3.15)$$

The state feedback controller is in the form of $\underline{U}_i = K_{1i}\underline{R}_t$.

Theorem 6.2. *The uncertain switched system Θ_{42} is robust stochastically stabilizable* *with state, if for each $i \in \mathcal{S}$ there exist matrices X_i, Y_i, $\tilde{Q} > 0$, B_i, D_i, $\epsilon_i > 0$, such that the following LMI conditions hold for $\forall(i,j) \in \mathcal{S} \times \mathcal{S}$:*

$$\begin{bmatrix} -X_i & \Pi_1 & 0 & 0 & \Pi_2 & X_i + D_i^T & \tau_{\max} D_i^T \\ \Pi_1^T & \Pi_3 & e\tilde{Q}E_i & M_i & 0 & B_i^T & \tau_{\max} B_i^T \\ 0 & eE_i^T\tilde{Q} & -\tilde{Q} & 0 & -\tilde{Q}N_{2i}^T & 0 & 0 \\ 0 & M_i^T & 0 & -\epsilon_i I & 0 & 0 & 0 \\ \Pi_2^T & 0 & -\tilde{Q}^T N_{2i} & 0 & -\epsilon_i^{-1} I & 0 & 0 \\ X_i^T + D_i & B_i & 0 & 0 & 0 & -X_j & 0 \\ \tau_{\max} D_i & \tau_{\max} B_i & 0 & 0 & 0 & 0 & -U \end{bmatrix} < 0. \quad (6.3.16)$$

In this case, an appropriate robust stabilizing state feedback controller can be chosen as $\underline{U}_t = Y_i X_i^{-1} \underline{R}_t$, where

$$\Pi_1 = X_i[J_i - eE_i - I]^T - eY_i^T Z_i^T - D_i^T,$$

$$\Pi_2 = X_i[N_{1i} + N_{2i}]^T + Y_i^T N_{3i}^T,$$

$$\Pi_3 = -B_i - B_i^T,$$

Proof 6.2. From Theorem 6.1 and for $\sigma_t = i, \sigma_{t+1} = j$, we have that the sufficient condition for robust stability of system Θ_{42} is

$$\begin{aligned} \Phi_{ij} \;=\; & \begin{bmatrix} \Phi_1 & \Phi_2 & eL_i^T E_i \\ \Phi_2^T & \Phi_3 & eS_i^T E_i \\ eE_i^T L_i & eE_i^T S_i & -Q \end{bmatrix} + \begin{bmatrix} L_i^T M_i \\ S_i^T M_i \\ 0 \end{bmatrix} F_t \begin{bmatrix} N_{1i} + N_{2i} & 0 & -N_{2i} \end{bmatrix} \\ & + \begin{bmatrix} (N_{1i} + N_{2i})^T \\ 0 \\ -N_{2i}^T \end{bmatrix} F_t^T \begin{bmatrix} M_i^T L_i & M_i^T S_i & 0 \end{bmatrix}, i, j \in \mathcal{S}. \end{aligned} \quad (6.3.17)$$

where

$\Phi_1 = L_i^T[J_i - eE_i - eZ_iK_{1i} - I] + [J_i - eE_i - eZ_iK_{1i} - I]^T L_i + P_j - P_i$,

$\Phi_2 = P_j - L_i^T + [J_i - eE_i - eZ_iK_{1i} - I]^T S_i$,

$\Phi_3 = P_j + \tau_{\max}^2 Q - S_i - S_i^T$.

We set $X_i = P_i^{-1}$, $B_i = S_i^{-1}$, $D_i = -B_i L_i X_i$, $\tilde{Q} = Q^{-1}$ and $Y_i = K_{1i} X_i$. Based on the result in Sun et al. (2007) [67], we have LMI conditions (6.3.16) is equivalent with (6.3.17). Therefore, we have Θ_{42} is robust stochastically stable. The proof for Theorem 6.2 is completed. $\square$

6.4 Robust H_∞ stability and H_∞ Controller of system Θ_4

Here we consider the P-R system Θ_4 which take the impact of outside disturbance $\underline{w}_{t+1}$ t and controller into account.

$$\Theta_4 : \begin{cases} \underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1} \\ \underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \\ \underline{z}_t = C\underline{R}_t \end{cases}$$

and state feedback controller:

$$\underline{U}_t = K_{1i}\underline{R}_t.$$

Theorem 6.3. *The system Θ_4 is robust stabilizable with noise attenuation level γ, if there exist matrices X_i,Y_i, $\tilde{Q} > 0$, B_i, and D_i, such that the following conditions hold $\forall (i,j) \in \mathcal{S} \times \mathcal{S}$:*

$$\begin{bmatrix} -X_i & \Pi_1 & 0 & 0 & X_iC^T & 0 & \Pi_2 & X_i + D_i^T & \tau_{\max}D_i^T \\ \Pi_1^T & \Pi_3 & e\tilde{Q}E_i & I & 0 & M_i & 0 & B_i^T & \tau_{\max}B_i^T \\ 0 & eE_i^T\tilde{Q} & -\tilde{Q} & 0 & 0 & 0 & -\tilde{Q}N_{2i}^T & 0 & 0 \\ 0 & I & 0 & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 \\ CX_i^T & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & M_i^T & 0 & 0 & 0 & -\epsilon_i I & 0 & 0 & 0 \\ \Pi_2^T & 0 & -\tilde{Q}^T N_{2i} & 0 & 0 & 0 & -\epsilon_i^{-1} I & 0 & 0 \\ X_i^T + D_i & B_i & 0 & 0 & 0 & 0 & 0 & -X_j & 0 \\ \tau_{\max}D_i & \tau_{\max}B_i & 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{Q} \end{bmatrix} < 0. \tag{6.4.1}$$

In this case, an appropriate robust stabilizing state feedback controller can be chosen as $\underline{U}_t = Y_iX_i^{-1}\underline{R}_t$.

Proof 6.3. We redefine system Θ_4 as

$$\begin{cases} \underline{R}_{t+1} = \underline{R}_t + \underline{Y}_t, \\ \underline{Y}_t = \{J_i + \Delta J_{i,t} - e[E_i + \Delta E_{i,t}] - e[Z_i + \Delta Z_{i,t}]K_{1i} - I\}\underline{R}_t + \underline{w}_{t+1} \\ \qquad + e[E_i + \Delta E_{i,t}] \sum_{m=t-\tau_{\sigma_t}}^{t-1} \underline{Y}_m, \\ \underline{z}_t = C\underline{R}_t. \end{cases} \tag{6.4.2}$$

We can construct a switching Lyapunov Function same as (6.3.7). Similar to the proof of Theorem 6.2, we have LMI condition (6.4.1) indicate that system Θ_4 with $\underline{w}_{t+1} = 0$ is robust stabilizable. With the next step, our aim is to show that $||\underline{z}_t||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}$ holds for all nonzero $\underline{w}_t$ and $\gamma > 0$. To prove this, we need to define

$$T_{H_\infty} = \mathbb{E}\{\sum_{t=0}^{N}(\underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t)|\underline{R}_{0,\sigma_t} = 0\}. \tag{6.4.3}$$

With zero initial condition we know $V_{\sigma_0}(\underline{R}_0) = 0$. On the other hand, we have shown in Theorem 6.1 that $\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})] - \mathbb{E}[V_{\sigma_t}(\underline{R}_t)] \leq 0$. Therefore, for any time $\mathbb{T}$ we have that $\mathbb{E}\left(\sum_{t=0}^{\mathbb{T}} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0$ and $V_{\mathbb{T}}(\underline{R}_{\mathbb{T}}) \geq 0$, which after tending $\mathbb{T} \to \infty$ will give us

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0.$$

Using this relation and the definition of T_{H_∞}

$$\begin{aligned} T_{H_\infty} &= \mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\ &\quad -\mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t)]\right). \end{aligned} \tag{6.4.4}$$

Similar to the proof in Chapter 5 the inequalities conditions in Theorem 6.3 can guarantee that for each $(i, j) \in \mathcal{S} \times \mathcal{S}$, $T_{H_\infty} < 0$ under zero initial conditions. Then, the system is robust stochastically stable with an H_∞ norm bound γ. □

6.5 Numerical Application 4

In this section, a numerical application for illustrating the applicability of the theoretical results for arbitrary regime switching is shown. Let us recall that the same regime switching P-R system in Numerical Application 3 is run by an insurance company. We assume the initial state condition, fixed part of parameters and time delay are same with Numerical Application 3 in Chapter 5. However the regime switching signal is arbitrary signal instead of markovian signal, and the parameter uncertainties. Then, we use the result from the Theorem 6.3 to find out the H_∞ controller such that the total reserve process is stabilized with a particular disturbance attenuation level γ. The different element with Numerical Example 3 is described here:

- $\gamma = 21.8$. This is the value the maximum impact level of the disturbance to the reserves.
- The time-varying unknown parameter uncertainties $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$, $i \in [1, 2]$ are defined by:

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}],$$

where

$$M_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 0.002 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 0.005 & 0 \\ 0 & 0 & 0.004 \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 3 & 3 \\ 4 & 5 & 5 \end{bmatrix}, N_{21} = \begin{bmatrix} 6 & 3 & 4 \\ 5 & 6 & 5 \\ 3 & 2 & 5 \end{bmatrix}, N_{31} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 5 & 5 \\ 3 & 3 & 5 \end{bmatrix}.$$

$$N_{12} = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 3 & 3 \\ 4 & 5 & 5 \end{bmatrix}, N_{22} = \begin{bmatrix} 6 & 3 & 4 \\ 5 & 6 & 5 \\ 3 & 2 & 5 \end{bmatrix}, N_{32} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 5 & 5 \\ 3 & 3 & 5 \end{bmatrix}.$$

Here, the performance of system under different arbitrary switching signals is presented. The simulation results are provided for the time-period of $t = 52$ weeks.

By applying the result of the Theorem 6.3 , the H_∞-controller is derived, and we get the feedback controller for each regime separately under arbitrary switching signal are:

If system is in Regime 1:

$$K_{11} = \begin{bmatrix} 0.6783 & -0.9012 & -0.8369 \\ -0.7034 & 0.8533 & -0.6048 \\ 1.1697 & 1.2032 & 2.6863 \end{bmatrix}.$$

If system is in Regime 2:

$$K_{12} = \begin{bmatrix} 1.0600 & -0.5498 & -0.4322 \\ -0.8053 & 0.7840 & -0.6738 \\ 0.8661 & 0.8753 & 2.2867 \end{bmatrix}.$$

6.5.1 Summary

As mentioned in Chapter 5, regime switching linear P-R systems exhibit complex dynamical behavior which can be critical for their stability properties. In this chapter we analyse the stability of arbitrary regime switching linear systems and contribute to a better understanding of the stability properties along with markovian regime switching linear P-R systems.

Following same path in Chapter 5, a linear robust control theory for arbitrary regime switching systems has been implemented in the P-R model. The LMI conditions for the robust stabilization and a feasible $H\infty$ controller for system Θ_4 are derived through a series of Lemmas and Theorems. Thus, with the H_∞ controller, the premium is adjusted to reasonable levels for different modes (regimes). Both robust stochastic stability and a pre-specified disturbance attenuation level can be guaranteed for all admissible uncertainties. Corresponding results have been illustrated by introducing a numerical example.

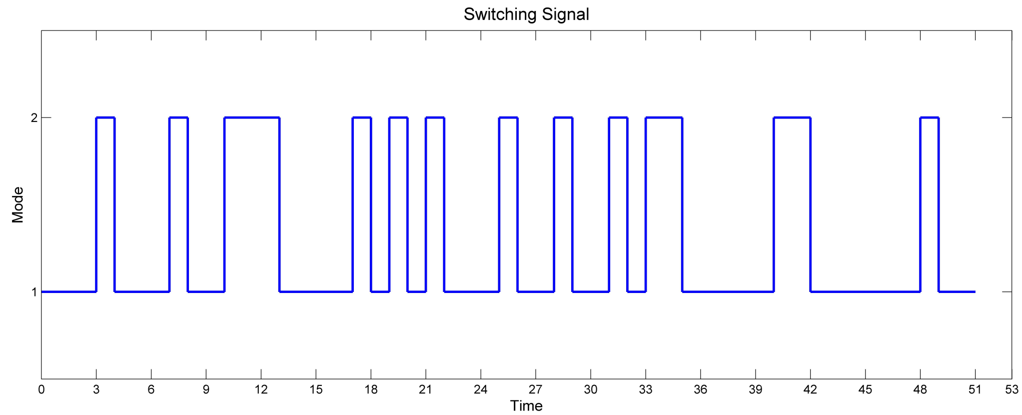

Figure 6.1: Arbitrary switching signal

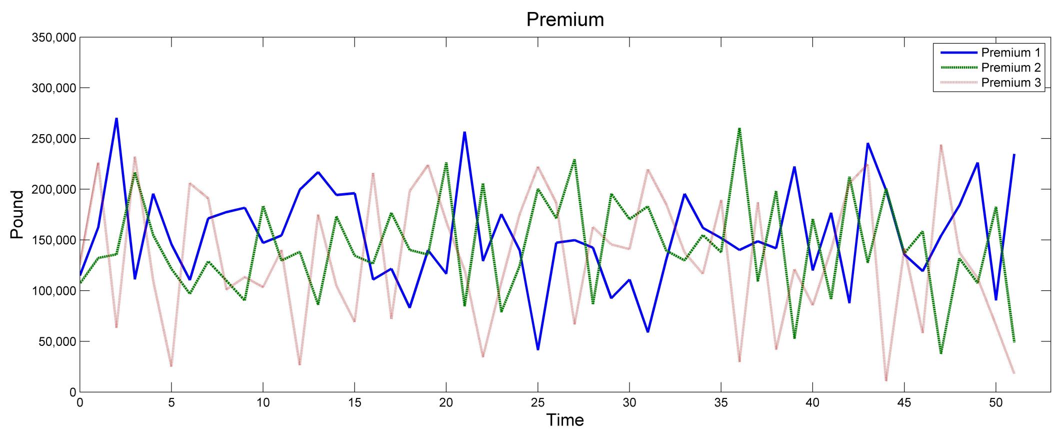

Figure 6.2: The evolution of the three Premiums under the arbitrary switching signal

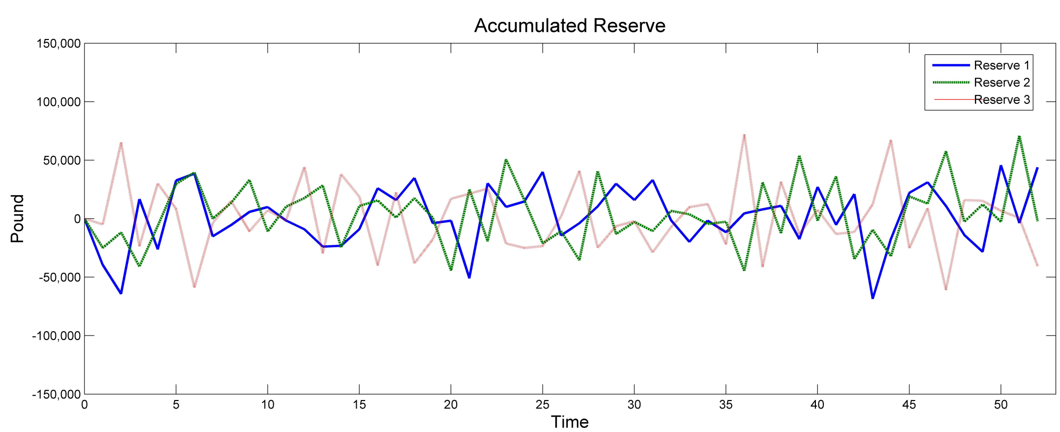

Figure 6.3: The evolution of the the three accumulated reserves account under the arbitrary switching signal

Chapter 7

H_∞ Robust guaranteed cost control

7.1 Introduction

In practical applications, the choice of control policy depends upon the optimization of some preassigned performance criteria. When designing a controller for a real system, it is often desirable to make the controlled system not only stable but also guarantee an adequate level of performance. To deal with such control problems, the so-called guaranteed cost control approach was first introduced by Chang and Peng (1972) [14]. The objective of this approach is to establish an upper bound on a given performance index so that the system performance degradation incurred by the uncertainties is guaranteed to be less than this bound. For guaranteed cost control, a great number of results on this topic have been reported in the literature and various approaches have been proposed. For example, in Petersen and McFarlane (1994) [54], notion of the quadratic guaranteed cost control was introduced to allow for a quadratic performance index and a Riccati equation approach was presented for designing quadratic guaranteed cost controllers, where the system was delay-free. In Yu and Chu (1999) [76], an LMI approach was proposed to deal with the guaranteed cost control problem for a class of linear time delay systems with time-varying norm-bounded parameter uncertainty, and a sufficient condition for the existence of memoryless state-feedback guaranteed cost controllers was derived. In Chen et al. (2003) [18], the solutions to the guaranteed cost control problem via state-feedback are presented for a class of uncertain Markovian jump systems with mode-dependent delays in LMI framework, and the delay dependent/independent sufficient conditions for the existence of guaranteed cost state-feedback controllers have been derived.

In recent years, multi-objectives design approach for control systems has received more and more attention. In modern control theory it is common to minimize a performance index which may be a generalized quadratic energy function, in many cases with some secondary constraints (or limitations on the range or character of the solution). This may be seen as a "natural" requirement as most systems in nature operate in such a way as to minimize energy consumption (Hendricks et al., 2008 [34]). In the P-R system, it could be desirable that the system can satisfy another characteristic besides the stability of accumulated reserve trajectory.

7.2 Model formulation

The assumption in this chapter is almost same with those in Chapter 3, except for the time-delay. In this chapter, positive integer τ_1 represents the state time delay of system. Unlike previous chapters, it is a fixed but unknown definite integer satisfying $0 \leq \tau_1 \leq \tau_{\max}$ with $\tau_{\max} \in \mathbb{N}$. So, considering a specific time-delay interval, at the end of each year $[t, t+1)$, we have the exact information up to the end of the year $t - \tau_1$. The value for $\tau_{\max}$ in this chapter can be estimated using past experience and statistical data. Consider the following system:

$$\begin{cases} \underline{R}_{t+1} = [J + \Delta J_t]\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_1} - e[Z + \Delta Z_t]U_t + \underline{w}_{t+1}, \\ \underline{z}_t = C\underline{R}_t, \\ \underline{R}_t = \varphi_t \ for \ t \in [-\tau_{\max}, 0]. \end{cases}$$

Also, after substituting the control input $\underline{U}_t = K\underline{R}_t$, our new closed loop P-R system becomes

$$\Theta_5 : \begin{cases} \underline{R}_{t+1} = \{[J + \Delta J_t] - e[Z + \Delta Z_t]K\}\underline{R}_t - e[E + \Delta E_t]\underline{R}_{t-\tau_1} + \underline{w}_{t+1}, \\ \underline{z}_t = C\underline{R}_t, \\ \underline{R}_t = \varphi_t \ for \ t \in [-\tau_{\max}, 0]. \end{cases}$$

Performance index for the P-R system is defined as follow:

$$PI = \sum_{t=0}^{\infty} [\underline{R}_t^T Q \underline{R}_t^T + \underline{U}_t^T R \underline{U}_t], \tag{7.2.1}$$

where Q and R are given positive definite weight symmetrical matrices.

7.3 Main result

Definition 7.1. For a given positive constant $\gamma > 0$ and symmetrical positive definite matrices Q and R, state feedback controller $\underline{U}_t$ is a robust H_∞ guaranteed cost controller for the P-R systems Θ_5, if the following conditions holds for all the admissible parameter uncertainties.

1. The closed-loop system Θ_5 is stable, when $\underline{w}_{t+1} = 0$
2. With the zero initial condition, the controlled output $\underline{z}_t$ satisfies

$$||\underline{z}_t||_{e_2} \leq \gamma ||\underline{w}_t||_{e_2},$$

3. In the case when $\underline{w}_t = 0$, the performance index for the P-R system is

$$PI = \sum_{t=0}^{\infty} [\underline{R}_t^T Q \underline{R}_t^T + \underline{U}_t^T R \underline{U}_t] < k$$

   where k is a positive number.

7.3.1 H_∞ guaranteed cost control

To get the result in this chapter, Lemma 3.2 and Lemma 5.1 are used.

Theorem 7.1. *For the given constant $\gamma > 0$ and the performance index (7.2.1), a sufficient condition for the existence of H_∞ guaranteed cost controller $\underline{U}_t = K\underline{R}_t$ for P-R systems Θ_5 is that it exists symmetrical positive definite matrices P, S_1 such that for all admissible parameter uncertainties, the following matrix inequality holds*

$$\begin{bmatrix} -P^{-1} & [J+\Delta J_t]-e[Z+\Delta Z_t]K & -e[E+\Delta E_t] & I & 0 \\ \{[J+\Delta J_t]-e[Z+\Delta Z_t]K\}^T & \Omega & 0 & 0 & C^T \\ -e\{E+\Delta E_t\}^T & 0 & -S_1 & 0 & 0 \\ I & 0 & 0 & \gamma I & 0 \\ 0 & C & 0 & 0 & I \end{bmatrix} < 0, \tag{7.3.1}$$

where $\Omega = -P + S_1 + K^T S_2 K + Q + K^T R K.$

Proof 7.1. Here the closed loop P-R system uses the controller $\underline{U}_t = K\underline{R}_t$. If there exist positive definite matrices P and S_1 such that LMI (7.3.1) holds, we can construct a generalized Lyapunov function $V_t(\underline{R}_t)$

$$V_t(\underline{R}_t) = \underline{R}_t^T P \underline{R}_t + \sum_{i=t-\tau_1}^{t-1} \underline{R}_i^T (S_1) \underline{R}_i, \tag{7.3.2}$$

in the case when $\underline{w}_{t+1} = 0$, the forward difference of $\underline{R}_t$ is

$$\begin{aligned} &\Delta V_t(\underline{R}_t) = V_{t+1}(\underline{R}_{t+1}) - V_t(\underline{R}_t) \qquad (7.3.3) \\ &= \underline{R}_{t+1}^T P \underline{R}_{t+1} + \underline{R}_t^T(-P+S_1)\underline{R}_t - \underline{R}_{t-\tau_1}^T S_1 \underline{R}_{t-\tau_1} \\ &= \begin{bmatrix} \underline{R}_t \\ \underline{R}_{t-\tau_1} \end{bmatrix}^T \left(\begin{bmatrix} \{[J+\Delta J_t]-e[Z+\Delta Z_t]K\}^T \\ -e\{E+\Delta E_t\}^T \end{bmatrix} P \begin{bmatrix} (J+\Delta J_t)-e(Z+\Delta Z_t)K & -e(E+\Delta E_t) \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} \Omega - Q - K^T R K & 0 \\ 0 & -S_1 \end{bmatrix} \right) \begin{bmatrix} \underline{R}_t \\ \underline{R}_{t-\tau_1}. \end{bmatrix} \end{aligned}$$

Considering the LMI (7.3.1) and Lemma 3.2 (Schur complements), we follow the same method in Theorem 3.1. It is easy to get

$$\begin{aligned} \Delta V_t(\underline{R}_t) &< -\underline{R}_t^T (Q + K^T R K) \underline{R}_t^T \\ &\leq -\lambda_{\min}(Q + K^T R K) ||\underline{R}_t||^2 < 0, \end{aligned} \tag{7.3.4}$$

where $\lambda_{\min}(\quad)$ is the minimum eigenvalue of respective matrix. Therefore, the closed-loop system Θ_5 is robust stable. Furthermore, if we have arbitrary disturbance $\underline{w}_{t+1} \neq$

0, same as the proof in Theorem 3.3 and Theorem 5.3, we can get

$$\Delta V_t(\underline{R}_t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t < 0. \tag{7.3.5}$$

By the zero initial condition, we can develop equation (7.3.5) to

$$\sum_{t=1}^{\infty} \underline{z}_t^T \underline{z}_t - \gamma^2 \sum_{t=1}^{\infty} \underline{w}_t^T \underline{w}_t < -V_\infty(\underline{R}_\infty) \leq 0. \tag{7.3.6}$$

Thus,

$$||\underline{z}_t||_{e_2} \leq \gamma ||\underline{w}_t||_{e_2}.$$

Sum time t from 0 to ∞ at both sides of equation (7.3.4), we get

$$PI \leq \underline{R}_0^T P \underline{R}_0 + \sum_{i=-\tau_1}^{-1} \underline{R}_i^T S_1 \underline{R}_i. \tag{7.3.7}$$

The proof is completed. $\square$

Remark 7.1. Noticing that the closed-loop performance upper bound obtained from inequality (7.3.7) depends on the initial condition of system Θ_5. To remove this dependence on the initial condition, we suppose that the initial state of system Θ_5 is unknown but all belongs to the set $S = \{\underline{R}_{-i} \in \mathbb{R}^m, \underline{R}_{-i} = Uo_i, o_i^T o_i \leq 1, i = [-\tau_{\max}]\}$, where U is a given constant matrix. Thus, we can get

$$PI \leq \lambda_{\max}(U^T P U) + \tau_1 \lambda_{\max}(U^T S_1 U). \tag{7.3.8}$$

Theorem 7.2. *For the given constant $\gamma > 0$ and system and system performance index PI , if there exists a positive scalar ϵ and symmetrical positive definite matrices X, L, T and matrix Y such that the following LMI holds*

$$\begin{bmatrix} T_1 & T_2 & T_3 \\ T_2^T & -\epsilon I & 0 \\ T_3^T & 0 & T_4 \end{bmatrix} < 0, \tag{7.3.9}$$

where

$$T_1 = \begin{bmatrix} -X+3\epsilon MM^T & JX-eZY & -eEL & 0 & I & 0 \\ (JX-eZY)^T & -X & 0 & 0 & 0 & (CX)^T \\ -eLE^t & 0 & -L & 0 & 0 & 0 \\ 0 & 0 & 0 & -T & 0 & 0 \\ I & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & CX & 0 & 0 & 0 & -I \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ XN_1^T & Y^T N_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & LN_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & X & 0 & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$T_4 = \begin{bmatrix} -L & 0 & 0 & 0 \\ 0 & -T & 0 & 0 \\ 0 & 0 & -Q^{-1} & 0 \\ 0 & 0 & 0 & -R^{-1} \end{bmatrix},$$

The state feedback controller $\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t$ *is a* H_∞ *guaranteed cost control law for systems* Θ_5 *and the corresponding closed-loop performance index satisfies:*

$$PI \leq \lambda_{\max}(U^T X^{-1} U) + \tau_{\max}\lambda_{\max}(U^T L^{-1} U).$$

Proof 7.2. The inequality (7.3.1) can be written as

$$\begin{bmatrix} -P^{-1} & J - eZK & -eE & I & 0 \\ \{J - eZK\}^T & \Omega & 0 & 0 & C^T \\ -eE^T & 0 & -S_1 & 0 & 0 \\ I & 0 & 0 & \gamma I & 0 \\ 0 & C & 0 & 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \Delta J_t - e\Delta Z_t K & -e\Delta E_t & 0 & 0 \\ \{\Delta J_t - e\Delta Z_t K\}^T & 0 & 0 & 0 & 0 \\ -e\Delta E_t^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0 \qquad (7.3.10)$$

Since

$$[\Delta J_t \quad - e\Delta E_t \quad - e\Delta Z_t] = MF_t[N_1 \quad N_2 \quad N_3].$$

By Lemma 3.2 (Schur comlement), Lemma 5.1 and following same approach in the proof of Theorem 3.2 and Theorem 3.4, we can prove that if there exists feasible positive definite matrices X, L, T and matrix Y satisfying LMI (7.3.9), then then $\underline{U}_t = K\underline{R}_t = YX^{-1}\underline{R}_t$ is a H_∞ guaranteed cost control law for system Θ_5 and the corresponding closed-loop performance index PI is upper bounded. We should notice here $X = P^{-1}$, $Y = KX$, $L = S_1^{-1}$. □

7.3.2 Optimal guaranteed cost controller

Substituting the given into LMI (7.3.9), and then by solving the following optimization problem, we can get the guaranteed cost control law such that the corresponding closed-loop performance index that upper bound is minimum.

$$
\begin{cases}
\quad \min(\alpha + \tau_{\max}\beta) \\
s.t \quad 1) \begin{bmatrix} T_1 & T_2 & T_3 \\ T_2^T & -\epsilon I & 0 \\ T_3^T & 0 & T_4 \end{bmatrix} < 0 \\
\quad 2) \begin{bmatrix} -\alpha I & U^T \\ U & -X \end{bmatrix} \\
\quad 3) \begin{bmatrix} -\beta I & U^T \\ U & -L \end{bmatrix}
\end{cases}
\tag{7.3.11}
$$

System (7.3.12) is a convex optimization problem, so we can get the global optimization solution by the mincx in LMI software toolbox (see Gahinet et al. (1995) [27]) for this optimization problem.

7.3.3 Numerical Example 5

In this section, we present a basic example to show how robust guaranteed cost control can be useful to solve some affiliated problem in P-R problem. We consider a simple system as:

$$\underline{R}_{t+1} = [J + \Delta J_t]\underline{R}_t - e[Z + \Delta Z_t]\underline{U}_t. \tag{7.3.12}$$

Here, we know the relevant parameters are:

$$
J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, -eZ = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \ N_1 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 15.045 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.009 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.006 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.955 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.991 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.994 \end{bmatrix},$$

$$[\Delta J_t \quad -e\Delta Z_t] = MF_t[N_1 \quad N_3].$$

Based on the result in this chapter, the controller for the basic system (7.3.12) is derived.

$$Y = \begin{bmatrix} -0.0001 & 0 & 0 & 0 \\ 0 & -0.0052 & 0 & 0 \\ 0 & 0 & -0.002 & 0 \\ 0.3789 & 0 & 0 & 0 \\ 0 & 0.0053 & 0 & 0 \\ 0 & 0 & 0.0117 & 0 \end{bmatrix}, \ X = \begin{bmatrix} 0.6124 & 0 & 0 & 0 \\ 0 & 0.1022 & 0 & 0 \\ 0 & 0 & 0.1089 & 0 \\ 0 & 0 & 0 & 0.3698 \end{bmatrix},$$

and the feedback controller is

$$\underline{U}_t = YX^{-1}\underline{R}_t = \begin{bmatrix} -0.002 & 0 & 0 & 0 \\ 0 & -0.0509 & 0 & 0 \\ 0 & 0 & -0.0184 & 0 \\ 0.6187 & 0 & 0 & 0 \\ 0 & 0.0519 & 0 & 0 \\ 0 & 0 & 0.1074 & 0 \end{bmatrix} \underline{R}_t.$$

The corresponding Performance Index has a upper bound $PI \leq 23.3044$.

Chapter 8

Conclusion and future research

The purpose of this thesis is to develop a sound approach for robust control of the P-R system in insurance, which form the basis of good reserve management and premium rating policy. As an extension of previous literature, we define comprehensive mathematical frameworks that adequately describe premium rating formation and accumulated reserve process. The P-R models in this thesis captures the essential factors that influence the trajectories of premium and reserve. In particular, factors which influence the stability of P-R system can be captured by the P-R models in this thesis. These can provide insurance company a new approach for financial strength analysis, solvency margin supervision and management of premium rating policy.

Modellers in actuarial science always face the difficulties from the complexity and uncertainty around the model. Ideally, the model should be sufficiently sophisticated so that it can appropriately capture real world behaviour. However, the classical models are often one-dimensional deterministic models. The parameter in those classical models is fixed. Since real world behaviour is involved, actuaries should instead consider more complicated model which captures those stochastic and uncertain factors. It is impossible to model all the real world characteristics, but it is desirable to capture the essential influential factors in an appropriate way, i.e. stochastic system and outside disturbance in Chapter 3, risky investment in Chapter 4, and regime switching impact in Chapter 5 and 6.

The disadvantage of such complex models is that they have difficulties in understanding and interpretation, because the stochastic and uncertain factors are not easy to be described at same time in the model.

On the contrary, the linear robust H_∞ control theory which is initially developed in engineering science can provide us powerful tools to describe real world behavior sufficiently well. Therefore, we are motivated to implement the linear robust H_∞ control theory into the classical P-R system problem. The results in this thesis give us a solid and reliable framework to analyze, understand and manage the impact of these uncertain and stochastic factors on the P-R system.

The models that we describe have been investigated by several researchers before, but most of them are restricted only in deterministic linear system. One main contribution of this thesis is illustrated in Chapter 3, where we extend the research on the stability of linear stochastic P-R system model. Under Chapter 3 the model is developed into a stochastic, discrete time framework and norm-bounded parameter uncertainties have been also incorporated.

In Chapter 4, model defined in Chapter 3 is modified by taking into account a predefined risky investment strategy, which makes the theorem more realistic in practice. Same as Chapter 3, robust H_∞ control problems for the P-R system are proposed using LMI criteria.

During the last two decades, applications of regime switching models in finance and macroeconomics have received a great attention among researchers and particularly, market practitioners. In Chapter 5 and 6, research has been done in regime switching framework. Chapter 5 is an attempt to consider how a linear Markovian regime switching system in discrete-time could be used to model the medium- and long- term reserves and the premiums of an insurer. Meanwhile, Chapter 6 considers the problems under the arbitrary regime switching assumption. The essence of those theorems is based on sufficient LMI criteria.

The applicability of those theorems is demonstrated by numerical examples. In numerical examples, we assume an insurance company runs a non-life insurance portfolio containing multiple products, which may be exposed to outside financial and economic disturbances, parameter uncertainties, etc.

The basic model in Chapter 7 can be viewed as an important cornerstone towards a comprehensive multiple objects optimization problem. In this chapter, we care not only the stability of P-R system but also minimization of cost function.

In future, we plan to extend the result in Chapter 3 and 4 by considering a model with multiple independent stochastic factors. This would enable us to avoid predefined fixed investment plan for the portfolio and switch to dynamic portfolio investment. This means we can control the allocation strategy between risk-free and risky investments and possibly find an approach to create an optimal portfolio. Moreover, markovian regime switching problem could be analyzed in stochastic framework and the affiliated problems will be solved through robust guaranteed cost control approach. We have a very strong assumption that the contract between the insurance company and policyholder will last for a very long time. This is not very realistic situation in competitive market. So in future it is possible to implement a game theoretic model, and the relaxation of this assumption will be considered. Also, we could implement output feedback mechanism instead of state feedback mechanism in P-R system.

Last but importantly, when we try to apply the models to solve real world P-R system process problems, we should always keep in mind that we need to translate the real world problem in an appropriate way. In order to effectively use the results in this thesis, one needs to have a good understanding of the relevant insurance and financial products. This includes a good understanding of external factors like economic developments (monetary policy, economic growth, insurance and financial markets, interest rate behaviour, inflation, legal and political changes), environmental factors (natural hazards, scientific developments, etc.), the insurance contracts itself, policyholder behaviour, management actions, etc. Therefore, besides the theoretical aspect, in the future we need also to have a deeper understanding for the connection between the models in this thesis and the real world actuarial applications. That means we should give a reasonable approach to define the practical meaning and determine specific value of uncertain parameters.

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