Classifications of the Free Fermionic Heterotic String Vacua

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

by

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Bu tezi sevgili aileme armağan ediyorum:
Annem Cennet'e, babam Hüseyin'e, ablam Derya'ya ve abim Murat'a.

Dedicated to my beloved family:
My mother Cennet, my father Hüseyin, my sister Derya and my brother Murat.

Abstract

The existence of discrete properties is shown in the landscape of the Free-Fermionic Heterotic-String vacua. These were discovered via the classification of the $SO(10)$ GUT gauge group and its subgroups, such as, the Pati-Salam, the Flipped $SU(5)$ and the $SU(4) \times SU(2) \times U(1)$ models. The classification is carried out by fixing a set of basis vectors and then varying the GGSO projection coefficients entering the one-loop partition function. The analysis of the models is facilitated by deriving algebraic expressions for the GGSO projections to enable a computerised analysis of the entire String spectrum and the scanning of large spaces of vacua. The analysis reveals an abundance of 3 generation models with exophobic String vacua. This is observed with the $SO(10)$ and the Pati-Salam models. Contrary to this, the Flipped $SU(5)$ models contained no exophobic vacua with an odd number of generations. Moreover, it is also observed that the $SU(4)_C \times SU(2)_L \times U(1)_L$ models are substantially more constrained and that no generations exist. The analysis of the $SU(3)_C \times U(1)_C \times SU(2)_L \times U(1)_L$ and the $SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)_R$ models are being examined, which is work in progress, that are expected to generate further interesting phenomenology.

Publication List

This thesis contains material that has appeared in the following publications by the author:

1. Laura Bernard, Alon E. Faraggi, Ivan Glasser, John Rizos and Hasan Sonmez, *"String Derived Exophobic SU(6) × SU(2) GUTs"*, Nuclear Physics **B868** Pages 1-15 (2013) [arXiv:1208.2145 hep-th].

2. Alon E. Faraggi, John Rizos and Hasan Sonmez, *"Classification of Flipped SU(5) Heterotic-String Vacua"*, Nuclear Physics **B886** Pages 202-242 (2014) [arXiv:1403.4107 hep-th].

3. Alon E. Faraggi and Hasan Sonmez, *"Classification of SU(4) × SU(2) × U(1) Heterotic-String Models"*, Physical Review **D91** 066006 (2015) [arXiv:1412.2839 hep-th].

4. Hasan Sonmez, *"Exotica and Discreteness in the Classification of String Spectra"*, Journal of Physics: Conference Series **631** 012081 (2015) [arXiv:1503.01193 hep-th].

5. Johar M. Ashfaque, Panos Athanasopoulos, Alon E. Faraggi and Hasan Sonmez, *"Non-Tachyonic Semi-Realistic Non-Supersymmetric Heterotic String Vacua"*, arXiv:1506.03114 hep-th (2015).

Declaration

I hereby declare the work described in this thesis is the result of my research unless otherwise acknowledged within the text. All work was carried out in the Department of Mathematical Sciences at the University of Liverpool from October 2011 to September 2015. It has not been submitted previously for a degree at this or any other University.

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Chapter 1

Introduction

The LHC discovery of a Higgs-like resonance [1, 2] lends further support to the viability of the Standard Model as the effective parameterisation of all observational subatomic data. However, the Standard Model only consists of three of the four known forces in physics: the electromagnetic, weak and strong forces. On the other hand, nature's fourth force, gravity, is not included as it is only described by its classical effects with general relativity. A promising attempt to solve this problem is by using String theory, as it is also a consistent theory of quantum gravity. As String theory is yet to be proven experimentally, one method is to focus on the consequences of String models for the effective low-energy limit. Alternatively, the consequences of the minimal low-energy requirements for String theory can also be considered. Therefore, String theories have been formulated for all dimensions $D < 10$, in particular directly in four space-time dimensions. In this thesis, the results in the Free-Fermionic construction of String theory within these four space-time dimensions are presented. The low-energy requirements such as $\mathcal{N} = 1$ space-time supersymmetry and chiral space-time fermions are considered. The consequences of these results for the models derived from the Free-Fermionic construction of String theory are then discussed.

In this chapter, the Standard Model will be briefly reviewed as to some of its theoretical shortcomings. Following this, the Grand Unified Theories (GUTs), with some elaboration on the $SU(5)$ GUT models, will then be discussed, as a possible extension of the Standard Model. To finalize, a thesis outline of each of the remaining chapters will be given.

1.1 The Standard Model

The Standard Model gives an excellent description of nature in agreement with experimental data up to the energy scales of 8.1 TeV. It encompasses 3 of the 4 known interactions between elementary particles: the strong, the weak and the elec-

tromagnetic interactions. Gravity is negligible at the present accessible experimental scales and is therefore not included. The matter content of the Standard Model is summarized in the following table:

Generation	**Fermion**	**Symbol**	**Electric Charge**
1^{st}	*Up quark*	u	$+\frac{2}{3}$
	Down quark	d	$-\frac{1}{3}$
	Electron	e	-1
	Electron neutrino	ν_e	0
2^{nd}	*Charm quark*	c	$+\frac{2}{3}$
	Strange quark	s	$-\frac{1}{3}$
	Muon	μ	-1
	Muon neutrino	ν_μ	0
3^{rd}	*Top quark*	t	$+\frac{2}{3}$
	Bottom quark	b	$-\frac{1}{3}$
	Tau	τ	-1
	Tau neutrino	ν_τ	0

All the above particle content of the Standard Model has been detected in a number of experiments carried out by the various colliders over the decades. Similarly, the spin-1 vector bosons of the Standard Model that mediate the interactions, are summarized in the following table:

Force	**Boson**	**Symbol**	**Electric Charge**
Eletromagnetic	*Photon*	γ	0
Weak	*W-boson*	$W^{\pm}$	± 1
	Z-boson	Z^0	0
Strong	*Gluon*	$g^{1,\ldots,8}$	0

These interactions are all described by gauge field theories. The electromagnetic and the weak interactions are combined in the $SU(2)_L \times U(1)_Y$ electroweak gauge theory, whereas the strong interactions are described by the gauge group $SU(3)_C$. Therefore, the Standard Model is often denoted as $SU(3)_C \times SU(2)_L \times U(1)_Y$. In addition to the above particles, the Standard Model contains a $SU(2)_L$ doublet, $SU(3)_C$ singlet, called the Higgs boson

$$h = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}.$$

1.2 Grand Unified Theories

The Standard Model has been extremely successful, having survived all experimental tests in the last five decades. Despite this, it is considered to be incomplete. In regard to this, in what follows, some of the shortcomings will be assessed. Since, these shortcomings are not based on any discrepancy between theory and experiment or any theoretical inconsistencies, their evaluation is highly subjective. Yet, given the desire for a unified and simple theory of the fundamental processes in nature, the Standard Model is considered to be unsatisfactory. An example of a classical unified theory is Maxwell's theory of electromagnetism. Here, the electric and magnetic fields $\overrightarrow{E}$, $\overrightarrow{B}$ are specific aspects of one physical quantity, the field strength tensor $F^{\alpha\beta}$ defined as

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

The equations of motion or Maxwell's equations are then written in terms of $F^{\alpha\beta}$,

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c} J^\beta, \\ \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} &= 0, \end{aligned}$$

with one current term J^β. In quantum electrodynamics this translates to the fact that all electric and magnetic interactions are parametrized by the same coupling constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$[1], known as the fine structure constant. On the other hand, in the $SU(3)_C \times SU(2)_L \times U(1)_Y$ theory, there are 3 gauge coupling parameters: α_S, α_W and α, where the parameter α here is the analogue of the QED fine structure constant stated above. Thus, the three interactions are physically distinct. Another major deficiency of the Standard Model, with respect to unification, is that the fourth known interaction gravity is not included. There is no common framework for the gauge field theories of the Standard Model and the classical theory of gravity. A further unsatisfactory point of the Standard Model is its lack of simplicity, it has many free parameters. Therefore, although the Standard Model provides a good explanation of the current experimental observations, nevertheless it cannot be a fundamental theory, and the search for such a theory is pressing and open issue in particle physics today.

[1]The value observed at low energies

In addition to the aesthetical questions of unification and simplicity, the structure of the Standard Model provokes many unanswered questions, of which only a few will be listed: Why is electric charge quantized? Why is there a hierarchy of fermion masses? What is the origin of the family structure? How many fermion generations are there? A proposed candidate to solve some of the problems of the Standard Model is to consider GUTs [3], an example being the $SU(5)$ models. Grand unified theories, as the name suggests, focuses on the problem of unification in the Standard Model, postulating on an enlarged internal symmetry to achieve this goal. Ever since the development of the theories of special and general relativity, symmetries have played an essential role in the construction of physical theories. The main symmetry of the Standard Model and the foundation of its success is the gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$. The central idea of GUTs is to assume that $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ are distinct subgroups of a larger gauge symmetry group in which formerly disconnected fermions of a family, or bosons of different gauge groups, transform in larger fermionic or bosonic multiplets. This larger symmetry is unbroken above a yet-to-be-determined mass scale M_χ, that must be broken at presently accessible energies, as it is not observed.

1.3 SU(5) GUT Models

In the $SU(5)$ models, the fermions of a family transform as the $\overline{\mathbf{5}} + \mathbf{10}$, where

$$\overline{\mathbf{5}} = (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{\mathbf{2}}{\mathbf{3}}} \oplus (\mathbf{1}, \mathbf{2})_{+\mathbf{1}},$$
$$\mathbf{10} = (\mathbf{3}, \mathbf{2})_{+\frac{\mathbf{1}}{\mathbf{3}}} \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{\mathbf{4}}{\mathbf{3}}} \oplus (\mathbf{1}, \mathbf{1})_{+\mathbf{2}}.$$

Here, the right hand side of the equations denotes the $SU(3)_C \times SU(2)_L \times U(1)_Y$ decomposition. Looking at this decomposition, with a specific set of $U(1)_Y$ quantum numbers, the 15 fermions of the first family are distributed over these representations uniquely as

$$\overline{\mathbf{5}}: \psi_{\mathbf{iL}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu_e \end{pmatrix},$$

$$\mathbf{10}: \psi_{\mathbf{L}}^{\mathbf{jk}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix},$$

where the superscript c refers to the complex conjugate fields. The gauge bosons transform globally under the **24** adjoint representation of the $SU(5)$, given by

$$\mathbf{24} = (\mathbf{8},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3},\mathbf{2})_{-\frac{\mathbf{5}}{\mathbf{3}}} \oplus (\overline{\mathbf{3}},\mathbf{2})_{+\frac{\mathbf{5}}{\mathbf{3}}},$$

with the notation as before. The first three multiplets denote the gauge bosons of the Standard Model. The last two multiplets $(\mathbf{3},\mathbf{2})_{-\frac{\mathbf{5}}{\mathbf{3}}}$ and $(\overline{\mathbf{3}},\mathbf{2})_{+\frac{\mathbf{5}}{\mathbf{3}}}$ are new to GUTs. They have non-trivial quantum numbers under both the $SU(3)_C$ and $SU(2)_L$ gauge groups. Therefore, they can mediate transitions between quarks and leptons, as well as between quarks and anti-quarks (i.e. the d^c and the e in the $\overline{\mathbf{5}}$ and the u^c and the d in the **10**, for example).

In order to accomplish the spontaneous symmetry breaking, the $SU(5)$ models uses a **24** and $\overline{\mathbf{5}}$ of scalar Higgs particles: Φ_{24} and Φ_5 respectively. The Φ_{24} can break the $SU(5)$ gauge group to the $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry at a scale $M_\chi \sim \langle\Phi_{24}\rangle$. This also splits Φ_5 into a $SU(3)_C$ triplet $H_{(3,1)}$ and a $SU(2)_L$ doublet $h_{1,2}$. The latter is the Higgs doublet of the Standard Model which spontaneously breaks the electroweak theory. In this way, the Standard Model is retrieved at low energies, as required by a generalized correspondence principle. This completes the particle content of the $SU(5)$ models, with the only new particles being the extra gauge bosons and the extra Higgs scalars.

A central predication of all GUTs is the decay of the proton via the lepto-quark gauge bosons. Experimentally the lifetime of the proton τ_P [4] is known to be

$$\tau_P \geq 10^{34} \text{ years}.$$

The minimal $SU(5)$ models predicts $\tau_P \sim O(10^{31})$ years and is therefore most likely

ruled out experimentally. However, other models such as $SO(10)$, E_6 and the Flipped $SU(5)$ are consistent with this experiment. For this reason, there is an interest in studying the Flipped $SU(5)$ models, instead of the straight $SU(5)$ models. The Flipped $SU(5)$ models differ from the straight $SU(5)$ models, with the right-hand neutrino being embedded in the **10** representation of the Flipped $SU(5)$, rather than the singlet. Therefore, the GUT can be broken with the **10** representation instead of the adjoint representation. A detailed discussion of the Flipped $SU(5)$ models are the topic of discussion in chapter 4.

1.4 Thesis Outline

The chapters of this thesis are organised as follows:

- **Chapter 2:** The superstring theory is reviewed, where the classical and quantum dynamics of the string are discussed.

- **Chapter 3:** The set up of the Free-Fermionic construction in the Heterotic-String is shown. The one-loop partition function is defined at an arbitrary point in the moduli space. This enables the derivation of specific constraints deduced from modular invariance. These constraints lead to the ABK rules, where String model building can be achieved.

- **Chapter 4:** The Flipped $SU(5)$ classification of the Free-Fermionic Heterotic-String vacua is presented. The constructed models are given by breaking the $SO(10)$ GUT symmetry at the String scale to the Flipped $SU(5)$ subgroup. A set of basis vectors defined by the boundary conditions assigned to the free fermions, is fixed. Then the enumeration of the String vacua that is obtained in terms of the Generalised GSO (GGSO) projection coefficients entering the one-loop partition function is shown. The total scanned models is 10^{12} GGSO configurations. Contrary to the previous Free-Fermionic classifications, no exophobic Flipped SU(5) vacua with odd numbers of generations are found. However, other interesting properties are presented. This chapter contains material that has appeared in publication [5] presented by the author.

- **Chapter 5:** The classification is extended to the $SU(4) \times SU(2) \times U(1)$ Heterotic-String models. These models are obtained from the $SO(10)$ symmetry breaking to the Pati-Salam subgroup, where the $SU(2)_R$ symmetry breaking to the $U(1)_L$ gauge group takes place. However, it will be shown that in these class of Free-Fermionic models, three chiral generations cannot be produced.

This chapter contains material that has appeared in publication [6] presented by the author.

- **Chapter 6:** The Free-Fermionic landscape is presented by considering the various GUT model classifications. These models descend from the E_6 symmetry, broken to the $SO(10)$ gauge group and then its subgroup. Discussion takes place on the Pati-Salam, the Flipped $SU(5)$, the $SU(4) \times SU(2) \times U(1)$ and the Standard-Like models. It will be shown that discrete symmetries emerge, in addition to there being an abundance of three generation models in the Free-Fermionic setting. This chapter contains material that has appeared in publication [7] presented by the author.

Chapter 2

String Theory

In this chapter, a review is given of the classical and quantum dynamics of the Superstring. For the classical Superstring, the Superstring action is first introduced and then followed by a discussion of the symmetries, supersymmetry, equations of motion, boundary conditions and mode expansions. For the quantum Superstring, the classical analog is quantized using the canonical quantization. In order to find the critical values of the normal ordering constants a_R and a_{NS} and the dimension D, the light-cone quantization is then examined. Further to this, the GSO[2] projection of the spectrum is taken and then is used to project out the Tachyon.

String theory [9, 10, 11, 12, 13, 14] is a popular research area in modern theoretical physics. It has been the leading candidate over the past decades for a theory that consistently unifies all fundamental forces of nature, including gravity. The theory predicts gravity and gauge symmetries around flat space. The elementary objects are one-dimensional Strings whose vibration modes correspond to the usual elementary particles.

2.1 Classical Superstring Dynamics

In String theory, fundamental particles are no longer seen as point particles, instead they are depicted as Strings that give rise to vibrational modes. The String is described by the variable $X^\mu(\sigma, \tau)$, where $\mu = 0, 1, ..., D-1$ and X^μ is the position of a point of the String parametrized by the time-like coordinate τ and the space-like coordinate σ. As the String propagates, it sweeps out an area called the worldsheet, that is a surface in two-dimensional space-time. Hence, the String moves in a D-dimensional Minkowski space-time. The action of the String, namely the Polyakov

[2]Gliozzi, Scherk and Olive [8]

action, is given by

$$S = -\frac{T}{2}\int d\sigma d\tau \sqrt{-h} h^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X^\mu,$$

where $\alpha, \beta = \sigma, \tau$, $h^{\alpha\beta}$ is the metric of the worldsheet, h is the determinant of $h^{\alpha\beta}$ and $T = \frac{1}{2\pi\alpha'}$ is the tension of the String. This action is invariant under the following transformations:

- **Poincaré invariance**
  This is a global symmetry on the worldsheet of the Lorentz transformations and translations given by

  $$X^\mu(\sigma,\tau) = \Lambda^\mu{}_\nu X^\nu(\sigma,\tau) + c^\mu,$$

  where $\Lambda^{\mu\nu} = -\Lambda^{\nu\mu}$ and c^μ is a constant.

- **Reparameterization invariance**
  This is a gauge symmetry on the worldsheet defined by

  $$\sigma^\alpha \longrightarrow \tilde{\sigma}^\alpha(\sigma,\tau),$$

  where the fields X^μ and the metric $h^{\alpha\beta}$ transforms as follows

  $$\begin{aligned} X^\mu(\sigma,\tau) &\longrightarrow \tilde{X}^\mu(\tilde{\sigma},\tilde{\tau}) = X^\mu(\sigma,\tau), \\ h^{\alpha\beta}(\sigma,\tau) &\longrightarrow \tilde{h}^{\alpha\beta}(\tilde{\sigma},\tilde{\tau}) = \frac{\partial\sigma_\gamma}{\partial\tilde{\sigma}_\alpha}\frac{\partial\sigma_\delta}{\partial\tilde{\sigma}_\beta}h^{\gamma\delta}(\sigma,\tau). \end{aligned}$$

- **Weyl invariance**
  This is a gauge symmetry on the worldsheet defined by

  $$h^{\alpha\beta}(\sigma,\tau) \longrightarrow e^{2\omega(\sigma,\tau)}h^{\alpha\beta}(\tilde{\sigma},\tilde{\tau}).$$

Using these invariance properties of the Polyakov action, the action can be rewritten as

$$S = -\frac{T}{2}\int d\sigma d\tau \partial_\alpha X_\mu \partial^\alpha X^\mu.$$

As this action only describes a theory with bosonic fields X^μ, it would be desirable to incorporate fermionic fields ψ^μ. This can be achieved by relating the bosons and the fermions to each other by a supersymmetric transformation, such as

$$\begin{aligned} \delta X^\mu &= i\bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= \rho^\alpha \partial_\alpha X^\mu \epsilon. \end{aligned}$$

Here ϵ is a constant and an infinitesimal Majorana spinor that consists of anti-commuting Grassmann numbers, ρ^α for $\alpha = 0, 1$ are the two-dimensional Dirac matrices, where for a Majorana representation this is given by:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \rho^3 = -\rho^0\rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It should be noted that these matrices satisfy the Dirac algebra known as the Clifford algebra $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$. The symmetry of the worldsheet theory is global, since ϵ is independent of the worldsheet coordinates σ and τ. Using these supersymmetric transformations, the action of the Superstring can be written as

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right), \tag{2.1.0.1}$$

where $\psi^\mu = \psi^\mu(\sigma, \tau)$ is the two-dimensional fermionic field represented by a Majorana spinor

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix},$$

which satisfies the canonical fermionic anti-commutation relations that is in agreement with the spin-statistics given by

$$\{\psi^\mu_A(\tau, \sigma), \psi^\nu_B(\tau, \sigma')\} = \pi\, \delta_{AB}\, \delta(\sigma - \sigma')\, \eta^{\mu\nu}.$$

Taking a Majorana basis, it can be deduced that the conjugate fermion spinor is given as

$$\overline{\psi}^\mu = \begin{pmatrix} \psi^\mu_+ \\ -\psi^\mu_- \end{pmatrix}.$$

2.1.1 Boundary Conditions and Mode Expansions

The boundary conditions for the bosonic fields X^μ, are given by the variation of the first term in the Superstring action (2.1.0.1) as

$$\begin{aligned} \delta S_b &= -\frac{1}{2\pi\alpha'} \int d\tau \int d\sigma \left(\partial_\sigma X^\mu \delta X_\mu \right) \\ &= -\frac{1}{2\pi\alpha'} \int d\tau \int d\sigma \left[\partial_\sigma X^\mu \delta X_\mu \right]_{\sigma=0}^{\sigma=\pi} \\ &= 0. \end{aligned}$$

Taking $\partial_\sigma X^\mu \delta X_\mu = 0$ at $\sigma = 0$ and $\sigma = \pi$ implies

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma). \tag{2.1.1.1}$$

This is a periodic boundary condition that describes closed Strings, in the light-cone gauge it is the tensor product of the left-moving and the right-moving bosons, whereas the open Strings take the boundary conditions:

$$\begin{aligned} \delta X^\mu &= 0 \quad \text{(Dirichlet)}, \\ \partial_\sigma X^\mu &= 0 \quad \text{(Neumann)}. \end{aligned}$$

Similarly, the variation of the second term in the Superstring action (2.1.0.1) in the light-cone gauge, gives the boundary conditions of the fermionic fields ψ^μ as

$$\delta S_f = -T \int d\tau d\sigma \left(\delta\psi_-^\mu \partial_+ \psi_{-\mu} + \delta\psi_+^\mu \partial_- \psi_{+\mu} \right) - \frac{T}{2} \int d\tau \left[\psi_+^\mu \delta\psi_{+\mu} - \psi_-^\mu \delta\psi_{-\mu} \right]_{\sigma=0}^{\sigma=\pi}. \tag{2.1.1.2}$$

Therefore, the equations of motion are

$$\partial_+ \psi_- = 0 \ \text{ and } \ \partial_- \psi_+ = 0.$$

These are the Weyl conditions for spinors in two-dimensions. The fields ψ_- and ψ_+ are thus Majorana-Weyl spinors. These equations imply that

$$\psi_-^\mu = \psi_-^\mu(\tau - \sigma) \ \text{ and } \ \psi_+^\mu = \psi_+^\mu(\tau + \sigma).$$

The requirement of the vanishing of the boundary terms when varying the action from (2.1.1.2) yields

$$[\psi_+^\mu(\tau, \sigma) \delta\psi_{+\mu}(\tau, \sigma) - \psi_-^\mu(\tau, \sigma) \delta\psi_{-\mu}(\tau, \sigma)]_{\sigma=0}^{\sigma=\pi} = 0,$$

leading to two types of boundary conditions for the Superstring.

Closed Superstrings

Considering the closed Superstring, where periodic or anti-periodic boundary conditions can be imposed, the boundary conditions can also be imposed separately for the left- and right-movers:

$$\begin{aligned} \psi_-^\mu(\tau, \sigma) &= \pm \psi_-^\mu(\tau, \sigma + \pi) \ , \\ \psi_+^\mu(\tau, \sigma) &= \pm \psi_+^\mu(\tau, \sigma + \pi) \ . \end{aligned}$$

The plus sign corresponds to the Ramond (R) boundary conditions, which are periodic, while the minus sign corresponds to the Neveu-Schwartz (NS) boundary conditions, which are anti-periodic. Therefore, for the closed Superstring, the possible sectors are: NS-NS, R-R, NS-R and R-NS, where the left-movers are written on the left. Here the sectors NS-NS and R-R corresponds to states that are space-time bosons, whereas the sectors NS-R and R-NS corresponds to states that are space-time fermions. It is important to note here, that the open Strings are just tensor products of the closed Strings, such that R-NS is R$\otimes$NS. The Fourier modes for the functions $\psi^\mu_\pm$ are as follows:

- **Ramond Boundary Conditions:**

$$\begin{aligned} \psi^\mu_-(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)} \ , \\ \psi^\mu_+(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)} \ . \end{aligned}$$

- **Neveu-Schwartz Boundary Conditions:**

$$\begin{aligned} \psi^\mu_-(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir(\tau-\sigma)} \ , \\ \psi^\mu_+(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)} \ . \end{aligned}$$

In quantum theory, the coefficients d^μ_n and b^μ_n are called raising operators for $n<0$ and lowering operators for $n>0$. However, the expansion of X^μ is more complicated, since X^μ_R and X^μ_L are a priori not periodic, as only their derivatives are periodic and can be computed as

$$\partial_- X^\mu_R(\tau,\sigma) \ = l_s \textstyle\sum_{r\in\mathbb{Z}} \alpha^\mu_n e^{-2in(\tau-\sigma)}.$$

Similarly, this follows for the left-movers. Although, without decoupling X^μ, the bosonic fields for the closed Strings have the following mode expansion

$$X^\mu(\tau,\sigma) \ = x^\mu + \alpha' p^\mu \tau \sqrt{\tfrac{\alpha'}{2}} i \textstyle\sum_{n\neq 0} \left(\frac{\alpha^\mu_n}{n} e^{-in\pi\sigma^+} + \frac{\tilde{\alpha}^\mu_n}{n} e^{-in\pi\sigma^-} \right), \qquad (2.1.1.3)$$

with periodicity from (2.1.1.1).

Open Superstrings

For open Strings, the end-points are given as $\sigma=0$, $\sigma=\pi$. This implies that the

boundary terms $\sigma = 0$ and $\sigma = \pi$ vanish independently. Therefore, this requirement is satisfied if at each end of the String, the following holds

$$\psi_+ = \pm\psi_-.$$

The overall sign between ψ_+ and ψ_- is a matter of convention. Without loss of generality, this can be chosen at $\sigma = 0$ to be

$$\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau).$$

For the endpoints with $\sigma = \pi$, there are two possibilities:

$$\begin{aligned}
&\psi_+^\mu(\sigma,\tau) = +\psi_-^\mu(\sigma,\tau) \quad \text{(Ramond (R) Boundary Condition)},\\
&\psi_+^\mu(\sigma,\tau) = -\psi_-^\mu(\sigma,\tau) \quad \text{(Neveu-Schwarz (NS) Boundary Condition)}.
\end{aligned}$$

Looking into the mode expansions, the R-sector is given as

$$\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau\pm\sigma)} \ .$$

Similarly, the NS-sector is as follows

$$\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau\pm\sigma)} \ .$$

Additionally, the Majorana condition requires the fermionic fields to be real and therefore $d_{-n}^\mu = (d_n^\mu)^\dagger$ is taken. For the bosonic fields, the following mode expansion for the open String is given

$$x^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'} i \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{inr} cosn\sigma.$$

This is given with either the Dirichlet or Neumann boundary condition.

2.2 Quantum Superstring Dynamics

In this section, the quantization of the superstrings will be discussed. Recall that there are two different types of quantization methods from Bosonic String theory; one being the canonical quantization and the other being the light-cone quantization. In the canonical approach, Lorentz invariance was evident. However, as equally observed, there were drawbacks such as having negative-norm states and the inability

to prove the number of space-time dimensions and the constant a. As to the light-cone approach, no negative-norm states were observed, in addition, to the derivation of the space-time dimension value D and the constant a being facilitated. Similarly, there is also a drawback here, as Lorentz invariance was not evident. Furthermore, as the two quantization methods are equivalent [10], the canonical method will be used to quantize the superstrings. Then to get the space-time dimension and the constant values, the light-cone approach will be used.

2.2.1 Canonical Quantization

In quantum theory, the fields X^μ, ψ^μ are promoted to the operators $\hat{X}^\mu$, $\hat{\psi}^\mu$. These operators are referred to as the boson operator for $\hat{X}^\mu$ and the fermion operator for $\hat{\psi}^\mu$. The oscillator modes in the mode expansions $d_n^\mu, \tilde{d}_n^\mu, b_r^\mu,$ etc, are therefore also given as operators. However, for future purposes these hats will be dropped. The fermions and bosons here, are described by a two-dimensional field theory. Since the fermions are anti-commuting Grassmann variables, an anti-commutation relation is required for them. Whereas for the bosons, the standard commutation relation is used, thus the boson operators satisfy:

$$\begin{aligned}
[\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] &= i\delta(\sigma-\sigma')\eta^{\mu\nu}, \\
[\hat{X}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] &= [\hat{P}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = 0,
\end{aligned}$$

where $\hat{P}^\nu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\partial_\tau \hat{X}^\mu(\tau,\sigma)$. Then using the mode expansions it can be shown that

$$\begin{aligned}
[a_n^\mu, a_m^\nu] &= [\tilde{a}_n^\mu, \tilde{a}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \\
[x^\mu, p^\nu] &= i\eta^{\mu\nu},
\end{aligned}$$

where $n, m \in \mathbb{Z}$ and all other commutators vanish. Moreover, the anti-commutator relation is given by

$$\{\hat{\psi}_A^\mu(\tau,\sigma), \hat{\psi}_B^\nu(\tau,\sigma')\} = \pi\delta(\sigma-\sigma')\eta^{\mu\nu}\delta_{A,B}, \tag{2.2.1.1}$$

where A and B are the spin indices, $\pm$. Then using the mode expansions it can be shown that

$$\begin{aligned}
\{d_n^\mu, d_m^\nu\} &= \eta^{\mu\nu}\delta_{n+m,0}, \\
\{b_r^\mu, b_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s,0}, \\
\{\tilde{d}_n^\mu, \tilde{d}_m^\nu\} &= \eta^{\mu\nu}\delta_{n+m,0}, \\
\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s,0},
\end{aligned}$$

where $r, s \in \mathbb{Z} + \frac{1}{2}$, $n, m \in \mathbb{Z}$ and all other anti-commutators vanish.

Fock Space

The spectrum of states in the quantum Superstring theory is defined by a Fock space. Here, a ground state $|0\rangle$ is constructed to be annihilated by all of the annihilation operator modes given by

$$\begin{aligned}
a_n^\mu|0\rangle_R = d_n^\mu|0\rangle_R = 0, \qquad & \forall n > 0, \\
a_n^\mu|0\rangle_{NS} = b_r^\mu|0\rangle_{NS} = 0, \qquad & \forall n, r > 0,
\end{aligned}$$

where $|0\rangle_R$ and $|0\rangle_{NS}$ are the ground states in the R/NS-Sectors respectively. The zero modes should act on the ground state, in particular the momentum p^μ. The ground state is an eigenstate of p^μ

$$\hat{p}^\mu|0\rangle = p^\mu|0\rangle.$$

To be precise, it should be noted that the ground state $|0\rangle$ is $|0, p\rangle$ in both sectors. A Fock space of multi-particle states is constructed by acting on the ground state with the creation operators: $d_{-n}^\mu, \tilde{d}_{-n}^\mu, b_{-r}^\mu, \ldots\ \tilde{a}_{-m}^\mu$, for example

$$d_{-1}^\mu \tilde{d}_{-1}^\mu|0\rangle_R \quad , \quad a_{-2}^\nu|0\rangle_{NS} \quad , \quad etc.$$

When the mode operators act on a state, they raise or lower the energy (mass), depending on the operator (creation and annihilation respectively). In the NS-sector, the b_r^μ operator changes the energy by half-integer units, thus bosons have half-integer spacings; whereas in the R-sector, the fermions are all integer units apart. This asymmetry between fermions and bosons is removed by the GSO projection, which will be discussed later. The integer units in the R-sector can take the values 0 or 1, where the zero modes d_0^μ gives rise to a degeneracy in the spectrum. However, for the half-integer units in the NS-sector, there are no zero modes. In fact, the

ground state $|0\rangle_{\rm NS}$ is non-degenerate, hence it can be identified as a spin zero state. Since a_n^μ , b_r^μ are vector operators in space-time, only states with integer spin can be obtained. Therefore, the NS-sector only describes bosonic integer spin modes. As to the R-sector, the ground state $|0\rangle_{\rm R}$ is degenerate, as there is a unique set of operators d_0^μ which satisfy

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}.$$

This is a Clifford algebra (in D-dimensions and with a factor 2 missing), hence d_0^μ can be identified as the gamma matrices Γ^μ , which obey

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu},$$

giving

$$\Gamma^\mu = \sqrt{2}\, d_0^\mu.$$

The zero mode d_0^μ , does not actually raise or lower the energy of a state, leaving M^2 invariant, where M^2 is the space-time mass-squared of a physical state.

Super-Virasoro Generators and Algebra

Having constructed a Fock space, the constraints are now considered. These are the super-Virasoro generators that are the mode expansions of the energy-momentum tensor and the supercurrent that are defined as:

$$\begin{aligned} \hat{T}_{\pm\pm} &= \partial_\pm \hat{X}^\mu \partial_\pm \hat{X}_\mu + \tfrac{i}{2}\hat{\psi}_\pm^\mu \partial_\pm \hat{\psi}_\pm^\nu \eta_{\mu\nu}, \\ \hat{J}_\pm &= \hat{\psi}_\pm^\mu \partial_\pm \hat{X}_\mu. \end{aligned}$$

Considering the left-moving oscillator modes, the energy-momentum tensor can be written as

$$\begin{aligned} \hat{T}_{++} &= \frac{1}{4}\sum_{n,m} \eta_{\mu\nu} a_n^\mu a_m^\nu e^{-i(n+m)\sigma^+} + \frac{1}{4}\sum_{n,m} m\eta_{\mu\nu} d_n^\mu d_m^\nu e^{-i(n+m)\sigma^+} \\ &= \frac{1}{2}\sum_k l_k e^{-ik\sigma^+}, \end{aligned}$$

where

$$\begin{aligned} l_n &= l_n^{(a)} + l_n^{(d)} \\ &= \frac{1}{2}\sum_m a_{n-m}^\mu a_m^\nu \eta^{\mu\nu} + \frac{1}{2}\sum_m m d_{n-m}^\mu d_m^\nu \eta^{\mu\nu}. \end{aligned}$$

Similarly in the NS-sector the following is deduced

$$l_n = l_n^{(a)} + l_n^{(b)},$$

where

$$l_n^{(b)} = \frac{1}{2}\sum_r r b_{n-r}^{\mu} b_r^{\nu} \eta^{\mu\nu}.$$

For the supercurrent $\hat{J}_+$ in the R-sector, the following can be obtained

$$\begin{aligned} J_+ &= \frac{1}{4}\sum_{n,m} \eta_{\mu\nu} a_n^{\mu} \cdot d_m^{\nu} e^{-i(n+m)\sigma^+} \\ &= \frac{1}{4}\sum_n F_n e^{-in\sigma^+}, \end{aligned}$$

where

$$F_n = \sum_m a_m^{\mu} \cdot d_{n-m}^{\nu} \eta^{\mu\nu}.$$

Similarly in the NS-sector, the following is deduced

$$G_r = \sum_n a_n^{\mu} \cdot b_{r-n}^{\nu} \eta^{\mu\nu}.$$

Turning to the case of the right-moving oscillator modes, switching the modes to the tilde modes (i.e $\tilde{F}_n$ and $\tilde{G}_r$) is sufficient. All operators should be normal ordered here, where the annihilation operators always appear to the right of the creation operators. It should be noted that $F_n =: F_n :$ and $G_r =: G_r :$, since l_n is defined by

$$L_n =: l_n :$$

Furthermore, proceeding to the R-sector, the following super-Virasoro algebra can be obtained:

$$\begin{aligned} [L_m, L_n] &= (m-n)\, L_{m+n} + \tfrac{D}{8} m^3 \delta_{m+n,0}, \\ [L_m, F_n] &= \left(\tfrac{m}{2} - n\right) F_{m+n}, \\ \{F_m, F_n\} &= 2L_{m+n} + \tfrac{D}{2} m^2 \delta_{m+n,0}\,. \end{aligned} \tag{2.2.1.2}$$

Similarly, in the NS-sector, the following super-Virasoro algebra is obtained:

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \tfrac{D}{8} m \left(m^2 - 1\right) \delta_{m+n,0}, \\
[L_m, G_r] &= \left(\tfrac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \tfrac{D}{2} \left(r^2 - \tfrac{1}{4}\right) \delta_{r+s,0}.
\end{aligned}$$

The expressions for the tilde modes $\tilde{L}_n\,,\, \tilde{G}_r\,,\, \tilde{F}_m$ follow similarly. Looking at the NS-sector, a physical state $|\phi\rangle$ is required to satisfy:

$$\begin{aligned}
G_r|\phi\rangle &= 0\ ,\ r > 0, \\
L_n|\phi\rangle &= 0\ ,\ n > 0, \\
(L_0 - a_{\mathrm{NS}})\,|\phi\rangle &= 0.
\end{aligned}$$

Here a_{NS} is a normal ordering constant, arising when normal ordering l_0. This is regularized, to avoid the constant from being infinite. Similarly, in the R-sector, a physical state $|\psi\rangle$ is required to satisfy:

$$\begin{aligned}
L_n|\psi\rangle &= 0\ ,\ n > 0, \\
F_n|\psi\rangle &= 0\ ,\ n > 0, \\
(L_0 - a_{\mathrm{R}})\,|\psi\rangle &= 0.
\end{aligned}$$

Again a_{R} is a normal ordering constant. There is also a condition involving the zero mode F_0, since normal ordering has no effect on F_0, then the following is given

$$F_0|\psi\rangle = 0.$$

From the third super-Virasoro algebra equation in (2.2.1.2), the case $n = m = 0$ gives

$$\{F_0, F_0\} = 2L_0 + 0 \ \Rightarrow\ L_0 = F_0^2.$$

Thus for a physical state $|\psi\rangle$, the following is given

$$L_0|\psi\rangle = F_0\big(F_0|\psi\rangle\big) = 0$$

Hence, this implies $a_{\mathrm{R}} = 0$. However, a_{NS} can't be derived in this way, which will be discussed in the next section.

2.2.2 Light-Cone Quantization

In Bosonic String theory, consistency requires that the normal ordering constant is $a = 1$. It can also be shown that the space-time dimension is $D = 26$. More precisely, by studying certain states and requiring that $a = 1$, it was shown that these states are physical and have zero norm. The search for the zero norm states can be performed in the Superstring theory. It can then be shown that the value of a_{NS} is $\frac{1}{2}$ and the space-time dimension is $D = 10$. Here, the light-cone gauge is used to find the critical values of a_{NS} and D. Although, the light-cone gauge is a choice, Lorentz invariance of the theory is not apparent. It is the requirement of Lorentz invariance at the quantum level, that gives a restriction on a and D. The same can be done in Superstrings, therefore, this is the approach that is presented here.

The String can be described by Noether's theorem from the following conserved current associated with the Poincare symmetry given by

$$M_{\alpha}^{\mu\nu} = -\frac{1}{2\pi\alpha'}\left(X^{\mu}\partial_{\alpha}X^{\nu} - X^{\nu}\partial_{\alpha}X^{\mu} - i\overline{\psi}^{\mu}\rho_{\alpha}\psi^{\nu}\right).$$

This gives the following conserved charges

$$M^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^{\pi} d\sigma\, M_{\tau}^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu},$$

where the terms correspond to:

- The orbital angular momentum:

$$l^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}.$$

- The spin of the String:

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(a_{-n}^{\mu}a_{n}^{\nu} - a_{-n}^{\nu}a_{n}^{\mu}\right).$$

- The contribution from the fermionic modes:

$$K^{\mu\nu} = -i\frac{1}{2\pi\alpha'}\int_0^{\pi} d\sigma\, \overline{\psi}^{\mu}\rho_0\psi^{\nu}. \tag{2.2.2.1}$$

Taking the NS-sector, the following is deduced

$$\begin{aligned}\overline{\psi}^{\mu}\rho_0\psi^{\nu} &= \begin{pmatrix} \psi_-^{\mu} & \psi_+^{\mu} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_-^{\nu} \\ \psi_+^{\nu} \end{pmatrix} \\ &= \psi_-^{\mu}\psi_-^{\nu} + \psi_+^{\mu}\psi_+^{\nu}.\end{aligned}$$

Plugging in the mode expansions to the contribution from the fermionic modes in (2.2.2.1), it is computed that

$$\begin{aligned}K^{\mu\nu} &= -i\frac{1}{2\pi\alpha'}\int_0^{\pi}\frac{\alpha'}{2}\sum_{r,s}\left(b_r^{\mu}b_s^{\nu}e^{-i(r+s)(\tau-\sigma)} + b_r^{\mu}b_s^{\nu}e^{-i(r+s)(\tau+\sigma)}\right) \\ &= -i\frac{1}{2\pi}\sum_{r,s}b_r^{\mu}b_s^{\nu}e^{-i(r+s)\tau}\int_0^{\pi}cos(r+s)\sigma\, d\sigma \\ &= -i\sum_{r=\frac{1}{2}}^{\infty}\left(b_{-r}^{\mu}b_r^{\nu} - b_{-r}^{\nu}b_r^{\mu}\right).\end{aligned}$$

This is sufficient to begin formulating the theory in the light-cone gauge, since the worldsheet time-coordinate can be chosen to be any of the space-time coordinates X^i, where $i = 1, .., D-1$. In fact, there are many choices, however, it is usually defined as

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right),$$

where the following can be taken

$$X^+ = x^+ + \frac{1}{2}p^+\tau = x^+ + \frac{1}{4}p^+\sigma^+.$$

Here x^+ and p^+ are just c-numbers, even after quantization. Additionally, it should be noted that X^+ satisfies the two-dimensional wave equation. However, in Superstring theory, there are supersymmetric transformations that preserve this gauge choice for the superpartner ψ_A^+, of X^+. This implies that

$$\psi_A^+ = \left(\psi_A^0 + \psi_A^{D-1}\right) = 0.$$

To omit the spinor index A, the following is defined

$$\psi^{\mu}(\tau,\sigma) = \begin{cases} \psi_+^{\mu}(\tau,\sigma) & , \quad \sigma > 0\ , \\ \psi_-^{\mu}(\tau,-\sigma)\ , & \sigma < 0\ . \end{cases}$$

This ensures the spinor index is free. Here, $\psi^+ = 0$ would imply $\delta X^+ = \bar{\epsilon}\psi^+ = 0$, thus, the X^+ gauge choice is not altered in the process. Taking the space-time metric

$\eta_{\mu\nu}$, the components are given by

$$\eta_{-+} = \eta_{+-} = -1 \ , \qquad \eta_{ij} = \delta_{ij},$$

where $i, j = 1, ..., D-2$. Using the constraints $J_{\pm} = T_{\pm\pm} = 0$, the terms X^- and ψ^- are given by:

$$\psi^- = \frac{4}{p^+}\psi^i \partial_+ X^j \delta_{ij}.$$
$$X^- = x^- + p^- \tau + i \sum_{n \neq 0} \frac{a_n^-}{n} e^{-in\sigma^+},$$

where

$$p^- = 2a_0^-.$$

As a result of this, quantization can now be carried out. Therefore, normal ordering b_r^-, a_n^-, only a_0^- is affected as follows

$$: a_0^- := \frac{1}{2p^+} \left(\sum_m : a_{-m}^i a_m^j : + \sum_r r : b_{-r}^i b_r^j : \right) \delta_{ij} - \frac{1}{2p^+} a_{\rm NS}.$$

Here, $M_{\mu\nu}$ are promoted to hermitian operators, since they implement Lorentz transformations. However, normal ordering also gives rise to the anomaly terms. Thus, in order for Lorentz covariance to hold, these terms must vanish. Consequently, this is the restriction needed to compute the values $a_{\rm NS}$ and D. In particular, the following is demanded

$$[M^{i-}, M^{j-}] = 0,$$

where $M^{i-} = l^{i-} + E^{i-} + K^{i-}$. Each component here is given by

$$\begin{aligned}
E^{i-} &= -i \sum_{n=1}^{\infty} \frac{1}{n} \left(a_{-n}^i a_n^- - a_{-n}^- a_n^i \right), \\
K^{i-} &= -i \sum_{r=\frac{1}{2}}^{\infty} \left(b_{-r}^i b_r^- - b_{-r}^- b_r^i \right), \\
l^{i-} &= x^i p^- - x^- p^i \\
&= \tfrac{1}{2} \left(x^i p^- + p^- x^i \right) - \tfrac{1}{2} x^- p^i \left(x^- p^i + p^i x^- \right) \\
&= \tfrac{1}{2} \left(x^i p^- - x^- p^i + p^- x^i + p^i x^- \right).
\end{aligned}$$

Further analysis leads to the following

$$[M^{i-}, M^{j-}] = -\frac{1}{(p^+)^2} \sum_{m=1}^{\infty} \Delta_m^2 \left(a_{-m}^i a_m^j - a_{-m}^j a_m^i \right),$$

where

$$\Delta_m^2 = \Delta_m^1 - m = \frac{Z(m)}{m^2} - m.$$

Here

$$Z(m) = \frac{D-2}{8} m \left(m^2 - 1 \right) + 2m a_{\mathrm{NS}}.$$
$$\Delta_m^1 = \frac{Z(m)}{m^2} = \frac{D-2}{8} m + \left(2a_{\mathrm{NS}} - \frac{D-2}{8} \right) \frac{1}{m}.$$

Therefore, taking $[M^{i-}, M^{j-}] = 0$, implies $\Delta_m^2 = 0 \ \forall m \neq 0$, given by

$$\Delta_m^2 = \left(\frac{D-2}{8} - 1 \right) m + \left(2a_{\mathrm{NS}} - \frac{D-2}{8} \right) \frac{1}{m} = 0.$$

Thus, $D = 10$ and $a_{\mathrm{NS}} = \frac{1}{2}$ is deduced. Similarly in the R-sector, values $D = 10$ and $a_{\mathrm{R}} = 0$ can be computed.

2.2.3 String Spectrum

In this section, the spectrum of Superstring theory will be examined. Here the NS and R-Sectors will be considered independently, where the number operator is used to determine the mass-squared of a physical state. The number operator N in the R-sector is given by

$$N = N^a + N^d.$$

where

$$N^a = \sum_{n=1}^{\infty} \eta_{\mu\nu} a_{-n}^{\mu} a_n^{\nu}.$$
$$N^d = \sum_{m=1}^{\infty} m d_{-m}^{\mu} d_m^{\nu} \eta_{\mu\nu}.$$

For the NS-sector, similarly, the following can be written

$$N = N^a + N^b, \tag{2.2.3.1}$$

where

$$N^b = \sum_{r=\frac{1}{2}}^{\infty} r b^{\mu}_{-r} b^{\nu}_{r} \eta_{\mu\nu}.$$

The commutator relations satisfied by these number operators are given as follows:

$$\begin{aligned} [N^a, a^{\lambda}_{-m}] &= m a^{\lambda}_{-m} \quad , \quad m > 0, \\ [N^d, d^{\lambda}_{-m}] &= m d^{\lambda}_{-m} \quad , \quad m > 0, \\ [N^b, b^{\lambda}_{-r}] &= r b^{\lambda}_{-r} \quad , \quad r > 0. \end{aligned}$$

To recall, a physical state $|\phi\rangle$ in the NS-sector satisfies the constraint $(L_0 - a_{\mathrm{NS}})\,|\phi\rangle = 0$, i.e $\left(L_0 - \frac{1}{2}\right)|\phi\rangle = 0$. Expanding the operator $L_0 - \frac{1}{2}$ in terms of oscillators, it can be found that

$$\begin{aligned} L_0 - \frac{1}{2} &= \frac{1}{2}\sum_{n} : a^{\mu}_{-n} a^{\nu}_{n} : \eta_{\mu\nu} + \frac{1}{2}\sum_{r} r : b^{\mu}_{-r} b^{\nu}_{r} : \eta_{\mu\nu} - \frac{1}{2} \\ &= \frac{1}{2} a^{\mu}_{0} a^{\nu}_{0} \eta_{\mu\nu} + \frac{1}{2}\sum_{n=1}^{\infty} a^{\mu}_{-n} a^{\nu}_{n} \eta_{\mu\nu} + \frac{1}{2}\sum_{r=\frac{1}{2}}^{\infty} r b^{\mu}_{-r} b^{\nu}_{r} \eta_{\mu\nu} - \frac{1}{2} \\ &= \alpha' p^{\mu} p_{\mu} + N - \frac{1}{2}, \end{aligned} \tag{2.2.3.2}$$

where $a^{\mu}_{0} = \sqrt{2\alpha'} p^{\mu}$ for open Strings. Similarly, for the R-sector with the constraint $(L_0 - a_{\mathrm{R}})|\phi\rangle = 0$, i.e $L_0|\phi\rangle = 0$, the operator L_0 expansion, in terms of oscillators gives

$$L_0 = \alpha' p^{\mu} p_{\mu} + N, \tag{2.2.3.3}$$

Comparing (2.2.3.2) with the Klein-Gordon equation in the NS-sector, the space-time mass-squared of a physical state is the eigenvalue of the operator

$$M^2 = \frac{1}{\alpha'}\left(N - \frac{1}{2}\right). \tag{2.2.3.4}$$

Similarly, for the R-sector using (2.2.3.3), the space-time mass-squared of a physical state is an eigenvalue of

$$M^2 = \frac{1}{\alpha'} N.$$

As for the case of closed Strings, $a^{\mu}_{0} = \sqrt{\frac{\alpha'}{2}} p^{\mu}$ is used, therefore, a factor of 4 is present in the equations for M^2 above. At level zero, the ground states $|0\rangle_{\mathrm{NS}}$ and $|0\rangle_{\mathrm{R}}$ are given, where $N = 0$, thus equation (2.2.3.5) shows that the mass-squared of $|0\rangle_{\mathrm{R}}$ is massless; this can also be seen from the constraint equation $F_0|0\rangle_{\mathrm{R}} = 0$. Inserting

the mode expansion for F_0 gives

$$\eta_{\mu\nu} a_0^\mu d_0^\nu |0\rangle_{\rm R} = 0, \tag{2.2.3.5}$$

since a_0^μ is proportional to p^μ and d_0^μ is proportional to the gamma matrices Γ^μ, this implies that (2.2.3.5) is just the massless Dirac equation.
However, a major problem is that, the mass-squared of $|0\rangle_{\rm NS}$ is negative, hence, $|0\rangle_{\rm NS}$ is a Tachyon. This can be removed using the GSO projection, as discussed in the next section.

2.2.4 GSO Projections

In the previous section, it was shown that the Superstring spectrum admits an imaginary mass, such as, the Tachyon states, indicating that the vacuum is unstable. However, the Tachyon state is eliminated, by using the GSO projection [8]. The GSO projection is a truncation (or projection) of the spectrum, which reduces the number of states in the theory, thus removing potential problems, such as the Tachyon states. Further to this, two additional major problems arise in the spectrum, one being that the spectrum is not space-time supersymmetric. There is for example, no fermion in the spectrum with the same mass as the Tachyon. Alternatively, there are space-time bosons that corresponds to both worldsheet bosons and fermions; the same is also true of the space-time fermions. For a sensible theory, these problems should not arise. Therefore, using the GSO projection these unnecessary states in the Superstring spectrum can be removed, thus, obtaining a consistent, interacting Superstring theory.

Taking the NS-sector, the GSO projections are defined by keeping states with an odd number of fermion oscillator excitations and removing those with an even number. This is the requirement, when replacing the physical states $|\psi\rangle$ according to

$$|\psi\rangle \longmapsto P_{GSO}|\psi\rangle,$$

where the parity operator P_{GSO} is defined as

$$P_{GSO} = \tfrac{1}{2}\left(1 - (-1)^F\right),$$

determining the states in the theory. Here, F is the fermion number operator, that determines whether a state has an even or odd number of fermion excitations. This

is defined as

$$F = \sum_{r=1/2}^{\infty} \eta_{\mu\nu} b^{\mu}_{-r} b^{\nu}_{r},$$

which also obeys

$$\{(-1)^F, b^{\mu}\} = 0.$$

Thus, only half-integer values of the level number (2.2.3.1) are possible, therefore the spectrum of allowed physical masses are integral multiples of $\frac{1}{\alpha'}$ given as

$$M^2 = 0, \frac{1}{\alpha'}, \frac{2}{\alpha'}, \ldots.$$

The spin-0 ground state of the NS-sector therefore, is now massless and the spectrum no longer contains a Tachyon.

Chapter 3

Free-Fermionic Construction

In this chapter, the set up of the Free-Fermionic construction [15, 16, 17] in the Heterotic-String is shown. The partition function at an arbitrary point in the moduli space is derived, with the description of the Heterotic-String at the self dual point under T-duality. Then the partition function at the self-dual point is written in the most general way, enabling the derivation of the constraints on the form of the partition function. This follows with the derivation of all the necessary constraints for the construction of the Free-Fermionic models in the Heterotic-String. Having derived the tools for the construction, a summary of the ABK rules is given. The chapter then concludes with a simple application of these rules.

3.1 Heterotic Strings

In the Heterotic-String theory [9, 10, 11, 12], $\mathcal{N} = 1$ supersymmetry is obtained by decoupling the left- and right-moving modes. The supersymmetric charges are carried by the left-moving currents, where the Superstring fields X^μ_+ and ψ^μ_+ for $\mu = 0, \ldots, 9$ are considered. On the other hand, the right-moving worldsheet fields are described by the formalism of the Bosonic String, where the ten bosonic right-movers X^μ_- for $\mu = 0, \ldots, 9$ are taken. Since a space-time boson contributes a unit to the central charge and a free-fermion contributes half a unit, 32 Majorana-Weyl right-moving free-fermions λ^i_- are needed to cancel the conformal anomaly $c = -26$ in the Bosonic String. The theory is still ten-dimensional because the space-time indices $\mu = 0, \ldots, 9$ are carried by the coordinates X^μ in both left- and right-moving sectors, while the internal fermions λ^i_- do not carry a space-time indice. In summary,

the following fields are considered:

$$\begin{aligned} &X_+^\mu \ \text{ and } \ \psi_+^\mu \ \text{ in the left-moving sector,} \\ &X_-^\mu \ \text{ and } \ \lambda_-^i \ \text{ in the right-moving sector,} \end{aligned}$$

where $\mu = 0, \ldots, 9$ and $i = 1, \ldots, 32$. The action of the Heterotic-String in the light-cone gauge with the above fields is given by

$$S = \frac{1}{\pi} \int d^2\sigma \left(2\partial_- X_\mu \partial_+ X^\mu + i\psi^\mu \partial_- \psi_\mu + i \sum_{i=1}^{32} \lambda^i \partial_+ \lambda^i \right).$$

3.2 Free-Fermionic Formalism

To construct a four-dimensional space-time theory, the conformal anomaly needs to be cancelled on both the left- and right-sectors. This is given by the following equations:

$$\begin{aligned} C_L &= -26 + 11 + D + \frac{D}{2} + \frac{N_{f_L}}{2} = 0, \\ C_R &= -26 + D + \frac{N_{f_R}}{2} = 0, \end{aligned}$$

where for $D = 4$, the left-sector is cancelled when 18 Majorana-Weyl fermionic left-moving fields are imposed, whereas for the right-sector 44 Majorana-Weyl fermionic right-moving fields are imposed. These degrees of freedom can be seen as free fermions propagating on the String worldsheet. Moreover, the total set of fields is now:

$$\begin{aligned} &X_+^\mu \ , \ \psi_+^\mu \ \text{ and } \ \lambda_+^j \ \text{ in the left-moving sector,} \\ &X_-^\mu \ \text{ and } \ \lambda_-^i \ \text{ in the right-moving sector,} \end{aligned}$$

where $\mu = 0, \ldots, 3$, $i = 1, \ldots, 44$ and $j = 1, \ldots, 18$. Adopting complex coordinates defined by

$$z = \tau + i\sigma \ \text{ and } \ \bar{z} = \tau - i\sigma,$$

the worldsheet fields can now be defined as functions of z and $\bar{z}$ given by the following:

$$\begin{aligned} X^\mu(z,\bar{z})\ ,&\ \ \mu = 1,2,\\ \psi^\mu(z)\ ,&\ \ \mu = 1,2,\\ \lambda^i(z)\ ,&\ \ i = 1,\ldots,18,\\ \bar{\lambda}^j(\bar{z})\ ,&\ \ j = 1,\ldots,44. \end{aligned}$$

In the light-cone gauge the space-time bosons and fermions have only two degrees of freedom, namely the transverse coordinates, where the Heterotic action can now take the form

$$S = \frac{1}{\pi}\int d^2z\Big(\partial_z X_\mu(z,\bar{z})\partial_{\bar{z}}X^\mu(z,\bar{z}) - 2i\psi^\mu(z)\partial_z\psi_\mu(z) \\ - 2i\sum_{i=1}^{18}\lambda^i(z)\partial_z\lambda^i(z) - 2i\sum_{j=1}^{44}\bar{\lambda}^j(\bar{z})\partial_{\bar{z}}\bar{\lambda}^j(\bar{z})\Big),$$

where $\psi^\mu = \psi^\mu(z)$ and $\overline{\psi}^\mu = \overline{\psi}^\mu(\overline{z})$ corresponds to the left- and right-moving fermionic fields respectively. The worldsheet field content is described as follows:

Left-moving:	$X_L^\mu(z)$	$\mu = 1,2$	2 transverse coordinates,
	$\psi_L^\mu(z)$	$\mu = 1,2$	their superpartners,
	$\chi_I(z), y_I(z), w_I(z)$	$I = 1\ldots6$	18 internal real fermions.
Right-moving:	$X_R^\mu(\bar{z})$	$\mu = 1,2$	2 transverse coordinates,
	$\bar{\lambda}^i(\bar{z})$	$i = 1\ldots44$	44 internal real fermions.

This formalism based on the Heterotic-String theory is the Free-Fermionic construction.

3.3 Partition Function

In the Polyakov picture, String theory is formulated as a perturbative sum over a path integral on the String worldsheet, a genus-g Riemann surface. The free fermions propagate around the non-contractible loops on this surface, thus, boundary conditions need to be specified for each worldsheet fermion. In addition, the worldsheet supersymmetry should be preserved, which imposes that the supercurrent T_F must be uniquely defined up to a sign, under the transformation of the worldsheet fermions.

The supercurrent is defined as

$$T_F = \psi^\mu \partial X_\mu + i \sum_{I=1}^{6} \chi^I y^I w^I,$$

where $\chi^I(z), y^I(z)$ and $w^I(z)$ for $I = 1, ..., 6$ transform as the adjoint representation of $SU(2)^6$. The transport properties of the left- and right-moving fermions around a non-contractible loop, show that any configuration of boundary conditions in some basis consisting of 64 fermions, is realised. These fermions can be real or complex, where two real fermions with the same boundary conditions in each basis vector can pair into a complex fermion in the following way:

$$\begin{aligned} \lambda_{ij} &= \frac{1}{\sqrt{2}}(\lambda_i + i\lambda_j), \\ \lambda^*_{ij} &= \frac{1}{\sqrt{2}}(\lambda_i - i\lambda_j). \end{aligned} \tag{3.3.0.1}$$

In the models considered in this thesis, the following notations for the real and complex basis fermions will be used:

Real Left Fermions:

$\{\psi^1, \psi^2, \chi^1, y^1, w^1, \chi^2, y^2, w^2, \chi^3, y^3, w^3, \chi^4, y^4, w^4, \chi^5, y^5, w^5, \chi^6, y^6, w^6\}$

Real Right Fermions:

$\{\overline{y}^1, \overline{w}^1, \overline{y}^2, \overline{w}^2, \overline{y}^3, \overline{w}^3, \overline{y}^4, \overline{w}^4, \overline{y}^5, \overline{w}^5, \overline{y}^6, \overline{w}^6\}$

Complex Left Fermions:

$\{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}\}$

Complex Right Fermions:

$\{\overline{\psi}^1, \overline{\psi}^2, \overline{\psi}^3, \overline{\psi}^4, \overline{\psi}^5, \overline{\eta}^1, \overline{\eta}^2, \overline{\eta}^3, \overline{\Phi}^1, \overline{\Phi}^2, \overline{\Phi}^3, \overline{\Phi}^4, \overline{\Phi}^5, \overline{\Phi}^6, \overline{\Phi}^7, \overline{\Phi}^8\}$.

Here, the first 4 complex left and the last 16 complex right fermions are given in complex form and the remaining fermions y and w are not paired. This is due to the fact that their boundary conditions do not always allow a pairing.

3.3.1 Torus and Modular Invariance

In String theory, the String amplitude is calculated by the path integral

$$A_n = \sum_{g=0}^{\infty} \int DhDX^\mu \int \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n)\, e^{-S[h, X^\mu]},$$

where S is the conformal worldsheet action, h is the metric on the worldsheet, the sum is over all the physically inequivalent paths and V_i are vertex operators of external String states on the genus-g Riemann surface that are defined by the worldsheet. The total String amplitude is a sum over all possible Riemann surfaces, moded out by conformal invariance, similar to the sum over all Feynman graphs in Quantum Field Theory. The conformal invariance maps the tree level String topology to the sphere and the one-loop topology to the torus. At tree level all reparametrizations are local and quantum corrections are not taken into account, however, at higher loops further constraints will arise. Hence, it is instructive to look at the one-loop vacuum to vacuum amplitude, with no external states. This is the one-loop partition function.

The one-loop String amplitude is a sum over all non-equivalent tori. To determine what the non-equivalent tori are, the symmetries of a torus need to be investigated. The torus can be mapped to the complex plane by cutting it along its two non-

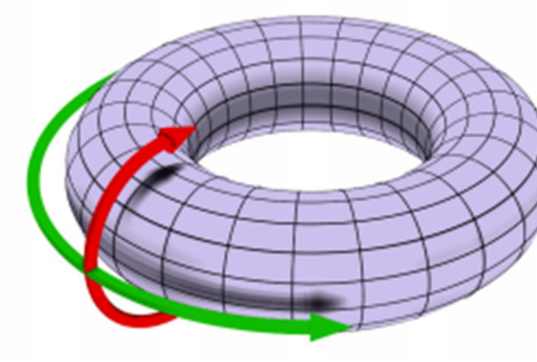

Figure 3.1: *Two non-contractible loops of the torus.*

contractible loops, as shown in the Figure 3.1. It can be characterized by specifying two finite and non-zero periods in the complex plane λ_1, λ_2 with a non-real ratio:

$$z \sim z + \lambda_1, \quad z \sim z + \lambda_2.$$

The torus is then identified with the complex plane modulo, a two-dimensional lattice $\Lambda_{(\lambda_1,\lambda_2)}$, where $\Lambda_{(\lambda_1,\lambda_2)} = \{m\lambda_1 + n\lambda_2,\ m, n \in \mathbb{Z}\}$. Using the reparametrization $z \to \frac{z}{\lambda_2}$, the torus is equivalent to one whose periods are 1 and $\tau = \frac{\lambda_1}{\lambda_2}$, as shown in Figure 3.2. In other words, the torus is left invariant by the following two transformations:

$T : \tau \to \tau + 1$ redefines the same torus,

$S : \tau \to -\dfrac{1}{\tau}$ swaps the two coordinates and reorientates the torus.

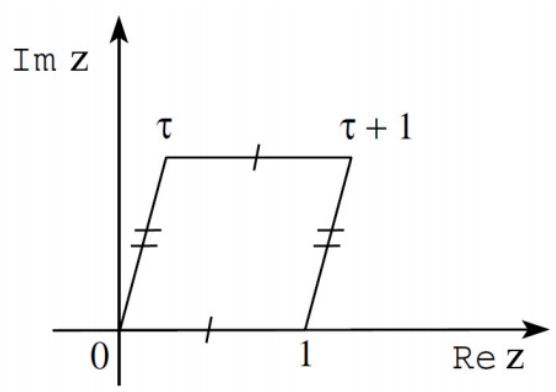

Figure 3.2: *Torus mapped to the complex plane, when the opposite edges of the parallelogram are identified.*

These transformations span a group of transformations known as the modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z},\ ab - cd = 1,$$

where any function invariant under these transformations is called modular invariant. The modular group is $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, where the division by $\mathbb{Z}_2$ takes the equivalence of an $SL(2,\mathbb{Z})$ matrix and its negative into account. The moduli space $\mathscr{M}$ of the torus or the space of conformally inequivalent tori is

$$\mathscr{M} \cong \mathbb{C}/PSL(2,\mathbb{Z}).$$

The fundamental domain can be taken as

$$\mathcal{F} = \{\tau || \tau| \geq 1, |\mathrm{Re}\ \tau| \leq 1/2, \mathrm{Im}\ \tau > 0\}.$$

Therefore, the partition function is a sum over this domain in order to integrate over all conformally inequivalent tori. The $SL(2,\mathbb{Z})$ invariant measure over the fundamental domain is given by

$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^2}.$$

Consequently, the modular transformations spanned by T and S are invariant. Additionally, it is required that the partition function does not depend on the parametrization of the tori.

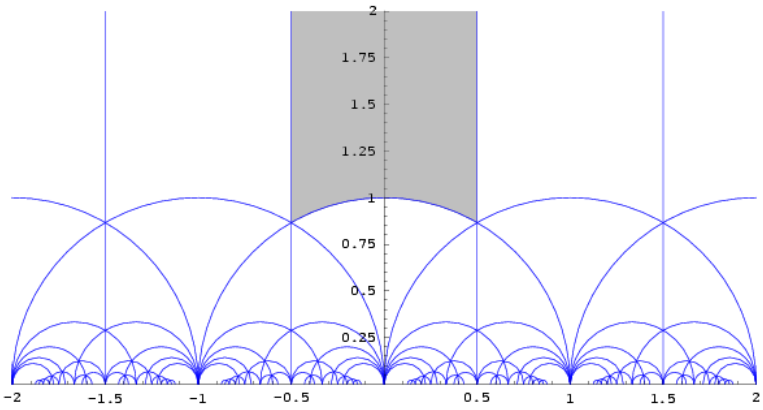

Figure 3.3: *The fundamental domain of the modular group of the torus indicated by the shaded area on the complex plane.*

3.3.2 Boundary Conditions

In order to provide predictions in a perturbative theory, the interest lies in the partition function of a one-loop diagram, which corresponds to the worldsheet being a torus, the vacuum to vacuum String amplitude. On this worldsheet, two boundary conditions for the non-contractible loops of the torus for each Free-Fermionic field needs to be specified. These conditions express the shifts of the phases of the fermionic fields under parallel transport around a non-contractible loop given by

$$f \to -e^{i\pi\alpha(f)} f.$$

where f is the fermionic field, $\alpha(f) = 0$ or 1 for Neveu-Schwarz and Ramond real fermions respectively, and $\alpha(f) \in (-1, 1]$ for complex fermions. Since there are two non-contractible loops on a torus, the complete phase assignment for a fermion can be written as a set of two phases

$$\begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}.$$

A set of specified phases for all basis fermions for one non-contractible loop is called a spin-structure and is expressed as a 64-dimensional vector

$$\alpha = \left\{ \alpha(\psi^1), \ldots, \alpha(\overline{\Phi}^8) \right\}.$$

To complete the spin-structure assignment for all the fermions on the torus, two vectors can then be defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

3.3.3 One-Loop Partition Function

Looking at the partition function, thinking of the path integral on a torus of parameter $\tau = \tau_1 + i\tau_2$, as formed by a field on a circle that has been evolved for Euclidean time $2\pi\tau_2$, translated by $2\pi\tau_1$, and identified with the initial circle. The generator of translations in time is the Hamiltonian $H = L_0 + \bar{L}_0 + \frac{1}{24}$, whereas the generator of translation in space is the momentum operator $P = L_0 - \bar{L}_0$. The identification of the ends of the cylinder thus formed, is realized by taking the trace over the Hilbert space of states

$$\begin{aligned} Z(\tau_1, \tau_2) &= \sum_{s \in \mathcal{H}} \left\langle s \left| e^{2\pi i \tau_1 P} e^{-2\pi i \tau_2 H} \right| s \right\rangle \\ &= \operatorname*{Tr}_{\mathcal{H}} e^{2\pi i \tau_1 P} e^{-2\pi i \tau_2 H}, \end{aligned}$$

which can be rewritten using $q \equiv e^{2\pi i \tau}$ as

$$Z(\tau) = q^{-1/48} \bar{q}^{-1/48} \operatorname*{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0}. \tag{3.3.3.1}$$

As it is known how L_0 acts on the states space, this can be calculated for each fermion. If the time boundary condition is anti-periodic (NS), then the partition function is just given by the trace with L_0 acting on the appropriate R or NS Fock space:

$$\begin{aligned} Z^{NS}_{NS}(\tau) &= \mathrm{Tr}_{NS}\ q^{L_0 - 1/48}, \\ Z^{NS}_{R}(\tau) &= \mathrm{Tr}_{R}\ q^{L_0 - 1/48}. \end{aligned} \tag{3.3.3.2}$$

When the time boundary condition is periodic (R) the definition of the trace is modified:

$$\begin{aligned} Z^{R}_{NS}(\tau) &= \mathrm{Tr}_{NS}(-1)^F q^{L_0 - 1/48}, \\ Z^{R}_{R}(\tau) &= \mathrm{Tr}_{R}(-1)^F q^{L_0 - 1/48}, \end{aligned} \tag{3.3.3.3}$$

where F is the fermion number operator, defined by the relations

$$
\begin{aligned}
F(f) &= +1, \text{ if f is a fermionic oscillator,} \\
F(f) &= -1, \text{ if f is the complex conjugate of a fermionic oscillator,} \\
F\left|+\right\rangle_R &= 0, \\
F\left|-\right\rangle_R &= -1,
\end{aligned}
$$

where $\left|+\right\rangle_R = \left|0\right\rangle$ is the state of a degenerated vacuum without an oscillator and $\left|-\right\rangle_R = f_0^\dagger \left|0\right\rangle$ is the state of a degenerated vacuum with zero mode oscillator. The partition function must include all possible combinations of boundary conditions, it is therefore a sum over all spin-structures. All the previous work now leads to the complete partition function

$$
Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}\, Z_B^2 \sum_{\substack{\text{spin} \\ \text{structure}}} C\binom{\alpha}{\beta} \prod_{f=1}^{64} Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix},
$$

where each term is described as follows:

- $\frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2}$ is the invariant measure under the modular transformations of the torus.
- Z_B is the bosonic contribution

  $$
  Z_B = \frac{1}{\sqrt{|\mathrm{Im}\tau|}\eta(\tau)},
  $$

  where

  $$
  \eta(\tau) = q^{\frac{1}{12}} \prod_n \left(1 - q^{2n}\right) \text{ with } q = e^{2\pi i \tau}.
  $$

- The $C\binom{\alpha}{\beta}$ are coefficients on the spin-structures that are yet to be determined.
- $Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}$ is the contribution of the fermion f, which depends on its boundary conditions $\alpha(f)$ and $\beta(f)$. It can be calculated using (3.3.3.1) to obtain the

following results:

$$
\begin{aligned}
Z_F \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \sqrt{\frac{\vartheta_3}{\eta}}, \\
Z_F \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \sqrt{\frac{\vartheta_2}{\eta}}, \\
Z_F \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \sqrt{\frac{\vartheta_4}{\eta}}, \\
Z_F \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \sqrt{\frac{\vartheta_1}{\eta}},
\end{aligned}
$$

where ϑ_i are defined as:

$$
\vartheta_1 = \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vartheta_2 = \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vartheta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vartheta_4 = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

and

$$
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{n \in \mathbb{Z}} q^{\frac{(n-a/2)}{2}^2} e^{2\pi i (n-b/2)(n-a/2)}.
$$

These formulae should be complex conjugated for the right moving fermions.

3.3.4 Modular Invariance Constraints

The invariance of the partition function under modular transformations gives further constraints for model building. Since the measure element and the bosonic contribution are modular invariant, imposing modular invariance on the remaining terms in the partition function results in additional constraints. Under $\tau \to \tau + 1$, the following transformations are given:

$$
\begin{aligned}
\eta &\longrightarrow e^{i\pi/12}\eta, \\
\vartheta_1 &\longrightarrow e^{i\pi/4}\vartheta_1, \\
\vartheta_2 &\longrightarrow e^{i\pi/4}\vartheta_2, \\
\vartheta_3 &\longleftrightarrow \vartheta_4,
\end{aligned}
$$

and under $\tau \to -\frac{1}{\tau}$:

$$\begin{aligned}
\eta &\longrightarrow (-i\tau)^{1/2}\eta, \\
\frac{\vartheta_1}{\eta} &\longrightarrow e^{-i\pi/2}\frac{\vartheta_1}{\eta}, \\
\frac{\vartheta_2}{\eta} &\longleftrightarrow \frac{\vartheta_4}{\eta}, \\
\frac{\vartheta_3}{\eta} &\longrightarrow \frac{\vartheta_3}{\eta}.
\end{aligned}$$

Since the partition function is a product of the spin-structures of 64 fermions, the modular transformations will take the spin-structure from one to another. Modular invariance requires that both spin-structures related by these transformations need to be present in the partition function with equal weight. This gives the constraints:

$$C\binom{\alpha}{\beta} = e^{i\frac{\pi}{4}(\alpha\cdot\alpha+\mathbf{1}\cdot\mathbf{1})}\, C\binom{\alpha}{\beta-\alpha+1}, \tag{3.3.4.1}$$

$$C\binom{\alpha}{\beta} = e^{i\frac{\pi}{2}\alpha\cdot\beta}\, C\binom{\beta}{\alpha}^{*}, \tag{3.3.4.2}$$

where $\mathbf{1}$ is the vector corresponding to periodic boundary conditions for all fermions and the product $\alpha\cdot\beta$ is defined by

$$\alpha\cdot\beta = \left\{\frac{1}{2}\sum_{\text{real left}} + \sum_{\text{complex left}} - \frac{1}{2}\sum_{\text{real right}} - \sum_{\text{complex right}}\right\}\alpha(f)\beta(f),$$

where 2 real fermions are equivalent to 1 complex fermion as given in equation (3.3.0.1). Another constraint arises when considering higher order loops,

$$C\binom{\alpha}{\beta}C\binom{\alpha'}{\beta'} = \delta_\alpha\delta_{\alpha'}e^{-i\frac{\pi}{2}\alpha\cdot\alpha'}C\binom{\alpha}{\beta+\alpha'}C\binom{\alpha'}{\beta'+\alpha}, \tag{3.3.4.3}$$

where δ_α is the space-time spin statistics index defined as

$$\delta_\alpha = e^{i\pi\alpha(\psi^\mu_{1,2})} = \begin{cases} +1, & \text{if} \quad \alpha(\psi^\mu_{1,2}) = 0, \\ -1, & \text{if} \quad \alpha(\psi^\mu_{1,2}) = 1. \end{cases}$$

These constraints can be used to derive the rules for constructing a model. Using (3.3.4.2) and (3.3.4.3) with $\alpha' = \alpha$ and $\beta = 0$, implies that

$$C\binom{\alpha}{0}^2 = \delta_\alpha C\binom{\alpha}{0}C\binom{0}{0},$$

thus, either $C\binom{\alpha}{0} = 0$ or $C\binom{\alpha}{0} = \delta_\alpha$ is taken, where $C\binom{0}{0} = 1$ is normalized. A set of vectors Ξ is then defined by

$$\Xi = \left\{ \alpha \,\middle|\, C\binom{\alpha}{0} = \delta_\alpha \right\}.$$

Using (3.3.4.2) and (3.3.4.3), Ξ is taken to be an Abelian additive group, and the spin-structures contributing to the partition function are pairs of elements in Ξ. Furthermore, if Ξ is taken to be finite and therefore the boundary conditions to be rational, it is in fact isomorphic to

$$\Xi \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{N_i},$$

which means that Ξ is generated by a set of basis vectors $\{b_1, \ldots, b_k\}$, such that

$$\sum_{i=1}^{k} m_i b_i = 0 \Leftrightarrow m_i = 0 \mod N_i \ \forall i,$$

where N_i is the smallest positive integer where $N_i b_i = 0$. Taking the three vectors α, β, $\gamma \in \Xi$, (3.3.4.3) can be rewritten as

$$C\binom{\alpha}{\beta+\gamma} = \delta_\alpha C\binom{\alpha}{\beta} C\binom{\alpha}{\gamma}. \tag{3.3.4.4}$$

Equation (3.3.4.1) with $\alpha = \beta$ gives

$$C\binom{\alpha}{\alpha} = e^{-i\frac{\pi}{4}\alpha\cdot\alpha} C\binom{\alpha}{\mathbf{1}}. \tag{3.3.4.5}$$

Manipulating (3.3.4.2), (3.3.4.3), (3.3.4.4) and using the fact that β generates a finite group of order N_β, if N_{ij} is the least common multiple of N_i and N_j, it must satisfy

$$N_{ij} b_i \cdot b_j = 0 \mod 4.$$

When $i = j$, this constraint holds if N_i is odd. However, if N_i is even, then the following stronger constraint holds

$$N_i b_i^2 = 0 \mod 8.$$

When all the constraints derived in this section are satisfied, the modular invariance condition is also satisfied, and thus, there is no further obstruction to consistently

assigning coefficients to pairs of elements of Ξ.

3.3.5 Hilbert Space

The equations (3.3.3.2) and (3.3.3.3) can be rewritten in the general case as

$$Z_F \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} = \mathrm{Tr}_\alpha \ \left[q^{H_\alpha} e^{\pi i \beta \cdot F_\alpha} \right],$$

where H_α is the Hamiltonian and F_α the fermion number operator in the Hilbert space sector $\mathscr{H}_\alpha$, defined by the vector α. The partition function can then be written as a sum over sectors, using the fact that the basis vectors b_i are generators of a discrete group $\mathbb{Z}_{N_i}$ and applying equation (3.3.4.4),

$$Z = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} \ Z_B^2 \ \sum_{\alpha \in \Xi} \delta_\alpha \mathrm{Tr} \left\{ \prod_{b_i} \left(\delta_\alpha C \binom{\alpha}{b_i} e^{i\pi b_i \cdot F_\alpha} + \cdots \right. \right.$$
$$\left. \left. \cdots + \left\{ \delta_\alpha C \binom{\alpha}{b_i} e^{i\pi b_i \cdot F_\alpha} \right\}^{N_i - 1} + 1 \right) e^{i\pi\tau H_\alpha} \right\}.$$

The only states that appear in the partition function are those that realise a generalised GSO projection

$$e^{i\pi b_i \cdot F_\alpha} \left| S \right\rangle_\alpha = \delta_\alpha C \binom{\alpha}{b_i}^* \left| S \right\rangle_\alpha .$$

The full Hilbert space is therefore given as

$$\mathscr{H} = \bigoplus_{\alpha \in \Xi} \prod_{i=1}^{k} \left\{ e^{i\pi b_i \cdot F_\alpha} = \delta_\alpha C \binom{\alpha}{b_i}^* \right\} \mathscr{H}_\alpha .$$

3.4 Free-Fermionic Construction Rules

It was shown in the previous section that for each consistent Heterotic Superstring model, there exists a partition function defined by a set of vectors with boundary conditions and a set of coefficients associated to each pair of these vectors. Now that all the constraints are derived for model building, using the Free-Fermionic construction, it will now be shown that for each set of such vectors and the set of associated coefficients, a set of general rules can be summarised for any Free-Fermionic models. These rules were originally derived by Antoniadis, Bachas and Kounnas in [15, 16], which are known as the ABK rules[3]. First, these rules are

[3]These rules were also developed with a different formalism by Kawai, Lewellen and Tye in [17]

shown and then an example model is given. This will also be the working tool set for all the classifications carried out in this thesis. For further convenience, the vectors containing the boundary conditions used to define a model are called the basis vectors and the coefficients are called the one-loop phases that appear in the partition function.

3.4.1 ABK Rules

The first requirement is a set of basis vectors that defines Ξ, the space of all sectors. For each sector $\beta \in \Xi$ there is a Hilbert space of states. Each basis vector b_i consists of a set of boundary conditions for each fermion, written as

$$b_i = \left\{\alpha(\psi^{\mu}_{1,2}), \ldots, \alpha(w^6)|\alpha(y^1), \ldots, \alpha(\bar{\phi}^8)\right\},$$

where $\alpha(f)$ is defined by

$$f \to -e^{i\pi\alpha(f)} f.$$

The b_i have to form an additive group and satisfy the constraints derived in the previous section. If N_i is the smallest positive integer for which $N_i b_i = 0$ and N_{ij} is the least common multiple of N_i and N_j, the following rules must hold:

1. $\sum_{i=1}^{k} m_i b_i = 0 \Leftrightarrow m_i = 0 \mod N_i \ \forall i,$
2. $\mathbf{1} \in \Xi,$
3. $N_{ij} b_i \cdot b_j = 0 \mod 4,$
4. $N_i b_i^2 = 0 \mod 8,$
5. Even number of real fermions.

Note, all the above constraints need to be obeyed by the basis vectors in order to preserve modular invariance.

3.4.2 One-Loop Coefficients Rules

Once the space of states is defined, the phases $C\binom{b_i}{b_j}$ for all intersection of basis vectors have to be specified. These coefficients are required to obey the constraints (3.3.4.2), (3.3.4.4) and (3.3.4.5), which can be rewritten as:

$$
\begin{aligned}
&1. \qquad C\binom{b_i}{b_j} = \delta_{b_i}\; e^{\frac{2\pi i}{N_j}n} = \delta_{b_j}\; e^{\frac{\pi i}{2}b_i\cdot b_j}\; e^{\frac{2\pi i}{N_i}m},\\
&2. \qquad C\binom{b_i}{b_i} = e^{-i\frac{\pi}{4}b_i\cdot b_i}\; C\binom{b_i}{\mathbf{1}},\\
&3. \qquad C\binom{b_i}{b_j} = e^{i\frac{\pi}{2}b_i\cdot b_j}\; C\binom{b_j}{b_i}^{*},\\
&4. \qquad C\binom{b_i}{b_j+b_k} = \delta_{b_i}\; C\binom{b_i}{b_j}\; C\binom{b_i}{b_k},
\end{aligned}
$$

where n and m are natural integers. These are all the coefficients that appear in the partition function.

3.4.3 GGSO Projections

To complete the Free-Fermionic construction, another condition on the physical states are imposed, called the generalised GSO projection. This is a consequence of modular invariance and the partition function as discussed earlier, where a state is either projected in or out or undergoes a truncation given by the following

$$
e^{i\pi b_i\cdot F_\alpha}\,|S\rangle_\alpha = \delta_\alpha C\binom{\alpha}{b_i}^{*}|S\rangle_\alpha \quad \forall i, \tag{3.4.3.1}
$$

here $|S\rangle_\alpha$ is a state generated by the Hilbert space in the sector $\alpha \in \Xi$, $\delta_\alpha = e^{i\pi\alpha(\psi^\mu)}$ and the scalar product $b_i \cdot F_\alpha$ is defined as

$$
b_i \cdot F_\alpha = \left\{ \sum_{\text{Left-Movers}} - \sum_{\text{Right-Movers}} \right\} b_i(f) \times F_\alpha(f),
$$

for all basis vectors b_i and the sum runs for all real and complex fermions.

3.4.4 Massless String Spectrum

Now that these conditions are satisfied, the following formulae can be used to analyse the complete spectrum. The mass of a state in the sector $\mathscr{H}_\alpha$ is given by the zero-moment Virasoro gauge conditions:

$$
\begin{aligned}
M_L^2 &= -\frac{1}{2} + \frac{\alpha_L\cdot\alpha_L}{8} + \sum_{\text{left-movers}},\\
M_R^2 &= -1 + \frac{\alpha_R\cdot\alpha_R}{8} + \sum_{\text{right-movers}},\\
M_L^2 &= M_R^2,
\end{aligned} \tag{3.4.4.1}
$$

where α_L and α_R are the boundary conditions defined by the basis vector α for the left and right moving fermions respectively. The frequencies of the fermionic oscillators depending on their boundary conditions is taken to be:

$$f \to -e^{i\pi\alpha(f)} f \quad , \quad f^* \to -e^{-i\pi\alpha(f)} f^*.$$

The frequency for the fermions is then given by

$$\nu_f = \frac{1+\alpha}{2} \quad , \quad \nu_f^* = \frac{1-\alpha}{2}.$$

Each complex fermion f generates a $U(1)$ current, with a charge with respect to the unbroken Cartan generators of the four-dimensional gauge group given by

$$\begin{aligned} Q_\nu(f) &= \nu - \frac{1}{2}, \\ &= \frac{\alpha(f)}{2} + F. \end{aligned}$$

With all the necessary tools for the construction of the Heterotic Free-Fermionic model building, an example will now be provided.

3.4.5 A Simple Example

To construct a simple model in order to understand how this formalism works, the basis vector $\mathbf{1}$, where all boundary conditions are periodic is used, since it is required to be in Ξ. Then there exists two sectors: $\Xi = \{\mathbf{1}, 2\cdot\mathbf{1} = \mathbf{0}\}$. Here, the notation NS for the sector $\mathbf{0}$ is used, which is the Neveu-Schwartz sector. Given $2\cdot\mathbf{1} \pmod 2 = \mathbf{0}$, then $N_{\mathbf{1}} = 2$ and $\mathbf{1}\cdot\mathbf{1} = -12$, therefore the rules on the basis vectors are satisfied. Since the states with a mass at the String scale would have a mass of the order of the Planck mass, they are not phenomenologically acceptable. Therefore, only the massless String spectrum is considered, hence $M = 0$ must be satisfied. Moreover, all the particle content of the Standard Model should exist in the spectrum. Applying (3.4.4.1) the simplest model gives:

For the sector $\mathbf{1}$

$$M_L^2 = -\frac{1}{2} + \frac{10}{8} + \sum_{left-movers} \nu \quad > \quad 0,$$

Thus, this sector contains no massless states and should be excluded.

For the NS-sector

$$M_L^2 = -\frac{1}{2} + \frac{0}{8} + \sum_{left-movers} \nu, \tag{3.4.5.1}$$
$$M_R^2 = -1 + \frac{0}{8} + \sum_{right-movers} \nu,$$

where for the fermions the frequency is given by

$$\nu_{f,f^*} = \frac{1 \pm 0}{2} = \frac{1}{2}.$$

Recall, the condition $M_L^2 = M_R^2$ must be satisfied. Thus either a Tachyonic negative mass $-\frac{1}{2}$ is given by acting on the NS vacuum with 1 fermionic right-moving oscillator, or a massless state is given by acting with 1 left-moving fermionic oscillator and either 2 right-moving fermionic oscillators or 1 right-moving bosonic oscillator. The massless states are then given as follows:

- $\psi^\mu_{1/2}\partial\bar{X}^\nu_1 \left|0\right\rangle_{NS}$, where $\partial\bar{X}^\nu_1$ is the bosonic creation operator. These states correspond to the Graviton, the Dilaton and the antisymmetric tensor.

- $\psi^\mu_{1/2}\bar{\Phi}^a_{1/2}\bar{\Phi}^b_{1/2}\left|0\right\rangle_{NS}$, $a, b \in \{1, \ldots, 44\}$: Gauge bosons in the adjoint representation of $SO(44)$.

- $\{\chi^i_{1/2}, y^i_{1/2}, w^i_{1/2}\}\partial\bar{X}^\nu_1\left|0\right\rangle_{NS}$, $i \in \{1, \ldots, 6\}$: Gauge bosons in the adjoint representation of $SU(2)^6$.

- $\{\chi^i_{1/2}, y^i_{1/2}, w^i_{1/2}\}\bar{\Phi}^a_{1/2}\bar{\Phi}^b_{1/2}\left|0\right\rangle_{NS}$, $i \in \{1, \ldots, 6\}$: Scalars in the adjoint representation of $SU(2)^6 \times SO(44)$.

The Tachyonic states are $\overline{\Phi}^a_{1/2}\left|0\right\rangle_{NS}$ with a mass $M^2 = -\frac{1}{2}$. Now the GSO projection for each state is performed. Thus, the first state is given as

$$e^{i\pi\mathbf{0}\cdot F_\mathbf{1}}\psi^\mu_{1/2}\partial\bar{X}^\nu_1\left|0\right\rangle_{NS} = -\psi^\mu_{1/2}\partial\bar{X}^\nu_1\left|0\right\rangle_{NS},$$

and using the rules (3.4.3.1), it can be computed that $C\binom{NS}{\mathbf{1}} = \delta_\mathbf{1} = -1$, therefore

$$\delta_{NS}C\binom{NS}{\mathbf{1}}^*\psi^\mu_{1/2}\partial\bar{X}^\nu_1\left|0\right\rangle_{NS} = -\psi^\mu_{1/2}\partial\bar{X}^\nu_1\left|0\right\rangle_{NS}.$$

This state survives the GSO projection. Similarly, all other states in this model, including Tachyons, survive the GSO projection. To eliminate the Tachyon, an additional basis vector with appropriate phases is added, in order to project it out of the spectrum with the corresponding GSO projections. It is also desired to include the particle content of the Standard Model and reduce the gauge group. This will be discussed in the next chapter.

Chapter 4

Classification of the Flipped SU(5) Heterotic-String Vacua

In this chapter, the Flipped $SU(5)$ classification [5] of the Free-Fermionic Heterotic-String vacua is presented. Here, the constructed models are given by breaking the SO(10) GUT symmetry at the String scale to the Flipped SU(5) subgroup [18, 19, 20, 21, 22, 23]. In the classification method, the set of basis vectors defined by the boundary conditions assigned to the free fermions, is fixed. The enumeration of the String vacua is obtained in terms of the Generalised GSO (GGSO[4]) projection coefficients entering the one-loop partition function. Then algebraic expressions for the GGSO projections are derived for all the physical states appearing in the sectors generated by the set of basis vectors. This enables the analysis of the entire String spectrum to be programmed into a computer code that is used to perform a statistical sampling in the space of $2^{44} \approx 10^{13}$ Flipped SU(5) vacua. The scan was carried out for 10^{12} GGSO configurations, for that purpose, two independent codes were developed based on JAVA and FORTRAN95. All the results presented here are confirmed by the two independent routines. Contrary to the previous Free-Fermionic classifications, no exophobic Flipped SU(5) vacua with odd numbers of generations were found. Therefore, the structure of exotic states appearing in the three generation models containing a viable Higgs spectrum, are studied.

4.1 Flipped $SU(5)$ Free-Fermionic Models

The Free-Fermionic construction provides an elegant approach to studying phenomenologically viable properties of the String vacua. The matter content arises from the **27** of the E_6 symmetry, which breaks to the $SO(10)$ symmetry at the String

[4]Note: The abbreviation GGSO will be used for Generalised GSO in this chapter

scale, decomposing under the **16** spinorial and **10** vectorial representations. The **16** consists of all the left- and right-handed fermions known and predicted whereas the **10** contains the Higgs states. The $SO(10)$ gauge group is further broken at the String scale to one of its subgroups and therefore the gauge group in the effective low energy field theory is given to be a subgroup of the $SO(10)$.

4.1.1 Free-Fermionic Construction

Recalling chapter 3, the four-dimensional Free-Fermionic construction in the light-cone gauge is represented by 20 left-moving and 44 right-moving real worldsheet fermions. These worldsheet fermions acquire a phase as they are parallel transported along the non-contractible loops of the vacuum to vacuum amplitude. The usual light-cone gauge notation of the fermions are given by: $\psi^{\mu}_{1,2}, \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6}$ (left-movers) and $\overline{y}^{1,\ldots,6}, \overline{\omega}^{1,\ldots,6}$, $\psi^{1,\ldots,5}$, $\overline{\eta}^{1,2,3}$, $\overline{\phi}^{1,\ldots,8}$ (right-movers). Under the modular invariance constraints [15, 16], each model is defined by a particular choice of phases for the fermions that can be spanned by a set of basis vectors $v_1, \ldots, v_N$

$$v_i = \left\{\alpha_i(f_1), \ldots, \alpha_i(f_{20}) | \alpha_i(\overline{f}_1), \ldots, \alpha_i(\overline{f}_{44})\right\}.$$

The basis vectors generate a space Ξ which produces the String spectrum consisting of 2^N sectors. Each sector here is given as a linear combination of all the basis vectors

$$\xi = \sum_{i=1}^{N} m_j v_i, \qquad m_j = 0, 1, \ldots, N_j - 1,$$

where $N_j \cdot v_j = 0 \bmod 2$. They also describe the transformation properties of each fermion on the worldsheet, which is given by

$$f_j \to -e^{i\pi\alpha_i(f_j)}\, f_j, \qquad j = 1, \ldots, 64.$$

The basis vectors also induces the generalised GSO projections, with an action on any given String state $|S_\xi>$. This can be written as

$$e^{i\pi v_i \cdot F_\xi} |S_\xi> = \delta_\xi \, C\binom{\xi}{v_i}^{*} |S_\xi> \tag{4.1.1.1}$$

where $\delta_\xi = \pm 1$ is the index for the space-time spin statistics and F_ξ is the fermion number operator. Varying the different set of GGSO projection coefficients $C\binom{\xi}{v_i} = \pm 1$ produces distinct String models. In summary, a Free-Fermionic model is constructed by a set of basis vectors $v_1, \ldots, v_N$ together with a set of $2^{N(N-1)/2}$

independent GGSO projection coefficients $C\binom{v_i}{v_j}, i > j$ consistent with modular invariance.

4.1.2 SO(10) Models

The $SO(10)$ GUT models are generated by a set of 12 basis vectors, these are given as follows:

$$\begin{aligned}
v_1 = \mathbf{1} &= \{\psi^\mu,\ \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6}| \\
&\qquad \overline{y}^{1,\ldots,6}, \overline{\omega}^{1,\ldots,6}, \overline{\eta}^{1,2,3}, \overline{\psi}^{1,\ldots,5}, \overline{\phi}^{1,\ldots,8}\}, \\
v_2 = S &= \{\psi^\mu, \chi^{1,\ldots,6}\}, \\
v_{2+i} = e_i &= \{y^i, \omega^i|\overline{y}^i, \overline{\omega}^i\},\ i = 1, \ldots, 6, \\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56}|\overline{y}^{34}, \overline{y}^{56}, \overline{\eta}^1, \overline{\psi}^{1,\ldots,5}\}, \\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56}|\overline{y}^{12}, \overline{y}^{56}, \overline{\eta}^2, \overline{\psi}^{1,\ldots,5}\}, \\
v_{11} = z_1 &= \{\overline{\phi}^{1,\ldots,4}\}, \\
v_{12} = z_2 &= \{\overline{\phi}^{5,\ldots,8}\}.
\end{aligned} \tag{4.1.2.1}$$

The first basis vector **1** above is a requirement of the ABK rules [15, 16] in order to preserve modular invariance. This generates the $SO(44)$ gauge symmetry together with Tachyons in the massless String spectrum as discussed in chapter 3. With the addition of vector S, the $SO(44)$ gauge group is preserved and the Tachyons are all projected out of the massless spectrum as a $\mathcal{N} = 4$ supersymmetric theory is constructed. The following six vectors: e_1,...,e_6 all correspond to the possible symmetric shifts of the six internal coordinates, therefore, this breaks the $SO(44)$ group to $SO(32) \times U(1)^6$ that preserves $\mathcal{N} = 4$ space-time supersymmetry. The vectors b_1 and b_2 break $\mathcal{N} = 4$ to $\mathcal{N} = 1$ space-time supersymmetry. These vectors also break the $U(1)^6$ symmetry giving rise to the $SO(10) \times U(1)^2 \times SO(18)$ gauge symmetry. The states coming from the hidden sector are produced by the vectors z_1 and z_2, which are given by the remaining fermions: $\overline{\phi}^{1,\ldots,8}$, that were not affected by the action of the previous vectors on the GGSO projection given in equation (4.1.1.1). These vectors together with the others generate the following adjoint representation of the gauge symmetry: $SO(10) \times U(1)^3 \times SO(8) \times SO(8)$. Here, $SO(10) \times U(1)^3$ is the observable gauge group that gives rise to the matter states arising from the twisted sectors that are charged under the $U(1)$s, whereas $SO(8) \times SO(8)$ is the hidden gauge group, where all the matter states are neutral under the $U(1)$s.

4.1.3 SO(10) Subgroups

The $SO(10)$ GUT models generated by equation (4.1.2.1) can be broken to a subgroup by the boundary condition assignment on the complex fermions $\overline{\psi}^{1,\ldots,5}$. For the Flipped $SU(5)$ and Pati-Salam cases, one additional basis vector is required, denoted as $v_N \equiv \alpha$, to break the $SO(10)$ symmetry. In these cases, the $SO(6)\times SO(4)$ models, utilises solely periodic and anti-periodic boundary conditions. Whereas in the Flipped $SU(5)$ case, the boundary conditions includes the $1/2$ assignments. In order to construct the $SU(4) \times SU(2) \times U(1)$ and Standard-Like models, the Pati-Salam breaking is required as well as an additional $SO(10)$ breaking basis vector. These gauge groups can be constructed from the following basis vectors:

Pati-Salam Models:

$$v_{13} \;= \alpha \;= \{\overline{\psi}^{4,5}, \overline{\phi}^{1,2}\}. \tag{4.1.3.1}$$

Flipped SU(5) Models:

$$v_{13} \;= \alpha \;= \{\overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\psi}^{1,\ldots,5} = \frac{1}{2}, \overline{\phi}^{1,\ldots,4} = \frac{1}{2}, \overline{\phi}^{5} = 1\}. \tag{4.1.3.2}$$

SU(4) x SU(2) x U(1) Models:

$$\begin{aligned} v_{13} &= \alpha = \{\overline{\psi}^{4,5}, \overline{\phi}^{1,2}\}, \\ v_{14} &= \beta = \{\overline{\psi}^{4,5} = \frac{1}{2}, \overline{\phi}^{1,\ldots,6} = \frac{1}{2}\}. \end{aligned} \tag{4.1.3.3}$$

Standard-Like Models:

$$\begin{aligned} v_{13} &= \alpha = \{\overline{\psi}^{4,5}, \overline{\phi}^{1,2}\}, \\ v_{14} &= \beta = \{\overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\psi}^{1,\ldots,5} = \frac{1}{2}, \overline{\phi}^{1,\ldots,4} = \frac{1}{2}, \overline{\phi}^{5} = 1\}. \end{aligned} \tag{4.1.3.4}$$

Hereafter, the study of the Flipped $SU(5)$ models are considered and the classification methodology is presented using this $SO(10)$ subgroup. The $SU(4) \times SU(2) \times U(1)$ models will be discussed in chapter 5, whereas the Pati-Salam and Standard-Like models are discussed in chapter 6.

4.1.4 Flipped $SU(5)$ Models

The Flipped $SU(5)$ models are achieved from the breaking of the $SO(10)$ GUT generated by the basis vectors in (4.1.2.1) using the vector α in (4.1.3.2). The rational boundary condition assignment of the complex right-moving fermions $\overline{\psi}^{1,\ldots,5} = \pm\frac{1}{2}$

makes this possible. However, the basis vector α in (4.1.3.2), used to assist this breaking, is in fact not unique. As in the cases of other Free-Fermionic Flipped $SU(5)$ models constructed to date [21, 23], only the assignment of $\overline{\psi}^{1,\ldots,5}$ to the case of positive 1/2 boundary conditions are considered here. Furthermore, the assignment of the 3 complex worldsheet fermions $\overline{\eta}^{1,2,3} = 1/2$ is fixed by the modular invariance constraint $b_j \cdot \alpha = 0 \,\mathrm{mod}\, 1$. Consequently, it follows that the assignment of the boundary conditions of the eight worldsheet complex fermions $\overline{\psi}^{1,\ldots,5}, \overline{\eta}^{1,2,3}$ is unique and the variation is in the boundary conditions of the worldsheet fermions $\overline{\phi}^{1,\ldots,8}$. Modular invariance constraints, restrict the possibilities to assigning 1/2 boundary conditions of $\overline{\phi}^{1,\ldots,8}$ worldsheet fermions to 0, 4 or 8. The null case been given by

$$\alpha = \{\overline{\psi}^{1,\ldots,5} = \tfrac{1}{2}, \overline{\eta}^{1,2,3} = \tfrac{1}{2}, \overline{\phi}^{1,2} = 1, \overline{\phi}^{3,4} = 1, \overline{\phi}^{5} = 0, \overline{\phi}^{6,7} = 0, \overline{\phi}^{8} = 0\},$$

is automatically excluded because the sector $x = 2\alpha = \{\overline{\psi}^{1,\ldots,5}, \overline{\eta}^{1,2,3}\}$ enhances the $SU(5) \times U(1)$ gauge group back to the $SO(10)$ symmetry. The condition $z_{1,2} \cdot \alpha = 0 \,\mathrm{mod}\, 1$, imposes the assignment of 1/2 boundary conditions to 0, 2 or 4 of each of the groups of worldsheet fermions $\overline{\phi}^{1,\ldots,4}$ and $\overline{\phi}^{5,\ldots,8}$. The possible choices of v_{13} are then given by

$$\begin{aligned}
\alpha_1 &= \{\overline{\psi}^{1,\ldots,5} = \tfrac{1}{2}, \overline{\eta}^{1,2,3} = \tfrac{1}{2}, \overline{\phi}^{1,2} = \tfrac{1}{2}, \overline{\phi}^{3,4} = \tfrac{1}{2}, \overline{\phi}^{5} = 1, \overline{\phi}^{6,7} = 0, \overline{\phi}^{8} = 0\,\}, \\
\alpha_2 &= \{\overline{\psi}^{1,\ldots,5} = \tfrac{1}{2}, \overline{\eta}^{1,2,3} = \tfrac{1}{2}, \overline{\phi}^{1,2} = \tfrac{1}{2}, \overline{\phi}^{3,4} = \tfrac{1}{2}, \overline{\phi}^{5} = \tfrac{1}{2}, \overline{\phi}^{6,7} = \tfrac{1}{2}, \overline{\phi}^{8} = \tfrac{1}{2}\}, \\
\alpha_3 &= \{\overline{\psi}^{1,\ldots,5} = \tfrac{1}{2}, \overline{\eta}^{1,2,3} = \tfrac{1}{2}, \overline{\phi}^{1,2} = \tfrac{1}{2}, \overline{\phi}^{3,4} = 0, \overline{\phi}^{5} = 1, \overline{\phi}^{6,7} = \tfrac{1}{2}, \overline{\phi}^{8} = 0\}.
\end{aligned} \quad (4.1.4.1)$$

These α's require that the sets of basis vectors are linearly independent. This does not hold for the cases with α_1 and α_2, therefore the following equations are obtained:

$$\begin{aligned}
1 &= S + \sum_{i=0}^{6} e_i + 2\alpha_1 + z_2, \\
1 &= S + \sum_{i=0}^{6} e_i + 2\alpha_2 \ .
\end{aligned}$$

In order to keep the set of basis vectors in (4.1.2.1), in addition to α_1 or α_2 being linearly independent, the basis vector $\mathbf{1}$ is removed, leaving the set of 12 vectors $\{S, e_1, e_2, e_3, e_4, e_5, e_6, b_1, b_2, z_1, z_2, \alpha_i\}$, where i = 1 or 2. In the case with α_3, the set in (4.1.2.1) is linearly independent, giving the set $\{1, S, e_1, e_2, e_3, e_4, e_5, e_6, b_1, b_2, z_1, z_2, \alpha_3\}$.

In the remainder of the chapter, a comprehensive view of the methodology is given with an insight into the classification of the $SU(5) \times U(1)$ models, using the α_1 basis. The classification was carried out by use of two independent codes, the first being the JAVA and the second being the FORTRAN95 code. The classification for α_2 and

α_3 will be reported in a future publication.

4.1.5 GGSO Projections

In order to define the String vacua, the GGSO projection coefficients appearing in the one-loop partition function $c\binom{v_i}{v_j}$ need to be specified. Taking the coefficients to span a 12×12 matrix, only the elements $i \geq j$ are independent. Modular invariance dictates that the 66 lower triangle elements of the matrix are fixed by the corresponding 66 upper triangle elements. Adding the remaining 12 diagonal terms, 78 independent coefficients are left, which corresponds to $2^{78} \approx 3 \times 10^{23}$ different String vacua. Moreover, requiring that the models possess $N = 1$ space–time supersymmetry, 11 of the coefficients are fixed. Without loss of generality, the following associated GGSO projection coefficients are set

$$C\binom{S}{S} = C\binom{S}{e_i} = C\binom{S}{b_k} = C\binom{S}{z_1} = C\binom{S}{\alpha} = -1, \qquad i = 1, ..., 6,\ k = 1, 2. \tag{4.1.5.1}$$

Modular invariance imposes additional constraints on the diagonal terms. In this case, where the vector $\mathbf{1}$ is composite, they are given by:

$$\begin{aligned}
C\binom{S}{z_2} &= -\prod_{i=1}^{6} C\binom{S}{e_i}, \\
C\binom{e_k}{z_2} &= \prod_{\substack{i=1 \\ i\neq k}}^{6} C\binom{e_k}{e_i}, \quad k = 1 \ldots 6, \\
C\binom{b_k}{b_k} &= -\prod_{i=1}^{6} C\binom{b_k}{e_i} C\binom{b_k}{z_2}, \quad k = 1, 2, \\
C\binom{z_1}{z_1} &= -\prod_{i=1}^{6} C\binom{z_1}{e_i} C\binom{z_1}{z_2}, \\
C\binom{\alpha}{\alpha} &= -\prod_{i=1}^{6} C\binom{\alpha}{e_i} C\binom{\alpha}{z_2},
\end{aligned} \tag{4.1.5.2}$$

where $C\binom{z_2}{z_2}$ is independent of any term. Further analysis of the GGSO projections of interest, show that there are additional phases that do not affect the properties of the String spectrum. As a result, the following coefficients are fixed in the ensuing analysis

$$C\binom{e_i}{e_i} = C\binom{e_3}{b_1} = C\binom{e_4}{b_1} = C\binom{e_1}{b_2} = C\binom{e_2}{b_2} = C\binom{b_1}{b_2} = C\binom{z_2}{z_2} = 1, \quad (4.1.5.3)$$

where $i = 1, ..., 6$. Taking the equations (4.1.5.1), (4.1.5.2) and (4.1.5.3), 44 independent coefficients are left, which can take two discrete values ± 1, except in the cases $C\binom{\alpha}{b_1}$, $C\binom{\alpha}{b_2}$ and $C\binom{\alpha}{z_2}$, where they take the values $\pm i$ since $\alpha \cdot b_1 = -3$ (odd), $\alpha \cdot b_2 = -3$ (odd) and $\alpha \cdot z_2 = -1$ (odd). Furthermore, a simple counting gives $2^{44} \approx 1.76 \times 10^{13}$ vacua in this class of Superstring models. It should be noted that there may still exist some degeneracies in this space of vacua with regard to the characteristics of the low energy effective field theory, and in particular with respect to the observable massless states. For instance, the 3 twisted sectors of $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal Orbifolds possess a cyclic permutation symmetry. Nevertheless, some of the vacua that may seem identical in the low energy effective field theory limit of the observable sector, differ by other properties such as the massive spectrum, the superpotential couplings and the hidden sector matter states and are therefore distinct.

4.2 String Spectrum

The vector bosons from the untwisted sector generate the

$$SU(5) \times U(1) \times U(1)^3 \times SU(4) \times U(1) \times U(1) \times SO(6)$$

gauge symmetry. Depending on the choices of the GGSO projection coefficients, extra space–time vector bosons may be obtained from the following 12 sectors

$$\mathbf{G} = \left\{ \begin{array}{cccc} z_1, & z_2, & z_1 + z_2, & z_1 + 2\alpha, \\ \alpha, & z_1 + \alpha, & z_2 + \alpha, & z_1 + z_2 + \alpha, \\ 3\alpha, & z_1 + 3\alpha, & z_2 + 3\alpha, & z_1 + z_2 + 3\alpha \end{array} \right\}. \quad (4.2.0.1)$$

The projections on the sectors 3α, $z_1 + 3\alpha$, $z_2 + 3\alpha$, $z_1 + z_2 + 3\alpha$ can be inferred from the projections on the sectors α, $z_1 + \alpha$, $z_2 + \alpha$, $z_1 + z_2 + \alpha$ respectively. Therefore, the details will not be discussed here. The gauge bosons that are obtained from the sectors in (4.2.0.1) enhance the untwisted gauge symmetry. Additionally, only the gauge bosons arising from the untwisted sector are imposed. Now, the gauge groups

in these models are defined as Observable and Hidden given as follows:

$$\begin{aligned} \text{Observable} \;:\quad & SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \\ \text{Hidden} \;:\quad & SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid} \end{aligned}$$

The NS-sector matter spectrum is common in these models and consists of 3 pairs of $\mathbf{5}$ and $\overline{\mathbf{5}}$ representations of the observable $SU(5) \times U(1)_5$ gauge group and 12 that are singlets under the non–Abelian gauge symmetries.

4.2.1 Observable Matter Spectrum

The chiral matter spectrum arises from the twisted sectors. The method of classification enables a straightforward enumeration of all the twisted sectors that produce massless states and the GGSO projection that operate on them. Below, the details of the method in the case of α_1 in (4.1.4.1) is provided. The chiral spinorial representations of the observable $SU(5) \times U(1)_5$ arise from the following sectors:

$$\begin{aligned} B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\qquad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1, \overline{\psi}^{1,\ldots,5}\}, \qquad (4.2.1.1) \\ B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6, \\ B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4, \end{aligned}$$

where $p, q, r, s = 0, 1$ and $b_3 = b_1 + b_2 + 2\alpha + z_1$. These 48 sectors give rise to $\mathbf{16}$ and $\overline{\mathbf{16}}$ multiplets of $SO(10)$, decomposed under $SU(5) \times U(1)$, which are given by

$$\begin{aligned} \mathbf{16} &= \left(\overline{\mathbf{5}}, -\tfrac{3}{2}\right) + \left(\mathbf{10}, +\tfrac{1}{2}\right) + \left(\mathbf{1}, +\tfrac{5}{2}\right), \\ \overline{\mathbf{16}} &= \left(\mathbf{5}, +\tfrac{3}{2}\right) + \left(\overline{\mathbf{10}}, -\tfrac{\mathbf{1}}{\mathbf{2}}\right) + \left(\mathbf{1}, -\tfrac{5}{2}\right). \end{aligned}$$

Additionally, vector–like representations of the observable $SU(5)\times U(1)_5$ gauge group arise from the sectors

$$\begin{aligned} B^{(4)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + 2\alpha \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\qquad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^{2,3}\}, \\ B^{(5,6)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + 2\alpha. \end{aligned}$$

These sectors contain four periodic worldsheet right–moving complex fermions. The massless states are obtained by acting on the vacuum with a Neveu–Schwarz right–moving fermionic oscillator. Furthermore, if the oscillator is given by $\{\overline{\psi}^{1,\ldots,5}\}$ or $\{\overline{\psi}^{*1,\ldots,5}\}$, then some of the 48 twisted sectors can give rise to the vectorial **10** representation of $SO(10)$, decomposed under $SU(5)\times U(1)$, which is given by

$$\mathbf{10} \;=\; \left(\overline{\mathbf{5}},+1\right)+\left(\mathbf{5},-1\right).$$

These states are identified with light Higgs representations that are used to break the Standard Model gauge symmetry to $SU(3)\times U(1)_{\rm e.m.}$. Additional states which are singlets under the observable $SU(5)\times U(1)_5$ might also arise from any of the 48 sectors in (4.2.1.2), given by the following representations:

- $\{\overline{\eta}^{i}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\eta}^{*i}\}|R\rangle^{(4,5,6)}_{pqrs}$, $i=1,2,3$, where $|R\rangle^{(4,5,6)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(4,5,6)}_{pqrs}$ sector. These states transform as vector–like representations under the $U(1)_i$'s.
- $\{\overline{\phi}^{1,\ldots,4}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations of $SU(4)\times U(1)_4$.
- $\{\overline{\phi}^{5}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*5}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations under the $U(1)_5$'s.
- $\{\overline{\phi}^{6,7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*6,7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vectorial representations of $SO(6)$.

4.2.2 Hidden Matter Spectrum

The sectors which produce states that transform under representations of the hidden gauge group are singlets of the observable $SO(10)$ GUT gauge group. These states are hidden matter states that are obtained in the String model, that are not exotic with respect to the Standard Model gauge charges. The 48 sectors in $B^{1,2,3}_{pqrs}+2\alpha$ produce states that transforms under the $\left(\overline{\mathbf{4}},+1\right)$, $(\mathbf{4},-1)$, $(\mathbf{6},0)$, $(\mathbf{1},+2)$ and $(\mathbf{1},-2)$ representations of the $SU(4)\times U(1)$ hidden gauge group and are given by

$$\begin{aligned}
B^{(7)}_{pqrs} &= B^{(1)}_{pqrs}+2\alpha \\
&= \{\psi^{\mu},\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\
&\qquad (1-r)y^5\overline{y}^5,r\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^{2,3},\overline{\phi}^{1,\ldots,4}\}, \qquad (4.2.2.1)\\
B^{(8,9)}_{pqrs} &= B^{(2,3)}_{pqrs}+2\alpha.
\end{aligned}$$

In addition, there are also the following 48 sectors

$$\begin{aligned} B^{(10)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + 2\alpha \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\quad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^{2,3}, \overline{\phi}^{5,\ldots,8}\}, \quad (4.2.2.2) \\ B^{(11,12)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + 2\alpha, \end{aligned}$$

which produce states in the **4** and $\overline{\mathbf{4}}$ spinorial representations of the hidden $SO(6)$ gauge group.

4.2.3 Exotic Matter Spectrum

In the String spectrum, additional sectors exist which produce fractionally charged states under the $SU(5) \times U(1)$ symmetry. These sectors arise when massless states exist, which are produced from a linear combination of basis vectors that include the vector α, resulting in the breaking of $SO(10)$ symmetry. Moreover, these sectors produce states that do not fall into representations of the underlying $SO(10)$ GUT symmetry. Specifically, they possess fractionally charged assignments with respect to the $U(1)$ symmetry in the decomposition $SO(10) \longrightarrow SU(5) \times U(1)$. Consequently, provided that the weak hypercharge has the canonical $SO(10)$ GUT embedding and the canonical GUT prediction $\sin^2\theta_w = 3/8$, these sectors produce states that carry fractional electric charges. This is a generic feature of String compactifications [24, 25, 26], that may have interesting phenomenological implications [27, 28, 29], as electric charge conservation implies that the lightest of those exotic states is necessarily stable. Many experimental searches for fractionally charged matter have been conducted [30]. However, no reported observation of any such particles has ever been confirmed and there are strong upper bounds on their abundance [30]. This implies that such exotic states in String models should be either confined into integrally charged states [21], or be sufficiently heavy and diluted in the cosmological evolution of the universe [27, 28, 29]. The first of these solutions is problematic, due to the effect of the charged states on the renormalisation group running of the weak–hypercharge and gauge coupling unification. The preferred solution is therefore, for the fractionally charged states to become sufficiently massive, *i.e.* with a mass which is larger than the GUT scale. In this case the fractionally charged states can be diluted by the inflationary evolution of the universe. Due to their heavy mass they will not be reproduced during reheating and the experimental constraints can be evaded. 3 generation Pati–Salam Heterotic-String models, in which the fractionally charged states arise only in the massive String spectrum, were constructed in [31, 32], are dubbed as the semi-

realistic exophobic Pati–Salam String models. A particular question of interest in the current work, is the existence of semi-realistic Flipped $SU(5)$ Heterotic-String models. It should also be noted that the sectors appearing in (4.2.2.1) and (4.2.2.2) contain the combination 2α and do not break the $SO(10)$ symmetry. Therefore, these sectors do not produce exotic states under the $SU(5) \times U(1)$ gauge symmetry.

In the Free-Fermionic construction here, the sectors that produce exotic states are classified according to the product $\xi_R \cdot \xi_R = 4$, 6, or 8. In the first case, massless states are obtained by acting on the vacuum with a Neveu–Schwarz fermion or with two oscillators with 1/4 frequencies. In the second case, oscillators with 1/4 frequency are needed to produce massless states. In the third case no oscillators are used to produce massless states, which are given by the following 96 sectors:

$$\begin{aligned} B^{(13)}_{pqrs} &= B^{(1)}_{pqrs} + z_2 + \alpha \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\quad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\ &\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\ldots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\ldots,4} = \tfrac{1}{2}, \overline{\phi}^{6,7,8}\}, \\ B^{(14,15)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_2 + \alpha, \end{aligned}$$

$$\begin{aligned} B^{(16)}_{pqrs} &= B^{(1)}_{pqrs} + z_1 + z_2 + \alpha \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\quad r(1-r)y^5\overline{y}^5, \omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\ &\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\ldots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\ldots,4} = -\tfrac{1}{2}, \overline{\phi}^{6,7,8}\}, \\ B^{(17,18)}_{pqrs} &= B^{(2,3)}_{pqrs} + z_1 + z_2 + \alpha. \end{aligned}$$

These produce states that are singlets under the observable $SU(5)$, which are charged under the $U(1)_5$ and are given by $\left(\mathbf{1}, -\frac{5}{4}\right)$ and $\left(\mathbf{1}, +\frac{5}{4}\right)$. The second case that consists of oscillators with one 1/4 frequency giving rise to additional massless vector–like states given by the following 48 sectors:

$$\begin{aligned} B^{(19)}_{pqrs} &= B^{(1)}_{pqrs} + \alpha \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\quad r(1-r)y^5\overline{y}^5, \omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1 = -\tfrac{1}{2}, \\ &\quad \overline{\eta}^{2,3} = \tfrac{1}{2}, \overline{\psi}^{1,\ldots,5} = -\tfrac{1}{2}, \overline{\phi}^{1,\ldots,4} = \tfrac{1}{2}, \overline{\phi}^{5}\}, \\ B^{(20,21)}_{pqrs} &= B^{(2,3)}_{pqrs} + \alpha. \end{aligned} \tag{4.2.3.1}$$

As an example, the sectors in $B^{(19)}_{pqrs}$ produce the following states:

- $\{\overline{\eta}^1\}|R\rangle^{(19)}_{pqrs}$, where $|R\rangle^{(19)}_{pqrs}$ is the degenerate Ramond vacuum of the $B^{(19)}_{pqrs}$ sector. These states transform as vector–like representations under the $U(1)_1$.
- $\{\overline{\eta}^{*2}\}|R\rangle^{(19)}_{pqrs}$ and $\{\overline{\eta}^{*3}\}|R\rangle^{(19)}_{pqrs}$. These states transform as vector–like representations under the $U(1)_{2/3}$.
- $\{\overline{\psi}^{1,\ldots,5}\}|R\rangle^{(19)}_{pqrs}$. These states transform as $\left(\overline{\mathbf{5}},+\frac{1}{4}\right)$ and $\left(\mathbf{5},-\frac{1}{4}\right)$ representations of $SU(5)\times U(1)$.
- $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle^{(19)}_{pqrs}$. These states transform as vector–like representations of $SU(4)\times U(1)$.

Similarly the sectors in $B^{(20)}_{pqrs}$ and $B^{(21)}_{pqrs}$ also produce the states as above. What is more, similar states appear in the following 48 sectors:

$$\begin{aligned} B^{(22)}_{pqrs} &= B^{(1)}_{pqrs}+z_1+\alpha \\ &= \{\psi^\mu,\chi^{12},(1-p)y^3\overline{y}^3,p\omega^3\overline{\omega}^3,(1-q)y^4\overline{y}^4,q\omega^4\overline{\omega}^4, \\ &\qquad r(1-r)y^5\overline{y}^5,\omega^5\overline{\omega}^5,(1-s)y^6\overline{y}^6,s\omega^6\overline{\omega}^6,\overline{\eta}^1=-\tfrac{1}{2}, \\ &\qquad\qquad \overline{\eta}^{2,3}=\tfrac{1}{2},\overline{\psi}^{1,\ldots,5}=-\tfrac{1}{2},\overline{\phi}^{1,\ldots,4}=-\tfrac{1}{2},\overline{\phi}^5\}, \\ B^{(23,24)}_{pqrs} &= B^{(2,3)}_{pqrs}+z_1+\alpha. \end{aligned} \tag{4.2.3.2}$$

The only difference between the sectors in (4.2.3.1) and (4.2.3.2) is the sign of the 1/2 boundary condition of the worldsheet fermion $\overline{\phi}^{1,\ldots,4}$. This changes some of the $U(1)$ charges arising in (4.2.3.1) compared to those arising in (4.2.3.2). The structure and type of states are similar to those listed above. Finally, the first case of exotic states arise in the sectors α and $z_1+\alpha$. These exotic states can be eliminated by the same conditions that eliminate the space–time vector bosons arising in these sectors which will be discussed in section 4.4.

4.3 Twisted Matter Spectrum

The counting of spinorial and vector–like representations in the given String vacua is realised by utilizing the so called projectors. Each sector B^i_{pqrs}, corresponds to a projector, $P^i_{pqrs}=0,1$, which is expressed in terms of GGSO coefficients and determines whether a given sector survives the GGSO projections. Therefore, the computational analysis is facilitated by rewriting the projectors in an analytic form. These are written as algebraic conditions, for the individual states arising in the String spectrum, in terms of the GGSO phases of the basis vectors. The algebraic

expressions are inserted into the computer code, which enables the scan of the large space of models spanned by the basis GGSO phases.

4.3.1 Observable Spinorial States

In order to get the particle content for the representations for the sectors in (4.2.1.1), the following normalisations are used for the hypercharge and the electromagnetic charge:

$$\begin{aligned} Y &= \frac{1}{3}(Q_1+Q_2+Q_3)+\frac{1}{2}(Q_4+Q_5), \\ Q_{em} &= Y+\frac{1}{2}(Q_4-Q_5). \end{aligned}$$

Where the Q_i charges of a state, arise due to ψ^i for $i=1,...,5$. The following table summarises the charges of the colour $SU(3)$ and electroweak $SU(2)\times U(1)$ Cartan generators, of the states which form the $SU(5)\times U(1)$ matter representations

Representation	$\overline{\psi}^{\mathbf{1,2,3}}$	$\overline{\psi}^{\mathbf{4,5}}$	**Y**	$\mathbf{Q_{em}}$
	$(+,+,+)$	$(+,-)$	1/2	1,0
$(\mathbf{5}\,,\,+\frac{3}{2})$	$(+,+,-)$	$(+,+)$	2/3	2/3
	$(+,-,-)$	$(-,-)$	-2/3	-2/3
$(\overline{\mathbf{5}}\,,\,-\frac{3}{2})$	$(-,-,-)$	$(+,-)$	-1/2	-1,0
	$(+,+,+)$	$(-,-)$	0	0
$(\mathbf{10},+\frac{1}{2})$	$(+,-,-)$	$(+,+)$	1/3	1/3
	$(+,+,-)$	$(+,-)$	1/6	-1/3,2/3
	$(+,+,-)$	$(-,-)$	-1/3	-1/3
$(\overline{\mathbf{10}},-\frac{1}{2})$	$(+,-,-)$	$(+,-)$	-1/6	1/3,-2/3
	$(-,-,-)$	$(+,+)$	0	0
$(\mathbf{1}\,,\,+\frac{5}{2})$	$(+,+,+)$	$(+,+)$	1	1
$(\mathbf{1}\,,\,-\frac{5}{2})$	$(-,-,-)$	$(-,-)$	-1	-1

Here " $+$ ", and " $-$ ", label the contribution of an oscillator with fermion number $F=0$, or $F=-1$, to the degenerate vacuum. For example $(+,+,-)$ under $\overline{\psi}^{1,2,3}$ corresponds to a part of the Ramond vacuum, formed by two oscillators with fermion number $F=0$ and 1 oscillator with fermion numbers $F=-1$. These states correspond to particles of the Standard Model. More precisely these representations can

be decomposed under $SU(3) \times SU(2) \times U(1)$:

$$\begin{aligned}
\left(\overline{\mathbf{5}}, -\frac{3}{2} \right) &= \left(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3} \right)_{u^c} + \left(\mathbf{1}, \mathbf{2}, -\frac{1}{2} \right)_{L} , \\
\left(\mathbf{10}, +\frac{1}{2} \right) &= \left(\mathbf{3}, \mathbf{2}, +\frac{1}{6} \right)_{Q} + \left(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3} \right)_{d^c} + (\mathbf{1}, \mathbf{1}, 0)_{\nu^c} , \\
\left(\mathbf{1}, +\frac{5}{2} \right) &= (\mathbf{1}, \mathbf{1}, +1)_{e^c} ,
\end{aligned}$$

where L is the lepton–doublet; Q is the quark–doublet; d^c, u^c, e^c and ν^c are the quark and lepton singlets. Due to the α–projection, which projects on incomplete **16** and $\overline{\mathbf{16}}$ representations, complete families and anti–families are formed by combining states from different sectors.

4.3.2 Chirality Operators

A phenomenologically viable model consists of 3 families of chiral **16** representations of $SO(10)$, decomposed under $SU(5) \times U(1)$. Therefore, the number of **16**s and $\overline{\mathbf{16}}$s are counted. The choice of GGSO coefficients determine the model considered and therefore the number of families. In order to be able to distinguish between **16** and $\overline{\mathbf{16}}$, it is necessary to define operators that determine the representations in which the states of each observable sector fall into. The operators $X_{pqrs}^{(1,2,3)_{SO(10)}} = \pm 1$, defines the $SO(10)$ chirality (**16** or $\overline{\mathbf{16}}$) for B_{pqrs}^1, B_{pqrs}^2 and B_{pqrs}^3, which are given by:

$$\begin{aligned}
X_{pqrs}^{(1)_{SO(10)}} &= C\begin{pmatrix} B_{pqrs}^{(1)} \\ b_2 + (1-r)e_5 + (1-s)e_6 \end{pmatrix}, \\
X_{pqrs}^{(2)_{SO(10)}} &= C\begin{pmatrix} B_{pqrs}^{(2)} \\ b_1 + (1-r)e_5 + (1-s)e_6 \end{pmatrix}, \\
X_{pqrs}^{(3)_{SO(10)}} &= C\begin{pmatrix} B_{pqrs}^{(3)} \\ b_1 + (1-r)e_3 + (1-s)e_4 \end{pmatrix}.
\end{aligned}$$

The components in the **16** and $\overline{\mathbf{16}}$, need to be determined that survive the α projection from breaking $SO(10)$ to $SU(5) \times U(1)$. In this respect, it should be noted that the α projection operates identically on the $\mathbf{1} \equiv (\mathbf{1}, +5/2)$ and $\overline{\mathbf{5}} \equiv (\overline{\mathbf{5}}, -3/2)$ states and similarly on the conjugate representations $\overline{\mathbf{1}} \equiv (\mathbf{1}, -5/2)$ and $\mathbf{5} \equiv (\mathbf{5}, +3/2)$. The surviving components are determined by defining the operators $X_{pqrs}^{(1,2,3)_{SU(5)}} = \pm 1$, where $X_{pqrs}^{(i)_{SU(5)}} = 1$ indicates survival of the $(\mathbf{1}, +5/2)$ and $(\overline{\mathbf{5}}, -3/2)$ pair and $X_{pqrs}^{(1,2,3)_{SU(5)}} = -1$ indicates survival of the $(\mathbf{10}, +1/2)$ states. The operator $X_{pqrs}^{(i)_{SU(5)}}$

acts similarly on the $\overline{\mathbf{16}}$ of $SO(10)$. These conditions are expressed as

$$\begin{aligned} X^{(1)_{SU(5)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{\alpha}, \\ X^{(2)_{SU(5)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{\alpha}, \\ X^{(3)_{SU(5)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{\alpha}. \end{aligned}$$

4.3.3 Projectors

The states in the sectors in $B^{(1)}_{pqrs}$, as given in (4.2.1.1), can be projected in or out of the String spectrum depending on the GGSO projections of the vectors e_1, e_2, z_1 and z_2. Likewise for $B^{(2)}_{pqrs}$ and $B^{(3)}_{pqrs}$, a projector P is defined so that the states survive when $P = 1$ and are projected out when $P = 0$, which are given as:

$$\begin{aligned} P^{(1)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_1}{B^{(1)}_{pqrs}}\right).\left(1 - C\binom{e_2}{B^{(1)}_{pqrs}}\right).\left(1 - C\binom{z_1}{B^{(1)}_{pqrs}}\right).\left(1 - C\binom{z_2}{B^{(1)}_{pqrs}}\right), \\ P^{(2)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_3}{B^{(2)}_{pqrs}}\right).\left(1 - C\binom{e_4}{B^{(2)}_{pqrs}}\right).\left(1 - C\binom{z_1}{B^{(2)}_{pqrs}}\right).\left(1 - C\binom{z_2}{B^{(2)}_{pqrs}}\right), \\ P^{(3)}_{pqrs} &= \frac{1}{16}\left(1 - C\binom{e_5}{B^{(3)}_{pqrs}}\right).\left(1 - C\binom{e_6}{B^{(3)}_{pqrs}}\right).\left(1 - C\binom{z_1}{B^{(3)}_{pqrs}}\right).\left(1 - C\binom{z_2}{B^{(3)}_{pqrs}}\right). \end{aligned}$$

These projectors can be expressed as a system of linear equations with p, q, r and s as unknowns. The solutions of such a system of equations yield the different combinations of p, q, r and s for which sectors survive the GGSO projections. The analytic expressions for each of the different projectors $P^{1,2,3}_{pqrs}$ are given in a matrix form $\Delta^i W^i = Y^i$, where

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{pmatrix} = Y^1,$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{pmatrix} = Y^2,$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{pmatrix} = Y^3.$$

Here the GGSO phases are defined as

$$C\begin{pmatrix} v_i \\ v_j \end{pmatrix} = e^{i\pi(v_i|v_j)} \ ,$$

where v_i and v_j refer to the basis vectors and the GGSO projections are defined as in (4.1.1.1). The corresponding algebraic expressions, for the states from the remaining sectors above, are enumerated in the appendix. Furthermore, the projectors presented in the appendix determine the number of surviving observable, hidden and exotic states in each model.

4.4 Gauge Group Enhancements

The $SU(5) \times U(1)$ gauge symmetry generated by the untwisted space–time vector bosons, may be enhanced by the vector bosons that arise from the sectors listed in (4.2.0.1). Here, it is imposed that all the additional space–time vector bosons are projected out. The gauge symmetry is therefore identical in all the models scanned, though the occurrence of models with enhancements are approximately about 23.8% of the total models. The String models in the classification differ by the String spectrum that arises from the twisted sectors. In the classification method, the GGSO projections coefficients are transformed in terms of algebraic equations, which are applied to all the sectors listed in section 4.2.

The gauge bosons of any given sector in (4.2.0.1), transform under a subgroup of

the Neveu–Schwarz gauge group. If they survive the GGSO projections, then the NS gauge group is enhanced. The classification here is restricted to the cases without enhancement, by identifying when the gauge bosons survive the GGSO projections and generalizing the formulae to eliminate them. Below, the different types of enhancements are presented that can occur within the String spectrum from the sectors given in (4.2.0.1). In addition, it is assumed that only one set of conditions are satisfied from any one given sector in (4.2.0.1).

4.4.1 Observable Gauge Group Enhancements

There is one sector contributing only to the enhancement of the observable gauge group i.e. $SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3$. This is the sector $z_1 + 2\alpha$, given by the conditions:

- $z_1 + 2\alpha = \{\overline{\psi}^{1,\ldots,5}, \overline{\eta}^{1,2,3}\}$

Sector Condition
$(z_1 + 2\alpha\|e_i) = (z_1 + 2\alpha\|z_k) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + 2\alpha\|\alpha) = (z_1 + 2\alpha\|b_2)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_\zeta$ $\longrightarrow SU(6) \times SU(2)$
$(z_1 + 2\alpha\|\alpha) \neq (z_1 + 2\alpha\|b_2)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_\zeta$ $\longrightarrow SO(10) \times U(1)$

where $i = 1, \ldots, 6$ and $k = 1, 2$ and $U(1)_\zeta$ is a linear combination of $U(1)_1$, $U(1)_2$ and $U(1)_3$.

4.4.2 Hidden Gauge Group Enhancements

The vector bosons arising from the untwisted sector produce the hidden gauge symmetry, which is given as $SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$. Similar to the observable sector, there is one sector that enhances only the untwisted hidden sector gauge symmetry and is given by the sector $z_1 + z_2$, where the conditions are given by:

- $z_1 + z_2 = \{\overline{\phi}^{1,\ldots,8}\}$

Sector Condition
$(z_1 + z_2|e_i) = (z_1 + z_2|b_k) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + z_2|z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(8) \times U(1)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

4.4.3 Mixed Gauge Group Enhancements

The additional sectors in (4.2.0.1), produce vector bosons coming from the mixture of the observable and hidden sector gauge groups. The mixed gauge group enhancements are formed from the untwisted symmetries of the observable and hidden gauge group. These are given from the sectors z_1, z_2, α, $z_1 + \alpha$, $z_2 + \alpha$ and $z_1 + z_2 + \alpha$. The conditions are as follows:

- $z_2 = \{\overline{\phi}^{5,\ldots,8}\}$

Sector Condition
$(z_2|e_i) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_2|z_1) = 1$ $(z_2|b_k) = 0$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(8) \times U(1)$
$(z_2|z_1) = 0$ $(z_2|b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid} \times U(1)_{hid}$ $\longrightarrow SU(5) \times U(1)$
$(z_2|z_1) = 0$ $(z_2|b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid} \times U(1)_5$ $\longrightarrow SU(9) \times U(1)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

- $z_1 = \{\overline{\phi}^{1,\ldots,4}\}$

Enhancement Condition	**Resulting Enhancement**
$(z_1\|e_i) = (z_1\|b_k) = (z_1\|z_1) = (z_1\|\alpha) = 0$ $(z_1\|z_2) = 1$	$SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid}$ $\longrightarrow SO(8) \times SO(8)$
$(z_1\|e_i) = (z_1\|b_k) = (z_1\|z_1) = 0$ $(z_1\|z_2) = (z_1\|\alpha) = 1$	$SU(4)_{hid} \times U(1)_4 \times U(1)_{hid} \times SO(6)_{hid}$ $\longrightarrow SO(12) \times SO(4)$
$(z_1\|e_i) = (z_1\|z_2) = 0$ $(z_1\|b_k) \neq 1$ $(z_1\|z_1) = 1$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4$ $\longrightarrow SU(5) \times U(1)$
$(z_1\|e_i) = (z_1\|z_2) = 0$ $(z_1\|b_k) = (z_1\|z_1) = 1$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4$ $\longrightarrow SU(9) \times U(1)$
$(z_1\|e_j) = (z_1\|z_1) = (z_1\|z_2) = (z_1\|\alpha) = 0$ $(z_1\|e_i) = 1$ AND $(z_1\|b_1) = 0,\ i = 1, 2$ or $(z_1\|b_2) = 0,\ i = 3, 4$ or $(z_1\|b_1) = (z_1\|b_2),\ i = 5, 6$	$U(1)_4 \longrightarrow SO(3)$
$(z_1\|e_j) = (z_1\|z_1) = (z_1\|z_2) = 0$ $(z_1\|e_i) = (z_1\|\alpha) = 1$ AND $(z_1\|b_1) = 0,\ i = 1, 2$ or $(z_1\|b_2) = 0,\ i = 3, 4$ or $(z_1\|b_1) = (z_1\|b_2),\ i = 5, 6$	$SU(4)_{hid} \longrightarrow SO(7)$

where $i, j = 1, \ldots, 6$, $i \neq j$ and $k = 1, 2$.

- $\alpha \oplus 3\alpha = \{\overline{\psi}^{1,\ldots,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,\ldots,4} = \frac{1}{2}, \overline{\phi}^{5}\} \oplus \{\overline{\psi}^{1,\ldots,5} = -\frac{1}{2}, \overline{\eta}^{1,2,3} = -\frac{1}{2}, \overline{\phi}^{1,\ldots,4} = -\frac{1}{2}, \overline{\phi}^{5}\}$

  The states that give two $\frac{1}{4}$ oscillators are produced from the following conditions:

Sector Condition
$(\alpha\|e_i) = 0$ $(\alpha\|z_2) \neq (\alpha\|\alpha)$

Enhancement Condition	**Resulting Enhancement**
$(\alpha\|z_1) = (\alpha\|b_k) = 0$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(10) \times SO(8) \times U(1)$
$(\alpha\|z_1) = 0$ $(\alpha\|b_k) \neq 0$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_{hid}$ $\longrightarrow SU(6) \times SU(2) \times U(1)^3$
$(\alpha\|z_1) = 1$ $(\alpha\|b_k) \neq 1$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(5) \times U(1)^2$
$(\alpha\|z_1) = (\alpha\|b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(9) \times U(1)^2$

where $i = 1, \ldots, 6$ and $k = 1, 2$. Additionally, the states that give one $\frac{1}{2}$ oscillators are produced from the following conditions:

Sector Condition
$(\alpha\|z_1) = 0$

Enhancement Condition	**Resulting Enhancement**
$(\alpha\|e_i) = (\alpha\|b_k) = 0$ $(\alpha\|z_2) \neq (\alpha\|\alpha)$	$SO(6)_{hid} \longrightarrow SO(7)$
$(\alpha\|z_2) = (\alpha\|\alpha)$ $(\alpha\|e_j) = 0$ $(\alpha\|e_i) = 1$ AND $(\alpha\|b_1) = 0,\ i = 1, 2$ or $(\alpha\|b_2) = 0,\ i = 3, 4$ or $(\alpha\|b_1) = (\alpha\|b_2),\ i = 5, 6$	$U(1)_{hid} \longrightarrow SO(3)$

where $i, j = 1, \ldots, 6$, $i \neq j$ and $k = 1, 2$.

- $z_1+\alpha\oplus z_1+3\alpha = \{\overline{\psi}^{1,\ldots,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,\ldots,4} = -\frac{1}{2}, \overline{\phi}^{5}\}\oplus\{\overline{\psi}^{1,\ldots,5} = -\frac{1}{2}, \overline{\eta}^{1,2,3} = -\frac{1}{2}, \overline{\phi}^{1,\ldots,4} = \frac{1}{2}, \overline{\phi}^{5}\}$

  The states that give two $\frac{1}{4}$ oscillators are produced from the following conditions:

Sector Condition
$(z_1 + \alpha\|e_i) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + \alpha\|z_1) = (z_1 + \alpha\|b_k) = 0$ $(z_1 + \alpha\|z_2) \neq (z_1 + \alpha\|\alpha)$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SO(10) \times SO(8) \times U(1)$
$(z_1 + \alpha\|z_1) = 0$ $(z_1 + \alpha\|b_k) \neq 0$ $(z_1 + \alpha\|z_2) \neq (z_1 + \alpha\|\alpha)$	$SU(5)_{obs} \times U(1)_5 \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_{hid}$ $\longrightarrow SU(6) \times SU(2) \times U(1)^3$
$(z_1 + \alpha\|z_1) = 1$ $(z_1 + \alpha\|b_k) \neq 1$ $(z_1 + \alpha\|z_2) = (z_1 + \alpha\|\alpha)$	$U(1)_{1/2/3} \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(5) \times U(1)^2$
$(z_1 + \alpha\|z_1) = (z_1 + \alpha\|b_k) = 1$ $(z_1 + \alpha\|z_2) = (z_1 + \alpha\|\alpha)$	$SU(5)_{obs} \times U(1)_5 \times SU(4)_{hid} \times U(1)_4 \times U(1)_{hid}$ $\longrightarrow SU(9) \times U(1)^2$

where $i = 1, \ldots, 6$ and $k = 1, 2$. Additionally, the states that give one $\frac{1}{2}$ oscillators are produced from the following conditions:

Sector Condition
$(z_1 + \alpha \vert z_1) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1 + \alpha \vert e_i) = (z_1 + \alpha \vert b_k) = 0$ $(z_1 + \alpha \vert z_2) \neq (z_1 + \alpha \vert \alpha)$	$SO(6)_{hid} \longrightarrow SO(7)$
$(z_1 + \alpha \vert z_2) = (z_1 + \alpha \vert \alpha)$ $(z_1 + \alpha \vert e_j) = 0$ $(z_1 + \alpha \vert e_i) = 1$ AND $(z_1 + \alpha \vert b_1) = 0,\ i = 1, 2$ or $(z_1 + \alpha \vert b_2) = 0,\ i = 3, 4$ or $(z_1 + \alpha \vert b_1) = (z_1 + \alpha \vert b_2),\ i = 5, 6$	$U(1)_{hid} \longrightarrow SO(3)$

where $i, j = 1, \ldots, 6$, $i \neq j$ and $k = 1, 2$.

- $z_2+\alpha \oplus z_2+3\alpha = \{\overline{\psi}^{1,..,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,..,4} = \frac{1}{2}, \overline{\phi}^{6,7,8}\} \oplus \{\overline{\psi}^{1,..,5} = -\frac{1}{2}, \overline{\eta}^{1,2,3} = -\frac{1}{2}, \overline{\phi}^{1,..,4} = -\frac{1}{2}, \overline{\phi}^{6,7,8}\}$

Sector Condition
$(z_2 + \alpha \vert e_i) = 0$ $(z_2 + \alpha \vert \alpha) = \frac{1}{2}$

Enhancement Condition	**Resulting Enhancement**
$(z_2 + \alpha \vert z_1) = 0$ $(z_2 + \alpha \vert b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid}$ $\longrightarrow SU(9)$
$(z_2 + \alpha \vert z_1) = 0$ $(z_2 + \alpha \vert b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid}$ $\longrightarrow SU(5)$
$(z_2 + \alpha \vert b_k) = 0$ $(z_2 + \alpha \vert z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid}$ $\longrightarrow SU(8)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

- $z_1+z_2+\alpha \oplus z_1+z_2+3\alpha = \{\overline{\psi}^{1,..,5} = \frac{1}{2}, \overline{\eta}^{1,2,3} = \frac{1}{2}, \overline{\phi}^{1,..,4} = -\frac{1}{2}, \overline{\phi}^{6,7,8}\} \oplus \{\overline{\psi}^{1,..,5} = -\frac{1}{2}, \overline{\eta}^{1,2,3} = -\frac{1}{2}, \overline{\phi}^{1,..,4} = \frac{1}{2}, \overline{\phi}^{6,7,8}\}$

Sector Condition
$(z_1+z_2+\alpha\|e_i) = 0$

Enhancement Condition	**Resulting Enhancement**
$(z_1+z_2+\alpha\|z_1) = 0$ $(z_1+z_2+\alpha\|\alpha) = \frac{1}{2}$ $(z_1+z_2+\alpha\|b_k) = 1$	$SU(5)_{obs} \times U(1)_5 \times SO(6)_{hid}$ $\longrightarrow SU(9)$
$(z_1+z_2+\alpha\|z_1) = 0$ $(z_1+z_2+\alpha\|\alpha) = \frac{1}{2}$ $(z_1+z_2+\alpha\|b_k) \neq 1$	$U(1)_{1/2/3} \times SO(6)_{hid}$ $\longrightarrow SU(5)$
$(z_1+z_2+\alpha\|b_k) = 0$ $(z_1+z_2+\alpha\|\alpha) = -\frac{1}{2}$ $(z_1+z_2+\alpha\|z_1) = 1$	$SU(4)_{hid} \times U(1)_4 \times SO(6)_{hid}$ $\longrightarrow SU(8)$

where $i = 1, \ldots, 6$ and $k = 1, 2$.

Finally, recalling in section 4.2.3, the sectors α, $z_1+\alpha$, $z_2+\alpha$ and $z_1+z_2+\alpha$ may also give rise to exotic states, when the left–moving ψ^μ oscillator is replaced by a left–moving χ^i oscillator. Moreover, it should be noted that the GGSO projections of the basis vectors $e_{1,\ldots,6}$, $z_{1,2}$ and α do not distinguish between ψ^μ and χ^i, which can therefore, be used to project both the enhancements, as well as the exotic states arising from the sectors α, $z_1+\alpha$, $z_2+\alpha$ and $z_1+z_2+\alpha$.

4.5 Classification

By use of the algebraic expressions given in the sections previously, as well as in the appendix, the entire massless spectrum is analysed for a given choice of configuration of GGSO projection coefficients. These expressions were transformed into matrix equations which were then programmed into a computer code, that were used to scan the space of the String vacua. The number of possible configurations is $2^{44} \approx 10^{13}$, which is very large in order for a classification of the entire String vacua. For this purpose, a random number generator algorithm was used and the characteristics of the models for each set of random GGSO projection coefficients were extracted. From the generated sample, a model with the desired phenomenological criteria can be attained. This procedure was followed in [31, 32, 33], which produced 3 generation Pati–Salam Heterotic-String models that did not contain any exotic massless states

with fractional electric charge. In this thesis, this methodology is used to classify the Flipped $SU(5)$ Free-Fermionic String models with respect to some phenomenological criteria. For example, a question of interest is the existence of viable 3 generation exophobic Flipped $SU(5)$ vacua. The observable sector of a Heterotic-String Flipped $SU(5)$ model is characterised by 15 integers, which are listed as:

$$
\begin{aligned}
n_1 &= \#\text{ of } (\mathbf{1}, +\tfrac{5}{2}),\\
n_{\overline{1}} &= \#\text{ of } (\mathbf{1}, -\tfrac{5}{2}),\\
n_{5s} &= \#\text{ of } (\mathbf{5}, +\tfrac{3}{2}),\\
n_{\overline{5}s} &= \#\text{ of } (\overline{\mathbf{5}}, -\tfrac{3}{2}),\\
n_{10} &= \#\text{ of } (\mathbf{10}, +\tfrac{1}{2}),\\
n_{\overline{10}} &= \#\text{ of } (\overline{\mathbf{10}}, -\tfrac{1}{2}),\\
n_g &= n_{10} - n_{\overline{10}} = n_{\overline{5}} - n_5 = \#\text{ of generations},\\
n_{10H} &= n_{10} + n_{\overline{10}} = \#\text{ of non-chiral heavy Higgs pairs},\\
n_{\overline{5}v} &= \#\text{ of } (\overline{\mathbf{5}}, +1),\\
n_{5v} &= \#\text{ of } (\mathbf{5}, -1),\\
n_{5h} &= n_{5v} + n_{\overline{5}v} = \#\text{ of non-chiral light Higgs pairs},\\
n_{1e} &= \#\text{ of } (\mathbf{1}, -\tfrac{5}{4}) \quad \text{(exotic)},\\
n_{\overline{1}e} &= \#\text{ of } (\mathbf{1}, +\tfrac{5}{4}) \quad \text{(exotic)},\\
n_{5e} &= \#\text{ of } (\mathbf{5}, -\tfrac{1}{4}) \quad \text{(exotic)},\\
n_{\overline{5}e} &= \#\text{ of } (\overline{\mathbf{5}}, +\tfrac{1}{4}) \quad \text{(exotic)}.
\end{aligned}
$$

The numbers above are all relevant for the classification of the String vacua. As recalled in section 4.3.2, the α projection dictates that $n_{\overline{1}} = n_{5s}$ and $n_1 = n_{\overline{5}s}$. Therefore, the counting of n_5 and $n_{\overline{5}}$ is sufficient for the number of generations. There is also a distinction to be made between the $\mathbf{5}$ and $\overline{\mathbf{5}}$ representations arising from the spinorial $\mathbf{16}$ representation of $SO(10)$ decomposed under $SU(5) \times U(1)$, denoted by $n_{5s}, n_{\overline{5}s}$. Whereas, the $\mathbf{5}$ and $\overline{\mathbf{5}}$ that arise from its vectorial $\mathbf{10}$ representation are denoted by $n_{5v}, n_{\overline{5}v}$. While the former gives rise to the Standard Model up–type quark electroweak singlet and lepton–doublet, the latter accommodates the light electroweak Higgs doublets. In the Flipped $SU(5)$ models they are distinguished by their charges under the $U(1)_5$ symmetry. Moreover, using the methodology outlined in section 4.3, analytic formulas were obtained for all these quantities. In order to extract a String spectrum from the phenomenologically viable models of the Flipped

$SU(5)$, the following is needed:

$n_g = 3$	3 light chiral of generations,
$n_{10H} \geq 1$	At least 1 heavy Higgs pair to break the $SU(5) \times U(1)$ symmetry,
$n_{5h} \geq 1$	At least 1 pair of light Minimal SM Higgs doublets,
$n_{1e} = n_{\overline{1}e} \geq 0$	Heavy mass can be generated for vector–like exotics,
$n_{5e} = n_{\overline{5}e} \geq 0$	Heavy mass can be generated for vector–like exotics.

Here, it should be noted that the constraints $n_{\overline{5}h} = n_{5h}$, $n_{1e} = n_{\overline{1}e}$ and $n_{5e} = n_{\overline{5}e}$ were imposed, in order to sustain anomaly free Flipped $SU(5)$ models.

4.5.1 Minimal Exophilic Models

Compared to the case of the Pati–Salam classification [32], which yielded 3 generation models that are completely free of massless exotic states, no such models were found in the scan of the Flipped $SU(5)$ models. It must be emphasised, that this does not indicate that exophobic Free-Fermionic Flipped $SU(5)$ vacua do not exist, and that, they did not exist in the space of vacua that were explored. Nevertheless, it did show that large spaces of vacua may not contain exophobic models, which is in line with related searches [34]. Further to this, models with minimal number of exotic states that were found in the scan, had the following properties: $n_g = 3$, $n_{\overline{5}s} = 3$, $n_{5s} = 0$, $n_{10} = 4$, $n_{\overline{10}} = 1$, $n_{10H} = 1$, $n_{5h} = 4$, $n_{1e} = 2$ and $n_{5e} = 0$. In these particular minimal models, exotics states still exist in the spectrum consisting of 4 states. For instance, an example of a minimal model is given by the following GGSO coefficients matrix:

$$
(v_i|v_j) \;=\; \begin{array}{c} \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1/2 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & -1/2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

Further elaboration on the structure of the exotic states in the Flipped $SU(5)$ models will be discussed in section 4.5.3.

4.5.2 Results and Discussions

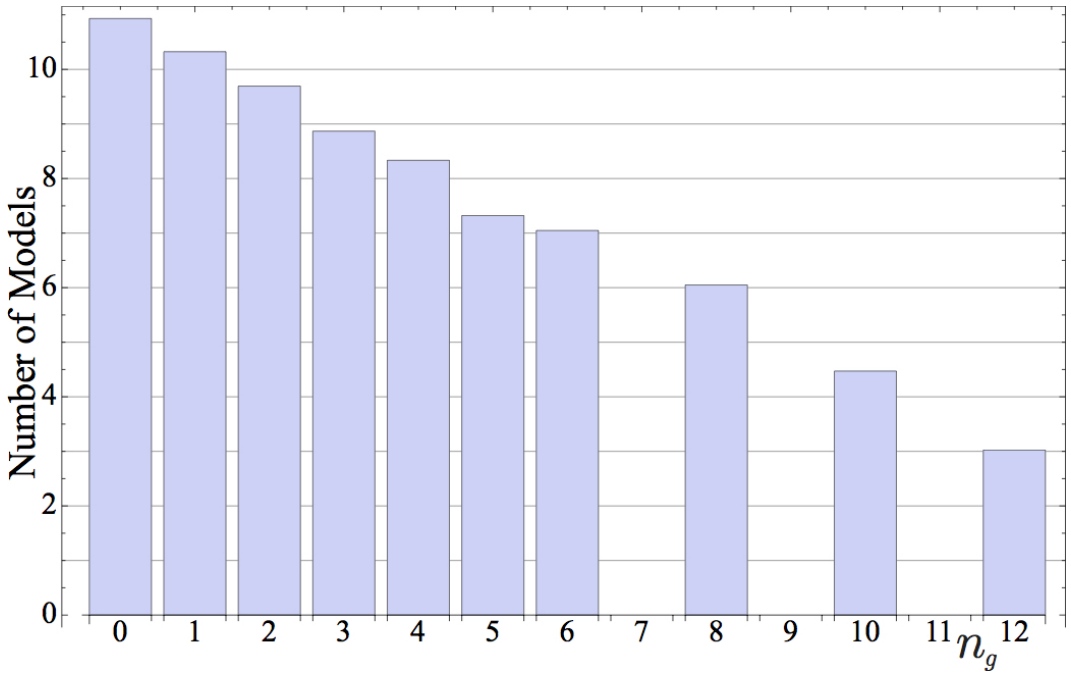

Figure 4.1: *Logarithm of the number of models in relation to the number of generations (n_g), in a random sample of* 10^{12} *Flipped* $SU(5)$ *configurations.*

To accomplish the classification of the Free-Fermionic Flipped $SU(5)$ String vacua,

$\mathbf{n_{5h}/n_{10H}}$	**0**	**1**	**2**	**3**
0	281477	28518	0	0
1	3626622	275967	8197	651
2	630727	61910	2092	0
3	23924485	63774	5901	0
4	78959	67900	0	0
5	139642	12380	0	0

Table 4.1: *Number of 3 generation models as a function of the Flipped $SU(5)$ breaking Higgs pairs (n_{10H}) in relation to the SM breaking Higgs pairs (n_{5h}), in a random sample of 10^{10} models.*

a statistical sampling was carried out in a space of 10^{12} models out of the 2^{44} possibilities. For this purpose, it was necessary to develop two independent computer codes, which were required to cross check the vast data. The results from this are presented in figures 4.1 - 4.3 and table 4.1 - 4.3.

The number of models in relation to the number of generations is indicated in figure 4.1. This is in agreement with the results of [32, 35, 36, 37, 38, 39], where the number of models peaks for the 0 generation and decreases as the number of generations increases. In this figure also, the absence of models with 7, 9, 11 and greater than 12 generations can be seen. These results can be understood in light of the corresponding results in the $SO(10)$ classification [37, 38, 39]. Recalling that the α projection which breaks the $SO(10)$ symmetry to the Flipped $SU(5)$, truncates the number of generations by two. Therefore, by examining the corresponding figure in the $SO(10)$ classification, the absence of the models with double the number of generations is observed, *i.e.* no models with 14, 18, 22 and more than 24 generations. Remark, that this result is applicable to the case in which all gauge group enhancements are projected out, as discussed in section 4.4. Thus, models with the excluded number of generations may occur, when the hidden gauge group is enhanced. However, comparing figure 4.1 to the corresponding figure in [32], the existence of models with 16 generations is noticed, since not all of the hidden gauge group enhancements where projected out there. Whereas, from section 4.4, this is not the case for the basis vector α_1 used here. Hence, some of these models descend from $SO(10)$ GUTs with enhanced gauge group. These do not arise in the case of the Flipped $SU(5)$ models studied here.

The number of 3 generation models in relation to the number of pairs of light and heavy Higgs representations appearing in the models is shown in table 4.1, with the light and heavy pairs being $\mathbf{5}+\overline{\mathbf{5}}$ and $\mathbf{10}+\overline{\mathbf{10}}$ representations of $SU(5)$, respectively. The null cases are not viable phenomenologically and the minimal cases are models with one pair of each. In models with a larger number of light Higgs pairs, it may be

easier to accommodate the Standard Model fermion mass textures, whereas models with a larger number of heavy Higgs pairs, may facilitate gauge coupling unification at the String scale [23, 40, 41].

As seen in section 4.2.3, some of the exotic matter states in the models transform into vector–like representations of the hidden sector non–Abelian group factors. They carry fractional electric charge and must be sufficiently massive or confined. These exotic states may nevertheless have interesting phenomenological implications. In table 4.2, the structure of the exotic states arising in the models is explored. These are labelled by the four integers: $\mathbf{n}^e_5, \mathbf{n}^e_1, \mathbf{n}^e_4$ and $\mathbf{n}^e_{4'}$, where $\mathbf{n}^e_5 = n_{5e}+n_{\bar{5}e}$ is the number of exotic states that transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$ of the observable $SU(5)$; $\mathbf{n}^e_1 = n_{1e} + n_{\bar{1}e}$ is the number of exotic states that transform as singlets of all non–Abelian group factors; $\mathbf{n}^e_4 = n_{4e} + n_{\bar{4}e}$ is the number of exotic states that transform as $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the hidden $SU(4)$; $\mathbf{n}^e_{4'} = n_{4'e} + n_{\bar{4}'e}$ is the number of exotic states that transform as $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the hidden $SO(6)$ gauge group.

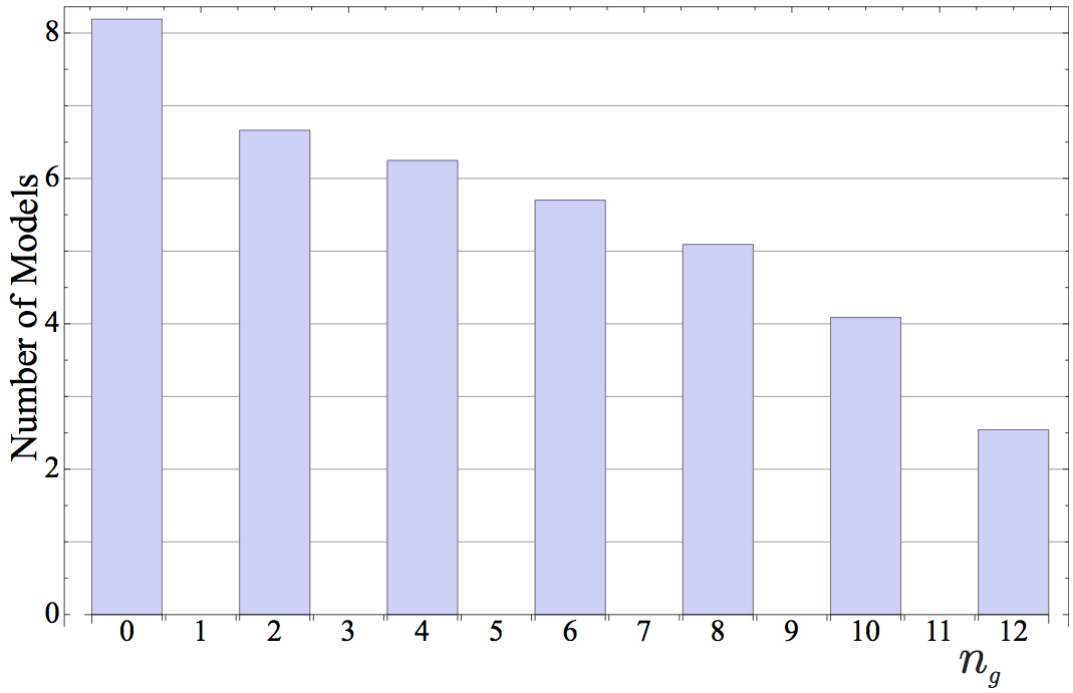

Figure 4.2: *Logarithm of the number of exophobic models in relation to the number of generations (n_g), in a random sample of* 10^{12} *Flipped* $SU(5)$ *configurations.*

The number of exophobic models in relation to the number of generations is displayed in figure 4.2. The striking feature in this figure is the absence of models with 3 chiral generations. This is in contrast to the case of the Pati–Salam models that yielded numerous 3 generation exophobic models. Figure 4.2 also reveals the absence of any exophobic odd generation models with 1, 3, 5, 7, 9, 11, whereas exophobic models arise for the even numbers of generations, up to 12. As a result, exophobic

models in this class arise in configurations with even numbers of generations and not in models with odd numbers of generations. It should be emphasized, that these results hold in the space of models that were explored here and may not indicate absence of 3 generation exophobic Flipped $SU(5)$ models. In figure 4.3, the number of 3 generation models in relation to the number of exotic multiplets is displayed, note that the minimal number of exotic multiplets is 4.

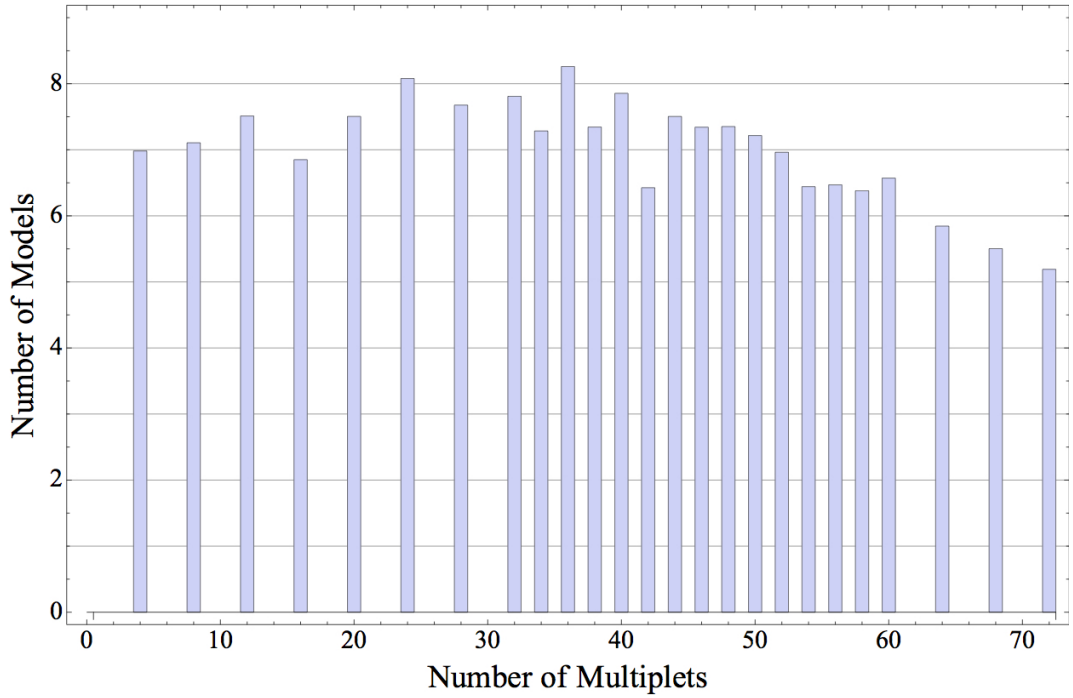

Figure 4.3: *Logarithm of the number 3 generation models in relations to the number of exotic multiplets* $(n_{1e}, n_{\overline{1}e}, n_{5e}, n_{\overline{5}e})$*, in a random sample of* 10^{12} *Flipped* $SU(5)$ *configurations.*

4.5.3 Structure of Exotic States

One of the main highlights of the classification method in the case of the Pati–Salam Heterotic-String models, has been the discovery of the exophobic Heterotic-String models, in which all exotic states are limited to the massive spectrum and do not appear among the massless states. As shown in figures 4.2 and 4.3, in the class of 10^{12} Flipped $SU(5)$ models that were analysed here, there are no exophobic 3 generation vacua with a statistical frequency larger than $1 : 10^{12}$. The structure of the exotic states arising in the models are analysed further in table 4.2. All the models given in this table contain 3 chiral generations of which at least one–pair is the light Higgs states and at least one pair is the heavy Higgs states. Thus, in all these models the gauge symmetry can be broken to the Standard Model in the effective

field theory limit, which also contains all the fields required for viable Standard Model phenomenology.

The occurrence of models in which all exotic states transform in representations of an hidden non–Abelian gauge group, is noted in table 4.2. In this case, the exotic states are confined into integrally charged states and produce the so–called Crypton states [42, 43, 44]. As a further note from this table, is the existence of models with equal numbers of $\mathbf{4}$ and $\mathbf{4}'$ states. This suggests the possible existence of the Free-Fermionic models that admit the Race-Track mechanism to stabilise the vacuum expectation value of the Dilaton field [45, 46]. Moreover, table 4.2 reveals interesting observations and directions for future research. The first eleven models in the table contain only states that transform in non–trivial representations of an hidden non–Abelian gauge group. Thus, this class of models may give rise to the so–called Crypton states that are confined into integrally charged states. It was seen that there is an abundance of such models. There are also numerous models with a small number of Crypton states that may remain asymptotically free and therefore, confined at some scale. A well known example of a model that gives rise only to Crypton like states is given in [21]. The table shows the existence of a large space of models with similar characteristics. One notable difference between the vacua in this table and the one of [21], is the fact that the model in [21] uses asymmetric internal shifts, whereas the models in this table only use symmetric internal shifts. The models in the six and twelfth rows of the table, with $n_4 = n_{4'} = 2$ are interesting to study for implementation of the Race-Track mechanism [45, 46].

In examination of the other types of exotic states. The non–Abelian singlet states that are counted in the second column, are fractionally charged and must decouple from the light spectrum or be sufficiently diluted. The fields counted in the first column transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$ of the observable $SU(5)$ and carry $1/2$ of the hypercharge compared to the standard Flipped $SU(5)$ states. Such states do not arise in the Flipped $SU(5)$ model studied in [21], however, their colour triplet and electroweak doublet components arise generically in the Standard–Like Heterotic-String models [40, 41, 47, 48, 49, 50, 51]. These fields may be instrumental as intermediate matter states to resolve the conflict between Heterotic-String scale unification and the low scale gauge coupling experimental data [40, 41]. The models appearing in the 13^{th} and 25^{th} rows in table 4.2, are interesting examples of Flipped $SU(5)$ models admitting such states. The models in the 25^{th} row with $n_5 = 2$, $n_1 = 6$, $n_4 = 2$ and $n_{4'} = 2$ may accommodate both the intermediate matter thresholds and the Race-Track mechanism, therefore maybe of particular interest.

The model given in (4.5.3.1) is analysed to illustrate the exotic spectrum appearing in the Flipped $SU(5)$ models. The twisted sectors of the model given here

produce: 3 chiral generations; one pair of heavy Higgs states; one pair of light Higgs representations. Therefore, this model may yield viable Standard Model phenomenology.

$$(v_i|v_j) \;=\; \begin{array}{c} \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array} \begin{array}{c} \begin{array}{cccccccccccc} S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\ \left(\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1/2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array} \right) \end{array} \tag{4.5.3.1}$$

These model contain the following states that transform under the hidden $SU(4)$ gauge group: six non–exotic pairs of $(\mathbf{4}+\overline{\mathbf{4}})$; one non–exotic state transforming in the vectorial $\mathbf{6}$ representation; one pair of exotic states transforming as $(\mathbf{4}+\overline{\mathbf{4}})$. Additionally, these model also contain the following states that transform under the hidden $SU(4)'$ gauge group: four non–exotic pairs of $(\mathbf{4}+\overline{\mathbf{4}})$; one non–exotic state transforming in the vectorial $\mathbf{6}$ representation; one pair of exotic states transforming as $(\mathbf{4}+\overline{\mathbf{4}})$. Thus, the β–functions of the $SU(4)$ and $SU(4)'$ hidden gauge groups are $\beta_4 = -4$ and $\beta_{4'} = -6$, respectively. Depending on the mass scales for the hidden sector matter states, this model may therefore, provide a workable example for implementing the Race-Track mechanism. This model also contains one pair of exotic $(\mathbf{5}+\overline{\mathbf{5}})$ states of the observable Flipped $SU(5)$ group that can be used to mitigate the gauge coupling unification problem.

$\mathbf{n_5}$	$\mathbf{n_1}$	$\mathbf{n_4}$	$\mathbf{n'_4}$	#	$\mathbf{n_5}$	$\mathbf{n_1}$	$\mathbf{n_4}$	$\mathbf{n'_4}$	#	$\mathbf{n_5}$	$\mathbf{n_1}$	$\mathbf{n_4}$	$\mathbf{n'_4}$	#
0	0	0	2	12627	2	10	3	3	9311	4	12	4	3	4889
0	0	0	4	3561	2	10	4	5	668	4	12	4	5	5720
0	0	0	6	1630	2	10	5	1	1614	4	12	4	6	965
0	0	0	8	187	2	10	5	2	4074	4	12	5	2	1479
0	0	2	0	16329	2	10	5	4	906	4	12	5	4	6105
0	0	2	2	18381	2	10	7	2	1745	4	12	6	4	608
0	0	2	4	2409	2	14	2	5	1474	4	12	8	0	153
0	0	4	0	6814	2	14	4	5	873	4	16	1	1	11395
0	0	4	2	3722	2	14	5	2	1412	5	7	2	4	1352
0	0	6	0	2338	2	14	5	4	1040	5	7	4	2	1323
0	0	8	0	356	2	18	1	1	2966	5	11	2	2	9462
0	8	2	2	1166	3	9	2	4	9505	5	11	2	5	2675
1	3	1	1	45575	3	9	3	3	2670	5	11	3	4	2491
1	3	2	8	4343	3	9	4	2	9949	5	11	4	3	2828
1	3	4	6	12465	3	13	2	2	9367	5	11	4	5	5164
1	3	6	4	12858	3	13	3	4	2562	5	11	5	2	2432
1	3	8	2	4287	3	13	4	3	2599	5	11	5	4	5074
1	11	2	4	1135	3	13	4	5	1909	5	15	2	5	4163
1	11	4	2	1336	3	13	5	4	1630	5	15	2	7	1069
2	6	0	0	13014	4	4	2	2	1231	5	15	4	5	1937
2	6	0	2	17622	4	8	1	5	1331	5	15	5	2	2605
2	6	0	4	6164	4	8	2	5	993	5	15	5	4	1170
2	6	0	6	3942	4	8	3	3	7915	5	15	7	2	670
2	6	2	0	14550	4	8	4	5	649	6	14	1	1	9970
2	6	2	2	25235	4	8	5	1	1443	6	18	0	2	2171
2	6	2	4	5864	4	8	5	2	1298	6	18	2	0	1499
2	6	3	5	10824	4	8	5	4	981	6	18	2	2	2272
2	6	4	0	5593	4	12	0	2	3255	6	18	2	4	799
2	6	4	2	4924	4	12	0	4	6986	6	18	4	2	490
2	6	4	4	2712	4	12	0	8	383	7	13	2	5	1311
2	6	4	6	1870	4	12	2	0	1795	7	13	2	7	758
2	6	5	3	10858	4	12	2	2	7064	7	13	5	2	849
2	6	6	0	2699	4	12	2	4	3139	7	13	7	2	428
2	6	6	4	2099	4	12	2	5	1269	8	12	1	1	2755
2	10	1	5	1522	4	12	3	4	5217	8	24	0	4	397
2	10	2	5	2794	4	12	4	0	3237	8	24	4	0	163
2	10	2	7	1199	4	12	4	2	2489	-	-	-	-	-

Table 4.2: *Number of 3 generation models consisting of* $n_{10H} \geq 1$ *and* $n_{5h} \geq 1$*, in relation to the exotic multiplets* (n_5,n_1,n_4,n'_4)*, in a random sample of* 10^{10}.

	Constraints	**Total Models**	**Probability**	**Number of Models**
	No Constraints	1000000000000	1	1.76×10^{13}
(1)	+ No Enhancements	762269298719	7.62×10^{-1}	1.34×10^{13}
(2)	+ Anomaly Free Flipped $SU(5)$	139544182312	1.40×10^{-1}	2.45×10^{12}
(3)	+ 3 Generations	738045321	7.38×10^{-4}	1.30×10^{10}
(4a)	+ SM Light Higgs	706396035	7.06×10^{-4}	1.24×10^{10}
(4b)	+ Flipped $SU(5)$ Heavy Higgs	46470138	4.65×10^{-5}	8.18×10^{8}
(5)	+ SM Light Higgs + & Heavy Higgs	43624911	4.36×10^{-5}	7.67×10^{8}
(6a)	+ Minimal Flipped $SU(5)$ Heavy Higgs	42310396	4.23×10^{-5}	7.44×10^{8}
(6b)	+ Minimal SM Light Higgs	25333216	2.53×10^{-5}	4.46×10^{8}
(7)	+ Minimal Flipped $SU(5)$ Heavy Higgs + & Minimal SM Light Higgs	24636896	2.46×10^{-5}	4.33×10^{8}
(8)	+ Minimal Exotic States	1218684	1.22×10^{-6}	2.14×10^{7}

Table 4.3: *Statistics for the Flipped $SU(5)$ models, with respect to phenomenological constraints.*

4.5.4 Phenomenological Constraints

The number of models with sequential imposition of phenomenological constraints is displayed in table 4.3 . The total number of models in the sample is 10^{12}. Firstly, the condition of there being no enhancements of the four-dimensional gauge symmetry were imposed, which approximated to 76.2% of the total models. Secondly, the condition that the Flipped $SU(5)$ models are anomaly free with respect to the $U(1)_5$ group factor were imposed, and about 14% of the total models satisfied this criterion. A further reduction by 3 orders of magnitude, resulted from the restriction to have 3 chiral generation model. Then, imposing the existence of both the heavy and light Higgs states to break the Flipped $SU(5)$ gauge symmetry, to the Standard Model gauge group and the electroweak breaking, respectively, leads to a further reduction of one order of magnitude. Finally, imposing the minimal number of massless exotic states resulted in the reduction of the number of models by a further order of magnitude. Therefore, the number of String vacua in the space of models scanned, reduced from 10^{12} to 10^{6}, that satisfies all the constraints that were imposed. This left a substantial number to accommodate further phenomenological constraints. To conclude, regarding the results obtained by using α_2 and α_3 in (4.1.4.1) to break the $SO(10)$ symmetry. The preliminary results are not that substantially different compared to the classification with α_1 and it seems that there are also no 3 generation exophobic vacua in these cases.

Chapter 5

Classification of the SU(4) x SU(2) x U(1) Heterotic-String Vacua

In this chapter, the classification is extended and given for the class of $SU(4) \times SU(2) \times U(1)$ (SU421[5]) Heterotic-String models [6]. The classification methodology will be identical to the Flipped $SU(5)$ models, as discussed in chapter 4. However, here it will be shown that breaking the $SO(10)$ symmetry to the $SU(4)\times SU(2)\times U(1)$ subgroup [52] cannot produce 3 chiral generations in the Free-Fermionic construction.

5.1 SU(4) x SU(2) x U(1) Free-Fermionic Models

The SU421 symmetry results from the breaking of the SO(10) gauge group to the Pati-Salam models, followed by the breaking of the $SU(2)_R \to U(1)_L$ symmetry, all directly at the String scale. Therefore, the SU421 models, admit the $SO(10)$ embedding with the chiral states, that are obtained from the spinorial **16** representations of $SO(10)$ that decomposes in the following way:

$$\begin{aligned} F_L^i &= (4,2,\ 0) = (3,2,\ \frac{1}{3},\ 0) + (1,2,-1,\ 0) = \begin{pmatrix} u \\ d \end{pmatrix}^i + \begin{pmatrix} \nu \\ e \end{pmatrix}^i, \\ U_R^i &= (\overline{4},1,-1) = (\overline{3},1,-\frac{1}{3},-\frac{1}{2}) + (1,1,+1,-\frac{1}{2}) = u^{ci} + N^{ci}, \quad (5.1.0.1) \\ D_R^i &= (\overline{4},1,+1) = (\overline{3},1,-\frac{1}{3},+\frac{1}{2}) + (1,2,+1,+\frac{1}{2}) = d^{ci} + e^{ci}. \end{aligned}$$

The first and second equalities show the decomposition under $SU(4)_C \times SU(2)_L \times U(1)_R$ and $SU(3)_C \times SU(2)_L \times U(1)_{B-L} \times U(1)_R$, respectively. It should be noted that F_L produces the quarks and leptons weak doublets, whereas U_R and D_R produces the

[5]Note: The abbreviation SU421 for the gauge group $SU(4) \times SU(2) \times U(1)$ will be used in this chapter

right–handed weak singlets. In these models the electroweak $U(1)_Y$ current is given by

$$U(1)_Y = \frac{1}{2}U(1)_{B-L} + U(1)_R.$$

As for the two Higgs multiplets of the Minimal Supersymmetric Standard Model, h^u and h^d, appearing in the SU421 models are given by the following states:

$$\begin{aligned} h^d &= (1, 2, -1), \\ h^u &= (1, 2, +1). \end{aligned}$$

Additionally, the heavy Higgs states that are responsible for breaking the $SU(4)_C \times SU(2)_L \times U(1)_L$ gauge symmetry to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ Standard Model gauge groups are given by the following fields:

$$\begin{aligned} H &= (\overline{4}, 1, -1), \\ \overline{H} &= (4, 1, +1). \end{aligned}$$

These SU421 models may also contain states that transform as

$$(6, 1, 0) = (3, 1, \frac{1}{3}, 0) + (\overline{3}, 1, -\frac{1}{3}, 0).$$

Here, these multiplets arise from the vectorial **10** representation of $SO(10)$ and are the coloured states.

5.1.1 $SU(4) \times SU(2) \times U(1)$ Construction

In the following, the necessary tools for the classification of the SU421 Free-Fermionic Heterotic-String models are given. The analysis is similar to the one performed in the classification of the Flipped $SU(5)$ models [5], as discussed in chapter 4. The novelty compared to these cases, is that the SU421 models employ two basis vectors that break the $SO(10)$ symmetry, whereas the Flipped $SU(5)$ models use only one. The basis vectors that generate the $SU(4) \times SU(2) \times U(1)$ Heterotic-String

models are given by the following 14 basis vectors:

$$\begin{aligned}
v_1 = \mathbf{1} &= \{\psi^\mu,\ \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6} | \overline{y}^{1,\ldots,6}, \overline{\omega}^{1,\ldots,6}, \overline{\eta}^{1,2,3}, \overline{\psi}^{1,\ldots,5}, \overline{\phi}^{1,\ldots,8}\},\\
v_2 = S &= \{\psi^\mu, \chi^{1,\ldots,6}\},\\
v_{2+i} = e_i &= \{y^i, \omega^i | \overline{y}^i, \overline{\omega}^i\},\ i = 1,\ldots,6,\\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \overline{y}^{34}, \overline{y}^{56}, \overline{\eta}^1, \overline{\psi}^{1,\ldots,5}\},\\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \overline{y}^{12}, \overline{y}^{56}, \overline{\eta}^2, \overline{\psi}^{1,\ldots,5}\},\\
v_{11} = z_1 &= \{\overline{\phi}^{1,\ldots,4}\},\\
v_{12} = z_2 &= \{\overline{\phi}^{5,\ldots,8}\},\\
v_{13} = \alpha &= \{\overline{\psi}^{4,5}, \overline{\phi}^{1,2}\},\\
v_{14} = \beta &= \{\overline{\psi}^{4,5} = \tfrac{1}{2}, \overline{\phi}^{1,\ldots,6} = \tfrac{1}{2}\}.
\end{aligned} \tag{5.1.1.1}$$

The first 12 basis vectors considered here, were discussed in chapter 4 section 4.1.2. Recall, these gave rise to the $SO(10) \times U(1)^3 \times SO(8) \times SO(8)$ gauge group. Furthermore, taking the combined projection of the basis vectors α and β here breaks the $SO(10)$ GUT symmetry to the $SU(4) \times SU(2) \times U(1)$ gauge group, where α is identical to the basis vector used in the classification of the Pati–Salam models, that breaks the $SO(10)$ symmetry to $SO(6) \times SO(4)$. Then using the β basis vector with fractional boundary conditions reduces the $SO(10)$ gauge symmetry to the $SU(4) \times SU(2) \times U(1)$ symmetry.

5.1.2 String Spectrum

The space–time vector bosons that are obtained from the Neveu–Schwarz (NS) sector, that survive the GGSO projections as defined by the basis vectors in (5.1.1.1), generate the following observable and hidden gauge groups:

$$\begin{aligned}
\text{Observable} &: \quad SU(4) \times SU(2)_L \times U(1)_L \times U(1)^3\\
\text{Hidden} &: \quad SU(2)_A \times U(1)_A \times SU(2)_B \times U(1)_B \times SU(2)_C \times U(1)_C \times SO(4)_2
\end{aligned}$$

All the String states in the spectrum transform under these gauge group factors. However, additional space–time vector bosons may also arise in the massless String spectrum. These will enhance the NS observable and/or hidden gauge groups. In order to preserve the above gauge groups, all these additional space–time vector

bosons need to be projected out, that are given in the following 36 sectors:

$$\mathbf{G_{Enh}} = \left\{ \begin{array}{ccc} z_1, & z_1+\beta, & z_1+2\beta, \\ z_1+\alpha, & z_1+\alpha+\beta, & z_1+\alpha+2\beta, \\ z_2, & z_2+\beta, & z_2+2\beta, \\ z_2+\alpha, & z_2+\alpha+\beta, & z_2+\alpha+2\beta, \\ z_1+z_2, & z_1+z_2+\beta, & z_1+z_2+2\beta, \\ z_1+z_2+\alpha, & z_1+z_2+\alpha+\beta, & z_1+z_2+\alpha+2\beta, \\ \beta, & 2\beta, & \alpha, \\ \alpha+\beta, & \alpha+2\beta, & x, \\ z_1+x+\beta, & z_1+x+2\beta, & z_1+x+\alpha, \\ z_1+x+\alpha+\beta, & z_2+x+\beta, & z_2+x+\alpha+\beta, \\ z_1+z_2+x+\beta, & z_1+z_2+x+2\beta, & z_1+z_2+x+\alpha+\beta, \\ x+\beta, & x+\alpha, & x+\alpha+\beta, \end{array} \right\},$$

where $x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2$.

5.1.3 Matter Content

The observable matter states in the Heterotic-String vacuum are embedded in the **27** representation of E_6. In the Free-Fermionic construction adopted here and using the basis vectors in (5.1.1.1), the E_6 is first broken to the $SO(10) \times U(1)$ symmetry. The decomposition of the **27** of E_6 under the $SO(10)$ gauge group is taken, which is given as

$$\mathbf{27} \;=\; \mathbf{16} + \mathbf{10} + \mathbf{1},$$

where, the **16** transforms under the spinorial representation of $SO(10)$ and the **10** transforms in the vectorial representation of the $SO(10)$, and similarly for $\overline{\mathbf{27}}$. Further to this, the spinorial **16** and $\overline{\mathbf{16}}$ states of $SO(10)$ in the String spectrum, are produced in the following 48 sectors:

$$\begin{aligned} B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\quad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^1, \overline{\psi}^{1,\ldots,5}\}, \\ B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6, \\ B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4, \end{aligned}$$

where $p,q,r,s=0,1$ and $b_3=b_1+b_2+x$. In order to distinguish between the spinorial **16** and $\overline{\mathbf{16}}$ representations given in these sectors, the following chirality operators are needed:

$$\begin{aligned}
X^{(1)_{SO(10)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{b_2+(1-r)e_5+(1-s)e_6},\\
X^{(2)_{SO(10)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{b_1+(1-r)e_5+(1-s)e_6},\\
X^{(3)_{SO(10)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{b_1+(1-r)e_3+(1-s)e_4}.
\end{aligned}$$

Here the chirality $X^{(1,2,3)_{SO(10)}}_{pqrs}=1$, indicates that the state is the **16** of $SO(10)$, similarly, $X^{(i)_{SO(10)}}_{pqrs}=-1$ corresponds to a state giving the $\overline{\mathbf{16}}$ of $SO(10)$. Since, states can be projected in or out depending on the GGSO projections, it should be noted that the basis vectors $e_1,....,e_6$, z_1 and z_2 can be used to define projectors $P^{(1,2,3)}$, so that $P^{(1,2,3)}=1$ implies that the states are projected in and $P^{(1,2,3)}=0$ implies that the states are projected out. These projectors $P^{(1,2,3)}$ are:

$$\begin{aligned}
P^{(1)}_{pqrs} &= \tfrac{1}{16}\left(1-C\binom{e_1}{B^{(1)}_{pqrs}}\right).\left(1-C\binom{e_2}{B^{(1)}_{pqrs}}\right).\left(1-C\binom{z_1}{B^{(1)}_{pqrs}}\right).\left(1-C\binom{z_2}{B^{(1)}_{pqrs}}\right),\\
P^{(2)}_{pqrs} &= \tfrac{1}{16}\left(1-C\binom{e_3}{B^{(2)}_{pqrs}}\right).\left(1-C\binom{e_4}{B^{(2)}_{pqrs}}\right).\left(1-C\binom{z_1}{B^{(2)}_{pqrs}}\right).\left(1-C\binom{z_2}{B^{(2)}_{pqrs}}\right),\\
P^{(3)}_{pqrs} &= \tfrac{1}{16}\left(1-C\binom{e_5}{B^{(3)}_{pqrs}}\right).\left(1-C\binom{e_6}{B^{(3)}_{pqrs}}\right).\left(1-C\binom{z_1}{B^{(3)}_{pqrs}}\right).\left(1-C\binom{z_2}{B^{(3)}_{pqrs}}\right),
\end{aligned}$$

Furthermore, these projectors can be expressed as matrix equations given in the following form:

$$\begin{pmatrix}(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_1|b_1)\\(e_2|b_1)\\(z_1|b_1)\\(z_2|b_1)\end{pmatrix},$$

$$\begin{pmatrix}(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_3|b_2)\\(e_4|b_2)\\(z_1|b_2)\\(z_2|b_2)\end{pmatrix}, \qquad (5.1.3.1)$$

$$\begin{pmatrix}(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_5|b_3)\\(e_6|b_3)\\(z_1|b_3)\\(z_2|b_3)\end{pmatrix}.$$

Expressing these projectors as matrix equations, entail solving systems of linear equations. These algebraic equations can then be solved using a computerised code, that can be used to scan a vast space of models. Similar to the spinorial representations, the singlets and vectorial **10** representations of $SO(10)$ are obtained from the following 48 sectors:

$$\begin{aligned} B^{(4)}_{pqrs} &= B^{(1)}_{pqrs} + x \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\overline{y}^3, p\omega^3\overline{\omega}^3, (1-q)y^4\overline{y}^4, q\omega^4\overline{\omega}^4, \\ &\qquad (1-r)y^5\overline{y}^5, r\omega^5\overline{\omega}^5, (1-s)y^6\overline{y}^6, s\omega^6\overline{\omega}^6, \overline{\eta}^{2,3}\}, \qquad (5.1.3.2) \\ B^{(5,6)}_{pqrs} &= B^{(2,3)}_{pqrs} + x. \end{aligned}$$

The massless states that arise in these sectors are obtained by acting on the vacuum with the NS oscillator. The type of states, therefore, depend on the type of oscillator, that may correspond to singlets or the vectorial **10** representations of the $SO(10)$ symmetry. Here, the vectorial **10** state is needed for electroweak symmetry breaking. As for the different types of $SO(10)$ singlets, that arises from equation (5.1.3.2) are:

- $\{\overline{\eta}^i\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\eta}^{*i}\}|R\rangle^{(4,5,6)}_{pqrs}$, $i = 1,2,3$, where $|R\rangle^{(4,5,6)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(4,5,6)}_{pqrs}$ sector. These states transform as vector–like representations under the $U(1)_i$'s.
- $\{\overline{\phi}^{1,2}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*1,2}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations of $SU(2)_A \times U(1)_A$.
- $\{\overline{\phi}^{3,4}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*3,4}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations of $SU(2)_B \times U(1)_B$.
- $\{\overline{\phi}^{5,6}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*5,6}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations of $SU(2)_C \times U(1)_C$.
- $\{\overline{\phi}^{7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$ or $\{\overline{\phi}^{*7,8}\}|R\rangle^{(4,5,6)}_{pqrs}$. These states transform as vector–like representations of $SO(4)$.

Similarly, for the matrix equations given in equation (5.1.3.1), algebraic equations can also be written for the sectors in equation (5.1.3.2) as follows:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+x) \\ (e_2|b_1+x) \\ (z_1|b_1+x) \\ (z_2|b_1+x) \end{pmatrix},$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ (z_1|b_2+x) \\ (z_2|b_2+x) \end{pmatrix},$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ (z_1|b_3+x) \\ (z_2|b_3+x) \end{pmatrix}.$$

5.2 Observable Matter Spectrum

The basis vectors α and β given in equation (5.1.1.1) are used to break the $SO(10)$ symmetry to the $SU(4) \times SU(2)_L \times U(1)_L$ gauge group. Taking the α and β GGSO projections, the decomposition of the spinorial **16** and $\overline{\mathbf{16}}$ representations of $SO(10)$, under the $SU(4) \times SU(2)_L \times U(1)_L$ gauge group is given as:

$$\begin{aligned} \mathbf{16} &= (\mathbf{4},\mathbf{2},0) + \left(\overline{\mathbf{4}},\mathbf{1},-1\right) + \left(\overline{\mathbf{4}},\mathbf{1},+1\right), \\ \overline{\mathbf{16}} &= \left(\overline{\mathbf{4}},\mathbf{2},0\right) + (\mathbf{4},\mathbf{1},-1) + (\mathbf{4},\mathbf{1},+1). \end{aligned}$$

To break the $SU(4) \times SU(2)_L \times U(1)_L$ gauge group to the Standard Model group, the heavy Higgs pair is required, where this pair is given by

$$\left(\overline{\mathbf{4}},\mathbf{1},-1\right) + (\mathbf{4},\mathbf{1},-1).$$

The vectorial representation **10** of the $SO(10)$ symmetry, is also needed for the electroweak breaking as discussed before. These are given by the following decomposition of the $SU(4) \times SU(2)_L \times U(1)_L$ gauge group

$$\mathbf{10} = (\mathbf{6},\mathbf{1},0) + (\mathbf{1},\mathbf{2},-1) + (\mathbf{1},\mathbf{2},+1).$$

As for the normalizations of the hypercharge and electromagnetic charge, the following is taken:

$$\begin{aligned} Y &= \frac{1}{3}(Q_1+Q_2+Q_3)+\frac{1}{2}(Q_4+Q_5), \\ Q_{em} &= Y+\frac{1}{2}(Q_4-Q_5), \end{aligned}$$

where the Q_i charges of a state arise due to ψ^i for $i = 1, ..., 5$. The following table, summarizes the charges of the colour $SU(3)$ and electroweak $SU(2) \times U(1)$ Cartan generators of the states which form the $SU(4) \times SU(2)_L \times U(1)_L$ matter representations:

Representation	$\overline{\psi}^{\mathbf{1,2,3}}$	$\overline{\psi}^{\mathbf{4,5}}$	**Y**	$\mathbf{Q_{em}}$
	$(+,+,-)$	$(+,-)$	1/6	2/3
	$(+,+,-)$	$(-,+)$	1/6	-1/3
$(\mathbf{4},\mathbf{2},0)$	$(-,-,-)$	$(+,-)$	-1/2	0
	$(-,-,-)$	$(-,+)$	-1/2	-1
	$(+,-,-)$	$(-,-)$	-2/3	-2/3
$(\overline{\mathbf{4}},\mathbf{1},-1)$	$(+,+,+)$	$(-,-)$	0	0
	$(+,-,-)$	$(+,+)$	1/3	1/3
$(\overline{\mathbf{4}},\mathbf{1},+1)$	$(+,+,+)$	$(+,+)$	1	1
	$(+,-,-)$	$(+,-)$	-1/6	-2/3
	$(+,-,-)$	$(-,+)$	-1/6	1/3
$(\overline{\mathbf{4}},\mathbf{2},0)$	$(+,+,+)$	$(+,-)$	1/2	0
	$(+,+,+)$	$(-,+)$	1/2	1
	$(+,+,-)$	$(+,+)$	2/3	2/3
$(\mathbf{4},\mathbf{1},-1)$	$(-,-,-)$	$(+,+)$	0	0
	$(+,+,-)$	$(-,-)$	-1/3	-1/3
$(\mathbf{4},\mathbf{1},+1)$	$(-,-,-)$	$(-,-)$	-1	-1

Here " $+$ " and " $-$ ", label the contribution of an oscillator with fermion number $F = 0$ or $F = -1$, to the degenerate vacuum. These states correspond to particles of the Standard Model. More precisely these representations are decomposed under

$SU(3) \times SU(2) \times U(1)$ as:

$$
\begin{aligned}
(\mathbf{4},\mathbf{2},0) &= \left(\mathbf{3},\mathbf{2},+\frac{1}{6}\right)_{Q} + \left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{L}, \\
\left(\overline{\mathbf{4}},\mathbf{1},-1\right) &= \left(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\right)_{u^c} + (\mathbf{1},\mathbf{1},0)_{\nu^c}, \\
\left(\overline{\mathbf{4}},\mathbf{1},+1\right) &= \left(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3}\right)_{d^c} + (\mathbf{1},\mathbf{1},+1)_{e^c},
\end{aligned}
$$

where L is the lepton–doublet; Q is the quark–doublet; $d^c,\ u^c,\ e^c$ and ν^c are the quark and lepton singlets. Due to the α- and β-projections, which projects on incomplete **16** and $\overline{\mathbf{16}}$ representations, complete families and anti–families are formed by combining states from different sectors.

5.3 Nonviability of the $SU(4) \times SU(2) \times U(1)$ Models

Recall that the matter content comes from the **16** of $SO(10)$. However, with the addition of the α and β basis vectors from equation (5.1.1.1), the **16** representation is broken by the GGSO projections that are in general given by

$$
e^{i\pi v_i \cdot F_\xi}|S_\xi\rangle = \delta_\xi \; C\binom{\xi}{v_i}^{*}|S_\xi\rangle. \tag{5.3.0.1}
$$

Here $\delta_\xi = \pm 1$ is a space-time spin statistics index and F_ξ is the fermion number operator. In the SU421 models spanned by equation (5.1.1.1) the GGSO projection coefficients $C\binom{\xi}{v_i}$ can take the values $\pm 1; \pm i$. Therefore, firstly considering the α GGSO projection, the **16**s are decomposed into the Pati-Salam group representation. Using the following chirality operators:

$$
\begin{aligned}
X^{(1)_{SO(6)}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{\alpha}, \\
X^{(2)_{SO(6)}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{\alpha}, \\
X^{(3)_{SO(6)}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{\alpha},
\end{aligned}
$$

the operators $X^{(i)_{SO(6)}}_{pqrs} = 1$ gives rise to the $Q_R \equiv (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$ states under $SU(4) \times SU(2)_L \times SU(2)_R$ symmetry, whilst the $Q_L \equiv (\mathbf{4},\mathbf{2},\mathbf{1})$ states correspond to

$X^{(i)_{SO(6)}}_{pqrs} = -1$. Secondly, considering the β GGSO projection, the operators:

$$\begin{aligned} X^{(1)_{421}}_{pqrs} &= C\binom{B^{(1)}_{pqrs}}{\beta}, \\ X^{(2)_{421}}_{pqrs} &= C\binom{B^{(2)}_{pqrs}}{\beta}, \\ X^{(3)_{421}}_{pqrs} &= C\binom{B^{(3)}_{pqrs}}{\beta}, \end{aligned}$$

determine the decomposition of the Q_L and Q_R states under the $SU(4)\times SU(2)\times U(1)$ symmetry. Using the product $\beta \cdot B^{pqrs}_j = -1$ for $j = 1, 2, 3$, and applying the ABK rules, the chirality operator must satisfy $X^{(1,2,3)_{421}}_{pqrs} = \pm i$. Hence, this implies that the states cannot be completed to form a family of generation. Since, to complete the **16**, the states: $(\mathbf{4}, \mathbf{2}, 0)$, $(\overline{\mathbf{4}}, \mathbf{1}, -1)$ and $(\overline{\mathbf{4}}, \mathbf{1}, +1)$ under the $SU(4) \times SU(2)_L \times U(1)_L$ group, all need to survive the GGSO projections. However, the states: $(\overline{\mathbf{4}}, \mathbf{1}, -1)$ and $(\overline{\mathbf{4}}, \mathbf{1}, +1)$ can only survive the GGSO projection, when the operator $X^{(1,2,3)_{421}}_{pqrs} = \pm 1$ is satisfied. This is forbidden, as modular invariance is preserved. Evidently, using the combinatorial notation in [53, 54], the decomposition of the **16** representations is given by

$$\begin{aligned} \mathbf{16} &\equiv \left[\binom{5}{0} + \binom{5}{2} + \binom{5}{4}\right] \\ &\equiv \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0} + \binom{2}{2}\right] + \left[\binom{3}{1}\right]\left[\binom{2}{1}\right] \\ &\equiv \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0}\right] + \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{2}\right] + \left[\binom{3}{1}\right]\left[\binom{2}{1}\right]. \end{aligned} \tag{5.3.0.2}$$

Here, the combinatorial factor counts the number of periodic fermions in the $|-\rangle$ state. The second line in this equation in (5.3.0.2), corresponds to the decomposition of the **16** under the Pati–Salam subgroup, whereas in the third line it shows its decomposition under the SU421 subgroup. A crucial point that can be observed here is the even number of fermions, in the $|-\rangle$ vacuum of the Q_R states. This results in ± 1 projections on the left–hand side of the GGSO projection equation in (5.3.0.1), whereas the right–hand side is fixed by the product $\beta \cdot B^{pqrs}_j = -1$ to be $\pm i$. Thus, the exclusion arises because the β projection fixes the chirality of the vacuum of the world–sheet fermions $\overline{\psi}^{4,5}$ that generate the $SU(2)_L \times U(1)_L$ symmetry. It is to be noted that, the situation here, is different from the case of the SU421 models in [52], where a similar argument followed projecting out all the 3 generation models. The reason is that the classification method here, only allows for symmetric boundary conditions for the set of internal fermions $\{y, \omega|\overline{y}, \overline{\omega}\}^{1,\cdots,6}$,

whereas the models in [52] introduce additional freedom by allowing asymmetric boundary conditions. Thus, while the NAHE–based models in [52] did not yield any model with 3 complete generations, they contained both the Q_L and Q_R states in their spectra. On the other hand, vacua with only symmetric boundary conditions, with respect to the set $\{y,\omega|\overline{y},\overline{\omega}\}^{1,\cdots,6}$, do not contain Q_R states and are therefore categorically excluded. Also note that in the case of the Left-Right symmetric models, the chirality of the $Q_L + L_L$ and $Q_R + L_R$ is similarly affected [55, 56]. However, in that case it is compensated by the chirality of the $\overline{\eta}^j$ worldsheet fermions, leading to opposite charges under the $U(1)_j$ gauge symmetries. Furthermore, the Standard-Like models [47, 48, 49, 50, 51], which are obtained by combining the Pati-Salam and Flipped $SU(5)$ breaking vectors, can also produce complete **16** which decompose under the Standard-Like group with equal $U(1)_j$ charges here. The SU421 class of models is an exception in that it is excluded in vacua with symmetric internal boundary conditions.

Chapter 6

Conclusions

The four-dimensional Free-Fermionic construction [15, 16, 17] of the Heterotic-String provides a worldsheet approach to analysing semi-realistic String vacua. The models constructed to date, corresponding to symmetric and asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold compactifications, represent some of the most realistic String models consisting of 3 generations. The early semi-realistic examples built since the late eighties were composed of asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold compactifications. These models corresponded to compactifications with $\mathcal{N} = (2, 0)$ super-conformal worldsheet symmetry. Here, the observable symmetry was taken to be a E_8 gauge group and then broken down to a specific $SO(10)$ symmetry subgroup. The cases consisted of $SU(5) \times U(1)$ (Flipped $SU(5)$) [21, 23], $SU(3) \times SU(2) \times U(1)^2$ (Standard-Like) [47, 48, 49, 50, 51], $SU(3) \times SU(2)^2 \times U(1)$ (Left-Right symmetric) [55, 56, 57], and $SO(6) \times SO(4)$ (Pati-Salam) [58, 59]. Some of these models shared the NAHE-based structure [60], which was used to develop the contemporary research in the Free-Fermionic model building; focusing on exploring large classes of String vacua. Towards the end of the nineties, tools for the classification of the Free-Fermionic symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifolds were derived for type II superstrings [61], during the last decade, this was extended in the Heterotic-String construction [35, 36, 37, 38, 39].

The classification of the Heterotic-String vacua with the unbroken E_6 and $SO(10)$ GUT gauge groups, revealed the existence of a symmetry called the spinor-vector duality in the space of $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ String models. This symmetry under the $SO(10)$ gauge group, generates the exchange of the spinorial **16** plus anti-spinorial $\overline{\mathbf{16}}$ with the vectorial **10** representations [37, 38, 39, 53, 54, 62, 63, 64, 65, 66]. The classification was then extended, where the $SO(10)$ symmetry was broken to the Pati-Salam subgroup in [31, 32]. This revealed that exophobic String vacua existed for all the sectors containing the massless states, where the exotic fractionally charged fermions appeared only in the massive spectrum. A 3 generation Pati-Salam

model studied in [33], was shown to be phenomenologically viable. Consequently, the classification method was employed to enhance the Pati-Salam gauge group in [33] to the $SU(6) \times SU(2)$ [67] models, which is the maximal subgroup of the E_6 symmetry. Here, an exophobic model was found that admitted an additional anomaly free family universal $U(1)$ symmetry beyond the $U(1)$ generators of the $SO(10)$ GUT gauge group [68, 69, 70, 71]. However, further studies involving the classifications of the Flipped $SU(5)$ [5] and the $SU(4) \times SU(2) \times U(1)$ [6] subgroups of $SO(10)$ were shown to contain no 3 generation exophobic String vacua. The Flipped $SU(5)$ models here were shown to only produce exophobic models with even generations, whereas 3 generation models only existed within exophilic vacua. On the other hand, the $SU(4) \times SU(2) \times U(1)$ models were shown to produce no 3 generation models, as all right-handed particles were projected out of the massless spectrum[6].

The methodologies developed in the classifications of [5, 6, 32, 35, 36, 37, 38, 39, 61, 66] provided a vital tool to analyse the phenomenological properties of large classes of String vacua[7]. In this thesis, the techniques used to classify these large classes of String vacua were discussed. Firstly, the $SO(10)$ models were constructed with the required basis vectors consistent with the ABK rules [15, 16], containing only periodic and anti-periodic fermion boundary conditions. Then, the additional $SO(10)$ symmetry breaking basis vectors were added to form the subgroups. The Flipped $SU(5)$ subgroup was discussed in chapter 4, whereas in chapter 5 the $SU(4) \times SU(2) \times U(1)$ subgroup was discussed. Finally, to conclusion, in the following, the landscape of the Free-Fermionic models [7] will be discussed.

Initially, in the Free-Fermionic construction, the $SO(10)$ models were a success, as it was shown to have an abundance of 3 generation models, as given in Figure 6.1. When the classification was done for a random sample of 10^{11} String vacua, discrete properties began to emerge. Here, it was observed that the odd generations above 5 vanished, whereas the even generations above 12 were incremented by 4 integers. With the success of the $SO(10)$ models, the Pati-Salam models were investigated, the classification results are given in Figure 6.2. Here, the Pati-Salam models contained identical discrete properties as the $SO(10)$ models, with the exclusion of all the 24 generations, since they were all projected out. In this case, as the spinorial **16** representation of the $SO(10)$ was broken to the **4** and $\overline{\mathbf{4}}$ representations transforming under the $SU(4)$ of the Pati-Salam, therefore, instead of one, two states were required to complete family of generations. However, the Pati-Salam models also contained many 3 generations that were exophobic. Further to this, the Flipped $SU(5)$ models were

[6]Similarly in [52] it was shown that for the $SU(4) \times SU(2) \times U(1)$ gauge group, 3 generation models are forbidden in NAHE based basis vectors

[7]Other groups have also performed analysis of large sets of String vacua [72]

then explored as discussed in chapter 4. Contrary to the Pati-Salam classification consisting of exophobic 3 generation models, the Flipped $SU(5)$ models contained only 3 generation models with fractionally charged exotic states, in a random sample of 10^{12} String vacua scanned[8], as given in Figure 6.3. An additional property of the Flipped $SU(5)$ models, is that it was more constrained, as the generations following a logarithmic distribution together with all the odd generations being projected out. Finally, the classification of the $SU(4) \times SU(2) \times U(1)$ models were considered. It was revealed that these models were even more constrained than the Flipped $SU(5)$ models. However, this was anticipated, due to there being two $SO(10)$ breaking basis vectors, which forbid complete generations, as given in Figure 6.4. In actual fact, this was a rare occurrence in the Free-Fermionic classifications, as the second $SO(10)$ breaking basis vector was unique and the GGSO projection on the **16** of $SO(10)$ projected out all the right-handed particles. Thus, whole family of generations were incomplete. The next stage will be the classification of the Standard-Like models, which are similar to the $SU(4) \times SU(2) \times U(1)$ models, as they also require two $SO(10)$ breaking basis vectors. This GUT model is a working progress, preliminary scans show interesting results.

To conclude, the current status of the unification of gravity and the gauge interactions are heavily motivated by String derived models and theories, which continue to provide a viable contemporary framework. Consequently, 3 generation models need to be obtained for phenomenological purposes, however, a detailed example is still work in progress. Nevertheless, String theory provides a sea of well established semi-realistic examples that are explored as toy models for achieving a theory of everything.

[8]Note: although this does not prove that in the Free-Fermionic Flipped $SU(5)$ vacua, exophobic models do not exist, they do not exist in the space of 10^{12} vacua explored with the given basis vector in equation (4.1.3.2).

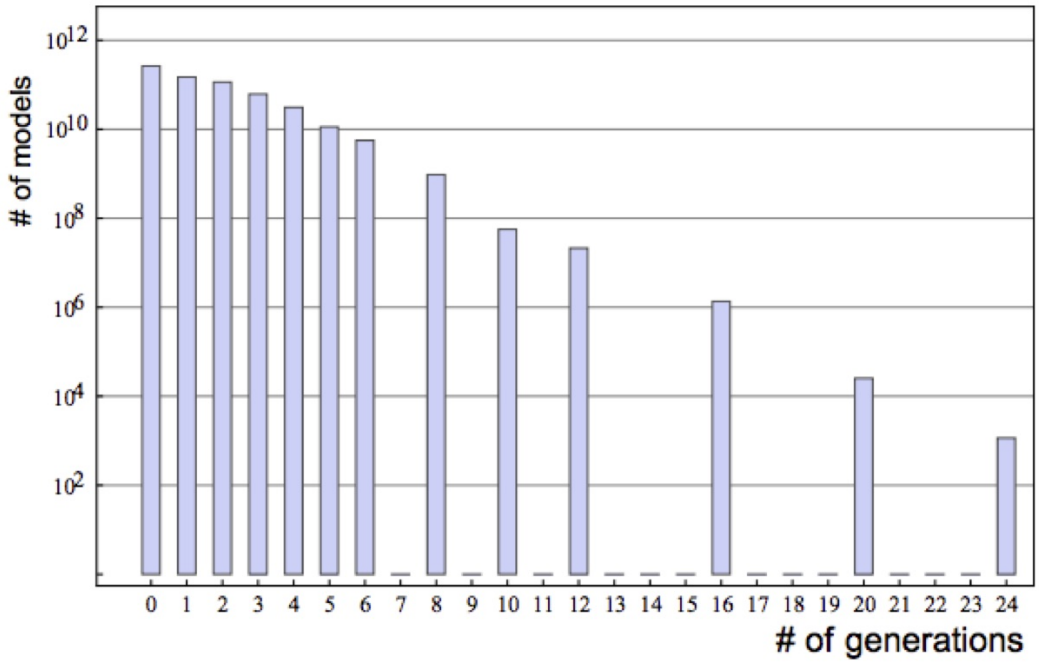

Figure 6.1: *Number of exophobic models in relation to the number of generations, in a random sample of* 10^{11} $SO(10)$ *configurations, as given in Figure 1 in [66].*

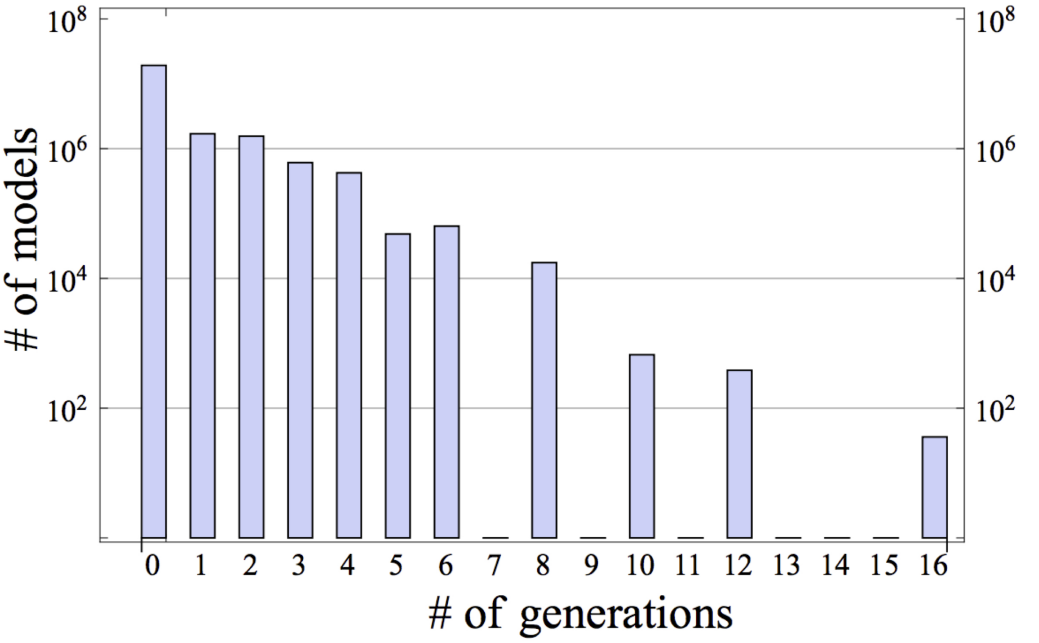

Figure 6.2: *Number of exophobic models in relation to the number of generations, in a random sample of* 10^{11} *Pati-Salam configurations, as given in Figure 3 in [32].*

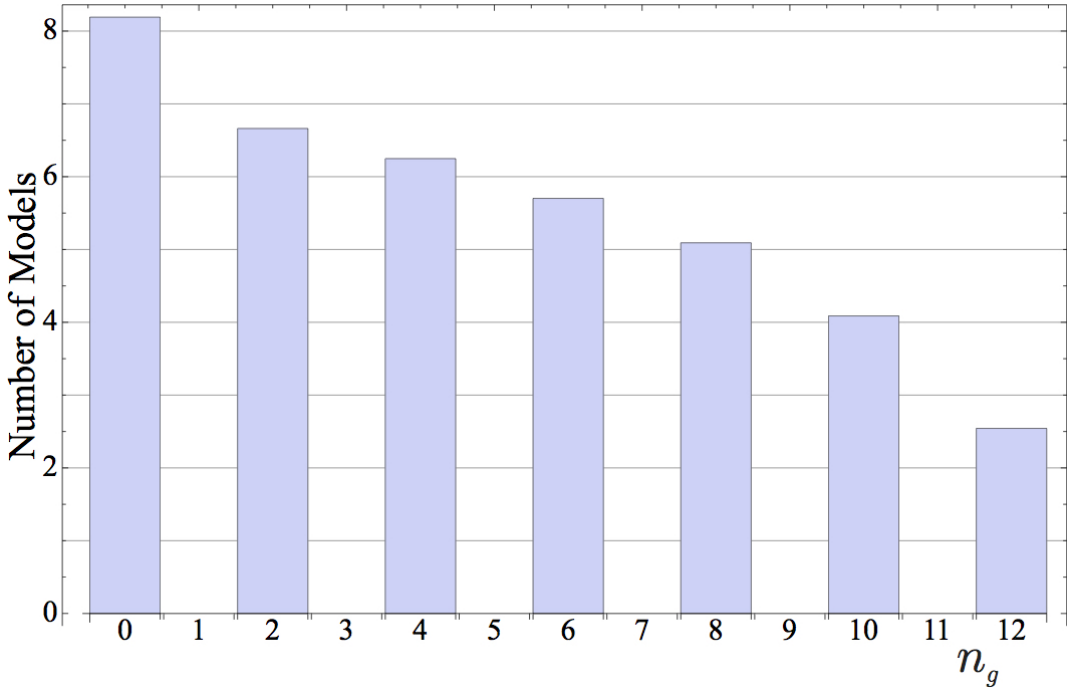

Figure 6.3: *Logarithm of the number of exophobic models in relation to the number of generations, in a random sample of* 10^{12} $SU(5) \times U(1)$ *configurations.*

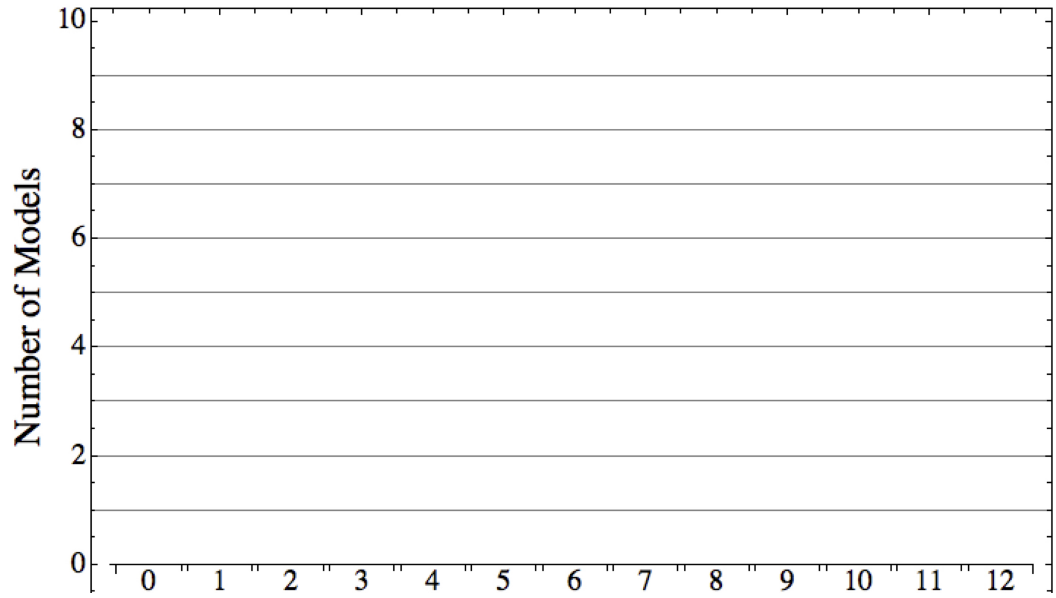

Figure 6.4: *Number of models in relation to the number of generations, in the* $SU(4) \times SU(2) \times U(1)$ *vacua.*

Appendix A

Projectors and Matrix Formalism

The algebraic expressions corresponding to states in the Flipped $SU(5)$ String spectrum are given by: $B_{pqrs}^{(4,5,6)}$ from (4.2.1.2), which produce light Higgs and hidden vectorial states; $B_{pqrs}^{(7,8,9)}$ and $B_{pqrs}^{(10,11,12)}$ given in (4.2.2.1) and (4.2.2.2) respectively, which produce spinorial hidden matter states; $B_{pqrs}^{(13,14,15)}$ and $B_{pqrs}^{(16,17,18)}$ given in (4.2.3.1) and (4.2.3.1) respectively, which produce spinorial exotic states; $B_{pqrs}^{(19,20,21)}$ and $B_{pqrs}^{(22,23,24)}$ given in (4.2.3.1) and (4.2.3.2) respectively, which produce vectorial exotic states. Now the enumeration of these projectors with their corresponding algebraic expressions and matrix equations are given as follows.

A.1 Vectorial Representations

The sectors in (4.2.1.2) produce vectorial states in the observable and hidden sector. These sectors are obtained from the combinations

$$B_{pqrs}^{(4,5,6)} = B_{pqrs}^{(1,2,3)} + z_1 + 2\alpha$$

The following is a list of the states produced in these sectors and the projectors that act on them:

- **States:** $\{\overline{\eta}^{1,2,3}\}|R\rangle$, $\{\overline{\eta}^{*1,2,3}\}|R\rangle$, $\{\overline{\psi}^{1,\ldots,5}\}|R\rangle$ **and** $\{\overline{\psi}^{*1,\ldots,5}\}|R\rangle$

  This gives rise to the states that transform under the $SU(5)\times U(1)_5$ or $U(1)_{1/2/3}$ gauge group. The projectors are given by:

$$\begin{aligned}
P^{(4)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_1}{B^{(4)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(4)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(4)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(4)}_{pqrs}}\right)\\
P^{(5)(\overline{\eta}^2,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_3}{B^{(5)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(5)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(5)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(5)}_{pqrs}}\right)\\
P^{(6)(\overline{\eta}^3,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_5}{B^{(6)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(6)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(6)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2}{B^{(6)}_{pqrs}}\right)
\end{aligned}$$

  The corresponding matrix equations are given as:

$$\begin{pmatrix}(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_1|b_1+z_1)\\(e_2|b_1+z_1)\\(z_1|b_1+z_1)\\(z_2|b_1+z_1)+1\end{pmatrix}$$

$$\begin{pmatrix}(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_3|b_2+z_1)\\(e_4|b_2+z_1)\\(z_1|b_2+z_1)\\(z_2|b_2+z_1)+1\end{pmatrix}$$

$$\begin{pmatrix}(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_5|b_1+b_2)\\(e_6|b_1+b_2)\\(z_1|b_1+b_2)\\(z_2|b_1+b_2)\end{pmatrix}$$

- **States:** $\{\overline{\phi}^{1,\ldots,4}\}|R\rangle$ **and** $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle$

  These states transform under the $SU(4) \times U(1)_4$ hidden gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(4)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right) \\
P^{(5)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_3 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_4 \\ B^{(5)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2 \\ B^{(5)}_{pqrs}\end{pmatrix}\right) \\
P^{(6)(\overline{\phi}^{1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1 - C\begin{pmatrix} e_5 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_6 \\ B^{(6)}_{pqrs}\end{pmatrix}\right) \\
&\quad\cdot\left(1 + C\begin{pmatrix} z_1 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_2 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1) \\ (e_2|b_1+z_1) \\ (z_1|b_1+z_1)+1 \\ (z_2|b_1+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1) \\ (e_4|b_2+z_1) \\ (z_1|b_2+z_1)+1 \\ (z_2|b_2+z_1)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\
(z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_1+b_2) \\ (e_6|b_1+b_2) \\ (z_1|b_1+b_2)+1 \\ (z_2|b_1+b_2)
\end{pmatrix}
$$

- **State:** $\{\overline{\phi}^{5,\ldots,8}\}|R\rangle$ **and** $\{\overline{\phi}^{*5,\ldots,8}\}|R\rangle$

  These states transform under the $U(1)_{hid}$ or $SO(6)$ gauge groups. The projectors on these states are given by:

$$
\begin{aligned}
P^{(4)(\overline{\phi}^{5,\ldots,8})}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(4)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 + C\begin{pmatrix} z_2 \\ B^{(4)}_{pqrs}\end{pmatrix}\right) \\
P^{(5)(\overline{\phi}^{5,\ldots,8})}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_3 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_4 \\ B^{(5)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(5)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 + C\begin{pmatrix} z_2 \\ B^{(5)}_{pqrs}\end{pmatrix}\right) \\
P^{(6)(\overline{\phi}^{5,\ldots,8})}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_5 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_6 \\ B^{(6)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 + C\begin{pmatrix} z_2 \\ B^{(6)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+z_1) \\ (e_2|b_1+z_1) \\ (z_1|b_1+z_1) \\ (z_2|b_1+z_1) \end{pmatrix}
$$

$$
\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+z_1) \\ (e_4|b_2+z_1) \\ (z_1|b_2+z_1) \\ (z_2|b_2+z_1) \end{pmatrix}
$$

$$
\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1+b_2) \\ (e_6|b_1+b_2) \\ (z_1|b_1+b_2) \\ (z_2|b_1+b_2)+1 \end{pmatrix}
$$

A.2 Hidden Sector Representations

The sectors $B^{(1,2,3)}_{pqrs} + 2\alpha$ and $B^{(1,2,3)}_{pqrs} + z_1 + z_2 + 2\alpha$ give rise to non–exotic states that transform under the hidden gauge group. The states in these sectors descend from the **16** vectorial representation of the hidden $SO(16)$ gauge group, decomposed under the final unbroken hidden sector gauge group. The sectors

$$B^{(7,8,9)}_{pqrs} = B^{(1,2,3)}_{pqrs} + 2\alpha$$

produce states that transform under the $SU(4) \times U(1)_4$ hidden gauge group. The projectors on states arising in these sectors are given by:

$$\begin{aligned}
P^{(7)}_{pqrs} &= \frac{1}{8}\left(1 - C\binom{e_1}{B^{(7)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_2}{B^{(7)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(7)}_{pqrs}}\right)\\
P^{(8)}_{pqrs} &= \frac{1}{8}\left(1 - C\binom{e_3}{B^{(8)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_4}{B^{(8)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(8)}_{pqrs}}\right)\\
P^{(9)}_{pqrs} &= \frac{1}{8}\left(1 - C\binom{e_5}{B^{(9)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_6}{B^{(9)}_{pqrs}}\right)\cdot\left(1 - C\binom{z_2}{B^{(9)}_{pqrs}}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix} (e_1|b_1)\\ (e_2|b_1)\\ (z_2|b_1)+1 \end{pmatrix}$$

$$\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix} (e_3|b_2)\\ (e_4|b_2)\\ (z_2|b_2)+1 \end{pmatrix}$$

$$\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix} (e_5|b_1+b_2+z_1)\\ (e_6|b_1+b_2+z_1)\\ (z_2|b_1+b_2+z_1) \end{pmatrix}$$

The sectors

$$B^{(10,11,12)}_{pqrs} = B^{(1,2,3)}_{pqrs} + z_1 + z_2 + 2\alpha$$

give rise to states that transform under the hidden $SO(6)$ gauge group. The projectors acting on these states are given by:

$$\begin{aligned}
P^{(10)}_{pqrs} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_1 \\ B^{(10)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_2 \\ B^{(10)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(10)}_{pqrs}\end{pmatrix}\right) \\
P^{(11)}_{pqrs} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_3 \\ B^{(11)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_4 \\ B^{(11)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(11)}_{pqrs}\end{pmatrix}\right) \\
P^{(12)}_{pqrs} &= \frac{1}{8}\left(1 - C\begin{pmatrix} e_5 \\ B^{(12)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} e_6 \\ B^{(12)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - C\begin{pmatrix} z_1 \\ B^{(12)}_{pqrs}\end{pmatrix}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + z_1 + z_2) \\ (e_2|b_1 + z_1 + z_2) \\ (z_1|b_1 + z_1 + z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2 + z_1 + z_2) \\ (e_4|b_2 + z_1 + z_2) \\ (z_1|b_2 + z_1 + z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1 + b_2 + z_2) \\ (e_6|b_1 + b_2 + z_2) \\ (z_1|b_1 + b_2 + z_2) \end{pmatrix}$$

A.3 Exotics Sector Representations

The exotic states are obtained from sectors containing the $SO(10)$ breaking vector α. As mentioned in section 4.2.3, the sectors that give rise to exotic states are classified according to their vacuum in the right–moving sector. For a given sector ξ with $\xi_R \cdot \xi_R = 6$, a right–moving oscillator of a world–sheet fermion with $\pm 1/4$ boundary conditions acting on the vacuum is needed to obtain a massless state. Sectors with $\xi_R \cdot \xi_R = 8$ produce massless states without an oscillator. The first type of sectors can therefore produce states that transform as $\mathbf{5}$ and $\overline{\mathbf{5}}$, as well as states that transform as singlets under the observable $SU(5)$ gauge group. The second type of sectors gives rise to states that transform as singlets of the observable $SU(5)$ gauge symmetry. All the exotic states transform into the standard representations under the observable $SU(5)$ gauge group (including singlets) carrying exotic charges under the observable $U(1)_5$ gauge group. The sectors

$$B_{pqrs}^{(13,14,15)} = B_{pqrs}^{(1,2,3)} + z_2 + \alpha$$

produce states that transform under the $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the $SO(6)$ hidden gauge group. The projectors acting on these states are given by:

$$\begin{aligned}
P_{pqrs}^{(13)} = \frac{1}{16} &\left(1 - C\binom{e_1}{B_{pqrs}^{(13)}}\right) \cdot \left(1 - C\binom{e_2}{B_{pqrs}^{(13)}}\right) \\
&\left(1 + C\binom{z_1}{B_{pqrs}^{(13)}}\right) \cdot \left(1 - C\binom{\alpha}{B_{pqrs}^{(13)}}\right) \\
P_{pqrs}^{(14)} = \frac{1}{16} &\left(1 - C\binom{e_3}{B_{pqrs}^{(14)}}\right) \cdot \left(1 - C\binom{e_4}{B_{pqrs}^{(14)}}\right) \\
&\left(1 + C\binom{z_1}{B_{pqrs}^{(14)}}\right) \cdot \left(1 - C\binom{\alpha}{B_{pqrs}^{(14)}}\right) \\
P_{pqrs}^{(15)} = \frac{1}{16} &\left(1 - C\binom{e_5}{B_{pqrs}^{(15)}}\right) \cdot \left(1 - C\binom{e_6}{B_{pqrs}^{(15)}}\right) \\
&\left(1 + C\binom{z_1}{B_{pqrs}^{(15)}}\right) \cdot \left(1 - C\binom{\alpha}{B_{pqrs}^{(15)}}\right)
\end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+z_2+\alpha) \\ (e_2|b_1+z_2+\alpha) \\ (z_1|b_1+z_2+\alpha)+1 \\ (\alpha|b_1+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+z_2+\alpha) \\ (e_4|b_2+z_2+\alpha) \\ (z_1|b_2+z_2+\alpha)+1 \\ (\alpha|b_2+z_2+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3+z_2+\alpha) \\ (e_6|b_3+z_2+\alpha) \\ (z_1|b_3+z_2+\alpha)+1 \\ (\alpha|b_3+z_2+\alpha) \end{pmatrix}$$

Similar exotic states are produced from the sectors:

$$B^{(16,17,18)}_{pqrs} = B^{(1,2,3)}_{pqrs} + z_1 + z_2 + \alpha$$

The projectors acting on these states are given by:

$$\begin{aligned} P^{(16)}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_1 \\ B^{(16)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_2 \\ B^{(16)}_{pqrs} \end{pmatrix}\right) \\ &\left(1 + C\begin{pmatrix} z_1 \\ B^{(16)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} \alpha \\ B^{(16)}_{pqrs} \end{pmatrix}\right) \\ P^{(17)}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_3 \\ B^{(17)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_4 \\ B^{(17)}_{pqrs} \end{pmatrix}\right) \\ &\left(1 + C\begin{pmatrix} z_1 \\ B^{(17)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} \alpha \\ B^{(17)}_{pqrs} \end{pmatrix}\right) \\ P^{(18)}_{pqrs} = \frac{1}{16} &\left(1 - C\begin{pmatrix} e_5 \\ B^{(18)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - C\begin{pmatrix} e_6 \\ B^{(18)}_{pqrs} \end{pmatrix}\right) \\ &\left(1 + C\begin{pmatrix} z_1 \\ B^{(18)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 + C\begin{pmatrix} \alpha \\ B^{(18)}_{pqrs} \end{pmatrix}\right) \end{aligned}$$

The corresponding matrix equations are given as:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+z_1+z_2+\alpha) \\ (e_2|b_1+z_1+z_2+\alpha) \\ (z_1|b_1+z_1+z_2+\alpha)+1 \\ (\alpha|b_1+z_1+z_2+\alpha)+1 \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2+z_1+z_2+\alpha) \\ (e_4|b_2+z_1+z_2+\alpha) \\ (z_1|b_2+z_1+z_2+\alpha)+1 \\ (\alpha|b_2+z_1+z_2+\alpha)+1 \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_1+b_2+z_2+\alpha) \\ (e_6|b_1+b_2+z_2+\alpha) \\ (z_1|b_1+b_2+z_2+\alpha)+1 \\ (\alpha|b_1+b_2+z_2+\alpha)+1 \end{pmatrix}$$

The sectors

$$B_{pqrs}^{(19,20,21)} = B_{pqrs}^{(1,2,3)} + \alpha$$

produce massless states that are obtained by acting on the vacuum with a fermionic oscillator. Below is the list of the type of states that are produced and the projectors that act on them.

- **States:** $\{\overline{\eta}^1\}|R\rangle$ **and** $\{\overline{\psi}^{1,\ldots,5}\}|R\rangle$

  These transform as either singlets or **5** or $\overline{\mathbf{5}}$ under the observable $SU(5)$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(19)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(19)}_{pqrs}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(19)}_{pqrs}}\right)\\
P^{(20)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_3}{B^{(20)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(20)}_{pqrs}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B^{(20)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(20)}_{pqrs}}\right)\\
P^{(21)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_5}{B^{(21)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(21)}_{pqrs}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B^{(21)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(21)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+\alpha)\\
(e_2|b_1+\alpha)\\
(z_1|b_1+\alpha)+1\\
(z_2|b_1)+(\alpha|b_1+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+\alpha)\\
(e_4|b_2+\alpha)\\
(z_1|b_2+\alpha)+1\\
(z_2|b_2)+(\alpha|b_2+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s\end{pmatrix}
=
\begin{pmatrix}
(e_5|b_3+\alpha)\\
(e_6|b_3+\alpha)\\
(z_1|b_3+\alpha)+1\\
(z_2|b_3)+(\alpha|b_3+z_2+\alpha)+1
\end{pmatrix}
$$

  where $\delta = z_2+\alpha$.

- **States:** $\{\overline{\eta}^{*2,3}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and are charged under $U(1)_{2/3}$. The projectors are given by:

$$\begin{aligned}
P_{pqrs}^{(19)(\overline{\eta}^{*2,3})} &= \frac{1}{16}\left(1-C\binom{e_1}{B_{pqrs}^{(19)}}\right)\cdot\left(1-C\binom{e_2}{B_{pqrs}^{(19)}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B_{pqrs}^{(19)}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B_{pqrs}^{(19)}}\right)\\
P_{pqrs}^{(20)(\overline{\eta}^{*2,3})} &= \frac{1}{16}\left(1-C\binom{e_3}{B_{pqrs}^{(20)}}\right)\cdot\left(1-C\binom{e_4}{B_{pqrs}^{(20)}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B_{pqrs}^{(20)}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B_{pqrs}^{(20)}}\right)\\
P_{pqrs}^{(21)(\overline{\eta}^{*2,3})} &= \frac{1}{16}\left(1-C\binom{e_5}{B_{pqrs}^{(21)}}\right)\cdot\left(1-C\binom{e_6}{B_{pqrs}^{(21)}}\right)\\
&\quad\cdot\left(1+C\binom{z_1}{B_{pqrs}^{(21)}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B_{pqrs}^{(21)}}\right)
\end{aligned}$$

  The corresponding matrix equations are given as:

$$\begin{pmatrix}(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_1|b_1+\alpha)\\ (e_2|b_1+\alpha)\\ (z_1|b_1+\alpha)+1\\ (z_2|b_1)+(\alpha|b_1+z_2+\alpha)\end{pmatrix}$$

$$\begin{pmatrix}(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_3|b_2+\alpha)\\ (e_4|b_2+\alpha)\\ (z_1|b_2+\alpha)+1\\ (z_2|b_2)+(\alpha|b_2+z_2+\alpha)\end{pmatrix}$$

$$\begin{pmatrix}(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_5|b_3+\alpha)\\ (e_6|b_3+\alpha)\\ (z_1|b_3+\alpha)+1\\ (z_2|b_3)+(\alpha|b_3+z_2+\alpha)\end{pmatrix}$$

  where $\delta = z_2 + \alpha$.

- **States:** $\{\overline{\phi}^{*1,\ldots,4}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and transform in non–trivial representation of the hidden $SU(4)\times U(1)_4$ gauge group. The projectors are given by:

$$\begin{aligned}
P^{(19)(\overline{\phi}^{*1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(19)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(19)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B^{(19)}_{pqrs}}\right)\\
P^{(20)(\overline{\phi}^{*1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_3}{B^{(20)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(20)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(20)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B^{(20)}_{pqrs}}\right)\\
P^{(21)(\overline{\phi}^{*1,\ldots,4})}_{pqrs} &= \frac{1}{16}\left(1-C\binom{e_5}{B^{(21)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(21)}_{pqrs}}\right)\\
&\quad\cdot\left(1-C\binom{z_1}{B^{(21)}_{pqrs}}\right)\cdot\left(1-C\binom{z_2+\alpha}{B^{(21)}_{pqrs}}\right)
\end{aligned}$$

  The corresponding matrix equations are given as:

$$\begin{pmatrix}(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_1|b_1+\alpha)\\(e_2|b_1+\alpha)\\(z_1|b_1+\alpha)\\(z_2|b_1)+(\alpha|b_1+z_2+\alpha)\end{pmatrix}$$

$$\begin{pmatrix}(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_3|b_2+\alpha)\\(e_4|b_2+\alpha)\\(z_1|b_2+\alpha)\\(z_2|b_2)+(\alpha|b_2+z_2+\alpha)\end{pmatrix}$$

$$\begin{pmatrix}(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix}=\begin{pmatrix}(e_5|b_3+\alpha)\\(e_6|b_3+\alpha)\\(z_1|b_3+\alpha)\\(z_2|b_3)+(\alpha|b_3+z_2+\alpha)\end{pmatrix}$$

  where $\delta = z_2+\alpha$.

The remaining sectors

$$B^{(22,23,24)}_{pqrs} = B^{(1,2,3)}_{pqrs} + z_1 + \alpha$$

produce the following vector-like states:

- **States:** $\{\overline{\eta}^1\}|R\rangle$ **and** $\{\overline{\psi}^{1,\ldots,5}\}|R\rangle$

  These transform as either singlets or $\mathbf{5}$ or $\overline{\mathbf{5}}$ under the observable $SU(5)$ gauge group. The projectors are given by:

$$
\begin{aligned}
P_{pqrs}^{(22)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})} &= \frac{1}{16}\left(1-C\begin{pmatrix}e_1\\B_{pqrs}^{(22)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}e_2\\B_{pqrs}^{(22)}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix}z_1\\B_{pqrs}^{(22)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}z_2+\alpha\\B_{pqrs}^{(22)}\end{pmatrix}\right)\\
P_{pqrs}^{(23)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})} &= \frac{1}{16}\left(1-C\begin{pmatrix}e_3\\B_{pqrs}^{(23)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}e_4\\B_{pqrs}^{(23)}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix}z_1\\B_{pqrs}^{(23)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}z_2+\alpha\\B_{pqrs}^{(23)}\end{pmatrix}\right)\\
P_{pqrs}^{(24)(\overline{\eta}^1,\overline{\psi}^{1,\ldots,5})} &= \frac{1}{16}\left(1-C\begin{pmatrix}e_5\\B_{pqrs}^{(24)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}e_6\\B_{pqrs}^{(24)}\end{pmatrix}\right)\\
&\quad\cdot\left(1+C\begin{pmatrix}z_1\\B_{pqrs}^{(24)}\end{pmatrix}\right)\cdot\left(1-C\begin{pmatrix}z_2+\alpha\\B_{pqrs}^{(24)}\end{pmatrix}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_1|b_1+z_1+\alpha)\\ (e_2|b_1+z_1+\alpha)\\ (z_1|b_1+z_1+\alpha)+1\\ (z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)\end{pmatrix}
$$

$$
\begin{pmatrix}(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_3|b_2+z_1+\alpha)\\ (e_4|b_2+z_1+\alpha)\\ (z_1|b_2+z_1+\alpha)+1\\ (z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)\end{pmatrix}
$$

$$
\begin{pmatrix}(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\ (\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}(e_5|b_3+z_1+\alpha)\\ (e_6|b_3+z_1+\alpha)\\ (z_1|b_3+z_1+\alpha)+1\\ (z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)+1\end{pmatrix}
$$

  where $\delta = z_2 + \alpha$.

- **States** $\{\overline{\eta}^{*2,3}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and are charged under $U(1)_{2/3}$. The projectors are given by:

$$
\begin{aligned}
P^{(22)(\overline{\eta}^{*2,3})}_{pqrs} = \frac{1}{16}&\left(1 - C\binom{e_1}{B^{(22)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_2}{B^{(22)}_{pqrs}}\right)\\
&\cdot\left(1 + C\binom{z_1}{B^{(22)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(22)}_{pqrs}}\right)\\
P^{(23)(\overline{\eta}^{*2,3})}_{pqrs} = \frac{1}{16}&\left(1 - C\binom{e_3}{B^{(23)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_4}{B^{(23)}_{pqrs}}\right)\\
&\cdot\left(1 + C\binom{z_1}{B^{(23)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(23)}_{pqrs}}\right)\\
P^{(24)(\overline{\eta}^{*2,3})}_{pqrs} = \frac{1}{16}&\left(1 - C\binom{e_5}{B^{(24)}_{pqrs}}\right)\cdot\left(1 - C\binom{e_6}{B^{(24)}_{pqrs}}\right)\\
&\cdot\left(1 + C\binom{z_1}{B^{(24)}_{pqrs}}\right)\cdot\left(1 + C\binom{z_2+\alpha}{B^{(24)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix}
(e_1|b_1+z_1+\alpha)\\
(e_2|b_1+z_1+\alpha)\\
(z_1|b_1+z_1+\alpha)+1\\
(z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix}
(e_3|b_2+z_1+\alpha)\\
(e_4|b_2+z_1+\alpha)\\
(z_1|b_2+z_1+\alpha)+1\\
(z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix} =
\begin{pmatrix}
(e_5|b_3+z_1+\alpha)\\
(e_6|b_3+z_1+\alpha)\\
(z_1|b_3+z_1+\alpha)+1\\
(z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)
\end{pmatrix}
$$

  where $\delta = z_2 + \alpha$.

- **States:** $\{\overline{\phi}^{1,\ldots,4}\}|R\rangle$

  These transform as singlets under the observable $SU(5)$ gauge group and transform in non–trivial representations of the the hidden $SU(4)\times U(1)_4$ gauge group. The projectors are given by:

$$
\begin{aligned}
P^{(22)(\overline{\phi}^{1,\ldots,4})}_{pqrs} = \frac{1}{16}&\left(1-C\binom{e_1}{B^{(22)}_{pqrs}}\right)\cdot\left(1-C\binom{e_2}{B^{(22)}_{pqrs}}\right)\\
&\cdot\left(1-C\binom{z_1}{B^{(22)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(22)}_{pqrs}}\right)\\
P^{(23)(\overline{\phi}^{1,\ldots,4})}_{pqrs} = \frac{1}{16}&\left(1-C\binom{e_3}{B^{(23)}_{pqrs}}\right)\cdot\left(1-C\binom{e_4}{B^{(23)}_{pqrs}}\right)\\
&\cdot\left(1-C\binom{z_1}{B^{(23)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(23)}_{pqrs}}\right)\\
P^{(24)(\overline{\phi}^{1,\ldots,4})}_{pqrs} = \frac{1}{16}&\left(1-C\binom{e_5}{B^{(24)}_{pqrs}}\right)\cdot\left(1-C\binom{e_6}{B^{(24)}_{pqrs}}\right)\\
&\cdot\left(1-C\binom{z_1}{B^{(24)}_{pqrs}}\right)\cdot\left(1+C\binom{z_2+\alpha}{B^{(24)}_{pqrs}}\right)
\end{aligned}
$$

  The corresponding matrix equations are given as:

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6)\\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6)\\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_3) & (\delta|e_4) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1+z_1+\alpha)\\
(e_2|b_1+z_1+\alpha)\\
(z_1|b_1+z_1+\alpha)\\
(z_2|b_1+z_1)+(\alpha|b_1+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6)\\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_5) & (\delta|e_6)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix}
(e_3|b_2+z_1+\alpha)\\
(e_4|b_2+z_1+\alpha)\\
(z_1|b_2+z_1+\alpha)\\
(z_2|b_2+z_1)+(\alpha|b_2+z_1+z_2+\alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
(e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4)\\
(e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4)\\
(z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4)\\
(\delta|e_1) & (\delta|e_2) & (\delta|e_3) & (\delta|e_4)
\end{pmatrix}
\begin{pmatrix} p\\ q\\ r\\ s \end{pmatrix}
=
\begin{pmatrix}
(e_5|b_3+z_1+\alpha)\\
(e_6|b_3+z_1+\alpha)\\
(z_1|b_3+z_1+\alpha)\\
(z_2|b_1+b_2)+(\alpha|b_1+b_2+z_2+\alpha)+1
\end{pmatrix}
$$

  where $\delta = z_2+\alpha$.

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