DENSITY OF HYPERBOLICITY FOR CLASSES OF REAL TRANSCENDENTAL ENTIRE FUNCTIONS AND CIRCLE MAPS

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Abstract. We prove density of hyperbolicity in spaces of (i) real transcendental entire functions, bounded on the real line, whose singular set is finite and real and (ii) transcendental functions $f\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}\setminus\{0\}$ that preserve the circle and whose singular set (apart from $0,\infty$) is finite and contained in the circle. In particular, we prove density of hyperbolicity in the famous Arnol'd family of circle maps and its generalizations, and solve a number of other open problems for these functions, including three conjectures of de Melo, Salomão and Vargas [MSV].

We also prove density of (real) hyperbolicity for certain families as in (i) but without the boundedness condition. Our results apply, in particular, when the functions in question have only finitely many critical points and asymptotic singularities, or when there are no asymptotic values and the degree of critical points is uniformly bounded.

1. Introduction

Among dynamical systems, those that are *hyperbolic*, a property also called *Axiom A*, have particularly simple behaviour and are the easiest to understand (for a definition in our context see below). For this reason, *density of hyperbolicity* – the question whether any system in a given parameter space can be perturbed to a hyperbolic one – is one of the central problems of one-dimensional dynamics. (It has been known for about 50 years that the answer is negative in higher dimensions; for references and recent results, see for example [PS] and [Bo].)

Recently, there has been major progress on this problem in the real setting. Lyubich [L] and, independently, Graczyk and Świątek [GŚ] solved the problem for the real quadratic family $x\mapsto x^2+c$, while it was solved by Kozlovski, Shen and the second author for real polynomials with real critical points in [KSS1] and for general interval maps and circle maps in [KSS2]. For a discussion of related results, see [vS2].

As an example of questions that are left open by these theorems, let us consider the most famous family of circle maps: the *Arnol'd family*

$$F_{\mu_1,\mu_2}(t) = t+\mu_1+\mu_2\sin(2\pi t);\quad \mu_1\in\mathbb{R}, \mu_2>0.$$

This family describes the behaviour of a periodically forced nonlinear oscillator, and has been used to model a variety of physical and biological systems.

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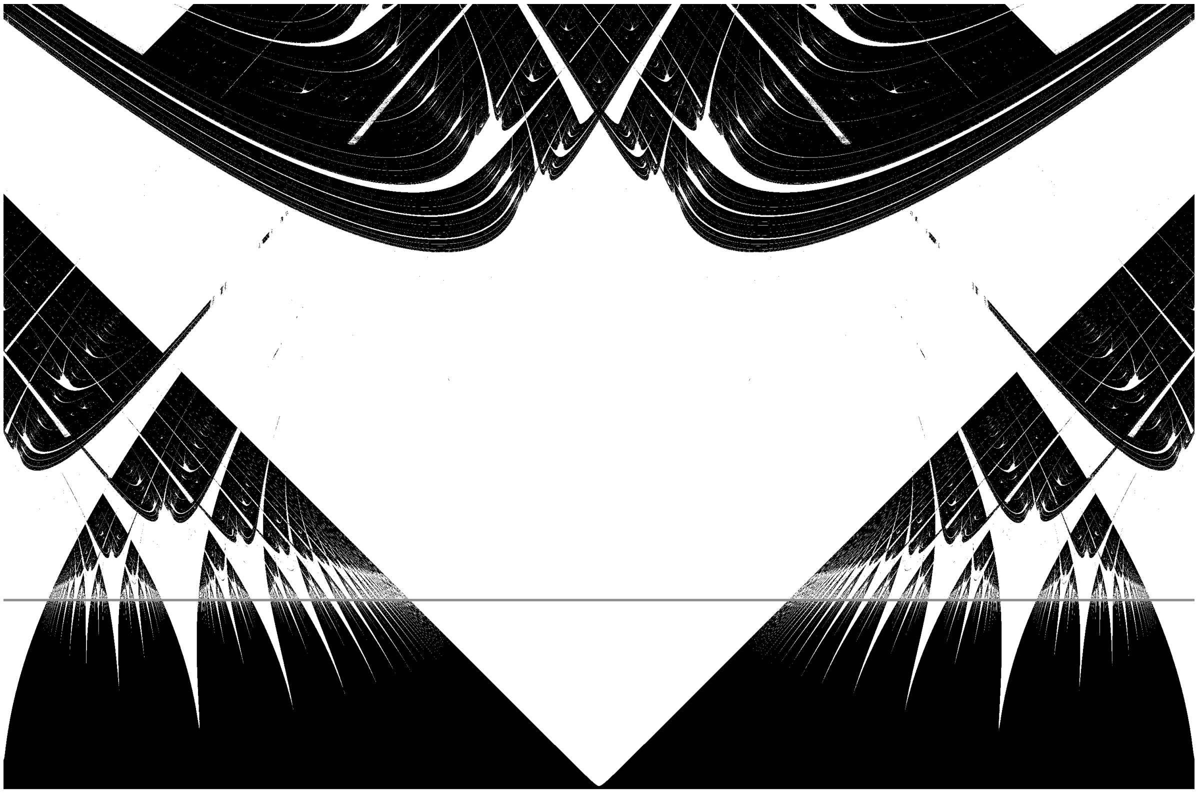

FIGURE 1. Parameter space for the Arnol'd family in the region $(\mu_1, \mu_2) \in (-1/2, 1/2) \times (0, 1/2 + 1/(2\pi))$. White regions correspond to points where a numerical experiment indicates that both critical points belong to attracting basins. The critical line $\mu_2 = 1/(2\pi)$ is indicated in grey; note that the Arnol'd tongues in the invertible region lead to hyperbolic components above the critical line, but that other hyperbolic regions exist also.

It is well-known that hyperbolicity is dense in the region where the map is a circle diffeomorphism, i.e. for $\mu_2 < 1/(2\pi)$. In the non-invertible case, $\mu_2 > 1/(2\pi)$, [KSS2, Theorem 2] implies that $F_{\mu_1,\mu_2}(t)$ can be perturbed to a hyperbolic circle map, and indeed to a hyperbolic trigonometric polynomial of high degree. However, we would like the perturbation to remain within the same family; that is, we ask whether the set of parameters (μ_1, μ_2) for which both critical points belong to the basins of periodic attractors is dense in the region $\mu_2 > 1/(2\pi)$ (Figure 1). This question, and in fact even the density of *structural stability* (see below), had remained open prior to our work.

As a further example, we discuss the families of *real cosine maps* and *degenerate standard maps*:

$$C_{a,b}(x) := a\sin(x) + b\cos(x), \quad (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\},$$
$$S_{a,b}(x) := axe^x + b, \quad a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

These are natural families of transcendental entire functions. (The study of transcendental dynamics, which goes back to Fatou, has received increasing attention recently,

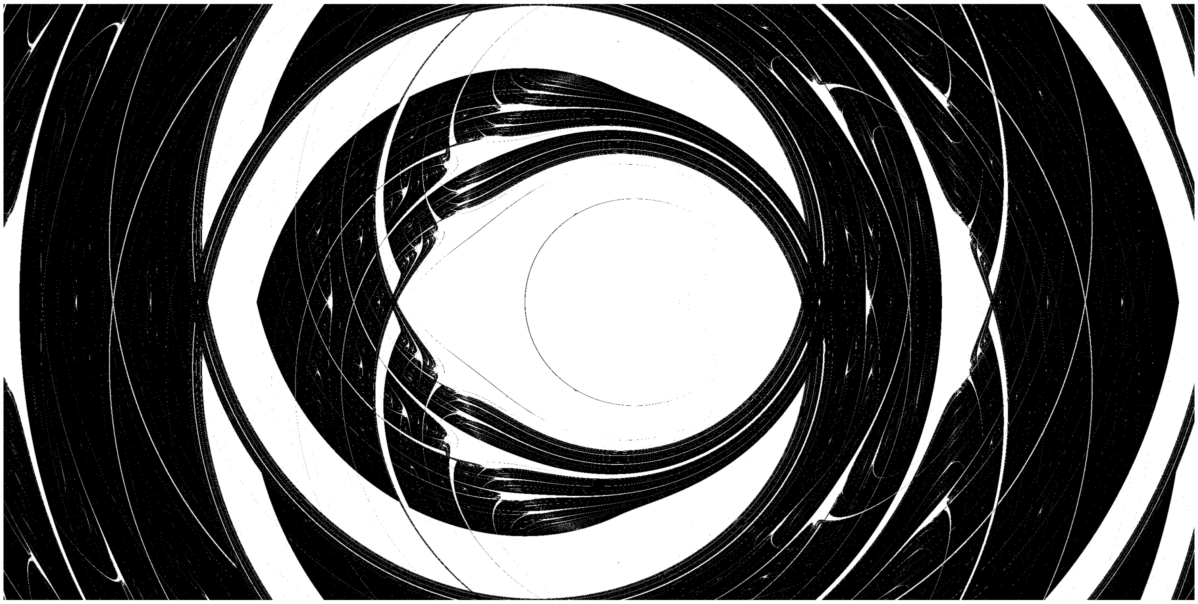

FIGURE 2. The real cosine family. As in Figure 1, white regions correspond to hyperbolic maps.

partly due to the discovery of deep connections with polynomial and rational dynamics. We refer e.g. to [R^{3}S] for some examples.) Again, [KSS2] implies that each of the functions above can be approximated by a hyperbolic *polynomial* of high degree, but we should look for hyperbolic perturbations within the families; see Figure 2. (In the case of maps such as $S_{a,b}$, which are unbounded on the real axis, we shall need to take care to use the right notion of hyperbolicity; see Definition 1.4 below.)

We give positive answers to all of the above questions and in fact establish density of hyperbolicity for a large general class of parameter spaces of transcendental entire functions and circle maps. To do so, we must abandon the proof strategy of [KSS2], which relies heavily on the use of *polynomial-like mappings*. Instead, we return to the methods of [KSS1]. The difficulty here is that we require global rigidity statements, in particular regarding the *absence of invariant line fields* on the complex plane, which become more difficult to establish given that our functions are transcendental. In order to prove our results, we shall need to combine and adapt a number of ingredients:

(a) rigidity results for maps of the interval and circle maps;
(b) rigidity results for the dynamics of transcendental entire functions near infinity;
(c) an argument to establish the absence of invariant line fields on certain subsets of the complex plane;
(d) the function-theoretic construction of natural parameter spaces.

As far as we know, density of hyperbolicity had not previously been established in any nontrivial family of transcendental functions.

Statement of results for real transcendental functions. If $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function, we denote by $S(f)$ the *set of (finite) singular values* of f.

That is, $S(f) \subset \mathbb{C}$ is the smallest closed set such that

$$f : f^{-1}(\mathbb{C} \setminus S(f)) \to \mathbb{C} \setminus S(f)$$

is an unbranched covering.

Let f belong to the *Speiser class* $\mathcal{S}$ of transcendental entire functions for which $S(f)$ is finite. We say that f is *hyperbolic* if every singular value belongs to a basin of attraction; as in the rational case, this definition implies uniform expansion on the Julia set (see [RS, Theorem C] or [R, Lemma 5.1]). When studying density of hyperbolicity, it is reasonable to restrict to the class $\mathcal{S}$. Indeed, for maps with infinite sets of singular values, the associated natural parameter spaces will be infinite-dimensional, the number of periodic attractors may become infinite, there might exist wandering domains, and even density of structural stability may fail. In fact, it is not entirely clear whether "hyperbolicity" is a notion that makes sense when the set $S(f)$ is unbounded.

Since we are interested in real dynamics, we consider only *real* transcendental entire functions; i.e. those that satisfy $f(\mathbb{R}) \subset \mathbb{R}$. Furthermore, we assume that all singular values are also real; i.e. we study the class

$$\mathcal{S}_{\mathbb{R}} := \{f : \mathbb{C} \to \mathbb{C} \text{ real transcendental entire} : S(f) \text{ is finite and contained in } \mathbb{R}\}.$$

This is a reasonable restriction if our goal is to study hyperbolicity in the complex sense. It seems sensible to expect that density of hyperbolicity *on the real line* also holds without the assumption that $S(f) \subset \mathbb{R}$, but our current methods will not yield this. We note that a function $f \in \mathcal{S}_{\mathbb{R}}$ may have non-real critical *points*, but only real critical *values*.

To study density of hyperbolicity we must first clarify what perturbations we allow. It is natural to require these to preserve the global properties of the original map: for example, if a function is bounded on the real line, the approximating map should have the same property. It turns out that the correct notion is to seek perturbations of a map f that are entire functions of the form $\psi \circ f \circ \varphi^{-1}$, where ψ and φ belong to the class $\mathrm{Homeo}_{\mathbb{R}}$ of orientation-preserving homeomorphisms of the complex plane that commute with complex conjugation and restrict to order-preserving homeomorphisms of the real line. Our first result concerns maps $f \in \mathcal{S}_{\mathbb{R}}$ that are bounded on the real axis.

1.1. Theorem (Perturbation of bounded functions)**.**
Suppose that $f \in \mathcal{S}_{\mathbb{R}}$ is bounded on the real axis. Then there exist $\varphi, \psi \in \mathrm{Homeo}_{\mathbb{R}}$ arbitrarily close to the identity such that $g := \psi \circ f \circ \varphi^{-1}$ is entire and hyperbolic.

Another (more practical) point of view is to study perturbations that belong to natural families of functions in $\mathcal{S}_{\mathbb{R}}$. Using Theorem 1.1, we can deduce the following result in this spirit. Let $\mathrm{Möb}_{\mathbb{R}} \subset \mathrm{Homeo}_{\mathbb{R}}$ denote the set of all affine maps $M(z) = az + b$, $a > 0$, $b \in \mathbb{R}$.

1.2. Theorem (Density of hyperbolicity in families of bounded functions)**.**
Let $n \geq 1$ and let N be an n-dimensional (topological) manifold. Suppose that $(f_\lambda)_{\lambda \in N}$ is a continuous family of functions $f_\lambda \in \mathcal{S}_{\mathbb{R}}$ such that

(a) *$f_\lambda|_{\mathbb{R}}$ is bounded for all $\lambda \in N$;*
(b) *$\#S(f_\lambda) \leq n$ for all $\lambda \in N$;*
(c) *no two maps f_{λ_1} and f_{λ_2} are conjugate by a map $M \in \mathrm{Möb}_{\mathbb{R}}$.*

Then the set $\{\lambda \in N : f_\lambda$ *is hyperbolic*$\}$ *is open and dense in* N*.*

Assumption (b) is needed: as in [vS1] it is not hard to construct d-parameter families with $d < n$ so that *no* map within this family is hyperbolic.

We note that it is possible to embed every $f \in \mathcal{S}_\mathbb{R}$ with $\#S(f) = n$ in an n-dimensional family f_λ satisfying (b) and (c) in a natural fashion (see Section 7). Furthermore, if $f|_\mathbb{R}$ is bounded, then all elements of this family will also be bounded.

As a particular case, the above theorems imply density of hyperbolicity in the real cosine family mentioned above. It also holds for general real trigonometric polynomials for which all critical values are real. (See also Corollary 1.12 below for a more general statement regarding circle maps.)

1.3. Corollary (Density of hyperbolicity for trigonometric polynomials)**.**
The set of parameters (a, b) *for which the cosine map* $C_{a,b}$ *is hyperbolic forms an open and dense subset of* $\mathbb{R}^2$*.*

More generally, let $n \geq 1$*. Then hyperbolicity is dense in the space of real trigonometric polynomials*

$$f(x) = a_0 + \sum_{j=1}^{n} \big(a_j \cos(jx) + b_j \sin(jx)\big) \tag{1.1}$$

for which all critical values are real.

Proof. All functions $C_{a,b}$ belong to the class $\mathcal{S}_\mathbb{R}$, with exactly two critical values and no asymptotic values. Furthermore, no two different maps $C_{a,b}$ are conjugate by a Möbius transformation $z \mapsto \alpha z + \beta$, $\alpha > 0$, $\beta \in \mathbb{R}$ (Lemma 2.4).

We note that if f is a trigonometric polynomial and $g = \psi \circ f \circ \varphi^{-1}$ is entire with ψ and φ close to the identity, then g is conformally conjugate to a trigonometric polynomial of the same degree whose coefficients are close to those of f (Lemma 2.7).

Hence the corollary indeed follows from Theorems 1.2 and 1.1. ∎

For functions that are unbounded along the real axis, such as the family $S_{a,b}$, we need to relax our notion of hyperbolicity somewhat. The reason is that here some singular values may "escape to infinity" (i.e., converge to infinity under iteration). In this case, the function is not hyperbolic in the complex sense, as ∞ is not a hyperbolic attractor. However, such a singular value cannot be perturbed into an attracting basin by a *real* perturbation. For example, consider the real exponential family, $E_a(x) = \exp(x) + a$, $a \in \mathbb{R}$. For $a < 1$, $E_a(z)$ is hyperbolic, but for $a > 1$, the singular value a, and indeed every real starting value x, converges to ∞ under iteration. These maps are not hyperbolic in the complex plane – indeed, the Julia set is the whole complex plane, the maps are far from uniformly expanding, and their topological dynamics is still not completely understood – but it seems reasonable to describe their action on $\mathbb{R}$ as hyperbolic, motivating the following definition.

1.4. Definition (Real-hyperbolicity of maps in $\mathcal{S}_\mathbb{R}$)**.**
A function $f \in \mathcal{S}_\mathbb{R}$ *is called* real-hyperbolic *if every singular value either belongs to a basin of attraction or tends to infinity under iteration.*

When $f|_{\mathbb{R}}$ is bounded, this corresponds to the usual definition of hyperbolicity. We should note that, if f_λ is a family of functions in $\mathcal{S}_{\mathbb{R}}$ for which the number of singular values is constant, then any real-hyperbolic parameter λ_0 for which there are *no critical relations* is *real-structurally stable* within the family. By this we mean that any nearby map f_λ is conjugate to f_{λ_0} on the real line. (However, they are not necessarily conjugate in the complex plane; indeed $\exp(x)+a$ and $\exp(x)+b$ are not topologically conjugate for $a > b > -1$, see [DG].) Here we say that f has no critical relations if no critical point or asymptotic value of f is eventually mapped onto a critical point. Indeed, real-structural stability follows from the fact that f_{λ_0} and a nearby map will be combinatorially and hence topologically conjugate on the real line (see Lemma 3.5).

It is reasonable to conjecture that real-hyperbolicity is dense in every full parameter space in $\mathcal{S}_{\mathbb{R}}$. In this paper, we establish this conjecture for functions satisfying quite general additional "geometrical" properties, by which we mean that these properties depend on the function-theoretic behaviour of the maps (rather than their dynamics). There are two such conditions, each of which will allow us to establish density of real-hyerbolicity. The first of these relates to the geometry of the finite singular values of f: We shall say that a function $f \in \mathcal{S}$ has *bounded criticality* if f has no asymptotic values and the degree of critical points of f is uniformly bounded. This class of entire functions appears in work of Mihaljević-Brandt [M-B], which shows that these maps often have particularly nice dynamical properties. Bishop [Bi] has recently presented methods that allow the construction of a vast array of entire functions with bounded criticality.

When our functions do have asymptotic values, or critical points of unbounded multiplicity, we will impose some geometric conditions concerning the singular value at ∞. Essentially, the following condition says that the set of points where f is large is sufficiently thick near the real axis.

1.5. Definition (Sector condition)**.**
Let f be a real transcendental entire function and define

$$\Sigma := \{\sigma \in \{+,-\} : \textit{there is some } x \in \mathbb{R} \textit{ whose orbit accumulates on } \sigma\infty\}$$

We say that f satisfies the sector condition *if, for every $M > 0$ and $\sigma \in \Sigma$, there exist $\vartheta > 0$ and $x_0 > 0$ such that*

$$|f(\sigma x + iy)| > M$$

whenever $x \geq x_0$ and $|y| \leq \vartheta x$.

For $f \in \mathcal{S}_{\mathbb{R}}$, the sector condition is equivalent to requiring that there are constants $r, K > 0$ such that

$$\frac{|f'(\sigma x)|}{|f(\sigma x)|} \leq K \cdot \frac{\log|f(\sigma x)|}{x} \tag{1.2}$$

for all $x \geq r$ and all $\sigma \in \Sigma$ [MiRe, Theorem 6.1]. It is satisfied for most explicit transcendental entire functions that have finite order of growth, such as $z \mapsto ze^z$. In [MiRe], this condition is used to exclude the existence of wandering domains for certain real transcendental functions.

1.6. Theorem (Density of real-hyperbolicity)**.**
Let $n \geq 1$ and let N be an n-dimensional (topological) manifold. Suppose that $(f_\lambda)_{\lambda\in N}$ is a continuous family of functions $f_\lambda \in \mathcal{S}_\mathbb{R}$ such that the following three conditions hold:

(a) *$\#S(f_\lambda) \leq n$ for all $\lambda \in N$;*
(b) *no two maps f_{λ_1} and f_{λ_2} are conjugate by a map $M \in \text{Möb}_\mathbb{R}$;*
(c) *f_λ has bounded criticality for every $\lambda \in N$,* or *f_λ satisfies the sector condition for every $\lambda \in N$.*

Then the set $\{\lambda \in N : f_\lambda$ is real-hyperbolic$\}$ is open and dense in N.

Remark 1. If f is bounded along the real axis, then it trivially satisfies the sector condition, so Theorem 1.6 contains Theorem 1.2 as a special case.

Remark 2. Again, there is an analogous statement to Theorem 1.1: any map $f \in \mathcal{S}_\mathbb{R}$ that has bounded criticality or satisfies the sector condition can be perturbed to a real-hyperbolic function $g \in \mathcal{S}_\mathbb{R}$ by pre- and post-composition with some $\varphi, \psi \in \text{Homeo}_\mathbb{R}$ close to the identity.

To describe some families to which the preceding result applies, let $f \in \mathcal{S}_\mathbb{R}$, choose $\varepsilon > 0$ smaller than one-half the minimal distance between two different singular values of f, and set

$$W := \{z \in \mathbb{C} : \text{dist}(z, S(f)) < \varepsilon\}.$$

Every component of $f^{-1}(W)$ is mapped either as a finite-degree branched covering or as an infinite-degree covering map by f. We say that f has k *singularities* if there are exactly k components of $f^{-1}(W)$ on which f is not one-to-one. (In particular, f has at most k critical points.)

If $f \in \mathcal{S}_\mathbb{R}$ has only a finite number of singularities, then f is of the form

$$f(z) = \int P(w)e^{Q(w)}dw,$$

where P and Q are real polynomials with $P \not\equiv 0$ and $\deg Q \geq 1$. It is well-known that such functions satisfy the sector condition; see Lemma 2.3.

1.7. Corollary (Density of real-hyperbolicity)**.** (a) *For each k, real-hyperbolicity is dense in the space of functions $f \in \mathcal{S}_\mathbb{R}$ that have k singularities.*
(b) *Real-hyperbolicity is dense in the family*

$$S_{a,b} : x \mapsto axe^x + b, \quad a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}.$$

QC-rigidity for maps in $\mathcal{S}_\mathbb{R}$. As is usual, our proof of the above results proceeds along three steps:

(a) *QC rigidity:* Two functions that are topologically (or combinatorially) conjugate are in fact quasiconformally conjugate;
(b) *Absence of line fields*: The functions under consideration support no nontrivial quasiconformal deformations on the Julia set;
(c) *Parameter space arguments*: Density of hyperbolicity is deduced from the first two statements by performing suitable perturbations in parameter space.

Traditionally, the first step of this program has been the hardest to achieve. In our context, we are able to solve it completely, i.e. *without* assuming the sector condition or bounded criticality. This is accomplished by combining the solution of the rigidity problem by the second author in joint work with Trevor Clark, see [CvS], with recent results by the first author on the dynamics of entire functions near infinity [R].

1.8. Theorem (QC Rigidity for maps in $\mathcal{S}_{\mathbb{R}}$)**.**
Suppose that $f, g \in \mathcal{S}_{\mathbb{R}}$ are topologically conjugate on the complex plane, and that the conjugacy takes the real axis to itself. Then f and g are quasiconformally conjugate.

An immediate corollary is:

1.9. Corollary (Connected conjugacy classes)**.**
Take $f \in \mathcal{S}_{\mathbb{R}}$. Then the conjugacy class of f (i.e. the set of maps that are topologically conjugate to f on the complex plane) is connected with respect to the topology of locally uniform convergence.

Remark. This statement is true even if one takes the topology coming from the natural parameter space $M_f^{\mathbb{R}}$. (For the definition of this space see Section 7.)

Step (b) contains an additional complication in the case of transcendental maps: it is necessary to rule out the existence of invariant line fields on the set of escaping points as well as the set of points that tend to escaping singular orbits under iteration. (Both sets are contained in the Julia set.) While the first issue was resolved in [R], we can deal with the second only by assuming either the sector condition or bounded criticality.

Statement of corresponding results for circle maps and trigonometric polynomials. As usual in one-dimensional real dynamics, our results for real functions have analogues for circle maps. Here it is natural to consider transcendental (non-rational) analytic self-maps of the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ that preserve the unit circle. For such a function f, we can define the set of singular values $S(f) \subset \mathbb{C}^*$ analogously to the case of entire functions. The natural class to consider for our purposes is

$$\mathcal{S}_{S^1} := \{f : \mathbb{C}^* \to \mathbb{C}^* \text{ transcendental: } f(S^1) \subset S^1, S(f) \subset S^1, \#S(f) < \infty\}.$$

We note that every map $f \in \mathcal{S}_{S^1}$ has at least one critical point on the circle; see Lemma 9.1. Again, $f \in \mathcal{S}_{S^1}$ is called hyperbolic if every singular value belongs to a basin of attraction of a periodic point in S^1.

1.10. Theorem (Density of hyperbolicity for circle maps)**.**
Let $n \geq 1$ and let N be an n-dimensional (topological) manifold. Suppose that $(f_\lambda)_{\lambda \in N}$ is a continuous family of functions $f_\lambda \in \mathcal{S}_{S^1}$ such that

(a) *$\#S(f_\lambda) \leq n$ for all $\lambda \in N$ (recall that $S(f_\lambda) \subset \mathbb{C}^*$ by definition; i.e. this count does not include 0 or ∞);*
(b) *no two maps f_{λ_1} and f_{λ_2} are conjugate by a rotation.*

Then the set $\{\lambda \in N : f_\lambda \text{ is hyperbolic}\}$ is open and dense in N.

As before, there is an associated rigidity statement:

1.11. Theorem (QC rigidity for maps in $\mathcal{S}_{S^1}$)**.**
Suppose that $f, g \in \mathcal{S}_{S^1}$ are topologically conjugate on $\mathbb{C}^*$, and that the conjugacy preserves the unit circle. Then f and g are quasiconformally conjugate. Furthermore, the dilatation of the map is supported on the Fatou set.

A natural family of degree D circle maps that are $2m$-multimodal can be described as follows. For $\mu \in \mathbb{R}^{2m}$, consider the generalized trigonometric polynomial

$$F_\mu(t) = D \cdot t + \mu_1 + \mu_{2m} \sin(2\pi m t) + \sum_{j=1}^{m-1} (\mu_{2j} \sin(2\pi j t) + \mu_{2j+1} \cos(2\pi j t)). \tag{1.3}$$

F_μ induces a circle map $f_\mu : \mathbb{S}^1 \to \mathbb{S}^1$ (via the covering map $\mathcal{P}(t) = e^{2\pi i t}$). Note that if $\mu, \mu' \in \mathbb{R}^{2m}$ with $\mu_1 - \mu'_1 \in \mathbb{Z}$ and $\mu_j = \mu'_j$ for $j \neq 1$, then $f_\mu = f_{\mu'}$. So it is natural to consider f_μ as parametrized by $\mu = (\mu_1, \ldots, \mu_{2m}) \in \Delta$, where

$$\Delta := \{\mu \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m-1} \ : \ \mu_{2m} > 0 \text{ and } f_\mu \text{ is } \ 2m-\text{multimodal } \}. \tag{1.4}$$

More generally we could require that f_μ has precisely $2m$ critical points on the circle (counting multiplicities). Under these assumptions, the map f_μ belongs to the class $\mathcal{S}_{S^1}$; see Lemma 2.5.

1.12. Corollary (Density of hyperbolicity and rigidity in the trigonometric family)**.**
The set of parameters in Δ for which f_μ is hyperbolic is dense. Furthermore, let $\mu_0 \in \Delta$.

(a) *Consider the set $[\mu_0]$ of parameters μ for which f_μ is topologically conjugate to f_{μ_0} by an order-preserving homeomorphism of the circle. Then $[\mu_0]$ has at most m components.*
(b) *If f_{μ_0} has no periodic attractors on the circle, then each component of $[\mu_0]$ is equal to a point.*

This answers Conjectures 1, 2 and 3 posed by de Melo, Salomão and Vargas in [MSV]; in particular, it establishes density of hyperbolicity in the Arnol'd family mentioned at the beginning of this introduction (for $D = 1$ and $m = 1$). In [BR] the family $F_{a,b}(x) = 2x + a + b\sin(2\pi x)$, $a \in \mathbb{R}$, $b = 1/\pi$, was discussed. In this case, the corresponding circle map $f_{a,1/\pi}$ has a single cubic critical point and belongs to $\mathcal{S}_{S^1}$; see Lemma 2.6. Thus Theorem 1.10 implies that the set of values for which $f_{a,1/\pi}$ is hyperbolic is dense; this fact also follows already from [LvS, Theorem C]. When $b < 1/\pi$, the critical points do not belong to the circle and $f_{a,b} \notin \mathcal{S}_{S^1}$ is a covering map of degree 2. In this case, by Mañé's theorem there is a dense set of parameters for which $f_{a,b}$ is hyperbolic as a map of the circle (i.e., expanding on the complement of the—potentially empty—union of attracting basins on the circle). For $b > 1/\pi$, the map $f_{a,b}$ has two critical points on the circle, and our results imply density of hyperbolicity as well as various conjectures stated in [MaT] and [ELT], as we will discuss in [RvS2].

We remark that the proofs can also be applied to obtain the corresponding results for families of finite Blaschke products

$$B(z) = e^{2\pi i a_0} z^{k_0} \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \bar{a}_j z} \right)^{k_j}, \quad |a_1|, \ldots, |a_n| < 1, a_0 \in \mathbb{R}, k_0 \neq 0, k_j \in \mathbb{Z}$$

for which all critical values, apart from 0 and ∞ (which have period ≤ 2), lie on the circle. Similarly, although we stated the results for entire functions only in the transcendental case for simplicity, they can also be applied verbatim to polynomial families – e.g. for two-dimensional families of quartic polynomials having two distinct critical values, both of which are real. Of course, here there is no need to use tools from transcendental dynamics – the only new ingredients compared to [KSS1] are our discussion of parameter spaces and the rigidity theorem from [CvS].

Further directions. The rigidity results in this paper can also be used, similarly as in [BvS], to prove monotonicity of entropy in families of real transcendental functions. For example, it can be deduced that the topological entropy of maps within the family

$$\mathbb{R} \ni x \mapsto a\sin(2\pi x) \in \mathbb{R}$$

increases with $a \geq 0$. Similar results hold for families of trigonometric polynomials; these questions will be discussed in a sequel to this paper, [RvS2].

Similarly, Theorem 1.11 implies Conjecture B in [ELT] for the family $f_{a,b}(x) = x + a + b\sin(2\pi x)$, $a \in [0,1)$. This conjecture states that the set of parameters $(a,b) \in (0,1) \times \mathbb{R}$ so that the rotation interval of $f_{a,b}$ is equal to a given interval with irrational boundary points, is equal to a single point. (It was already shown in [ELT] that this set is contractible.) We will discuss how this follows in [RvS2]. A similar kind of question was raised in [MiRo, Section 5] for the family of double standard maps: $x \mapsto 2x + a + b\sin(2\pi x)$, $a \in (0,1)$ and will also be discussed in [RvS2].

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2. Preparatory definitions and remarks

Organisation of the paper. In the remainder of this section we will collect notation and some simple facts. In Sections 3–5 we prove Theorem 1.8, that topologically conjugate entire functions in $\mathcal{S}_{\mathbb{R}}$ are quasiconformally conjugate. This relies on two deep results. The first ingredient (Theorem 3.8) is a theorem on real analytic interval maps $f\colon [0,1] \to [0,1]$. Assume that two such maps are topologically conjugate and that the conjugacy maps hyperbolic periodic points to hyperbolic points, and critical points to critical points of the same order. Then these maps are quasisymmetrically conjugate. This follows from results of Trevor Clark and the second author of the current paper [CvS] that apply in fact to much more general functions (even C^3 mappings). This work builds on earlier results of Kozlovski, Shen and van Strien [KSS1]. The second ingredient (Theorem 4.6) uses rigidity of escaping dynamics for transcendental entire functions, a result which was proved by the first author in [R]. In order to prove Theorem 1.8, we will show how to apply and combine these two ingredients in our setting.

We then show in Section 6 that two maps that are quasiconformally conjugate, via a conjugacy that is conformal on the Fatou set, are in fact affinely conjugate. To do this, we show that the maps we consider cannot carry measurable invariant line fields on their Julia sets.

In Section 7, we introduce a natural parameter space $M_f^{\mathbb{R}}$, and discuss kneading sequences and analytic invariants. Using our rigidity results, these can be used to characterize conformal conjugacy classes within the family. In Section 8 we then derive density of hyperbolicity for the families in $\mathcal{S}_{\mathbb{R}}$. In Section 9 we discuss how to adapt our results to circle maps. In an appendix, we further clarify the structure of the parameter space $M_f^{\mathbb{R}}$.

Definitions. Throughout this article, with the exception of Section 9, $f : \mathbb{C} \to \mathbb{C}$ will be a transcendental entire function that maps the real line to itself. We recall that $S(f)$ denotes the set of *singular values* of f.

Let $\mathrm{Crit}_{\mathbb{R}}(f)$ denote the set of real critical points of f, and $\mathrm{CV}_{\mathbb{R}}(f) := f(\mathrm{Crit}_{\mathbb{R}}(f))$. We say that α is a *real-asymptotic value* if $f(x) \to \alpha$ as $x \to \infty$ or as $x \to -\infty$. Let $S_{\mathbb{R}}(f)$ be the *set of real-singular values* of $f|\mathbb{R}$, i.e. the union of $\mathrm{CV}_{\mathbb{R}}(f)$ and the real-asymptotic values. For any $X \subset \mathbb{C}$, we define the orbit

$$O_f^+(X) := \bigcup_{n \geq 0} f^n(X).$$

The *postsingular set* of f is defined as

$$\mathcal{P}(f) := \overline{O_f^+(S(f))}.$$

We also denote the escaping set of f by $I(f) = \{z : |f^n(z)| \to \infty \text{ as } n \to \infty\}$ and set $I_{\mathbb{R}}(f) = I(f) \cap \mathbb{R}$.

Recall that $\mathcal{S}_{\mathbb{R}}$ denotes the class of real transcendental entire functions for which $S(f)$ is a finite subset of the real axis. Also recall that $\mathrm{Homeo}_{\mathbb{R}}$ denotes the set of all homeomorphisms $\psi : \mathbb{C} \to \mathbb{C}$ that commute with complex conjugation and are increasing on the real axis, and that $\text{Möb}_{\mathbb{R}} \subset \mathrm{Homeo}_{\mathbb{R}}$ consists of the affine maps $z \mapsto az + b$, $a > 0$, $b \in \mathbb{R}$. If $\psi \in \mathrm{Homeo}_{\mathbb{R}}$ is quasiconformal, we call ψ a *real-quasiconformal homeomorphism*.

We denote Euclidean distance by dist and spherical distance by $\mathrm{dist}^{\#}$. If $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, then we denote by

$$B_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

the Euclidean ball of radius ε around z_0. We also denote the unit disk by $\mathbb{D} := B_1(0)$.

Quasiconformal maps and invariant line fields. Throughout the article, we assume familiarity with the theory of quasiconformal mappings of the plane; compare e.g. [A2].

We also use the notion of *invariant line fields*. This is a standard concept in holomorphic dynamics, but notation sometimes varies, so we give a concise summary here. A *measurable line field* on a measurable set $A \subset \mathbb{C}$ is a measurable function ℓ from A to the projective plane. (More precisely, ℓ takes each point $z \in A$ to a point in the projective tangent bundle at z; i.e. it represents a measurable choice of a real line in the tangent bundle.)

A line field is *invariant* under f if, for almost every z, the pushforward of the tangent line $\ell(z)$ is given by $\ell(f(z))$. In other words, for almost every z,

$$\ell(f(z)) = f'(z) \cdot \ell(z)$$

(note that the derivative acts on tangent lines by multiplication as long as z is not a critical point).

Invariant line fields are related to invariant Beltrami differentials (ellipse fields): if μ is an invariant Beltrami differential with $\mu(z) \neq 0$ almost everywhere on A, then e.g. the direction of the major axes of the ellipses described by μ will provide an invariant line field. Similarly, if ℓ is an invariant line field, we can find a corresponding non-zero invariant Beltrami differential on A. (See also [McM, Section 3.5].)

In particular, we have the following fact: If f and g are quasiconformally but not conformally conjugate, then there is an f-invariant line field supported on some set of positive measure.

The Koebe Distortion Theorem. We frequently use the following classical theorem in our proofs.

2.1. Theorem (Koebe Distortion Theorem)**.**
For any univalent map $f\colon \mathbb{D} \to \mathbb{C}$ and any $z \in \mathbb{D}$,

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2} \quad \textit{and}$$
$$|f'(0)|\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)|\frac{1+|z|}{(1-|z|)^3}.$$

In particular, $f(\mathbb{D}) \supset B_{|f'(0)|/4}(f(0))$.

For a proof, see for example [P, Theorem 1.3].

Functions with finitely many singularities and the sector condition. We note two standard facts regarding entire functions with finitely many singularities (compare [E]), which show that Corollary 1.7 indeed follows from Theorem 1.6.

2.2. Lemma (Functions with finitely many singularities)**.**
Suppose that f is a real transcendental entire function. Then f has only finitely many singularities if and only if there are real polynomials P and Q with $P \not\equiv 0$ and $\deg Q \geq 1$ such that

$$f'(z) = P(z)e^{Q(z)}.$$

Sketch of proof. First suppose that f has only finitely many singularities. Then f' has only finitely many zeroes. So if we let P be a real polynomial having the same zeroes as f' (counting multiplicities), we can write

$$f'(z) = P(z)e^{g(z)}$$

for some nonconstant real entire function g. The function g cannot be transcendental, as otherwise one could show that the function f has infinitely many singularities. So g must be a real polynomial.

The converse is trivial, as one can check by hand that any function of the stated form has only finitely many singularities; compare (2.1). □

2.3. Lemma (Sector condition)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$ be a function of the form

$$f(z) = \int P(w)e^{Q(w)}dw,$$

where P and Q are real polynomials with $P \not\equiv 0$ and $\deg Q \geq 1$.

Then f satisfies the sector condition (Definition 1.5).

Sketch of proof. This can be checked by direct calculation. Indeed, the function f satisfies

$$f(z) = \left(\frac{P(z)}{Q'(z)} + O\big(|z|^{\deg(P)-\deg(Q)}\big) \right) e^{Q(z)} + O(1) \tag{2.1}$$

as $z \to \infty$. (See [He, Lemma 4.1]) The claim follows easily from this estimate. □

Explicit families. To conclude this section, we collect some simple facts that are needed to deduce Corollaries 1.3 and 1.12, concerning explicit families of trigonometric polynomials and circle maps, from the more general Theorems 1.2, 1.10 and 1.11. These results are all well-known and easy to prove, but we include the short arguments for completeness.

2.4. Lemma (Cosine maps and standard maps)**.**
Let $(a,b),(c,d) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $(a,b) \neq (c,d)$. Then the cosine maps $C_{a,b}$ and $C_{c,d}$ are not conjugate by an affine map $M \in \text{Möb}_{\mathbb{R}}$.

The analogous statement holds for the family $S_{a,b}$.

Proof. We prove the contrapositive, so suppose that $C_{c,d} = M \circ C_{a,b} \circ M^{-1}$ for some affine map $M(x) = \alpha x + \beta$, $\alpha > 0$. Since both maps have period 2π, we must have $\alpha = 1$. Furthermore, we note that the image of the real axis under a map $C_{a,b}$ is an interval that is symmetric around the origin. This implies that $\beta = 0$, and hence $M = \text{id}$ and $(a,b) = (c,d)$.

We note that $S_{a,b} = axe^x + b$ has a critical point at -1 and an asymptotic value at b, and no other critical or asymptotic values. Furthermore, $z = 0$ is the unique preimage of the asymptotic value b. If we have a conjugacy $M \in \text{Möb}_{\mathbb{R}}$ between $S_{a,b}$ and $S_{c,d}$, it follows that M fixes 0, -1 and ∞. Hence $M = \text{id}$ and $(a,b) = (c,d)$. ■

2.5. Lemma (Number of critical points of Arnol'd-type maps)**.**
Let F_μ be a (generalized) trigonometric polynomial as in (1.3). Then the corresponding circle map f_μ has exactly $2m$ critical points in $\mathbb{C}^*$, counted with multiplicities. Moreover, f_μ has no asymptotic values in $\mathbb{C}^*$.

Proof. This is a classical fact. Indeed, note that F'_μ is a trigonometric polynomial of degree m, and hence

$$F'_\mu(z) = R(e^{2\pi i z}),$$

where R is a rational function of degree $2m$. Thus F_μ has exactly $2m$ critical points in every vertical strip of width 1 (counting multiplicities). The claim follows.

It is also elementary to see that F_μ has no finite asymptotic values (and hence f_μ has no asymptotic values in $\mathbb{C}^*$). Let us sketch the argument. Suppose by contradiction that $\gamma : [0,\infty) \to \mathbb{C}$ was a curve to infinity with $F_\mu(\gamma(t)) \to a \in \mathbb{C}$. We must have $|\operatorname{Im}\gamma(t)| \to \infty$. If $D = 0$, this follows from periodicity and otherwise from the fact that $F_\mu(z) = Dz + O(1)$ when restricted to any horizontal strip. Similarly, we must have $|\operatorname{Re}\gamma(t)| \to \infty$, as $|F_\mu(z)|$ grows like $|\mu_{2m}| \cdot e^{2\pi m|\operatorname{Im} z|}/2$ in any vertical strip.

For $\zeta \in \gamma$, we can write

$$F_\mu(\zeta) = D\zeta + \mu_{2m}e^{\pm 2\pi i m\zeta}/2i + o(e^{2\pi i m\zeta}).$$

For sufficiently large ζ, the argument of ζ will be contained in a fixed interval of length $\pi/2$, while the second term keeps "spiralling" to infinity. It follows that we must have $\limsup |F_\mu(\gamma(t))| = \infty$, a contradiction.

(The claim can also be deduced directly from the celebrated Denjoy-Carleman-Ahlfors theorem.) ∎

2.6. Lemma (Conformal conjugacy classes)**.**
Let $D \geq 0$, $m \geq 1$ be integers. Consider trigonometric polynomials as in (1.3) and let f_μ be the corresponding map of the circle S^1. Suppose that $M(z) = e^{2\pi i\beta}z$ is a rotation. Then M conjugates the map f_μ to some map $f_{\mu'}$ in the same family if and only if $\beta = p/m$ where $p \in \mathbb{Z}$. In particular, each affine conjugacy class (via rotations) consists of at most m maps.

Proof. Assume that $f_\mu, f_{\mu'}$ are conjugate by a rotation $M(z) = e^{2\pi i\beta}z$. Note that the lift of $M \circ f_\mu \circ M^{-1}(t)$ is equal to

$$\begin{aligned} F_\mu(t-\beta) + \beta = Dt - (D-1)\beta + \mu_1 + \mu_{2m}\sin(2\pi m(t-\beta)) \\ + \sum_{j=1}^{m-1} \left(\mu_{2j}\sin(2\pi j(t-\beta)) + \mu_{2j+1}\cos(2\pi j(t-\beta))\right). \end{aligned}$$

Using the addition theorems for sine and cosine, we see that this map is in the form (1.3) if and only if $m\beta = 0 \mod 1$. The lemma follows. ∎

Remark 1. Consider $t \mapsto 3t+\sin(4\pi t)+\epsilon\sin(2\pi t)$. Conjugating this with $t \mapsto t+1/2$ gives the map $t \mapsto 3t+1+\sin(4\pi t)-\epsilon\sin(2\pi t)$. These maps are both close to $t \mapsto 3t+\sin(4\pi t)$ as circle maps, so taking the quotient of the set Δ defined in (1.4) by conjugacy classes results in a space with an orbifold structure, not a manifold structure.

Remark 2. If $D \neq 1$, then for each map F_μ as in (1.3), one can find $M \in \text{Möb}_\mathbb{R}$ so that $M \circ F_\mu \circ M^{-1}$ is equal to

$$t \mapsto Dt + \sum_{j=1}^{m}(\mu'_{2j}\sin(2\pi jt) + \mu'_{2j-1}\cos(2\pi jt)) \tag{2.2}$$

by taking $M(t) = t + \mu_1/(D-1)$. (And, vice versa, each map as in (2.2) can be affinely conjugated to one with as in (1.3) by a translation $M(t) = t + \beta$ with β chosen so that $\mu_{2m-1}\cos(2\pi\beta) + \mu_{2m}\sin(2\pi\beta) = 0$.)

2.7. Lemma (Trigonometric polynomials)**.**
Suppose f is a trigonometric polynomial of degree n as in (1.1), and let $\varphi_n, \psi_n \in \text{Homeo}_\mathbb{R}$ with $\varphi_n, \psi_n \to \text{id}$ such that $g_n := \psi_n \circ f \circ \varphi_n^{-1}$ are entire functions for all n.

Then there is a sequence $\alpha_n > 0$ with $\alpha_n \to 1$ such that $f_n = g_n(\alpha_n z)/\alpha_n$ is a trigonometric polynomial for every n. (Furthermore, since $f_n \to f$, the Fourier coefficients of f_n converge to those of f.)

Proof. Let $n \in \mathbb{N}$ and define $\vartheta_n(z) = \varphi_n(\varphi_n^{-1}(z) + 2\pi)$. Then ϑ_n is a homeomorphism. For purposes of legibility we suppress the subscript n in the following. Note that

$$g(\vartheta(z)) = g(\varphi(\varphi^{-1}(z) + 2\pi)) = \psi(f(\varphi^{-1}(z) + 2\pi)) = \psi(f(\varphi^{-1}(z))) = g(z).$$

It follows that ϑ is holomorphic, and hence an affine map $\vartheta(z) = z + \beta$, where $\beta = \beta_n = \vartheta_n(0) \to 2\pi$.

So each g_n is periodic with period β_n, and we are done if we set $\alpha_n := \beta_n/2\pi$. ■

3. Quasisymmetric rigidity on the bounded part of the real dynamics

In this section we consider the following class of functions, which is more general than those considered in the introduction.

3.1. Definition (The class $\mathcal{B}_{\mathrm{real}}$)**.**
We denote by $\mathcal{B}_{\mathrm{real}}$ the set of all real transcendental entire functions with bounded singular sets. (Note that we do not *require that all singular values are real.)*

If $f \in \mathcal{B}_{\mathrm{real}}$, then either $\lim_{x\to+\infty} |f(x)| \to \infty$ or $\sup_{x\geq 0} |f(x)| < \infty$ (and similarly either $\lim_{x\to-\infty} |f(x)| = \infty$ or $\sup_{x\leq 0} |f(x)| < \infty$). For this class of functions one has the following:

3.2. Lemma.
Let $f \in \mathcal{B}_{\mathrm{real}}$, and let $\sigma \in \{+, -\}$. Suppose that $\lim_{x\to\sigma\infty} |f(x)| = \infty$. Then

$$\liminf_{x\to\sigma\infty} \frac{\log\log|f(x)|}{\log|x|} \geq \frac{1}{2}.$$

Proof. This is a standard consequence of the Ahlfors distortion theorem as stated in [A1, Corollary to Theorem 4.8]. Indeed, let $M > 0$ be chosen sufficiently large to ensure that $S(f) \cup \{f(0)\} \subset B_M(0)$, and let $V \subset \mathbb{C} \setminus \{0\}$ be the component of the preimage of $E(M) := \mathbb{C} \setminus \overline{B_M(0)}$ that contains σx for sufficiently large x. Then $f : V \to E(M)$ is a universal covering map, and by the Ahlfors distortion theorem, we have

$$\log|\log f(z)| \geq \frac{1}{2}\log|z| + O(1)$$

for $z \in V$, where $\log f$ is a branch of the logarithm of $f|_V$. The claim follows, since $|\log f(x)| = \log|f(x)| + O(1)$ as $x \to \sigma\infty$. (Compare [AB, Formula (1.2)] for further discussion.) ■

Hence we see that, if $f(\sigma x)$ is unbounded as $x \to +\infty$, then $|f(\sigma x)| > 2x$ for sufficiently large x. In particular, either

- $|f^2(\sigma x)|$ is bounded as $x \to +\infty$ (which can occur either if $|f(\sigma x)|$ remains bounded, or if $f(\sigma x) \to -\sigma\infty$ and $f(-\sigma x)$ remains bounded); or
- $|f^n(\sigma x)| \geq 2^n|x|$, and hence $\sigma x \in I(f)$, for sufficiently large x.

Combinatorial conjugacy.

3.3. Definition (The partition $\mathrm{Part}(f)$)**.**
Let $f \in \mathcal{B}_{\mathrm{real}}$. We denote by $\mathrm{Part}(f) \subset \mathbb{R}$ the set consisting of

(a) *the real critical points $\mathrm{Crit}_{\mathbb{R}}(f)$ of f,*
(b) *the real hyperbolic attracting periodic points of f and*
(c) *the real parabolic periodic points of f.*

3.4. Definition (Combinatorial conjugacy on the real line)**.**
Two functions $f, g \in \mathcal{B}_{\mathrm{real}}$ are called combinatorially conjugate on the real line *if there is an order-preserving bijection*

$$h : \mathrm{Part}(f) \cup O_f^+(S_{\mathbb{R}}(f)) \to \mathrm{Part}(g) \cup O_g^+(S_{\mathbb{R}}(g))$$

that satisfies $h \circ f = g \circ h$, maps points as in (a)-(c) above to corresponding points and preserves the degree of critical points. Furthermore, asymptotic values in $S_{\mathbb{R}}(f)$ and $S_{\mathbb{R}}(g)$ should correspond to each other, in the following sense: for $\sigma \in \{+, -\}$, we have $\lim_{x\to\sigma\infty} f(x) = a \in \mathbb{R}$ if and only if $\lim_{x\to\sigma\infty} g(x) = h(a) \in \mathbb{R}$.

3.5. Lemma (Combinatorial conjugacy and topological conjugacy on the real line)**.**
If $f, g \in \mathcal{B}_{\mathrm{real}}$ are combinatorially conjugate on the real line then they are topologically conjugate on the real line. Moreover, the topological conjugacy h extends the combinatorial conjugacy and hencesatisfies the following properties:

(a) *for each $n \geq 1$ and each $x \in \mathbb{R}$, x is a critical point of f of order n iff $h(x)$ is a critical point of g of order n and*
(b) *$x \in \mathbb{R}$ is a parabolic periodic point of f iff $h(x)$ is a parabolic periodic point of g.*

Furthermore, the extension h is uniquely determined outside of the union of real attracting and parabolic basins.

Proof. Since $f \in \mathcal{B}_{\mathrm{real}}$, Lemma 3.2 implies that if $\lim_{x\to\infty} f(x) = \infty$ then $f\colon \mathbb{R} \to \mathbb{R}$ can be extended to a continuous map $\hat{f}\colon (-\infty, \infty] \to (-\infty, \infty]$ having ∞ as an attracting fixed point. More generally, if $f^2|\mathbb{R}$ is unbounded, then either $\lim_{x\to\infty} f^2(x) = \infty$, $\lim_{x\to-\infty} f^2(x) = -\infty$ or both. So in the latter case, we can extend f to a continuous map $\hat{f}\colon (-\infty, \infty] \to (-\infty, \infty]$, to $\hat{f}\colon [-\infty, \infty) \to [-\infty, \infty)$ or to $\hat{f}\colon [-\infty, \infty] \to [-\infty, \infty]$ having respectively, ∞, $-\infty$ or $-\infty, \infty$ as attracting fixed points or attracting periodic two points. It follows that the only difference between a map $f \in \mathcal{B}_{\mathrm{real}}$ and a multimodal map on a compact interval is that f can have infinitely many turning points. Recall that a point c is called a *turning point* of an interval map $f\colon I \to I$ if the map has a local extremum at c and c is in the interior of I. The assumption in Definition 3.4 about the way asymptotic values are mapped by h ensures, furthermore, that two combinatorially conjugate maps extend in the same manner.

So assume that $f, g \in \mathcal{B}_{\mathrm{real}}$ are combinatorially conjugate on the real line. Since f and g are real analytic,

(i) f and g have no wandering intervals (see [dMvS, Theorem IV.A]);
(ii) f and g have at most finitely many real periodic points that are not hyperbolic and repelling (see [dMvS, Theorem IV.B']); in particular, f and g have no intervals consisting entirely of periodic points;

(iii) each periodic turning point is attracting.

(Note that point (i) implies that any extension of h as in the statement of the lemma is uniquely determined outside of attracting and parabolic basins.)

Let us denote the union of the immediate basins of the finitely many real periodic attractors of f and g by $B_0(f)$ and $B_0(g)$, respectively. Since the combinatorial conjugacy h sends periodic (parabolic) attractors to periodic (parabolic) attractors, we can extend h to a conjugacy between $f\colon B_0(f) \to B_0(f)$ and $g\colon B_0(g) \to B_0(g)$ (mapping iterates of singular values to corresponding iterates of singular values). This implies that assumption (iv) of [dMvS, Theorem II.3.1] is also satisfied, and one can easily check that the proof of that theorem goes through verbatim in our context. (Alternatively, we could apply the latter theorem directly to a restriction of f or a modification of such a restriction as in the proof of Theorem 3.6 below.) ■

Quasisymmetric rigidity. One of the main technical ingredients in this paper is the following:

3.6. Theorem (Quasisymmetric rigidity on the bounded part of the real dynamics)**.**
Let $f \in \mathcal{B}_{\text{real}}$. Then there exists a compact interval $J \subset \mathbb{R}$ (possibly empty or consisting of only one point) with the following properties.

(a) *If $x \in J$ and $f(x) \notin J$, then $x \in I_{\mathbb{R}}(f)$.*
(b) *For every $x \in \mathbb{R}$, either $x \in I_{\mathbb{R}}(f)$ or $f^j(x) \in J$ for all $j \geq 2$.*
(c) *The set of points $z \in \mathbb{C}$ whose ω-limit set is contained in J and which do not belong to an attracting or parabolic basin has empty interior and does not support any invariant line fields.*
(d) *If $g \in \mathcal{B}_{\text{real}}$ is combinatorially conjugate on the real line to f, then there is an interval $\tilde{J}$, which has the corresponding properties for g, and a quasisymmetric conjugacy between $f|_J$ and $g|_{\tilde{J}}$ that agrees with the combinatorial conjugacy.*

This theorem is essentially proved in [CvS], but the setting there is slightly different from ours (in [CvS], the functions have compact domains). Hence the remainder of this section is devoted to showing how to obtain the required intervals J and $\tilde{J}$ under the assumptions of Theorem 3.6, using the results of [CvS].

Anchored interval maps.

3.7. Definition (The class ARAIM of anchored maps)**.**
Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be real-analytic (by which we mean that f is real-analytic on an open interval containing $[a, b]$). If $f(\{a, b\}) \subset \{a, b\}$, then (following [MiT]*) we say that $f\colon [a, b] \to \mathbb{R}$ is an* anchored real-analytic interval map *(ARAIM).*

An ARAIM $f\colon [a, b] \to \mathbb{R}$ and an ARAIM $g\colon [\tilde{a}, \tilde{b}] \to \mathbb{R}$ are said to be *topologically conjugate* if there exists an order-preserving homeomorphism $h\colon [a, b] \to [\tilde{a}, \tilde{b}]$ so that $h \circ f(x) = g \circ h(x)$ for each $x \in [a, b]$ for which $f(x) \in [a, b]$ or $g(h(x)) \in [\tilde{a}, \tilde{b}]$. In [CvS], the following rigidity result is established.

3.8. Theorem (Quasisymmetric rigidity).
Suppose that f and g are ARAIM, and that f and g are topologically conjugate via a conjugacy h. Assume moreover that

(a) *for each $n \geq 1$ and each $x \in \mathbb{R}$, x is a critical point of f of order n iff $h(x)$ is a critical point of g of order n and*
(b) *$x \in \mathbb{R}$ is a parabolic periodic point of f iff $h(x)$ is a parabolic periodic point of g.*

Then the topological conjugacy between f and g extends to a quasisymmetric homeomorphism on the real line.

This theorem was announced by the second author in 2009. Since then, a significantly stronger result than Theorem 3.8 was established in joint work with Trevor Clark, and partly with Sofia Trejo, and hence the original manuscript remains unpublished. In particular, one of the key ingredients in the argument, the proof of *complex bounds* (even for C^3 maps), has been substantially unified and simplified and appears in [CST]. The work on rigidity has also been extended to the setting of C^3 mappings, and only this more general result [CvS] will be submitted for publication (in the near future).

Since the latter manuscript is not yet available, let us comment briefly on the proof of Theorem 3.8. We emphasize that, under the additional assumption that f and g are polynomials without parabolic periodic points whose critical points all have even order and are all real, this theorem already appears in [KSS1] (see the *Rigidity Theorem* and *Rigidity Theorem'* on page 751 of that paper). The proof there uses the following steps:

(a) associate to both f and g an induced first return map, called a *complex box mapping*, near iterates of the critical points;
(b) the main step in the proof is then to show that for a certain subsequence (the *enhanced nest*) these complex box mappings satisfy some a priori *complex bounds* (these are uniform estimates on the moduli of certain annuli);
(c) use the *qc-criterion* from [KSS1, Appendix] to show that these a priori bounds imply that these first return maps are quasiconformally conjugate;
(d) spread this quasiconformal conjugacy to the entire complex plane.

In [CST] it is shown that such complex bounds hold for arbitrary real analytic maps (and even more generally), a fact which was known previously in a large class of special cases, see [KSS1] and [She]. Exactly as in [KSS1], we can then use the qc-criterion which states that bounded geometry implies quasiconformal rigidity to deduce that the complex box mappings are quasiconformally conjugate. (The proof of this criterion in [KSS1] exploits recent results on quasiconformal maps due to Heinonen and Koskela [HK]. We note that, prior to [KSS1], their theorem and its variations were used to prove rigidity results in [PR], [Ha], [GSm], [LvS] and [Sm], where in the last work, the author explicitly stated that a bounded shape property of puzzle pieces implies rigidity for non-renormalizable unicritical maps.)

Once this is done, we can extend the conjugacy to the real line in a quasisymmetric manner. Here the fact that the map is non-polynomial means that one no longer can use Böttcher coordinates, and so one needs to paste together (essentially by hand) the conjugacies which are constructed near different critical points. (This final step is carried out in detail in [CvS], where additional arguments are required to deal also with the C^3 setting.)

We shall also require a result on the absence of invariant line fields. As mentioned above, the existence for complex box mappings for real-analytic maps is proved in [CST] (and in many cases in previous papers). The fact that such a complex box mapping does not support invariant line fields is by now well-known, see for example [KSS1, KvS, CST] and also [CvS]. Thus we immediately obtain the following theorem.

3.9. Theorem (Absence of invariant line fields)**.**
Let $f : [a, b] \to \mathbb{R}$ be an ARAIM, and let $U \supset [a, b]$ be an open subset of $\mathbb{C}$ on which f is analytic. Then the set of points $z \in U$ for which $\operatorname{dist}(f^n(z), [a, b]) \to 0$ as $n \to \infty$ and that do not belong to attracting or parabolic basins has empty interior and does not support any invariant line fields.

A function $f \in \mathcal{B}_{\text{real}}$ which is unbounded both for $x \to +\infty$ and $x \to -\infty$ has a restriction that is an ARAIM, and hence in this situation Theorem 3.6 follows from the statements above. In the case where f is bounded to the left or to the right, this is not necessarily the case, so we will need to be slightly more careful in showing how to deduce Theorem 3.6. However, there are no new dynamical phenomena in this setting, and we will show that we can modify f outside an interval that contains all relevant dynamics to obtain an ARAIM. (Instead, we could also observe that the proof from [CvS] goes through in this slightly modified setting.)

Proof of Theorem 3.6 from Theorems 3.8 and 3.9. By Lemma 3.5, the maps f and g from assumption (c) in the statement of the theorem are in fact topologically conjugate. Let us distinguish a few cases.

Case 1. $\mathbb{R} \setminus I_{\mathbb{R}}(f)$ contains at most one point.

In this case the set $J = \mathbb{R} \setminus I_{\mathbb{R}}(f)$ satisfies all the requirements of the theorem. So from now on we assume in the proof that $\mathbb{R} \setminus I_{\mathbb{R}}(f)$ contains several points.

Case 2. f is unbounded in both directions on the real line.

In this case, let a and b be the smallest resp. largest non-escaping points under f. Then clearly $f(\{a, b\}) \subset \{a, b\}$, so if we set $J := [a, b]$, then the restriction $f|_J$ is a an ARAIM. So the theorem follows from Theorems 3.8 and 3.9.

It remains to deal with the cases where at least one of $f|_{(-\infty,0]}$ and $f|_{[0,+\infty)}$ is bounded.

Case 3. $f|_{\mathbb{R}}$ is bounded.

In this case, there may not exist suitable points $a, b \in \mathbb{R}$ so that $f|_{[a,b]}$ becomes an ARAIM. Therefore we will modify f as follows. Set

$$\alpha := \inf_{x \in \mathbb{R}} f(x) \quad \text{and} \quad \beta := \sup_{x \in \mathbb{R}} f(x)$$

and choose numbers $A < \alpha - 1$ and $B > \beta + 1$ that are not critical points of f.

Choose $\varepsilon > 0$ such that

$$A + 1 < \operatorname{Re} f(z) < B - 1$$

whenever

$$z \in U := \{x + iy : x \in [A, B], |y| < \varepsilon\}.$$

We may also assume that $\varepsilon > 0$ is chosen sufficiently small that f is injective on the boundary segments $[A - i\varepsilon, A + i\varepsilon]$ and $[B - i\varepsilon, B + i\varepsilon]$.

Set $C := A - 1$ and $D := B + 1$. We now define a quasiregular extension $\tilde{f} : V \to \mathbb{C}$ of the restriction $f|_U$, where

$$V := \{x + iy : x \in [C - \varepsilon, D + \varepsilon], |y| < \varepsilon\}.$$

This extension will be chosen to have the following properties:

(a) $\tilde{f}$ commutes with complex conjugation;
(b) $\tilde{f}(\{C, D\}) \subset \{C, D\}$;
(c) $\tilde{f}$ is monotone (without critical points) on $[C - \varepsilon, A]$ and $[B, D + \varepsilon]$;
(d) If $\tilde{f}$ is not holomorphic at $z \in V$, then $\operatorname{Re} f(z) \in [A, B]$.

Such an extension is simple to construct. Indeed, we first determine $\tilde{f}(C)$ and $\tilde{f}(D)$ according to (b) and (c). Then we choose $\tilde{f}$ to be a linear map on $[C-\varepsilon, C+\varepsilon] \times [-\varepsilon, \varepsilon]$ whose image is $[\tilde{f}(C) - 1, \tilde{f}(C) + 1] \times [-1, 1]$, and similarly for D. Finally, we use a diffeomorphism to interpolate between this map and $f|_U$.

Note that $f([A, B]) \subset [A, B]$ so that the orbit of any $z \in V$ enters the region where $\tilde{f}$ is not holomorphic at most once under iteration of $\tilde{f} \colon V \to \mathbb{C}$. This means that $\tilde{f}$ has an invariant Beltrami field on V. Extend this Beltrami field μ to $\mathbb{C}$ by setting it to zero outside V. Now use the Measurable Riemann Mapping Theorem to straighten $\tilde{f}$ to an analytic map F; i.e. let $F = h_\mu^{-1} \circ \tilde{f} \circ h_\mu$ where h_μ is so that $\bar{\partial} h / \partial h = \mu$. Then F is holomorphic and an ARAIM, when restricted to a suitable interval $[a, b]$. Furthermore, the conjugacy h_μ between F and $\tilde{f}$ (and hence f) is conformal on U, so it follows from Theorem 3.9 that f supports no invariant line fields on the set of points whose ω-limit set is contained in $[\alpha, \beta]$.

It is also clear that we can apply the same procedure to a function that is topologically conjugate on the real line to f to obtain an ARAIM that is topologically conjugate on the real line to F. Hence we can apply Theorem 3.8. This completes the proof of the theorem in the case where f is bounded.

Case 4. f is unbounded in one direction, and bounded in the other.

Let us assume without loss of generality that $|f(x)| \to \infty$ as $x \to +\infty$ and that $\limsup_{x \to -\infty} |f(x)| < \infty$. If $f(x) \to -\infty$ as $x \to +\infty$, then f^2 is bounded, and we can apply the previous argument to this iterate. Hence we may suppose that $f(x) \to +\infty$ as $x \to +\infty$, in which case we see as above by Lemma 3.2 that $f(x) \in I(f)$ for sufficiently large x. Let $b \in \mathbb{R}$ be the largest real nonescaping point of f; then b is a (repelling or parabolic) fixed point. We now distinguish three further subcases.

(i) If $\liminf_{x \to -\infty} f(x) > b$, we can choose a as the smallest preimage of b. Then $f|_{[a,b]}$ is an ARAIM, and $J := [a, b]$ has the desired properties.
(ii) If f has infinitely many preimages of b on the real axis, we can pick a as such a preimage chosen small enough that $a < \alpha := \inf_{x \in \mathbb{R}} f(x)$. Again, we can set $J := [a, b]$ and $f|_J$ is an ARAIM.
(iii) In the remaining case, $f(x) < b$ whenever x is sufficiently negative. We can choose $A < \alpha - 1$ in such a way that A is not a critical point and modify the function f to the left of A, exactly as above, to obtain a quasiregular map that

straightens to a holomorphic map whose restriction to a suitable interval is an ARAIM. So we are done also in this case, setting $J := [A, b]$. ■

4. Gluing and extending quasiconformal homeomorphisms dynamically

Topological equivalence. To ensure that not only the order relation of the critical points and critical values of f and g on the real line are the same, but that they are also compatible in the complex plane we use the notion of topological equivalence from [EL].

4.1. Definition (Real-topological equivalence)**.**
Two maps $f, g \in \mathcal{S}_\mathbb{R}$ are called real-topologically equivalent *if there are functions $\varphi, \psi \in$ $\mathrm{Homeo}_\mathbb{R}$ such that*

$$\psi(f(z)) = g(\varphi(z))$$

for all $z \in \mathbb{C}$.

The set of all functions g that are real-topologically equivalent to f is denoted by $M_f^\mathbb{R}$.

Remark 1. If f and g are real-topologically equivalent, then they are in fact *real-quasiconformally equivalent*; i.e. the maps ψ and φ can be chosen to be quasiconformal. Indeed, suppose that maps $\varphi, \psi \in \mathrm{Homeo}_\mathbb{R}$ as in the definition are given. Because $S(f)$ is finite, we can find a quasiconformal homeomorphism $\tilde{\psi} \in \mathrm{Homeo}_\mathbb{R}$ such that ψ and $\tilde{\psi}$ are isotopic relative $S(f) \cup \infty$. We can lift the homotopy to a homotopy between φ and a map $\tilde{\varphi}$ such that $\tilde{\psi} \circ f = g \circ \tilde{\varphi}$. Because f and g are holomorphic, it follows that $\tilde{\varphi}$ is also quasiconformal.

Remark 2. The set $M_f^\mathbb{R}$ can naturally be given the structure of a $q+2$-dimensional real-analytic manifold, where $q = \#S(f)$, as we discuss in Section 7. For now, we only consider $M_f^\mathbb{R}$ as a set of entire functions.

Note that the maps φ and ψ might not be uniquely determined. When we speak of two real-topologically equivalent functions, we always implicitly assume that a specific choice of φ and ψ is given. Another way of saying this is that we *mark* the singular values and the critical points.

One important consequence of f, g being real-topologically equivalent is that if c is a critical point of f then $\varphi(c)$ is a critical point of g of the *same order*.

Several notions of conjugacy. Let $f, g \in \mathcal{S}_\mathbb{R}$ be real-topologically equivalent, with a suitable choice of φ and ψ as above. We will now discuss a number of different important notions of conjugacies: *combinatorial*, *topological*, *quasiconformal* and *conformal*.

First we modify the definition of combinatorial conjugacy on the real line (recall Definition 3.4). The point of this modification is that, when we look at functions in the complex plane, we should restrict to those that are real-topologically equivalent. Given such a real-topological equivalence, represented by maps φ and ψ, we have a natural correspondence between the critical points of f and g (via φ) and the singular values of f and g (via ψ). Our combinatorial conjugacy should respect this information; i.e. map corresponding critical points and singular values to each other. Furthermore, if we wish to relate our maps f and g also in the complex plane, then we must consider not only the behaviour of points in $\mathcal{S}_\mathbb{R}(f)$, but include *all* singular values in the definition.

4.2. Definition (Combinatorial conjugacy for maps in $\mathcal{S}_{\mathbb{R}}$)**.**
Two functions $f, g \in \mathcal{S}_{\mathbb{R}}$ are called combinatorially conjugate (in $\mathbb{C}$) *if they are real-topologically equivalent, say $\psi \circ f = g \circ \varphi$, and there exists an order-preserving bijection*

$$h\colon \operatorname{Part}(f) \cup O_f^+(S(f)) \to \operatorname{Part}(g) \cup O_f^+(S(g))$$

such that

(a) $h \circ f = g \circ h$,
(b) $h|_{\operatorname{Crit}_{\mathbb{R}}(f)} = \varphi|_{\operatorname{Crit}_{\mathbb{R}}(f)}$,
(c) $h|_{S(f)} = \psi|_{S(f)}$ *and*
(d) *h maps each nonrepelling periodic point to a nonrepelling periodic point of the same type (i.e. hyperbolic to hyperbolic, and parabolic to parabolic).*

The reason we say that f and g are combinatorially conjugate in $\mathbb{C}$ (rather than combinatorially conjugate on the real line) is that the assumption that $f, g \in \mathcal{S}_{\mathbb{R}}$ are real-topologically equivalent implies that f, g are topologically conjugate on the complex plane whenever the combinatorial conjugacy $h\colon \mathbb{R} \to \mathbb{R}$ is quasisymmetric, see Theorem 4.7.

4.3. Proposition (Combinatorial conjugacy in $\mathbb{C}$ implies topological conjugacy in $\mathbb{R}$)**.**
If $f, g \in \mathcal{S}_{\mathbb{R}}$ are combinatorially conjugate in $\mathbb{C}$, then these maps are combinatorially conjugate on the real line (in the sense of Definition 3.4) and therefore topologically conjugate on the real line. Furthermore, the topological conjugacy can be chosen to agree with the combinatorial conjugacy from Definition 4.2.

Proof. Property (b) and the fact that f, g are real-topologically equivalent imply that h sends critical points of f to critical points of g of the same order. Also, the condition on asymptotic values is automatically satisfied: if $\lim_{x \to \sigma\infty} f(x) = a$, then

$$\lim_{x \to \sigma\infty} g(x) = \lim_{x \to \sigma\infty} g(\varphi(x)) = \psi\big(\lim_{x \to \sigma\infty} f(x)\big) = \psi(a).$$

The proposition therefore follows from Lemma 3.5. We note that, a priori, the topological conjugacy provided by this lemma is an extension of the real combinatorial conjugacy, which in general is a *restriction* of our original map h. However, the extension will automatically agree with our original map on points that do not belong to attracting or parabolic basins (due to absence of wandering intervals), and can easily be arranged to respect the finitely many remaining orbits. ∎

Combinatorial conjugacy can also be expressed alternatively in terms of *kneading sequences*, which is an idea that we use later.

4.4. Definition (Topological and QC conjugacy)**.**
Two maps $f, g \in \mathcal{S}_{\mathbb{R}}$ are called real-topologically conjugate *if there is a homeomorphism $\vartheta \in \operatorname{Homeo}_{\mathbb{R}}$ such that $\vartheta \circ f = g \circ \vartheta$ on $\mathbb{C}$. (The prefix "real" in this notation is to express that ϑ preserves the real line.)*

If this homeomorphism ϑ is quasiconformal, we say that f and g are real-quasiconformally conjugate.

Finally, let us turn to a notion of conjugacy on escaping sets. Recall that $I_{\mathbb{R}}(f) = \{x \in \mathbb{R}\colon |f^n(x)| \to \infty\}$.

4.5. Definition (Escaping conjugacy)**.**
Let $f, g \in \mathcal{S}_{\mathbb{R}}$ be real-topologically equivalent. We say that f and g are escaping conjugate *if there is an order-preserving homeomorphism $j : I_{\mathbb{R}}(f) \to I_{\mathbb{R}}(g)$ such that:*

(a) *$j \circ f = g \circ j$ on $I_{\mathbb{R}}(f)$;*
(b) *j agrees with φ on $\mathrm{Crit}_{\mathbb{R}}(f) \cap I_{\mathbb{R}}(f)$ and with ψ on $S(f) \cap I_{\mathbb{R}}(f)$, and*
(c) *for every closed and forward-invariant set $K \subset I_{\mathbb{R}}(f)$, $j|_K$ extends to a quasisymmetric homeomorphism of the real line.*

The article [R] provides a simple way of encoding when two maps are escaping conjugate. We discuss this below. For now, we only need the following fact.

4.6. Theorem (Escaping rigidity)**.**
If $f, g \in \mathcal{S}_{\mathbb{R}}$ are real-topologically conjugate, then they are escaping conjugate.

Proof. Let h be the real-topological conjugacy between f and g; we set $j = h$. The first two conditions in the definition of escaping conjugacy are trivially satisfied (recall that we can take $\varphi = \psi = h$ in the definition of topological equivalence). So let $K \subset I_{\mathbb{R}}(f)$ be a closed and forward-invariant set; we must show that $h|_K$ extends to a quasisymmetric homeomorphism of the real line. This is stated explicitly in [R, Theorem 1.3] for the case where the set K is the union of finitely many escaping orbits (which is in fact sufficient for the purposes of this paper).

In general, we can deduce this claim from the results of [R] as follows. First observe that, for every $R > 0$ there is $n_0 \in \mathbb{N}$ such that $f^n(K) \cap [-R, R] = \emptyset$ for all $n \geq n_0$. Indeed, by the disussion following Lemma 3.2, we may assume that R is sufficiently large to ensure that $|f^n(x)| \geq R$ whenever $|f^2(x)| \geq R$. Hence we need to prove the claim only for the set $K \cap [-R, R]$, where it is trivial by compactness.

Now [R, Theorem 1.1. and Corollary 4.2] imply that $h|_{f^n(K)}$ extends to a real-quasiconformal homeomorphism ψ. We may choose ψ in such a way that it agrees with h on the set of singular values of f^n. Hence ψ and h are isotopic relative to $S(f^n)$, and we may lift the homotopy to obtain a map φ such that $\psi \circ f^n = g^n \circ \varphi$. By construction, the real-quasiconformal map φ extends $h|_K$, as desired. ■

Promoting conjugacies: the pullback argument. The following is a version of a well-known argument of promoting combinatorial conjugacies to quasiconformal ones, provided that one has control on the postsingular set.

4.7. Theorem (The pullback argument)**.**
Suppose that $f, g \in \mathcal{S}_{\mathbb{R}}$ are combinatorially conjugate (in $\mathbb{C}$) and that the combinatorial conjugacy h extends to a quasisymmetric homeomorphism $h : \mathbb{R} \to \mathbb{R}$.

Then f and g are real-quasiconformally conjugate, where the conjugacy ϑ can be chosen to agree with h on $\mathrm{Part}(f) \cup O^+_f(S(f))$.

Proof. Since the map h is quasisymmetric, it extends to a real-quasiconformal map $\vartheta_0 : \mathbb{C} \to \mathbb{C}$. Let φ and ψ be the maps from the definition of real-topological equivalence. By the definition of a combinatorial conjugacy, the map ϑ_0 is isotopic to ψ relative $S(f)$.

Furthermore, it follows from the assumption that f and g are combinatorially conjugate, or alternatively from the quasisymmetry of h, that every attracting cycle of f

maps to an attracting cycle of g, and every parabolic cycle of f maps to a parabolic cycle of g under h.

Also note that, in the class $\mathcal{S}_{\mathbb{R}}$, every attracting direction of a parabolic point must be aligned with the real axis, so there are only three possibilities for parabolic points: a parabolic point with one fixed attracting petal (corresponding to a saddle-node $z \mapsto z + z^2$) a parabolic point with two fixed attracting petals (as for $z \mapsto z - z^3$), or one with a 2-cycle of attracting petals (corresponding to a fixed point with eigenvalue -1 as in the period-doubling bifurcation). Since each attracting petal must contain some critical point, the combinatorial conjugacy must map each parabolic point to one of the same type.

It is then easy to see that we can choose the map ϑ_0 in such a way that ϑ_0 is a conjugacy between f and g in some linearizing neighbourhood or attracting petal for every attracting periodic point or parabolic attracting direction. This can be done as in Section 5 of [CvS].

By the covering homotopy theorem, we can find a map ϑ_1, isotopic to φ relative $f^{-1}(S(f))$, such that $\vartheta_0 \circ f = g \circ \vartheta_1$. Here we use that φ agrees with h on $\mathcal{P}(f)$ and that h maps critical points of f to critical points of g of the same order. Since φ preserves the real line, and f and g are real, we also get that $\vartheta_1(\mathbb{R}) = \mathbb{R}$.

We claim that ϑ_1 agrees with the original map h on the postsingular set. Indeed, let $v \in \mathcal{P}(f)$. By construction,

$$g(\vartheta_1(v)) = \vartheta_0(f(v)) = h(f(v)) = g(h(v)),$$

so $\vartheta_1(v)$ and $h(v)$ both have the same image. Since $\vartheta_1 = \varphi_1 = h$ on the set of critical points of f, we see that $\vartheta_1(v)$ and $h(v)$ belong to the same interval of $\mathbb{R} \setminus \operatorname{Crit}(g)$, and since g is injective on each of these intervals, we have $\vartheta_1(v) = h(v)$ as desired.

In particular, ϑ_1 is also isotopic to ψ, and we can repeat the above procedure to obtain maps ϑ_j with

$$\vartheta_j \circ f = g \circ \vartheta_{j+1},$$

and such that ϑ_j is isotopic to ψ relative to the postsingular set and isotopic to φ relative $\operatorname{Crit}_{\mathbb{R}}(f)$.

Note that the maps ϑ_j and ϑ_{j+1} agree on the j-th preimages of the union of the postsingular set with the originally chosen linearizing neighbourhoods and parabolic petals. Also note that their maximal dilatation does not increase with j. Hence ϑ_j converges to a suitable quasiconformal function h, which is the desired conjugacy. ■

5. Rigidity

In this section, we establish our main rigidity theorem.

5.1. Theorem (From combinatorial to quasiconformal conjugacy)**.**
Let $f, g \in \mathcal{S}_{\mathbb{R}}$ be combinatorially conjugate (in $\mathbb{C}$) and escaping conjugate. Then f and g are real-quasiconformally conjugate (and the conjugacy can be chosen to be an extension of the original combinatorial conjugacy).

Remark. For functions that are bounded on the real line, that have no asymptotic values and for which all critical points are real, the statement simplifies as follows: If f and g

are real-topologically combinatorially equivalent and topologically conjugate on the real line, then they are real-quasiconformally conjugate in the complex plane.

Proof. Let

$$h\colon \operatorname{Part}(f) \cup O_f^+(S(f)) \to \operatorname{Part}(g) \cup O_f^+(S(g))$$

be the combinatorial conjugacy between f and g. We write $\operatorname{dom}(h) := \operatorname{Part}(f) \cup O_f^+(S(f))$ for the domain of h.

Claim. There exists a quasisymmetric extension of h to $h : \mathbb{R} \to \mathbb{R}$.

Proof. Theorem 3.6 asserts that there exists a compact interval $J \subset \mathbb{R}$ (possibly empty or consisting of only one point) with the following properties.

(a) For every $x \in \mathbb{R}$, either $x \in I_{\mathbb{R}}(f)$ or $f^j(x) \in J$ for all $j \geq 2$.
(b) The set of points $z \in \mathbb{C}$ whose ω-limit set is contained in J and which do not belong to an attracting or parabolic basin does not support any invariant line fields.
(c) If $g \in \mathcal{B}_{\text{real}}$ is combinatorially conjugate on the real line to f, then there exists an interval $\tilde{J}$ with corresponding properties and an order-preserving quasisymmetric homeomorphism $h_1\colon \mathbb{R} \to \mathbb{R}$ which maps J onto $\tilde{J}$ and so that $h_1 \circ f = g \circ h_1$ on J and such that $h_1 = h$ on $\operatorname{dom}(h) \cap J$.

Let I_+ and I_- denote the two components of $\mathbb{R}\setminus J$, and let $\tilde{I}_+$ and $\tilde{I}_-$ be the corresponding components of $\mathbb{R} \setminus \tilde{J}$, i.e. $\tilde{I}_\sigma = h_1(I_\sigma)$. Fix $\sigma \in \{+, -\}$.

Subclaim. The restriction of h to $\operatorname{dom}(h) \cap I_\sigma$ can be extended to an order-preserving quasisymmetric homeomorphism $h_\sigma : \mathbb{R} \to \mathbb{R}$.

To see this, note that $\operatorname{dom}(h) \cap I_\sigma$ is a closed and discrete subset of the real line. We distinguish three cases:

- If $\operatorname{dom}(h) \cap I_\sigma$ is finite, then the subclaim is trivial.
- If I_σ contains infinitely many postsingular points, then $|f|$ is unbounded as $x \to \sigma\infty$, and in particular $\operatorname{dom}(h) \cap I_\sigma$ consists of finitely many escaping singular orbits (possibly together with finitely many additional points). The subclaim follows from the assumption that f and g are escaping conjugate.
- If $\operatorname{dom}(h) \cap I_\sigma$ is infinite but contains only finitely many postsingular points, it must contain infinitely many critical points. The subclaim follows from the fact, remarked after Definition 4.1, that the restriction of φ to the set of critical points of f extends to a quasisymmetric homeomorphism. This completes the proof of the subclaim.

Because $\operatorname{dom}(h) \cap I_\sigma$ is a closed set, we can construct the desired extension $h : \mathbb{R} \to \mathbb{R}$ by interpolating between h_-, h_1 and h_+. (E.g., h agrees with h_1 on J and with each h_σ on a closed subinterval of I_σ which contains $\operatorname{dom}(h) \cap I_\sigma$ and is linear on the complement of these intervals.) This completes the proof of the claim. △

The assertion in the theorem now follows from the pullback argument (Theorem 4.7). ■

Proof of Theorem 1.8. Suppose that $f, g \in \mathcal{S}_{\mathbb{R}}$ are topologically conjugate by a conjugacy h that preserves the real axis. We may assume that h commutes with complex conjugation (otherwise, replace h on the lower half-plane by the map $h(z) := \overline{h(\bar{z})}$). Hence either f and g are real-topologically conjugate (if $h|_{\mathbb{R}}$ is order-preserving) or f and $\tilde{g}(z) := -g(-z)$ are real-topologically conjugate (otherwise). So the claim follows from the previous theorem and Theorem 4.6. ■

6. Absence of Invariant Line Fields

Absence of invariant line fields in $\mathcal{S}_{\mathbb{R}}$. In this section, we are concerned with showing that the functions $f \in \mathcal{S}_{\mathbb{R}}$ we consider do not support any invariant line fields on their Julia sets. (Recall the definitions from Section 2.) As mentioned in the introduction, we will do so by decomposing the Julia set in a number of dynamically distinct sets and treat each separately.

So let $f \in \mathcal{S}_{\mathbb{R}}$ and define

$$\mathcal{P}_B(f) := \{z \in \mathcal{P}(f) : O^+(z) \text{ is bounded}\} \quad \text{and}$$
$$\mathcal{P}_I(f) := \mathcal{P}(f) \cap I_{\mathbb{R}}(f) = \mathcal{P}(f) \setminus \mathcal{P}_B(f).$$

We consider the following subsets of the complex plane:

(a) The radial Julia set $J_r(f)$ (by definition this is the set of all points $z \in J(f)$ with the following property: there is some $\delta > 0$ such that, for infinitely many $n \in \mathbb{N}$, the disk $\mathbb{D}^{\#}_{\delta}(f^n(z))$ can be pulled back univalently along the orbit of z).
(b) The escaping set $I(f) = \{z \in \mathbb{C} : |f^n(z)| \to \infty \text{ as } n \to \infty\}$.
(c) The set $L_B(f)$ of points $z \in J(f) \setminus J_r(f)$ with $\operatorname{dist}(f^n(z), \mathcal{P}_B(f)) \to 0$.
(d) The set $L_I(f)$ of points $z \in J(f) \setminus (J_r(f) \cup I(f))$ with $\operatorname{dist}^{\#}(f^n(z), \mathcal{P}_I(f)) \to 0$.

6.1. Lemma (Partition of the Julia set)**.**
For any $f \in \mathcal{S}_{\mathbb{R}}$, we have $J(f) = J_r(f) \cup I(f) \cup L_B(f) \cup L_I(f)$.

Proof. Any point with $\limsup \operatorname{dist}^{\#}(f^n(z), \mathcal{P}(f)) > 0$ belongs to $J_r(f)$. So it remains to show that an orbit cannot accumulate both on bounded and on escaping singular orbits.

This follows from continuity of f. Indeed, consider the spherical distance $\delta := \operatorname{dist}^{\#}(\mathcal{P}_B(f), \mathcal{P}_I(f) \cup \{\infty\})$. Since the set of singular values is finite, the sets $\mathcal{P}_B(f)$ and $\mathcal{P}_I(f) \cup \{\infty\}$ are both compact subsets of $\bar{\mathbb{C}}$, hence we have $\delta > 0$.

Then there exists $\varepsilon \in (0, \delta/2)$ such that $\operatorname{dist}^{\#}(\mathcal{P}_B(f), f(z)) < \delta/2$ for any point $z \in \mathbb{C}$ with $\operatorname{dist}(\mathcal{P}_B(f), z) < \varepsilon$. If $z \in \mathbb{C} \setminus J_r(f)$, then $\operatorname{dist}^{\#}(f^n(z), \mathcal{P}(f)) < \varepsilon$ for sufficiently large n. We then have either $\operatorname{dist}^{\#}(f^n(z), \mathcal{P}_B(f)) \geq \varepsilon$ for all such n, or $\operatorname{dist}^{\#}(f^n(z), \mathcal{P}_B(f)) < \varepsilon$ for all sufficiently large n. In the former case, we must have $z \in I(f) \cup L_I(f)$, while in the latter, $z \in L_B(f)$. ■

6.2. Theorem (No invariant line fields on radial, escaping and bounded orbits)**.**
Suppose that $f \in \mathcal{S}_{\mathbb{R}}$. Then the sets $J_r(f)$, $I(f)$ and $L_B(f)$ support no invariant line fields. Furthermore, if f has bounded criticality, then the set $L_I(f)$ has zero Lebesgue measure, and hence also does not support invariant line fields.

Proof. The set $J_r(f)$ (of any transcendental meromorphic function) does not support any invariant line field by [RvS1, Corollary 7.1]. (Compare also [MaR].) The set $I(f)$ does not support any invariant line fields; in fact, this is true for all transcendental entire functions for which $S(f)$ is bounded [R]. The set $L_B(f)$ supports no invariant line fields by Theorem 3.6 above.

Finally, let $z \in L_I(f)$, and let $v \in \mathcal{P}_I(f)$ be a limit point of the orbit of z; say $f^{n_i}(z) \to v$. If D is a small disk around v, and D_i is the component of $f^{-n_i}(D)$ containing z, then the assumption of bounded criticality implies that the restrictions $f^{n_i} : D_i \to D$ are proper maps of bounded degree (independent of i). By Lemma [RvS1, Lemma 3.6], the set of such points has Lebesgue measure zero, as claimed. (Compare also the proof of Claim 1 in the proof of Theorem 6.3 below.) ■

Absence of invariant line fields on points asymptotic to singular orbits. We now come to the main new result of this section.

6.3. Theorem (Absence of invariant line fields on $L_I(f)$)**.**
Suppose that $f \in \mathcal{S}_{\mathbb{R}}$ satisfies the sector condition (Definition 1.5). Then $L_I(f)$ supports no invariant line fields.

Proof. Suppose by contradiction that $L_I(f)$ supports a measurable invariant line field μ. As mentioned, this means that there exists a set $A \subset L_I(f)$ of positive Lebesgue measure so that $A \ni z \mapsto \mu(z)$ is a measurable choice of a (real) line through z (i.e. it is a measurable map from A into the projective plane).

The rough idea of the proof is as follows. First of all, we let z be a point of continuity of the line field μ, and will observe that (unless z belongs to a set of measure zero) its orbit must accumulate at some point $v \in \mathcal{P}_I(f)$, passing either through transcendental singularities or through neighbourhoods of critical points of high degree. This will allow us to conclude that v has circular neighbourhoods in which the line field μ looks almost like a *radial line field* $\vartheta(z) = \rho z/|z|$, where $\rho \in \mathbb{C}$ with $|\rho| = 1$. (See Figure 1.) More precisely, we show:

Claim 1. For almost every $z \in A$, the following holds. Let $v \in \mathcal{P}_I(f)$ be an accumulation point of the orbit of z. Then there exists a sequence $\delta_i \to 0$ of radii such that the rescalings

$$\tilde{\mu}_i(z) := \mu(\delta_i z + v), \quad z \in \mathbb{D} = B_1(0),$$

converge to a radial line field $\vartheta(z) = \rho z/|z|$ on $\mathbb{D}$. (Here, convergence means that for any $\epsilon > 0$ there exists a set $X_\epsilon \subset \mathbb{D}$ so that the Lebesgue measure of $\mathbb{D} \setminus X_\epsilon$ is less than ϵ and so that $\tilde{\mu}_i$ is defined on X_ϵ and converges uniformly to ϑ on X_ϵ.)

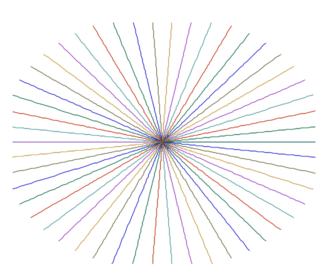

Figure 3. Near the value v the line field μ is almost radial.

Once this claim is established, we take forward iterates of the disk $B_{\delta_i}(v)$, until it stretches many times over some large annulus. Given what we know about the line field on A_i, we can derive a contradiction.

To make this idea more precise, we use the *logarithmic change of variable* [EL, Section 2]. If v is a limit point as in Claim 1, then $|f^n(v)| \to \infty$ as $n \to \infty$. Because the set of singular values of f is finite (and hence bounded), there is $\sigma \in \{+, -\}$ so that $f^n(v) \to \sigma\infty$ or so that $(-1)^n f^n(v) \to \sigma\infty$. In the first situation $\lim_{x\to\sigma\infty} f(x) = \sigma\infty$ and in the second case $\lim_{x\to\sigma\infty} f(x) = -\sigma\infty$. To fix our ideas, let us suppose that we are in the former case; the arguments in the latter are analogous. (Note, however, that the sector condition is not preserved under iteration, so we cannot simply reduce the second case to the first by considering f^2 instead of f.) Thus we assume that $\lim_{x\to+\infty} f(x) = +\infty$ and that there exists a point v as in Claim 1 such that $f^n(v) \to +\infty$. (In particular, we have $+ \in \Sigma$, where Σ is the set from the sector condition.)

Choose $M > 0$ large enough such that $M > |f(0)|$, such that $f(x) > x$ for $x \geq M$ and such that

$$E(M) := \{z \in \mathbb{C}; |z| > M\}$$

contains no singular values of f. Let V be the component of $f^{-1}(E(M))$ that contains $[M, \infty)$.

Since $E(M)$ contains no singular values, $f\colon V \to E(M)$ is a covering map. Since f is transcendental, V is simply-connected and $f\colon V \to E(M)$ is a universal covering. Set $r := \log M$, $\mathbb{H}_r := \{z \in \mathbb{C} : \operatorname{Re} z > r\}$ and let W be the component of $\exp^{-1}(V)$ that contains $[r, \infty)$. Because $\exp : \mathbb{H}_r \to E(M)$ is also a universal covering map, and $\exp : W \to V$ is univalent, there is a conformal isomorphism $F\colon W \to \mathbb{H}_r$ such that $\exp \circ F = f \circ \exp$ and $F(W \cap \mathbb{R}) \subset \mathbb{R}$.

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, "\exp"'] & \mathbb{H}_r \arrow[d, "\exp"] \\ V \arrow[r, "f"] & E(M) \end{tikzcd}$$

It is well-known that the map F is strongly expanding, see equation (6.3) below, and we will use this, together with the sector condition, to blow up the almost radial line field from Claim 1 to a large scale (in logarithmic coordinates). More precisely, we use the following.

Claim 2. There exist constants $r_1 > r$ and $c < 1$ such that, for every $K \geq c > 0$, there is $\delta_0 = \delta_0(K)$ with the following property.

Let $w \geq r_1$ and $\delta \leq \delta_0$. Then there exist $\tilde{\delta} \leq \delta$ with $\tilde{\delta} \geq c \cdot \delta/K$ and a number $n \geq 0$ such that F^n is defined and univalent on $B_{\tilde{\delta}}(w)$ and

$$F^n(B_{\tilde{\delta}}(w)) \supset B_K(F^n(w)).$$

To show how these two claims, together, yield the theorem, let $v \in \mathcal{P}_I(f)$ be a point as in Claim 1 such that $f^n(v) \to +\infty$. By passing to a forward iterate, if necessary, we can assume that $f^n(v) > e^{r_1}$ for all $n \geq 0$, where r_1 is as in Claim 2. So $f^n(v) \in E(M)$ for all $n \geq 0$. Set $w := \log v \in \mathbb{R}$.

Let μ' be the line field on $\mathbb{H}_r$ defined by pulling back μ under $\exp\colon \mathbb{H}_r \to E(M)$. Then μ' is $2\pi i$-periodic by definition. It follows from Claim 1 that there is a sequence $\delta_i \to 0$ of radii such that the rescalings of μ' on the disks $B_{\delta_i}(w)$ converge to a radial line field.

If we let K_i be a sequence that tends to infinity sufficiently slowly, then for any choice of ε_i between $4\pi \cdot c \cdot \delta_i/K_i^2$ and δ_i, the rescalings of μ' on the disks $B_{\varepsilon_i}(w)$ will also converge to a radial line field. (Here c is the constant from Claim 2.) We may also assume that $K_i > 4\pi$ and $\delta_0(K_i) > \delta_i$ for all i.

Now we apply Claim 2 to obtain numbers $\tilde{\delta}_i$ with $c\cdot\delta_i/K_i \leq \tilde{\delta}_i \leq \delta_i$ as well as numbers n_i such that F^{n_i} is defined and univalent on $B_{\tilde{\delta}_i}(w)$ and covers $B_{K_i}(F^{n_i}(w))$. If we set $\varepsilon_i := 4\pi\cdot\tilde{\delta}_i/K_i$, then $F^{n_i}(B_{\varepsilon_i}(w)) \supset B_{4\pi}(F^{n_i}(w))$. To see this, apply the Schwarz Lemma to the branch of $F^{-n_i}|B_{K_i}(F^{n_i}(w))$ mapping into $B_{\tilde{\delta}_i}(w)$.

Since $F^{n_i}(B_{\varepsilon_i}(w))$ can be much larger than $B_{4\pi}(F^{n_i}(w))$, we define $\kappa_i > 0$ to be the largest integer so that $\varphi_i(\mathbb{D}) \supset B_{4\pi}(0)$ where

$$\varphi_i : \mathbb{D} \to \mathbb{C}; \quad \zeta \mapsto \frac{F^{n_i}(w + \varepsilon_i \cdot \zeta) - F^{n_i}(w)}{\kappa_i}.$$

Passing to a subsequence again if necessary, we can assume that the φ_i converge uniformly to a non-constant linear map. (Recall that each φ_i extends to a conformal map on a disk whose radius tends to ∞ as $i \to \infty$.) Now consider the sequence of line fields on the disk $D = B_{4\pi}(0)$ obtained by first rescaling the line field μ' on the disk $D := B_{\varepsilon_i}(w)$ as above, and then pushing forward under the map φ_i. By construction, these push-forwards converge to the radial line field on D. On the other hand, by invariance of μ', these push-forwards are each obtained from μ' by a translation and a rescaling by a factor of $1/\kappa_i$, and hence are all $2\pi i$-periodic. (Here we use that κ_i is an integer, and consequently μ' is $2\pi\kappa_i$-periodic). But then we obtain that the radial line field on the disk D is also $2\pi i$-periodic, which is absurd.

It remains to establish Claims 1 and 2. To prove the former, let z be a Lebesgue density point of A which is also a point of continuity of μ. This means that for each $\epsilon > 0$ there exists $\delta > 0$ and a fixed line μ_0 so that $\operatorname{dens}(A, B_\delta(z)) \geq 1 - \epsilon$ and so that $|\{z \in B_\delta(z) \cap A; |\mu(z) - \mu_0| \leq \epsilon\}|/|B_\delta(z)| \geq 1 - \epsilon$. Here ,

$$\operatorname{dens}(A, B) := \frac{\operatorname{meas}(A \cap B)}{\operatorname{meas}(B)}$$

denotes the density of A in B and $|\mu(z) - \mu_0|$ denotes the angle between the lines $\mu(z)$ and μ_0.

Let $v \in \mathcal{P}_I(f)$ be a limit point of the orbit of z; say $f^{n_i}(z) \to v$. Since the set of singular values of f is finite, we can take $r > 0$ so small that the set $U := B_r(v)$ does not intersect $\mathcal{P}(f) \setminus \{v\}$. We may assume that $f^{n_i}(z) \in U$ for all i. Let U_i be the component of $f^{-n_i}(U)$ that contains z. Let us also denote by U_i^* the component of $f^{-n_i}(U \setminus \{v\})$ contained in U_i.

Then U_i is simply connected, and since $z \in J(f)$, we have

$$\operatorname{dist}(z, \partial U_i) \to 0. \tag{6.1}$$

Furthermore, $f^{n_i} : U_i \to U$ is either a finite-to-one covering map of some degree $d_i < \infty$ (branched only over v) or $f^{n_i} : U_i \to U \setminus \{v\}$ is a universal covering (of degree $d_i = \infty$). Note that $U_i^* = U_i$ when $d_i = \infty$, whereas otherwise $U_i \setminus U_i^*$ consists of a single iterated preimage v_i of v. The set of points $z \in L_I(f)$ for which the sequence d_i does not tend to infinity has Lebesgue measure zero by [RvS1, Lemma 3.6]. So we may assume that z was chosen such that $d_i \to \infty$.

Let $\mathbb{H} = \{z \in \mathbb{C}; \operatorname{Re}(z) > 0\}$ denote the right half plane, and define

$$E : \mathbb{H} \to U \setminus \{v\}; \quad z \mapsto v + r \cdot e^{-z}.$$

Since E is a universal covering, there exists a covering map $\psi_i : \mathbb{H} \to U_i^*$ with $f^{n_i} \circ \psi_i = E$. Since $f^{n_i} \colon U_i^* \to U \setminus \{v\}$ has degree d_i, ψ_i will be injective when restricted to any horizontal strip of height $2\pi d_i$.

$$\begin{tikzcd} U_i^* \arrow[dr, "f^{n_i}"] & \mathbb{H} \arrow[l, "\psi_i"'] \arrow[d, "E"] \\ & U \setminus \{v\} \end{tikzcd}$$

Let ζ_i be a preimage of $f^{n_i}(z)$ under E, and define

$$R_i := \operatorname{Re} \zeta_i.$$

Since $f^{n_i}(z) \to v$, we have $R_i \to \infty$.

Proof of Claim 1. Let $\Delta_i < 2\pi d_i$ be a sequence that tends to infinity sufficiently slowly (to be fixed below). For simplicity let us also require that Δ_i is a multiple of 2π. Consider the squares

$$Q_i := \zeta_i + \left[\frac{-\Delta_i}{2}, \frac{\Delta_i}{2}\right] \times \left[-i\frac{-\Delta_i}{2}, i\frac{\Delta_i}{2}\right]$$

with sides of length Δ_i and centre ζ_i. Note that ψ_i is injective on Q_i. Indeed, if

$$S_i := \{a + ib : a > 0, |b - \operatorname{Im} \zeta_i| < \pi d_i\}$$

is the horizontal strip of height $2\pi d_i$ centered at ζ_i, then ψ_i is injective on S_i, as mentioned above. Furthermore, if Δ_i grows sufficiently slowly, then $\operatorname{mod}(S_i \setminus Q_i) \to \infty$, and hence

$$\operatorname{mod}(\psi_i(S_i) \setminus \psi_i(Q_i)) \to \infty. \tag{6.2}$$

Let ν be the line field on Q_i that is obtained by pulling back μ under ψ_i. Using (6.2), the Koebe Distortion Theorem and that z is a point of continuity of the line field μ, we see that ν is an almost constant line field on Q_i. More precisely, there is a sequence $\eta_i \to 0$ such that, for each i, there are a subset $\hat{Q}_i$ of Q_i and a constant line field ν_0 so that $\operatorname{dens}(\hat{Q}_i, Q_i) \geq 1 - \eta_i$ and $|\nu(z) - \nu_0| \leq \eta_i$ for each $z \in \hat{Q}_i$. Moreover, if we decrease Δ_i, then the bound for η_i from the Koebe Theorem improves. This means that we may assume that Δ_i tends to infinity sufficiently slowly to ensure that

$$\Delta_i \cdot \eta_i \to 0.$$

Let us determine μ on $A_i = E(Q_i) = f^{n_i}(\psi_i(Q_i))$, using the fact that $\mu|_{A_i} = E_*(\nu|_{Q_i})$. Note that A_i is a round annulus centered around v with $\operatorname{mod}(A_i) \to \infty$; let r_i denote its outer radius. Also note that $\vartheta = E_*(\nu_0|Q_i)$ is the radial line field $z \mapsto \rho z/|z|$ where $\rho \in \mathbb{C}$ with $|\rho| = 1$ is constant; see Figure 3. Furthermore,

$$\operatorname{dens}(E(Q_i \setminus \hat{Q}_i), A_i) \leq \frac{1}{1 - e^{-\Delta_i}} \Delta_i \cdot \eta_i \to 0.$$

Indeed, considering the map $E\colon Q_i \to E(Q_i)$ on a horizontal segment L, and writing $T_i := (Q_i \setminus \hat{Q}_i) \cap L$, we have

$$\frac{\text{length}(E(T_i))}{\text{length}E(Q_i \cap L)} \leq \frac{\text{length}(T_i)}{1-e^{-\Delta_i}} = \text{dens}(T_i, Q_i \cap L) \cdot \frac{\Delta_i}{1-e^{-\Delta_i}}.$$

This completes the proof of Claim 1. △

Now let us turn to the proof of Claim 2. We begin by reformulating the sector condition in logarithmic coordinates.

Claim 3. There exists $\epsilon_0 \in (0,1)$ and $r_2 > r+1$ so that $W \supset H(\epsilon_0, r_2)$ where

$$H(\epsilon_0, r_2) := \{z \in \mathbb{C}; z = x+iy, x, y \in \mathbb{R} \text{ with } x > r_2 \text{ and } |y| < \epsilon_0\}.$$

Proof. Note that f satisfies the sector condition, and therefore V contains a sector of the form

$$S(\vartheta, M_2) := \{z \in \mathbb{C}; z = x+iy, x, y \in \mathbb{R} \text{ with } |y| < \vartheta|x|, x \geq M_2\}$$

where $\vartheta > 0$ and $M_2 > 0$ is some large number. There exist $\epsilon_0 \in (0,1)$ and $r_2 > 0$ so that $S(\vartheta, r_2) \supset \exp(H(\epsilon_0, M_2))$ concluding the proof of the claim. △

Proof of Claim 2. By [EL, Lemma 1] (which is an application of Koebe's theorem), the map F is expanding:

$$|F'(z)| \geq \frac{1}{4\pi}(\operatorname{Re} F(z) - r) \tag{6.3}$$

for all $z \in W$. (Recall that $r = \log M$.) In particular, if $r_1 > r_2 + \varepsilon_0$ is sufficiently large, then for every $x \geq r_2$ and all $j \geq 0$, there exists a branch of F^{-j} that takes $F^j(x)$ to x and is defined on the disk of radius $3\varepsilon_0$ around $F^j(x)$.

Now let $w \geq r_1$, $K > 0$ and $\delta > 0$. We set $w_j := F^j(w)$, $D := B_\delta(w)$ and $D_j := F^j(D)$. Let $m \geq 0$ be minimal such that D_m is not contained in the strip $H(\varepsilon_0/4, r_2)$. Then $F^m : D \to D_m$ is a conformal isomorphism.

We claim that there is a universal constant C such that

$$\frac{\max_{\zeta \in \partial D_m} |\zeta - w_m|}{\min_{\zeta \in \partial D_m} |\zeta - w_m|} \leq C. \tag{6.4}$$

This is trivial if $m = 0$. Otherwise, let φ be the branch of $F^{-(m-1)}$ that takes w_{m-1} to w and is defined on the disk of radius $3\varepsilon_0$. By definition of m, there is some point $\zeta \in \partial D_{m-1}$ with $|\zeta - w_{m-1}| \leq \varepsilon_0/4$. If $\omega \in \partial B_{\varepsilon_0/2}(w_{m-1})$, we see by the Koebe distortion Theorem 2.1 that

$$|\varphi(\omega) - w| \geq \frac{\frac{1}{6}}{(1+\frac{1}{6})^2} \cdot \frac{(1-\frac{1}{12})^2}{\frac{1}{12}} \cdot |\varphi(\zeta) - w| > |\varphi(\zeta) - w| = \delta.$$

Thus it follows that $D_{m-1} \subset B_{\varepsilon_0/2}(w_{m-1})$, and (6.4) follows from the Koebe Distortion Theorem (using the fact that F is univalent on the disk $B_{\varepsilon_0}(w_{m-1})$).

If $B_K(w_m) \subset D_m$, then we set $n := m$ and are done. Otherwise, define $R_1 := \max_{\zeta \in \partial D_m} |\zeta - w_m|$ and $R_2 := \min_{\zeta \in \partial D_m} |\zeta - w_m|$, so that $R_1/R_2 \leq C$ and $R_2 < K$. We set

$$\tilde{\delta} := \frac{\delta \cdot \varepsilon_0}{R_1},$$

$\tilde{D} := D_{\tilde{\delta}}(w)$ and $\tilde{D}_m := F^m(\tilde{D})$. Note that

$$\tilde{\delta} > \frac{\varepsilon_0}{C} \cdot \frac{\delta}{K}.$$

To prove Claim 2, we define $c := \epsilon_0/C$ and need to check that $F^{m+1}(\tilde{D}) \supset B_K(F^{m+1}(w))$. To see this, notice that $F^m(B_\delta(w)) \supset B_{R_1}(F^m(w))$. Hence by Schwarz and by the choice of $\tilde{\delta}$, $\tilde{D}_m = F^m(B_{\tilde{\delta}}(w)) \subset H(\varepsilon_0, r_2)$. It follows that F^{m+1} is defined and univalent on $\tilde{D}$. Furthermore, by Koebe's theorem, $\tilde{D}_m$ contains the disk $B_{C_1 \cdot \varepsilon_0}(w_m)$ for a universal constant C_1. It follows, again using Koebe's theorem and the estimate (6.3) that

$$F(\tilde{D}_m) \supset B_{K'}(w_{m+1}),$$

where

$$K' = C_1 \cdot \varepsilon_0 \cdot |F'(w_m)|/4 \geq \frac{C_1 \cdot \varepsilon_0}{16\pi} \cdot (\operatorname{Re} w_{m+1} - r).$$

Note that, as $\delta \to 0$, we have $m \to \infty$ and hence $\operatorname{Re} w_{m+1} \to \infty$. Thus we can choose δ_0 sufficiently small that $\delta < \delta_0$ implies $K' \geq K$, which completes the proof. △

■

7. Parameter spaces

Recall that, given $f \in \mathcal{S}_\mathbb{R}$, we denote by $M_f^\mathbb{R}$ the set of functions real-topologically equivalent to f (Definition 4.1). As we have already mentioned, this space can naturally be given the structure of a real-analytic manifold; this follows from work of Eremenko and Lyubich [EL] (who treated the complex-analytic case). More precisely:

7.1. Proposition (Manifold structure)**.**
Let $f \in \mathcal{S}_\mathbb{R}$ and set $q := \#S(f)$. Then the set $M_f^\mathbb{R}$ can be given the structure of a real-analytic manifold of dimension $q+2$ in such a way that:

- *A sequence $f_n \in M_f^\mathbb{R}$ converges to f in the manifold topology of $M_f^\mathbb{R}$ if and only if there are sequences of homeomorphisms $\psi_n, \varphi_n \in \mathrm{Homeo}_\mathbb{R}$ converging to the identity as $n \to \infty$ such that $f_n = \psi_n \circ f \circ \varphi_n^{-1}$.*
- *The inclusion from $M_f^\mathbb{R}$ (as a real-analytic manifold) to the space of entire functions is real-analytic.*

In the following, we will always assume $M_f^\mathbb{R}$ to be equipped with this topology and real-analytic structure. If we wish to make the distinction, we will refer to this as the "manifold topology", and the induced topology from the space of entire functions as the "locally uniform topology".

The fact that the dimension of $M_f^\mathbb{R}$ is $q+2$ (rather than q) reflects the fact that the group $\text{Möb}_\mathbb{R}$ of order-preserving real affine maps acts on $M_f^\mathbb{R}$ by conjugacy. We can quotient $M_f^\mathbb{R}$ by the action of this group:

7.2. Proposition (The quotient $\widetilde{M_f^\mathbb{R}}$)**.**
Let $\widetilde{M_f^\mathbb{R}}$ be the quotient of $M_f^\mathbb{R}$ by $\text{Möb}_\mathbb{R}$. Then $\widetilde{M_f^\mathbb{R}}$ is a real-analytic manifold of dimension $q = \#S(f)$, and the projection $\pi : M_f^\mathbb{R} \to \widetilde{M_f^\mathbb{R}}$ is real-analytic.

To prove our main results, we shall first establish density of hyperbolicity in $\widetilde{M_f^{\mathbb{R}}}$ (provided f satisfies the sector condition). The following fact then implies that the same is true under the (a priori) more general hypotheses given in the introduction.

7.3. Proposition (Continuous families and the manifold topology)**.**
Let $n, k \in \mathbb{N}$ and suppose that $(f_t)_{t\in[-1,1]^k}$ is a continuous family of functions $f_t \in \mathcal{S}_{\mathbb{R}}$ such that $\#S(f_t) = n$ for all t. Then there exist continuous families $\varphi_t, \psi_t \in \mathrm{Homeo}_{\mathbb{R}}$ such that

$$f_t = \psi_t \circ f_0 \circ \varphi_t^{-1}$$

for all $t \in [-1,1]^k$.

In other words, $f_t \in M_{f_0}^{\mathbb{R}}$ for all $t \in [-1,1]^k$ and f_t depends continuously on t in the topology of $M_{f_0}^{\mathbb{R}}$.

For completeness and future reference, we provide a proof of the preceding propositions in Appendix A. In fact, we will give a very explicit topological description of the spaces $M_f^{\mathbb{R}}$ and $\widetilde{M_f^{\mathbb{R}}}$.

7.4. Corollary (Connected conjugacy classes)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$. Then the set of functions $g \in \mathcal{S}_{\mathbb{R}}$ that are real-topologically conjugate to f is a connected subset of $M_f^{\mathbb{R}}$ with the manifold topology.

Proof. This follows from Theorem 1.8, by considering the Beltrami coefficient μ of the quasiconformal conjugacy h. Taking h_t to be the quasiconformal map associated to $t\mu$ (normalized appropriately), we obtain a family of maps $f_t = h_t^{-1} \circ f \circ h_t$ in $\mathcal{S}_{\mathbb{R}}$ that connects f and g. ∎

Remark. This implies Corollary 1.9.

Finally, we require the fact that, within any given parameter space $M_f^{\mathbb{R}}$, any parabolic point can be perturbed to an attracting one.

7.5. Proposition (Perturbations of parabolic points)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$, and suppose that f has a parabolic periodic point z_0. Then there exists a function $f \in M_f^{\mathbb{R}}$, arbitrarily close to f in the manifold topology, such that g has an attracting periodic point close to z_0.

Sketch of proof. We could prove this using Epstein's transversality results for finite-type maps. Since these are currently unpublished, we shall instead sketch how to prove the proposition along the lines of Shishikura's argument in [Shi], see also [EL, Theorem 5].

Let $w \in \mathbb{R}$ be a point which belongs to the parabolic basin of f. Let $\varepsilon > 0$ and let φ be a real-quasiconformal map such that:

(a) $|\varphi(z) - z| \leq \varepsilon$ for all z;
(b) the complex dilatation of φ is bounded by ε;
(c) φ fixes the orbit of z_0, as well as the points ∞ and w;
(d) φ is conformal on $\{z \in \mathbb{C} : |z - w| > \varepsilon\}$;
(e) $\varphi'(z_0) \in (0,1)$, and $\varphi' = 1$ on other points of the orbit of z_0.

To define this map, note that outside a small neighbourhood of w, we can let $\varphi(z) = z + \delta \cdot R(z)$, where R is a real rational function with a single pole at w, having zeros along the orbit of z_0 and at ∞, and such that $R'(z_0) < 0$ and $R'(z) = 0$ at other points of the orbit of z_0. Provided $\delta > 0$ is sufficiently small, φ is a conformal diffeomorphism from $\{z \in \mathbb{C} : |z - w| > \varepsilon\}$ onto its image, and one can extend φ to a neighbourhood of w as a quasiconformal homeomorphism of the Riemann sphere $\overline{\mathbb{C}}$ with small dilatation. (Compare [Shi, Lemma 2].)

Next consider the quasiregular map $F := \varphi \circ f$. Then z_0 is a periodic point of period n for F, and $(F^n)'(z_0) = (f^n)'(z_0) \cdot \varphi'(z_0)$, so z_0 is an attracting periodic point. Moreover, let U be a small attracting petal for z_0 as a periodic point of f, with $w \notin \overline{U}$. Provided that $\delta > 0$ is sufficiently small, and that R is chosen to have zeros of sufficiently high order along the forward iterates of z_0 (except z_0), the set U will be contained in the basin of attraction of z_0 for F.

If $\varepsilon > 0$ is sufficiently small, then $\overline{f^n(B_\varepsilon(w))} \subset U$ for some suitable n. Decreasing $\varepsilon > 0$ further, if necessary, the same will be true for F. Hence no orbit of F passes through $B_\varepsilon(w)$ more than once, and we can apply quasiconformal surgery to obtain the desired function g, compare [Shi, Lemma 1]. $\square$

Combinatorial and analytic data. We now introduce data that will allow us to encode when two real-topologically equivalent maps in $\mathcal{S}_\mathbb{R}$ are conformally conjugate (using Theorem 1.8). The notions of *kneading sequences*, which essentially determine combinatorial equivalence classes and of coordinates to ensure conformal conjugacy on attracting and parabolic basins are standard tools from the polynomial setting; we define and review them here briefly for completeness. To deal with escaping singular orbits, we will also require a new tool: *escaping coordinates*, which are provided by the results of [R].

7.6. Definition (Itineraries and kneading sequences)**.**
Let $f \in \mathcal{S}_\mathbb{R}$, and let $\mathcal{I}$ denote the set of connected components of $\mathbb{R} \setminus \mathrm{Crit}(f)$. The itinerary *of a point $x \in \mathbb{R}$ is the sequence $\underline{s} = s_0 s_1 s_2 \dots$, where $s_m = I_j$ if $I_j \in \mathcal{I}$ with $f^m(x) \in I_j$, or $s_m = f^m(x)$ if $f^m(x)$ is a critical point of f.*

Let $v_1 < v_2 < \dots < v_k$ be the singular values of f; the kneading sequence *of f is the collection $(\underline{s}^1, \underline{s}^2, \dots, \underline{s}^k)$ of the itineraries of the v_j, together with the information which v_j converge to an attracting cycle or to infinity.*

If two maps f and g are real-topologically equivalent, the map φ allows us to relate the itineraries of f and g. Hence it makes sense to speak of two such maps having 'the same kneading sequence'. More formally:

7.7. Definition (The notion of having the same kneading sequences)**.**
Let $f \in \mathcal{S}_\mathbb{R}$, let $v_1, \dots, v_k$ be the singular values of f and let $\underline{s}^1, \dots, \underline{s}^k$ be their itineraries.

Let $g = \psi \circ f \circ \varphi^{-1}$ be real-topologically equivalent to f. Then we say that f and g have the same kneading sequence *if*

$$g^m(\psi(v_j)) \in \varphi(s^j_m)$$

for all $m \geq 0$ and $1 \leq j \leq k$.

Remark. Note that the definition depends on the maps ψ and φ, not only on the function g. We suppress this in the notation, which should not cause any confusion.

As stated above, our goal is to use kneading sequences to identify maps that are conformally conjugate; however, to do so we need to augment these with some analytic data. Indeed, for example the conformal conjugacy class of a map with attracting periodic orbits is not determined by the kneading sequence, since there will be invariant line fields on the basins of attraction. Similar issues are associated with parabolic orbits and escaping singular orbits. This can be dealt with in a straightforward manner by introducing *attracting, parabolic and escaping coordinates.*

More precisely, let $f \in \mathcal{S}_{\mathbb{R}}$, and suppose that f has an attracting periodic point $p \in \mathbb{R}$. Let Ψ denote the linearizing coordinates for p (defined on the entire basin of the periodic attractor by the obvious functional relation), normalized such that $\Psi'(p) = 1$. Then the *attracting coordinates for f at p* consist of the multiplier μ of p together with the point

$$[\Psi(s_1) : \Psi(s_2) : \cdots : \Psi(s_k)] \in \mathbb{CP}^{k-1},$$

where $s_1 < s_2 < \cdots < s_k$ are the singular values of f that are attracted by the cycle of p. The *attracting coordinates for f* consists of the attracting coordinates at all attracting cycles of f, together with the information of which singular values are attracted to which attracting orbit.

If f belongs to a real-analytic family f_λ, say $f = f_{\lambda_0}$, then for every λ near λ_0 there will be an attracting periodic point $p(\lambda)$ of f_λ with $p(\lambda_0) = p$, depending real-analytically on λ. The linearizing coordinates for $p(\lambda)$ also depend analytically on λ, which implies that the corresponding attracting coordinates depend analytically on λ (provided we ignore any additional singular values of f_λ that may be attracted to $p(\lambda)$).

Similarly, one can define *parabolic coordinates* at parabolic points, which consist of the attracting Fatou coordinates of singular values, up to a translation. These will actually not be used in our proof of density of hyperbolicity, but we include them for completeness, to state the theorem below in full generality.

Finally, we also need to introduce analytic coordinates for singular values that are contained in the real part $I_{\mathbb{R}}(f)$ of the escaping set. Such coordinates are given by [R]:

7.8. Theorem (Escaping coordinates)**.**
Let M be a real-analytic manifold with base point $\lambda_0 \in M$. Also let $(f_\lambda)_{\lambda \in M}$ be a continuous family of functions in $\mathcal{S}_{\mathbb{R}}$, all of which are real-topologically equivalent, i.e. $\psi_\lambda \circ f_\lambda = f \circ \varphi_\lambda$ with ψ_λ and φ_λ depending continuously on λ.

Let $K \subset M$ and let R be sufficiently large (depending on K). Then for every $\lambda \in K$, there exists a quasisymmetric map $h_\lambda : \mathbb{R} \to \mathbb{R}$ such that

$$h_\lambda(f_{\lambda_0}(x)) = f_\lambda(h_\lambda(x))$$

whenever $x \in I_{\mathbb{R}}(f_{\lambda_0})$ has the property that $f_{\lambda_0}^k(x) \geq R$ for all $k \geq 0$.

Furthermore, $h_\lambda(x)$ depends real-analytically on λ for fixed x.

Using this map h_λ, we can now also define what it means that two functions f and $g \in M_f^{\mathbb{R}}$ have the *same escaping coordinates*: sufficiently large iterates of escaping singular values of f should be carried to the corresponding iterates for g using this conjugacy on the escaping set.

Using these concepts, we can state the following result.

7.9. Theorem (QC rigidity and conformal rigidity on the Fatou set).
Suppose that $f, g \in \mathcal{S}_{\mathbb{R}}$ are real-topologically equivalent.

Suppose also that f and g have the same kneading sequence and the same attracting, parabolic and escaping coordinates.

Then f and g are quasiconformally conjugate via a real-quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ which is conformal on the Fatou set of f.

Proof. The assumption implies that f and g are combinatorially conjugate and escaping conjugate, and that the combinatorial conjugacy can be chosen to be analytic in a neighbourhood of the part of the postsingular set that belongs to the Fatou set. Now it follows as in Theorem 5.1 that this conjugacy promotes to a quasiconformal conjugacy, and this conjugacy is conformal on the Fatou set. ■

8. Density of Hyperbolicity

The basic idea in our proof of density of hyperbolicity is to create more and more critical relations of a suitable type near a starting parameter, and restrict to a submanifold where this critical relation is persistent. To make this work, we need to know that we can carry out this process in such a way that the dimension of the manifold is not reduced by more than one would expect. We could do so by applying deep transversality results due to Adam Epstein (though it is not entirely clear how to apply these e.g. to work with escaping coordinates). Instead, we use the following, much softer statement.

8.1. Theorem (Finding submanifolds).
Let $U \subset \mathbb{R}^n$ be an open ball, and let

$$\rho : U \to \mathbb{R}$$

be real-analytic. Suppose that $x_1, x_2 \in U$ satisfy $\rho(x_1) \neq \rho(x_2)$, and let $\nu \in \mathbb{R}$ be a value between $\rho(x_1)$ and $\rho(x_2)$.

Then there exists $w \in U$ with $\rho(w) = \nu$ such that $\rho^{-1}(\nu)$ is a real-analytic $(n-1)$-dimensional manifold near w.

Proof. By continuity of ρ, the set $A := \rho^{-1}(\nu)$ separates U. This means that A has topological dimension at least $n - 1$ [HW].

Now the zero set of a real analytic function is a subanalytic, and indeed semianalytic, set. Subanalytic sets can be written as a locally finite union of real-analytic submanifolds, see [BM]. So A contains a real-analytic manifold of the same dimension as its topological dimension. Compare also Lojasiewicz's structure theorem for real-analytic sets [KP, Theorem 6.3.3]. ■

8.2. Proposition (Abundance of critical relations).
Let $f \in \mathcal{S}_{\mathbb{R}}$ and let M be a real-analytic submanifold of $M_f^{\mathbb{R}}$ of dimension n. Suppose that no two maps $f, g \in M$ have the same kneading sequence. Then, given any $f_0 \in M$, there exists some $f \in M$, arbitrarily close to f_0, such that f satisfies n non-persistent critical relations of the form $f(v) = c$, where v is a singular value and c is a real critical point.

Remark. Strictly speaking, our statement is ambiguous, since "having the same kneading sequence" depends on the choice of φ and ψ from the definition of real-topological equivalence. Since our conclusion is local, the assumption should also be understood locally: for $f_0 \in M$ we can find a neighbourhood $U \subset M$ on which the maps φ and ψ can be chosen to depend continuously, and no two maps in U should have the same kneading sequence with respect to this choice.

Proof. We will prove the theorem by induction on n. If $n = 0$, then there is nothing to prove. So suppose the theorem holds when the dimension of M is $n - 1$. Now assume M has dimension n. Let $f_0 \in M$; by a small perturbation, we may assume that f_0 does not satisfy any non-persistent critical relation of the form $f(v) = c$. Let U be a small neighbourhood of f_0 as in the above remark; in particular, the critical and asymptotic values of $f \in U$ depend continuously and real-analytically (with the respect to the manifold structure). We can choose U to be real-analytically diffeomorphic to an open ball in $\mathbb{R}^n$.

Since no two maps in U have the same kneading sequence, there must be maps $f_1, f_2 \in U$, as well as a critical point $c = c(f)$ and a critical value $v(f)$ such that $f_1^n(v(f_1)) - c(f_1)$ and $f_2^n(v(f_2)) - c(f_2)$ have opposite signs for some $n \in \mathbb{N}$.

Set

$$\rho(f) := f^n(v(f)) - c(f);$$

then ρ is real-analytic, and we can apply Theorem 8.1 to U, ρ, $x_1 := f_1$, $x_2 := f_2$ and $\nu = 0$. We obtain an $(n - 1)$-dimensional analytic submanifold N of M, contained in U, such that all maps $f \in N$ satisfy $\rho(f) = 0$; i.e., they satisfy a critical relation of the desired form which is non-persistent in M, but persistent in N.

Applying the induction hypothesis, we find a map $f \in N$ that satisfies $n - 1$ critical relations which are non-persistent in N, and hence n critical relations which are non-persistent in M, as desired. ■

Recall that $L_I(f)$ is the set of points $z \in J(f) \setminus (J_r(f) \cup I(f))$ whose orbits accumulate on escaping singular orbits under iteration. The following theorem shows density of hyperbolicity provided this set does not support invariant line fields.

8.3. Theorem (Density of hyperbolicity when L_I has no invariant line fields)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$. Let $U \subset \widetilde{M_f^{\mathbb{R}}}$ be open with the property that no $g \in U$ has an invariant line field on the set $L_I(g)$.

Then U contains a real-hyperbolic function.

Proof. Let $f_1 \in U$ be such that the number of singular values that belong to attracting basins is locally maximal near f_1. Then there is an open neighbourhood $U' \subset U$ of f_1 such that all $g \in U'$ have the same number, say k_1, of such singular values. By Proposition 7.5, this implies that no function in U' has any parabolic periodic points.

Now, similarly, pick $f_2 \in U'$ such that the number k_2 of singular values that tend to infinity under iteration is locally maximal, and let U'' be an open neighbourhood of f_2 such that all maps in U'' have k_2 such singular values.

Set $q := \#S(f)$; recall that $\widetilde{M_f^{\mathbb{R}}}$, and hence U'' has dimension q. We may assume that U'' is chosen sufficiently small that there is a real-analytic section $U'' \to M_f^{\mathbb{R}}$. (In fact, if f is not periodic, then there is even a global section $\widetilde{M_f^{\mathbb{R}}} \to M_f^{\mathbb{R}}$, see Appendix A.) So we can identify U'' with a q-dimensional submanifold of $M_f^{\mathbb{R}}$ in which no two maps are conformally conjugate.

Applying Theorem 8.1 repeatedly, we can find a manifold $M \subset U''$ of dimension $q' := q - k_1 - k_2$ on which the attracting and escaping coordinates are constant.

By Theorem 7.9, any two maps in M that have the same kneading sequence would be quasiconformally conjugate, and the dilatation would be supported on the Julia set. However, by the assumption, Lemma 6.1 and Theorems 6.2 and 6.3, this means that the maps would be conformally conjugate, and hence equal. Thus no two maps in M have the same kneading sequence. Note that within M the $k_1 + k_2$ singular values in attracting basins or tending to infinity do not satisfy any non-persistent relations. (Two singular values are said to have a relation if they have the same grand orbit.)

Now we apply Proposition 8.2, to obtain a function $g \in M$ that satisfies q' non-persistent critical relations of the form $g(v) = c$, where v is a singular value and c is a real critical point. This means that every singular value is eventually either mapped to a superattracting cycle or to one of the $k_1 + k_2$ singular values that belong to attracting basins or to $I_{\mathbb{R}}(g)$ (and which, by assumption, do not satisfy any singular relations). Thus all singular values of g belong to attracting basins or converge to infinity, as claimed. ■

8.4. Corollary (Density of hyperbolicity for bounded functions)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$ and suppose that one of the following holds:

(a) *$f|\mathbb{R}$ is bounded;*
(b) *f satisfies the sector condition; or*
(c) *f has bounded criticality.*

Then if (a) holds then hyperbolicity is dense in $\widetilde{M_f^{\mathbb{R}}}$. If (b) or (c) hold then real-hyperbolicity is dense in $\widetilde{M_f^{\mathbb{R}}}$.

Proof. This is an immediate consequence of the previous result. (Note that each of the conditions (a), (b) and (c) is invariant under real-quasiconformal equivalence, and hence no function $g \in M_f^{\mathbb{R}}$ supports an invariant line field on $L_I(g)$ by Theorems 6.2 and 6.3.) ■

This proves Theorem 1.1 (and the corresponding statement for maps satisfying the sector condition). We now deduce Theorem 1.6 (of which Theorem 1.2 is a special case).

Proof of Theorem 1.6. Real-hyperbolicity is an open property, so we need only prove that real-hyperbolicity is dense. Let $\lambda_0 \in N$ and set $f := f_{\lambda_0}$. We may assume (perturbing λ_0 if necessary) that the number $m := \#S(f)$ is locally maximal. For λ sufficiently close to λ_0, f_λ must have at least m singular values (see Lemma A.2). So we may assume, by shrinking N if necessary, that $\#S(f_\lambda) = m$ for all $\lambda \in N$.

By Proposition 7.3, all f_λ belong to $M_f^{\mathbb{R}}$, and f_λ is a continuous family with respect to the topology of $M_f^{\mathbb{R}}$. Via the natural projection $M_f^{\mathbb{R}} \to \widetilde{M_f^{\mathbb{R}}}$, we obtain a continuous map

$$\Phi : N \to \widetilde{M_f^{\mathbb{R}}}$$

such that f_λ and (any representative of) $\Phi(f_\lambda)$ are conformally conjugate. In particular, the map Φ is injective. Since $m \leq n$, we must in fact have $m = n$ and Φ is locally surjective (by the Invariance of Domain Theorem). The claim now follows from the preceding theorem. ■

Proof of Corollary 1.7. This follows from Corollary 8.4 and the fact that the number of singularities is preserved under topological equivalence. ■

9. Circle maps

The adaptation of our results to circle maps is straightforward, and essentially in complete analogy with the case of bounded functions $f \in \mathcal{S}_{\mathbb{R}}$. To formulate results for transcendental maps and Blaschke products at the same time, let us denote by X the union of $\mathcal{S}_{S^1}$ with the set of all Blaschke products of degree at least two which preserve $\{0, \infty\}$ and for which all critical values are contained in $S^1 \cup \{0, \infty\}$.

9.1. Lemma (Circle maps without critical points)**.**
If $f \in X$ has no critical points in S^1, then $f(z) = z^d$ with $d \neq 0$.

Proof. Let $Z = f^{-1}(S^1 \cup \{0, \infty\})$. Since the critical values of f are on the circle but S^1 contains no critical points, S^1 is one of the connected components of Z. We claim that it is the only nontrivial component (i.e. consisting of more than one point) of Z. Indeed, otherwise there is at least one multiply-connected component V of $\mathbb{C} \setminus Z$ that is not a punctured disk. But $f|_V$ is a covering whose image is either $\mathbb{D} \setminus \{0\}$ or $\mathbb{C} \setminus \overline{D}$, which is impossible. It follows that f has no singular values in $\mathbb{C}^*$, and hence $f(z) = z^d$ for some $d \in \mathbb{Z} \setminus \{0\}$. ■

9.2. Theorem (QC rigidity for circle maps)**.**
Suppose that two maps $f, g \in X$ are S^1-topologically equivalent and combinatorially conjugate. Then the two maps are S^1-quasiconformally conjugate via a conjugacy that agrees with the combinatorial conjugacy on the postsingular set.

(Here the notions of S^1-topological equivalence as well as combinatorial and S^1-qc conjugacy are defined in analogy to the real case. Note that by definition a combinatorial conjugacy sends parabolic to parabolic points.)

Proof. Note that from the previous lemma f and g have at least one critical point (unless f, g are of the form $z \mapsto z^d$). It follows from [CvS] that the two maps are quasisymmetrically conjugate on the circle. Applying a pullback argument yields a quasiconformal conjugacy on the entire complex plane. ■

The second ingredient is the absence of invariant line fields theorem:

9.3. Theorem (No invariant line fields)**.**
A map $f \in X$ does not support any invariant line fields on its Julia set.

Here, once again, the absence of invariant line fields on the set of points with bounded orbits follows from [CvS]. The absence of invariant line fields on the radial Julia set does not follow directly from [RvS1], since the function f is not necessarily meromorphic in the plane but can be proved in the same manner. Alternatively, the result of [RvS1] is generalized in [MaR] to arbitrary *Ahlfors islands maps*, and this result can be applied directly to f.

Finally, for $f \in \mathcal{S}_{S^1}$, absence of invariant line fields on the "escaping set"

$$I(f) := \{z \in \mathbb{C} : \omega(z) \subset \{0, \infty\}\}$$

follows from the following theorem, which is proved completely analogously to the corresponding result from [R].

9.4. Theorem (No invariant line fields on the escaping set)**.**
Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a transcendental self-map of the punctured plane, with essential singularities at 0 and ∞. Suppose that the set $S(f) \setminus \{0, \infty\}$ is compactly contained in $\mathbb{C}^*$.

Then the set $I(f)$ does not support invariant line fields.

Again, analogously to the case of $\mathcal{S}_{\mathbb{R}}$, the set $M_f^{S^1}$ of functions S^1-topologically equivalent to f has the structure of a real-analytic manifold of dimension $q + 1$, where $q = \#S(f)$. As we saw in Remark 1 below Lemma 2.6, its quotient $\widetilde{M_f^{S^1}}$ by conjugation by rotations is not a manifold. However, it is a q-dimensional orbifold. We then obtain by the same proof as for functions in $\mathcal{S}_{\mathbb{R}}$:

9.5. Theorem (Density of hyperbolicity for circle maps)**.**
Let $f \in X$. Then hyperbolicity is dense in $\widetilde{M_f^{S^1}}$.

The theorems on circle maps stated in the introduction follow from the preceding result in the same manner as for real entire functions:

Proof of Theorem 1.10. This follows from the preceding theorem in the same manner as for real entire function in $\mathcal{S}_{\mathbb{R}}$. ■

Proof of Theorem 1.11. This is a special case of Theorems 9.2 and 9.3. ■

Proof of Corollary 1.12. The first statement of this corollary follows Theorem 9.4. Part (a) follows from Theorems 9.2 and Lemma 2.6 using the same argument as in the proof of Corollary 7.4. Part (b) follows from Theorems 9.3 and 9.4. ■

Appendix A. More on parameter spaces

A.1. Proposition (Real-analytic structure of parameter spaces)**.**
Let $f \in \mathcal{S}_{\mathbb{R}}$ and set $q := \#S(f)$. If f is not periodic (i.e., there is no $\kappa \in \mathbb{R} \setminus \{0\}$ with $f(x + \kappa) = f(x)$ for all x), then

$$M_f^{\mathbb{R}} \simeq \mathbb{R}^{q+2} \quad \text{and} \quad \widetilde{M_f^{\mathbb{R}}} \simeq \mathbb{R}^q$$

(where $\simeq$ denotes real-analytic isomorphism). Otherwise,

$$M_f^{\mathbb{R}} \simeq \mathbb{R}^q \times S^1 \times S^1 \quad \text{and} \quad \widetilde{M_f^{\mathbb{R}}} \simeq \mathbb{R}^{q-1} \times S^1.$$

More precisely, let us set

$$\Lambda := \{(a_1, \dots, a_q) \in \mathbb{R}^q : a_1 < a_2 < \dots < a_q\}.$$

Then there exists a real-analytic covering map

$$\Theta : \Lambda \times (0, \infty) \times \mathbb{R} \to M_f^{\mathbb{R}}$$

with the following properties.

(a) *If* $\lambda = (a_1, \ldots, a_q) \in \Lambda$, $a > 0$ *and* $b \in \mathbb{R}$, *then the singular values of* $g := \Theta(\lambda, a, b)$ *are exactly* $a_1, \ldots, a_q$. *Furthermore, if* f *is periodic, then* g *is also periodic of minimal period* $a \cdot \kappa$, *where* κ *is the minimal period of* f.
(b) *Let* $\lambda_0 = (s_1, \ldots, s_q)$ *be the singular values of* f. *Then* $f = \Theta(\lambda_0, 1, 0)$.
(c) *If* f *is not periodic,* Θ *is a diffeomorphism. Otherwise,* $\Theta(\lambda, a, b) = \Theta(\lambda', a', b')$ *if and only if* $\lambda = \lambda'$, $a = a'$ *and* b *and* b' *differ by a multiple of* $a \cdot \kappa$.
(d) *Fix* $a > 0$ *and* $b \in \mathbb{R}$. *If* f *is not periodic, then* $\Theta(\lambda, a, b)$ *is not conformally conjugate (via a map from* $\text{Möb}_{\mathbb{R}}$*) to* $\Theta(\lambda', a, b)$ *for* $\lambda \neq \lambda'$. *Otherwise, these maps are conjugate if and only if there is* $m \in \mathbb{Z}$ *such that* λ' *is obtained from* λ *by adding* $m \cdot a \cdot \kappa$ *to all entries.*

Remark. A somewhat related theorem appears in [dMvS, Theorem 4.1] where it is shown that one can parametrise the space of real polynomials (as in Theorem 3.8 anchored at 0 and 1) with d distinct critical points $c_1, \ldots, c_d$ – all of which are assumed to be real – by its critical values $v_1, \ldots, v_d$. The difference is that in that theorem we allow critical values to coincide.

Proof. The idea is to start with a family of quasiconformal functions $\psi_\lambda \in \text{Homeo}_{\mathbb{R}}$, $\lambda \in \Lambda$, where ψ_λ takes λ_0 to λ, and then solve the Beltrami equation to obtain φ_λ such that $\psi_\lambda \circ f \circ \varphi_\lambda^{-1}$ is an entire function. There is a choice of normalization of φ_λ, which gives rise to the additional two real parameters come from.

If there are at least two real preimages of singular values, then it is easy to obtain a natural normalization of φ_λ. In order to obtain a construction that works in all cases, we will proceed in a slightly more ad-hoc manner.

If f is not periodic, let us set $\kappa := 1$, otherwise κ is the minimal period of f as defined above.

We define a real-analytic family $\psi_\lambda : \mathbb{C} \to \mathbb{C}$ with $\psi_\lambda(s_j) = a_j$, where $\lambda = (a_1, \ldots, a_q) \in \Lambda$. If $a_1 = s_1$ and $a_q = s_q$, let $h : \mathbb{R} \to \mathbb{R}$ be the unique map with $h(s_j) = a_j$ that is linear on every component of $\mathbb{R} \setminus S(f)$ and asymptotic to the identity at ∞. We define $\psi_\lambda(x + iy) := h(x) + iy$. In particular, $\psi_{\lambda_0} = \text{id}$.

Otherwise, set

$$A(z) := (z - s_1) \cdot \frac{a_q - a_1}{s_q - s_1} + a_1$$

and $\tilde{\lambda} := (A^{-1}(a_1), \ldots, A^{-1}(a_q))$. We define $\psi_\lambda(z) := A(\psi_{\tilde{\lambda}}(z))$.

By construction, the family ψ_λ has the following property:

Let $\lambda_1 = (a_1, \ldots, a_q) \in \Lambda$ and $A(z) = az + b$ be a real-affine map. If we set $\lambda_2 := (A(a_1), \ldots, A(a_q))$, then $\psi_{\lambda_2} \circ \psi_{\lambda_1}^{-1} = A$.

Let μ_λ be the complex dilatation of ψ_λ. We can pull back under f to obtain a complex structure $\nu_\lambda := f^*(\mu_\lambda)$. By the Measurable Riemann Mapping theorem, see for example [A2], we can find a quasiconformal homeomorphism $\varphi_{\lambda,a,b} : \mathbb{C} \to \mathbb{C}$ whose complex dilatation is given by ν_λ. This map is uniquely determined if we require that $\varphi_{\lambda,a,b}(0) = a \cdot b$ and $\varphi_{\lambda,a,b}(\kappa) = a \cdot (b + \kappa)$.

The functions $\varphi_{\lambda,a,b}$ depend real-analytically on ν_λ, and hence on λ, as well as on a and b. By a well-known argument, see for example [BC, Page 21], the family

$$\Phi(\lambda, a, b) := f_{\lambda,a,b} := \psi_\lambda^{-1} \circ f \circ \varphi_{\lambda,a,b}$$

also depends analytically on (λ, a, b). Clearly we have $f_{\lambda_0,1,0} = f$ and $S(f_{(a_1,\dots,a_q),a,b}) = \{a_1, \dots, a_q\}$.

Since f is real, the Beltrami differential ν_λ is symmetric with respect to the real axis (i.e. $\nu_\lambda(\bar{z}) = \overline{\nu_\lambda(z)}$), and hence the normalization ensures that $\varphi_{\lambda,a,b}$ restricts to an orientation-preserving homeomorphism of the real line. Thus $f_{\lambda,a,b} \in M_f^{\mathbb{R}}$ for all λ.

Similarly, if f is periodic, then ν_λ is periodic with period κ, and the normalization ensures that $\varphi_{\lambda,a,b}(z + \kappa) = \varphi_{\lambda,a,b}(z) + a\kappa$ for all z. Thus each $f_{\lambda,a,b}$ is periodic with period $b\kappa$. We can apply the same argument to see that $b\kappa$ is the minimal period of $f_{\lambda,a,b}$. Indeed, we write $f = \psi_\lambda \circ f_{\lambda,a,b} \circ \varphi_{\lambda,a,b}^{-1}$. If $b\kappa' \leq b\kappa$ is a period of $f_{\lambda,a,b}$, then we see as above that $\varphi_{\lambda,a,b}^{-1}(z + b\kappa') = \varphi_{\lambda,a,b}^{-1}(z) + c$ for some $c > 0$. Clearly $c \leq \kappa$, and by construction c is a period of f. Thus $c = \kappa = \kappa'$, as claimed.

The remaining claims follow from the construction. Indeed, suppose that $f_{\lambda,a,b} = f_{\lambda',a',b'}$. Then $\lambda = \lambda'$ (because these are the singular values) and $a = a'$ (because this is the period). By construction, we have $f_{\lambda,a,b'}(z) = f_{\lambda,a,b}(z + ab - ab')$. Hence $a(b - b')$ is a period of $f_{\lambda,a,b}$, and hence $b - b'$ is a multiple of κ (if f is periodic) or $b = b'$ (otherwise).

Now fix a and b and suppose now that $f_\lambda := f_{\lambda,a,b}$ and $f_{\lambda'} := f_{\lambda',a,b}$ are conformally conjugate by some real-affine map $A(z) = \alpha z + \beta$, $\alpha > 0$, $\beta \in \mathbb{R}$. Then it follows from the property stated above that $\psi_{\lambda'} \circ \psi_\lambda^{-1} = A$, and hence $A \circ f_\lambda \circ A^{-1} = f_{\lambda'} = A \circ f_{\lambda'}$. In particular, we must have $\alpha = 1$ and β is a period of f_λ; i.e., f is periodic and β is an integer multiple of $a \cdot \kappa$. ■

A.2. Lemma (Dependence of singular values)**.**
Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and let f_n be entire functions with $f_n \to f$ locally uniformly. If $a \in S(f)$, then for sufficiently large n, there is $a_n \in S(f_n)$ such that $a_n \to a$.

Proof. See e.g. [KK]. ■

A.3. Proposition (All continuous families arise from QC equivalence)**.**
Let $n, k \in \mathbb{N}$ and suppose that $(f_t)_{t \in [-1,1]^k}$ is a continuous family of functions $f_t \in \mathcal{S}_\mathbb{R}$ such that $\#S(f_t) = n$ for all t. Then there exist continuous families $\varphi_t, \psi_t \in \mathrm{Homeo}_\mathbb{R}$ such that

$$f_t = \psi_t \circ f_0 \circ \varphi_t^{-1}$$

for all $t \in [-1, 1]$.

Sketch of proof. We first note that the assumption implies that the singular values of f_t move continuously by Lemma A.2. That is, there are continuous functions $s_1, \dots, s_n : [-1, 1]^k \to \mathbb{R}$ with $s_1(t) < s_2(t) < \cdots < s_n(t)$ for all t and $S(f_t) = \{s_1(t), \dots, s_n(t)\}$. We set $s_j := s_j(0)$ (where 0 denotes the origin in $[-1, 1]^k$).

Choose a continuous family ψ_t of real-quasiconformal homeomorphisms such that $\psi_t(s_j) = s_j(t)$ for all $t \in [-1,1]$ and $j \in \{1,\dots,n\}$, such that $\psi_0 = \mathrm{id}$, and such that $\psi_t(z) = z$ whenever $|\operatorname{Im} z| \geq 1$. (For example, we can define ψ_t in a piecewise linear manner.)

By solving the Beltrami equation (similarly as above), we can also find a continuous family φ_t of real-quasiconformal homeomorphisms such that

$$g_t := \psi_t^{-1} \circ f_t \circ \varphi_t \tag{A.1}$$

is an entire function for every t. The function φ_t is uniquely determined up to precomposition by an element of $\text{Möb}_{\mathbb{R}}$.

The idea is to normalize φ_t in such a way as to ensure that g_t agrees with f_0 at a chosen base point, and conclude that $g_t = f_0$ for all t. It is not a priori clear that such a normalization can always be carried out globally (that this is true shall follow *a posteriori* from the proof), but locally this follows from the argument principle. Hence the following claim (which requires a specific normalisation for the quasiconformal maps φ_t) provides a local version of our proposition:

Claim. Let $D \subset [-1,1]^k$ be a connected subset with $0 \in D$, and suppose that there is a continuous function $t \mapsto \zeta_t \in \mathbb{C}$, defined on D, such that $f_t(\zeta_t) = i$ for all $t \in D$ (where i, as usual, denotes the imaginary unit). Define $(g_t)_{t\in D}$ by (A.1), where φ_t is normalized such that $\varphi_t(\zeta_0) = \zeta_t$. Then $g_t = f_0$ for all $t \in D$.

Proof. We use the concept of *line complexes* from classical function theory. Fix $n+1$ pairwise disjoint arcs $\gamma_0, \dots, \gamma_n$ connecting i and $-i$ and each intersecting the real axis in precisely one point, in such a way that different arcs intersect at $\pm i$, and such that the arcs intersect the real axis in different intervals of $\mathbb{R} \setminus \{s_1, \dots, s_n\}$. We can assume that the arcs are ordered such that their intersection points with $\mathbb{R}$ are listed in increasing order; then $\gamma_j \cup \gamma_{j-1}$ is a Jordan curve separating s_j from ∞ and all other $s_{j'}$. The *line complex* $LC(g_t)$ is the preimage of $\bigcup \gamma_j$ under g_t.

More precisely, we can think of $LC(g_t)$ as an abstract graph with a base point and colored edges. The vertices are the elements of the set $g_t^{-1}(\{i,-i\})$, and the base point is the vertex represented by ζ_0. Two vertices z_1 and z_2 are connected by an edge of color $j \in \{0,\dots,n\}$ if and only if there is a component of $g_t^{-1}(\gamma_j)$ that connects z_1 and z_2. The following two facts are classical:

- The line complex $LC(g_t)$ depends continuously on t as a graph. (By this we mean that, for any fixed N, the part of $LC(g_t)$ within distance at most N of ζ_0 is locally constant.) Hence, since $[-1,1]$ is connected, it follows that all the abstract graphs $LC(g_t)$ are isomorphic.
- With the above normalization, the function g_t is uniquely determined by its line complex $LC(g_t)$.

The first of these is elementary: It follows from the fact that the analytic continuation of f_t^{-1} along a fixed composition of the curves γ_j will depend continuously on t. To reconstruct the function g_t from its line complex, we need only build the Riemann surface of g_t^{-1} by pasting together copies of the upper and lower half plane as specified by the line complex. The resulting entire function is determined uniquely up to precomposition by a map in $\text{Möb}_{\mathbb{R}}$; in other words, the function is fixed once we require that the base point of its line-complex be placed at ζ_0. For details, compare [GO]. △

We observe that, in this article, we only ever use a local version of Proposition A.3 that asserts the existence of the maps ψ_t and φ_t in a neighbourhood of the origin; this version follows immediately from the Claim.

To also deduce the global statement, note that we can apply our claim near any given base point $t_0 \in [-1,1]^k$. This implies that the full set of preimages $f_t^{-1}(i)$ moves continuously near every point t_0. Since $[-1,1]^k$ is simply-connected, it follows that this set moves continuously thorughout the entire family. In particular, we can take $D = [-1,1]^k$ in the Claim, and are done. □

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