Non-extremal and non-BPS extremal five-dimensional black strings from generalized special real geometry

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Abstract

We construct non-extremal as well as extremal black string solutions in minimal five-dimensional supergravity coupled to vector multiplets using dimensional reduction to three Euclidean dimensions. Our method does not assume that the scalar manifold is a symmetric space, and applies as well to a class of non-supersymmetric theories governed by a generalization of special real geometry. We find that five-dimensional black string solutions correspond to geodesics in a specific totally geodesic para-Kähler submanifold of the scalar manifold of the dimensionally reduced theory, and identify the subset of geodesics that corresponds to regular black string solutions in five dimensions. BPS and non-BPS extremal solutions are distinguished by whether the corresponding geodesics are along the eigendirections of the para-complex structure or not, a characterization which carries over to non-supersymmetric theories. For non-extremal black strings the values of the scalars at the outer and inner horizon are not independent integration constants but determined by certain functions of the charges and moduli. By lifting solutions from three to four dimensions we obtain non-extremal versions of small black holes, and find that while the outer horizon takes finite size, the inner horizon is still degenerate.

Contents

1 Introduction

Black holes provide an important testing ground for ideas of quantum gravity. In the context of string theory and supergravity BPS solutions have been studied extensively since the discovery of the attractor mechanism [1] and of the quantitative matching between microscopic and macroscopic entropy [2, 3, 4, 5]. It was

realized early that many macroscopic features of BPS black holes, in particular the attractor mechanism, do not strongly depend on supersymmetry and can be understood as a consequence of the field equations [6]. More recently the study of non-BPS solutions has received increasing attention starting with [7, 8], and the attractor mechanism for general extremal black holes has been formulated using the entropy function formalism [9]. The knowledge of non-extremal solutions is more limited and less systematic, although many examples of non-extremal black hole and black brane solutions in higher dimensions and in compactified solutions have been known for quite some time [10, 11, 12, 13, 14, 15]. More recently it has been observed that, like BPS and non-BPS extremal solutions, some non-extremal solutions can be obtained by reducing the equations of motion to first order form [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In this paper we will further develop a complementary approach to non-BPS and non-extremal solutions which aims at directly solving the second order field equations using dimensional reduction and the special geometry of supergravity theories with eight supercharges and their time-reduced (Euclidean) versions. The special geometry of Euclidean supergravities has been developed in [26, 27, 28, 29], and applied to extremal five-dimensional black holes [30], non-extremal five-dimensional black holes [31, 32] and extremal four-dimensional black holes [33]. Our formalism does not assume that the scalar target space is a symmetric space, but aims to exploit the fact that for vector multiplets all couplings are encoded in a single homogeneous function, which is real in five dimensions and holomorphic in four dimensions. In five dimensions one can consider models with a degree of homogeneity different from three, which is the degree dictated by supersymmetry, and thus obtain a generalization of the special real geometry of five-dimensional vector multiplets [34], which was dubbed 'generalized special real geometry' in [30, 32]. The non-supersymmetric theories covered by this formalism allow one to make manifest which features of a supergravity theory do not depend on supersymmetry *per se*, but on certain features of the scalar manifolds which supersymmetric theories share with a larger class of theories.

The specific type of solution we investigate in this paper is magnetically

charged black string solutions, both extremal and non-extremal, for five-dimensional supergravity and non-supersymmetric theories described by generalized special real geometry. In five dimensions magnetic charges with respect to vector fields are carried by strings, so that black strings are the 'magnetic partners' of black holes, which only carry electric charges. For minimal five-dimensional supergravity coupled to vector multiplets, BPS black string solutions were constructed in [35]. Like their electric BPS partners they exhibit attractor behaviour, and the Killing spinor equations give rise to generalized stabilization equations which allow one to express solutions in terms of harmonic functions. Static multi-centred BPS solutions can be obtained by choosing multi-centred harmonic functions. More recently non-BPS extremal and non-extremal black string solutions have been found using the FGK formalism [25, 24], which, following the observations of [6], employs an effective potential. In this paper we approach the same problem using the formalism described above. Our main interest is to understand the systematics and general properties of solutions. One aspect is the relation between geodesic curves and totally geodesic submanifolds of the scalar manifold $\mathcal{M}_{(3)}$ of the three-dimensional Euclidean theory, and solutions of the original five-dimensional theory. Dimensional reduction reduces the problem of finding the equations of motion to the problem of finding harmonic maps from the reduced three-dimensional space-'time' (in our case a Riemannian space with positive signature) into $\mathcal{M}_{(3)}$ [36, 28]. Solutions can sometimes be constructed by identifying suitable totally geodesic submanifolds $S \subset \mathcal{M}_{(3)}$, and then finding harmonic maps from the reduced space-time into them. In terms of scalar fields corresponding to local coordinates, finding totally geodesic submanifolds is equivalent to consistently truncating the equations of motion by setting part of the scalar fields to constant values. Since we are interested in black strings in this paper, we truncate out some of the degrees of freedom of the five-dimensional theory from the start. We then show that the manifold obtained by dimensional reduction to three dimensions takes the form

$$S = N \times \mathbb{R} \subset \mathcal{M}_{(3)},$$

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where N is a para-Kähler manifold that can be identified with the cotangent bundle T^*M of a Hessian manifold M, which encodes the couplings of the original five-dimensional theory. While we restrict ourselves to the submanifold S relevant for black strings in this paper, the reduction of five-dimensional supergravity without and with vector multiplets to three Euclidean dimensions will be studied in depth in two companion papers [37, 29].

Single-centred black string solutions correspond to geodesic curves on S, which are space-like for non-extremal and null for extremal solutions. In the extremal case one can also find multi-centred solutions which correspond to totally geodesic, totally isotropic submanifolds. However, not all geodesics correspond to regular black string solutions, and the question of which geodesics do is related to the question of how many independent integration constants a general regular black string solution depends on. We address this question using cases where solutions can be obtained in closed form in terms of harmonic functions. While this is always possible for BPS solutions in supergravity, and a distinguished class of extremal solutions in non-supersymmetric theories, dubbed 'BPS-type solutions', the required decoupling of the scalar equations does not happen automatically for non-extremal and non-BPS extremal solutions. Similar to the case of five-dimensional black holes discussed in [31], we show that explicit non-extremal (and, as well, non-BPS extremal) solutions can be obtained whenever the scalar metric of the reduced three-dimensional theory admits a block decomposition and thus is compatible with a constant charge rotation matrix. The 'best case', with a maximal number of independent non-constant scalar fields expressible in terms of harmonic functions, are diagonal models, which include the ST^2 and STU models of supergravity and a class of STU-like models of non-supersymmetric theories. For these we find explicit solutions, which for the ST^2 models have been derived previously using the FGK formalism [24]. We use these explicit solutions to investigate which geodesics lift to regular black string solutions. It turns out that the necessary boundary conditions ensuring regularity at infinity and at the horizon always reduce the number of integration constants by a factor of $\frac{1}{2}$. This resembles the attractor mechanism for extremal

solutions, which is indeed recovered in the extremal limit. As in [31], where the same behaviour was observed for five-dimensional black holes, we refer to this phenomenon as the 'deformed attractor mechanism'. We add for clarification that for non-extremal solutions the behaviour of solutions at the horizon remains dependent on the values of the scalars at infinity, so that there is no fixed-point behaviour in the strict sense. However, there are no independent integration constants related to the horizon values of the scalars. Moreover, the values of the scalars at the outer and inner horizon depend on simple functions of the charges and moduli which we dub 'horizon charges'.

We also investigate extremal solutions, where we observe that there exists a distinguished class of solutions which corresponds to null geodesic curves evolving along the eigendistributions ('eigendirections') of the para-complex structure of N. This type of extremal solution exists in both supersymmetric and non-supersymmetric theories, and in supersymmetric theories these are precisely the BPS solutions. Therefore we refer to them as 'BPS-type solutions'. They require certain restrictions on the signs of the magnetic charges. In particular, for models where the scalar manifold is given by inequalities of the form $h^I > 0$, all magnetic charges must either be positive or negative. A second, 'non-BPS-type' of solution can be constructed explicitly if a charge rotation matrix with certain properties exists in the given model. Geometrically such solutions correspond to null geodesic curves which do not evolve along the eigendistributions of the para-complex structure. In supersymmetric models these solutions are extremal, but not BPS. In models with scalar manifolds of the form $h^I > 0$, such solutions carry magnetic charges which are not all positive or all negative. For a class of models which includes the ST^2 and STU models of supergravity, as well as STU-like solutions of non-supersymmetric theories, we show explicitly that charge rotation matrices giving rise to all possible choices of signs exist. For generic models our observation explains geometrically why non-BPS extremal solutions are harder to find than BPS solutions.

The structure of this paper is as follows. In Section 2 we briefly review black string solutions in five-dimensional Einstein-Maxwell theory. In Section 3 we

review the special real geometry of five-dimensional vector multiplets and carry out the reduction of the relevant part of the theory to three dimensions. We observe that the target manifold is the product of a para-Kähler manifold with a one-dimensional factor. A short self-contained proof of the para-Kähler property is relegated to Appendix A. In Section 4 we solve the three-dimensional Einstein equations and observe that the three-dimensional line element is universal and coincides with the reduced line element of a five-dimensional 'Reissner-Nordström string'. In Section 5 we solve the scalar field equations, while in Section 6 we discuss the resulting five-dimensional non-extremal black string solutions. In Section 7 we obtain extremal black string solutions and compare BPS and non-BPS-type solutions. For the ST^2 and STU models we compare our method to the FGK formalism used in [24, 25]. In Section 9 we present the generalization to non-supersymmetric theories and uncover the relation between the BPS condition and eigendirections of the para-complex structure of the scalar manifold. Our conclusions are given in Section 10. Appendix A contains a short proof that the submanifold $N \subset N \times \mathbb{R} = \mathcal{M}_{(3)}$ of the scalar manifold of the reduced theory is para-Kähler.

2 Black strings in five-dimensional Einstein-Maxwell theory

For reference we briefly review the basic black string solution of five-dimensional Einstein-Maxwell theory, which might be viewed as a variant of the four-dimensional Reissner-Nordström solution. This solution is a special example of a Reissner-Nordström type black brane solution, which exist in various dimensions and which are reviewed, for example, in [24]. A RN (Reissner-Nordström) type black string solution has an isometry group which contains a static time-like Killing vector field and space-like translational Killing vector field which commute with one another and with the transverse rotation group $SO(3)$:

$$\text{Isom} \supset \mathbb{R}_t \times \mathbb{R}_y \times SO(3).$$

When using adapted coordinates $(t, y, \rho, \theta, \phi)$, the line element can be brought to the form [24]

$$ds^2_{(5)} = H^{-1}(\rho) \left[-W(\rho)dt^2 + dy^2 \right] + \frac{H^2(\rho)}{W(\rho)} \left[d\rho^2 + W(\rho)\rho^2 d\Omega^2_{(2)} \right], \quad (1)$$

where $d\Omega^2_{(2)}$ is the line element of the round unit 2-sphere, and where

$$H = 1 + \frac{p}{\rho} , \quad W = 1 - \frac{2c}{\rho} .$$

The two parameters p, c are non-negative: $p \geq 0$, $c \geq 0$. The solution has an outer horizon at $\rho = 2c$ and an inner horizon at $\rho = 0$. To explore the region inside the inner horizon one can choose different coordinates, see for example [24], but the coordinate system above will be convenient later. For $c = 0$ one obtains the extremal limit where both horizons coincide, thus identifying c as the non-extremality parameter. The second parameter p is related to the magnetic charge of the black string. The non-vanishing component of the field strength is

$$F_{\theta\phi} \simeq \pm p \sin\theta ,$$

which implies that the magnetic charge is $\tilde{p} = \pm p$. Observe that the magnetic charge can be positive or negative, whereas the parameter p must be non-negative. For negative p the coefficients of the line element will have additional zeroes and infinities, which correspond to naked singularities, see for example [38]. We remark that the overall sign between the magnetic charge $\tilde{p}$ and the parameter p is not determined by the field equations, so that choosing this sign is part of specifying the solution.

We finally recall that black string solutions are subject to an extremality bound of the form

$$\mathcal{T} \geq \text{Const}|\tilde{p}| ,$$

where $\mathcal{T}$ is the ADM tension, see [38] for more details. For static BPS string solutions in five-dimensional supergravity, this extremality bound is implied by the BPS bound, which takes the form

$$\mathcal{T} \geq \text{Const}|\mathcal{Z}_m|,$$

where $\mathcal{Z}_m$ is the 'magnetic central charge' [39, 40, 35] of the string. As for black holes, supersymmetric theories can also have extremal solutions which are not BPS, i.e. solutions which satisfy the extremality bound but not the BPS bound.

In the following our goal is to construct non-BPS solutions, both non-extremal and extremal, in five-dimensional supergravity with vector multiplets and, more generally, five-dimensional Einstein-Vector-Scalar type theories where the couplings are determined by 'generalized special real geometry' as defined in [30, 31, 32]. As the solutions are in general non-BPS, we need to solve the full field equations. This is done by dimensional reduction to three space-like dimensions using the existence of two commuting Killing vector fields corresponding to staticity and translations along the string. We then use the formalism of 'generalized special geometry' and exploit the fact that all couplings are encoded in a single function, the Hesse potential.

3 Dimensional reduction

We begin with the action for minimal five-dimensional supergravity coupled to some number, $n_V^{(5)}$, of vector multiplets [34]. In the conventions of [28], the bosonic part of the action takes the form

$$
\begin{aligned}
S_5 \;\; = \;\; & \int d^5 x \left[\sqrt{\hat{g}} \left(\frac{\hat{R}}{2} - \frac{3}{4} a_{ij}(h) \partial_{\hat{\mu}} h^i \partial^{\hat{\mu}} h^j - \frac{1}{4} a_{ij}(h) \mathcal{F}_{\hat{\mu}\hat{\nu}}^i \mathcal{F}^{j|\hat{\mu}\hat{\nu}} \right) \right. \\
& \left. + \frac{1}{6\sqrt{6}} c_{ijk} \epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}} \mathcal{F}_{\hat{\mu}\hat{\nu}}^i \mathcal{F}_{\hat{\rho}\hat{\sigma}}^j \mathcal{A}_{\hat{\lambda}}^k \right].
\end{aligned}
\tag{2}
$$

Here $\hat{\mu}, \hat{\nu}, \dots$ are five-dimensional Lorentz indices and $i = 1, \dots, n_V^{(5)} + 1$ labels the five-dimensional gauge fields. The scalars h^i are understood to satisfy the constraint

$$
H(h) = c_{ijk} h^i h^j h^k = 1,
\tag{3}
$$

which defines an $n_V^{(5)}$-dimensional submanifold $\mathcal{H} \subset M$, where M is a real manifold of dimension $n_V^{(5)} + 1$. The fields h^i can be interpreted as coordinates for M and as homogeneous coordinates for the hypersurface $\mathcal{H}$.

The symmetric, positive definite tensor field $a_{ij}(h)$ appearing in the action (2) is obtained by taking the second derivatives

$$a_{ij} = \frac{\partial^2 \tilde{H}}{\partial h^i \partial h^j} \ ,$$

(4)

of the Hesse potential

$$\tilde{H} = -\frac{1}{d} \log H.$$

(5)

The tensor $a_{ij}(h)$ defines a positive definite Hessian metric $ds_M^2 = a_{ij}dh^i dh^j$ on M. One property of Hessian metrics which we use later is that the first derivatives $\partial_k a_{ij}$, and therefore also the Christoffel symbols of the first kind, are totally symmetric in all three indices. We will also use that the metric coefficients a_{ij} are homogeneous functions of degree -2 with respect to the co-ordinates h^i. Recall that a homogeneous function $f(h^i)$ of degree n satisfies the Euler relation $h^i \partial_i f = nf$. The metric coefficients $a_{ij} = \partial^2_{i,j}H$ of a metric with a Hesse potential H that is homogeneous of degree n are themselves homogeneous of degree $n-2$. If one takes the Hesse potential $\tilde{H}$ to be proportional to the logarithm of a homogeneous function H (of any degree), as in (5), then $\tilde{H}$ itself is not a homogeneous function. However, its kth derivatives ($k > 1$) are homogeneous functions of degree $-k$ and, in particular, the metric coefficients of the corresponding Hessian metric (4) are homogeneous of degree -2.

While the vector couplings are given by restricting the tensor a_{ij} to the hypersurface $H = 1$, the couplings of the physical (independent) scalars are given by the pullback of a_{ij} to $\mathcal{H}$. To make this explicit one can solve (3) in terms of $n_V^{(5)}$ independent scalars, which then provide (inhomogeneous) coordinates for $\mathcal{H}$. For us it is more convenient to work with the dependent scalars h^i for reasons that will become clear later.

We remark that the formalism we use in the following only depends on the fact that H is a homogeneous function, and not on the more specific condition that it is a polynomial and has degree three. These additional conditions follow from imposing that the theory is supersymmetric. By allowing a non-polynomial function with degree of homogeneity different from three, one obtains a more

general class of non-supersymmetric theories of vector and scalar fields (and possibly fermions) coupled to gravity. The formalism of generalized special geometry developed in [30, 31, 32] allows one to solve the field equations within this larger class in precisely the same way as in supergravity. For concreteness we will in the following focus on supergravity. The generalization to general homogeneous H is however completely straightforward and will be discussed in Section 9.

We are interested here in five-dimensional string-like solutions which are static and magnetically-charged under the gauge fields $\mathcal{A}_\mu^i$. As such our solutions will admit one timelike and one spacelike isometry (along the direction of the string) and so we can use the techniques of dimensional reduction over one timelike and one spacelike direction to generate solutions.

In particular we impose that the line element takes the form

$$ds^2_{(5)} = -\epsilon_1 e^{2\sigma} \left(dx^0\right)^2 - \epsilon_2 e^{2\phi-\sigma} \left(dx^4\right)^2 + e^{-2\phi-\sigma} ds^2_{(3)}, \qquad (6)$$

where the two as yet undetermined functions σ and ϕ only depend on the coordinates of the reduced three-dimensional space with as yet undetermined line element $ds^2_{(3)}$. Our parametrization has been chosen such that σ and ϕ are the Kaluza-Klein scalars of the dimensional reductions from the five-dimensional to the four-dimensional Einstein frame, and from the four-dimensional to the three-dimensional Einstein frame, respectively. The parameters $\epsilon_{1,2}$ take the values -1 for reduction over a spacelike direction and $+1$ for reduction over a timelike direction[1]. Note that we can take either x^0 or x^4 to be timelike. The seemingly asymmetric treatment of $\{x^0, x^4\}$ stems from the fact that we first perform a reduction (taken to be either timelike or spacelike depending on the sign of ϵ_1) over x^0 and *then* a reduction over x^4. Our parametrization allows us to postpone the decision as to whether we first reduce over time and then over space, or vice versa. While it will turn out that when restricting to those fields which are non-trivial for black string solutions this choice is not relevant, the

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[1]Note that in the case at hand we reduce over one timelike and one spacelike direction, so will always take $\epsilon_2 = -\epsilon_1$. However, we leave the general case for convenience.

distinction becomes relevant when considering all fields. This will be discussed in a separate publication [37].

Furthermore, restricting ourselves to magnetic solutions leads us to impose the ansatz for the gauge fields $\mathcal{A}^i_{\hat{\mu}}$,

$$\mathcal{A}^i_{\hat{\mu}} dx^{\hat{\mu}} = A^i_\mu dx^\mu, \tag{7}$$

where x^μ with $\mu = 1, 2, 3$ are coordinates transverse to the string. In other words, we set $\mathcal{A}^i_0 = \mathcal{A}^i_4 = 0$.

For this class of solutions, the resulting three-dimensional Euclidean action is

$$S_3 = \int d^3x \sqrt{g} \left[\frac{R}{2} - \hat{g}_{ij}(y)\partial_\mu y^i \partial^\mu y^j - (\partial\phi)^2 + e^{-2\phi-3\sigma}\hat{g}^{ij}(y)\partial_\mu s_i \partial^\mu s_j \right], \tag{8}$$

where R is the three-dimensional Ricci scalar which does not give rise to local dynamics.

The dynamical fields are the $2n_V^{(5)} + 3$ scalar fields (y^i, s_i, ϕ), which have the following five-dimensional origin: the scalars y^i encode the degrees of freedom of the original (constrained) scalars h^i and the Kaluza-Klein scalar from the five-to-four reduction, σ, via

$$y^i = 6^{\frac{1}{3}} e^\sigma h^i,$$

and are therefore unconstrained; the scalar ϕ arises as the Kaluza-Klein scalar in the reduction from four to three dimensions; the axions s_i are obtained by dualizing the gauge fields A^i_μ after reduction to three dimensions. Finally, using homogeneity we can express the metric a_{ij} in terms of the rescaled fields y^i. Including a constant overall factor we obtain [28]

$$\hat{g}_{ij}(y) = -\frac{3}{2} \left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2} \frac{(cyy)_i (cyy)_j}{(cyyy)^2} \right) = -\frac{1}{4} \partial^2_{y^i, y^j} \log \left(c_{klm} y^k y^l y^m \right). \tag{9}$$

We note that this metric is Hessian, and homogeneous of degree -2. For later use, we also note the identity $\hat{g}_{ij}(y)y^i y^j = \frac{3}{4}\hat{g}_{ij}(y)$.

The action (8) can be simplified by a further field redefinition

$$w^i = e^{-\phi-\frac{3}{2}\sigma} y^i, \quad \xi = \phi - \frac{3}{2}\sigma,$$

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which gives

$$S_3 = \int d^3x\sqrt{g}\left[\frac{R}{2} - \hat{g}_{ij}(w)\partial_\mu w^i \partial^\mu w^j + \hat{g}^{ij}(w)\partial_\mu s_i \partial^\mu s_j - \frac{1}{4}(\partial\xi)^2\right], \quad (10)$$

where we now take (w^i, ξ, s_i) to be the dynamical fields. These fields parametrize a $(2n_V^{(5)}+3)$-dimensional submanifold S of the full $(4n_V^{(5)}+8)$-dimensional manifold $\mathcal{M}_{(3)}$ which is obtained if all five-dimensional degrees of freedom are kept and dualized into scalars. As we will show in separate publications [37, 29], the full manifold $\mathcal{M}_{(3)}$ is a para-quaternionic Kähler manifold. Here we restrict ourselves to investigating the geometry of the submanifold S. The manifold S is a totally geodesic submanifold of $\mathcal{M}_{(3)}$, since it is obtained by solving the equations of motion for $2n_V^{(5)}+5$ out of $4n_V^{(5)}+8$ scalars by setting them to constant values. The fields which are truncated out are (i) three out of five degrees of freedom of the five-dimensional metric, see (6), or, equivalently, the scalars corresponding to the Kaluza-Klein vectors of the two reduction steps, and (ii) $2(n_V^{(5)}+1)$ out of $3(n_V^{(5)}+1)$ degrees of freedom of the five-dimensional vector fields, see (7), or, equivalently, the corresponding three-dimensional scalars. The line element of the submanifold S takes the form

$$ds_S^2 = \hat{g}_{ij}(w)dw^i dw^j - \hat{g}^{ij}(w)ds_i ds_j + \frac{1}{4}(d\xi)^2 .$$

The metric on S is the product of a one-dimensional factor parametrized by ξ and a $2(n_V^{(5)}+1)$-dimensional manifold N, which can be identified with the cotangent bundle of the manifold M of the five-dimensional theory, $N \simeq T^*M$. Moreover, since $ds_M^2 = \hat{g}_{ij}(w)dw^i dw^j$ is a Hessian metric, it follows that $ds_N^2 = \hat{g}_{ij}(w)dw^i dw^j - \hat{g}^{ij}(w)ds_i ds_j$ is a para-Kähler metric on N, as we show in Appendix A.

We next observe that for the subsector of fields relevant for black string solutions the parameters ϵ_1 and ϵ_2 do not appear explicitly in the action (10). Thus this subsector is manifestly insensitive to whether we first reduce over time or over space. As we will discuss in [37], [29], this is different when the full set of fields is considered.

For later reference, we list the relations between the three-dimensional fields

and our original five-dimensional fields. Specifically,

$$ds^2_{(5)} = e^{\xi + 2\sigma} \left[-\epsilon_1 e^{-\xi} (dx^0)^2 - \epsilon_2 e^{\xi} (dx^4)^2 \right] + e^{-2(\xi + 2\sigma)} ds^2_{(3)}, \qquad (11)$$

for the metric, and

$$h^i = e^{\xi + 2\sigma} w^i, \qquad F^i_{\mu\nu} = -\frac{1}{\sqrt{2}} \epsilon_{\mu\nu\rho} \hat{g}^{ij}(w) \partial^\rho s_j, \qquad (12)$$

for the remaining fields.

4 Solving the three-dimensional Einstein equations

We now turn our attention to the three-dimensional equations of motion coming from the action (10). The Einstein equations (after taking a trace and back-substituting) read

$$\frac{1}{2} R_{\mu\nu} - \hat{g}_{ij}(w) \partial_\mu w^i \partial_\nu w^j + \hat{g}^{ij}(w) \partial_\mu s_i \partial_\nu s_j - \frac{1}{4} \partial_\mu \xi \partial_\nu \xi = 0. \qquad (13)$$

We will look primarily for solutions describing a single static black string and which therefore possess spherical symmetry in the three-dimensional transverse space. We remark that one could dispense with spherical symmetry when considering extremal solutions, thus allowing for the possibility of multi-centred solutions. While this is not the main focus of this work, we will come back to this point later when we discuss extremal solutions.

Any spherically symmetric line element in 3 dimensions can be brought to the form

$$ds^2_{(3)} = e^{4A(\tau)} d\tau^2 + e^{2A(\tau)} \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \qquad (14)$$

where τ is a radial coordinate [22]. Spherical symmetry of the field configuration then imposes that the scalar fields (w^i, s_i, ξ) are independent of the angular coordinates (θ, φ).

Plugging this ansatz into (13) with $m, n \neq \tau$ we find

$$1 - e^{-2A} \ddot{A} = 0, \qquad (15)$$

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where here $\dot{X}$ denotes differentiation with respect to τ. Multiplying (15) through by $2e^{2A}\dot{A}$ we obtain

$$\frac{d}{d\tau}(e^{2A} - \dot{A}^2) = 0,$$

which can be integrated to find

$$\dot{A}^2 = e^{2A} + \mu, \tag{16}$$

for some integration constant μ. Taking the square root and multiplying through by $-e^{-A}$ gives the differential equation

$$\frac{d}{d\tau}e^{-A} = \sqrt{1 + \mu e^{-2A}}, \tag{17}$$

which can be solved to find an expression for $e^{A(\tau)}$ provided we make a choice for the sign of μ. If we choose the integration constant to be positive, $\mu = c^2 > 0$, we obtain the general solution

$$e^{A(\tau)} = \frac{c}{\sinh(c\tau)}, \tag{18}$$

where the real constant c is chosen positive, $c > 0$, for concreteness. We will see later that solutions with $c = 0$ are well-defined and correspond to the extremal limit, thus identifying c as the non-extremality parameter. Since (18) is manifestly invariant under $c \to -c$, we do not need to consider $c < 0$. In solutions with negative values $\mu < 0$ of the integration constant the hyperbolic function appearing in (18) is replaced by a trigonometric function. In this case the 'radial' coordinate τ is periodic, and such solutions cannot lift to asymptotically flat black string solutions. We therefore discard solutions with $\mu < 0$.

With this, the three-dimensional part of the metric (14) becomes

$$ds^2_{(3)} = \frac{c^4}{\sinh^4(c\tau)}d\tau^2 + \frac{c^2}{\sinh^2(c\tau)}d\Omega_2^2. \tag{19}$$

Returning now to the remaining equations (13), namely those with $m = n = \tau$, we obtain

$$c^2 - \hat{g}_{ij}(w)\dot{w}^i\dot{w}^j + \hat{g}^{ij}(w)\dot{s}_i\dot{s}_j - \frac{1}{4}\dot{\xi}^2 = 0. \tag{20}$$

This relation is often called the Hamiltonian constraint. If one imposes spherical symmetry at the level of the action and reduces the action to one dimension,

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this equation no longer follows from the variational principle and thus has to be imposed as an additional condition. We instead obtained it as a field equation because we imposed spherical symmetry on the three-dimensional field equations, and not on the action itself. The Hamiltonian constraint allows the following interpretation in terms of the scalar manifold S. A spherically symmetric solution corresponds to a geodesic curve C on S, parametrized by τ, with tangent vector $(\dot{w}^i, \dot{s}_i, \dot{\xi})$. The Hamiltonian constraint implies that this tangent vector has constant scalar product $\mu = c^2$ with itself. Therefore the radial coordinate τ is an affine curve parameter. Moreover curves with $\mu = c^2 > 0$ are space-like while curves with $c = 0$ are light-like (null). We will see later that geodesics with $c^2 > 0$ satisfying appropriate boundary conditions lift to non-extremal black string solutions, while geodesics with $c^2 = 0$ lift to extremal black string solutions. As we have seen above, space-like geodesics ($\mu < 0$) do not lift to black string solutions.

It is useful to introduce a new radial coordinate

$$\rho = \frac{ce^{c\tau}}{\sinh(c\tau)}, \tag{21}$$

which no longer corresponds to an affine coordinate on the geodesic curve C on S.

In terms of ρ the line element (19) takes the form

$$ds_{(3)}^2 = d\rho^2 + W\rho^2 d\Omega_2^2, \tag{22}$$

where

$$W := 1 - \frac{2c}{\rho} = e^{-2c\tau}, \tag{23}$$

is harmonic in the three-dimensional transverse space. This is exactly the same as the three-dimensional part of the line element of the standard five-dimensional RN-type black string (1). Thus, as for five-dimensional black holes (see for example [31]), the geometry of this three-dimensional part is universal and remains the same when the solution is deformed by allowing a non-trivial profile for scalar fields.

We also observe[2] that the range $0 < \tau < \infty$ of the 'affine' radial coordinate τ corresponds to the range $\infty > \rho > 2c$ of the standard radial coordinate, which covers the region between the asymptotically flat limit $\tau \to 0 \Leftrightarrow \rho \to \infty$ and the outer horizon at $\tau \to \infty \Leftrightarrow \rho \to 2c$. As in [31] one can therefore use the coordinate ρ to continue the solution to the region between the outer horizon at $\rho = 2c$ and the inner horizon $\rho = 0$. Given that we used dimensional reduction over time it is clear that we should only expect to obtain a solution valid up to the outer horizon, because the Killing vector field ∂_t is not time-like but space-like for $2c > \rho > 0$. Thus in this region one would have to use a dimensional reduction with respect to two space-like directions, leading to a different auxiliary three-dimensional theory.

5 Solving the three-dimensional scalar equations of motion

We now turn to the equations of motion for the scalar fields (w^i, s_i, ξ), which by assumption of spherical symmetry only depend on the radial coordinate τ. The equations of motion for these $2n_v^{(5)} + 3$ fields are of second order. Therefore the general solution, which is guaranteed to exist at least locally, will depend on $2(2n_V^{(5)} + 3)$ integration constants. Geometrically, solutions correspond to geodesic curves on S and the integration constants correspond to the initial position and initial 'velocity' (tangent vector). Since the norm-squared of the tangent vector is fixed by the non-extremality parameter c, one integration constant is determined by c. Equivalently, we can regard c as being determined by the integration constants of the scalar equations.

Geodesics which lift to regular black string solutions need to satisfy specific boundary conditions. This will reduce the number of independent integration constants. For static solutions, irrespective of whether they are BPS or non-BPS, we expect that solutions depend on $2n_V^{(5)} + 1$ integration constants, namely the $n_V^{(5)} + 1$ magnetic charges and the initial values of the $n_V^{(5)}$ physical scalar

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[2]Here we anticipate that the following discussion is not modified by the presence of non-constant scalar fields. This is justified by the discussion at the end of Section 6.

fields at infinity. Due to the attractor mechanism, the values of the scalars at the horizon are fixed in terms of the magnetic charges, and therefore the number of integration constants in the second order equations of motion is reduced by a factor of $\frac{1}{2}$. We will show that for certain models we can construct explicit non-extremal solutions which depend on one additional parameter, namely the non-extremality parameter c. The interpretation of the remaining integration constants will be discussed in Section 6, where we lift three-dimensional solutions to five dimensions.

5.1 The equation of motion for ξ

The equation of motion for ξ is the easiest to deal with. It reads $\ddot{\xi} = 0$, which is solved by

$$\xi(\tau) = a\tau + b \, ,$$

with two arbitrary constants a, b. However, there are additional conditions which must be satisfied if the three-dimensional solution lifts to a regular five-dimensional black string. Transverse asymptotic flatness of (11) implies that $e^{2\sigma}$ and e^{ξ} must independently approach unity for $\tau \to 0 \Leftrightarrow \rho \to \infty$. For ξ this implies that we must choose $b = 0$ and hence we have $\xi = a\tau$ [3]. Next, let us look at the near-horizon geometry $\tau \to \infty$. In this regime, the three-dimensional metric (19) behaves as

$$ds_{(3)}^2 \sim (2c)^4 e^{-4c\tau} d\tau^2 + (2c)^2 e^{-2c\tau} d\Omega_2^2 \quad \text{as } \tau \to \infty,$$

so in this regime the full five-dimensional metric (11) will look like

$$\begin{aligned} ds_{(5)\text{hor}}^2 \quad \sim \quad & e^{2\sigma(\tau)+a\tau} \left[-\epsilon_1 e^{-a\tau}(dx^0)^2 - \epsilon_2 e^{a\tau}(dx^4)^2 \right] \\ & + (2c)^2 e^{-4\sigma(\tau)-2(a+c)\tau} \left[(2c)^2 e^{-2c\tau} d\tau^2 + d\Omega_2^2 \right] . \end{aligned} \tag{24}$$

The horizon of the black string has topology $S^2 \times \mathbb{R}$. In order to have a finite horizon size, both the metric coefficient of the "S^2-factor" and of the "$\mathbb{R}$-factor", must be finite. Looking at the coefficient of the $d\Omega_{(2)}^2$-term, we see that

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[3]The required asymptotics of σ at infinity imposes conditions on the solutions for the other scalar fields to which we will return later.

we must require

$$2\sigma(\tau) = 2\sigma_{\text{hor}} - (a + c)\tau \quad \text{as} \ \tau \to \infty,$$

so that the line element becomes

$$
\begin{aligned}
ds^2_{(5)\text{hor}} &= e^{2\sigma_{\text{hor}} - c\tau} \left[-\epsilon_1 e^{-a\tau} (dx^0)^2 - \epsilon_2 e^{a\tau} (dx^4)^2 \right] \\
&\quad + (2c^2) e^{-4\sigma_{\text{hor}}} \left[(2c)^2 e^{-2c\tau} d\tau^2 + d\Omega_2^2 \right].
\end{aligned}
\tag{25}
$$

Depending on whether we take x^0 or x^4 as the spatial coordinate along the string, we then need to take $a = -c$ or $a = c$ to have a finite coefficient for the dx^0-term or dx^4-term, respectively. This condition can be written in universal form as $a = \epsilon_1 c$, where $\epsilon_1 = -1 = -\epsilon_2$ corresponds to space-time reduction, while $\epsilon_1 = 1 = -\epsilon_2$ corresponds to time-space reduction. The solution for $\xi(\tau)$ is

$$\xi(\tau) = \epsilon_1 c\tau. \tag{26}$$

Note that the integration constants a, b have been determined in terms of the non-extremality parameter c by imposing boundary conditions. Thus the number of independent parameters has been reduced by 2.

5.2 The equations of motion of s_i

We now move on to the equation of motion for the scalars s_i, which were obtained by dualizing the three-dimensional gauge fields:

$$\frac{d}{d\tau} \left(\hat{g}^{ij}(w) \dot{s}_j \right) = 0.$$

Integrating, we find

$$\dot{s}_i = \hat{g}_{ij}(w) \tilde{p}^j. \tag{27}$$

In terms of the corresponding five-dimensional gauge fields we have

$$F^i_{\theta\varphi} = -\frac{1}{\sqrt{2}} \tilde{p}^i \sin\theta, \tag{28}$$

and therefore we will refer to the parameters $\tilde{p}^i$ as the magnetic charges carried by the string. While further integrating (27) will introduce another $n_V^{(5)} + 1$

integration constants, the metric is invariant under constant shifts of the s_i, and therefore solutions where these integration constants are chosen differently are related by isometries. From the five-dimensional point of view such solutions are related by gauge transformations, and therefore we will not count these integration constants as relevant parameters.

We note that, by substituting in (27), the Hamiltonian constraint (20) becomes

$$\frac{3}{4}c^2 - \hat{g}_{ij}(w)\left(\dot{w}^i\dot{w}^j - \tilde{p}^i\tilde{p}^j\right) = 0. \tag{29}$$

This will be useful in the following.

5.3   The equation of motion of the w^i

Finally, the equation of motion for the scalars w^i reads, after making use of (27),

$$\frac{d}{d\tau}\left(\hat{g}_{ij}(w)\dot{w}^j\right) - \frac{1}{2}\left(\partial_i\hat{g}_{jk}(w)\right)\left(\dot{w}^j\dot{w}^k + \tilde{p}^j\tilde{p}^k\right) = 0. \tag{30}$$

Using the fact that $\hat{g}_{ij}$ is Hessian, this becomes

$$\hat{g}_{ij}(w)\ddot{w}^j + \frac{1}{2}\partial_i\hat{g}_{jk}\left(\dot{w}^j\dot{w}^k - \tilde{p}^j\tilde{p}^k\right) = 0. \tag{31}$$

Due to the explicit dependence on $\hat{g}_{ij}(w)$ and its derivatives, it is difficult to solve this equation explicitly in a model-independent way. We will proceed as in [31] and find a class of explicit solutions, which depending on the model might even be the general solution, and which at least always contains a solution which recovers the standard RN black string (with arbitrary charges but constant five-dimensional scalar fields).

To obtain this class of solutions we contract (31) with w^i. Using the fact that $\hat{g}_{ij}$ is homogeneous of degree -2, we find

$$\hat{g}_{ij}(w)\left(w^i\ddot{w}^j - \dot{w}^i\dot{w}^j + \tilde{p}^i\tilde{p}^j\right) = 0. \tag{32}$$

Then, using (29) and the identity $\hat{g}_{ij}(w)w^iw^j = -\frac{3}{4}$, we arrive at the equation

$$\hat{g}_{ij}(w)w^i\left(\ddot{w}^j - c^2 w^j\right) = 0. \tag{33}$$

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This equation still contains $\hat{g}_{ij}(w)$ but we can obtain a class of universal, model-independent solutions by setting $\ddot{w}^j - c^2 w^j = 0$, which results in[4]:

$$w^i(\tau) = A^i \cosh(c\tau) + \frac{B^i}{c} \sinh(c\tau), \tag{34}$$

where A^i, B^i are constants. It remains of course to show that the full scalar equation of motion (31) and the Hamiltonian constraint (29) are solved.

Substituting (34) into the Hamiltonian constraint (29) gives

$$\hat{g}_{ij} \left(c^2 A^i A^j - B^i B^j + \tilde{p}^i \tilde{p}^j \right) = 0 \; .$$

Similarly, using $\ddot{w}^j - c^2 w^j = 0$ the full scalar equation of motion (31) becomes

$$\frac{1}{2} \partial_k \hat{g}_{ij} \left(c^2 w^i w^j - \dot{w}^i \dot{w}^j + \tilde{p}^i \tilde{p}^j \right) = 0 \; ,$$

and substituting in the explicit solution (34) gives

$$\frac{1}{2} \partial_k \hat{g}_{ij} \left(c^2 A^i A^j - B^i B^j + \tilde{p}^i \tilde{p}^j \right) = 0 \; .$$

Thus both remaining equations impose relations between the integration constants, which have to hold for each value of τ separately, because the relations contain $\hat{g}_{ij}(w)$.

At this point any further analysis depends on the form of $\hat{g}_{ij}(w)$. For 'diagonal models', where $\hat{g}_{ij}$ and $\partial_k \hat{g}_{ij}$ are diagonal in (i,j), we can solve both the Hamiltonian constraint and the scalar equation of motion by imposing the $n_V^{(5)} + 1$ relations

$$c^2 (A^i)^2 - (B^i)^2 + (\tilde{p}^i)^2 = 0 \; . \tag{35}$$

Thus we are left with $2n_V^{(5)} + 2$ independent non-trivial integration constants for the scalar equations of motion. Apart from fixing the integration constants for ξ the number of integration constants for (w^i, s_i) were reduced by a factor of $\frac{1}{2}$, by discarding the irrelevant initial values of s_i and by imposing (35). By later investigation of the resulting five-dimensional black string solutions we will see that (35) can be viewed as a deformed version of the black hole

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[4]The factors have been chosen for later convenience with regard to taking the extremal limit.

attractor mechanism, which determines half of the integration constants of the five-dimensional scalars in terms of the magnetic charges. The diagonal models include the ST^2 and STU models of five-dimensional supergravity and STU-like models in non-supersymmetric theories constructed using generalized special real geometry.

For non-diagonal models the ansatz (34) only yields solutions with a reduced number of integration constants. In the most generic case, where $\hat{g}_{ij}$ and its derivatives (when evaluated on the solution) do not allow a simultaneous block decomposition, the only model-independent way to make the ansatz (34) work is to impose the stronger condition

$$c^2 A^i A^j - B^i B^j + \tilde{p}^i \tilde{p}^j = 0 \ .$$

The additional off-diagonal relations can still be solved by imposing

$$\frac{A^i}{A^j} = \frac{B^i}{B^j} = \frac{\tilde{p}^i}{\tilde{p}^j},$$

but this has the effect that the ratios $\frac{w^i(\tau)}{w^j(\tau)}$ are constant, so that all scalar fields $w^i(\tau)$ are proportional to one another. From the formulae given below it will be clear that in this case the five-dimensional metric is just the one of the standard RN-type black string. The physical five-dimensional scalars, which can be chosen to be parametrized by the $n_V^{(5)}$ independent ratios of the fields w^i, are constant for this universal solution.

In between these extremes are models where $\hat{g}_{ij}$ and $\partial_k \hat{g}_{ij}$ admit a simultaneous decomposition into k different blocks ($k = 1$ is the most generic indecomposable case discussed in the previous paragraph). For such models we obtain k sets of non-proportional scalars w^i. Thus for $k > 1$ the solutions will admit $k - 1 > 0$ independent non-constant five-dimensional scalars, and the five-dimensional metric will be different from the standard RN-type black string metric.

Furthermore, the ansatz (34) might still yield non-trivial solutions for indecomposable scalar metrics, if one can restrict the solution to a totally geodesic

submanifold of S, on which the metric becomes block-decomposable. Examples of this phenomenon were observed in [33].

6 Non-extremal black string solutions

We now proceed to investigate the black string solutions obtained by the ansatz (34). To prepare for this we rewrite (34) in terms of the new radial coordinate ρ defined in (21):

$$w^i(\rho) = \left(A^i + \frac{p^i}{\rho}\right) W^{-\frac{1}{2}} := \mathcal{H}^i(\rho) W^{-\frac{1}{2}}. \tag{36}$$

Here we have used the definition of the function $W(\rho)$ given in (23), and introduced $p^i := B^i - cA^i$. At this point it is convenient also to introduce the quantity $\bar{p}^i := p^i + 2cA^i$. We will see later on that p^i and $\bar{p}^i$ are related to the values of the scalar fields $h^i(\rho)$ at the inner and outer horizons respectively. We also note for later reference that in terms of the charges $p^i, \bar{p}^i, \tilde{p}^i$ the Hamiltonian constraint takes the form

$$\hat{g}_{ij}(\tilde{p}^i\tilde{p}^j - p^i\bar{p}^j) = 0 \ .$$

We now express the solution in terms of five-dimensional quantities. Using (12) and the hypersurface constraint (3), we see that

$$e^{\xi+2\sigma} = H(w)^{-\frac{1}{3}} = H(\mathcal{H})^{-\frac{1}{3}} W^{\frac{1}{2}},$$

so, using also (26) and (23), the five-dimensional metric (11) becomes

$$ds^2_{(5)} = H(\mathcal{H})^{-\frac{1}{3}} \left(-W dt^2 + dy^2\right) + H(\mathcal{H})^{\frac{2}{3}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega_2^2\right), \tag{37}$$

where $\{t, y\}$ are the time-like and space-like directions corresponding to the worldvolume of the string. We note that this form of the solution is *independent* of which order (space-then-time or time-then-space) we perform the reduction. The metric (37) is a generalization of the standard RN black string metric, where the single harmonic function $\mathcal{H}$ has been replaced by the function $(H(\mathcal{H}^i))^{1/3}$, which depends on $n_V^{(5)} + 1$ harmonic functions $\mathcal{H}^i(\rho)$. The standard RN-type

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string is recovered when all these harmonic functions are proportional to one another.

The (constrained) five-dimensional scalar fields are given by

$$h^i(\rho) = H(\mathcal{H})^{-\frac{1}{3}} \mathcal{H}^i(\rho). \tag{38}$$

Transverse asymptotic flatness of the metric implies that $H(\mathcal{H}) \to 1$ for $\rho \to \infty$. Therefore the constant term A^i in the harmonic function $\mathcal{H}^i$ specifies the value of the scalar h^i at transverse infinity, $A^i = h^i_\infty$, and

$$\mathcal{H}^i(\rho) = h^i_\infty + \frac{p^i}{\rho}.$$

The condition of transverse asymptotic flatness $H \to 1$ can be written as $H(h_\infty) = 1$ by taking the limit. This imposes one relation between the $n_V^{(5)} + 1$ integration constants h^i_∞. Obviously, this condition is precisely the hypersurface constraint and takes into account the fact that there are only $n_V^{(5)}$ independent five-dimensional scalars for which we can impose boundary values at infinity. One convenient way to parametrize the independent five-dimensional scalars is to use $n_V^{(5)}$ independent ratios, for example $\phi^x = \frac{h^x}{h^0} = \frac{w^x}{w^0}$ [31, 32].

To interpret the integration constants p^i (equivalently B^i) we consider the limits $\rho \to 2c$ and $\rho \to 0$, which correspond to the outer and inner horizons respectively. We see that, in these cases, the scalars $h^i(\rho)$ satisfy

$$h^i \xrightarrow[\rho \to 2c]{} \left(H(\bar{p})(2c)^{-3} \right)^{-\frac{1}{3}} \frac{\bar{p}^i}{2c} = H(\bar{p})^{-\frac{1}{3}} \bar{p}^i,$$

and

$$h^i \xrightarrow[\rho \to 0]{} \left(H(p)\rho^{-3} \right)^{-\frac{1}{3}} \frac{p^i}{\rho} = H(p)^{-\frac{1}{3}} p^i.$$

Here we use $p^i := B^i - cA^i$ and $\bar{p}^i := p^i + 2cA^i$.

This is the same "dressed attractor behaviour" as noted in [31] for five-dimensional black holes and motivates calling $\bar{p}^i$ and p^i the outer and inner "horizon charges" respectively. It remains to clarify how these "horizon charges" are related to the physical magnetic charges $\tilde{p}^i$. To do this recall that the magnetic charges are (the non-trivial half of) the integration constants of the

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scalars s_i and appear in the five-dimensional gauge fields as

$$F^i = -\frac{1}{\sqrt{2}}\tilde{p}^i \sin\theta \, d\theta \wedge d\varphi. \tag{39}$$

As observed at the end of the previous section, the Hamiltonian constraint takes the form

$$\hat{g}_{ij}(w)\left(\tilde{p}^i\tilde{p}^j - p^i\bar{p}^j\right) = 0 \ . \tag{40}$$

For diagonal models we solve this by imposing

$$(\tilde{p}^i)^2 - p^i\bar{p}^i = 0 \ , \tag{41}$$

which is (35) expressed in terms of p^i and $\bar{p}^i = p^i + 2ch_\infty^i$. This can be used to express the horizon charges p^i (and, hence, $\bar{p}^i$) in terms of $\tilde{p}^i$, h_∞^i and c:

$$p^i = -ch_\infty^i \pm \sqrt{(\tilde{p}^i)^2 + c^2(h_\infty^i)^2}.$$

The sign is to be chosen such that the metric is regular outside the horizon[5]. We have now identified the number and interpretation of the independent integration constants for solutions (34) for diagonal models. There are $2n_V^{(5)} + 2$ independent integration constants, namely $n_V^{(5)} + 1$ magnetic charges $\tilde{p}^i$, the $n_V^{(5)} + 1$ constants h_∞^i which are subject to one constraint and encode the asymptotic values of the $n_V^{(5)}$ five-dimensional scalars at infinity, and the non-extremality parameter c.

A priori, one might have expected $n_V^{(5)}$ further integration constants, corresponding to the initial velocities of the five-dimensional scalars at infinity, or, equivalently, their values at the outer or inner horizon. However, these values are determined by the condition which generalizes the attractor mechanism known from extremal black holes. While we do not have proper fixed point behaviour, i.e. the values of the scalars at the horizons are not determined exclusively by the charges, but also depend on their values at infinity, it is still true that there are no independent integration constants related to the horizon values, but rather they are determined by other data. This suggests that the

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[5]We will come back to questions of regularity at the end of Section 6.

solution can be obtained from a reduction of the scalar field equations to first order form, similar to BPS equations. As discussed in [32] for the similar case of five-dimensional black holes, the deformed attractor mechanism guarantees that the physical scalar fields take finite values on the horizon.

Let's now turn our attention to some further properties of the solution.

In order to explore the geometry near the outer horizon, we introduce the variable $u^2 = \rho - 2c$, and look at the region $u^2 \approx 0$. Then (37) becomes

$$ds^2_{(5)} = \frac{2c}{H(\bar{p})^{\frac{1}{3}}} dy^2 + H(\bar{p})^{\frac{2}{3}} d\Omega_2^2 + \frac{2H(\bar{p})^{\frac{2}{3}}}{c} \left(du^2 - \frac{c}{2H(\bar{p})} u^2 dt^2 \right). \qquad (42)$$

Introducing $v^2 = \rho$ and concentrating on the region $v^2 \approx 0$, we find that the metric near the inner horizon takes the form

$$ds^2_{(5)} = \frac{2c}{H(p)^{\frac{1}{3}}} dt^2 + H(p)^{\frac{2}{3}} d\Omega_2^2 + 2\frac{H(p)^{\frac{2}{3}}}{c} \left(-du^2 + \frac{c}{2H(p)} u^2 dy^2 \right). \qquad (43)$$

In both cases, the first two factors give an $\mathbb{R} \times S^2$, with the size of the S^2 determined by the horizon charges $p^i, \bar{p}^i$, whilst the rest of the metric takes the form of a two-dimensional Rindler space.

From these expressions we can read off that the entropy of the inner and outer horizons are given, respectively, by

$$S_- = \pi H(p)^{\frac{2}{3}}, \qquad S_+ = \pi H(\bar{p})^{\frac{2}{3}},$$

whilst the temperatures associated to each horizon are

$$T_- = \frac{\sqrt{2c}}{4\pi} H(p)^{-\frac{1}{2}}, \qquad T_+ = \frac{\sqrt{2c}}{4\pi} H(\bar{p})^{-\frac{1}{2}},$$

which vanish as expected in the extremal limit. The combination

$$T_\pm S_\pm^{\frac{3}{4}} = \frac{\sqrt{2c}}{4} \pi^{-\frac{1}{4}},$$

depends only on the non-extremality parameter.

The tension of the solution is

$$\mathcal{T} = \frac{1}{2} c_{ijk} h_\infty^i h_\infty^j \bar{p}^k,$$

where we are using the normalization of [24].

We conclude our discussion by pointing out that in order to obtain regular black string solutions one might need to impose further conditions in addition to the restrictions that guarantee asymptotic flatness and a regular solution on the horizon. The line element is modified compared to the standard RN black string by replacing the single harmonic function $\mathcal{H}(\rho)$ by $(H(\mathcal{H}^i))^{1/3}$, which is a rational function of several harmonic functions. Therefore it may happen that, for some choices of integration constants, $(H(\mathcal{H}^i))^{1/3}$ takes the values zero or infinity at finite $\rho > 2c$, generically resulting in a naked singularity even if the behaviour at $\rho \to \infty$ and $\rho = 2c$ is regular. This phenomenon was studied for five-dimensional BPS black holes and five-dimensional domain walls in [41] and [38]. It was observed in particular that naked singularities can occur even though the scalar fields take finite values within the scalar manifold along the whole solution. For M-theory compactifications on Calabi-Yau threefolds naked singularities cannot occur for domain walls and BPS black holes as long as the scalar fields take values within the extended Kähler cone, which is the modified scalar manifold relevant for M-theory [38]. However, apart from this there are no model-independent results we are aware of. For the case at hand, we should therefore add the condition that the integration constants (h^i_∞, p^i) have to be chosen such that $(H(\mathcal{H}^i))^{1/3}$ does not have zeros or infinities for $\rho > 2c$, and, if we want to continue the solution to the inner horizon, for $\rho > 0$. The existence of such solutions is guaranteed because the standard RN black string is always contained in our class of solutions. Sufficiently small deformations away from this solution will not introduce zero or infinities for $(H(\mathcal{H}^i))^{1/3}$ and therefore give rise to regular solutions with non-constant scalar fields. However, it cannot be excluded without model-by-model investigation that large deformations away from the RN black string lead to singular solutions.

7 Extremal black strings

Extremal solutions can be obtained by either taking the limit $c \to 0$ of non-extremal solutions, or by directly solving the equations of motion for $c = 0$. To illustrate the drastic simplification occurring in this limit, observe that for $c = 0$ the Hamiltonian constraint simplifies to

$$\hat{g}_{ij}(\dot{w}^i \dot{w}^j - \tilde{p}^i \tilde{p}^j) = 0 \; ,$$

which can be solved, for any $\hat{g}_{ij}$, by

$$\dot{w}^i = p^i = \pm\tilde{p}^i \; ,$$

so that the solution of (30) is simply

$$w^i = A^i + p^i \tau = h^i_\infty + \frac{p^i}{\rho} = \mathcal{H}^i(\rho) \; .$$

Since $W = 1$ there is only one horizon, and the horizon charges are equal to one another and, up to an overall sign, equal to the magnetic charges: $p^i = \bar{p}^i = \pm\tilde{p}^i$. Further simplified relations include $\rho = \frac{1}{\tau}$ and $\xi = 0$.

At the horizon, the values of the scalars are determined by the charges $p^i = \pm\tilde{p}^i$:

$$h^i \to H(p)^{-1/3} p^i \; , \quad \text{for} \quad \rho \to 0 \; .$$

This is the attractor mechanism for BPS solutions.

The ADM tension carried by an extremal string is

$$\mathcal{T} = \frac{1}{2} c_{ijk} h^i_\infty h^j_\infty p^k,$$

where p^k are the parameters appearing in the solution for the scalar fields, while the magnetic central charge is [39, 40]

$$\mathcal{Z}_m = h_i(\infty)\tilde{p}^i = c_{ijk} h^i_\infty h^j_\infty \tilde{p}^k \; ,$$

where $\tilde{p}^k$ are the magnetic charges.

Solutions where $p^i = \pm\tilde{p}^i$ saturate the supersymmetric mass bound

$$\mathcal{T} \geq \frac{1}{2}|\mathcal{Z}_m|,$$

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and are therefore BPS solutions. Promoting $\mathcal{Z}_m$ to a space-time field by setting $\mathcal{Z}_m = h_i \tilde{p}^i$, where $h_i = c_{ijk} h^j h^k$, one finds

$$\mathcal{Z}_m \to \frac{1}{\rho^2} c_{ijk} \tilde{p}^i \tilde{p}^j \tilde{p}^k = \frac{1}{\rho^2} H(\tilde{p}),$$

so that the attractor mechanism takes the form

$$\mathcal{Z}_m h^i \to p^i \quad \text{for} \quad \rho \to 0 \,.$$

In general, only $p^i = \pm \tilde{p}^i$ is guaranteed to give a solution of the Hamiltonian constraint and of the field equations. But further solutions arise whenever the scalar metric $\hat{g}_{ij}$ (when evaluated on the solution) admits a non-trivial 'charge rotation matrix.' This observation was made in the context of first order flow equations [42, 43], but can be applied to the second order formalism used here as previously in [30, 33]. A charge rotation matrix is a constant matrix which relates the horizon charges p^i and the magnetic charges $\tilde{p}^i$ by

$$\tilde{p}^i = R^i_j p^j \,,$$

and satisfies $\hat{g}_{ij} R^i_k R^j_l = \hat{g}_{kl}$ so that the Hamiltonian constraint (and the full field equations (30)) is solved. Such solutions are extremal, i.e. have $c = 0$ and a single horizon located at $\rho = \frac{1}{\tau} = 0$, but they are not BPS because $\mathcal{T} \neq \frac{1}{2} |\mathcal{Z}_m|$.

For extremal solutions the assumption of three-dimensional spherical symmetry is not necessary, and by relaxing it we can obtain multi-centred solutions. First note that for extremal solutions we have $\xi = 0$, so that the scalar ξ can already be truncated out at the level of the action (10). In this case the target space of the three-dimensional theory reduces to the para-Kähler manifold $N = T^* M$. We can then proceed essentially as in [30], with the minor modification that there the para-Kähler manifold was TM, the tangent bundle of a Hessian manifold M, rather than the cotangent bundle. Imposing the "extremal instanton ansatz"

$$\partial_\mu w^i = R^i_j \hat{g}^{jk}(w) \partial_\mu s_k,$$

the equations of motion for w^i reduce to

$$\Delta w^i = 0,$$

where Δ is the flat three-dimensional Laplacian. Taking the solutions to be multi-centred Harmonic functions,

$$w^i(\vec{x}) = \mathcal{H}^i(\vec{x}) \equiv h_\infty^i + \sum_n \frac{p_n^i}{|\vec{x} - \vec{x}_n|},$$

where $\vec{x} = (x^\mu) = (x^1, x^2, x^3)$, we obtain static multi-centred black string solutions with horizons located at $\vec{x}_n$ in transverse space. The spherically symmetric solutions are recovered by restricting to solutions with one centre. The near horizon asymptotics of each centre is the same as for the corresponding single-centred solutions. We do not give any further details but refer to the analogous case of black holes which was analysed in detail in [30].

7.1   Example: ST^2 model

We now choose a particular Hesse potential (3) describing the one-dimensional special real manifold $h^0(h^1)^2 = 1$. Since BPS and non-BPS black string solutions for this model have already been discussed in [24], we keep the presentation brief, with the main purpose of comparing our formalism to the FGK formalism used there. In order for the hypersurface $h^0(h^1)^2 = 1$ to be well-defined we must take $h^0 > 0$. There are then two disjoint patches in which h^1 can take values, namely $\{h^1 > 0\}$ and $\{h^1 < 0\}$. Working out the associated metric $\hat{g}_{ij}$, we find

$$\hat{g}_{ij} = \frac{1}{4} \begin{pmatrix} (h^1)^4 & 0 \\ 0 & 2h^0 \end{pmatrix}.$$

It turns out that there are 4 possible "R-matrices" satisfying $R^T \hat{g} R = \hat{g}$, namely $R = \pm R_{(\sigma)}$, where

$$R_{(\sigma)} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix},$$

and $\sigma = \pm 1$.

For this model the ADM tension $\mathcal{T}$ is given by

$$6\mathcal{T} = (h_\infty^1)^2 p^0 + 2h_\infty^0 h_\infty^1 p^1,$$

where p^0, p^1 are the horizon charges, while the magnetic central charge $\mathcal{Z}_m$ is given by

$$3\mathcal{Z}_m = (h_\infty^1)^2 \tilde{p}^0 + 2h_\infty^0 h_\infty^1 \tilde{p}^1,$$

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where $\tilde{p}^0$, $\tilde{p}^1$ are the magnetic charges. Let us discuss the range of values that the parameters $h_\infty^i, p^i, \tilde{p}^i$ can take. The magnetic charges $\tilde{p}^i$ can independently be positive or negative. In contrast the parameters h_∞^i, p^i are restricted by the fact that the scalar fields

$$h^i \simeq \mathcal{H}^i = h_\infty^i + \frac{p^i}{\rho},$$

must take values inside the scalar manifold for $\infty > \rho > 0$. For definiteness, consider the connected component $\{h^0 > 0, h^1 > 0\}$. Then we must impose that all four parameters are positive: $h_\infty^0 > 0$, $h_\infty^1 > 0$, $p^0 > 0$, $p^1 > 0$. This implies immediately that solutions where $R = \pm R_{(1)}$ saturate the BPS bound $\mathcal{T} = \frac{1}{2}|\mathcal{Z}_m|$ while for $R = \pm R_{(-1)}$ we have $\mathcal{T} > \frac{1}{2}|\mathcal{Z}_m|$. Thus the solutions generated by a non-trivial charge rotation matrix are non-BPS. We note that on the component $\{h^0 > 0, h^1 > 0\}$ of the scalar manifold BPS solutions have magnetic charges with the same sign (i.e. both positive or both negative) while non-BPS solutions have magnetic charges with opposite signs.

One can also consider the second connected component $\{h^0 > 0, h^1 < 0\}$. On this component BPS solutions have opposite signs of the magnetic charge while non-BPS solutions have magnetic charges with the same sign. Our results for the ST^2 model are consistent with those of [24]. One distinct feature of our formalism, which we view as an advantage, is that we can perform the whole analysis using the homogeneous coordinates (h^0, h^1), without making a choice for a physical scalar parametrizing the hypersurfaces. The FGK formalism used in [24] requires such a choice, in order to minimize the effective potential, describe attractor behaviour, and to identify the different branches corresponding to BPS and non-BPS solutions. In contrast we can obtain the same information more easily working in homogeneous coordinates.

7.2   M-theory compactifications on Calabi-Yau threefolds

We remark that there is an important class of models where the domain of the scalar fields can be chosen of the form $\{h^i > 0\}$, namely compactifications of M-theory on toric Calabi-Yau threefolds. In this case the scalar manifold is the

hypersurface of the Kähler cone of the Calabi-Yau manifold obtained by fixing the volume. For toric Calabi-Yau threefolds the Kähler cone is a 'strongly convex finite polyhedral cone', which admits a parametrization of the above form. We refer to [38] and references therein for details. In this parametrization, all charges will be either positive or negative for BPS solutions, while non-BPS solutions, if they exist, will have a mixture of positive and negative charges. In general the metric will not have a block decomposition, so that we cannot guarantee the existence a non-trivial charge rotation matrix and, hence, of explicit non-BPS solutions.

7.3   Example: STU model

As a final illustration of our method for constructing BPS and non-BPS extremal solutions, we consider the case of the STU model, which has Hesse potential $H(h) = h^0 h^1 h^2$. Again, we keep the discussion brief as the extremal BPS and non-BPS solutions to this model have been discussed before using the FGK formalism in [25].

The equation $h^0 h^1 h^2 = 1$ defines a two-dimensional projective special real manifold, which consists of four disjoint patches depending on the signs of (say) h^0 and h^1. The metric $\hat{g}_{ij}$ is

$$
\hat{g}_{ij} = \frac{1}{4} \begin{pmatrix} (h^1 h^2)^2 & 0 & 0 \\ 0 & (h^0 h^2)^2 & 0 \\ 0 & 0 & (h^0 h^1)^2 \end{pmatrix}.
$$

The eight possible charge rotation matrices satisfying $R^T \hat{g} R = \hat{g}$ in this case are given by $R = \pm R_{(\sigma,\tau)}$, where

$$
R_{(\sigma,\tau)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \tau \end{pmatrix},
$$

and σ and τ can each take the values ± 1.

The ADM tension $\mathcal{T}$ and magnetic central charge $\mathcal{Z}_m$ are given, respectively, by

$$
6\mathcal{T} = h^0_\infty h^1_\infty p^2 + h^0_\infty h^2_\infty p^1 + h^1_\infty h^2_\infty p^0,
$$

and

$$3\mathcal{Z}_m = h^0_\infty h^1_\infty \tilde{p}^2 + h^0_\infty h^2_\infty \tilde{p}^1 + h^1_\infty h^2_\infty \tilde{p}^0.$$

Taking, for concreteness, the patch $\{h^0 > 0, h^1 > 0\}$, we find again that solutions with $R = \pm R_{(1,1)}$ saturate the BPS bound $\mathcal{T} = \frac{1}{2}|\mathcal{Z}_m|$, whilst for the six other choices of R we have $\mathcal{T} > \frac{1}{2}|\mathcal{Z}_m|$.

We note that for any diagonal model one can always find an R-matrix which flips the sign of any of the charges $\tilde{p}^i$. Thus for diagonal models we cannot only find explicit non-extremal solutions, but also explicit extremal solutions with any choice of signs for the charges. Moreover, this does not only apply to supergravity models, but also to non-supersymmetric models with couplings determined by generalized special real geometry, as we will see in Section 9.

8 Small black holes

The method of dimensional oxidation employed in Section 6 to obtain black string solutions of the original five-dimensional action (2) can also be used to generate a class of four-dimensional black hole solutions to the spacelike reduction of (2).

In particular, we can take the three-dimensional solutions constructed in Section 5 and lift them over a single timelike direction, thereby obtaining a solitonic solution to a four-dimensional action. The line element we obtain is

$$ds^2_{(4)} = -H(\mathcal{H})^{-\frac{1}{2}} W dt^2 + H(\mathcal{H})^{\frac{1}{2}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2_2 \right), \qquad (44)$$

which corresponds to a black hole solution having an inner horizon at $\rho = 0$ and an outer horizon at $\rho = 2c$. The area of the outer horizon is

$$A_+ = 4\pi\sqrt{2c}\, H(\bar{p})^{\frac{1}{2}},$$

whereas the area of the inner horizon vanishes. In the extremal limit $c \to 0$ the outer horizon shrinks to zero size, so we are left with what has been dubbed a 'small' black hole. In the context of string theory, black hole solutions are modified by higher derivative corrections to the effective action [5, 44], which has

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the effect that small (extremal) black holes obtain a finite horizon [45]. The non-extremal black holes solutions obtained above are non-extremal deformations of such small black holes. Our solutions show that while non-extremality makes the outer horizon of small black holes finite, the inner horizon still remains singular in the absence of higher derivative corrections.

9 Generalized special geometry

As emphasized in Section 3, the formalism we have used above in constructing non-extremal black string solutions depends on H being a homogeneous function, but not on its degree or polynomial nature. In the previous section we took H to be of degree three for concreteness. Let us now see what changes if we take H to have a different degree.

To start with, the five-dimensional vector kinetic coupling a_{ij} is still given by (4) in terms of a homogeneous function H and is thus homogeneous of degree -2. Moreover, the physical scalar manifold is still given by the level set $\{H = 1\}$. However the constant factor c_{ijk} in front of the Chern-Simons term is no longer related to the function H. In supergravity theories supersymmetry relates the Chern-Simons term to other terms in the action and forces the coefficient to be given by the third derivatives of H. Gauge symmetry (up to a surface term) then implies that the third derivatives of H must be constant, thus forcing H to be a homogeneous degree three polynomial. If we change the degree of homogeneity and thus give up supersymmetry, gauge symmetry still forces c_{ijk} to be constant, but it is no longer encoded by the function H and becomes an independent set of parameters. As far as purely magnetic black string solutions are concerned (or purely electric black hole solutions) these parameters are however irrelevant, because the Chern-Simons term does not contribute to purely magnetic (or purely electric) solutions. Thus for this class of solutions the only input needed is the function H, which we take to be homogeneous of degree n.

The dimensional reduction proceeds as before with some changes of numerical factors in some formulae. The explicit expression for $\hat{g}_{ij}(y)$ given in (9) will

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be modified, although it will still take the form

$$\hat{g}_{ij}(y) = -\frac{1}{4}\partial^2_{ij} \log H(y),$$

with H a homogeneous function. The expression (10) for the reduced action remains valid, and since $(M, \hat{g}_{ij})$ is a Hessian manifold, the target space S of the reduced theory is still the product of the para-Kähler manifold $N \simeq T^*M$ with a one-dimensional factor parametrized by ξ. While this follows from known results [46, 28, 30], we give a short self-contained proof in Appendix A.

We can then follow through the construction of non-extremal black string solutions as in Sections 5 and 6 above. The main difference is in the form of the line element (37), which becomes

$$ds^2_{(5)} = H(\mathcal{H})^{-\frac{1}{n}} \left(-W dt^2 + dy^2 \right) + H(\mathcal{H})^{\frac{2}{n}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2_2 \right). \qquad (45)$$

For example, one could consider the 'STU-like' models introduced in [30], which have the Hesse potential

$$H(h) = h^1 \dots h^n.$$

In this case the line element (45) takes the form

$$ds^2_{(5)} = \frac{1}{(\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{1}{n}}} \left(-W dt^2 + dy^2 \right) + (\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{2}{n}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2_2 \right), \quad (46)$$

where each of the $\mathcal{H}^i(\rho)$ are harmonic functions. The scalar fields $h^i(\rho)$ are given by

$$h^i(\rho) = \frac{\mathcal{H}^i(\rho)}{(\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{1}{n}}}. \qquad (47)$$

For the case where all of the $\mathcal{H}^i \propto \mathcal{H}$ are proportional to one another, we find that the scalar fields $h^i(\rho)$ take constant values, and the line element collapses to that of the RN black string (1).

As in the supersymmetric case, all models admit generic 'BPS-type' extremal solutions where $p^i = \pm\tilde{p}^i$, while further explicit solutions can be found whenever a charge rotation matrix exists. All such extremal solutions admit non-spherical, multi-centred versions. For STU-like models the discussion given

for ST^2 and STU model can be adapted. For these models there exist charge rotation matrices which allow one to find explicit solutions for any choice of signs for the charges.

Since all this is completely analogous to the case of five-dimensional black hole solutions discussed in [30], we refrain from giving more details or working through explicit examples, but instead discuss the relation between geodesics in the manifold $S = N \times \mathbb{R}$ and five-dimensional black string solutions from a general geometrical point of view. To start, let us remember that while non-extremal solutions correspond to space-like geodesics in $N \times \mathbb{R}$, extremal solutions correspond to null geodesics in N. If we do not assume the existence of a charge rotation matrix, we can still always find explicit extremal solutions which satisfy the same relation

$$\partial_\mu w^i = \pm \hat{g}^{ij} \partial_\mu s_j, \tag{48}$$

as BPS solutions in supersymmetric theories. We refer to such solutions as BPS-type solutions. Using the information about the para-Kähler geometry of the manifold N collected in Appendix A, we obtain a geometric characterisation of BPS-type solutions, which does not make use of supersymmetry and applies to BPS-type solutions of non-supersymmetric theories as well. Comparing (48) to formula (50) in Appendix A it is manifest that the BPS-type solutions evolve along the 'eigendirections' (eigendistributions) of the para-complex structure of N. As explained in [30] the integral submanifolds tangent to these eigendirections are not only isotropic and totally geodesic (hence solving the equations of motion) but even flat, which explains why the solution can be written in terms of harmonic functions.

If the metric admits a non-trivial charge rotation matrix we can explicitly construct further extremal solutions, which satisfy

$$\partial_\mu w^i = R^i_j \hat{g}^{jk} \partial_\mu s_k \ ,$$

with $R^i_j \neq \pm \delta^i_j$. For supersymmetric theories such extremal solutions are non-BPS. Geometrically, these 'non-BPS-type' solutions are characterized by null

geodesics, or, for multi-centred solutions, totally geodesic, totally isotropic submanifolds, where the tangent vectors do not belong to the eigendistributions of the para-complex structure. This provides a geometrical characterization of 'non-BPS-type' solutions, which applies to supersymmetric and as well non-supersymmetric theories. A non-trivial charge rotation matrix allows one to explicitly construct totally geodesic, totally isotropic submanifolds starting from the eigendistributions of the para-complex structure. We remark that from this point of view the existence of non-BPS (type) is less generic (or at least less obvious) than the existence of BPS (type) solutions.

10 Conclusions

By dimensional reduction from five to three Euclidean dimensions we have shown that non-extremal black string solutions correspond to space-like geodesics in the manifold $S = N \times \mathbb{R}$, where $N \simeq T^*M$ is a para-Kähler manifold which can be identified with the cotangent bundle of the manifold M encoding the couplings of the original five-dimensional theory. Extremal black string solutions correspond to null geodesics in N. Our construction is not limited to minimal supergravity coupled to abelian vector multiplets but applies as well to Einstein-Maxwell-Scalar theories where all couplings are encoded by a single homogeneous function.

For BPS-type extremal solutions, where the null geodesics are contained in the eigendistributions of the para-complex structure, we can always find explicit solutions where all five-dimensional scalar fields are independent, with horizon values determined by the attractor mechanism in terms of the magnetic charges. These solutions involve $n_V^{(5)}$ real scalars and $n_V^{(5)} + 1$ vector fields and depend on $2n_V^{(5)} + 1$ independent integration constants, namely the values of the scalars at infinity and the magnetic charges. For supergravity theories we recover the known BPS string solutions of [35].

Non-extremal solutions and a second type of extremal solutions, dubbed non-BPS-type solutions can be found *explicitly* if the metric of the scalar manifold

admits a non-trivial charge rotation matrix. The 'best case' is provided by di-
agonal models, where the metric is diagonal and charge rotation matrices allow
one to flip all charges independently. For this case we have found explicit non-
extremal solutions depending on $2n_V^{(5)} + 2$ independent parameters, which can
be taken to be the values of the scalars at infinity, the magnetic charges and the
non-extremality parameter, and extremal solutions depending on $2n_V^{(5)} + 1$ inde-
pendent parameters. While the non-BPS-type extremal solutions are of course
subject to the attractor mechanism we observe a deformed attractor mechanism
at work for non-extremal solutions: while the horizon values of the scalars are
no longer determined by the magnetic charges alone, they do not become in-
dependent integration constants. Moreover, the functional dependence of the
horizon values of the scalars was cast in the form of 'horizon charges', both for
the inner and outer horizon.

While diagonal models constitute a special, non-generic, class of models, this
class contains interesting models, such as the ST^2 and STU models of super-
gravity and STU-like models in non-supersymmetric theories. These examples
were analysed in some detail. For non-diagonal models some non-extremal and
non-BPS-type extremal solutions can still be constructed *explicitly* if the metric
admits a block decomposition. One important problem left for future work is
to find explicit non-extremal and non-BPS-type extremal solutions without the
need of a charge rotation matrix compatible with the metric. Note that the
relation between non-extremal, non-BPS-type extremal and BPS-type extremal
solutions with particular types of geodesic curve in $S = N \times \mathbb{R}$ holds irrespective
of whether we are able to find solutions explicitly. The distinguished feature of
BPS-type solutions, namely that one can always find explicit solutions in terms
of harmonic functions, corresponds to the existence of a distinguished class of to-
tally isotropic, totally geodesic submanifolds associated with the eigendirections
of the para-complex structure. This explains why non-BPS extremal solutions
are harder to find explicitly (unless the metric has special properties), despite
the fact that one might expect that one 'just needs to flip signs of charges'.
While this is true for 'double-extreme' solutions with constant scalar fields, the

scalar equations become in general more complicated because they no longer decouple.

We finish by pointing out some directions for future research. Understanding the precise relation between higher-dimensional solutions and geodesic curves and, more generally, totally geodesic submanifolds of the scalar manifold of a reduced effective theory should be helpful in analysing the spectrum of BPS and non-BPS solutions of string theory and M-theory compactifications in the generic case, where the scalar manifold is not a symmetric space. One part of the problem is to characterize submanifolds of the full scalar manifold which are relevant for a particular type of higher-dimensional solution, as we did here for five-dimensional black strings. Another part is to investigate which additional conditions one has to impose on a geodesic curve or totally geodesic submanifold in order that they lift to regular higher-dimensional solutions. This determines the number of parameters the higher-dimensional solution depends on, and has allowed us in this paper to recover the attractor mechanism and understand in which sense it survives in a deformed form for non-extremal solutions. We have seen that we could also obtain the non-extremal versions of small black holes by lifting up to four rather than five dimensions. One might then ask which other types of regular solutions can be obtained by lifting geodesics with different boundary conditions.

For extremal solutions we observed that it is always possible to give up transverse spherical symmetry and to replace single-centred by multi-centred harmonic functions. Geometrically such solutions do not correspond to null geodesic curves but to totally isotropic totally geodesic submanifolds of S. Apart from BPS-type solutions, where these submanifolds are contained within the integrable eigendistributions of the para-complex structure, we can find explicit non-BPS-type multi-centred solutions whenever a non-trivial charge rotation matrix exists. Applying such a matrix corresponds to an overall change of charges at all centres. However, in the context of superstring compactifications described by effective supergravity with symmetric target spaces it is known that there is a more intricate system of multi-centred solutions which is not

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covered by this single operation [47, 48, 49, 50, 51]. A deeper understanding of totally geodesic submanifolds and their relation to multi-centred solutions will be useful for extending these results to generic models with non-symmetric target spaces.

Our work has been restricted to purely magnetic, non-rotating black strings. More general types of black strings have been studied in detail for pure five-dimensional supergravity in [52, 53]. Extending these results to models with vector multiplets would be another possible extension of the work presented in this paper.

A   From Hessian manifolds to para-Kähler manifolds

In this appendix we give a simple self-contained proof that the metric on the space $N \simeq T^*M$ appearing in our construction is a para-Kähler metric given that M carries a Hessian metric.

Let (M, g) be a Hessian manifold. A coordinate-free definition can be found in [46]. For our purposes we assume that M is a domain which is covered by a single system of affine coordinates w^i, $i = 1, \ldots, n$. We refer to such Hessian manifolds as Hessian domains. In affine coordinates the metric takes the form $g = g_{ij}(w)dw^i dw^j$, where $g_{ij}(w) = \partial^2_{i,j} h(w)$ for some function $h(w)$, the Hesse potential[6].

Define a new manifold $N = M \times \mathbb{R}^n$ with coordinates (w^i, s_i). This manifold can be interpreted as the (trivial) cotangent bundle of M: $N = T^*M$. Next, define a pseudo-Riemannian metric on N by

$$g_N = g_{ij}(w)dw^i dw^j - g^{ij}(w)ds_i ds_j \;,$$

where $g^{ij}(w)$ is the inverse matrix of $g_{ij}(w)$. The metric g_N obviously has signature (n, n). We claim the following statement: *Under the above assumptions, $N = T^*M$ is a para-Kähler manifold.*

---

[6]In the main part of this paper, the metric coefficients are denoted $\hat{g}_{ij}(w)$.

We refer the reader to [26] for the relevant definitions and theorems on para-complex and para-Kähler manifolds. We introduce the frames $F = (\theta_A) = (\partial_{w^i}, \partial_{s_i})$ for TN and $F^* = (\theta^A) = (dw^i, ds_i)$ for T^*N. The components of g_N are

$$g_N = g_{AB}\theta^A\theta^B = g_{AB}\theta^A \otimes \theta^B , \quad (g_{AB}) = \begin{pmatrix} g & 0 \\ 0 & -g^{-1} \end{pmatrix} .$$

Here we use a block-matrix notation where $g = (g_{ij}(w))$ and $g^{-1} = (g^{ij}(w))$ are the coefficients of the metric of M and of its inverse with respect to the coordinate system w^i. Our convention for the symmetrized tensor product is $\theta^A\theta^B = \frac{1}{2}(\theta^A \otimes \theta^B + \theta^B \otimes \theta^A)$.

We define an endomorphism field J on TN

$$J = g^{ij}\partial_{w^i} \otimes ds_j + g_{ij}\partial_{s_i} \otimes dw^j = J_B^A\theta_A \otimes \theta^B .$$

This acts on the frame F as

$$J(\partial_{w^i}) = g_{ij}\partial_{s_j}, \quad J(\partial_{s_i}) = g^{ij}\partial_{w^j} .$$

The components of J with respect to the frame F are

$$(J_B^A) = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} .$$

It follows immediately that $J^2 = \mathbb{1}$. Thus J is an almost para-complex structure on N.

The action of J on T^*N with respect to the dual frame F^* is

$$J^*(dw^i) = g^{ij}ds_j , \quad J^*(ds_i) = g_{ij}dw^j .$$

From these expressions it is clear that the para-complex structure J acts anti-isometrically on the metric $g_N = g_{ij}dw^i dw^j - g^{ij}ds_i ds_j$, $J^*g = -g$. Therefore (N, g, J) is almost para-Hermitian. We define the fundamental form

$$\omega = g_N(J\cdot, \cdot) = \omega_{AB}\theta^A \otimes \theta^B = \frac{1}{2}\omega_{AB}\theta^A \wedge \theta^B .$$

Our convention for the exterior product is $\theta^A \wedge \theta^B = \theta^A \otimes \theta^B - \theta^B \otimes \theta^A$. Evaluate ω in the frame F^*:

$$\omega = ds_i \otimes dw^i - dw^i \otimes ds_i = -dw^i \wedge ds_i, \quad (\omega_{AB}) = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{49}$$

We note that ω is the canonical symplectic form on T^*N. Since ω is closed, it follows that (N, g, J) is almost para-Kähler.

It remains to show that J is integrable, which is equivalent to showing that the two eigendistributions are involutive. Since $J^2 = \mathbb{1}$, the eigenvalues of J are ± 1. A basis for the corresponding eigenvectors is

$$X_\pm^i := \frac{1}{\sqrt{2}} \left(\partial_{w^i} \pm g_{ij} \partial_{s_j} \right) \;,$$

since

$$J(X_\pm^i) = \frac{1}{\sqrt{2}} J(\partial_{w^i} \pm g_{ij} \partial_{s_j}) = \frac{1}{\sqrt{2}} g_{ij} \partial_{s_j} \pm g_{ij} g^{jk} \partial_{w^k}$$

$$= \pm \frac{1}{\sqrt{2}} \left(\partial_{w^i} \pm g_{ij} \partial_{s_j} \right) = \pm X_\pm^i \;.$$

The eigenvectors $X_\pm^i$ span the eigendistributions $\mathcal{D}_\pm$ of J. We compute the Lie brackets between the eigenvectors:

$$[X_\pm^i, X_\pm^j] = \frac{1}{2} [\partial_{w^i} \pm g_{ij} \partial_{s_j}, \partial_{w^k} \pm g_{kl} \partial_{s_l}] = \pm \frac{1}{2} \partial_{w^i} g_{kl} \partial_{s_l} \mp \frac{1}{2} \partial_{w^k} g_{ij} \partial_{s_j} = 0 \;,$$

where we used the fact that $\partial_{w^i} g_{jk}$ is totally symmetric for the Hessian metric g_{jk}. Thus eigenvectors $X_\pm^i$ belonging to the same eigendistribution commute, $[X_+^i, X_+^j] = [X_-^i, X_-^j] = 0$. This implies that the eigendistributions $\mathcal{D}_+$ and $\mathcal{D}_-$ are both involutive, therefore J is integrable and (N, g_N, J) is para-Kähler. This completes the proof.

For some purposes it is useful to use a frame and co-frame with respect to which the para-complex structure is diagonal. Such a frame might be called an 'eigenframe' or isotropic frame (as it is spanned by null vectors). We already saw that $F' = (X_+^i, X_-^i)$ is an eigenframe. The associated co-frame is

$$F'^* = \left(\frac{1}{\sqrt{2}} \left(dw^i + g^{ij} ds_j \right) \;, \frac{1}{\sqrt{2}} \left(dw^i - g^{ij} ds_j \right) \right).$$

With respect to these frames the components of the metric and of the para-complex structure are

$$(g'_{AB}) = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \;, \quad (J'^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{50}$$

These expressions make manifest that tangent vectors of the form $(\dot{w}^i, \pm g^{ij}\dot{s}_j)$ are isotropic and moreover are contained in the eigendistributions $\mathcal{D}_\pm$ of the para-complex structure. This provides a characterization of BPS in contrast to non-BPS extremal solutions, which generalizes to non-supersymmetric theories.

We remark that it is clear that the following more general statement is true: *The cotangent bundle of a Hessian manifold carries a natural para-Kähler structure.* In other words one can drop the assumption that the Hessian manifold is a domain covered by a single affine coordinate system. This can be shown by adapting the results of [46], where it was proven that the tangent bundle of a Hessian manifold carries a natural Kähler structure. Replacing complex by para-complex structures amounts to systematically changing certain signs, see [26, 28]. In addition one has to replace the tangent bundle by the cotangent bundle using the natural isomorphism provided by the metric.

We further remark that a similar situation arises in the case of the supergravity r-map and its generalization to non-supersymmetric theories. As shown in [30], the dimensional reduction with respect to time of (not necessarily supersymmetric) five-dimensional Einstein-Maxwell-Scalar theories encoded by a homogeneous Hesse potential relates Hessian manifolds (M, g) to para-Kähler manifolds $(\tilde{N}, g_{\tilde{N}})$, where $\tilde{N}$ can be identified with the tangent bundle TM of M. This reduction has been used to construct the 'electric cousins' of the black strings found in this paper, see [30, 31, 32].

Acknowledgements

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