Variational Inequalities and Optimization Problems

Thesis submitted in accordance with the requirements of
the Department of Mathematical Sciences of the
University of Liverpool for the degree of Doctor in Philosophy
by

Yina Liu

October 5, 2015

Abstract

The main purpose of this thesis is to study weakly sharp solutions of a variational inequality and its dual problem. Based on these, we present finite convergence algorithms for solving a variational inequality problem and its dual problem. We also construct the connection between variational inequalities and engineering problems.

We consider a variational inequality problem on a nonempty closed convex subset of $\mathbb{R}^n$. In order to solve this variational inequality problem, we construct the equivalence between the solution set of a variational inequality and optimization problems by using two gap functions, one is the primal gap function and the other is the dual gap function. We give properties of these two gap functions. We discuss sufficient conditions for the subdifferentiability of the primal gap function of a variational inequality problem. Moreover, we characterize relations between the Gâteaux differentiabilities of primal and dual gap functions. We also build some results for the Lipschitz and locally Lipschitz properties of primal and dual gap functions as well.

Afterwards, several sufficient conditions for the relevant mapping to be constant on the solution set of a variational inequality has been obtained, including the relations between solution sets of a variational inequality and its dual problem as well as the optimal solution sets to gap functions. Based on these, we characterize weak sharpness of the solution set of a variational inequality by its primal gap function g and its dual gap function G. In particular, we apply error bounds of g, G and $g + G$ on C.

We also construct finite convergence of algorithms for solving a variational inequality by considering the convergence of a local projection. We carry out these results in terms of the weak sharpness of solution sets of a variational inequality as well as the error bounds of gap functions of a variational inequality problem.

Keywords. variational inequality, gap functions, Gâteaux differentiability, locally Lipschitz property, weakly sharp solution, error bound, finite convergence of algorithms, projection, image processing

Contents

Notation

VIP(C, F)	the variational inequality problem
DVIP(C, F)	the dual variational inequality problem
C^*	the solution set to VIP(C, F)
C_*	the solution set to DVIP(C, F)
$g(x)$	the primal gap function
$G(x)$	the dual gap function
$f'(x; v)$	directional derivative of f at x in the direction v
$\nabla g(x)$	gradient of g at x
$\partial g(x)$	the subdifferential of g at x
int C	interior of C
$N_C(x)$	the normal cone to C at x
$T_C(x)$	the tangent cone to C at x
A°	the polar set of A
$d_C(x)$	the distance from x to C
$P_C(x)$	projection of x onto C
$\|v\|$	Euclidean norm
dom$f$	$\{x \in \mathbb{R}^n : f(x) < +\infty\}$
$\mathbb{R}_\infty$	$\mathbb{R} \cup \{\pm\infty\}$
$\inf C$	greatest lower bound of C
$\sup C$	least upper bound of C

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. Zili Wu, for his support and help during my Ph.D. study. Six years ago, it was my first time to see Dr. Zili Wu in the class of the module of Engineering Mathematics II. From his teaching, I fell in love with mathematics and I decided to involve in mathematics in the future. Fortunately, I successfully applied for a Ph.D. program supervised by him in the Department of Mathematical Sciences of the University of Liverpool. From September 2011, I began conducting research following him based at Xi'an Jiaotong-Liverpool University. During these four years, his cautious attitudes towards science have inspired me a lot and it has totally changed my attitudes for work. Under his guidance, I learnt not only the knowledge but also the way of thinking. In addition, with the support of Dr. Zili Wu, I have the chance to attend the fourth Asian conference on nonlinear analysis and optimization which was held in Taipei to exchange ideas with researchers from all over the world. I offer my deepest appreciation to him for all of his patient advice, warm help and comments and suggestions to the papers written over last four years.

I would like to thank all the teachers during my Ph.D. period. In particular, Dr. Alexei B. Piunovskiy, my co-supervisor from the University of Liverpool, gave me a lot of help when I was in United Kingdom for a three-month research visit. Dr. Keith Nuttall, previous academic staff in the Department of Electrical and Electronic Engineering of Xi'an Jiaotong-Liverpool University, taught me many modules during my undergraduate study. With his great support and beautiful reference letter, I could apply for the Ph.D. easily. Prof. Tammam Tillo, supervisor of my final year project of the undergraduate study, gave me a lot of caring about my research progress during my Ph.D. period.

I would like to thank my friends and the fellow Ph.D. students in our department. My friends, Miss Jin Zheng and Lu Zong, who are also my fellow Ph.D. students, helped me a lot in terms of the living aspect. I would also like to thank Mr. Yichen Liu for answering too many academic questions of mine. I wish to thank Miss Lili Xu from Xi'an Jiaotong University for her encouragement during the early stage of my Ph.D. study.

I would like to thank my parents and grandparents for their support and love. Without their supports, I cannot make such an achievement.

I would like to thank my boyfriend, Mr. Zhen Wen, who is a Ph.D. student of Zhejiang University. He is now studying at Georgia Institute of Technology. I wish to thank his encouragement.

I wish to thank the anonymous referees from two journals, who have given me valuable comments, suggestions and supports. Thanks to their work, we have got a better presentation of certain results.

Last but not least, thanks are also due to Xi'an Jiaotong-Liverpool University for offering me financial support for last four years and to the University of Liverpool for offering me a chance to have a three-month research visit there. I wish to thank academic staffs of Xi'an Jiaotong-Liverpool University, especially Dr. Hayk Mikayelyan, Dr. Yafang Wang and Prof. Tammam Tillo for their recommendation letters for finding a job. I shall always appreciate the research environment of the Department of Mathematical Sciences of Xi'an Jiaotong-Liverpool University and the University of Liverpool.

Chapter 1

Introduction

The first chapter introduces the background of variational inequalities and some results related to their weakly sharp solutions. In particular, we summarize some earlier work of weakly sharp results of variational inequality problems.

1.1 Background of variational inequalities

The subject of variational inequalities could be traced back to the calculus of variations combined with the minimization of infinite-dimensional functions. The systematic study of the subject began in the early 1960s with the influential work of Hartman and Stampacchia [28]. They used a variational inequality [1] as an analytic tool for solving partial differential equations with applications of mechanics in infinite-dimensional spaces. This work was expanded by Stampacchia in some of the earliest papers related to variational inequalities, see [53, 59, 84, 86]. Stampacchia [85] first proved the existence and uniqueness of the solutions of variational inequalities. For the applications of variational inequalities in infinite-dimensional spaces, the reader can refer to the book of Kinderlehrer and Stampacchia [41]. For the detail of a numerical treatment of variational inequalities, the reader can refer to an early book by Glowinski, Lions and Trémolière [23].

The finite-dimensional variational inequality problem is a generalization of the nonlinear complementarity problem (NCP) which is a system consisting of finitely many nonlinear inequalities in finitely many nonnegative variables together with a special equation expressing the complementary relations between the variables and their corresponding inequalities. The systematic study of the finite-dimensional variational inequality began in the mid-1960s and achieved its

[1] To find $u_0 \in \Re \subseteq \mathbb{R}^n$ such that $\langle A(u_0), v - u_0 \rangle \geq \langle f(u_0), v - u_0 \rangle$ for all $v \in \Re$, where $A(u)$ is bounded and linear and f is a mapping from $\Re$ into $\mathbb{R}^n$.

status as a fruitful area of research in mathematical programming. The variational inequality problem was first applied in finite-dimensional spaces by Smith who formulated the traffic assignment problem as a finite-dimensional variational inequality problem in [83]. Actually, Smith did not realize that his formulation was exactly a variational inequality until Dafermos recognized this in [15] in 1980. Since the appearance of these papers, many models of variational inequalities were used in practice, including a rich mathematical theory, some interesting connections to numerous disciplines and a wide range of important applications in engineering and economics. Moreover, variational inequalities provide us with a tool for a wide variety of problems in mathematical programming, including systems of nonlinear equations, optimization problems and fixed point theorems. Variational inequalities are systematically used in many practical problems related to "equilibrium", see [35]. For detailed statement of the theory, algorithms and applications of finite-dimensional variational inequalities, the reader can refer to [16] and [27].

As discussed before, the finite-dimensional variational inequality was born in the domain of mathematical programming. In the following, we give a more detailed introduction of the evolutionary process of the field, covering some of its major events and notable highlights based on a survey reference [16].

In the 1960s, Lemke and Howson [49] formulated an algorithm for solving a bimatrix game as a linear complementarity problem (LCP) and Lemke [48] extended it as a general LCP. Since then much attention has been paid to the study of this general LCP. Cottle, Pang and Stone [13] presented a comprehensive treatment of the LCP in 1992, which contains an extensive bibliography of the LCP up to 1990 and detailed historical accounts of this fundamental problem. Even in recent years, research on the LCP still remains active and its new applications continue to be studied.

In 1967, Scarf [82] presented the first constructive iterative method for approximating a fixed point of a continuous mapping. Because of Scarf's work, the entire area of fixed-point methods developed greatly as well as the computation of economic equilibria. Moreover, his work led the appearance of the field of equilibrium programming. The "equilibrium programming" refers to the modeling, analysis and computation of equilibria via the method of mathematical programming. Clearly, complementarity problems and equilibrium problems have close relations. In essence, all the equilibrium problems which can be solved by fixed-point methods are variational inequalities. The subject of fixed-point homotopy methods dominated much of the early research of equilibrium programming. The major advantage of the methods is their global convergence. Because of this advantage, many scholars have made contributions to this subject. This work has come to a turning point until the fixed-point method to the computation of equilibria is replaced by a contemporary variational inequality. The reader can refer to [96] for the detail of this approach.

In the same period, Karamardian developed an extensive existence theory for the NCP in a series of papers. In particular, he developed the connection between the CP and the variational inequality [39]. In the 1970s, there are a lot of fundamental papers on the variational inequality appearing. Although the early developments of infinite-dimensional variational inequalities and finite-dimensional variational inequalities followed different paths, there are some attempts to bring these two fields closely, see [12]. Hence the 1970s is marked as the beginning of the finite-dimensional variational inequality. In this time period, a large-scale variational inequality appeared which was solved by an iterative algorithm. At the same time, Smith [83] formulated the traffic equilibrium problem as a variational inequality.

The above contents show the developments of the variational inequality. In addition, we also introduce some major events which propelled this subject as a useful discipline in mathematical programming.

In order to solve variational inequalities, there are several popular approaches. The initial methods were known as fixed point methods which were based on Lemke and Howson [49]. The first algorithm to approximate a fixed point of a continuous mapping was proposed by Scarf [82]. Many applications have been made of these methods . In particular, there are a large number of new results for solving variational inequalities by applying the idea of fixed point and iterate methods, see [8, 33, 73, 87, 93, 97, 98, 99].

Another traditional approach to solving variational inequalities is nonlinear optimization, that is, to reformulate the variational inequality problem into equivalent optimization problems. This approach is based on so-called gap or merit functions for variational inequalities. Some related results are due to [1, 2, 20, 32, 52, 63, 80, 89, 91, 95, 100]. For these approaches discussed in the papers above, different classes of gap functions for variational inequalities were applied which were known as primal [47]; dual [63]; regularized [1, 102]; "difference" [80, 95] and Giannessi's [22]. Moreover, the duality aspects for variational inequalities also play an important role in solving variational inequalities. They were first proposed by Mosco [71] and extended by Chen, Goh and Yang in [9].

In addition, variational inequalities can also be solved by proximal point algorithm (PPA) method. This algorithm was introduced by Martinet [64, 65] and developed by Rockafellar [81] to a more general setting, including convex programs and variational inequality problems. It is known that one of the classical iterate schemes of PPA for solving variational inequalities is called the exact PPA. This approach has been studied and extended by many scholars, see [3, 17, 18, 37, 43, 45, 58, 74, 75, 76, 94] and the references therein. However, there are very few results about finite convergence of this algorithm, that is, an exact solution of a variational inequality problem can be found in a finite number of iterations. Rockafellar [81] was the first to prove a solution of the variational inequality problem in a finite number of iterations. However, the assumption of his result is quite strong since it implies that the solution set of the variational inequality problem is a singleton. Based on this result, Luque [58] obtained the same termination property under relaxed conditions that the solution set of the variational inequality problem is not necessarily a singleton. Since then this algorithm has been widely studied, see [4, 5, 18, 25, 40, 44, 60, 92].

Moreover, there are some other algorithms used for solving variational inequalities. The earliest algorithm is the extragradient method introduced by Korpelevich [46] and extended in [36, 38, 61]. The gradient projection method [24, 50] and the hybrid method [34, 72] can also be applied.

In this thesis, we characterize weakly sharp solutions of a variational inequality problem by using its primal and dual gap functions. Some finite convergence algorithms for solving variational inequalities are also included. The primal gap functions were introduced by Auslender [2] and the concepts of the dual gap functions were first presented by Marcotte and Dussault [62]. Any solution to the variational inequality is a global minimum of gap functions, the reader can refer to [21, 29, 47, 78] for the detailed information of gap functions for variational inequalities.

The notion of a sharp, or strongly unique, minimum solution is referred to the work of Cromme [14]. Afterwards, Burke and Ferris [7] defined this notion for the possibility of a non-unique solution set. Burke and Deng [6] extended and refined the results of [7] in a number of ways by weakening some of the assumptions. Based on the inclusion proved by Burke and Ferris [7] for the weakly sharp minima of a function, Patriksson [79] extended this inclusion as a definition of weakly sharp solutions to a variational inequality. According to [7], $\overline{S} \subseteq \mathbb{R}^n$ is said to be a set of weakly sharp minima for the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$

relative to the set $S \subseteq \mathbb{R}^n$ if there exists a positive α such that

$$f(x) \geq f(y) + \alpha d_{\overline{S}}(x) \quad \text{for all } x \in S \text{ and } y \in \overline{S},$$

where $\overline{S} \subseteq S$ and $d_{\overline{S}}(x) = \min\{\|s - x\| : s \in \overline{S}\}$. Therefore, the notion of weakly sharp minima can be interpreted as a type of error bound. Such an estimate is often used in sensitivity analysis of mathematical programming and in convergence analysis of some algorithms. More and more error bound results have appeared in literature since Hoffman [31] showed that a linear inequality system had a global error bound. The existence of error bounds has been studied for nonlinear inequality systems under some conditions. Most of the early results of error bounds were related to a continuous or convex system in $\mathbb{R}^n$, see [51, 56, 57, 88]. We can also refer to the recent independent survey paper of Pang [77] and the references therein for a summary of the theory and applications of error bounds.

1.2 Sharp solutions to a variational inequality

One important application of error bounds concerns about a variational inequality problem VIP(C, F) arising from optimization, in which one finds $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in C,$$

where C is a closed and convex set in $\mathbb{R}^n$ and F is a mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. The solution set C^* of the VIP(C, F) is said to be weakly sharp provided that $-F(x^*) \in \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$ for each $x^* \in C^*$. We say that the dual gap function G has an error bound on C if there exists $\mu > 0$ such that $d_{C^*}(x) \leq \mu G(x)$ for all $x \in C$. Recently, Marcotte and Zhu [63] have presented the sufficiency for the weak sharpness of the solution set of a variational inequality which was in terms of the error bound of the dual gap function $G(x) = \sup\{\langle F(c), x - c\rangle : c \in C\}$ with $x \in \mathbb{R}^n$ under the condition that F is continuous and pseudomonotone$^+$ on the compact set C. These assumptions are too strong and have been relaxed by Zhang, Wan and Xiu [100] to the case that F is continuous and pseudomonotone on C. Under certain conditions expressed by G instead of F, Wu and Wu [89] have further showed that the mapping F is not necessarily continuous to get the same result. Under conditions of the Gâteaux differentiability and locally Lipschitz property of g on C^*, we [55] show that if F is monotone on $\mathbb{R}^n$ and constant on $\Gamma(x^*)$ for some $x^* \in C^*$, then C^* is weakly sharp if and only if

g has an error bound on C, where $g(x) = \sup\{\langle F(x), x - c\rangle : c \in C\}$ and $\Gamma(x) = \{c \in C : \langle F(x), x - c\rangle = g(x)\}$. Moreover, under the condition that F is constant on C^* and $g + G$ is Gâteaux differentiable and locally Lipschitz on C^*, if $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$, then the existence of the error bound of $g + G$ implies the weak sharpness of C^*, see [54].

Based on these results, some finite convergence of algorithms for solving a variational inequality problem have also been investigated. Marcotte and Zhu [63] have shown that $\Gamma(x_n) \subseteq C^*$ for sufficiently large n either with the convergence of the sequence $\{x_n\}$ or the convergence of $d_{C^*}(x_n)$ under some condition. Xiu and Zhang [92] have relaxed the conditions of this result. However, their result still requires the convergence of the sequence $\{x_n\}$. We [55] show that this result still follows even if $\{x_n\}$ is not convergent but with some other restrictions. Meanwhile, Marcotte and Zhu [63] constructed that x_n is a solution to the variational inequality for every large enough n with generating the convergence sequence $\{x_n\}$ under the condition that the sequence $\{x_n\}$ of a local projection goes to zero and F is continuous and pseudomonotone$^+$ on C. Later Xiu and Zhang [92] have refined their result to the case that F is continuous and pseudomonotone on C. This came to a challenging result until Zhou and Wang [101] found that, under conditions of the continuity of F and weak sharpness of C^*, if $\{x_n\} \subseteq C$ is bounded and all its accumulation points belong to C^*, then the convergence of a local projection implies that $x_n \in C^*$ for sufficiently large n. We also show this result but under different conditions. Furthermore, some authors extended finite convergence of algorithms into Banach and Hilbert spaces, see [32, 66, 67, 68, 69].

As described, the work in the thesis builds on the earlier work of Marcotte and Zhu [63], Wu and Wu [89, 90, 91], Xiu and Zhang [92], Zhang, Wan and Xiu [100] and a short paper of Zhou and Wang [101] by extending and refining their results in a number of ways while weakening some of assumptions or obtaining the same results by using different conditions.

1.3 Aim and scope

From existing results about VIP(C, F), we see that the dual gap function G is used more often than the primal gap function g. However, g is usually easier to be calculated than G. This advantage motivates us to consider the following questions in this thesis:

(i) What relations between g and G can be explored and applied for discussing their error bounds' problems?

(ii) Do we have the following relations?

(a) $g(x)$ has an error bound on $C \Rightarrow$ the solution set C^* of VIP(C, F) is weakly sharp?

(b) C_* is weakly sharp $\Rightarrow C^*$ is weakly sharp?

(c) Either $g(x)$ has an error bound or C_* is weakly sharp $\Rightarrow$ there exist some finite convergence results for other algorithms for VIP(C, F)?

(iii) Since $\max\{d_{C^*}(x), d_{C_*}(x)\} \leq d_{C^* \cap C_*}(x)$ for $x \in C$, can we consider

$$d_{C^* \cap C_*}(x) \leq \mu[g(x) + G(x)]$$

for the weak sharpness of C^*?

(iv) If F is monotone, what properties can be obtained for gap functions g and G of VIP(C, F)?

(v) It is known that VIP(C, F) occurs in Engineering. How can we apply the existing results in this field?

This thesis is constructed as follows. We summarize existing results for characterizing properties of the dual gap function in Chapter 3. We present similar properties of the primal gap function under certain conditions. By considering relations between primal and dual gap functions, their Gâteaux differentiabilities and locally Lipschitz properties are also investigated.

The introduced definition for the weak sharpness of the solution set to the variational inequality problem provides a convenient way of exploring relations between the weak sharpness of the solution set to the variational inequality and its dual problem. Based on this definition, our approach to study the weak sharpness results is extended to apply error bounds of both primal and dual gap functions.

As done for characterizing weakly sharp solutions of variational inequalities, some finite convergence of algorithms for solving variational inequalities are proposed as well. Some of the results are shown under relaxed conditions without continuity and pseudomonotonicity of the relevant mapping F.

1.4 Comments on individual chapters

The thesis is organized as follows. It consists of six chapters, followed by an abstract and acknowledgement. The first chapter deals with the study of the background of variational inequalities. We summarize earlier work of weakly sharp solutions of variational inequalities. We describe the aim for our project. Moreover, we show some contributions of the thesis.

In the second chapter, we introduce the construction of a variational inequality problem (VIP(C, F)) and its dual problem (DVIP(C, F)). Afterwards, we show a two-dimensional box constrained VIP(C, F) as an example for calculating the solution set of the VIP(C, F) and that of its dual formulation. For readers' convenience, we refer to several relevant concepts about C and pseudomonotone operators which allow us to apply for the proofs later.

The third chapter is devoted to the characterization of gap functions for the VIP(C, F). First of all, we study the background of the construction of gap functions for VIP(C, F). In particular, we focus on that of primal and dual gap functions. We characterize properties of primal and dual gap functions. The Gâteaux differentiabilities and locally Lipschitz properties of primal and dual gap functions are discussed. We also show the relationships between the Gâteaux differentiabilities and locally Lipschitz properties of primal and dual gap functions.

Afterwards, we focus our attention on the investigation of weakly sharp solutions of VIP(C, F) and its dual problem. In order to describe weakly sharp solutions of VIP(C, F), we introduce its definition given by Patriksson [79] which is obtained based on the work of Burke and Ferris [7]. We study the constancy of F on the solution set of the VIP(C, F) since this is useful for characterizing weak sharpness of the solution sets of VIP(C, F) and its dual problem. In conclusion, the relations between the weak sharpness of the solution sets of the VIP(C, F) and that of its dual problem is constructed based on our extended definition. Some results of weakly sharp solutions of variational inequality problems are presented in terms of error bounds of primal and dual gap functions. Moreover, we also present some equivalent statements of the weak sharpness results.

The fifth chapter concentrates on finite convergence of algorithms for solving variational inequalities. We mainly introduce two finite convergence of algorithms for solving the VIP(C, F). We show that $\arg\min\{\langle F(x_n), y\rangle | y \in C\}$ is a subset of the solution set to the VIP(C, F) for sufficiently large n under certain conditions. For this result, neither the sequence $\{x_n\}$ is required to be convergent nor the relevant mapping F is continuous or pseudomonotone on C. The other algorithm implies that x_n is a solution to the VIP(C, F) for sufficiently large n in terms of a new projection generating x_n going to zero. Finally, these results are considered under equivalent conditions of the weak sharpness of the solution set of the VIP(C, F).

In the last chapter, we summarize our work and draw conclusions and remarks for the thesis. We also list the work needs to be done in the future for fulfilling this project since some results are hard to test using numerical examples. We also point out some applications of variational inequalities, especially for these applied in Engineering, e.g., image processing.

1.5 Contributions of the thesis

In this work, we characterize the weak sharpness of C^* by using the primal gap function g which hasn't been covered before. We also study Lipschitz and locally Lipschitz properties of gap functions g and G since they have seldom been characterized. Moreover, we extend the definition for the weak sharpness of C^* and C_* which makes it more convenient to discuss their relations. We conclude this work by finding some finite convergence of algorithms for solving variational inequalities under mild conditions.

We end this chapter by mentioning that the thesis is based on the following papers written by the author during the period of stay in the Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University and University of Liverpool as a graduate student:

1. Y. N. Liu and Z. L. Wu, "Characterization of weakly sharp solutions of a variational inequality by its primal gap function", Optimization Letters, 2015, DOI $10.1007/s11590-015-0882-7$

2. Y. N. Liu and Z. L. Wu, "Weakly sharp solutions of primal and dual variational inequality problems", to appear in: Pacific Journal of Optimization, 2015.

Chapter 2

Preliminaries

In this chapter, we introduce the construction of a variational inequality and a dual variational inequality. Some definitions, notations and basic results will be used later in the thesis are studied as well.

2.1 A variational inequality problem and its dual variational inequality problem

Throughout the paper, C denotes a nonempty closed convex subset of $\mathbb{R}^n$. For a mapping F from $\mathbb{R}^n$ into $\mathbb{R}^n$, the *variational inequality problem* (VIP(C, F)) is to find a vector $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in C. \tag{2.1}$$

The *dual variational inequality problem* (DVIP(C, F)) is to find a vector $x_* \in C$ such that

$$\langle F(x), x - x_* \rangle \geq 0 \quad \text{for all } x \in C, \tag{2.2}$$

where the *inner product* $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$ is defined as the bi-linear form

$$\langle y, x \rangle := \sum_{i=1}^{n} y_i x_i \quad \text{for any } x, y \in \mathbb{R}^n.$$

The solution set of the (VIP(C, F)) is denoted by C^* and that of the (DVIP(C, F)) by C_*. Obviously,

$$\langle F(x^*), x_* - x^* \rangle = 0 \quad \text{for all } (x^*, x_*) \in C^* \times C_*.$$

Next we refer to a numerical example for calculating C^* and C_*.

Example 2.1.1. We consider a two-dimensional box-constrained variational inequality with $C = [-1,0] \times [-2,-1]$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, $F(x) = (x_1 - 1, x_2 + 1)$.
Then the solution set C^* is to find $x^* = (x_1^*, x_2^*) \in C$ such that

$$(x_1^* - 1)(x_1 - x_1^*) + (x_2^* + 1)(x_2 - x_2^*) \geq 0 \quad \text{for all } (x_1, x_2) \in C,$$

that is, to find $(x_1^*, x_2^*) \in [-1,0] \times [-2,-1]$ such that

$$x_1(x_1^*-1) - x_1^*(x_1^*-1) + x_2(x_2^*+1) - x_2^*(x_2^*+1) \geq 0 \quad \text{for all } (x_1, x_2) \in [-1,0] \times [-2,-1].$$

It follows that $(x_1^*, x_2^*) \in [-1,0] \times [-2,-1]$ can be calculated as

$$-x_1^*(x_1^* - 1) - (x_2^* + 1) - x_2^*(x_2^* + 1) \geq 0.$$

So that $(x_1^* - \frac{1}{2})^2 + (x_2^* + 1)^2 \leq \frac{1}{4}$. Hence $C^* = \{(0,-1)\}$.

The solution set to the DVIP(C, F) C_* aims to find $x_* = (x_{1*}, x_{2*}) \in [-1,0] \times [-2,-1]$ such that

$$(x_1 - 1)(x_1 - x_{1*}) + (x_2 + 1)(x_2 - x_{2*}) \geq 0 \quad \text{for all } (x_1, x_2) \in C,$$

that is, $(x_{1*}, x_{2*}) \in C$ is obtained by

$$\left(x_1 - \frac{x_{1*} + 1}{2}\right)^2 - \frac{(x_{1*} - 1)^2}{4} + \left(x_2 + \frac{1 - x_{2*}}{2}\right)^2 - \frac{(x_{2*} + 1)^2}{4} \geq 0 \quad \text{for all } (x_1, x_2) \in C.$$

Therefore, (x_{1*}, x_{2*}) can be calculated as

$$(0 - \frac{1 + x_{1*}}{2})^2 - \frac{(1 - x_{1*})^2}{4} - \frac{(1 + x_{2*})^2}{4} \geq 0.$$

Thus $4x_{1*} - (x_{2*} + 1)^2 \geq 0$ and hence $C_* = \{(0,-1)\}$.

In this thesis, C^* and C_* are assumed to be nonempty. Obviously $C_* \subseteq C^*$ if F is continuous on C, as can be seen by the proof below.

Proposition 2.1.2. *If F is continuous on C, then $C_* \subseteq C^*$.*

Proof. Let $x \in C$ and $x_* \in C_*$. Suppose that $\{x_n\}$ is a sequence in C with

$$x_n = x_* + \frac{x - x_*}{n} \quad \text{for sufficiently large } n.$$

Then $\langle F(x_n), \frac{x - x_*}{n} \rangle \geq 0$, that is, $\langle F(x_n), x - x_* \rangle \geq 0$. Since F is continuous on C and $x_n \to x_*$, then $F(x_n) \to F(x_*)$, which implies

$$\langle F(x_*), x - x_* \rangle \geq 0.$$

Hence $x_* \in C^*$ and $C_* \subseteq C^*$. $\square$

However, $C_* \subseteq C^*$ does not imply the continuity of F on C which can be implied from the example as follows.

Example 2.1.3. *[91, Example 2.2(iii)]* $C_* \subseteq C^*$ *does not imply the continuity of* F *on* C.
Let $C = [0,1] \times [0,1]$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = \begin{cases} (1,1) & \text{if } (x_1,x_2) = (0,0); \\ (0,0) & \text{if } (x_1,x_2) \neq (0,0). \end{cases}$$

Then the solution set C^* is to find $(x_1^*, x_2^*) \in [0,1] \times [0,1]$ such that (2.1) holds. First, $(x_1^*, x_2^*) = (0,0)$ satisfies (2.1). If $(0,0) \neq (x_1^*, x_2^*) \in C$, then (x_1^*, x_2^*) satisfies (2.1). So $C = C^*$.
Similarly, C_* is to find $(x_{1*}, x_{2*}) \in [0,1] \times [0,1]$ such that (2.1) holds. We first find $x'_* = (x'_{1*}, x'_{2*}) \in [0,1] \times [0,1]$ such that

$$\langle F(x), x - x'_* \rangle \geq 0 \quad \text{for } x = (0,0),$$

that is,

$$-x'_{1*} - x'_{2*} \geq 0.$$

Therefore, $x'_* = (0,0)$. Moreover, $x'_* = (0,0)$ still satisfies

$$\langle F(x), x - x'_* \rangle \geq 0 \quad \text{for all } x \in C \setminus \{(0,0)\}.$$

Hence $C_* = \{(0,0)\}$. We have $C_* = \{(0,0)\} \subseteq C = C^*$ but F is not continuous at $x = (0,0)$.
So $C_* \subseteq C^*$ does not imply that F is continuous on C.

2.2 Definitions and notations

For further discussion, we need several relevant concepts about C and some notations in $\mathbb{R}^n$ as below.

As usual, the *normal cone* to C at $x \in \mathbb{R}^n$ is defined and denoted by

$$N_C(x) := \begin{cases} \{\xi \in \mathbb{R}^n : \langle \xi, c - x \rangle \leq 0 \quad \text{for all } c \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

It is easy to see that $N_C(x)$ is a closed and convex cone and $N_C(x) = \{0\}$ for $x \in \operatorname{int} C$ (interior of C). The *tangent cone* to C at $x \in \mathbb{R}^n$ is defined dually by $T_C(x) := [N_C(x)]^\circ$, where A° denotes the polar set of $A \subseteq \mathbb{R}^n$ given by

$$A^\circ := \{v \in \mathbb{R}^n : \langle v, x \rangle \leq 0 \quad \text{for all } x \in A\}.$$

Since C is closed and convex, Clarke [10] defined the tangent cone to C at $x \in C$ as follows:

$$T_C(x) = \{v \in \mathbb{R}^n : d'_C(x; v) = 0\},$$

where d_C is the *distance function* associated with C as

$$d_C(x) := \min\{\|c - x\| : c \in C\} = \|x - P_C(x)\|.$$

Here $P_C(x)$ denotes the projection of x onto C, that is, the point in C which is closest to x with respect to the Euclidean norm $\|\cdot\|$:

$$P_C(x) := \arg\min\{\|x - y\| : y \in C\},$$

and the *Euclidean norm* $\|\cdot\|$ is captured as

$$\|v\| := \langle v, v\rangle^{\frac{1}{2}} \quad \text{for any } v \in \mathbb{R}^n.$$

According to [30, pp. 46], the closedness of C plays the role of the existence of $P_C(x)$. And the convexity of C ensures the uniqueness of the point $P_C(x)$. What's more, the *projection operator*, that is, $x \to P_C(x)$, came to be an influential work until Hiriart-Urruty and Lemaréchal [30] characterized it as solving a so-called variational inequality as below.

Theorem 2.2.1. *[30, Theorem 3.1.1, pp. 47] A point $y_x \in C$ is the projection $P_C(x)$ if and only if*

$$\langle x - y_x, y - y_x\rangle \le 0 \quad \textit{for all } y \in C.$$

Therefore, the definition above pointed out that the normal cone is related to the problem of finding the projection of a point onto C, i.e., if $z = P_C(x)$, then $x - z \in N_C(z)$.

Based on the definition of the tangent cone given by Clarke [10], he also makes a convenient way for calculating tangents as below.

Theorem 2.2.2. *[10, Theorem 2.4.5] A direction d is tangent to C at $x \in C$ if and only if for every sequence $\{x_k\}$ in C converging to x and a positive sequence $\{t_k\}$ decreasing to 0, there is a sequence $\{d_k\}$ converging to d, such that $x_k + t_k d_k \in C$ for all k.*

Therefore, the tangent cone $T_C(x)$ is closed and convex, see [30, Propositions 5.1.3 and 5.2.1].

2.3 Pseudomonotone operators

The monotonicity and pseudomonotonicity of F plays an important role in characterizing weakly sharp solutions of the VIP(C, F)(to be introduced in Chapter 4). In this case, we study monotone and pseudomonotone operators in the section.

Definition 2.3.1. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone* on $C \subseteq \mathbb{R}^n$ if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \text{for all } x, y \in C.$$

The mapping F is *antimonotone* on $C \subseteq \mathbb{R}^n$ if

$$\langle F(x) - F(y), x - y\rangle \leq 0 \quad \text{for all } x, y \in C.$$

F is *strongly monotone* on C with modulus μ if

$$\langle F(x) - F(y), x - y\rangle \geq \mu\|x - y\|^2 \quad \text{for all } x, y \in C \text{ with some } \mu > 0.$$

Next we define the *pseudomonotonicity* of F which is a weaker condition than monotonicity. The mapping F is *pseudomonotone* at $x \in C$ if for each $y \in C$ we have

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

F is said to be *pseudomonotone* on C if it is pseudomonotone at each $x \in C$. Obviously the monotonicity of F on C implies the pseudomonotonicity of F on C. F is defined to be *pseudomonotone*$^+$ on C if it is pseudomonotone at each point in C and, for all $x, y \in C$,

$$\left.\begin{array}{ll} \langle F(y), x - y\rangle & \geq 0 \\ \langle F(x), x - y\rangle & = 0 \end{array}\right\} \Rightarrow F(x) = F(y).$$

We say that F is *pseudomonotone*$_*$ on C if it is pseudomonotone at each point in C and, for some $k > 0$ and all $x, y \in C$ we have

$$\left.\begin{array}{ll} \langle F(y), x - y\rangle & = 0 \\ \langle F(x), x - y\rangle & = 0 \end{array}\right\} \Rightarrow F(x) = kF(y).$$

So F is said to be *pseudomonotone*$_*^+$ on C if it is pseudomonotone$_*$ on C with $k = 1$. Therefore, the pseudomonotonicity$^+$ of F on C implies the pseudomonotonicity$_*^+$ of F on C.

Based on the construction of the VIP(C, F) and the definition of pseudomonotone functions, we end this section by showing an immediate result as below.

Proposition 2.3.2. *$C^* \subseteq C_*$ if and only if F is pseudomonotone on C^*.*

Proof. Let $C^* \subseteq C_*$. For any $x^* \in C^*$,

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad \text{for all } y \in C.$$

By assumption for all $y \in C$ we have $\langle F(y), y - x^* \rangle \geq 0$. Hence F is pseudomonotone on C^*.

Conversely, suppose that F is pseudomonotone on C^*. Let $x^* \in C^*$. Then for all $y \in C$ we have $\langle F(y), y - x^* \rangle \geq 0$. This implies that $x^* \in C_*$ and hence $C^* \subseteq C_*$. □

Chapter 3

Gap functions for variational inequalities

3.1 Introduction

This chapter deals with the construction of gap functions for variational inequality problems. The idea of this approach is to reformulate the variational inequalities into constrained optimization problems by utilizing different gap functions, which allows us to obtain the solutions of variational inequality problems by minimizing their gap functions.

In Section 3.2, the background of gap functions for variational inequalities are studied. We summarize the formulation of gap functions in Section 3.3. In Section 3.4, we focus on the construction of primal and dual gap functions for variational inequalities. Based on these, we state some properties of primal and dual gap functions for variational inequalities in Section 3.5. In particular, the Gâteaux differentiability of these two gap functions and their locally Lipschitz properties are discussed. In the last section, we draw the conclusion of this chapter.

3.2 Motivation and background of gap functions for variational inequalities

In order to solve the VIP(C, F), much work has been done to reformulate it as an optimization problem by using different gap functions.

We recall the definition of a merit function for the VIP(C, F) and state the equivalence between a merit function for the VIP(C, F) and a gap function for the VIP(C, F) as follows.

Definition 3.2.1. A function γ: $\mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a *merit function* for the VIP(C, F) if there exists a set $\Omega \subseteq \mathbb{R}^n$ such that

(i) $\gamma \geq 0 \quad$ for all $x \in \Omega$, and

(ii) x^* solves the VIP(C, F) if and only if $x^* \in \Omega$ and $\gamma(x^*) = 0$.

By Definition 3.2.1, a point in the solution set of the VIP(C, F) can be obtained by minimizing γ on Ω. Therefore, the VIP(C, F) can be reformulated as the following optimization problem:

$$\begin{aligned} &\text{minimize } \gamma(x) \\ &\text{subject to } x \in \Omega. \end{aligned}$$

Therefore, merit functions are the key concept for connecting the VIP(C, F) with optimization problems. The merit function γ may be expected to have desirable properties as below:

- γ is differentiable;
- Any stationary point of γ on Ω is also a global minimum of γ on Ω;
- γ provides a global error bound for the VIP(C, F), that is, for any given point $x \in \mathbb{R}^n$, the distance from x to the solution set of the VIP(C, F) is bounded by the value $\gamma(x)$ multiplied by some positive constant.

The above desirable properties of merit functions will be frequently used for characterizing the solution sets of variational inequalities in the thesis later. If the set Ω coincides with C, then a merit function is also known as a *gap function* for the VIP(C, F). So a gap function is also used as a measurement of the violation of the VIP(C, F) at a point $x \in C$. The reader can refer to [21] and [78] for details of gap functions for variational inequalities.

3.3 Saddle point formulation for gap functions

Auchmuty [1] has developed gap functions for the VIP(C, F) based on a function $L : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, +\infty)$ defined by

$$L(x, y) = f(x) - f(y) - \langle x - y, \nabla f(x) - F(x) \rangle, \tag{3.1}$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex, lower semicontinuous function [1] and it is continuously differentiable on C.

The Lagrangian L is said to have a saddle point (x^*, y^*) on $C \times C$ if it satisfies

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*) \quad \text{for all } (x, y) \in C \times C.$$

Auchmuty [1, Theorem 4] has shown that the existence of the saddle point (x^*, y^*) of the Lagrangian L implies that x^* is a solution of the VIP(C, F) under certain conditions.

3.4 The primal and dual gap functions for the VIP(C, F)

To study the important special case of Auchmuty's merit functions [1], we assume $f \equiv 0$ in (3.1). Therefore, the Lagrangian L becomes

$$L(x, y) := \langle F(x), x - y \rangle.$$

It is known that a solution to the VIP(C, F) is obtained by searching a saddle point of L, which was proved in 1960s that, under certain condition of F, a solution to the VIP(C, F) (x^*, y^*) can be searched such that

$$\begin{aligned} &\sup\{\inf\{\langle F(x), x - y \rangle : x \in C\} : y \in C\} = \langle F(x^*), x^* - y^* \rangle \\ &= \inf\{\sup\{\langle F(x), x - y \rangle : y \in C\} : x \in C\}. \end{aligned} \tag{3.2}$$

The *primal gap function* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined from the rightmost of (3.2), that is,

$$\begin{aligned} g(x) :&= \sup\{\langle F(x), x - c \rangle : c \in C\} \\ &= \langle F(x), x - c \rangle \quad \text{for } c \in \Gamma(x), \quad \text{where} \\ \Gamma(x) :&= \{c \in C : \langle F(x), x - c \rangle = g(x)\} \quad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Note that if $\Gamma(x) \neq \emptyset$, then

$$g(x) := \max\{\langle F(x), x - c \rangle : c \in C\}.$$

Larsson and Patriksson [47] summarized some important properties of g which will be used later.

[1] $f(x_0) \leq \liminf_{x \to x_0} f(x)$

Theorem 3.4.1. *[47, Theorem 3.1] For $x \in C$, the properties of g are summarized as follows.*

(i) *g is a gap function.*

(ii) *If $F \in C^1$ on C, then g is differentiable at $x \in C$ if $\Gamma(x) = \{y(x)\}$, with*

$$\nabla g(x) = F(x) + \langle \nabla F(x), x - y(x) \rangle.$$

(iii) *g is convex on C if $\langle F(x), x \rangle$ is convex on C and each component of F is concave on C.*

(iv) *(A fixed point characterization of C^*) $x \in C^* \Leftrightarrow x \in \Gamma(x)$.*

Hence if the solution of the VIP(C, F) is nonempty, then the solution set to

$$\inf\{g(x) : x \in C\}$$

equals C^*.

Turning to the leftmost of (3.2), we obtain the *dual gap function* as below.

$$G(x) := \inf\{\langle F(c), c - x \rangle : c \in C\}.$$

In order to coordinate with the format of the primal gap function g, we define the dual gap function G as follows.

$$\begin{aligned} G(x) :&= \sup\{\langle F(c), x - c \rangle : c \in C\} \\ &= \langle F(c), x - c \rangle \quad \text{for } c \in \Lambda(x), \quad \text{where} \\ \Lambda(x) :&= \{c \in C : \langle F(c), x - c \rangle = G(x)\} \quad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Similarly, if $\Lambda(x) \neq \emptyset$, then

$$G(x) := \max\{\langle F(c), x - c \rangle : c \in C\}.$$

For $f \equiv 0$, a result of Larsson and Patriksson [47] shows that G is a gap function for the VIP(C, F) under certain condition.

Theorem 3.4.2. *[47, Theorem 3.2] Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be pseudomonotone on C. Then*

(i) *G is a gap function.*

(ii) *G is convex on C.*

Since G is convex, there are many scholars discussing properties of G, see Section 3.5.

3.5 Properties of primal and dual gap functions

Recently, there are many papers discussing properties of the dual gap function G since it is convex on C, see [47, 63, 91, 100]. Although G is convex, it is usually more complicated to compute its values than g since, for a fixed point $x \in \mathbb{R}^n$, $G(x)$ is the maximum of a nonlinear program usually while $g(x)$ is that of a linear program. To see this, we consider the following example.

Example 3.5.1. *[55]* Let $C = \{(x_1, x_2) | -2 \leq x_1 \leq 2, -2 \leq x_2 \leq 2\}$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, $F(x) = (x_1^3, x_2^3)$.
Then we have

$$
\begin{aligned}
G(x) &= \max\{\langle F(c), x - c\rangle : c \in C\} = \max\{c_1^3 x_1 + c_2^3 x_2 - c_1^4 - c_2^4 : (c_1, c_2) \in C\}, \\
g(x) &= \max\{\langle F(x), x - c\rangle : c \in C\} = \max\{x_1^4 + x_2^4 - c_1 x_1^3 - c_2 x_2^3 : (c_1, c_2) \in C\}.
\end{aligned}
$$

Thus for this simple example in $\mathbb{R}^2$, four terms of variables $c = (c_1, c_2)$ in $G(x)$ are nonlinear and more complicated to be calculated than two linear terms of c in $g(x)$.

By definitions of g and G, we have easy but important results as below.

Proposition 3.5.2. *[89, Proposition 2.1] For $x^* \in C$,*

(i) $x^* \in C^* \Leftrightarrow g(x^*) = 0 \Leftrightarrow x^* \in \Gamma(x^*)$*;*

(ii) $x^* \in C_* \Leftrightarrow G(x^*) = 0 \Leftrightarrow x^* \in \Lambda(x^*)$.

This section is organized as follows. In Section 3.5.1, we summarize existing results of properties of the dual gap function G, i.e., several sufficient conditions for the Gâteaux differentiability of the dual gap function G proposed by Marcotte and Zhu [63] and Wu and Wu [91]. Motivated by Example 3.5.1, for a fixed $x \in \mathbb{R}^n$, $g(x)$ is usually easier to be calculated than G. In this case, we present some properties of g since it is also important for characterizing the solution set of the VIP(C, F). We also study the Gâteaux differentiability of the primal gap function g and the dual gap function G on C^* and C_* and discuss their relations under certain conditions. Moreover, like the Gâteaux differentiability of g and G, the locally Lipschitz property of these two gap functions are also very important. Thus, in Section 3.5.2, the locally Lipschitz property of g, G and $g + G$ are also presented.

3.5.1 Gâteaux differentiability of primal and dual gap functions

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$. Recall that the *directional derivative* of f at $x \in \mathbb{R}^n$ along the direction $d \in \mathbb{R}^n$ is

$$f'(x; d) := \lim_{t \searrow 0} \frac{f(x + td) - f(x)}{t},$$

which provides the limit of the right-hand side exists as $t \geq 0$ approaches zero. It is *Gâteaux differentiable* at x if the directional derivative $f'(x; d)$ exists for all directions $d \in \mathbb{R}^n$ and it is a linear function of d, that is,

$$f'(x; d) = \langle \nabla f(x), d \rangle \quad \text{for all } d \in \mathbb{R}^n,$$

where $\nabla f(x)$ denotes the gradient of f at x, see [26].

In this part, we study sufficient conditions for the Gâteaux differentiability of G on C^* by obtaining the result $\nabla G(x^*) = F(x^*)$ for $x^* \in C^*$. This result was first stated by Marcotte and Zhu [63]. Similarly, we propose that under some condition, the Gâteaux differentiability of g at $x^* \in C^*$ implies that $\partial g(x^*) = \{\nabla g(x^*)\} = \{F(x^*)\}$. Based on these two results, we show that under the condition $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$, the Gâteaux differentiability of g at $x^* \in C^*$ implies that of G at x^* as well, and $\{\nabla g(x^*)\} = \partial g(x^*) = \partial G(x^*) = \{\nabla G(x^*)\} = \{F(x^*)\}$. An example for discussing the assumption $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ is also presented. Moreover, we present some equivalent statements of the Gâteaux differentiability of G proposed by Wu and Wu [91]. Similar to their results, we also show some equivalent statements of the Gâteaux differentiability of g at $x^* \in C^*$. In addition, we show that the Gâteaux differentiability of G at $x_* \in C_*$ implies that of g at x_*, and $\{\nabla g(x_*)\} = \partial g(x_*) = \partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}$ by considering relevant relations between g and G. Finally, conditions for $\partial(g + G)(x^*) = \{\nabla(g + G)(x^*)\} = \{2F(x^*)\}$ are also stated.

We start with a result shown by Marcotte and Zhu which characterized the Gâteaux differentiability of G at $x^* \in C^*$ under the condition that F is continuous and pseudomonotone$^+$ on C and C is compact as below.

Theorem 3.5.3. *[63, Theorem 3.1] Let F be continuous and pseudomonotone$^+$ on C. Then*

(i) F is constant on C^*;

(ii) for any $x^* \in C^*$, F is constant and equal to $F(x^*)$ on $\Lambda(x^*)$;

(iii) $\Lambda(x^*) = C^*$ *for any* $x^* \in C^*$*;*

(iv) if C *is compact, then* G *is continuously differentiable on* C^**, and* $\nabla G(x^*) = F(x^*)$ *for any* $x^* \in C^*$*.*

In Theorem 3.5.3, the continuity and pseudomonotonicity of F implies $C^* = C_*$ and hence C^* is closed and convex in this case.

A useful result which will be used for characterizing the weak sharpness of C^* with $\nabla G(x^*) = F(x^*)$ for $x^* \in C^*$ is presented by Theorem 3.5.3 under the condition that F is continuous and pseudomonotone$^+$ on C.

Zhang, Wan and Xiu [100] have extended Theorem 3.5.3 for exploring some properties of G by applying the subdifferential of G.

Proposition 3.5.4. *[100, Proposition 3.13] Assume that* F *is continuous and pseudomonotone on* C*, and that* C^* *is nonempty.*

(i) If $d \in \mathbb{R}^n$*, then* $G'(\cdot; d)$ *is convex on* C^**.*

(ii) If $x^* \in C^*$*, then* $F(y^*) \in \partial G(x^*)$ *for any* $y^* \in \Lambda(x^*)$*.*

(iii) If G *is differentiable at some point* $x^* \in C^*$*, then* $C^* = \Lambda(x^*)$ *and* $\nabla G(x^*) = F(y^*)$ *for any* $y^* \in \Lambda(x^*)$*, i.e.,* F *is a constant vector on* $\Lambda(x^*)$*.*

(iv) If G *is differentiable at some point* $x^* \in C^*$*, then for any* $d \in \mathbb{R}^n$*,*

$$\langle \nabla G(x^*), d \rangle = \min\{G'(x; d) : x \in C^*\}.$$

(v) If C^* *is bounded and* G *is differentiable on* Ω^* *(*Ω^* *denotes the set of extreme points of* C^**), then* G *is differentiable on* C^* *and* ∇G *is a constant vector on* C^**.*

Wu and Wu [90] have studied the subdifferential of G in order to characterize its Gâteaux differentiability. We refer to their result in $\mathbb{R}^n$ as follows.

Proposition 3.5.5. *[90, Proposition 4.1] For* $x \in \mathbb{R}^n$*, if* $G(x) < +\infty$*, then* $\{F(y) : y \in \Lambda(x)\} \subseteq \partial G(x)$*, particularly,*
$\{F(y) : y \in C^* \cup \{x\}\} \subseteq \partial G(x), \quad$ *for each* $x \in C_*$*.*

Motivated by this result, we relate the subdifferential of g at $x^* \in C^*$ with $F(x^*)$ and to show that $\partial g(x^*) = \{\nabla g(x^*)\} = \{F(x^*)\}$ for $x^* \in C^*$.

Proposition 3.5.6. *[55, Proposition 1] Let $x^* \in C^*$ and let F be monotone on $\mathbb{R}^n$. Suppose that $g(x) < +\infty$ for all $x \in \mathbb{R}^n$ and Gâteaux differentiable at x^*. Then*

$$\partial g(x^*) = \{F(x^*)\}.$$

Proof. For all $y \in \mathbb{R}^n$, since F is monotone, we have

$$g(y) - g(x^*) \geq \langle F(y), y - x^* \rangle \geq \langle F(x^*), y - x^* \rangle.$$

Hence $F(x^*) \in \partial g(x^*)$.

Let $u \in \partial g(x^*)$. Then for any $v \in \mathbb{R}^n$ and $t > 0$, we have

$$g(x^* + tv) - g(x^*) \geq t\langle u, v \rangle.$$

The Gâteaux differentiability of g at x^* implies that $\langle \nabla g(x^*), v \rangle \geq \langle u, v \rangle$. This implies that $u = \nabla g(x^*)$. So that $\partial g(x^*) = \{\nabla g(x^*)\} = \{F(x^*)\}$. $\square$

Remark 3.5.7. Recall that if a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is Gâteaux differentiable at $x \in \mathbb{R}^n$, then it is subdifferentiable at x and $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f(x)$ is the gradient of f at x. By Proposition 3.5.6, although g is not convex, we can get the same result under certain conditions which imply the nonemptyness of $\partial g(x^*)$. Consequently, $\nabla g(x^*) = F(x^*)$ for each $x^* \in C^*$. This is similar to Theorem 3.5.3, in which $\nabla G(x^*) = F(x^*)$ under the condition that F is continuous and pseudomonotone$^+$ on a compact convex set C.

As mentioned in Remark 3.5.7, if G is Gâteaux differentiable at $x^* \in C^*$, then $\partial G(x^*) = \{\nabla G(x^*)\}$, where $\nabla G(x^*)$ is the gradient of G at x^*. Our next propose is to discuss relations between the Gâteaux differentiabilities of g and G. It is noted that the inequality $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ ensures the nonemptyness of the subdifferential of g at $x^* \in C^*$. In this case, we show that for $x^* \in C^*$ the Gâteaux differentiability of g at x^* implies that of G at x^*.

Proposition 3.5.8. *[54, Proposition 2.1] Let $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Suppose that g is Gâteaux differentiable at $x^* \in C^*$. Then G is Gâteaux differentiable at x^*,*

$$\{\nabla g(x^*)\} = \partial g(x^*) = \partial G(x^*) = \{\nabla G(x^*)\} = \{F(x^*)\}.$$

Proof. Let $x^* \in C^*$. Then by Proposition 3.5.2 we have $0 = g(x^*) \geq G(x^*)$. Since G is nonnegative on C, we obtain $G(x^*) = 0$ and hence $x^* \in C_*$. Therefore for all $x \in \mathbb{R}^n$ we have

$$g(x) - g(x^*) \geq G(x) - G(x^*) \geq \langle F(x^*), x - x^* \rangle,$$

which implies that $F(x^*) \in \partial g(x^*)$.

Let $\xi \in \partial g(x^*)$. Then for all $v \in \mathbb{R}^n$ and $t > 0$,

$$g(x^* + tv) - g(x^*) \geq t\langle \xi, v\rangle.$$

Since g is Gâteaux differentiable at x^*, $\langle \nabla g(x^*), v\rangle \geq \langle \xi, v\rangle$. This implies that $\xi = \nabla g(x^*)$. So $\{\ F(x^*)\} = \partial g(x^*) = \{\nabla g(x^*)\}$.

By assumption, we have

$$\begin{aligned}\langle F(x^*), v\rangle = \langle \nabla g(x^*), v\rangle &= \lim_{t\to 0} \frac{g(x^* + tv) - g(x^*)}{t} \\ &\geq \lim_{t\to 0} \frac{G(x^* + tv) - G(x^*)}{t} \geq \langle F(x^*), v\rangle,\end{aligned}$$

so

$$\lim_{t\to 0} \frac{G(x^* + tv) - G(x^*)}{t} = \langle F(x^*), v\rangle.$$

This implies that G is Gâteaux differentiable at x^* with $\nabla G(x^*) = F(x^*)$. Hence the proof is complete. □

We note that Proposition 3.5.8 may fail if the inequality $g(x) \geq G(x)$ holds only for $x \in C$ not for all $x \in \mathbb{R}^n$.

Example 3.5.9. *[54, Example 2.2]* Let $C = [0, 1]$ and

$$F(x) = \begin{cases} x & \text{for } x \in C; \\ -x & \text{for } x \notin C. \end{cases}$$

The solution set C^* is to find $x^* \in [0, 1]$ such that

$$F(x^*) \cdot (x - x^*) \geq 0 \quad \text{for all } x \in [0, 1],$$

that is, $x^* \cdot (x - x^*) \geq 0$ for all $x \in [0, 1]$. So $C^* = \{x^*\} = \{0\}$.

For $x \in \mathbb{R}$, $g(x) = \sup\{F(x) \cdot (x - c) : c \in [0, 1]\}$. If $x \in C$, then

$$g(x) = \sup\{x \cdot (x - c) : c \in [0, 1]\} = x^2.$$

For $x \notin C$, $g(x) = \sup\{-x \cdot (x - c) : c \in [0, 1]\} = \sup\{-x^2 + cx : c \in [0, 1]\}$. In this case, if $x < 0$, then $g(x) = -x^2$. Otherwise, $g(x) = -x^2 + x$. Therefore,

$$g(x) = \begin{cases} -x^2 & \text{for } x < 0; \\ x^2 & \text{for } x \in C; \\ -x^2 + x & \text{for } x > 1. \end{cases}$$

Moreover, for $x \in \mathbb{R}$ we have

$$
\begin{aligned}
G(x) &= \sup\{F(c)\cdot(x-c) : c\in[0,1]\} = \sup\{c\cdot(x-c) : c\in[0,1]\}\\
&= \sup\{-(c-\frac{1}{2}x)^2 + \frac{1}{4}x^2 : c\in[0,1]\}.
\end{aligned}
$$

It follows that

$$
G(x) = \begin{cases} 0 & \text{for } x < 0;\\ \frac{1}{4}x^2 & \text{for } 0 \le x \le 2;\\ x-1 & \text{for } x > 2. \end{cases}
$$

It is clear that $g(x) \geq G(x)$ holds for each $x \in C = [0,1]$ but not for $x \in (-\infty,0)\cup(1,+\infty)$. In this case, for $x^* \in C^*$, there exists no $\xi \in \mathbb{R}$ such that

$$
\langle \xi, x - x^* \rangle \le g(x) - g(x^*) \quad \text{for each } x \in \mathbb{R},
$$

which implies that $\partial g(x^*)$ is empty. This shows that the assumption $g(x) \geq G(x)$ for all $x \in C$ is not sufficient for $\partial g(x^*)$ to be nonempty.

Remark 3.5.10. Proposition 3.5.8 is an extension of Proposition 3.5.6 since the monotonicity of F on $\mathbb{R}^n$ is a special case of the assumption $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$, which can be easily observed from the following example.

Example 3.5.11. Let $C = [0,1]$ and

$$
F(x) = \begin{cases} -1 & \text{for } x < 0;\\ x & \text{for } x \in C;\\ \frac{1}{2}x & \text{for } x > 1. \end{cases}
$$

By definition, $g(x) = \sup\{F(x)\cdot(x-c) : c\in[0,1]\}$ for $x \in \mathbb{R}$. If $x \in C$, then $g(x) = \sup\{x(x-c) : c \in [0,1]\} = x^2$. For $x < 0$,

$$
g(x) = \sup\{-1(x-c) : c\in[0,1]\} = -x+1.
$$

If $x > 1$, then $g(x) = \sup\{\frac{1}{2}x(x-c) : c\in[0,1]\} = \frac{1}{2}x^2$. Therefore,

$$
g(x) = \begin{cases} -x+1 & \text{for } x < 0;\\ x^2 & \text{for } x \in C;\\ \frac{1}{2}x^2 & \text{for } x > 1. \end{cases}
$$

By the definition of G, we obtain that G is the same as in Example 3.5.9. It is easy to see that $g(x) \geq G(x)$ for all $x \leq 2$. For $x > 2$,

$$
(g-G)(x) = \frac{1}{2}x^2 - (x-1) = \frac{1}{2}(x-1)^2 + \frac{1}{2} > 0.
$$

So $g(x) \geq G(x)$ for all $x \in \mathbb{R}$. However, F is not monotone on $\mathbb{R}$. This shows that the assumption $g(x) \geq G(x)$ for all $x \in \mathbb{R}$ does not imply the monotonicity of f on $\mathbb{R}$. Conversely, if F is monotone on $\mathbb{R}$, then by definition of monotone functions in Section 2, this is sufficient for $g(x) \geq G(x)$ for all $x \in \mathbb{R}$. Therefore, it shows that Proposition 3.5.8 is an extension of Proposition 3.5.6.

Wu and Wu [91] have presented an equivalent statement of the Gâteaux differentiability of G in a Hilbert space. We recall their result in a finite-dimensional Hilbert space $\mathbb{R}^n$ as below.

Theorem 3.5.12. *[91, Theorem 2.2] Let $x \in \mathbb{R}^n$ and $\Lambda(x)$ be nonempty. Then G is Gâteaux differentiable at x iff F is constant on $\Lambda(x)$ and*

$$G'(x; v) = \sup\{\langle F(c), v\rangle : c \in \Lambda(x)\}$$

for all $v \in \mathbb{R}^n$.

Based on this, a special case of this result is studied.

Theorem 3.5.13. *[91, Theorem 2.3] For $x_* \in C_*$ the following are equivalent:*

(i) G is Gâteaux differentiable at x_*.

(ii) $C^* = \Lambda(x_*)$, F is constant on C^*, and
$G'(x_*; v) = \sup\{\langle F(c), v\rangle : c \in C^*\} = \langle F(x_*), v\rangle$ *for all $v \in \mathbb{R}^n$.*

Hence, if G is Gâteaux differentiable on C_*, then $C_* \subseteq C^*$ and ∇G is constant on C_*.

Remark 3.5.14. For $x^* \in C^*$, $\Lambda(x^*)$ is the solution set to maximize $f(x) = \langle F(x), x^* - x\rangle$ subject to $x \in C$. Also it is known that the assumption $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ implies $C^* \subseteq C_*$. Therefore, by Proposition 3.5.8 and Theorem 3.5.13, the solution set C^* can be obtained by $\Lambda(x^*)$ if g is Gâteaux differentiable at x^*. In this case, F is constant on $\Lambda(x^*)$ and $x^* \in C^* = \Gamma(x^*) \cap \Lambda(x^*)$. Furthermore, if each $x^* \in C^*$ and each $y^* \in \Gamma(x^*)$ satisfy

$$\{v \in \mathbb{R}^n : \langle F(x^*), v\rangle \geq 0\} = \{v \in \mathbb{R}^n : \langle F(y^*), v\rangle \geq 0\},$$

then $\Lambda(x^*) = C^* = C_* = \Gamma(x^*)$ from [89, Proposition 3.1] and hence F is constant on $\Gamma(x^*)$. So the solution set to the VIP(C, F) and the DVIP(C, F) can be determined either by $\Lambda(x^*)$ or by $\Gamma(x^*)$.

The above statements together with Proposition 3.5.8 and Theorem 3.5.13 imply the following two results.

Theorem 3.5.15. *[91, Theorem 2.4] Let $x_* \in C_*$. Then the following are equivalent:*

(i) G is Gâteaux differentiable at x_*.

(ii) C^* is nonempty, F is constant on $\Lambda(x_*) \cap \Gamma(x_*)$ and satisfies

$$\{v \in \mathbb{R}^n : \langle F(x), v\rangle \geq 0\} = \{v \in \mathbb{R}^n : \langle F(x_*), v\rangle \geq 0\} \quad \textit{for each } x \in C^*, \textit{ and}$$

$$G'(x_*; v) = \sup\{\langle F(c), v\rangle : c \in C^*\} \quad \textit{for each } v \in \mathbb{R}^n.$$

Similarly, the Gâteaux differentiability of g at $x^* \in C^*$ has the similar equivalent result as that of G in Theorem 3.5.15.

Proposition 3.5.16. *[54, Proposition 2.4] Let $x^* \in C^*$. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. Then the following are equivalent:*

(i) g is Gâteaux differentiable at x^*.

(ii) F is constant on $\Gamma(x^*) \cap \Lambda(x^*)$ and
$g'(x^*; v) = \sup\{\langle F(x), v\rangle : x \in C^*\}$ for all $v \in \mathbb{R}^n$.

Proof. Since $(i) \Rightarrow (ii)$ is direct from Proposition 3.5.8 and Remark 3.5.14, it suffices to prove $(ii) \Rightarrow (i)$.

By assmption, we have $C^* \subseteq C_*$. Therefore, $C^* \subseteq \Lambda(x^*)$ and $C^* \subseteq \Gamma(x^*)$ by [89, Proposition 2.3]. This implies that $C^* \subseteq \Lambda(x^*) \cap \Gamma(x^*)$ and F is constant on C^* from (ii). By the expression of $g'(x^*; v)$ in (ii), g is Gâteaux differentiable at x^* with $\nabla g(x^*) = F(x^*)$. □

Remark 3.5.17. Wu and Wu [91] have discussed some sufficient conditions for the result

$$G'(x; v) = \sup\{\langle F(c), v\rangle : c \in \Lambda(x)\} \quad \text{for } x \in \mathbb{R}^n.$$

By Theorem 3.5.13, if $C^* \subseteq C_*$, then the Gâteaux differentiability of G at $x^* \in C^*$ implies that

$$G'(x^*; v) = \sup\{\langle F(c), v\rangle : c \in C^*\}$$

since $C^* = \Lambda(x^*)$. From Proposition 3.5.16, it shows that the Gâteaux differentiability of g at $x^* \in C^*$ implies that $g'(x^*; v) = G'(x^*; v)$ for all $v \in \mathbb{R}^n$ under certain conditions.

Next we study relations between the Gâteaux differentiability of G at $x_* \in C_*$ and that of g at x_*.

Proposition 3.5.18. *[54, Proposition 2.5] Let $g(x) \le G(x)$ for all $x \in \mathbb{R}^n$. Suppose that G is Gâteaux differentiable at $x_* \in C_*$ and $\partial g(x_*) \neq \emptyset$. Then g is Gâteaux differentiable at x_*,*

$$\{\nabla g(x_*)\} = \partial g(x_*) = \partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}$$

and F is constant on C^*.

Proof. Since $g(x) \le G(x)$ for all $x \in \mathbb{R}^n$, by Proposition 3.5.2, we have $C_* \subseteq C^*$. Applying Theorem 3.5.13, the Gâteaux differentiability of G at x_* implies that $\partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}$ and F is constant on C^*.

Let $\xi \in \partial g(x_*)$. Then for all $v \in \mathbb{R}^n$ and $t > 0$,

$$\langle \xi, tv \rangle \le g(x_* + tv) - g(x_*) \le G(x_* + tv) - G(x_*),$$

from which we obtain that

$$\langle \xi, v \rangle \le \lim_{t \to 0} \frac{g(x_* + tv) - g(x_*)}{t} \le G'(x_*; v) = \langle F(x_*), v \rangle.$$

This implies that $\xi = F(x_*)$. Thus g is Gâteaux differentiable at x_* and

$$\{\nabla g(x_*)\} = \partial g(x_*) = \partial G(x_*) = \{\nabla G(x_*)\} = \{F(x_*)\}.$$

□

Remark 3.5.19. Propositions 3.5.8 and 3.5.18 state the relationships between the Gâteaux differentiability of g and that of G on C^* and C_* and present sufficient conditions for F to be constant on C^*. It is noted that in Proposition 3.5.8, the Gâteaux differentiability of g at $x^* \in C^*$ also implies that $F(c) = F(x^*)$ for all $c \in \Lambda(x^*)$. Therefore, Propositions 3.5.8 and 3.5.18 present sufficient conditions for the constancy of F on C^* as well.

Note that Proposition 3.5.8 implies that $g + G$ is Gâteaux differentiable at $x^* \in C^*$ and $\nabla(g + G)(x^*) = 2F(x^*)$. The following proposition presents weaker conditions for this result.

Proposition 3.5.20. *[54, Proposition 2.7] Let $g(x) \ge G(x)$ for all $x \in \mathbb{R}^n$. Suppose that $g + G$ is Gâteaux differentiable at $x^* \in C^*$. Then*

$$\partial(g + G)(x^*) = \{\nabla(g + G)(x^*)\} = \{2F(x^*)\}.$$

Proof. Let $x^* \in C^*$. Then, by assumption and definition, we have

$$g(x^*) = G(x^*) = 0,\ C^* \subseteq C_* \text{ and } g(x) \geq G(x) \geq \langle F(x^*), x - x^* \rangle \quad \text{for all } x \in \mathbb{R}^n,$$

from which we obtain that

$$(g + G)(x) - (g + G)(x^*) \geq \langle 2F(x^*), x - x^* \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

This implies that $2F(x^*) \in \partial(g + G)(x^*)$.

Let $\xi \in \partial(g + G)(x^*)$. Then for any $v \in \mathbb{R}^n$ and $t > 0$ we have

$$(g + G)(x^* + tv) - (g + G)(x^*) \geq t\langle \xi, v \rangle.$$

If $g + G$ is Gâteaux differentiable at x^*, then

$$\langle \nabla(g + G)(x^*), v \rangle = \lim_{t \to 0} \frac{(g + G)(x^* + tv) - (g + G)(x^*)}{t} \geq \langle \xi, v \rangle.$$

This implies that $\xi = \nabla(g + G)(x^*)$ since v is arbitrary. Thus

$$\{2F(x^*)\} \subseteq \partial(g + G)(x^*) \subseteq \{\nabla(g + G)(x^*)\},$$

which implies $\partial(g + G)(x^*) = \{\nabla(g + G)(x^*)\} = \{2F(x^*)\}$. □

3.5.2 Locally Lipschitz property of g and G

Like the Gâteaux differentiability of g and G, the locally Lipschitz property of these two gap functions are also very important for characterizing solutions of primal and dual variational inequalities.

Definition 3.5.21. Let f be a function from $\mathbb{R}^n$ into $\mathbb{R}^n$ and let $S \subseteq \mathbb{R}^n$. f is said to be *Lipschitz* on S if there exists a constant $K \geq 0$ such that

$$\|f(x) - f(y)\| \leq K\|x - y\| \quad \text{for all } x, y \in S.$$

It is *Lipschitz* near x_0 if for some neighbourhood V_{x_0} of x_0 there exists some $K_{x_0} \geq 0$ such that

$$\|f(x) - f(y)\| \leq K_{x_0}\|x - y\| \quad \text{for all } x, y \in V_{x_0}.$$

f is said to be *locally Lipschitz* on $U \subseteq \mathbb{R}^n$ if it is Lipschitz near x_0 for every $x_0 \in U$.

In this section, we prove that the locally Lipschitz property of F on C^* implies that of g. We present a sufficient condition for G to be Lipschitz. In addition, we discuss the result of Clarke [11] for sufficient conditions of the locally Lipschitz property of convex functions in $\mathbb{R}^n$. Based on this, we show relations between the locally Lipschitz property of g and G. Finally, the sufficiency for the locally Lipschitz property of $g + G$ is also studied.

We begin with the following result which presents a sufficient condition for locally Lipschitz property of g on C^*.

Lemma 3.5.22. *[55, Lemma 1] Let C be compact. If F is locally Lipschitz on C^*, then so is g.*

Proof. Since F is locally Lipschitz on C^*, for any $x^* \in C^*$ there exist $\delta > 0$ and $L_1 \geq 0$ such that

$$\|F(x)\| \leq L_1 \quad \text{and} \quad \|F(x) - F(y)\| \leq L_1\|x - y\| \quad \text{for all } x, y \in B(x^*, \delta).$$

Let $c \in \Gamma(x)$ with $x \in B(x^*, \delta)$. Then

$$\begin{aligned}
g(x) - g(y) &\leq \langle F(x), x - c\rangle - \langle F(y), y - c\rangle \\
&= \langle F(x), x - y\rangle + \langle F(x), y - c\rangle - \langle F(y), y - c\rangle \\
&= \langle F(x), x - y\rangle + \langle F(x) - F(y), y - c\rangle \\
&\leq \|F(x)\|\|x - y\| + \|F(x) - F(y)\|\|y - c\| \\
&\leq L_1\|x - y\| + L_1\|x - y\|\|y - c\|.
\end{aligned}$$

The compactness of C implies that there exists a constant $M \geq 0$ such that

$$\|y - c\| \leq M \quad \text{for all } y \in B(x^*, \delta) \text{ and } c \in C.$$

Thus taking $L = L_1 + L_1 M$, we obtain

$$g(x) - g(y) \leq L\|x - y\|.$$

This implies that g is Lipschitz near x^*. Hence g is locally Lipschitz on C^*. $\square$

The following proposition presents one sufficient condition for the Lipschitz property of G on $\mathbb{R}^n$.

Proposition 3.5.23. *[54, Proposition 3.1] Let F be bounded on C. Then G is Lipschitz.*

Proof. Let $y, z \in \mathbb{R}^n$. For $c \in \Lambda(y)$, we have

$$\begin{aligned} G(y) - G(z) &\leq \langle F(c), y - c\rangle - \langle F(c), z - c\rangle \\ &= \langle F(c), y - z\rangle \leq \|F(c)\|\|y - z\| \leq M\|y - z\|, \end{aligned}$$

where $M = \sup\{\|F(x)\| : x \in C\}$. This implies that G is Lipschitz. □

Since G is convex, its locally Lipschitz property can immediately be obtained by the following theorem of Clarke [11].

We first introduce two lemmas which will be used for characterizing locally Lipschitz property of convex functions.

Lemma 3.5.24. *[11, Lemma 1 in Theorem 2.34] Let $f : X \to \mathbb{R}_\infty$ be convex, and let C be a convex set such that, for certain positive constants δ and N, we have $|f(x)| \leq N$ for any $x \in C + \delta B$. Then f is Lipschitz on C of rank $2N/\delta$.*

Lemma 3.5.25. *[11, Lemma 2 in Theorem 2.34] Let x_0 be a point such that, for certain numbers M and $\varepsilon > 0$, we have $f(x) < M$ for all $x \in B(x_0, \varepsilon)$. Then, for any $x \in$ int dom$f$, there exists a neighborhood V of x and $N \geq 0$ such that $|f(y)| \leq N$ for all $y \in V$.*

Based on these, we obtain the following theorem for presenting locally Lipschitz property of convex functions.

Theorem 3.5.26. *[11, Theorem 2.34] Let $f : X \to \mathbb{R}_\infty$ be a convex function which admits a nonempty open set upon which f is bounded above. Then f is locally Lipschitz in the set int dom$f$.*

In Theorem 3.5.26, if X is assumed to be finite dimensional, i.e., $\mathbb{R}^n$, then we obtain the corollary as follows.

Corollary 3.5.27. *[11, Corollary 2.35] If X is finite dimensional, then any convex function $f : X \to \mathbb{R}_\infty$ is locally Lipschitz in the set int dom$f$.*

Proof. Without loss of generality, we may take $X = \mathbb{R}^n$. Let x_0 be any point in int dom$f$. By Theorem 3.5.26, it suffices to prove that f is bounded above in a neighborhood V of x_0. To see this, observe that, for some $r > 0$, we have

$$V := \text{co}\{x_0 \pm re_i\}_i \subset \text{dom}f,$$

where the $e_i (i = 1, 2, ..., n)$ are the canonical vectors in $\mathbb{R}^n$. Then, by the convexity of f, we deduce

$$f(y) \leq M := \max\{\|f(x_0+re_i)\|+\|f(x_0-re_i)\| : i \in \{1, 2, \cdots, n\}\} \quad \text{for any } y \in V.$$

The proof is complete. □

So if G bounded above on some set S and int$S$ is nonempty, then G is locally Lipschitz in int$S$. Based on this idea, we characterize the locally Lipschitz property of G. Moreover, we discuss relations between the locally Lipschitz property of g and that of $g + G$ since they are important for stating the weak sharpness of C^* and C_*.

Proposition 3.5.28. *[54, Proposition 3.3] Let $x^* \in C^*$. Suppose that there exists $\delta > 0$ such that $g(x) \geq G(x)$ for all $x \in B(x^*, \delta)$. Then the following hold:*

(i) If g is bounded in a neighbourhood of x^*, then G is Lipschitz near x^*.

(ii) $g + G$ is Lipschitz near x^* if and only if g is Lipschitz near x^*.

Proof. (i) Since g is bounded near x^*, there exist $0 < \delta_1 < \delta$ and $L \geq 0$ such that

$$\|g(x)\| \leq L \quad \text{for all } x \in B(x^*, \delta_1),$$

which implies that $\langle F(x^*), x - x^* \rangle \leq G(x) \leq L$ for all $x \in B(x^*, \delta_1)$. Then by Corollary 3.5.27, G is Lipschitz near x^*.

(ii) Since the sufficiency is direct from (i), it remains to show the necessity. If $g + G$ is Lipschitz near x^*, then there exist $0 < \delta_1 < \delta$ and $L \geq 0$ such that

$$2\langle F(x^*), x - x^* \rangle \leq 2G(x) \leq (g + G)(x) \leq L \quad \text{for all } x \in B(x^*, \delta_1).$$

So, by Corollary 3.5.27, G is Lipschitz near x^*. Hence $g = (g+G) - G$ is Lipschitz near x^*. □

Remark 3.5.29. Under the assumption of Proposition 3.5.28, G is locally Lipschitz on C^*. In this case, the locally Lipschitz property of $g + G$ is equivalent to that of g.

3.6 Summary

In this chapter, we introduce gap functions for variational inequalities. In particular, the primal gap function g and the dual gap function G are studied. We also characterize some properties of these two gap functions for variational inequality problems.

The relations between the Gâteaux differentiabilities of G and g on C^* and C_* are discussed (Propositions 3.5.8 and 3.5.18). In addition, it is proved that under some condition, the Gâteaux differentiability of $g + G$ on C^* implies that

$\nabla(g+G)(x^*) = 2F(x^*)$ for $x^* \in C^*$, see Proposition 3.5.20. Some equivalent statements of the Gâteaux differentiability of g are also studied (Proposition 3.5.16). Furthermore, we present the locally Lipschitz property of g on C^* (Lemma 3.5.22) as well as the Lipschitz property of G on $\mathbb{R}^n$(Proposition 3.5.23). By applying the results of the locally Lipschitz property of convex functions (Theorem 3.5.26 and Corollary 3.5.27) proposed by Clarke [11], we present the locally Lipschitz property of G as well as the equivalence between the locally Lipschitz property of $g + G$ and g.

An interesting question for further research is to find the sufficiency for the Gâteaux differentiability of g in terms of the relevant mapping F. We note that some of the results are obtained under the assumption that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$. In the future, we try to present the results by considering F and C.

Chapter 4

Weakly sharp solutions of primal and dual variational inequality problems

4.1 Introduction

In this chapter, we study weakly sharp solutions of variational inequality problems by primal and dual gap functions.

The chapter begins with recalling the concept of sharp solutions of a variational inequality [7] and that of the error bound of a function. We also propose the constancy of F on C^* since it is important for characterizing the weak sharpness of variational inequality problems. Moreover, we present sufficient conditions for the minimum principle sufficiency and maximum principle sufficiency properties introduced by Ferris and Mangasarian [19] and Wu and Wu [89]. Based on these, the weak sharpness of the solution sets to VIP(C, F) and DVIP(C, F) are discussed in terms of primal and dual gap functions.

4.2 Preliminaries

Let f be a mapping from $\mathbb{R}^n$ into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, f is said to have a sharp minimum at $\overline{x} \in \mathbb{R}^n$ if

$$f(x) \geq f(\overline{x}) + \alpha\|x - \overline{x}\| \quad \text{for all } x \text{ near } \overline{x} \text{ with some } \alpha > 0.$$

To the knowledge of the authors, Burke and Ferris [7] have extended the concept of a sharp minimum solution to the case of a nonunique solution set. They have indicated that the solution set $\overline{C}$ to

$$\min\{f(x) : x \in C\} \tag{4.1}$$

is weakly sharp if there exists $\alpha > 0$ such that

$$f(x) \geq f(\overline{x}) + \alpha d_{\overline{C}}(x) \quad \text{for all } x \in C \text{ and } \overline{x} \in \overline{C}. \tag{4.2}$$

The constant α and the set $\overline{C}$ are called the modulus and the domain of sharpness for f over C. Clearly $\overline{C}$ is a set of global minima for f over C. Moreover, if f is differentiable, closed and proper convex, the sets $\overline{C}$ and C are nonempty, closed and convex, then $\overline{C}$ is said to be weakly sharp if and only if

$$-\nabla f(\overline{x}) \in \operatorname{int} \bigcap_{x \in \overline{C}} [T_C(x) \cap N_{\overline{C}}(x)]^\circ \quad \text{for each } \overline{x} \in \overline{C},$$

see [7, Corollary 2.7]. And this conclusion reduces to

$$-\nabla f(\overline{x}) \in \operatorname{int} N_C(\overline{x}) \quad \text{for } \overline{x} \in \overline{C}$$

when $\overline{C}$ is a singleton. Burke and Ferris [7] found that, under the condition of the differentiability of f and closeness of the nonempty sets C and $\overline{C}$, the inclusion

$$\alpha B \subseteq \nabla f(\overline{x}) + [T_C(\overline{x}) \cap N_{\overline{C}}(\overline{x})]^\circ$$

holds at $\overline{x} \in \overline{C}$ if and only if

$$\langle \nabla f(\overline{x}), z \rangle \geq \alpha \|z\| \quad \text{for all } z \in T_C(\overline{x}) \cap N_{\overline{C}}(\overline{x}),$$

for which the reader can refer to [7, Corollary 2.7].

Since the VIP(C, F) lacks a natural objective function f, Patriksson has generalized the concept of the weak sharpness of a solution set to the VIP(C, F) in [79]. Following [79, pp. 108], a solution set of the VIP(C, F), C^*, is said to be *weakly sharp* provided that

$$-F(x^*) \in \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for each } x^* \in C^*. \tag{4.3}$$

This is equivalent to saying that for each $x^* \in C^*$ there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ,$$

where B denotes the closed unit ball in $\mathbb{R}^n$. If F is constant on C^*, then this α is unified.

For an inequality system, an *error bound* is an estimate for the distance from any point to the solution set of the inequality system. If the inequality system is described by $f(x) \le 0$ (where f is a function defined on $\mathbb{R}^n$) with the solution set S, then the system has an *error bound* provided that there exists a positive μ such that the distance function d_S satisfies

$$d_S(x) \le \mu f(x)_+ := \mu \max\{f(x), 0\} \quad \text{for } x \in \mathbb{R}^n.$$

Based on these and under certain conditions, for the VIP(C, F), G has an error bound on C provided that there exists some $\mu > 0$ such that

$$d_{C^*}(x) \le \mu G(x) \quad \text{for all } x \in C.$$

Similar to G, the primal gap function g is said to have an error bound on C if there exists some $\mu > 0$ such that

$$d_{C^*}(x) \le \mu g(x) \quad \text{for all } x \in C.$$

We note that the error bounds of gap functions have close relations with the weak sharpness of solution sets of variational inequalities, see [32], [52], [63], [89] and [91].

4.3 Constancy of F on C^*

In this section, we study the constancy of F on C^* since it is important for characterizing the weak sharpness of C^* and C_*. In addition to this characterization, the notion of *minimum principle sufficiency* property is also presented. Following Ferris and Mangasarian [19], the VIP(C, F) is said to have the *minimum principle sufficiency* property if

$$\Gamma(x^*) = C^* \text{ for each } x^* \in C^*.$$

Similar to the minimum principle sufficiency property, the VIP(C, F) has the *maximum principle sufficiency* property provided that

$$\Lambda(x^*) = C^* \text{ for each } x^* \in C^*.$$

Since this is also very useful for characterizing the weak sharpness of C^*, we will discuss the maximum principle sufficiency property later. Based on these, for $x \in C^*$, the relation

$$C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$$

is also studied under certain conditions. All results obtained in this section are true in a Hilbert space.

We begin with the following proposition which presents a sufficient condition for F to be constant on C^*.

Proposition 4.3.1. *[55, Proposition 2] F is constant on C^* if F is pseudomonotone$^+$ on C^*.*

Proof. Suppose that F is pseudomonotone$^+$ on C^* and $x_1, x_2 \in C^*$. Then

$$\langle F(x_2), x_1 - x_2 \rangle \geq 0 \quad \text{and} \quad \langle F(x_1), x_2 - x_1 \rangle \geq 0.$$

Since F is pseudomonotone on C^*, we have $\langle F(x_1), x_1 - x_2 \rangle \geq 0$ and hence $\langle F(x_1), x_1 - x_2 \rangle = 0$. Combining this with $\langle F(x_2), x_1 - x_2 \rangle \geq 0$, we obtain $F(x_1) = F(x_2)$. Thus F is constant on C^*. □

Remark 4.3.2. As we know, F is constant on C^* in the following cases:

(i) F is continuous and pseudomonotone$^+$ on C (Theorem 3.5.3).

(ii) F is continuous and pseudomonotone on C and G is Gâteaux differentiable at $x^* \in C^*$ (Proposition 3.5.4).

(iii) G is Gâteaux differentiable at $x_* \in C_*$ (Theorem 3.5.13).

We deduce from Theorem 3.5.3 that (i) implies (ii). Since the continuity and pseudomonotonicity of F implies $C^* = C_*$, it is obvious that (i) or (ii) implies that (iii) holds. Next we show that (iii) is different from the condition of the constancy of F discussed in Proposition 4.3.1.

Example 4.3.3. *[55, Examples 1 and 2]* (i) The pseudomonotonicity$^+$ of F on C^* does not imply the Gâteaux differentiability of G at $x_* \in C_*$.

Let $C = [-1, 0]$ and

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ -x + 1 & \text{if } x \geq 0. \end{cases}$$

The solution set C^* is to find $x^* \in [-1, 0]$ such that

$$F(x^*)(x - x^*) \geq 0 \quad \text{for all } x \in [-1, 0]. \tag{4.4}$$

Since $F(x) = 0$ for $x \in [-1, 0)$, all of $x^* \in [-1, 0)$ are the solution of (4.4). For $x^* = 0$ we have $F(x^*) = 1$. According to (4.4), $x^* = 0$ does not satisfy this.

Therefore, $C^* = [-1, 0)$.
Similarly, C_* aims to find $x_* \in [-1, 0]$ such that

$$F(x)(x - x_*) \geq 0 \quad \text{for all } x \in [-1, 0].$$

Let $\overline{x} \in [-1, 0]$ be fixed to satisfy $F(x)(x - \overline{x}) \geq 0$ for all $x \in [0, 1)$. Then $\overline{x}$ can be any number in $[-1, 0]$. Moreover, any $\overline{x} \in [-1, 0]$ satisfies $F(0)(0 - \overline{x}) \geq 0$. Hence $C_* = [-1, 0]$.

We claim that F is pseudomonotone$^+$ on $[-1, 0)$. Clearly F is pseudomonotone on $[-1, 0)$. If $x^* \in [-1, 0)$ and $c \in [-1, 0]$ are fixed to satisfy

$$F(x^*)(c - x^*) \geq 0 \quad \textit{and} \quad F(c)(c - x^*) = 0,$$

then $F(c) = F(x^*) = 0$. So our claim is proved.

By definition of G we have

$$G(x) = \max\{F(c)(x - c) : c \in [-1, 0]\} = \max\{G_1(x), G_2(x)\}, \quad \text{where}$$

$$G_1(x) = \max\{F(c)(x - c) : c \in [-1, 0)\} = 0 \quad \text{and} \quad G_2(x) = F(0) \cdot x = x.$$

Therefore,

$$G(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } x \geq 0. \end{cases}$$

Obviously G is not Gâteaux differentiable at $x = 0$. This can also be observed from Theorem 3.5.12 or Theorem 3.5.13 since

$$\Lambda(0) = \{c \in [-1, 0] : -F(c) \cdot c = 0\} = [-1, 0] \neq C^*.$$

This shows that the pseudomonotonicity$^+$ of F on C^* does not imply the Gâteaux differentiability of G at $x_* \in C_*$.

(ii) The Gâteaux differentiability of G at $x_* \in C_*$ does not imply the pseudomonotonicity$^+$ of F on C^*.

Let $C = [0, 1]$ and

$$F(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}; \\ x - 1 & \text{if } x > \frac{1}{2}. \end{cases}$$

Then C^* is to find $x^* \in [0, 1]$ such that

$$F(x^*)(x - x^*) \geq 0 \quad \text{for all } x \in [0, 1]. \tag{4.5}$$

For $x^* \in \left[0, \frac{1}{2}\right]$, $F(x^*) = 0$. So all $x^* \in \left[0, \frac{1}{2}\right]$ satisfies (4.5). For $x^* \in \left(\frac{1}{2}, 1\right]$, $F(x^*) = x^* - 1$. In this case, $x^* \in \left(\frac{1}{2}, 1\right]$ is obtained by

$$(x^* - 1)(x - x^*) \geq 0 \quad \text{for all } x \in [0, 1].$$

Therefore, $x^* = 1$ and hence $C^* = \left[0, \frac{1}{2}\right] \cup \{1\}$.
Similarly, the solution set C_* is to find $x_* \in [0,1]$ such that

$$F(x)(x - x_*) \geq 0 \quad \text{for all } x \in [0,1]. \tag{4.6}$$

We first find $\overline{x} \in [0,1]$ such that $F(x)(x - \overline{x}) \geq 0$ for all $x \in [0, \frac{1}{2}]$. In this case, $F(x) = 0$ and all $\overline{x} \in [0,1]$ satisfies (4.6). We then find $\overline{x} \in [0,1]$ such that $F(x)(x - \overline{x}) \geq 0$ for all $x \in (\frac{1}{2}, 1]$. Since $F(x) = x - 1 \leq 0$, we have $\overline{x} = 1$. Hence $C_* = \{x_*\} = \{1\}$.

By definition of G we have $G(x) = \max\{G_1(x), G_2(x)\}$, where

$$G_1(x) = \max\{F(c)(x-c) : c \in [0, \frac{1}{2}]\} = 0 \quad \text{and}$$

$$\begin{aligned} G_2(x) &= \sup\{F(c)(x-c) : c \in (\frac{1}{2}, 1]\} = \sup\{(c-1)(x-c) : c \in (\frac{1}{2}, 1]\} \\ &= \sup\{-(c - \frac{x+1}{2})^2 + \frac{(x-1)^2}{4} : c \in (\frac{1}{2}, 1]\} = \begin{cases} -\frac{1}{2}(x - \frac{1}{2}) & \text{if } x \leq 0; \\ \frac{(x-1)^2}{4} & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x > 1. \end{cases} \end{aligned}$$

This implies that

$$G(x) = \begin{cases} -\frac{1}{2}(x - \frac{1}{2}) & \text{if } x \leq 0; \\ \frac{(x-1)^2}{4} & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x > 1. \end{cases}$$

Based on these, G is Gâteaux differentiable at $x_* = 1$. However, it is clear that F is not pseudomonotone on C^*. This shows that the Gâteaux differentiability of G at $x_* \in C_*$ does not imply the pseudomonotonicity$^+$ of F on C^*.

Under the condition of the Gâteaux differentiability of G at $x_* \in C_*$, Wu and Wu [90] have shown some additional expressions of C^* and C_*. Next we discuss their results in $\mathbb{R}^n$.

Theorem 4.3.4. *[90, Theorem 4.1] Let G be Gâteaux differentiable at some $x_* \in C_*$. Denote*

$$\begin{aligned} C(x_*) :&= \{x \in C : \{v \in \mathbb{R}^n : \langle \xi, v\rangle \geq 0\} \\ &= \{v \in \mathbb{R}^n : \langle F(x_*), v\rangle \geq 0\}, \quad \textit{for some } \xi \in \partial G(x)\}. \end{aligned}$$

Then, $C_* = C_0 = C_1 = C_2 = C_3 = C_4 = C_5$, *where*

$$\begin{aligned}
C_0 &:= \{x \in C : \langle \xi, y - x\rangle \geq 0, \text{ for some } \xi \in \partial G(x) \text{ and each } y \in C\},\\
C_1 &:= \{x \in C : \langle F(x_*), x - x_*\rangle = 0, F(x_*) \in \partial G(x)\},\\
C_2 &:= \{x \in C(x_*) : \langle F(x_*), x - x_*\rangle = 0\},\\
C_3 &:= \{x \in C : \langle \xi, x - x_*\rangle = \langle F(x_*), x - x_*\rangle = 0, \text{ for some } \xi \in \partial G(x)\},\\
C_4 &:= \{x \in C : \langle \xi, x - x_*\rangle = 0, \text{ for some } \xi \in \partial G(x)\},\\
C_5 &:= \{x \in C : \langle \xi, x - x_*\rangle \leq 0, \text{ for some } \xi \in \partial G(x)\}.
\end{aligned}$$

Moreover, if $C^* = C_*$, *then*

$$C_* = D_0 = C_1 = D_2 = D_3 = D_4 = D_5 = \Lambda(x_*),$$

where $D_0, D_2, \ldots, D_5$ *denote the above sets* $C_0, C_2, \ldots, C_5$ *with* ξ *replaced with* $F(x)$.

Theorem 4.3.5. *[90, Theorem 4.2] Let* G *be Gâteaux differentiable at* $x_* \in C_*$. *Then,*

$$C^* \cap C_* = C_1 = C_2 = C_3,$$

where

$$\begin{aligned}
C_1 &= \{x \in C : \langle F(x), x - x_*\rangle = \langle F(x_*), x - x_*\rangle = 0, F(x) \in \partial G(x)\},\\
C_2 &= \{x \in C : \langle F(x), x - x_*\rangle = 0, F(x) \in \partial G(x) \cap \partial G(x_*)\},\\
C_3 &= \{x \in C : \langle F(x), x - x_*\rangle = 0, F(x) \in \partial G(x)\}.
\end{aligned}$$

Hence, if $C^* \subseteq C_*$, *then* $C^* = C_1 = C_2 = C_3$*; if* $C_* \subseteq C^*$, *then* $C_* = C_1 = C_2 = C_3$.

In Theorem 4.3.4, Wu and Wu have characterized the relations between C_* and $\Lambda(x_*)$ for $x_* \in C_*$. Next we show that $C^* = \Lambda(x^*)$ for $x^* \in C^*$ under certain condition.

Proposition 4.3.6. *[55, Proposition 3] For* $x^* \in C^*$, *if* F *is pseudomonotone*$^+$ *on* C, *then* F *is constant on* $\Lambda(x^*)$ *and* $C^* = \Lambda(x^*)$.

Proof. For $x^* \in C^*$ and $c \in C$, we have $\langle F(x^*), c - x^*\rangle \geq 0$. This with the pseudomonotonicity$^+$ of F on C yields $\langle F(c), c - x^*\rangle \geq 0$, that is, $G(x^*) = 0$. In particular, for $c \in \Lambda(x^*)$, we have

$$\langle F(c), c - x^*\rangle = -G(x^*) = 0 \quad \text{and hence} \quad F(c) = F(x^*).$$

Since $x^* \in C^* \subseteq C_*$, it follows from [89, Proposition 2.3 and Theorem 2.6] that $C^* = \Lambda(x^*)$. The proof is complete. □

Remark 4.3.7. Under the assumption that $C^* \subseteq C_*$, we deduce from Theorems 3.5.12 and 3.5.13 that the Gâteaux differentiability of G at $x^* \in C^*$ also implies that F is constant on $\Lambda(x^*)$ and $C^* = \Lambda(x^*)$ for $x^* \in C^*$. However, we can refer to an example to see that the pseudomonotonicity$^+$ of F on C does not imply the Gâteaux differentiability of G at $x^* \in C^*$.

Example 4.3.8. *[91, Example 2.1]*
Let $C = [0,1] \times [0,1]$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = (F_1(x), F_2(x)) = \begin{cases} (1,1) & \text{if } x \neq (0,0); \\ (0,0) & \text{if } x = (0,0). \end{cases}$$

By [91, Example 2.1] $C^* = \{(0,0)\}$. To show the pseudomonotonicity of F on $C\backslash\{(0,0)\}$, let (x_1, x_2) be an arbitrary point in $C\backslash\{(0,0)\}$. For any $(y_1, y_2) \in C$ satisfying

$$(F_1(x), F_2(x)) \cdot (y_1 - x_1, y_2 - x_2) \geq 0,$$

that is, $F_1(x)(y_1 - x_1) + F_2(x)(y_2 - x_2) = y_1 - x_1 + y_2 - x_2 \geq 0$. So $(y_1, y_2) \neq (0,0)$. Therefore,

$$\begin{aligned} (F_1(y), F_2(y)) \cdot (y_1 - x_1, y_2 - x_2) &= F_1(y)(y_1 - x_1) + F_2(y)(y_2 - x_2) \\ &= y_1 - x_1 + y_2 - x_2 \geq 0. \end{aligned}$$

Thus F is pseudomonotone on $C\backslash\{(0,0)\}$. Clearly F is also pseudomonotone at $x = (0,0)$. Hence F is pseudomonotone on C.

For all (x_1, x_2), $(y_1, y_2) \in C$ satisfying

$$F_1(y)(x_1 - y_1) + F_2(y)(x_2 - y_2) \geq 0 \quad \text{and} \quad F_1(x)(x_1 - y_1) + F_2(x)(x_2 - y_2) = 0,$$

we have $F_1(x) = F_1(y)$ and $F_2(x) = F_2(y)$. Thus F is also pseudomonotone$^+$ on C.

By the definition of G we have, for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} G(x) &= \sup\{F_1(c)(x_1 - c_1) + F_2(c)(x_2 - c_2) : (c_1, c_2) \in C\} \\ &= \max\{0, \sup\{x_1 - c_1 + x_2 - c_2 : (c_1, c_2) \in C\backslash\{(0,0)\}\}\} \\ &= \max\{0, x_1 + x_2\}. \end{aligned}$$

Hence

$$G(x) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 \geq 0; \\ 0 & \text{if } x_1 + x_2 < 0. \end{cases}$$

This shows that G is not Gâteaux differentiable at $x = (0,0)$ although F is pseudomonotone$^+$ on C.

Next we discuss some fundamental results stated by Wu and Wu [89] for discussing relations between C^*, C_*, $\Gamma(x^*)$ and $\Lambda(x^*)$, which have already been applied for characterizing the constancy of F on C^* in Proposition 4.3.6.

Proposition 4.3.9. *[89, Proposition 2.3] The following hold:*

(i) $C_* \subseteq \Gamma(x^*)$ *for each* $x^* \in C^*$.

(ii) $C^* \subseteq \Lambda(x^*)$ *for each* $x^* \in C_*$.

In particular, if F *is continuous on* C_*, *then* $C_* \subseteq \Gamma(x^*)$ *for each* $x^* \in C_*$*; if* F *is pseudomonotone on* C^*, *then* $C^* \subseteq \Lambda(x^*)$ *for each* $x^* \in C^*$.

We note that Wu and Wu [89] have stated the weak sharpness results by presenting a novel condition:

$$\{v \in \mathbb{R}^n : \langle F(x^*), v\rangle \geq 0\} = \{v \in \mathbb{R}^n : \langle F(y^*), v\rangle \geq 0\} \quad \text{for } x^*, y^* \in C, \tag{4.7}$$

that is, $F(x^*)$ and $F(y^*)$ have the same direction. Under this condition, $\langle F(x^*), x^* - y^*\rangle = 0$ is equivalent to $\langle F(y^*), x^* - y^*\rangle = 0$. Moreover, if one these two equalities holds, then x^* is a solution of the VIP(C, F) if and only if y^* is also a solution of this, see the proposition below.

Proposition 4.3.10. *[89, Proposition 2.5] Let* $x^* \in C$ *and* $y^* \in C$ *satisfy* (4.7).

(i) $\langle F(x^*), x^* - y^*\rangle = 0 \Leftrightarrow \langle F(y^*), x^* - y^*\rangle = 0$.

(ii) If either $\langle F(x^*), x^* - y^*\rangle = 0$ *or* $\langle F(y^*), x^* - y^*\rangle = 0$, *then*

$$x^* \in C^* \Leftrightarrow y^* \in C^*.$$

As a result of Propositions 4.3.9 and 4.3.10, the following result is immediate.

Theorem 4.3.11. *[89, Theorem 2.6] Let* $x^* \in C$ *and* $y^* \in C$ *satisfy* (4.7).

(i) $x^* \in C^*$ *and* $y^* \in \Gamma(x^*)$ *iff* $x^* \in \Gamma(x^*)$ *and* $y^* \in C^*$.

(ii) $x^* \in C^*$ *and* $y^* \in \Lambda(x^*)$ *iff* $x^* \in \Lambda(x^*)$ *and* $y^* \in C^*$.

Recall that the VIP(C, F) has the minimum principle sufficiency (MPS) property if

$$\Gamma(x^*) = C^* \quad \text{for each } x^* \in C^*.$$

Next we show a sufficient condition for this in terms of (4.7).

Proposition 4.3.12. *[89, Proposition 3.1]*

(i) If (4.7) *holds for $x^* \in C^*$ and all $y^* \in \Gamma(x^*)$, then $\Gamma(x^*) \subseteq C^*$.*

(ii) If (4.7) *holds for $x^* \in C_*$ and all $y^* \in \Gamma(x^*)$, then $x^* \in \Gamma(x^*) = C^*$.*

(iii) If (4.7) *holds for $x^* \in C^* \cup C_*$ and all $y^* \in C^*$, then $x^* \in C^* \subseteq \Gamma(x^*)$.*

Similarly, the VIP(C, F) has the maximum principle sufficiency property if $\Lambda(x^*) = C^*$ for each $x^* \in C^*$. By considering (4.7), Wu and Wu [89] have presented a sufficient condition for this as follows.

Theorem 4.3.13. *[89, Theorem 4.1] Let $C^* \neq \emptyset$.*

(i) If (4.7) *holds for $x^* \in C^* \cup C_*$ and all $y^* \in \Lambda(x^*)$, then $x^* \in \Lambda(x^*) = C^*$.*

(ii) If for each $x^* \in C^*$ there exits $y^* \in \Lambda(x^*)$ such that (4.7) *holds, then $C^* \subseteq C_*$.*

(iii) If for each $x^* \in C_*$ there exists $y^* \in C^*$ such that (4.7) *holds, then $C_* \subseteq C^*$.*

(iv) If (4.7) *holds for each $x^* \in C^* \cup C_*$ and each $y^* \in \Lambda(x^*)$, then*

$$C^* = \Lambda(x^*) = C_* \text{ for each } x^* \in C^* \cup C_*.$$

Based on the results above, we state some sufficient conditions for

$$C^* = C_* = \Gamma(x^*) = \Lambda(x^*) \quad \text{for } x^* \in C^*$$

under the assumption of the pseudomonotonicity of F on C.

Proposition 4.3.14. *[55, Proposition 4] Let F be pseudomonotone on C and $x^* \in C^*$. If F is constant on $\Gamma(x^*)$, then*

$$C^* = C_* = \Gamma(x^*) = \Lambda(x^*).$$

And hence F is constant on C^*.

Proof. By assumption and Proposition 4.3.9, we have $C^* \subseteq C_* \subseteq \Gamma(x^*)$. Since F is constant on $\Gamma(x^*)$, Proposition 4.3.12 implies that $\Gamma(x^*) \subseteq C^*$. So

$$C^* = \Gamma(x^*) = C_*.$$

For $c \in \Gamma(x^*)$, we have

$$\langle F(x^*), x^* - c\rangle = g(x^*) = 0,$$

so $\langle F(c), x^* - c\rangle = 0 = G(x^*)$. Therefore, $c \in \Lambda(x^*)$, which implies that $\Gamma(x^*) \subseteq \Lambda(x^*)$.

Now let $c \in \Lambda(x^*)$. Then

$$\langle F(c), x^* - c\rangle = G(x^*) = 0.$$

The pseudomonotonicity of F on C implies that $\langle F(x^*), x^* - c\rangle \geq 0$. In this case,

$$\langle F(x^*), x^* - c\rangle = 0 = g(x^*) \quad \text{since } x^* \in C^*.$$

Thus $c \in \Gamma(x^*)$ and hence $\Lambda(x^*) \subseteq \Gamma(x^*)$. Therefore,

$$C^* = C_* = \Lambda(x^*) = \Gamma(x^*).$$

□

Remark 4.3.15. From the definition of $\Gamma(x^*)$, we see that for $x^* \in C^*$, it is the solution to minimize $f(x) = \langle F(x^*), x - x^*\rangle$ subject to $x \in C$. Under the conditions of Proposition 4.3.14, the solution set C^* to the VIP(C, F) and C_* to the DVIP(C, F) can be determined by $\Gamma(x^*)$.

We note that Proposition 4.3.14 is presented under the assumption that F is pseudomonotone on C and F is constant on $\Gamma(x^*)$ for $x^* \in C^*$. We apply a simple example to see that, for $x^* \in C^*$, the constancy of F on $\Lambda(x^*)$ is not sufficient for $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ under the assumption of the pseudomonotonicity of F on C.

Example 4.3.16. Let $C = [0, 1]$ and $F(x) = x$. Then

$$g(x) = \sup\{x(x - c) : c \in [0, 1]\} = \begin{cases} x^2 - x & \text{if } x < 0; \\ x^2 & \text{if } x \geq 0. \end{cases}$$

The solution set C^* is to find $x^* \in [0, 1]$ such that

$$x^*(x - x^*) \geq 0 \quad \text{for all } x \in [0, 1].$$

So $C^* = \{0\}$. Similarly, the solution set of the DVIP(C, F) C_* consists of the vectors $x_* \in [0, 1]$ such that

$$x(x - x_*) \geq 0 \quad \text{for all } x \in [0, 1].$$

Therefore, $C_* = \{0\}$. Hence $G(0) = g(0) = 0$. In this case, we have

$$\Lambda(0) = \{c \in [0, 1] : c(0 - c) = G(0) = 0\} = \{0\}$$

and

$$\Gamma(0) = \{c \in [0, 1] : F(0)(0 - c) = g(0) = 0\} = C.$$

Obviously F is pseudomonotone on $[0,1]$ since F is monotone on $\mathbb{R}^n$. Moreover, F is constant on $\Lambda(0)$ but not on $\Gamma(0)$. This shows that under the assumption of the pseudomonotonicity of F, the constancy of F on $\Lambda(x^*)$ for $x^* \in C^*$ is not sufficient for $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$.

As a result of Propositions 4.3.6 and 4.3.14, we show two immediate results for the sufficiency for $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ for $x^* \in C^*$.

Proposition 4.3.17. *[55, Proposition 5] Let F be pseudomonotone$^+$ on C. Then, for $x^* \in C^*$, F is constant on $\Gamma(x^*)$ iff*

$$C^* = C_* = \Gamma(x^*) = \Lambda(x^*).$$

Proposition 4.3.18. *[55, Proposition 6] Let F be pseudomonotone$^+$ on C. Then the following are equivalent:*

(i) F is constant on $\Gamma(x^*)$ for each $x^* \in C^*$.

(ii) $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^*$.

(iii) $C^* = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^*$.

(iv) $C^* = \Gamma(x^*)$ for each $x^* \in C^*$.

We end up this section by characterizing some other expressions of the solution sets C^* and C_* as below.

Theorem 4.3.19. *[90, Theorem 4.3] Let $C^* \subseteq C_*$ and $x_* \in C_*$. Then,*

$$\begin{aligned} C^* &\subseteq \{x \in C : \langle F(x), x - x_*\rangle = 0, F(x) \in \partial G(x) \cap \partial G(x_*)\} \\ &= \{x \in C : \langle F(x), x - x_*\rangle \leq 0, F(x) \in \partial G(x) \cap \partial G(x_*)\} \\ &\subseteq \{x \in C : \langle F(x), x - x_*\rangle = 0, F(x) \in \partial G(x)\} \\ &= \{x \in C : \langle F(x), x - x_*\rangle \leq 0, F(x) \in \partial G(x)\} \subseteq C_*. \end{aligned}$$

If either G is Gâteaux differentiable on C_*, then the above six sets coincide with each other.

4.4 Weak sharpness of C^* and C_*

There are some existing results for characterizing the weak sharpness of C^* and C_* by using the dual gap functions G [63, 89]. In this section, we study weakly sharp results by utilizing the primal and dual gap functions g and G, respectively. We note that most of the previous results were discussed by using the dual gap function G, however, for a fixed point $x \in \mathbb{R}^n$, $g(x)$ is usually easier to be calculated since this is a linear program. Motivated by this, we study the weak sharpness results as follows.

We show the sufficiency for the weak sharpness of C^* in terms of the error bound of g in which similar proofs are used with those of [63, Theorem 4.1]. We extend the definition of the weak sharpness of C^* introduced in Section 4.2. Moreover, we define the weak sharpness of the solution set of the DVIP(C, F) as well. Based on these, we present the relations between the weak sharpness of C^* and C_* under certain conditions. As an application, we show the weak sharpness of C^* and C_* in terms of the error bound of $g + G$ on C. In addition, some equivalent sufficient conditions for the weak sharpness of C^* are also studied.

We begin with a result of Marcotte and Zhu [63] which shows that C^* is weakly sharp if and only if G has an error bound on C under some condition.

Theorem 4.4.1. *[63, Theorem 4.1] Let F be continuous and pseudomonotone$^+$ over the compact set C. Let the solution set C^* of the VIP be nonempty. Then C^* is weakly sharp if and only if there exists a positive α such that*

$$G(x) \geq \alpha d_{C^*}(x) \quad \text{for all } x \in C. \tag{4.8}$$

Remark 4.4.2. The locally Lipschitz property of G on C^* needs to be added as a part of the condition in this theorem since this is necessary for the result

$$\lim_{\substack{d_k \to d \\ t_k \to 0}} \frac{G(x^* + t_k d_k) - G(x^*)}{t_k} = \langle \nabla G(x^*), d \rangle.$$

Wu and Wu [89] have proved the same result of Theorem 4.4.1 under the condition that G is Gâteaux differentiable and locally Lipschitz on C^*.

Theorem 4.4.3. *[89, Theorem 5.1] Let G be Gâteaux differentiable and locally Lipschitz on C^*. Suppose that each $x^* \in C^* \cup C_*$ and each $y^* \in \Lambda(x^*)$ satisfy (4.7) and*

$$\langle F(x^*), x^* - y^* \rangle = 0 \text{ and } \langle F(y^*), x^* - y^* \rangle = 0 \Rightarrow F(y^*) = F(x^*).$$

Then C^* is weakly sharp iff there exists $\mu > 0$ such that

$$d_{C^*}(x) \leq \mu G(x) \quad \text{for each } x \in C. \tag{4.9}$$

Similar to Theorems 4.4.1 and 4.4.3, we characterize the weak sharpness of C^* in terms of the error bound of g on C under the condition of the Gâteaux differentiability and locally Lipschitz property of g since this is usually easier to be calculated.

Theorem 4.4.4. *[55, Theorem 1] Let F be monotone on $\mathbb{R}^n$ and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. Suppose that g is Gâteaux differentiable, locally Lipschitz on C^*, and $g(x) < +\infty$ for all $x \in \mathbb{R}^n$. Then C^* is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d_{C^*}(x) \leq g(x) \quad \text{for all } x \in C. \tag{4.10}$$

Proof. Under the given conditions, by Proposition 4.3.14, we have

$$C^* = C_* = \Gamma(x^*) = \Lambda(x^*).$$

If C^* is weakly sharp, then for any $x^* \in C^*$ there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ. \tag{4.11}$$

Since F is constant on $\Gamma(x^*)$, α satisfies (4.11) for all $x^* \in C^*$. It follows that

$$\langle F(x^*), z \rangle \geq \alpha \|z\| \quad \text{for any } z \in T_C(x^*) \cap N_{C^*}(x^*),$$

as proved in [63, Theorem 4.1]. For any $x \in C$, there exists a unique $\overline{x} \in C^*$ such that $\|x - \overline{x}\| = d_{C^*}(x)$, since C^* is convex and closed. Applying the definition of $T_C(\overline{x})$ and Theorem 2.2.1, we have

$$x - \overline{x} \in T_C(\overline{x}) \cap N_{C^*}(\overline{x}).$$

Thus the point $\overline{x} \in C^*$ satisfies:

$$\langle F(\overline{x}), x - \overline{x} \rangle \geq \alpha \|x - \overline{x}\| = \alpha d_{C^*}(x).$$

Since F is monotone on C, we conclude that for $c \in \Gamma(x)$ with $x \in C$,

$$g(x) = \langle F(x), x - c \rangle \geq \langle F(x), x - \overline{x} \rangle \geq \langle F(\overline{x}), x - \overline{x} \rangle \geq \alpha d_{C^*}(x).$$

Conversely, suppose that there exists $\alpha > 0$ such that

$$\alpha d_{C^*}(x) \leq g(x) \quad \text{for each } x \in C.$$

We claim that

$$\alpha B \subseteq F(x^*) + [T_C(x^*) \cap N_{C^*}(x^*)]^\circ \quad \text{for each} \quad x^* \in C^*. \tag{4.12}$$

It is evident that (4.12) holds if $T_C(x^*) \cap N_{C^*}(x^*) = \{0\}$ for $x^* \in C^*$. So it suffices to prove (4.12) to hold if $T_C(x^*) \cap N_{C^*}(x^*) \neq \{0\}$ for $x^* \in C^*$. Now let $0 \neq v \in T_C(x^*) \cap N_{C^*}(x^*)$. Then

$$\langle v, v\rangle > 0 \quad \text{and} \quad \langle v, y^* - x^*\rangle \leq 0 \quad \text{for each } y^* \in C^*,$$

which implies that C^* is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - x^*\rangle = 0\}.$$

Since $v \in T_C(x^*)$, by [10, Theorem 2.4.5], for each positive sequence $\{t_i\}$ decreasing to 0, there exists a sequence $\{v_i\}$ converging to v such that $x^* + t_i v_i \in C$ for sufficiently large i. Thus $\langle v, v_i\rangle > 0$ holds for sufficiently large i, and hence we suppose that $x^* + t_i v_i$ lies in the open set $\{x \in \mathbb{R}^n : \langle v, x - x^*\rangle > 0\}$. Therefore,

$$d_{C^*}(x^* + t_i v_i) \geq d_{H_v}(x^* + t_i v_i) = \frac{t_i\langle v, v_i\rangle}{\|v\|},$$

and hence, by (4.10),

$$g(x^* + t_i v_i) \geq \alpha d_{C^*}(x^* + t_i v_i) \geq \alpha t_i \frac{\langle v, v_i\rangle}{\|v\|}.$$

By Proposition 3.5.2 $g(x^*) = 0$ for any $x^* \in C^*$, so

$$g(x^* + t_i v_i) = g(x^* + t_i v_i) - g(x^*) \geq \alpha t_i \frac{\langle v, v_i\rangle}{\|v\|}.$$

Since g is Gâteaux differentiable and locally Lipschitz on C^*, there hold

$$\|g(x^* + t_i v) - g(x^* + t_i v_i)\| \leq L t_i \|v_i - v\|$$

for some $L > 0$ and all sufficiently large i and

$$\begin{aligned}\langle \nabla g(x^*), v\rangle &= \lim_{i\to\infty} \frac{g(x^* + t_i v) - g(x^*)}{t_i} \\ &= \lim_{i\to\infty} \frac{g(x^* + t_i v_i) - g(x^*)}{t_i} \geq \alpha\|v\|.\end{aligned}$$

By Proposition 3.5.6, $\nabla g(x^*) = F(x^*)$. Thus

$$\langle F(x^*), v\rangle \geq \alpha\|v\| \quad \text{for each } v \in T_C(x^*) \cap N_{C^*}(x^*).$$

This implies that for each $w \in B$,

$$\langle \alpha w - F(x^*), v\rangle = \langle \alpha w, v\rangle - \langle F(x^*), v\rangle \leq \alpha\|v\| - \alpha\|v\| = 0.$$

Hence $\alpha B - F(x^*) \subseteq [T_C(x^*) \cap N_{C^*}(x^*)]^\circ$, that is,

$$\alpha B \subseteq F(x^*) + [T_C(x^*) \cap N_{C^*}(x^*)]^\circ.$$

This shows that C^* is weakly sharp since F is constant on C^*. □

Theorem 4.4.4 presents the weak sharpness of C^* in terms of g instead of G.

Following the definition of the weak sharpness of C^* in (4.3) and according to [7], since

$$\operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \subseteq \operatorname{int} \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ,$$

we extend the definition of weak sharpness of the solution set of the VIP(C, F) as follows.

Definition 4.4.5. *[54, Definition 4.1]* C^* is said to be *weakly sharp* provided that

$$-F(x^*) \in \operatorname{int} \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ \quad \text{for each } x^* \in C^*. \tag{4.13}$$

This is equivalent to saying that for each $x^* \in C^*$ there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ,$$

where B denotes the closed unit ball in $\mathbb{R}^n$.

Similarly, C_* is said to be *weakly sharp* provided that

$$-F(x_*) \in \operatorname{int} \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ \quad \text{for each } x_* \in C_*.$$

The advantage of this extended definition is that the relationship between the weak sharpness of C^* and C_* can immediately be obtained as the following proposition states.

Proposition 4.4.6. *[54, Proposition 4.2]*

(i) Let $C^* \subseteq C_*$. If C_* is weakly sharp, then C^* is weakly sharp as well.

(*ii*) *Let* $C_* \subseteq C^*$. *If* C^* *is weakly sharp, then so is* C_*.

Proof. (*i*) Suppose that C_* is weakly sharp. Then by Definition 4.4.5

$$-F(x_*) \in \operatorname{int} \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ \quad \text{for each } x_* \in C_*.$$

Since $C^* \subseteq C_*$, it follows that

$$-F(x^*) \in \operatorname{int} \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ \quad \text{for each } x^* \in C^*,$$

Hence C^* is weakly sharp.

(*ii*) The proof is similar to (*i*), so it is omitted. □

Based on this extended definition for the weak sharpness of C^* and C_*, we use similar proofs of Theorem 4.4.4 to show their weak sharpness results in terms of the error bound of $g + G$ on C.

Theorem 4.4.7. *[54, Theorem 4.3] Let* F *be constant on* C^*. *Suppose that* $g(x) \geq G(x)$ *for all* $x \in \mathbb{R}^n$ *and that* $g + G$ *is Gâteaux differentiable and locally Lipschitz on* C^*. *If there exists* $\alpha > 0$ *such that*

$$\alpha d_{C^* \cap C_*}(x) \leq (g + G)(x) \quad \textit{for each } x \in C,$$

Then C^* *is weakly sharp. In particular, if* $C^* = C_*$, *then the above sufficient condition is also necessary.*

Proof. By assumption, we have $C^* \subseteq C_*$, so $C^* \cap C_* = C^*$ and $C^* \cup C_* = C_*$.

Suppose that there exists $\alpha > 0$ such that

$$\alpha d_{C^* \cap C_*}(x) \leq (g + G)(x) \quad \text{for each } x \in C.$$

Since F is constant on C^*, it suffices to show that there holds

$$\delta B \subseteq F(\overline{x}) + [T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})]^\circ \quad \text{for each } \overline{x} \in C^* \text{ with } \delta = \frac{\alpha}{2}. \tag{4.14}$$

It is obvious that (4.14) holds if $T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x}) = \{0\}$ for $\overline{x} \in C^*$.

If $0 \neq v \in T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})$ for $\overline{x} \in C^*$, then

$$\langle v, v \rangle > 0 \quad \text{and} \quad \langle v, \overline{y} - \overline{x} \rangle \leq 0 \quad \text{for each } \overline{y} \in C^* \cup C_*,$$

which implies that C^* is separated from $\overline{x} + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - \overline{x} \rangle = 0\}.$$

Since $v \in T_C(\overline{x})$, according to [10, Theorem 2.4.5], there exist a sequence $\{v_i\}$ converging to v and a positive sequence $\{t_i\}$ decreasing to 0 such that for each index i we have $\overline{x} + t_i v_i \in C$. Therefore,

$$d_{C^*}(\overline{x} + t_i v_i) \geq d_{H_v}(\overline{x} + t_i v_i) = t_i \frac{\langle v, v_i \rangle}{\|v\|}.$$

By assumption, we have

$$(g + G)(\overline{x} + t_i v_i) - (g + G)(\overline{x}) \geq \alpha d_{C^* \cap C_*}(\overline{x} + t_i v_i) = \alpha d_{C^*}(\overline{x} + t_i v_i).$$

Since $g + G$ is Gâteaux differentiable and locally Lipschitz on C^*, by Proposition 3.5.20, we have

$$\langle 2F(\overline{x}), v \rangle = \langle \nabla(g + G)(\overline{x}), v \rangle = \lim_{i \to \infty} \frac{(g + G)(\overline{x} + t_i v_i) - (g + G)(\overline{x})}{t_i} \geq \alpha \|v\|.$$

Let $w \in B$. Then

$$\left\langle \frac{\alpha}{2} w - F(\overline{x}), v \right\rangle = \frac{\alpha}{2} \langle w, v \rangle - \langle F(\overline{x}), v \rangle \leq \frac{\alpha}{2} \|v\| - \frac{\alpha}{2} \|v\| = 0.$$

Hence $\frac{\alpha}{2} B - F(\overline{x}) \subseteq [T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})]^\circ$.

Next if C^* is weakly sharp and $C^* = C_*$, then by Definition 4.4.5 there exists $\delta > 0$ such that

$$\delta B \subseteq F(x^*) + \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for each } x^* \in C^*$$

since F is constant on C^*. From the proof of [63, Theorem 4.1], this is equivalent to saying that

$$\langle F(x^*), z \rangle \geq \delta \|z\| \quad \text{for each } z \in T_C(x^*) \cap N_{C^*}(x^*) \text{ and each } x^* \in C^*.$$

Since C_* is closed and convex and $C^* = C_*$, for each $x \in C$ there exists unique $c^* \in C^*$ such that $d_{C^*}(x) = \|x - c^*\|$. It follows that

$$x - c^* \in T_C(c^*) \cap N_{C^*}(c^*).$$

Hence the point c^* satisfies

$$(g + G)(x) \geq 2G(x) \geq 2\langle F(c^*), x - c^* \rangle \geq 2\delta \|x - c^*\| = 2\delta d_{C^*}(x).$$

Taking $\alpha = 2\delta$, we have

$$\alpha d_{C^* \cap C_*}(x) = \alpha d_{C^*}(x) \leq (g + G)(x) \quad \text{for each } x \in C.$$

The proof is complete. $\square$

Remark 4.4.8. As mentioned above, Wu and Wu have characterized the weak sharpness of C^* in Theorem 4.4.3 under the condition that G is Gâteaux differentiable and locally Lipschitz on C^*. By presenting relations between g and G in Theorem 4.4.7, the same result was proposed in terms of the error bound of $g+G$ on C. Under the conditions of Theorem 4.4.7, the existence of positive μ satisfying $d_{C^*\cap C_*}(x) \leq \mu G(x)$ for all $x \in C$ is also sufficient for the weak sharpness of C^* since $G(x) \leq g(x)$ for $x \in \mathbb{R}^n$. In this case, $(g+G)(x) \leq 2g(x)$ for all $x \in \mathbb{R}^n$. So if the condition that $g+G$ is Gâteaux differentiable and locally Lipschitz on C^* is replaced by a stronger one that g is Gâteaux differentiable and locally Lipschitz on C^*, by Propositions 3.5.8 and 3.5.28, we have an immediate result that $d_{C^*\cap C_*}(x) \leq \mu g(x)$ for each $x \in C$ with some $\mu > 0$ is still a sufficient condition for the weak sharpness of C^*.

Corollary 4.4.9. *Let F be constant on C^*. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and that g is Gâteaux differentiable and locally Lipschitz on C^*. Then C^* is weakly sharp if there exists $\alpha > 0$ such that*

$$\alpha d_{C^*\cap C_*}(x) \leq g(x) \quad \text{for each } x \in C.$$

Clearly the condition of Theorem 4.4.9 is weaker than that of Theorem 4.4.4. In addition, similar proofs of Theorem 4.4.7 can be applied to the theorem below for discussing sufficient conditions for the weak sharpness of C_*.

Theorem 4.4.10. *[54, Theorem 4.5] Let $\partial g(x_*) \neq \emptyset$ for each $x_* \in C_*$. Suppose that $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$ and that G is Gâteaux differentiable and $g+G$ is locally Lipschitz on C_*. If there exists $\alpha > 0$ such that*

$$\alpha d_{C^*\cap C_*}(x) \leq (g+G)(x) \quad \text{for each } x \in C,$$

Then C_* is weakly sharp.

Proof. By assumption, we have $C_* \subseteq C^*$, that is, $C^* \cap C_* = C_*$.

Suppose that there exists $\alpha > 0$ such that

$$\alpha d_{C^*\cap C_*}(x) \leq (g+G)(x) \quad \text{for each } x \in C.$$

We claim that

$$\delta B \subseteq F(\overline{x}) + [T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x})]^\circ \quad \text{for each } \overline{x} \in C_* \text{ with } \delta = \frac{\alpha}{2}. \tag{4.15}$$

Obviously, (4.15) holds if $T_C(\overline{x}) \cap N_{C^*\cup C_*}(\overline{x}) = \{0\}$ for $\overline{x} \in C_*$.

If $0 \neq v \in T_C(\overline{x}) \cap N_{C^* \cup C_*}(\overline{x})$ for $\overline{x} \in C_*$, then

$$\langle v, v \rangle > 0 \text{ and } \langle v, \overline{y} - \overline{x} \rangle \leq 0 \quad \text{for all } \overline{y} \in C^* \cup C_* = C^*.$$

Therefore, C_* is separated from $\overline{x} + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - \overline{x} \rangle = 0\}.$$

Since $v \in T_C(\overline{x})$, there exist a sequence $\{v_i\}$ converging to v and a positive sequence $\{t_i\}$ decreasing to 0 such that for each index i there holds $\overline{x} + t_i v_i \in C$. Hence we have

$$(g+G)(\overline{x} + t_i v_i) - (g+G)(\overline{x}) \geq \alpha d_{C_*}(\overline{x} + t_i v_i) \geq \alpha d_{H_v}(\overline{x} + t_i v_i) = \alpha t_i \frac{\langle v, v_i \rangle}{\|v\|}.$$

By Proposition 3.5.18, the Gâteaux differentiability of G on C_* implies that g is Gâteaux differentiable on C_* with $\nabla g(x_*) = \nabla G(x_*) = F(x_*)$ for each $x_* \in C_*$ and F is constant on C_*, that is, we have

$$\nabla(g+G)(x_*) = 2F(x_*) \quad \text{for each } x_* \in C_*.$$

If $g + G$ is locally Lipschitz on C_*, then

$$\langle 2F(\overline{x}), v \rangle = \langle \nabla(g+G)(\overline{x}), v \rangle = \lim_{i \to \infty} \frac{(g+G)(\overline{x} + t_i v_i) - (g+G)(\overline{x})}{t_i} \geq \alpha \|v\|.$$

Let $u \in B$. Then

$$\left\langle \frac{\alpha}{2} u - F(\overline{x}), v \right\rangle = \frac{\alpha}{2} \langle u, v \rangle - \langle F(\overline{x}), v \rangle \leq \frac{\alpha}{2} \|v\| - \frac{\alpha}{2} \|v\| = 0.$$

Thus $\frac{\alpha}{2} B - F(\overline{x}) \subseteq [T_C(\overline{x}) \cap N_{C^* \cap C_*}(\overline{x})]^\circ$. This implies that (4.15) holds. And hence C_* is weakly sharp since F is constant on C_*. $\square$

Theorem 4.4.10 characterizes the weak sharpness of C_* under the assumption that $g(x) \leq G(x)$ for all $x \in \mathbb{R}^n$. Wu and Wu [89] have stated two equivalent statements for the weak sharpness of C^* since for $x^* \in C^*$ we have

$$T_{C^*}(x^*) = [N_{C^*}(x^*)]^\circ \subseteq [T_{C^*}(x^*) \cap N_{C^*}(x^*)]^\circ.$$

Theorem 4.4.11. *[89, Theorem 5.4] Let G be Gâteaux differentiable on C^*. Suppose that each $x^* \in C^* \cup C_*$ and each $y^* \in \Lambda(x^*)$ satisfy (4.7) and*

$$\langle F(x^*), x^* - y^* \rangle = 0 \text{ and } \langle F(y^*), x^* - y^* \rangle = 0 \Rightarrow F(y^*) = F(x^*).$$

Then the following are equivalent:

(*i*) $-F(x^*) \in \text{int} \bigcap_{x \in C^*} T_{C^*}(x)$ *for each* $x^* \in C^*$.

(*ii*) *There exists* $\mu > 0$ *such that*

$$d_{C^*}(x) \leq \mu G(x) \text{ for each } x \in \mathbb{R}^n.$$

Motivated by their results, we note that

$$-F(x^*) \in \text{int} \bigcap_{x \in C^* \cap C_*} T_{C^*}(x)$$

is also sufficient for the weak sharpness of C^* since

$$\text{int} \bigcap_{x \in C^* \cap C_*} T_{C^*}(x) = \text{int} \bigcap_{x \in C^* \cap C_*} [N_{C^*}(x)]^\circ \subseteq \text{int} \bigcap_{x \in C^* \cap C_*} [T_{C^*}(x) \cap N_{C^* \cup C_*}(x)]^\circ.$$

Then we use similar proofs of Theorem 4.4.11 to present this equivalence by considering both the error bounds of G and $g + G$ on $\mathbb{R}^n$.

Theorem 4.4.12. *[54, Theorem 4.6] Let* C^* *be closed and convex and* F *constant on* C^*. *Suppose that* $g(x) \geq G(x)$ *for all* $x \in \mathbb{R}^n$ *and that* $(g+G)(x)$ *is Gâteaux differentiable on* C^*. *Then the following are equivalent:*

(*i*) $-F(x^*) \in \text{int} \bigcap_{x \in C^* \cap C_*} T_{C^*}(x)$ *for each* $x^* \in C^*$.

(*ii*) *There exists* $\alpha > 0$ *such that* $\alpha d_{C^* \cap C_*}(x) \leq G(x)$ *for each* $x \in \mathbb{R}^n$.

(*iii*) *There exists* $\alpha > 0$ *such that*

$$\alpha d_{C^* \cap C_*}(x) \leq (g+G)(x) \quad \text{for each } x \in \mathbb{R}^n. \tag{4.16}$$

Proof. By assumption, we have $0 \leq G(x^*) \leq g(x^*) = 0$ for all $x^* \in C^*$, so $C^* \subseteq C_*$ and $C^* \cap C_* = C^*$.

(*i*) $\Rightarrow$ (*ii*) : Let (*i*) hold. Then since F is assumed to be constant on C^*, there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + T_{C^*}(x^*) = F(x^*) + [N_{C^*}(x^*)]^\circ \quad \text{for each } x^* \in C^*.$$

This implies that for each $x^* \in C^*$ and each $u \in B$ we have

$$\langle \alpha u - F(x^*), v \rangle \leq 0 \quad \text{for each } v \in N_{C^*}(x^*).$$

Let $u = \frac{v}{\|v\|}$ for $v \neq 0$. Then

$$\langle F(x^*), v \rangle \geq \alpha \|v\| \quad \text{for each } v \in N_{C^*}(x^*).$$

Since C^* is closed and convex, for each $x \in \mathbb{R}^n$ there exists a unique $\overline{x} \in C^*$ such that

$$d_{C^* \cap C_*}(x) = d_{C^*}(x) = \|x - \overline{x}\|,$$

which yields that $x - \overline{x} \in N_{C^*}(\overline{x})$. Therefore,

$$G(x) \geq \langle F(\overline{x}), x - \overline{x}\rangle \geq \alpha\|x - \overline{x}\| = \alpha d_{C^* \cap C_*}(x).$$

$(ii) \Rightarrow (iii)$ is immediate from the inequality $G(x) \leq g(x)$ for all $x \in \mathbb{R}^n$. It remains to prove $(iii) \Rightarrow (i)$.

Suppose that (4.16) holds for some $\alpha > 0$. We claim that

$$\delta B \subseteq F(x^*) + T_{C^*}(x^*) = F(x^*) + [N_{C^*}(x^*)]^\circ \tag{4.17}$$

for each $x^* \in C^*$ with $\delta = \frac{\alpha}{2}$.
It is clear that (4.17) holds for $x^* \in C^*$ if $N_{C^*}(x^*) = \{0\}$. It remains to prove that (4.17) holds for $x^* \in C^*$ with $N_{C^*}(x^*) \neq \{0\}$.

Let $0 \neq v \in N_{C^*}(x^*)$. Then

$$\langle v, v\rangle > 0 \text{ and } \langle v, y^* - x^*\rangle \leq 0 \quad \text{for each } y^* \in C^*.$$

Thus C^* is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in \mathbb{R}^n : \langle v, x - x^*\rangle = 0\}.$$

Therefore for each positive sequence $\{t_i\}$ decreasing to 0, $x^* + t_i v$ lies in the open set $\{x \in \mathbb{R}^n : \langle v, x - x^*\rangle > 0\}$. Hence

$$d_{C^* \cap C_*}(x^* + t_i v) = d_{C^*}(x^* + t_i v) \geq d_{H_v}(x^* + t_i v) = t_i\|v\|.$$

From (4.16) we have

$$(g + G)(x^* + t_i v) - (g + G)(x^*) \geq \alpha d_{C^* \cap C_*}(x^* + t_i v) \geq \alpha t_i\|v\|.$$

Since $(g + G)(x)$ is Gâteaux differentiable on C^*, by Proposition 3.5.20,

$$\langle 2F(x^*), v\rangle = \lim_{i \to \infty} \frac{(g + G)(x^* + t_i v) - (g + G)(x^*)}{t_i} \geq \alpha\|v\|.$$

Therefore for each $u \in B$ we have

$$\left\langle \frac{1}{2}\alpha u - F(x^*), v \right\rangle = \frac{1}{2}\alpha\langle u, v\rangle - \langle F(x^*), v\rangle \leq \frac{\alpha}{2}\|v\| - \frac{\alpha}{2}\|v\| = 0,$$

from which we obtain that $\delta B - F(x^*) \subseteq T_{C^*}(x^*)$. This implies that (4.17) holds since F is constant on C^*. The proof is complete. $\square$

Remark 4.4.13. In Theorem 4.4.12, we present the equivalence of three sufficient conditions for the weak sharpness of C^*. We note that the equivalence $(i) \Leftrightarrow (ii)$ in this theorem has been proved by Wu and Wu in Theorem 4.4.11 in terms of some restrictions of the relevant mapping F and the Gâteaux differentiability of G. By considering the Gâteaux differentiability of $g + G$ instead, we state that $(i) - (iii)$ are equivalent under the assumption that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$.

4.5 Summary and further research

In this chapter, we study weakly sharp solutions of primal and dual variational inequality problems.

We discuss sufficient conditions for the constancy of F on C^* in Section 4.3 (Propositions 4.3.1 and 4.3.6). We summarize some expressions of the solution sets C^* and C_*. In addition, the relation $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ for $x^* \in C^*$ is also presented since this is closely related to the weak sharpness of C^* and C_* (Propositions 4.3.14, 4.3.17 and 4.3.18). Based on these observations, the weak sharpness of C^* and C_* are proposed in Section 4.4. We state several sufficient conditions for the weak sharpness results of the VIP(C, F) and DVIP(C, F) (Theorems 4.4.4, 4.4.7 and 4.4.10). Moreover, several equivalent conditions for the weak sharpness of C^* are also studied.

An interesting question for future research is that if it is possible to show the differences between Theorems 4.4.3, 4.4.4 and 4.4.7 by some numerical examples.

Chapter 5

Convergence results for solving the VIP(C, F)

5.1 Introduction

This chapter concentrates on finite convergence of algorithms for solving variational inequality problems.

As mentioned before, there are various approaches for solving variational inequalities. In Chapter 4, we present sufficient conditions for $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ for $x^* \in C^*$. In this case, the solution sets to primal and dual variational inequality problems are determined by optimized sets related to gap functions. The iterative methods are also widely used for solving the VIP(C, F), i.e., the proximal point algorithm [65, 81], the extragradient method [46], the gradient projection method [24, 50] and the hybrid method [34, 72]. From the references mentioned, it shows that under some conditions the sequences generated by iterate schemes converge to a solution of the VIP(C, F). In this chapter, finite termination of the sequences which are generated by projection methods for solving the VIP(C, F) are presented.

There are many results of finite termination of projection-type methods for solving the VIP(C, F). One popular scheme of projection-type methods is proximal point algorithm (PPA) which was introduced by Martinet [65]. This algorithm has been refined and extended by Rockafellar [81] to variational inequality problems. The classical iterate method of PPA for solving the VIP(C, F) is exact PPA which is described as follows. Let $x_0 \in \mathbb{R}^n$ be given. For each successive $k \in \mathbf{K} := \{0, 1, 2, ...\}$, if $x_k \notin C^*$, then let

$$x_{k+1} = P_C[x_k - \alpha_k F(x_{k+1})],$$

where α_k is determined by some stepsize rule. Both numerical experiments and theoretical results show that the PPA has nice convergence properties. So it has been widely studied, see [17] and the references therein. However, there are very few results concerning finite convergence of this algorithm. It was Rockafellar who first solved the VIP(C, F) by applying the finite convergence of this algorithm in [81]. He showed that the sequence $\{x_k\}$ generated by the exact PPA converges globally to a solution $x^* \in C^*$ under the condition that F is continuous and monotone on C and $\{\alpha_k\}$ is bounded below. In his work, he also investigated that if $-F(x^*) \in \text{int}N_C(x^*)$, then $\{x_k\}$ reaches at exactly $x^* \in C^*$ after a finite number of iterations. However, this assumption is quite strong since it requires that x^* is the unique solution of the VIP(C, F). Luque [58] extended this result under weaker conditions that C^* is not necessarily a singleton.

The proximal point algorithm has been drawing great attention since the appearing of the seminal work of Ferris [18]. He gave a brief description of the notion of a proximal point which was proposed by Moreau [70] and he demonstrated that, under a weak sharpness condition, the algorithm terminates at a solution after a finite number of iterations. Burke and Ferris [7] further demonstrated that the generated sequences terminate finitely at weakly sharp minima under the condition that the sequence of projected gradients tends to zero. Since Patriksson [79] refined the notion of weakly sharp minima from a convex program into variational inequality problems, Marcotte and Zhu [63] established finite convergence of an algorithm for solving the VIP(C, F) based on the work of Patriksson [79] under the condition of the continuity and pseudomonotonicity$^+$ of F on the compact set C. We note that the condition of this result is quite strict and it has been improved by Xiu and Zhang [92] to the case that F is continuous and pseudomonotone on C. Moreover, Zhou and Wang [101] have further obtained the same result even without the restriction of the pseudomonotonicity of F on C. We also show this result but under different conditions. In addition, we apply an example to explain the advantages and disadvantages of our result.

Moreover, Hu and Song [32] have applied the notion of weak sharpness of C^* into Banach spaces to prove the finite termination of an algorithm. Motivated by their results, Matsushita and Xu [68] proved that the notion of weakly sharp minima is a sufficient condition for finite termination of the same algorithm. Recently, Matsushita and Xu [69] have established the finite termination of the algorithm for solving the VIP(C, F) in terms of the weak sharpness of C^* in Hilbert spaces.

5.2 Finite convergence of algorithms for solving the VIP(C, F)

In this section, we are interested in sufficient conditions for finite termination of algorithms for solving the VIP(C, F).

We present sufficient conditions for $\Gamma(x_n) \subseteq C^*$ for sufficiently large n with a given sequence $\{x_n\} \subseteq \mathbb{R}^n$ in terms of the weak sharpness of C^*. We also discuss the same result by considering error bounds of primal and dual gap functions. Moreover, we show that $x_n \in C^*$ for sufficiently large n under certain conditions.

We start with the following result given by Marcotte and Zhu [63] which presents a sufficient condition for a finite convergence result of an arbitrary algorithm for solving the VIP(C, F).

Theorem 5.2.1. *[63, Theorem 5.1] Let F be continuous and pseudomonotone$^+$ over the set C, and let the solution set C^* of the VIP be weakly sharp. Also let $\{x^k\}$ be a sequence in $\mathbb{R}^n$. If either*

(i) the sequence $\{d_{C^*}(x^k)\}$ converges to zero and the mapping F is uniformly continuous on an open set containing the sequence $\{x^k\}$ and the set C^*, or

(ii) the sequence $\{x^k\}$ converges to some $x^* \in C^*$,

then there exists a positive integer k_0 such that, for any index $k \geq k_0$, any solution of the linear program

$$\min\{\langle F(x^k), x\rangle : x \in C\}$$

is a solution of the VIP.

This result is established under the assumption that F is continuous and pseudomonotone$^+$ on C. Wu and Wu [89] have obtained the same result under weaker conditions as below.

Theorem 5.2.2. *[89, Theorem 5.5] Let F be continuous on C_* and let C^* be weakly sharp. Suppose that each $x^* \in C^*$ and each $y^* \in \Lambda(x^*)$ satisfy (4.7) and*

$$\langle F(x^*), x^* - y^*\rangle = 0 \quad \textit{and} \quad \langle F(y^*), x^* - y^*\rangle = 0 \Rightarrow F(y^*) = F(x^*).$$

If $\{x_k\}$ is a sequence in $\mathbb{R}^n$ satisfying either

(i) $d_{C^*}(x_k) \to 0$ and F is uniformly continuous on C^*, or

(ii) x_k converges to some $x^* \in C^*$,

then $\Gamma(x_k) \subseteq C^*$ *for sufficiently large* k.

As a result of Theorems 5.2.1 and 5.2.2, Xiu and Zhang [92] significantly refined their results under the condition that the convergence point of the sequence $\{x_k\}$ is a weakly sharp solution of the VIP(C, F).

Theorem 5.2.3. *[92, Theorem 3.1] Assume that* F *is continuous and pseudomonotone on* C. *If* $\{x^k\}$ *is a sequence produced by an algorithm for solving the VIP*(C, F)*, and if* $\lim_{k\to\infty} x^k = x^\infty \in C^*$ *and* (4.3) *holds at* x^∞*, then* $\Gamma(x^k) \subseteq C^*$ *for all sufficiently large* k*, where* $\Gamma(x) = \arg\min\{\langle F(x), y\rangle | y \in C\}$.

Next we construct the same result of Theorems 5.2.1, 5.2.2 and 5.2.3 but under weaker conditions.

Theorem 5.2.4. *[55, Theorem 2] Let* C^* *be closed and convex. Suppose that* $\{x_n\}$ *is a bounded sequence in* $\mathbb{R}^n$ *whose accumulation points belong to* C^*. *If for each convergent subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *there holds*

$$-F(x_{n_k}) \in \operatorname{int} \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \textit{for sufficiently large } k, \tag{5.1}$$

then $\Gamma(x_n) \subseteq C^*$ *for sufficiently large* n. *In particular, if for each convergent subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *there holds*

$$-F(x_{n_k}) \in \operatorname{int} \bigcap_{x\in C^*} [N_C(x) \cup T_{C^*}(x)] \quad \textit{for sufficiently large } k, \tag{5.2}$$

then we have $\Gamma(x_n) \subseteq C^*$ *for sufficiently large* n.

Proof. Suppose by contradiction that for each $k \in \mathbb{N}$ there exists $c_{n_k} \in \Gamma(x_{n_k})$ such that $c_{n_k} \notin C^*$. Since $\{x_{n_k}\}$ is bounded, we can assume, by passing to a subsequence if necessary, that $\{x_{n_k}\}$ is a convergent subsequence with limit $\overline{x} \in C^*$. Since C^* is closed and convex, for each $k \in \mathbb{N}$ there exists $c^*_{n_k} \in C^*$ such that $c^*_{n_k} = P_{C^*}(c_{n_k})$. So by definitions of tangent and normal cones we have

$$c_{n_k} - c^*_{n_k} \in T_C(c^*_{n_k}) \cap N_{C^*}(c^*_{n_k}) \quad \text{for all } k \in \mathbb{N}.$$

If $\{x_{n_k}\}$ is convergent and satisfies (5.1), then there exists a positive sequence $\{\delta_k\}$ such that

$$-F(x_{n_k}) + \delta_k B \subseteq \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \subseteq [T_C(c^*_{n_k}) \cap N_{C^*}(c^*_{n_k})]^\circ$$

for sufficiently large k. Therefore,

$$\left\langle -F(x_{n_k}) + \delta_k \frac{c_{n_k} - c^*_{n_k}}{\|c_{n_k} - c^*_{n_k}\|}, c_{n_k} - c^*_{n_k} \right\rangle \leq 0$$

for sufficiently large k. This with the inclusion $c_{n_k} \in \Gamma(x_{n_k})$ implies that

$$\delta_k \leq \langle F(x_{n_k}), \frac{c_{n_k} - c^*_{n_k}}{\|c_{n_k} - c^*_{n_k}\|} \rangle \leq 0,$$

which is a contradiction. Hence $\Gamma(x_n) \subseteq C^*$ for sufficiently large n.

Next, if $\{x_{n_k}\}$ is convergent and satisfies (5.2), then

$$-F(x_{n_k}) \in \operatorname{int} \bigcap_{x \in C^*} \{[T_C(x)]^\circ \cup [N_{C^*}(x)]^\circ\} \subseteq \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$$

for sufficiently large k, that is, (5.1) is satisfied. Hence we obtain the desired result. $\square$

Remark 5.2.5. As mentioned before, the conclusion of Theorem 5.2.4 has been established by Marcotte and Zhu for solving variational inequalities in Theorem 5.2.1 under the condition that F is continuous and pseudomonotone$^+$ on C. Wu and Wu have obtained the same result in Theorem 5.2.2 under different conditions. Both of these theorems are presented by considering the weak sharpness of C^*. Xiu and Zhang have presented a sufficient condition for this in terms of the continuity and pseudomonotonicity of F on C in Theorem 5.2.3. The advantage of their result is that F is not required to be pseudomonotone$^+$ on C. This implies that $\Gamma(x_n) \subseteq C^*$ for sufficiently large n remains to be true although F is not constant on C^*. We note that in Theorems 5.2.1, 5.2.2 and 5.2.3, the sequence $\{x_n\}$ are all assumed to converge to some $x^* \in C^*$. However, in Theorem 5.2.4, the sequence $\{x_n\}$ is only required to be bounded with its accumulation points in C^*.

Next we use an example to show that Theorem 5.2.4 makes more sense than Theorems 5.2.1, 5.2.2 and 5.2.3 in some cases.

Example 5.2.6. Let $C = [-1, 0]$ and

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ x - 1 & \text{if } x \geq 0. \end{cases}$$

Suppose that $\{x_n\}$ is a sequence in $\mathbb{R}$ with

$$x_n = \begin{cases} -1 - \frac{1}{2k+1} & \text{for } n = 2k+1; \\ \frac{1}{2k+2} & \text{for } n = 2k+2, \end{cases}$$

for $k \in \{0\} \cup \mathbb{N}$.

Then the solution set C^* is to find $x^* \in [-1, 0]$ such that

$$F(x^*)(x - x^*) \geq 0 \quad \text{for all } x \in [-1, 0]. \tag{5.3}$$

For $x^* \in [-1, 0)$ we have $F(x^*) = 0$. Clearly any $x^* \in [-1, 0)$ satisfies (5.3) for all $x \in [-1, 0]$. Moreover, $x^* = 0$ also satisfies (5.3) since $-x \geq 0$ for all $x \in [-1, 0]$. Hence $C^* = [-1, 0]$. Obviously C^* is closed and convex.

By definition,

$$\begin{aligned}
\Gamma(x_{2k+2}) &= \arg\min\{F(x_{2k+2}) \cdot y : y \in [-1, 0]\} \\
&= \arg\min\left\{(\frac{1}{2k+2} - 1) \cdot y : y \in [-1, 0]\right\} \\
&= \{0\}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(x_{2k+1}) &= \arg\min\{F(x_{2k+1}) \cdot y : y \in [-1, 0]\} \\
&= \arg\min\left\{F(-1 - \frac{1}{2k+1}) \cdot y : y \in [-1, 0]\right\} \\
&= \arg\min\{0 \cdot y : y \in [-1, 0]\} = [-1, 0].
\end{aligned}$$

Hence $\Gamma(x_n) \subseteq C^*$ for any $n \in \mathbb{N}$.

Alternatively, we apply Theorem 5.2.4 to verify our result while we prove that Theorems 5.2.1, 5.2.2 or 5.2.3 are not applicable.

We first calculate $\bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$. For $x = -1$ we have

$$N_C(-1) = \{v \in \mathbb{R} : v(c+1) \leq 0 \quad \text{for all } c \in [-1, 0]\} = \{v \in \mathbb{R} : v \leq 0\}.$$

This implies that

$$T_C(-1) = \{\xi \in \mathbb{R} : \xi \cdot v \leq 0 \quad \text{for all } v \leq 0\} = \{\xi \in \mathbb{R} : \xi \geq 0\}.$$

Since $C^* = C$, $N_{C^*}(-1) = N_C(-1) = \{v \in \mathbb{R} : v \leq 0\}$. Based on these,

$$T_C(-1) \cap N_{C^*}(-1) = \{0\} \text{ and hence } [T_C(-1) \cap N_{C^*}(-1)]^\circ = \mathbb{R}.$$

For $x \in (-1, 0)$,

$$N_C(x) = \{v \in \mathbb{R} : v(c - x) \leq 0 \quad \text{for all } c \in [-1, 0]\} = \{0\}.$$

So $T_C(x) = \mathbb{R}$ and hence $[T_C(x) \cap N_{C^*}(x)]^\circ = \mathbb{R}$ since $N_{C^*}(x) = N_C(x) = \{0\}$. Similarly, if $x = 0$, then $N_C(0) = \{v \in \mathbb{R} : v \geq 0\}$. So that

$$T_C(0) = \{\xi \in \mathbb{R} | \xi \leq 0\}.$$

Therefore, $[T_C(0) \cap N_{C^*}(0)]^\circ = \mathbb{R}$. Thus we conclude that

$$\bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ = \mathbb{R}.$$

Clearly F is not continuous on [-1, 0]. Furthermore, neither the sequence $\{x_n\}$ nor $d_{C^*}(x_n)$ is convergent. Consequently, Theorems 5.2.1, 5.2.2 and 5.2.3 are not applicable to this example. Since $\{x_n\}$ is bounded and its accumulation points 0 and -1 belong to $C^* = [-1, 0]$ and for each of the convergent subsequence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ there hold

$$-F(x_{2n}) \in \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{and} \quad -F(x_{2n+1}) \in \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$$

for any $n \in \mathbb{N}$, by Theorem 5.2.4 we conclude that $\Gamma(x_n) \subseteq C^*$ for sufficiently large n.

If C^* is weakly sharp and F is continuous on C^*, then, for the sequence $\{x_n\}$ in Theorem 5.2.4, (5.1) is satisfied. In this case, we have the following result.

Theorem 5.2.7. *[55, Theorem 3] Let $\{x_k\}$ be a bounded sequence in C such that $d_{C^*}(x_k)$ converges to zero. Suppose that F is pseudomonotone on C and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. If C^* is weakly sharp and F is continuous on C^*, then there exists a positive integer k_0 such that for any integer $k \geq k_0$,*

$$\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*.$$

Proof. On the given conditions, by Proposition 4.3.14, $C^* = C_*$, which implies that C^* is convex and closed. In addition, the boundedness of $\{x_k\}$ and the limit $d_{C^*}(x_k) \to 0$ imply that the accumulation points of $\{x_k\}$ belong to C^*.

Since C^* is weakly sharp and F is continuous on C^*, (5.1) in Theorem 5.2.4 is satisfied, so the conclusion desired follows. □

When C is compact, based on Lemma 3.5.22 and Theorem 4.4.4, C^* in next result is weakly sharp, so its conclusion is immediate from Proposition 4.3.14 and Theorem 5.2.4.

Corollary 5.2.8. *[55, Corollary 1] Let C be compact and $\{x_k\}$ a sequence in C such that $d_{C^*}(x_k)$ converges to zero. Suppose that F is monotone on $\mathbb{R}^n$, constant on $\Gamma(x^*)$ for some $x^* \in C^*$, and locally Lipschitz on C^* and that g is Gâteaux differentiable on C^*. If for some $\alpha > 0$ there holds $\alpha d_{C^*}(x) \leq g(x)$ for all $x \in C$, then there exists a positive integer k_0 such that for any interger $k \geq k_0$,*

$$\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*.$$

Remark 5.2.9. Theorem 5.2.7 is motivated by Theorem 5.2.1 and both theorems have the same conclusion but under slightly different conditions. In Theorem 5.2.7, F is pseudomonotone (instead of pseudomonotone$^+$) on C and continuous on C^* (instead of on C) but it is constant on $\Gamma(x^*)$. Finally Corollary 5.2.8 is expressed in terms of g instead of G.

Next we recall an algorithm $\arg\min\{v, x\rangle : x \in C\}$ established by Wu and Wu [89] for solving the VIP(C, F) as below.

Theorem 5.2.10. *[89, Theorem 3.2] Let C_1 be a nonempty closed convex subset of C and let*

$$K_1 := \operatorname{int} \bigcap_{x \in C_1} [T_C(x) \cap N_{C_1}(x)]^\circ$$

be nonempty. Then, for each $v \in K_1$, $\arg\max\{\langle v, x\rangle : x \in C\} \subseteq C_1$. Hence, if $C_1 = C_*$ and $-F(x^*) \in K_1$ for each $x^* \in C^*$, then

$$C^* \subseteq C_* = \Gamma(x^*) \textit{ for each } x^* \in C^*.$$

If F is also continuous on C_* or

$$\{v \in \mathbb{R}^n : \langle F(x^*), v\rangle \geq 0\} = \{v \in \mathbb{R}^n : \langle F(y^*), v\rangle \geq 0\},$$

for each $x^* \in C^*$ and each $y^* \in C_*$, then

$$C^* = C_* = \Gamma(x^*) \textit{ for each } x^* \in C^*.$$

Motivated by Theorems 4.4.10, 4.4.12, 5.2.1 and 5.2.10, we derive a finite convergence theorem for solving the VIP(C, F) under the condition that either C^* is weakly sharp or $g + G$ has an error bound on C.

Theorem 5.2.11. *[54, Theorem 4.7] Let $\{x_k\}$ be a sequence in C such that $d_{C^*}(x_k)$ converges to 0 and let F be constant on C^* and uniformly continuous on an open set containing $\{x_k\}$ and C_*. Suppose that $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and that $g + G$ is Gâteaux differentiable and locally Lipschitz on C^*. If*

(i) C^* *is weakly sharp, or*

(ii) *there exists* $\alpha > 0$ *such that* $\alpha d_{C^* \cap C_*}(x) \leq (g + G)(x)$ *for each* $x \in C$,

then $\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*$ *for sufficiently large* k.

Proof. Let (i) hold. Then there exists $\alpha > 0$ such that

$$-F(x^*) + \alpha B \subseteq \bigcap_{x \in C^* \cap C_*} [T_C(x) \cap N_{C^* \cup C_*}(x)]^\circ \quad \text{for each } x^* \in C^*$$

since F is constant on C^*. Under the given conditions, we have $C^* = C_*$ and since C_* is closed and convex, for each x_k there exists a unique $x_k^* \in C^*$ such that $d_{C^*}(x_k) = \|x_k - x_k^*\|$. Therefore the uniformly continuity of F on an open set containing $\{x_k\}$ and C^* implies that

$$\|F(x_k) - F(x^*)\| = \|F(x_k) - F(x_k^*)\| < \alpha \text{ for sufficiently large } k.$$

Thus $-F(x^*) + F(x^*) - F(x_k) \subseteq \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ$, that is,

$$-F(x_k) \in \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ.$$

By Theorem 5.2.10, $\arg\min\{\langle F(x_k), x\rangle : x \in C\} \subseteq C^*$ for sufficiently large k.

Now suppose that (ii) holds. Then, by Theorem 4.4.7, the weak sharpness of C^* can be proved. Hence we get the desired result. □

Remark 5.2.12. Under conditions of Theorem 5.2.11, (i), (ii) or (iii) in Theorem 4.4.12 implies that both (i) and (ii) in Theorem 5.2.11 hold. Hence under the same conditions of Theorem 5.2.11, (i), (ii) and (iii) in Theorem 4.4.12 are all sufficient for the finite convergence algorithm presented in Theorem 5.2.11.

Most of the above results concentrate on $\Gamma(x_n) \subseteq C^*$ for sufficiently large n under different conditions. We note that $\Gamma(x_n)$ does not necessarily contain x_n. Zhou and Wang [101] have shown that x_n is a solution to the weak sharp minima for (4.1) for sufficiently large n if $\lim_{n \to \infty} P_{T_C(x_n)}[-\nabla f(x_n)] = 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function.

Theorem 5.2.13. *[101, Theorem 2] Let* $\overline{C}$ *be a set of weak sharp minima for* (4.1)*. If* $\{x_n\} \subseteq C$*, then* $x_n \in \overline{C}$ *for all* n *sufficiently large if and only if*

$$\lim_{n \to \infty} P_{T_C(x_n)}[-\nabla f(x_n)] = 0.$$

If the notion of the weak sharp minima is applied to the variational inequality, then this result can also be used to solve the VIP(C, F) as well. In the following, we present the results proved by Marcotte and Zhu [63], Xiu and Zhang [92] and Zhou and Wang [101], which state that x_n must be a solution of the VIP(C, F) for sufficiently large n if $\lim_{n\to\infty} P_{T_C(x_n)}[-F(x_n)] = 0$.

We start with a result established by Hiriart-Urruty and Lemaréchal [30] for connecting a normal cone with a projection operator.

Lemma 5.2.14. *[30, Proposition 5.3.3, pp. 69] Let C be convex and $x \in C$. Then the following properties are equivalent:*

(i) $s \in N_C(x)$;

(ii) $x = P_C(x + s)$.

Recall that Burke and Ferris [7] stated a useful result which will be used in the proof of a geometric characterization of sequences achieving the finite identification of a solution to the VIP(C, F).

Lemma 5.2.15. *[7, Lemma 4.6] Let Q be any nonempty closed convex subset of the closed convex set $S \subseteq \mathbb{R}^n$. Then*

$$Q + \bigcap_{x\in Q} [T_S(x) \cap N_Q(x)]^\circ \subseteq \bigcup_{x\in Q} [x + N_S(x)]. \tag{5.4}$$

Based on this, Marcotte and Zhu [63] show that $x_n \in C^*$ for sufficiently large n, where $\{x_n\}$ is a sequence in C.

Theorem 5.2.16. *[63, Theorem 5.2] Let F be pseudomonotone$^+$ and continuous over the compact set C. Let the solution set C^* of the VIP be weakly sharp. Let $\{x_k\}$ be a subsequence with elements in C such that the real sequence $d_{C^*}(x_k)$ converges to zero. If F is uniformly continuous on an open set containing $\{x_k\}$ and C^*, then there exists a positive integer k_0 such that, for any index $k \geq k_0$, x_k is a solution of the VIP if and only if*

$$\lim_{k\to\infty} P_{T_C(x_k)}[-F(x_k)] = 0.$$

Xiu and Zhang [92] have refined this result significantly since their theorem does not require the pseudomonotonicity$^+$ of F as well as the compactness of C.

Theorem 5.2.17. *[92, Theorem 3.2] Assume that F is continuous and pseudomonotone on C. If $\{x_k\} \subseteq C$ is a sequence produced by an algorithm for solving the VIP(C, F) such that $\lim_{k\to\infty} x_k = x_\infty \in C^*$ and*

$$-F(x_\infty) \in int \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ,$$

then $x_k \in C^*$ for all sufficiently large k if and only if

$$\lim_{k\to\infty} P_{T_C(x_k)}[-F(x_k)] = 0.$$

Zhou and Wang [101] have further relaxed the assumptions of the generated sequence $\{x_k\}$ as follows.

Theorem 5.2.18. *[101, Theorem 3] Let F be continuous on C and let the solution set C^* be weakly sharp. If $\{x_k\} \subseteq C$ is bounded and all accumulation points belong to C^*, then $x_k \in C^*$ for all k sufficiently large if and only if*

$$\lim_{k\to\infty} P_{T_C(x_k)}[-F(x_k)] = 0.$$

Following the definition of the weak sharpness of C^* given by Patriksson [79], we note that x_n is still a solution of the VIP(C, F) under certain conditions if

$$\lim_{n\to\infty} \{x_n - P_{T_C(x_n)}[x_n - F(x_n)]\} = 0.$$

Theorem 5.2.19. *Let C^* be closed and convex and let it be weakly sharp. Suppose that $\{x_n\}$ is a sequence in C such that*

$$\lim_{n\to\infty} \{x_n - P_{T_C(x_n)}[x_n - F(x_n)]\} = 0. \tag{5.5}$$

Then $x_n \in C^*$ for sufficiently large n if either

(*i*) *F is monotone on C and constant on C^*, or*

(*ii*) *F is continuous on C^* and $\{x_n\}$ is bounded and all accumulation points belong to C^*.*

Proof. Suppose that for each $k \in \mathbb{N}$ there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \notin C^*$. Since C^* is closed and convex, for each $k \in \mathbb{N}$ there exists $c^*_{n_k} \in C^*$ such that $c^*_{n_k} = P_{C^*}(x_{n_k})$. By definitions of tangent and normal cones we have

$$x_{n_k} - c^*_{n_k} \in T_C(c^*_{n_k}) \cap N_{C^*}(c^*_{n_k}) \quad \text{and} \quad c^*_{n_k} - x_{n_k} \in T_C(x_{n_k}) \quad \text{for all } k \in \mathbb{N}.$$

Applying Moreau decomposition in [30, Theorem 3.2.5, pp. 51],

$$x_{n_k} - F(x_{n_k}) = P_{T_C(x_{n_k})}[x_{n_k} - F(x_{n_k})] + P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})] \quad \text{for all } k \in \mathbb{N}.$$

Thus by (5.5) we have

$$\lim_{k\to\infty} \{P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})] + F(x_{n_k})\} = 0. \tag{5.6}$$

Let (i) hold. Then it follows from the weak sharpness of C^* that

$$-F(c^*_{n_k}) \in \text{int} \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ.$$

Since F is constant on C^*, there exists $\alpha > 0$ such that

$$-F(c^*_{n_k}) + \alpha B \subseteq [T_C(c^*_{n_k}) \cap N_{C^*}(c^*_{n_k})]^\circ.$$

Therefore,

$$\left\langle -F(c^*_{n_k}) + \alpha \frac{x_{n_k} - c^*_{n_k}}{\|x_{n_k} - c^*_{n_k}\|}, x_{n_k} - c^*_{n_k} \right\rangle \le 0 \quad \text{for all } k \in \mathbb{N}.$$

Since F is monotone on C, it follows that for all $k \in \mathbb{N}$ we have

$$\begin{aligned}
\alpha &\le \left\langle -F(c^*_{n_k}), \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&= \left\langle F(x_{n_k}) - F(c^*_{n_k}), \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle + \left\langle P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})], \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&+ \left\langle -F(x_{n_k}) - P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})], \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&\le \left\langle -F(x_{n_k}) - P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})], \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&\le \|F(x_{n_k}) + P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})]\|.
\end{aligned}$$

Taking the limit as k approaches ∞, it follows from (5.6) that $\alpha \le 0$ which is a contradiction.

Suppose that (ii) holds. Since $\{x_{n_k}\}$ is bounded, we can assume that $\{x_{n_k}\}$ is a convergent subsequence with limit $x^* \in C^*$. In addition, the weak sharpness of C^* implies that there exists $\alpha > 0$ such that

$$-F(x^*) + \alpha B \subseteq \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \subseteq [T_C(c^*_{n_k}) \cap N_{C^*}(c^*_{n_k})]^\circ.$$

Therefore, $\left\langle -F(x^*) + \alpha \frac{x_{n_k} - c^*_{n_k}}{\|x_{n_k} - c^*_{n_k}\|}, x_{n_k} - c^*_{n_k} \right\rangle \leq 0$ and hence

$$
\begin{aligned}
\alpha &\leq \left\langle -F(x^*), \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&= \left\langle F(x_{n_k}) - F(x^*), \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle + \left\langle P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})], \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&+ \left\langle -F(x_{n_k}) - P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})], \frac{c^*_{n_k} - x_{n_k}}{\|c^*_{n_k} - x_{n_k}\|} \right\rangle \\
&\leq \|F(x_{n_k}) - F(x^*)\| + \|F(x_{n_k}) + P_{N_C(x_{n_k})}[x_{n_k} - F(x_{n_k})]\|.
\end{aligned}
$$

Since F is continuous on C^* and $\{x_{n_k}\}$ converges to x^*, $\|F(x_{n_k}) - F(x^*)\| \to 0$ as $k \to +\infty$. Combining this with (5.6) and taking the limit as $k \in \mathbb{N}$ approaches ∞, we obtain that $\alpha \leq 0$ which is a contradiction.

Hence $x_n \in C^*$ for all sufficiently large n. □

Under conditions of the continuity of F on C and weak sharpness of C^*, if the sequence $\{x_n\} \subseteq C$ is bounded and all accumulation points belong to C^*, then

$$\lim_{n\to\infty} P_{T_C(x_n)}[-F(x_n)] = 0$$

implies that $x_n \in C^*$ for sufficiently large n, see Theorem 5.2.18. Theorem 5.2.19 shows that x_n is still a solution to VIP(C, F) for sufficiently large n under the assumption of the convergence of a local projection.

If C^* is weakly sharp and the sequence $\{x_n\}$ in Theorem 5.2.19 satisfying $d_{C^*}(x_n) \to 0$, then we have the following result under certain conditions.

Theorem 5.2.20. *Let C^* be closed, convex and weakly sharp and let $\{x_k\}$ be a sequence in C such that $d_{C^*}(x_k)$ converges to zero. Suppose that F is constant on C^* and uniformly continuous on an open set containing $\{x_k\}$ and C^*. If*

$$\lim_{k\to\infty} \{x_k - P_{T_C(x_k)}[x_k - F(x_k)]\} = 0,$$

then there exists a positive integer k_0 such that $x_k \in C^*$ for all $k \geq k_0$.

Proof. Since C^* is weakly sharp and F is constant on C^*, there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(x^*) + \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for all } x^* \in C^*.$$

Let O be an open set containing $\{x_k\}$ and C^* such that F is uniformly continuous on O. Then there exists $0 < \delta < \frac{\alpha}{3}$ such that

$$\|F(x) - F(y)\| < \frac{\alpha}{3} \quad \text{for all } x, y \in O \text{ with } \|x - y\| < \delta.$$

In addition, since $d_{C^*}(x_k)$ converges to zero, there exist $x_k^* \in C^*$ and $k_1 \in \mathbb{N}$ such that $k \geq k_1$ implies $\|x_k - x_k^*\| < \delta$. Thus

$$\|F(x_k) - F(x_k^*)\| < \frac{\alpha}{3} \quad \text{for all } k \geq k_1.$$

By Moreau decomposition [30, Theorem 3.2.5, pp. 51], we have

$$x_k - F(x_k) = P_{T_C(x_k)}[x_k - F(x_k)] + P_{N_C(x_k)}[x_k - F(x_k)] \quad \text{for all } k \in \mathbb{N}.$$

If $\lim_{k\to\infty} \{x_k - P_{T_C(x_k)}[x_k - F(x_k)]\} = 0$, then

$$\lim_{k\to\infty} \{P_{N_C(x_k)}[x_k - F(x_k)] + F(x_k)\} = 0.$$

It follows that there is an integer $k_0 > k_1$ such that $k \geq k_0$ implies that

$$\|P_{N_C(x_k)}[x_k - F(x_k)] + F(x_k)\| < \frac{\alpha}{3}.$$

Hence for $k \geq k_0$ we have

$$\begin{aligned}
\|x_k \;&+\; P_{N_C(x_k)}[x_k - F(x_k)] - [x_k^* - F(x_k^*)]\| \\
&= \|(x_k - x_k^*) + [F(x_k^*) - F(x_k)] + F(x_k) + P_{N_C(x_k)}[x_k - F(x_k)]\| \\
&\leq \|x_k - x_k^*\| + \|F(x_k^*) - F(x_k)\| + \|F(x_k) + P_{N_C(x_k)}[x_k - F(x_k)]\| \\
&< \frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3} = \alpha,
\end{aligned}$$

that is,

$$x_k + P_{N_C(x_k)}[x_k - F(x_k)] - [x_k^* - F(x^*)] \in \alpha B \quad \text{for any } x^* \in C^*$$

due to the constancy of F on C^*. Thus we have

$$x_k + P_{N_C(x_k)}[x_k - F(x_k)] - [x_k^* - F(x^*)] \in F(x^*) + \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ,$$

from which it follows that

$$x_k + P_{N_C(x_k)}[x_k - F(x_k)] \in C^* + \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ.$$

Applying (5.4) to the sets C^* and C, we arrive at

$$C^* + \bigcap_{x\in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \subseteq \bigcup_{x\in C^*} [x + N_C(x)].$$

Consequently,

$$x_k + P_{N_C(x_k)}[x_k - F(x_k)] \in \bigcup_{x\in C^*}[x + N_C(x)] \quad \text{for } k \geq k_0.$$

Thus

$$P_C\{x_k + P_{N_C(x_k)}[x_k - F(x_k)]\} \in P_C\left\{\bigcup_{x\in C^*}[x + N_C(x)]\right\} \quad \text{for } k \geq k_0.$$

Applying Lemma 5.2.14, we obtain x_k satisfying

$$\begin{aligned} x_k &= P_C\{x_k + P_{N_C(x_k)}[x_k - F(x_k)]\} \\ &\in P_C\left\{\bigcup_{x\in C^*}[x + N_C(x)]\right\} \subseteq \cup_{x\in C^*}\{x\} = C^* \quad \text{for } k \geq k_0. \end{aligned}$$

The proof is complete. □

Remark 5.2.21. Theorems 5.2.19 and 5.2.20 present the same result as Theorems 5.2.16, 5.2.17 and 5.2.18. However, we present this by using a new local projection.

Based on Theorems 4.4.7 and 5.2.19, we obtain the next corollary which has the same result with Theorem 5.2.19 but in terms of the error bound of $g+G$ on C.

Corollary 5.2.22. *Let C^* be closed and convex and let $\{x_k\}$ be a sequence in C such that $d_{C^*}(x_k)$ converges to zero. Suppose that F is constant on C^* and uniformly continuous on an open set containing $\{x_k\}$ and C^*, $g(x) \geq G(x)$ for all $x \in \mathbb{R}^n$ and $g+G$ is Gâteaux differentiable and locally Lipschitz on C^*, and that for some $\alpha > 0$ there holds $\alpha d_{C^*}(x) \leq (g+G)(x)$ for each $x \in C$. If*

$$\lim_{k\to\infty}\{x_k - P_{T_C(x_k)}[x_k - F(x_k)]\} = 0,$$

then there exists a positive integer k_0 such that $x_k \in C^*$ for all $k \geq k_0$.

The following result will be obtained by considering some equivalent statements of the weak sharpness of C^*.

Corollary 5.2.23. *Let C^* be closed and convex and let $C^* \subseteq C_*$, F be continuous on C^* and G be Gâteaux differentiable on C_*. Suppose that $\{x_n\}$ is a sequence in C such that (5.5) holds and $\{x_n\}$ is bounded and all accumulation points belong to C^*. Then $x_n \in C^*$ for sufficiently large n if one of the following conditions holds:*

(*i*) *G is locally Lipschitz on* C_* *and there exists* $\mu > 0$ *such that* $d_{C^*}(x) \leq \mu G(x)$ *for each* $x \in C$,

(*ii*) *G is locally Lipschitz on* C_*, $\Gamma(x^*) = C^*$ *and there exists* $\alpha > 0$ *such that*

$$\alpha B \cap [F(x^*) + N_C(x)] = \emptyset \quad \textit{for each } x^* \in C^* \textit{ and each } x \in C \setminus C^*,$$

(*iii*) *there exists* $\mu > 0$ *such that* $d_{C^*}(x) \leq \mu G(x)$ *for each* $x \in \mathbb{R}^n$.

Proof. By [91, Proposition 5.1] we have $C^* = C_*$.

Let (i) hold. Then, by [91, Theorem 5.4], C^* is weakly sharp. Hence by Theorem 5.2.19 we have $x_n \in C^*$ for sufficiently large n.

Suppose that (ii) holds. Then, again by [91, Theorem 5.4], (i) follows. Hence the conclusion is still OK.

Finally, let (iii) hold. Then, by [91, Theorem 5.5],

$$-F(\overline{x}) \in \operatorname{int} \bigcap_{x \in C^*} T_{C^*}(x) = \operatorname{int} \bigcap_{x \in C^*} [N_{C^*}(x)]^\circ \subseteq \operatorname{int} \bigcap_{x \in C^*} [T_C(x) \cap N_{C^*}(x)]^\circ \quad \text{for each } \overline{x} \in C^*.$$

Therefore, C^* is weakly sharp and by applying Theorem 5.2.19, the desired result follows. □

5.3 Summary and future work

In this chapter, we discuss two finite convergence of algorithms for solving the VIP(C, F). We show that $\Gamma(x_n) \subseteq C^*$ for sufficiently large n under weaker conditions than Theorems 5.2.1, 5.2.2 and 5.2.3 (Theorem 5.2.4). In particular, we apply an example to show the advantage of our result (Example 5.2.6). In Theorem 5.2.4, it shows that F is not necessarily continuous and pseudomonotone on C. Moreover, even the sequence $\{x_n\}$ is not convergent, $\Gamma(x_n) \subseteq C^*$ for sufficiently large n still remains to be true. We also present a corollary of Theorem 5.2.4 (Corollary 5.2.8), which describes the establishment of $\Gamma(x_n) \subseteq C^*$ for sufficiently large n in terms of the error bound of g. Motivated by this corollary, we apply the error bound of $g + G$ for presenting this result as well (Theorem 5.2.11). The other part of this chapter is for the finite convergence of an algorithm as $x_n \in C^*$ for sufficiently large n. We first review some existing results of this algorithm for solving the VIP(C, F) (Theorems 5.2.16, 5.2.17 and 5.2.18). We study this result in terms of a new projection (Theorems 5.2.19 and 5.2.20). Finally, we show some corresponding corollaries of this result as well (Corollaries 5.2.22 and 5.2.23).

Since the result of Theorem 5.2.19 is quite abstract, for next step, we will try to make a numerical test for explaining the difference between the convergence of the local projection

$$\lim_{n\to\infty} \{x_n - P_{T_C(x_n)}[x_n - F(x_n)] = 0$$

and

$$\lim_{n\to\infty} P_{T_C(x_n)}[-F(x_n)] = 0.$$

Chapter 6

Conclusion

6.1 Future work for applying variational inequalities in image processing

Variational inequalities have a large amount of applications in various fields, from real engineering, public policy and strategic planning issues. Kitchener, Bouzerdoum and Phung [42] have formulated a generalized image restoration problem as a variational inequality problem.

Consider an ordered vector $\mathbf{g}$ of length $n = l \times m$ which represents a noisy blurred image:

$$\mathbf{g} = H\mathbf{f_0} + \mathbf{v},$$

where H is the blur matrix, $\mathbf{f_0}$ is the original image we seek to restore, and $\mathbf{v}$ is the additive noise vector. Kitchener, Bouzerdoum and Phung [42] transformed the image restoration problem into an optimization problem as

$$\begin{aligned} &\text{minimize } \|R\mathbf{f}\|_{k_1}^{k_1} \\ &\text{subject to } \|H\mathbf{f} - \mathbf{g}\|_{k_2}^{k_2} \leq \epsilon, \mathbf{f} \geq 0, \end{aligned} \tag{6.1}$$

where R is the operator which represents the quantity for minimizing, $\|\cdot\|_{k_1}$ and $\|\cdot\|_{k_2}$ are two given norms, and ϵ is a measure of noise. For simplification, let

$$M(\mathbf{f}) = \|R\mathbf{f}\|_{k_1}^{k_1} \quad \text{and} \quad N(\mathbf{f}) = \|H\mathbf{f} - \mathbf{g}\|_{k_2}^{k_2} - \epsilon.$$

Then the optimization problem can be written as follows:

$$\begin{aligned} &\text{minimize } M(\mathbf{f}) \\ &\text{subject to } N(\mathbf{f}) \leq 0, \mathbf{f} \geq 0. \end{aligned} \tag{6.2}$$

$M(\mathbf{f})$ and $N(\mathbf{f})$ are assumed to be convex functions. Kitchener, Bouzerdoum and Phung [42] have transformed this optimization problem as a problem for solving a variational inequality. Let $L(\mathbf{f},\lambda)$ be the Lagrangian function of (6.2) which is represented as below:

$$L(\mathbf{f},\lambda) = M(\mathbf{f}) + \lambda N(\mathbf{f}).$$

In sections 3.2 and 3.3, we have stated that the solution $(\mathbf{f}^*,\lambda^*)$ to (6.2), if exists, is a saddle point, which satisfies the following inequalities:

$$L(\mathbf{f}^*,\lambda) \leq L(\mathbf{f}^*,\lambda^*) \leq L(\mathbf{f},\lambda^*) \quad \text{for any } \mathbf{f} \geq 0 \text{ and } \lambda \geq 0.$$

Since $L(\mathbf{f},\lambda^*)$ is convex and admits a minimum at $\mathbf{f}^*$. Thus $\mathbf{f}^*$ is a solution to (6.2) if and only if there exists a $\lambda^* \geq 0$ such that

$$\begin{aligned} \langle \nabla M(\mathbf{f}^*) + \nabla N(\mathbf{f}^*)\lambda^*, \mathbf{f} - \mathbf{f}^* \rangle \geq 0 \text{ for any } \mathbf{f} \geq 0 \\ N(\mathbf{f}^*)(\lambda^* - \lambda) \geq 0 \text{ for any } \lambda \geq 0. \end{aligned} \tag{6.3}$$

Hence Equation (6.3) defines a *variational inequality problem* (VIP).

6.2 Conclusions and suggestions for future work

In this thesis, we study gap functions for the VIP(C,F) and propose some properties of gap functions. We characterize weakly sharp solutions of the VIP(C,F) and DVIP(C,F). We state sufficient conditions for the constancy of F on C^*. We also present the minimum principle sufficiency and maximum principle sufficiency properties of the VIP(C,F). In particular, we discuss sufficient conditions for $C^* = C_* = \Gamma(x^*) = \Lambda(x^*)$ for $x^* \in C^*$. Based on these, we obtain the weak sharpness of C^* and C_* by considering the error bounds of g, G and $g+G$. Finally, we introduce a finite convergence algorithm for solving the VIP(C,F) by considering the weak sharpness of C^*. Moreover, we apply an example for describing the improvement of this result compared with the earlier existing results. This finite convergence of algorithm is also derived by using some equivalent statements of the weak sharpness of the solution set of the VIP(C,F). What's more, we construct that x_n is still a solution to the VIP(C,F) for sufficiently large n by considering a new projection, where $\{x_n\}$ is a sequence in C.

Overall, we obtain some new results and methods for characterizing solutions of variational inequality problems. In particular, we characterize weakly sharp solutions of the VIP(C,F) by using the primal gap function g, which is a major breakthrough. However, some of our results are abstract and they need to be tested by some numerical examples in the future.

As discussed in section 6.1, the variational inequality methods have already been used in Engineering, i.e., image processing, and this formulated variational inequality problem is solved by a dynamic system approach in the work of Kitchener etc. [42].

In the future, we will attempt to do interesting work for the applications of variational inequality problems:

(*i*) We will attempt to construct proper gap functions for the variational inequality problem (6.3) to characterize its solutions.

(*ii*) In addition, it is noted that Kitchener, Bouzerdoum and Phung [42] applied the variational inequality approach for restoring cameraman images and we could also utilize this method for depth images in future research.

We will build these results by using the methods we have used before, or by introducing some new approaches for dealing with these problems.

Bibliography

[1] G. Auchmuty. Variational principles for variational inequalities. *Numer. Funct. Anal. Optim.*, 10:863–874, 1989.

[2] A. Auslender. *Optimization. Méthodes numériques. Maîtrise de Mathématiques et Applications Fondamentales.* Masson, Paris- New York-Barcelona, 1976.

[3] H. Brézis and P.-L. Lions. Produits infinis de résolvantes. *Israel J. Math.*, 29:329–345, 1978.

[4] R. S. Burachik and A. N. Iusem. *Set-valued Mappings and Enlargements of Monotone Operators.* Springer Optim. Appl., Springer, New York, 2008.

[5] R. S. Burachik and S. Scheimberg. A proximal point method for the variational inequality problem in Banach spaces. *SIAM J. Control Optim.*, 39:1633–1649, 2001.

[6] J. V. Burke and S. Deng. Weak sharp minima revisited, i: Basic theory. *Control Cybernet.*, 31:439–469, 2002.

[7] J. V. Burke and M. C. Ferris. Weak sharp minima in mathematical programming. *SIAM J. Control Optim.*, 31:1340–1359, 1993.

[8] S. S. Chang, H. W. J. Lee, and C. K. Chan. A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. *Nonlinear Anal.*, 70:3307–3319, 2009.

[9] G. Y. Chen, C. J Goh, and X. Q. Yang. On gap functions and duality of variational inequality problems. *J. Math. Anal. Appl.*, 214:658–673, 1997.

[10] F. H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley-Interscience, New York, 1983.

[11] F. H. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control.* Graduate Texts in Mathematics, Vol. 264, Springer-Verlag, London, 2013.

[12] R. W. Cottle. Complementarity and variational problems. *Sympos. Math.*, 19:177–208, 1976.

[13] R. W. Cottle, J. S. Pang, and R. E. Stone. *The Linear Complementarity Problem.* Academic Press, Boston, 1992.

[14] L. Cromme. Strong uniqueness. A far-reaching criterion for the convergence analysis of iterative procedures. *Numer. Math.*, 29:179–193, 1978.

[15] S. Dafermos. Traffic equilibria and variational inequalities. *Transportation Sci.*, 14:42–54, 1980.

[16] F. Facchinei and J. S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems.* Vols. I and II, Springer-Verlag, New York, 2003.

[17] N. Farouq. Pseudomonotone variational inequalities: convergence of proximal methods. *J. Optim. Theory Appl.*, 109:311–326, 2001.

[18] M. C. Ferris. Finite termination of the proximal point algorithm. *Math. Program.*, 50:359–366, 1991.

[19] M. C. Ferris and O. L. Mangasarian. Minimum principle sufficiency. *Math. Program.*, 57:1–14, 1992.

[20] M. Fukushima. Equivalent differentiable optimization problems and descent methods for asymmetic variational inequality problems. *Math. Program.*, 53:99–110, 1992.

[21] M. Fukushima. Merit functions for variational inequality and complementarity problems. *Book chapter of Nonlinear Optimization and Applications*, pages 155–170, 1996.

[22] F. Giannessi. Separation of sets and gap functions for quasi-variational inequalities. *Proceedings of a conference, Erice, Italy, New York, NY: Plenum*, pages 101–121, 1995.

[23] R. Glowinski, J. L. Lions, and R. Trémolière. *Analyse Numérique des Inéquations Variationelles, volumes I and II.* Dunod-Bordas, Paris, 1976.

[24] A. A. Goldstein. Convex programming in Hilbert space. *Bull. Am. Math. Soc.*, 70:709–710, 1964.

[25] O. Güler. On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.*, 29:403–419, 1991.

[26] O. Güler. *Fundations of Optimization.* Springer-Verlag, New York, 2010.

[27] P. Harker and J. S. Pang. Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Math. Program.*, 48:161–220, 1990.

[28] P. Hartman and G. Stampacchia. On some nonlinear elliptic differential functional equations. *Acta Math.*, 115:271–310, 1966.

[29] D. W. Hearn. The gap function of a convex program. *Oper. Res. Lett.*, 1:67–71, 1982.

[30] J. B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis.* Springer-Verlag, Berlin Heidelberg, 2001.

[31] A. J. Hoffman. On approximate solutions of systems of linear inequalities. *J. Research Nat. Bur. Standards*, 49:263–265, 1952.

[32] Y. H. Hu and W. Song. Weak sharp solutions of variational inequalities in Banach spaces. *J. Math. Anal. Appl.*, 374:118–132, 2011.

[33] H. Iiduka and W. Takahashi. Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. *Nonlinear Anal.*, 61:341–350, 2005.

[34] H. Iiduka, W. Takahashi, and M. Toyoda. Approximation of solutions of variational inequalities for monotone mappings. *Panam. Math. J.*, 14:49–61, 2004.

[35] G. Isac and M. G. Cojocaru. Variational inequalities, complementarity problems and pseudo-monotonicity. dynamical aspects. *Semin. Fixed Point Theory Cluj-Napoca*, 3:41–62, 2002.

[36] A. N. Iusem. An iterative algorithm for the variational inequality problem. *Comput. Appl. Math.*, 13:103–114, 1994.

[37] A. N. Iusem. A generalized proximal point algorithm for the variational inequality problem in Hilbert space. *SIAM J. Optim.*, 8:197–216, 1998.

[38] A. N. Iusem and B. F. Svaiter. A variant of Korpelevich's method for variational inequalities with a new search strategy. *Optimization*, 42:309–321, 1997.

[39] S. Karamardian. The complementarity problem. *Math. Program.*, 2:107–129, 1972.

[40] G. Kassay. The proximal point algorithm for reflexive Banach spaces. *Stud. Univ. Babes-Bolyai Math.*, 30:9–17, 1985.

[41] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*. Academic Press, New York, 1980.

[42] M. A. Kitchener, A. Bouzerdoum, and S. L. Phung. Adaptive regularization for image restoration using a variational inequality approach. *Proceedings of 2010 IEEE 17th International Conference on Image Processing*, pages 2513–2516, 2010.

[43] K. C. Kiwiel. Proximal minimization methods with generalized Bergman function. *SIAM J. Control Optim.*, 35:1142–1168, 1997.

[44] F. Kohsaka and W. Takahashi. Strong convergence of an iterative sequence for maximal monotone operators in a Banach space. *Abstr. Appl. Anal.*, 2004:239–249, 2004.

[45] I. V. Konnov. Application of the proximal point method to nonmonotone equilibrium problems. *J. Optim. Theory Appl.*, 119:317–333, 2003.

[46] G. M. Korpelevich. The extragradient method for finding saddle points and other problems. *Matecon*, 12:747–756, 1976.

[47] T. Larsson and M. Patriksson. A class of gap functions for variational inequalities. *Math. Program.*, 64:53–79, 1994.

[48] C. E. Lemke. Bimatrix equilibrium points and mathematical programming. *Management Sci.*, 11:681–689, 1965.

[49] C. E. Lemke and J. T. Howson. Equilibrium points of bimatrix games. *J. Soc. Indust. Appl. Math.*, 12:413–423, 1964.

[50] E. S. Levitin and B. T. Polyak. Constrained minimization methods. *Comput. Math. Math. Phys.*, 6:1–50, 1966.

[51] A. S. Lewis and J. S. Pang. Error bounds for convex inequality systems. *in: Generalized Convexity, Generalized Monotonicity: Recent Results, Luminy, 1996, Kluwer Acad. Publ., Dordrecht*, pages 75–110, 1998.

[52] J. Li, N. J. Huang, and X. Q. Yang. Weak sharp minima for set-valued vector variational inequalities with an application. *European J. Oper. Res.*, 205:262–272, 2010.

[53] J. L. Lions and G. Stampacchia. Variational inequalities. *Comm. Pure Appl. Math.*, 20:493–519, 1967.

[54] Y. N. Liu and Z. L. Wu. Weakly sharp solutions of primal and dual variational inequality problems. to appear in: Pacific Journal of Optimization.

[55] Y. N. Liu and Z. L. Wu. Characterization of wealy sharp solutions of a variational inequality by its primal gap function. *Optim. Lett.*, 2015. DOI 10.1007/s11590-015-0882-7.

[56] X. D. Luo and Z. Q. Luo. Extensions of Hoffman's error bound to polynomial systems. *SIAM J. Optim.*, 4:383–392, 1994.

[57] Z. Q. Luo and J. F. Sturm. Error bounds for quadratic systems. *in High Performance Optimization, H. Frenk, K. Roos, T. Terlaky, and S. Zhang, eds., Kluwer, Dordrecht, Netherlands*, pages 383–404, 2000.

[58] F. J. Luque. Asymptotic convergence analysis of the proximal point algorithm. *SIAM J. Control Optim.*, 22:277–293, 1984.

[59] O. G. Mancino and G. Stampacchia. Convex programming and variational inequalities. *J. Optim. Theory Appl.*, 9:3–23, 1972.

[60] O. L. Mangasarian. Error bounds for nondegenerate monotone linear complementarity problems. *Math. Program.*, 48:437–445, 1990.

[61] P. Marcotte. Application of Khobotov's algorithm to variational inequalities and network equilibrium problems. *Inform. Systems Oper. Res.*, 29:258–270, 1991.

[62] P. Marcotte and J. P. Dussault. A sequential linear programming algorithm for solving monotone variational inequalities. *SIAM J. Control Optim.*, 27:1260–1278, 1989.

[63] P. Marcotte and D. L. Zhu. Weak sharp solutions of variational inequalities. *SIAM J. Optim.*, 9:179–189, 1998.

[64] B. Martinet. Régularisation d'inéquations variationnelles par approximations successives. *Rev. Fr. Inform. Rech. Opér.*, 4:154–158, 1970.

[65] B. Martinet. Determination approchée d'un point fixe d'une application pseudo-contractante. *C. R. Acad. Sci. Paris*, 274:163–165, 1972.

[66] S. Y. Matsushita and L. Xu. On convergence of the proximal point algorithm in Hilbert spaces. *Proc. Amer. Math. Soc.*, 139:4087–4095, 2011.

[67] S. Y. Matsushita and L. Xu. Finite termination of the proximal point algorithm in Hilbert spaces. *J. Math. Anal. Appl.*, 387:765–769, 2012.

[68] S. Y. Matsushita and L. Xu. Finite convergence of the proximal point algorithm for variational inequality problems. *Set-Valued Var. Anal.*, 21:297–309, 2013.

[69] S. Y. Matsushita and L. Xu. On finite convergence of iterative methods for variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.*, 161:701–715, 2014.

[70] J. J. Moreau. Proximité et dualité dans un espace Hilbertien. *Bull. Soc. Math. France*, 93:273–299, 1965.

[71] U. Mosco. Dual variational inequalities. *J. Math. Anal. Appl.*, 40:202–206, 1972.

[72] N. Nadezhkina and W. Takahashi. Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. *SIAM J. Optim.*, 16:1230–1241, 2006.

[73] N. Nadezhkina and W. Takahashi. Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.*, 128:191–201, 2006.

[74] M. A. Noor. Proximal methods for mixed variational inequalities. *J. Optim. Theory Appl.*, 115:447–452, 2002.

[75] M. A. Noor. Some developments in general variational inequalities. *Appl. Math. Comput.*, 152:199–277, 2004.

[76] M. A. Noor, M. Akhter, and K. I. Noor. Inertial proximal method for mixed quasi variational inequalities. *Nonlinear Funct. Anal. Appl.*, 8:489–496, 2003.

[77] J. S. Pang. Error bounds in mathematical programming. *Math. Program.*, 79:299–332, 1997.

[78] M. Pappalardo, G. Mastroeni, and M. Passacantando. Merit functions: a bridge between optimization and equilibria. *4OR*, 12:1–33, 2014.

[79] M. Patriksson. *A Unified Framework of Descent Algorithms for Nonlinear Programs and Variational Ineuqlities*. Ph.D. thesis, Department of Mathematics, Linköping Institute of Technology, Linköping, Sweden, 1993.

[80] J. M. Peng. Equivalence of variational inequality problems to unconstrained minimization. *Math. Program.*, 78A:347–355, 1997.

[81] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.*, 14:877–898, 1976.

[82] H. E. Scarf. The approximation of fixed points of a continuous mapping. *SIAM J. Appl. Math.*, 15:1328–1342, 1967.

[83] M. J. Smith. The existence, uniqueness and stability of traffic equilibria. *Transpn. Res.*, 13B:295–304, 1979.

[84] G. Stampacchia. Variational inequalities. *Theory and Applications of Monotone Operators, (Proc. NATO Advanced Study Inst., Venice, 1968)*, pages 101–192.

[85] G. Stampacchia. Formes bilinéaries coercitives sur les ensembles convexes. *C.R. Acad. Sc. Paris*, 258:4413–4416, 1964.

[86] G. Stampacchia. Variational Inequalities. *Actes, Congres intern. Math.*, 2:877–883, 1970.

[87] W. Takahashi and M. Toyoda. Weak convergence theorems for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.*, 118:417–428, 2003.

[88] T. Wang and J. S. Pang. Global error bounds for convex quadratic inequality systems. *Optimization*, 31:1–12, 1994.

[89] Z. L. Wu and S. Y. Wu. Weak sharp solutions of variational inequalities in Hilbert spaces. *SIAM J. Optim.*, 14:1011–1027, 2004.

[90] Z. L. Wu and S. Y. Wu. Characterizations of the solution sets of convex programs and variational inequality problems. *J. Optim. Theory Appl.*, 130:339–358, 2006.

[91] Z. L. Wu and S. Y. Wu. Gâteaux differentiability of the dual gap function of a variational inequality. *European J. Oper. Res.*, 190:328–344, 2008.

[92] N. H. Xiu and J. Z. Zhang. On finite convergence of proximal point algorithms for variational inequalities. *J. Math. Anal. Appl.*, 312:148–158, 2005.

[93] H. K. Xu. Viscosity method for hierarchical fixed point approach to variational inequalities. *Taiwanese J. Math.*, 14:463–478, 2010.

[94] N. Yamashita and M. Fukushima. The proximal point algorithm with genuine superlinear convergence for the monotone complementarity problem. *SIAM J. Optim.*, 11:364–379, 2000.

[95] N. Yamashita, K. Taji, and M. Fukushima. Unconstrained optimization reformulations of variational inequality problems. *J. Optim. Theory Appl.*, 92:439–456, 1997.

[96] Z. Yang. A simplicial algorithm for computing robust stationary points of a continuous function on the unit simplex. *SIAM J. Control Optim.*, 34:491–506, 1996.

[97] Y. H. Yao, Y. C. Liou, and C. P. Chen. Algorithms construction for nonexpansive mappings and inverse-strongly monotone mappings. *Taiwanese J. Math.*, 15:1979–1988, 2011.

[98] Y. H. Yao, Y. C. Liou, and R. D. Chen. Convergence theorems for fixed point problems and variational inequality problems in Hilbert spaces. *Math. Nachr.*, 282:1827–1835, 2009.

[99] L. C. Zeng and J. C. Yao. Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. *Taiwanese J. Math.*, 10:1293–1303, 2006.

[100] J. Z. Zhang, C. Y. Wan, and N. H. Xiu. The dual gap function for variational inequalities. *Appl. Math. Optim.*, 48:129–148, 2003.

[101] J. Zhou and C. Wang. A note on finite termination of iterative algorithms in mathematical programming. *Oper. Res. Lett.*, 36:715–717, 2008.

[102] D. L. Zhu and P. Marcotte. An extended descent framework for variational inequalities. *J. Optim. Theory Appl.*, 80:349–366, 1994.