On the Efficiency of All-Pay Mechanisms

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Abstract. We study the inefficiency of mixed equilibria, expressed as the price of anarchy, of all-pay auctions in three different environments: combinatorial, multi-unit and single-item auctions. First, we consider item-bidding combinatorial auctions where m all-pay auctions run in parallel, one for each good. For fractionally subadditive valuations, we strengthen the upper bound from 2 [22] to 1.82 by proving some structural properties that characterize the mixed Nash equilibria of the game. Next, we design an all-pay mechanism with a randomized allocation rule for the multi-unit auction. We show that, for bidders with submodular valuations, the mechanism admits a unique, 75% efficient, pure Nash equilibrium. The efficiency of this mechanism outperforms all the known bounds on the price of anarchy of mechanisms used for multi-unit auctions. Finally, we analyze single-item all-pay auctions motivated by their connection to contests and show tight bounds on the price of anarchy of social welfare, revenue and maximum bid.

1 Introduction

It is a common economic phenomenon in competitions that agents make irreversible investments without knowing the outcome. *All-pay* auctions are widely used in economics to capture such situations, where all players, even the losers, pay their bids. For example, a lobbyist can make a monetary contribution in order to influence decisions made by the government. Usually the group invested the most increases their winning chances, but all groups have to pay regardless of the outcome. In addition, all-pay auctions have been shown useful to model rent seeking, political campaigns and R&D races. There is a well-known connection between all-pay auctions and *contests* [21]. In particular, the all-pay auction can be viewed as a single-prize contest, where the payments correspond to the effort that players make in order to win the competition.

In this paper, we study the efficiency of mixed Nash equilibria in all-pay auctions with complete information, from a worst-case analysis perspective, using the *price of anarchy* [16] as a measure. As social objective, we consider the *social welfare*, i.e. the sum of the bidders' valuations. We study the equilibria induced from all-pay mechanisms in three fundamental resource allocation scenarios; combinatorial auctions, multi-unit auctions and single-item auctions.

In a combinatorial auction a set of items are allocated to a group of selfish individuals. Each player has different preferences for different subsets of the items and this is expressed via a *valuation set* function. A multi-unit auction, can be considered as an important special case, where there are multiple copies of a single good. Hence the valuations of the players are not set functions, but depend only on the number of copies received. Multi-unit auctions have been extensively studied since the seminal work by Vickrey [23]. As already mentioned, all-pay auctions have received a lot of attention for the case of a single item, as they model all-pay contests and procurements via contests.

1.1 Contribution

Combinatorial Auctions. Our first result is on the price of anarchy of simultaneous all-pay auctions with item-bidding that was previously studied by Syrgkanis and Tardos [22]. For fractionally subadditive valuations, it was previously shown that the price of anarchy was at most 2 [22] and at least $e/(e-1) \approx 1.58$ [8]. We narrow further this gap, by improving the upper bound to 1.82. In order to obtain the bound, we come up with several structural theorems that characterize mixed Nash equilibria in simultaneous all-pay auctions.

Multi-unit Auctions. Our next result shows a novel use of all-pay mechanisms to the multi-unit setting. We propose an all-pay mechanism with a randomized allocation rule inspired by Kelly's seminal proportional-share allocation mechanism [15]. We show that this mechanism admits a *unique*, 75% efficient *pure* Nash equilibrium and no other mixed Nash equilibria exist, when bidders' valuations are submodular. As a consequence, the price of anarchy of our mechanism outperforms all current price of anarchy bounds of prevalent multi-unit auctions including uniform price auction [18] and discriminatory auction [14], where the bound is $e/(e-1) \approx 1.58$.

Single-item Auctions. Finally, we study the efficiency of a single-prize contest that can be modeled as a single-item all-pay auction. We show a tight bound on the price of anarchy for mixed equilibria which is approximately 1.185. By following previous study on the procurement via contest, we further study two other standard objectives, *revenue* and *maximum bid*. We evaluate the performance of all-pay auctions in the prior-free setting, i.e. no distribution over bidders' valuation is assumed. We show that both the revenue and the maximum bid of any mixed Nash equilibrium are at least as high as $v_2/2$, where v_2 is the second highest valuation. In contrast, the revenue and the maximum bid in some mixed Nash equilibrium may be less than $v_2/2$ when using reward structure other than allocating the entire reward to the highest bidder. This result coincides with the optimal crowdsourcing contest developed in [6] for the setting with prior distributions. We also show that in conventional procurements (modeled by first-price auctions), v_2 is exactly the revenue and maximum bid in the worst equilibrium. So procurement via all-pay contests is a 2-approximation to the conventional procurement in the context of worse-case equilibria.

1.2 Related work

The inefficiency of Nash equilibria in auctions has been a well-known fact (see e.g. [17]). Existence of efficient equilibria of simultaneous sealed bid auctions in full information settings was first studied by Bikhchandani [3]. Christodoulou, Kovács and Schapira [7] initiated the study of the (Bayesian) price of anarchy of simultaneous auctions with item-bidding. Several variants have been studied since then [2, 12, 11], as well as multi-unit auctions [14, 18].

Syrgkanis and Tardos [22] proposed a general smoothness framework for several types of mechanisms and applied it to settings with fractionally subadditive bidders obtaining several upper bounds (e.g., first price auction, all-pay auction, and multi-unit auction). Christodoulou et al. [8] constructed tight lower bounds for first-price auctions and showed a tight price of anarchy bound of 2 for all-pay auctions with subadditive valuations. Roughgarden [20] presented an elegant methodology to provide price of anarchy lower bounds via a reduction from the hardness of the underlying optimization problems.

All-pay auctions and contests have been studied extensively in economic theory. Baye, Kovenock and de Vries [1], fully characterized the Nash equilibria in single-item all-pay auction with complete information. The connection between all-pay auctions and crowdsourcing contests was proposed in [9]. Chawla et al. [6] studied the design of optimal crowdsourcing contest to optimize the maximum bid in all-pay auctions when agents' value are drawn from a specific distribution independently.

2 Preliminaries

In a *combinatorial auction*, n *players* compete on m *items* with unit supply. Every player (or *bidder*) $i \in [n]$ has a valuation function $v_i : \{0,1\}^m \rightarrow \mathbb{R}^+$ which is monotone and normalized, that is, $\forall S \subseteq T \subseteq [m]$, $v_i(S) \leq v_i(T)$, and $v_i(\emptyset) = 0$. The outcome of the auction is represented by a tuple of $(\mathbf{X}, \mathbf{p})$ where $\mathbf{X} = (X_1, \ldots, X_n)$ specifies the allocation of items (X_i is the set of items allocated to player i) and $\mathbf{p} = (p_1, \ldots, p_n)$ specifies the buyers payments (p_i is the payment of player i for the allocation $\mathbf{X}$). In the *simultaneous item-bidding* auction, every player $i \in [n]$ submits a non-negative bid b_{ij} for each item $j \in [m]$. The items are then allocated by independent auctions, i.e. the allocation and payment rule for item j only depend on the players' bids on item j. In a simultaneous *all-pay* auction the allocation and payment for each player is determined as follows: each item $j \in [m]$ is allocated to the bidder i^* with the highest bid for that item, i.e. $i^* = \arg\max_i b_{ij}$, and each bidder i is charged an amount equal to $p_i = \sum_{j \in [m]} b_{ij}$.

Definition 1 (Valuations). *Let $v : 2^{[m]} \rightarrow \mathbb{R}$ be a valuation function. Then v is called a)* additive*, if $v(S) = \sum_{j \in S} v(j)$; b)* submodular[1]*, if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$; c)* fractionally subadditive *or* XOS*, if v is determined by a finite set of additive valuations ξ_k such that $v(S) = \max_k \xi_k(S)$.*

The classes of the above valuations are in increasing order of inclusion.

[1] Equivalently, submodular valuations are the valuations with *decreasing marginal values*, i.e. $v(\{j\} \cup T) - v(T) \leq v(\{j\} \cup S) - v(S)$ holds for any item j and $S \subseteq T$.

Multi-unit Auction. In a multi-unit auction, m copies of an item are sold to n bidders. Here, bidder i 's valuation is a function that depends on the number of copies he gets. That is $v_i : \{0, 1, \dots, m\} \to \mathbb{R}^+$ and it is non-decreasing and normalized, with $v_i(0) = 0$. We say a valuation v_i is *submodular*, if it has non-increasing marginal values, i.e. $v_i(s+1) - v_i(s) \geq v_i(t+1) - v_i(t)$ for all $s \leq t$.

Nash equilibrium and price of anarchy. We use b_i to denote a pure strategy of player i and it might be a single value or a vector, depending on the auction. So, for the case of m simultaneous auctions, $b_i = (b_{i1}, \dots, b_{im})$. We denote by $\mathbf{b}_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ the strategies of all players except for i. Any *mixed strategy* B_i of player i is a probability distribution over b_i.

For any profile of strategies, $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{X}(\mathbf{b})$ denotes the allocation under the strategy profile $\mathbf{b}$. The valuation of player i for the allocation $\mathbf{X}(\mathbf{b})$ is denoted by $v_i(\mathbf{X}(\mathbf{b})) = v_i(\mathbf{b})$. The *utility* u_i of player i is defined as the difference between her valuation and payment: $u_i(\mathbf{X}(\mathbf{b})) = u_i(\mathbf{b}) = v_i(\mathbf{b}) - p_i(\mathbf{b})$.

Definition 2 (Nash equilibria). *A bidding profile* $\mathbf{b}$ *forms a pure Nash equilibrium if for every player* i *and all bids* b'_i, $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$. *Similarly, a mixed bidding profile* $\mathbf{B} = \times_i B_i$ *is a mixed Nash equilibrium if for all bids* b'_i *and every player* i, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$. *Clearly, any pure Nash equilibrium is also a mixed Nash equilibrium.*

Our global objective is to maximize the sum of the valuations of the players for their received allocations, i.e., to maximize the *social welfare* $SW(\mathbf{X}) = \sum_{i\in[n]} v_i(X_i)$. So $\mathbf{O}(\mathbf{v}) = \mathbf{O} = (O_1, \dots, O_n)$ is an *optimal allocation* if $SW(\mathbf{O}) = \max_{\mathbf{X}} SW(\mathbf{X})$. In Sect. 5, we also study two other objectives: the *revenue*, which equals the sum of the payments, $\sum_i p_i$, and the *maximum payment*, $\max_i b_i$. We also refer to the maximum payment as the *maximum bid*.

Definition 3 (Price of anarchy). *Let* $\mathcal{I}([n], [m], \mathbf{v})$ *be the set of all instances, i.e.* $\mathcal{I}([n], [m], \mathbf{v})$ *includes the instances for every set of bidders and items and any possible valuation functions. The mixed price of anarchy, PoA, of a mechanism is defined as*

$$PoA = \max_{I\in\mathcal{I}} \max_{\mathbf{B}\in\mathcal{E}(I)} \frac{SW(\mathbf{O})}{\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{X}(\mathbf{b}))]} ,$$

where $\mathcal{E}(I)$ *is the class of mixed Nash equilibria for the instance* $I \in \mathcal{I}$. *The pure PoA is defined as above but restricted in the class of pure Nash equilibria.*

Let $\mathbf{B} = (B_1, \dots, B_n)$ be a profile of mixed strategies. Given the profile $\mathbf{B}$, we fix the notation for the following *cumulative distribution functions (CDF):* G_{ij} is the CDF of the bid of player i for item j; F_j is the CDF of the highest bid for item j and F_{ij} is the CDF of the highest bid for item j if we exclude the bid of player i. Observe that $F_j = \Pi_k G_{kj}$ and $F_{ij} = \Pi_{k\neq i} G_{kj}$. We also use $\varphi_{ij}(x)$ to denote the probability that player i gets item j by bidding x. Then, $\varphi_{ij}(x) \leq F_{ij}(x)$. When we refer to a single item, we may drop the index j. Whenever it is clear from the context, we will use shorter notation for expectations, e.g. we use $\mathbb{E}[u_i(\mathbf{b})]$ instead of $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$, or even $SW(\mathbf{B})$ to denote $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[SW(\mathbf{X}(\mathbf{b}))]$.

3 Combinatorial Auctions

In this section we prove an upper bound of 1.82 for the mixed price of anarchy of simultaneous all-pay auctions when bidders' valuations are fractionally subadditive (XOS). This result improves over the previously known bound of 2 due to [22]. We first state our main theorem and present the key ingredients. Then we prove these ingredients in the following subsections.

Theorem 4. *The mixed price of anarchy for simultaneous all-pay auctions with fractionally subadditive (XOS) bidders is at most* 1.82.

Proof. Given a valuation profile $\mathbf{v} = (v_1, \dots, v_n)$, let $\mathbf{O} = (O_1, \dots, O_n)$ be a fixed optimal solution, that maximizes the social welfare. We can safely assume that $\mathbf{O}$ is a partition of the items. Since v_i is an XOS valuation, let $\xi_i^{O_i}$ be a maximizing additive function with respect to O_i. For every item j we denote by o_j item j's contribution to the optimal social welfare, that is, $o_j = \xi_i^{O_i}(j)$, where i is such that $j \in O_i$. The optimal social welfare is thus $SW(\mathbf{O}) = \sum_j o_j$. In order to bound the price of anarchy, we consider only items with $o_j > 0$, as it is without loss of generality to omit items with $o_j = 0$.

For a fixed mixed Nash equilibrium $\mathbf{B}$, recall that by F_j and F_{ij} we denote the CDFs of the maximum bid on item j among all bidders, with and without the bid of bidder i, respectively. For any item $j \in O_i$, let $A_j = \max_{x \geq 0} \{F_{ij}(x) o_j - x\}$.

As a key part of the proof we use the following two inequalities that bound from below the social welfare in any mixed Nash equilibrium $\mathbf{B}$.

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \left(A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx \right) \ , \tag{1}$$

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx \ . \tag{2}$$

Inequality (1) suffices to provide a weaker upper bound of 2 (see [8]). The proof of (2) is much more involved, and requires a deeper understanding of the equilibria properties of the induced game. We postpone their proofs in Sect. 3.1 (Lemma 5) and Sect. 3.2 (Lemma 6), respectively. By combining (1) and (2),

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \cdot \sum_j \left(A_j + \int_0^{o_j - A_j} \left(1 - F_j(x) + \lambda \cdot \sqrt{F_j(x)} \right) dx \right) \ , \tag{3}$$

for some $\lambda \geq 0$. It suffices to bound from below the right-hand side of (3) with respect to the optimal social welfare. For any cumulative distribution function F, and any positive real number v, let

$$R(F, v) \stackrel{\text{def}}{=} A + \int_0^{v-A} (1 - F(x)) dx + \lambda \cdot \int_0^{v-A} \sqrt{F(x)} dx \ ,$$

where $A = \max_{x \geq 0}\{F(x) \cdot v - x\}$. Inequality (3) can then be rewritten as $SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j)$. Finally, we show a lower bound of $R(F, v)$ that holds for any CDF F and any positive real v.

$$R(F,v) \geq \frac{3 + 4\lambda - \lambda^4}{6} \cdot v \ . \tag{4}$$

The proof of (4) is given in Sect. 3.3 (Lemma 9). Finally, we obtain that for any $\lambda > 0$,

$$SW(\mathbf{B}) \geq \frac{1}{1+\lambda} \sum_j R(F_j, o_j) \geq \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot \sum_j o_j = \frac{3 + 4\lambda - \lambda^4}{6\lambda + 6} \cdot SW(\mathbf{O}) \ .$$

By taking $\lambda = 0.56$, we conclude that the price of anarchy is at most 1.82. $\square$

3.1 Proof of Inequality (1)

This section is devoted to the proof of the following lower bound.

Lemma 5. $SW(\mathbf{B}) \geq \sum_{j \in [m]} (A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx)$.

Proof. Recall that $A_j = \max_{x_j \geq 0} \{F_{ij}(x) o_j - x_j\}$. We can bound bidder i's utility in the Nash equilibrium $\mathbf{B}$ by $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$. To see this, consider the deviation for bidder i, where he bids only for items in O_i, namely, for each item j, he bids the value x_j that maximizes the expression $F_{ij}(x_j) o_j - x_j$. Since for any obtained subset $T \subseteq O_i$, he has value $v_i(T) \geq \sum_{j \in T} o_j$, and the bids x_j must be paid in any case, the expected utility with these bids is at least $\sum_{j \in O_i} \max_{x_j \geq 0} (F_{ij}(x) o_j - x_j) = \sum_{j \in O_i} A_j$. With $\mathbf{B}$ being an equilibrium, we infer that $u_i(\mathbf{B}) \geq \sum_{j \in O_i} A_j$. By summing up over all bidders,

$$\begin{aligned} SW(\mathbf{B}) &= \sum_{i \in [n]} u_i(\mathbf{B}) + \sum_{i \in [n]} \sum_{j \in [m]} \mathbb{E}[b_{ij}] \geq \sum_{j \in [m]} A_j + \sum_{j \in [m]} \sum_{i \in [n]} \mathbb{E}[b_{ij}] \\ &\geq \sum_{j \in [m]} (A_j + \mathbb{E}[\max_{i \in [n]}\{b_{ij}\}]) \geq \sum_{j \in [m]} \left(A_j + \int_0^{o_j - A_j} (1 - F_j(x)) dx \right) \ . \end{aligned}$$

The first equality holds because $\sum_i \mathbb{E}_{\mathbf{b}}[v_i(\mathbf{b})] = \sum_i \mathbb{E}_{\mathbf{b}}[u_i(\mathbf{b}) + \sum_{j \in [m]} b_{ij}]$. The second inequality follows because $\sum_i b_{ij} \geq \max_i b_{ij}$ and the last one is implied by the definition of the expected value of any positive random variable. $\square$

3.2 Proof of Inequality (2)

Here, we prove the following lemma for any mixed Nash equilibrium $\mathbf{B}$.

Lemma 6. $SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx$.

First we show a useful lemma that holds for XOS valuations. We will further use the technical Proposition 8, whose proof is deferred to Appendix B.

Lemma 7. *For any fractionally subadditive (XOS) valuation function v,*

$$v(S) \geq \sum_{j \in [m]} \left(v(S) - v(S \setminus \{j\}) \right) \ .$$

Proof. Let ξ be a maximizing additive function of S for the XOS valuation v. By definition, $v(S) = \xi(S)$ and for every j, $v(S \setminus \{j\}) \geq \xi(S \setminus \{j\})$. Then, $\sum_{j \in [m]} (v(S) - v(S \setminus \{j\})) \leq \sum_{j \in S} (\xi(S) - \xi(S \setminus \{j\})) = \sum_{j \in S} \xi(j) = v(S)$. □

Proposition 8. *For any integer $n \geq 2$, any positive reals $G_i \leq 1$ and positive reals g_i, for $1 \leq i \leq n$,*

$$\sum_{i=1}^{n} \frac{g_i}{\sum_{k \neq i} \frac{g_k}{G_k}} \geq \sqrt{\prod_{i=1}^{n} G_i} \ .$$

We are now ready to prove Lemma 6. We only state a proof sketch here to illustrate the main ideas and defer the complete proof in Appendix A.

Proof (Sketch of Lemma 6). Recall that G_{ij} is the CDF of the bid of player i for item j. For simplicity, we assume $G_{ij}(x)$ is non-decreasing, continuous and differentiable, with $g_{ij}(x)$ being the PDF of player i's bid for item j. The general case is considered in the Appendix. First, we define the *expected marginal valuation* of item j w.r.t player i,

$$v_{ij}(x) \stackrel{\text{def}}{=} \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}} [v_i(X_i(\mathbf{b}) \cup \{j\}) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \ .$$

Given the above definition and a careful characterization of mixed Nash equilibria, we are able to show $F_{ij}(x) \cdot v_{ij}(x) = \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x]$ and $\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx}$ for any x in the support of G_{ij}. Let $g_{ij}(x)$ be the derivative of $G_{ij}(x)$. Using Lemma 7, we have

$$\begin{aligned} SW(\mathbf{B}) = \sum_i \mathbb{E}[v_i(X_i(\mathbf{b}))] &\geq \sum_i \sum_j \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})] \\ &\geq \sum_i \sum_j \int_0^{o_j - A_j} \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \cdot g_{ij}(x) dx \\ &\geq \sum_i \sum_j \int_0^{o_j - A_j} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) dx \ , \end{aligned}$$

where the second inequality follows by the law of total probability. By using the facts that $F_{ij}(x) = \prod_{k \neq i} G_{kj}(x)$ and $\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx}$, for any $x > 0$ such that $g_{ij}(x) > 0$ (x is in the support of player i) and $F_j(x) > 0$, we obtain

$$F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) = \frac{F_{ij}(x) \cdot g_{ij}(x)}{\frac{dF_{ij}}{dx}(x)} = \frac{\prod_{k \neq i} G_{kj}(x) \cdot g_{ij}(x)}{\sum_{k \neq i} \left(g_{kj} \cdot \prod_{s \neq k \wedge s \neq i} G_{sj} \right)} = \frac{g_{ij}(x)}{\sum_{k \neq i} \frac{g_{kj}(x)}{G_{kj}(x)}} \ .$$

For every $x > 0$, we use Proposition 8 only over the set S of players with $g_{ij}(x) > 0$. After summing over all bidders we get,

$$\sum_{i\in[n]} F_{ij}(x)\cdot v_{ij}(x)\cdot g_{ij}(x) \geq \sum_{i\in S} \frac{g_{ij}(x)}{\sum_{k\neq i,k\in S}\frac{g_{kj}}{G_{kj}}} \geq \sqrt{\prod_{i\in S} G_{ij}(x)} \geq \sqrt{F_j(x)} \ .$$

The above inequality also holds for $F_j(x) = 0$. Finally, by merging the above inequalities, we conclude that $SW(\mathbf{B}) \geq \sum_{j\in[m]}\int_0^{o_j - A_j}\sqrt{F_j(x)}dx$. □

3.3 Proof of Inequality (4)

In this section we prove the following technical lemma.

Lemma 9. *For any CDF F and any real $v > 0$, $R(F, v) \geq \frac{3+4\lambda-\lambda^4}{6}v$.*

In order to obtain a lower bound for $R(F, v)$ as stated in the lemma, we show first that we can restrict attention to cumulative distribution functions of a simple special form, since these constitute worst cases for $R(F, v)$. In the next lemma, for an arbitrary CDF F we will define a simple piecewise linear function $\hat{F}$ that satisfies the following two properties:

$$\int_0^{v-A}(1-\hat{F}(x))dx = \int_0^{v-A}(1-F(x))dx\ ;\quad \int_0^{v-A}\sqrt{\hat{F}(x)}dx \leq \int_0^{v-A}\sqrt{F(x)}dx\ .$$

Once we establish this, it is convenient to lower bound $R(\hat{F}, v)$ for the given type of piecewise linear functions $\hat{F}$.

Lemma 10. *For any CDF F and real $v > 0$, there always exists another CDF $\hat{F}$ such that $R(F, v) \geq R(\hat{F}, v)$ that, for $A = \max_{x\geq 0}\{F(x)\cdot v - x\}$, is defined by*

$$\hat{F}(x) = \begin{cases} 0 & ,\ \textit{if } x \in [0, x_0] \\ \frac{x+A}{v} & ,\ \textit{if } x \in (x_0, v-A] \end{cases} .$$

We give the proof of Lemma 10 in Appendix C. Now we are ready to proceed with the proof of Lemma 9.

Proof (of Lemma 9). By Lemma 10, for any fixed $v > 0$, we only need to consider the CDF's in the following form: for any positive A and x_0 such that $x_0 + A \leq v$,

$$F(x) = \begin{cases} 0 & ,\ \text{if } x \in [0, x_0] \\ \frac{x+A}{v} & ,\ \text{if } x \in (x_0, v-A] \end{cases} .$$

Clearly, $\max_{x\geq 0}\{F(x)\cdot v - x\} = A$. Let $t = \frac{A+x_0}{v}$. Then

$$\begin{aligned} R(F, v) &= A + \int_0^{v-A}(1-F(x))dx + \lambda\cdot\int_0^{v-A}\sqrt{F(x)}dx \\ &= v - \frac{v}{2}\cdot\left(\frac{x+A}{v}\right)^2\Bigg|_{x0}^{v-A} + \lambda\cdot\frac{2v}{3}\cdot\left(\frac{x+A}{v}\right)^{\frac{3}{2}}\Bigg|_{x_0}^{v-A} \\ &= v - \frac{v}{2}\cdot(1-t^2) + \lambda\cdot\frac{2v}{3}\cdot(1-t^{\frac{3}{2}}) = v\cdot\left(\frac{1}{2}(1+t^2) + \frac{2\lambda}{3}(1-t^{\frac{3}{2}})\right) \ . \end{aligned}$$

By optimizing over t, the above formula is minimized when $t = \lambda^2 \leq 1$. That is,

$$R(F, v) \geq v \cdot \left(\frac{1}{2}(1 + \lambda^4) + \frac{2\lambda}{3}(1 - \lambda^3) \right) = \frac{3 + 4\lambda - \lambda^4}{6} \cdot v \ . \qquad \square$$

4 Multi-unit Auctions

In this section, we propose a randomized all-pay mechanism for the multi-unit setting, where m identical items are to be allocated to n bidders. Markakis and Telelis [18] and de Keijzer et al. [14] have studied the price of anarchy for several multi-unit auction formats. The current best upper bound obtained was 1.58 for both pure and mixed Nash equilibria.

We propose a *randomized* all-pay mechanism that induces a *unique pure* Nash equilibrium, with an improved price of pnarchy bound of 4/3. We call the mechanism Random proportional-share allocation mechanism (PSAM), as it is a randomized version of Kelly's celebrated proportional-share allocation mechanism for divisible resources [15]. The mechanism works as follows (illustrated as Mechanism 1).

Each bidder submits a non-negative real b_i to the auctioneer. After soliciting all the bids from the bidders, the auctioneer associates a real number x_i with bidder i that is equal to $x_i = \frac{m \cdot b_i}{\sum_{i \in [n]} b_i}$. Each player pays their bid, $p_i = b_i$. In the degenerate case, where $\sum_i b_i = 0$, then $x_i = 0$ and $p_i = 0$ for all i.

We turn the x_i's to a random allocation as follows. Each bidder i secures $\lfloor x_i \rfloor$ items and gets one more item with probability $x_i - \lfloor x_i \rfloor$. An application of the Birkhoff-von Neumann decomposition theorem guarantees that given an allocation vector $(x_1, x_2, \ldots, x_n)$ with $\sum_i x_i = m$, one can always find a randomized allocation[2] with random variables $X_1, X_2, \ldots, X_n$ such that $\mathbb{E}[X_i] = x_i$ and $\Pr[\lfloor x_i \rfloor \leq X_i \leq \lceil x_i \rceil] = 1$ (see for example [10, 4]).

We next show that the game induced by the Random PSAM when the bidders have submodular valuations is *isomorphic* to the game induced by Kelly's mechanism for a single divisible resource when bidders have piece-wise linear concave valuations. For convenience, we review in Appendix D the definition of isomorphism between games as appears in Monderer and Shapley [19].

Theorem 11. *Any game induced by the Random PSAM applied to the multi-unit setting with submodular bidders is* isomorphic *to a game induced from Kelly's mechanism applied to a single divisible resource with piece-wise linear concave functions.*

Proof. For each bidder i's submodular valuation function $f_i : \{0, 1, \ldots, m\} \to R^+$, we associate a concave function $g_i : [0, 1] \to R^+$ such that,

$$\forall x \in [0, m], \ \ g_i(x/m) = f_i(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (f_i(\lfloor x \rfloor + 1) - f_i(\lfloor x \rfloor)) \ . \quad (5)$$

[2] As an example, assume $x_1 = 2.5, x_2 = 1.6, x_3 = 1.9$. One can define a random allocation such that assignments $(3, 2, 1)$, $(3, 1, 2)$ and $(2, 2, 2)$ occur with probabilities 0.1, 0.4, and 0.5 respectively.

Mechanism 1: Random PSAM

Input: Total number of items m and all bidders' bid $b_1, b_2, \ldots, b_n$
Output: Ex-post allocations $X_1, X_2, \ldots, X_n$ and payments $p_1, p_2, \ldots, p_n$
if $\sum_{i\in[n]} b_i > 0$ **then**
  foreach *bidder* $i = 1, 2, \ldots, n$ **do**
    $x_i \leftarrow \frac{m \cdot b_i}{\sum_{i\in[n]} b_i}$;
    $p_i \leftarrow b_i$;
  Sample $\{X_i\}_{i\in[n]}$ from $\{x_i\}_{i\in[n]}$ by using Birkhoff-von Neumann decomposition theorem such that $\lfloor x_i \rfloor \le X \le \lceil x_i \rceil$ and the expectation of sampling X_i is x_i;
else Set $\mathbf{X} = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$;
Return X_i and p_i for all $i \in [n]$;

Essentially, g_i is the piecewise linear function that comprises the line segments that connect $f_i(k)$ with $f_i(k+1)$, for all nonnegative integers k. It is easy to see that g_i is concave if f_i is submodular (see also Fig. D in Appendix D for an illustration). We use identity functions as the bijections ϕ^i of Definition 36. Therefore, it suffices to show that, for any pure strategy profile $\mathbf{b}$, $u_i(\mathbf{b}) = u_i'(\mathbf{b})$, where u_i and u_i' are the bidder i's utility functions in the first and second game, respectively. Let $x_i = \frac{m \cdot b_i}{\sum_i b_i}$, then

$$\begin{aligned} u_i(\mathbf{b}) &= (x_i - \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor + 1) + (1 - x_i + \lfloor x_i \rfloor) f_i(\lfloor x_i \rfloor) - b_i \\ &= f_i(\lfloor x_i \rfloor) + (x_i - \lfloor x_i \rfloor)(f_i(\lfloor x_i \rfloor + 1) - f_i(\lfloor x_i \rfloor)) - b_i \\ &= g_i\left(\frac{x_i}{m}\right) - b_i = g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i = u_i'(\mathbf{b}) \ . \end{aligned}$$

Note that $g_i\left(\frac{b_i}{\sum_i b_i}\right) - b_i$ is player i's utility, under $\mathbf{b}$, in Kelly's mechanism. $\square$

We next show an equivalence between the optimal welfares. We give the proof of Lemma 12 in Appendix D.

Lemma 12. *The optimum social welfare in the multi-unit setting, with submodular valuations* $\mathbf{f} = (f_1, \ldots, f_n)$*, is equal to the optimal social welfare in the divisible resource allocation with concave valuations* $\mathbf{g} = (g_1, \ldots g_n)$*, where* $\mathbf{g}$ *is derived from* $\mathbf{f}$ *as described in* (5)*.*

Theorem 11 and Lemma 12, allow us to obtain the existence and uniqueness of the pure Nash equilibrium, as well as the price of anarchy bounds of Random PSAM by the corresponing results on Kelly's mechanism for a single divisible resource [13]. Moreover, it can be shown that there are no other mixed equilibria by adopting the arguments of [5] for Kelly's mechanism. The main conclusion of this section is summarized in the following Corollary.

Corollary 13. *Random PSAM induces a* unique pure *Nash equilibrium when applied to the multi-unit setting with submodular bidders. Moreover, the price of anarchy of the mechanism is* exactly 4/3*.*

5 Single item auctions

In this section, we study mixed Nash equilibria in the single item all-pay auction. First, we measure the inefficiency of mixed Nash equilibria, showing tight results for the price of anarchy. En route (in the Appendix), we also show that the price of anarchy is 8/7 for two players. Then we analyze the quality of two other important criteria, the *expected revenue (the sum of bids)* and the quality of the expected *highest submission (the maximum bid)*, which is a standard objective in crowdsourcing contests [6]. For these objectives, we show a tight lower bound of $v_2/2$, where v_2 is the second highest value among all bidders' valuations. In the following, we drop the word expected while referring to the revenue or to the maximum bid.

We quantify the loss of revenue and the highest submission in the worst-case equilibria. We show that the all-pay auction achieves a 2-approximation comparing to the conventional procurement (modeled as the first price auction), when considering worst-case mixed Nash equilibria; we show in Appendix F that the revenue and the maximum bid of the conventional procurement equals v_2 in the worst case. We also consider other structures of rewards allocation and conclude that allocating the entire reward to the highest bidder is the only way to guarantee the approximation factor of 2. Roughly speaking, allocating all the reward to the top prize is the optimal way to maximize the maximum bid and revenue among all the prior-free all-pay mechanisms where the designer has no prior information about the participants' skills.

Due to the lack of space we give the proofs of theorems and lemmas of this section in Appendix E.

Theorem 14. *The mixed price of anarchy of the single item all-pay auction is* 1.185*.*

Theorem 15. *In any mixed Nash equilibrium of the single-item all-pay auction, the revenue and the maximum bid are at least half of the second highest valuation.*

Lemma 16. *There exists a mixed Nash equilibrium of the single-item all-pay auction, where the revenue and the maximum bid converges to $v_2/2$ when the number of players goes to infinity and v_2/v_1 approaches* 0*.*

Finally, the next theorem indicates that allocating the entire reward to the highest bidder is the best choice. In particular a prior-free all-pay mechanism is presented by a probability vector $\mathbf{q} = (q_i)_{i\in[n]}$, with $\sum_{i\in[n]} q_i = 1$, where q_i is the probability that the i^{th} highest bidder is allocated the item, for every $i \leq n$.

Theorem 17. *For any prior-free all-pay mechanism that assigns the item to the highest bidder with probability strictly less than* 1*, i.e. $q_1 < 1$, there exists a valuation profile and mixed Nash equilibrium such that the revenue and the maximum bid are strictly less than $v_2/2$.*

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A Proof of Lemma 6

Lemma 6 (restated). *$SW(\mathbf{B}) \geq \sum_{j\in[m]} \int_0^{o_j - A_j} \sqrt{F_j(x)}dx$.*

Recall that o_j is the contribution of item j to the optimum social welfare. If player i is the one receiving item j in the optimum allocation, then $A_j = \max_{x\geq 0}\{F_{ij}(x)\cdot o_j - x\}$. The proof of Lemma 6 needs a careful technical preparation that we divided into a couple of lemmas.

First of all, we define the expected marginal valuation of item j for player i. For given mixed strategy B_i, the distribution of bids on items in $[m]\setminus\{j\}$ depends on the bid b_{ij}, so one can consider the given conditional expectation:

Definition 18. *Given a mixed bidding profile $\mathbf{B} = (B_1, B_2, \ldots, B_n)$, the expected marginal valuation $v_{ij}(x)$ of item j for player i when $b_{ij} = x$ is defined as*

$$v_{ij}(x) \stackrel{\text{def}}{=} \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b})\cup\{j\}) - v_i(X_i(\mathbf{b})\setminus\{j\})|b_{ij} = x] \ .$$

For a given $\mathbf{B}$, let $\varphi_{ij}(x)$ denote the probability that bidder i gets item j when she bids x on item j. It is clear that φ_{ij} is non-decreasing and $\varphi_{ij}(x) \leq F_{ij}(x)$ (they are equal when no ties occur).

Lemma 19. *For a given $\mathbf{B}$, for any bidder i, item j and bids $x \geq 0$ and $y \geq 0$,*

$$\varphi_{ij}(y)\cdot v_{ij}(x) = \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x] \ ,$$

where $\mathbf{b}'$ is the modified bid of $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b}$ except that $b'_{ij} = y$.

Proof.

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x] \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\Pr(j\in X_i(\mathbf{b}')|b_{ij} = x) \\
&+ \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\notin X_i(\mathbf{b}')]\Pr(j\notin X_i(\mathbf{b}')|b_{ij} = x) \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\Pr(j\in X_i(\mathbf{b}')|b_{ij} = x) \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\cdot\varphi_{ij}(y) \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')\cup\{j\}) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\in X_i(\mathbf{b}')]\cdot\varphi_{ij}(y) \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}')\cup\{j\}) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x]\cdot\varphi_{ij}(y) \\
&= \varphi_{ij}(y)\cdot v_{ij}(x) \ .
\end{aligned}$$

The second equality is due to $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[v_i(X_i(\mathbf{b}'))) - v_i(X_i(\mathbf{b}')\setminus\{j\})|b_{ij} = x, j\notin X_i(\mathbf{b}')] = 0$; the third one holds because $b'_{ij} = y$, and that other players' bids have distribution $\times_{k\neq i}B_k$. The fourth one is obvious, since $X_i(\mathbf{b}') = X_i(\mathbf{b}')\cup\{j\}$ given that $j\in X_i(\mathbf{b}')$. The last two equalities follow from the fact that $v_i(X_i(\mathbf{b}')\cup\{j\}) - v_i(X_i(\mathbf{b}')\setminus\{j\})$ is independent of the condition $j\in X_i(\mathbf{b}')$ and of the player i's bid on item j. □

Definition 20. *Given a Nash equilibrium* $\mathbf{B}$*, we say a bid* x *is* good for bidder i and item j *(or* $b_{ij} = x$ *is* good*) if* $\mathbb{E}[u_i(\mathbf{b})] = \mathbb{E}[u_i(\mathbf{b})|b_{ij} = x]$*, otherwise we say* $b_{ij} = x$ *is* bad.

Lemma 21. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i *and any item* j*,* $\Pr[b_{ij} \text{ is bad }] = 0$*.*

Proof. The lemma follows from the definition of Nash equilibrium; otherwise we can replace the bad bids with good bids and improve the bidder's utility. $\square$

Lemma 22. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i*, item* j*, good bid* x *and any bid* $y \geq 0$*,*

$$\varphi_{ij}(x) \cdot v_{ij}(x) - x \geq \varphi_{ij}(y) \cdot v_{ij}(x) - y \ .$$

Moreover, for a good bid $x > 0$*,* $\varphi_{ij}(x) > 0$ *holds.*

Proof. Let $\mathbf{b}'$ be the modified bid of $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b}$ except that $b'_{ij} = y$.

$$\mathbb{E}[u_i(\mathbf{b})] = \mathbb{E}[u_i(\mathbf{b})|b_{ij} = x] \geq \mathbb{E}[u_i(\mathbf{b}')|b_{ij} = x] \ .$$

Now we consider the difference between the above two terms:

$$\begin{aligned}
0 &\leq \mathbb{E}[u_i(\mathbf{b})|b_{ij} = x] - \mathbb{E}[u_i(\mathbf{b}')|b_{ij} = x] \\
&= \mathbb{E}[v_i(X_i(\mathbf{b})) - b_{ij}|b_{ij} = x] - \mathbb{E}[v_i(X_i(\mathbf{b}')) - b'_{ij}|b_{ij} = x] \\
&= \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})|b_{ij} = x] \\
&\quad - \mathbb{E}[v_i(X_i(\mathbf{b}')) - v_i(X_i(\mathbf{b}') \setminus \{j\}|b_{ij} = x] + y - x \\
&= (\varphi_{ij}(x) \cdot v_{ij}(x) - x) - (\varphi_{ij}(y) \cdot v_{ij}(x) - y) \ .
\end{aligned}$$

The second equality holds since $X_i(\mathbf{b}) \setminus \{j\} = X_i(\mathbf{b}') \setminus \{j\}$; the third equality holds by Lemma 19.

Finally, $\varphi_{ij}(x) > 0$ for positive good bids follows by taking $y = 0$, since with $\varphi_{ij}(x) = 0$ the left hand side of the inequality would be negative. $\square$

Next, by using the above lemma, we are able to show several structural results for Nash equilibria.

Definition 23. *Given a mixed strategy profile* $\mathbf{B}$*, we say that a positive bid* $x > 0$ *is in bidder* i*'s* support on item j*, if for all* $\varepsilon > 0$*,* $G_{ij}(x) - G_{ij}(x - \varepsilon) > 0$*.*

Lemma 24. *Given a mixed strategy profile* $\mathbf{B}$*, if a positive bid* x *is in bidder* i*'s support on item* j*, then for every* $\varepsilon > 0$*, there exists* $x - \varepsilon < x' \leq x$ *such that* x' *is good.*

Proof. Suppose on the contrary that there is an $\varepsilon > 0$ such that for all x', such that $x - \varepsilon < x' \leq x$, x' is bad. Then $\Pr[b_{ij} \text{ is bad}] \geq G_{ij}(x) - G_{ij}(x - \varepsilon) > 0$ (given that x is in the support), which contradicts Lemma 21. $\square$

Lemma 25. *Given a Nash equilibrium* $\mathbf{B}$*, if* $x > 0$ *is in bidder* i*'s support on item* j*, then there must exist another bidder* $k \neq i$ *such that* x *is also in the bidder* k*'s support on item* j*, i.e. for all* $\varepsilon > 0$*,* $G_{kj}(x) - G_{kj}(x - \varepsilon) > 0$*.*

Proof. Assume on the contrary that for each player $k \neq i$, there exists $\varepsilon_k > 0$ such that $G_{kj}(x) - G_{kj}(x - \varepsilon_k) = 0$. Clearly, for $\varepsilon = \min\{\varepsilon_k | k \neq i\}$ it holds that $G_{kj}(x) - G_{kj}(x - \varepsilon) = 0$ for all bidders $k \neq i$. That is $\varphi_{ij}(x) = \varphi_{ij}(x - \varepsilon)$. By Lemma 24, there exists $x - \varepsilon < x' \leq x$ such that x' is good for player i. Since φ_{ij} is a non-decreasing function and $\varphi_{ij}(x) = \varphi_{ij}(x - \varepsilon)$, we have $\varphi_{ij}(x') = \varphi_{ij}(x - \varepsilon)$. By Lemma 22, $\varphi_{ij}(x') \cdot v_{ij}(x') - x' \geq \varphi_{ij}(x - \varepsilon) \cdot v_{ij}(x') - x + \varepsilon$ which contradicts the fact that $\varphi_{ij}(x') = \varphi_{ij}(x - \varepsilon)$ and $x' > x - \varepsilon$. □

Lemma 26. *Given a Nash equilibrium* $\mathbf{B}$*, for bidder* i *and item* j*, there are no* $x > 0$ *such that* $\Pr[b_{ij} = x] > 0$*, i.e. there are no mass points in the bidding strategy, except for possibly* 0*.*

Proof. Assume on the contrary that there exists a bid $x > 0$ such that $\Pr[b_{ij} = x] > 0$ for some bidder i and item j. By Lemma 21, x is good for bidder i and item j, and $\varphi_{ij}(x) > 0$ by Lemma 22.

According to Lemma 25, there must exist a bidder k such that x is in her support on item j. We can pick a sufficiently small ε such that $\varepsilon < (x - \varepsilon) \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]$. This can be done since $(x - \varepsilon)$ increases when ε decreases. Due to Lemma 24 there exists $x - \varepsilon < x' \leq x$ such that x' is good for bidder k and item j. Now we consider the following two cases for x'.

Case 1: $v_{kj}(x') \leq x'$. Then $\varphi_{kj}(x') \cdot v_{kj}(x') - x' \leq \varphi_{kj}(x') \cdot x' - x' \leq (1 - \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]) \cdot x' - x' < 0$, contradicting Lemma 22. The first inequality holds by the case assumption. The second holds because player k cannot get item j with bid x' whenever player i gets it by bidding x. The last inequality holds because both $\varphi_{ij}(x) > 0$ and $\Pr[b_{ij} = x] > 0$.

Case 2: $v_{kj}(x') > x'$. Then there exists a sufficiently small ε' such that $\varepsilon' \leq (x - \varepsilon) \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] - \varepsilon$. So $\varepsilon + \varepsilon' \leq x' \cdot \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]$. Then,

$$
\begin{aligned}
&\varphi_{kj}(x + \varepsilon') \cdot v_{kj}(x') - x - \varepsilon' \\
\geq &(\varphi_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x]) \cdot v_{kj}(x') - x - \varepsilon' \\
> &\varphi_{kj}(x') \cdot v_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] \cdot x' - x' - (x - x') - \varepsilon' \\
> &\varphi_{kj}(x') \cdot v_{kj}(x') + \varphi_{ij}(x) \cdot \Pr[b_{ij} = x] \cdot x' - x' - \varepsilon - \varepsilon' \\
\geq &\varphi_{kj}(x') \cdot v_{kj}(x') - x' \ ,
\end{aligned}
$$

which contradicts Lemma 22. Here the first inequality holds because the probability that player k gets the item with bid $x + \varepsilon'$ is at least the probablity that he gets it by bidding x' plus the probability that i bids x and gets the item (these two events for $\mathbf{b}_{-k}$ are disjoint). The second inequality holds by case assumption, and the rest hold by our assumptions on ε and ε'. □

Lemma 27. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i *and item* j*,* $\varphi_{ij}(x) = F_{ij}(x)$ *for all* $x > 0$*.*

Proof. The lemma follows immediately from Lemma 26. The probablity that some player $k \neq i$ bids exactly x is zero. Thus $F_{ij}(x)$ equals the probability that the highest bid of players other than i is strictly smaller than x, and $1 - F_{ij}(x)$ is the probability that it is strictly higher. Therefore $\varphi_{ij}(x) = F_{ij}(x)$. □

Lemma 28. *Given a Nash equilibrium* $\mathbf{B}$*, for any bidder* i*, item* j *and good bids* $x_1 > x_2 > 0$*,* $v_{ij}(x_1) \geq v_{ij}(x_2)$*.*

Proof. By Lemma 22, we have $(\varphi_{ij}(x_1) - \varphi_{ij}(x_2)) \cdot v_{ij}(x_1) \geq x_1 - x_2$ and $(\varphi_{ij}(x_2) - \varphi_{ij}(x_1)) \cdot v_{ij}(x_2) \geq x_2 - x_1$. Combining these two inequalities, we have

$$\frac{1}{v_{ij}(x_1)} \leq \frac{\varphi_{ij}(x_1) - \varphi_{ij}(x_2)}{x_1 - x_2} \leq \frac{1}{v_{ij}(x_2)} \ .$$

□

Lemma 29. *Given a Nash equilibrium* $\mathbf{B}$ *and item* j*, let* $T = \sup\{x | x$ *is in some bidder's support on item* $j\}$*. For any bid* $x < T$*,* x *is in some bidder's support on item* j*.*

Proof. Assume on the contrary that there exist a bid $x < T$ such that x is not in any bidder's support. Then there exists $\delta > 0$ such that $G_{ij}(x) = G_{ij}(x - \delta)$ for all bidder i. Let $y = \sup\{z | \forall i, G_{ij}(x) = G_{ij}(z)\}$. By Lemma 26, G_{ij} is continuous. So we have $G_{ij}(y) = G_{ij}(x) = G_{ij}(x - \delta)$ for any bidder i. That is $F_{ij}(y) = F_{ij}(x - \delta)$ for any bidder i.

By the definition of supremum, there exits a bidder k such that for any $\varepsilon > 0$, $G_{kj}(y + \varepsilon) > G_{kj}(x) = G_{kj}(y)$. By Lemma 21, there exists a good bid $y^+ \in (y, y + \varepsilon]$ for bidder k and item j. We pick a sufficient small ε such that $(F_{kj}(y^+) - F_{kj}(y)) \cdot v_{kj}(y^+) < \delta$. This can be done since F_{kj} is continuous by Lemma 26 and v_{kj} is non-decreasing by Lemma 28.

$$\begin{aligned}
&F_{kj}(x - \delta) \cdot v_{ij}(y^+) - x + \delta \\
=&F_{ij}(y) \cdot v_{ij}(y^+) - x + \delta \\
>&F_{ij}(y) \cdot v_{ij}(y^+) - y^+ + \delta \\
>&F_{ij}(y^+) \cdot v_{ij}(y^+) - y^+ \ ,
\end{aligned}$$

which contradicts Lemma 22 and Lemma 27. □

Lemma 30. *Given a Nash equilibrium* $\mathbf{B}$*, if* $x > 0$ *is a good bid for bidder* i *and item* j*, and* F_{ij} *is differentiable in* x*, then*

$$\frac{1}{v_{ij}(x)} = \frac{dF_{ij}(x)}{dx} \ .$$

Proof. Notice that $v_{ij}(x) \neq 0$ by Lemma 22. By Lemma 22 and 27, we have $F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(y) \cdot v_{ij}(x) - y$ for all $y \geq 0$. So for any $\varepsilon > 0$,

$$F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(x + \varepsilon) \cdot v_{ij}(x) - x - \varepsilon \ ,$$

$$F_{ij}(x) \cdot v_{ij}(x) - x \geq F_{ij}(x-\varepsilon) \cdot v_{ij}(x) - x + \varepsilon \ .$$

That is,

$$\frac{F_{ij}(x+\varepsilon) - F_{ij}(x)}{\varepsilon} \leq \frac{1}{v_{ij}(x)} \ ,$$

$$\frac{F_{ij}(x) - F_{ij}(x-\varepsilon)}{\varepsilon} \geq \frac{1}{v_{ij}(x)} \ .$$

The lemma follows by taking the limit when ε goes to 0. □

Proof (of Lemma 6). Since $G_{ij}(x)$ is non-decreasing, continuous (Lemma 26) and bounded by 1, $G_{ij}(x)$ is differentiable on almost all points. That is, the set of all non-differentiable points has Lebesgue measure 0. So it will not change the value of integration if we remove these points. Therefore it is without loss of generality to assume $G_{ij}(x)$ is differentiable for all x. Let $g_{ij}(x)$ be the derivative of $G_{ij}(x)$, i.e. probability density function for bidder i's bidding on item j. Using Lemma 7, we have

$$\begin{aligned}
SW(\mathbf{B}) &= \sum_i \mathbb{E}[v_i(X_i(\mathbf{b}))] \\
&\geq \sum_i \sum_j \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\})] \\
&\geq \sum_i \sum_j \int_0^{o_j - A_j} \mathbb{E}[v_i(X_i(\mathbf{b})) - v_i(X_i(\mathbf{b}) \setminus \{j\}) | b_{ij} = x] \cdot g_{ij}(x) dx \\
&\geq \sum_i \sum_j \int_0^{o_j - A_j} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) dx \ .
\end{aligned}$$

The second inequality follows by the law of total probability, and the third is due to Lemmas 19 and 27. By Lemma 30 and the fact that $F_{ij}(x) = \prod_{k \neq i} G_{kj}(x)$, if x is good, $g_{ij}(x) > 0$ and $G_{ij}(x) > 0$ we have for all j

$$\begin{aligned}
F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) =& \frac{F_{ij}(x) \cdot g_{ij}(x)}{\frac{dF_{ij}}{dx}(x)} \\
= \frac{\prod_{k \neq i} G_{kj}(x) \cdot g_{ij}(x)}{\sum_{k \neq i} \left(g_{kj} \cdot \prod_{s \neq k \wedge s \neq i} G_{sj} \right)} =& \frac{g_{ij}(x)}{\sum_{k \neq i} \frac{g_{kj}(x)}{G_{kj}(x)}} \ .
\end{aligned}$$

By concentrating on a specific item j, let S_x be the set of bidders so that x is in their support. We next show that $|S_x| \geq 2$ for all $x \in (0, o_j - A_j]$. Recall that $A_j = \max_x \{F_{ij}(x) \cdot o_j - x\}$ for the bidder i who receives j in $\mathbf{O}$. Let $h_{ij} = \min\{x | F_{ij} = 1\}$ (we use minimum instead of infimum, since, by Lemma 26, F_{ij} is continuous). By definition h_{ij} should be in some bidder's support. Moreover, $A_j \geq F_{ij}(h_{ij}) \cdot o_j - h_{ij} = o_j - h_{ij}$, resulting in $o_j - A_j \leq h_{ij}$. Therefore, by Lemma 29, for all $x \in (0, o_j - A_j]$, x is in some bidder's support and by Lemma 25, there are at least 2 bidders such that x is in their supports.

By the definition of derivative, for all $i \notin S_x$, $g_{ij}(x) = 0$. Similarly, we have $g_{ij}(x) > 0$ and $G_{ij}(x) > 0$ for all $i \in S_x$ by definition 23. Moreover, for every $i \in S_x$, x is good for bidder i and item j, since x is in their support. So, for any fixed $x \in (0, o_j - A_j]$, $\sum_{i \in [n]} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) = \sum_{i \in S_x} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x)$, and according to Proposition 8,

$$\sum_{i \in [n]} F_{ij}(x) \cdot v_{ij}(x) \cdot g_{ij}(x) \geq \sum_{i \in S_x} \frac{g_{ij}(x)}{\sum_{k \neq i, k \in S_x} \frac{g_{kj}}{G_{kj}}} \geq \sqrt{\prod_{i \in S_x} G_{ij}(x)} \geq \sqrt{\prod_{i \in [n]} G_{ij}(x)} \ .$$

Merging all these inequalities,

$$SW(\mathbf{B}) \geq \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{\prod_{i \in [n]} G_{ij}(x)} dx = \sum_{j \in [m]} \int_0^{o_j - A_j} \sqrt{F_j(x)} dx \ .$$

□

B Proof of Proposition 8

Proposition 8 (restated). For any integer $n \geq 2$, any positive real $G_i \leq 1$ and positive real g_i for $1 \leq i \leq n$,

$$\sum_{i=1}^{n} \frac{g_i}{\sum_{k \neq i} \frac{g_k}{G_k}} \geq \sqrt{\prod_{i=1}^{n} G_i} \ .$$

In order to prove the proposition, we will minimize the left hand side of the inequality over all G_i and g_i, such that

$$0 < G_i \leq 1 \qquad g_i > 0 \quad (i \in [n]) \quad \text{ where } \prod_{t=1}^{n} G_t \quad \text{is a constant} \ . \tag{6}$$

We introduce the following notation:

$$H = \sum_{i=1}^{n} \frac{g_i}{\sum_{t=1, t \neq i}^{n} \frac{g_t}{G_t}} \qquad \text{and} \qquad \forall i, \qquad H_i = \frac{g_i}{\sum_{t=1, t \neq i}^{n} \frac{g_t}{G_t}} \ .$$

Note that $H = \sum_{i=1}^{n} H_i$. Our goal is to minimize H over all possible variables G_i and g_i under the constraints (6), and eventually show $H \geq \sqrt{\prod_{i=1}^{n} G_i}$. We also use the notation $\mathbf{G} = (G_i)_i$, $\mathbf{g} = (g_i)_i$, $H = H(\mathbf{G}, \mathbf{g})$ and $H_i = H_i(\mathbf{G}, \mathbf{g})$, $\forall i$.

Lemma 31. *For every $\mathbf{G}$ and $\mathbf{g}$ that minimize $H(\cdot, \cdot)$ under constraints (6):*

1. *If $G_i < 1$ and $G_j < 1$, then $H_i = H_j$,*
2. *If $G_i = G_j = 1$ then $g_i = g_j$.*

We prove Lemma 31, by proving Lemmas 32 and 33.

Lemma 32. *Under constraints* (6)*, if* $\mathbf{G}$ *and* $\mathbf{g}$ *minimize* $H(\cdot,\cdot)$*, then for every* $G_i < 1$ *and* $G_j < 1$*,* $H_i(\mathbf{G},\mathbf{g}) = H_j(\mathbf{G},\mathbf{g})$*.*

Proof. For the sake of contradiction, suppose that there exist $G_i < 1$ and $G_j < 1$ such that (w.l.o.g.) $H_i(\mathbf{G},\mathbf{g}) > H_j(\mathbf{G},\mathbf{g})$. Let

$$r = \min\left\{\left(\frac{H_i(\mathbf{G},\mathbf{g})}{H_j(\mathbf{G},\mathbf{g})}\right)^{\frac{1}{2}}, \frac{1}{G_j}\right\} \ .$$

Notice that $r > 1$.

Claim: We claim that $H(\mathbf{G},\mathbf{g}) > H(\mathbf{G}',\mathbf{g}')$, where $\mathbf{G}' = (\frac{G_i}{r}, rG_j, \mathbf{G}_{-ij})$ and $\mathbf{g}' = (\frac{g_i}{r}, rg_j, \mathbf{g}_{-ij})$.
As usual $\mathbf{G}_{-ij}$ stands for $\mathbf{G}$ vector after eliminating G_i and G_j (accordingly for $\mathbf{g}_{-ij}$). Therefore $\mathbf{G}'$ and $\mathbf{g}'$ are the same as $\mathbf{G}$ and $\mathbf{g}$ by replacing G_i, G_j, g_i, g_j by $\frac{G_i}{r}$, rG_j, $\frac{g_i}{r}$, rg_j, respectively.

Proof of the claim: Notice that

$$\frac{g'_i}{G'_i} = \frac{g_i/r}{G_i/r} = \frac{g_i}{G_i} \ , \ \frac{g'_j}{G'_j} = \frac{rg_j}{rG_j} = \frac{g_j}{G_j} \quad \text{and} \quad \forall s \neq i,j, \ \ G'_s = G_s \ \text{ and } \ g'_s = g_s \ .$$

Therefore, $\forall s \neq i,j$, $H_s(\mathbf{G},\mathbf{g}) = H_s(\mathbf{G}',\mathbf{g}')$. So, we only need to show that $H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) > H_i(\mathbf{G}',\mathbf{g}') + H_j(\mathbf{G}',\mathbf{g}')$.

$$\begin{aligned}
&H_i(\mathbf{G}',\mathbf{g}') + H_j(\mathbf{G}',\mathbf{g}') \\
&= \frac{g'_i(x)}{\sum_{t=1,t\neq i}^{n} \frac{g'_t(x)}{G'_t(x)}} + \frac{g'_j(x)}{\sum_{t=1,t\neq j}^{n} \frac{g'_t(x)}{G'_t(x)}} \\
&= \frac{g_i(x)/r}{\sum_{t=1,t\neq i}^{n} \frac{g_t(x)}{G_t(x)}} + \frac{rg_j(x)}{\sum_{t=1,t\neq j}^{n} \frac{g_t(x)}{G_t(x)}} \\
&= \frac{H_i(\mathbf{G},\mathbf{g})}{r} + rH_j(\mathbf{G},\mathbf{g}) \\
&= \left(\frac{1}{r} - 1\right) H_i(\mathbf{G},\mathbf{g}) + (r-1)H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&\leq \left(\frac{1}{r} - 1\right) r^2 H_j(\mathbf{G},\mathbf{g}) + (r-1)H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&= -(r-1)^2 H_j(\mathbf{G},\mathbf{g}) + H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \\
&< H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \ .
\end{aligned}$$

In the above inequalities we used that $r > 1$ and $r^2 \leq \frac{H_i(\mathbf{G},\mathbf{g})}{H_j(\mathbf{G},\mathbf{g})}$. The claim contradicts the assumption that $H(\mathbf{G},\mathbf{g})$ is the minimum, so the lemma holds. ∎

Lemma 33. *Under constraints* (6), *if* $\mathbf{G}$ *and* $\mathbf{g}$ *minimize* $H(\cdot,\cdot)$, *then for every* $G_i = G_j = 1$, $g_i = g_j$.

Proof. For the sake of contradiction, suppose that there exist $G_i = G_j = 1$ such that $g_i \neq g_j$. We will prove that for $\mathbf{g}' = (\frac{g_i+g_j}{2}, \frac{g_i+g_j}{2}, g_{-ij})$ (i.e. for every $k \neq i,j$, $g'_k = g_k$, and $g'_i = g'_j = \frac{g_i+g_j}{2}$), $H(\mathbf{G},\mathbf{g}) > H(\mathbf{G},\mathbf{g}')$.

Notice that for every $k \neq i,j$, $H_k(\mathbf{G},\mathbf{g}') = H_k(\mathbf{G},\mathbf{g})$, since $g_i + g_j = g'_i + g'_j$ and $G_i = G_j = 1$. Hence it is sufficient to show that $H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) \geq H_i(\mathbf{G},\mathbf{g}') + H_j(\mathbf{G},\mathbf{g}')$. Let $A_{ij} = \sum_{t\neq j, t\neq i} \frac{g_t}{G_t}$.

$$
\begin{aligned}
&H_i(\mathbf{G},\mathbf{g}) + H_j(\mathbf{G},\mathbf{g}) - H_i(\mathbf{G},\mathbf{g}') - H_j(\mathbf{G},\mathbf{g}')\\
&= \frac{g_i}{g_j + A_{ij}} + \frac{g_j}{g_i + A_{ij}} - \frac{g_i}{\frac{g_i+g_j}{2} + A_{ij}} - \frac{g_j}{\frac{g_i+g_j}{2} + A_{ij}}\\
&= \frac{g_i}{g_j + A_{ij}} + \frac{g_j}{g_i + A_{ij}} - \frac{2g_i + 2g_j}{g_i + g_j + 2A_{ij}}\\
&= g_i \frac{(g_i + A_{ij})((g_i + g_j + 2A_{ij}) - 2(g_j + A_{ij}))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})}\\
&+ g_j \frac{(g_j + A_{ij})((g_i + g_j + 2A_{ij}) - 2(g_i + A_{ij}))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})}\\
&= \frac{g_i(g_i + A_{ij})(g_i - g_j) + g_j(g_j + A_{ij})(g_j - g_i)}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})}\\
&= \frac{(g_i - g_j)(g_i^2 - g_j^2 + A_{ij}(g_i - g_j))}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})}\\
&= \frac{(g_i - g_j)^2(g_i + g_j + A_{ij})}{(g_j + A_{ij})(g_i + A_{ij})(g_i + g_j + 2A_{ij})} > 0 \ ,
\end{aligned}
$$

which contradicts the assumption that $\mathbf{G}$ and $\mathbf{g}$ minimize $H(\cdot,\cdot)$. □

Lemma 34. *If* $H_i = H_j$, *then:*

1. $g_i = g_j \Leftrightarrow G_i = G_j$,
2. $(g_i = rg_j > 0$ *and* $r \geq 1) \Rightarrow G_i \geq r^2 G_j$.

Proof. Let $A_{ij} = \sum_{t\neq j, t\neq i} \frac{g_t}{G_t}$; then $H_i = \frac{g_i}{\frac{g_j}{G_j} + A_{ij}}$. By assumption:

$$
\begin{aligned}
\frac{g_i}{\frac{g_j}{G_j} + A_{ij}} &= \frac{g_j}{\frac{g_i}{G_i} + A_{ij}}\\
\frac{g_i^2}{G_i} + g_i A_{ij} &= \frac{g_j^2}{G_j} + g_j A_{ij}\\
(g_i - g_j)A_{ij} &= \frac{g_j^2}{G_j} - \frac{g_i^2}{G_i} \ .
\end{aligned}
$$

If $g_i = g_j$ then $\frac{1}{G_j} - \frac{1}{G_i} = 0$, so $G_i = G_j$.
If $G_i = G_j$ then $(g_i - g_j)(g_i + g_j + A_{ij}G_i) = 0$. Under constraints (6), $A_{ij}G_i > 0$

and $g_i, g_j > 0$, so $g_i - g_j = 0$ which results in $g_i = g_j$.
If $g_i = rg_j$, with $r \geq 1$ then $(g_i - g_j)A_{ij} \geq 0$ and so $\frac{1}{G_j} - \frac{r^2}{G_i} \geq 0$, which implies $G_i \geq r^2 G_j$. □

Lemma 35. *For n, k integers, $n \geq 2$, $1 \leq k \leq n$, $0 < a \leq 1$ and $g > 0$:*

$$L = \frac{kg}{(k-1)\frac{g}{a} + n - k} + \frac{n-k}{k\frac{g}{a} + n - k - 1} \geq a \quad .$$

Proof. We distinguish between two cases, 1) $k > \frac{1}{1-\sqrt{a}}$ and 2) $k \leq \frac{1}{1-\sqrt{a}}$.
Case 1 ($k > \frac{1}{1-\sqrt{a}}$): For $k = n$, $L = \frac{k}{k-1}a \geq a$. We next show that $\frac{dL}{dg} \leq 0$, for $n \geq 2$, $1 \leq k < n$, $0 < a \leq 1$ and $g > 0$.

$$\begin{aligned}
\frac{dL}{dg} = \frac{(n-k)k}{\left(\frac{(k-1)g}{a} + n - k\right)^2} - \frac{(n-k)k}{\left(\frac{kg}{a} + n - k - 1\right)^2 a} &\leq 0 \\
\left(\frac{(k-1)g}{a} + n - k\right)^2 - \left(\frac{kg}{a} + n - k - 1\right)^2 a &\geq 0 \\
\left(\frac{(k-1)g}{a} + n - k + \left(\frac{kg}{a} + n - k - 1\right) a^{\frac{1}{2}}\right) & \\
\cdot \left(\frac{(k-1)g}{a} + n - k - \left(\frac{kg}{a} + n - k - 1\right) a^{\frac{1}{2}}\right) &\geq 0 \\
\left(\frac{(k-1)g}{a} + n - k - \left(\frac{kg}{a} + n - k - 1\right) a^{\frac{1}{2}}\right) &\geq 0 \\
\left(\frac{g}{a}\left(k - 1 - ka^{\frac{1}{2}}\right) + (n-k)\left(1 - a^{\frac{1}{2}}\right) + a^{\frac{1}{2}}\right) &\geq 0 \\
k - 1 - ka^{\frac{1}{2}} &\geq 0 \quad ,
\end{aligned}$$

which is true by the case assumption. Therefore, L is non-increasing and so it is minimized for $g = \infty$. Hence, $L \geq \frac{k}{k-1}a \geq a$.
Case 2 ($k \leq \frac{1}{1-\sqrt{a}}$): L is minimized ($dL/dg(g^*) = 0$) for $g^* = \frac{a\left(\sqrt{a} + (n-k)\left(1-\sqrt{a}\right)\right)}{k\sqrt{a} - k + 1}$, therefore:

$$L \geq \frac{k^2\left(1-\sqrt{a}\right)^2 + k\left(a - n\left(1-\sqrt{a}\right)^2 - 1\right) + n)}{(n-1)} \quad ,$$

which is minimizes for $k = \frac{n}{2} + \frac{\left(1+\sqrt{a}\right)}{2\left(1-\sqrt{a}\right)}$. However, for $n \geq 2$, $\frac{n}{2} + \frac{\left(1+\sqrt{a}\right)}{2\left(1-\sqrt{a}\right)} \geq \frac{1}{1-\sqrt{a}}$. Notice, though, that for $k \leq \frac{1}{1-\sqrt{a}}$, L is decreasing, so it is minimized for $k = \frac{1}{1-\sqrt{a}}$. Therefore, $L \geq \sqrt{a} \geq a$. □

Proof. (Proposition 8)
Let $\mathbf{G}$ and $\mathbf{g}$ minimize $H(\cdot,\cdot)$ and also let $S = \{i | G_i < 1\}$ and $F = \prod_{t=1}^{n} G_t$.

Moreover, given Lemma 31, for $g_i = \hat{g}$ for every $i \notin S$ and $j = \arg\min_{i\in S} g_i$, $H(\mathbf{G}, \mathbf{g})$ can be written as:

$$H(\mathbf{G}, \mathbf{g}) = |S| \frac{g_j}{\sum_{t\in S, t\neq j} \frac{g_t}{G_t} + (n-|S|)\hat{g}} + (n-|S|) \frac{\hat{g}}{\sum_{t\in S} \frac{g_t}{G_t} + (n-|S|-1)\hat{g}} \ .$$

Let $g_i = r_i g_j$, for every $i \in S$. Since $j = \arg\min_{i\in S} g_i$, then for every $i \in S$, $r_i \geq 1$. By using Lemma 34:

$$\begin{aligned} H(\mathbf{G}, \mathbf{g}) &= \frac{|S| \cdot g_j}{\sum_{t\in S, t\neq j} \frac{r_t g_j}{G_t^{\frac{1}{2}} G_t^{\frac{1}{2}}} + (n-|S|)\hat{g}} + \frac{(n-|S|)\cdot \hat{g}}{\sum_{t\in S} \frac{r_t g_j}{G_t^{\frac{1}{2}} G_t^{\frac{1}{2}}} + (n-|S|-1)\hat{g}} \\ &\geq \frac{|S| \cdot g_j}{\sum_{t\in S, t\neq j} \frac{r_t g_j}{(r_t^2 G_j)^{\frac{1}{2}} G_t^{\frac{1}{2}}} + (n-|S|)\hat{g}} + \frac{(n-|S|)\cdot \hat{g}}{\sum_{t\in S} \frac{r_t g_j}{(r_t^2 G_j)^{\frac{1}{2}} G_t^{\frac{1}{2}}} + (n-|S|-1)\hat{g}} \\ &\geq \frac{|S| \cdot g_j}{\sum_{t\in S, t\neq j} \frac{g_j}{F^{\frac{1}{2}}} + (n-|S|)\hat{g}} + \frac{(n-|S|)\cdot \hat{g}}{\sum_{t\in S} \frac{g_j}{F^{\frac{1}{2}}} + (n-|S|-1)\hat{g}} \\ &= \frac{|S| \cdot g_j}{(|S|-1) \frac{g_j}{F^{\frac{1}{2}}} + (n-|S|)\hat{g}} + \frac{(n-|S|)\cdot \hat{g}}{|S| \frac{g_j}{F^{\frac{1}{2}}} + (n-|S|-1)\hat{g}} \ . \end{aligned}$$

Let $g = \frac{g_j}{\hat{g}}$, then:

$$H(\mathbf{G}, \mathbf{g}) \geq \frac{|S| \cdot g}{(|S|-1) \frac{g}{F^{\frac{1}{2}}} + n - |S|} + \frac{n-|S|}{|S| \frac{g}{F^{\frac{1}{2}}} + n - |S| - 1} \ .$$

If $|S| = 0$, $H(\mathbf{G}, \mathbf{g}) \geq \frac{n}{n-1} \geq 1 \geq \sqrt{F}$. Else, due to Lemma 35, $H(\mathbf{G}, \mathbf{g}) \geq \sqrt{F}$. □

C Proof of Lemma 10

LEMMA 10 (RESTATED). For any CDF F and real $v > 0$, there always exists another CDF $\hat{F}$ such that $R(F, v) \geq R(\hat{F}, v)$ that, for $A = \max_{x\geq 0}\{F(x)\cdot v - x\}$, is defined by

$$\hat{F}(x) = \begin{cases} 0 & \text{, if } x \in [0, x_0] \\ \frac{x+A}{v} & \text{,if } x \in (x_0, v-A] \end{cases} \ .$$

First notice that $\max_{x\geq 0}\{\hat{F}(x) \cdot v - x\} = A$. By the definition of Riemann integration, we can represent the integration as the limit of Riemann sums. For any positive integer l, let R_l be the Riemann sum if we partition the interval $[0, v-A]$ into small intervals of size $(v-A)/l$. That is

$$R_l(F, v) = A + \frac{v-A}{l} \cdot \left(\sum_{i=0}^{l-1} (1 - F(x_i)) + \lambda \cdot \sum_{i=0}^{l-1} \sqrt{F(x_i)} \right) \ ,$$

where $x_i = \frac{i}{l} \cdot (v - A)$. So we have $R(F, v) = \lim_{l \to \infty} R_l(F, v)$.

For any given l, let i^* be the index such that $\sum_{i > i^*} (x_i + A)/v < \sum_{i=0}^{l-1} F(x_i)$ and $\sum_{i >= i^*} (x_i + A)/v \geq \sum_{i=0}^{l-1} F(x_i)$. We define $\hat{F}_l$ as follows:

$$\hat{F}_l(x) = \begin{cases} 0 & , \text{if } x < x_{i^*} \\ \sum_{i=0}^{l-1} F(x_i) - \sum_{i > i^*} (x_i + A)/v & \text{if } x \in [x_{i^*}, x_{i^*+1}) \\ (x + A)/v & , \text{if } x \in [x_{i^*+1}, v - A] \end{cases} .$$

It is straight-forward to check that $\hat{F}(x) = \lim_{l \to \infty} \hat{F}_l(x)$, as described in the statement of the lemma. We will show that for any l, $R_l(F, v) \geq R_l(\hat{F}_l, v)$. Then the lemma follows by taking the limit, since $R_l(F, v) \to R(F, v)$, and $R_l(\hat{F}, v) \to R(\hat{F}, v)$. Figure 1(a) illustrates $\hat{F}(x)$ (when we take the limit of l to infinity).

By the construction of $\hat{F}_l$, it is easy to check that $\sum_{i=0}^{l-1} F(x_i) = \sum_{i=0}^{l-1} \hat{F}_l(x_i)$ and $\max_x\{\hat{F}_l(x) \cdot v - x\} = A$. Then in order to prove $R_l(F, v) \geq R_l(\hat{F}_l, v)$, it is sufficient to prove that $\sum_{i=0}^{l-1} \sqrt{F(x_i)} \geq \sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$. Let $\mathcal{Q}$ be the set of CDF functions such that $\forall Q \in \mathcal{Q}$, $\sum_{i=0}^{l-1} Q(x_i) = \sum_{i=0}^{l-1} F(x_i)$ and $A = \max_{x \geq 0}\{Q(x) \cdot v - x\}$, meaning further that $Q(x) \leq (x + A)/v$, for all $x \geq 0$. We will show that $\hat{F}_l(x)$ has the minimum value for the expression $\sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$ within $\mathcal{Q}$.

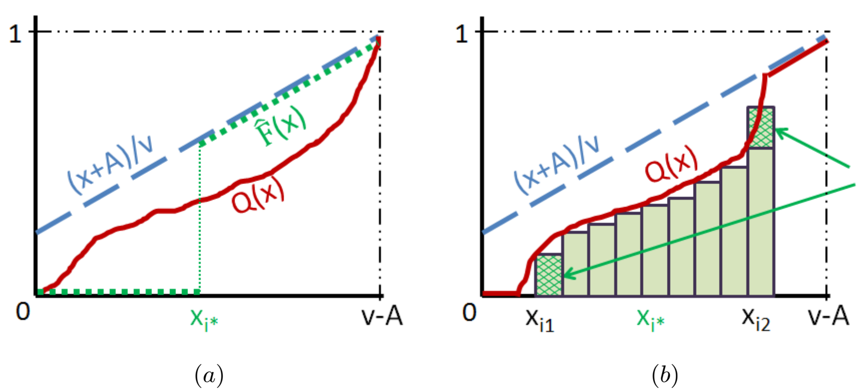

Fig. 1. Figure (a) illustrates $\hat{F}(x) = \lim_{l \to \infty} \hat{F}_l(x)$ and figure (b) shows how Q' is derived from Q.

Assume on the contrary that some other function $Q \in \mathcal{Q}$ has the minimum value for $\sum_{i=0}^{l-1} \sqrt{Q(x_i)}$ within $\mathcal{Q}$ and $Q(x_j) \neq \hat{F}_l(x_j)$ for some x_j. Let i_1 be the smallest index such that $Q(x_{i_1}) > 0$ and i_2 be the largest index such that $Q(x_{i_2}) < (x_{i_2} + A)/v$. By the monotonicity of Q, we have $i_1 \leq i_2$. Due to the assumption that $Q(x_j) \neq \hat{F}_l(x_j)$ for some x_j and $\sum_{i=0}^{l-1} \sqrt{Q(x_i)} \leq \sum_{i=0}^{l-1} \sqrt{\hat{F}_l(x_i)}$, we get $i_1 \neq i_2$. So $i_1 < i_2$ and $Q(x_{i_1}) < Q(x_{i_2})$ by the monotonicity of CDF functions. Now consider another CDF function Q' such that $Q'(x_i) = Q(x_i)$ for all $i \neq i_1 \wedge i \neq i_2$, $Q'(x_{i_1}) = Q(x_{i_1}) - \epsilon$ and $Q'(x_{i_2}) = Q(x_{i_2}) + \epsilon$ where $\epsilon = \min\{Q(x_{i_1}), (x_{i_2}+A)/v - Q(x_{i_2})\}$. Figure 1(b) shows how we modify Q to Q'. It is easy to check $Q' \in \mathcal{Q}$ and $\sum_{i=0}^{l-1} \sqrt{Q(x_i)} > \sum_{i=0}^{l-1} \sqrt{Q'(x_i)}$ which contradicts the optimality of Q. The inequality holds because of $\sqrt{a} + \sqrt{b} > \sqrt{a-c} + \sqrt{b+c}$ for all $0 < c < a < b$, which can be proved by simple calculations. $\square$

D Missing Parts of Section 4

Definition 36. (Isomorphism [19]). *Let Γ_1 and Γ_2 be games in strategic form with the same set of players $[n]$. For $k = 1, 2$, let $(A_k^i)_{i \in [n]}$ be the strategy sets in Γ_k, and let $(u_k^i)_{i \in [n]}$ be the utility functions in Γ_k. We say that Γ_1 and Γ_2 are isomorphic if there exists bijections $\phi^i : a_1^i \to a_2^i$, $i \in [n]$ such that for every $i \in [n]$ and every $(a^1, a^2, \ldots, a^n) \in \times_{i \in [n]} A_1^i$,*

$$u_1^i(a^1, a^2, \ldots, a^n) = u_2^i(\phi^1(a^1), \phi^2(a^2), \ldots, \phi^n(a^n)) \ .$$

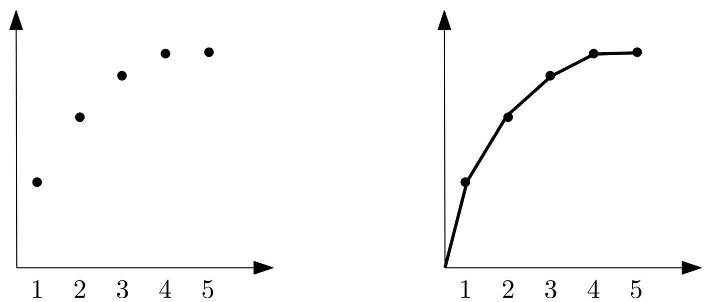

Fig. 2. The left part of the figure depicts some submodular function f, while the right part depicts the modified concave function g. One can verify that g is concave if f is submodular.

Proof of Lemma 12: For any valuation profile $\mathbf{v}$ and (randomized) allocation $\mathcal{A}$, we denote by $SW_{\mathbf{v}}(\mathcal{A})$ the social welfare of allocation $\mathcal{A}$ under the valuations $\mathbf{v}$. For any fractional allocation $\mathbf{x} = (x_1, \ldots, x_n)$, such that $\sum_i x_i = m$, let $\mathbf{X}(\mathbf{x}) = (X_1(\mathbf{x}), \ldots, X_n(\mathbf{x}))$ be the random allocation as computed by the Random PSAM given the fractional allocation $\mathbf{x}$. Also let $\mathbf{o} = (o_1, \ldots, o_n)$ and $\mathbf{O} = (O_1, \ldots, O_n)$ be the optimal allocations in the divisible resource allocation problem and in the multi-unit auction, respectively.

First we show that $SW_{\mathbf{g}}(\mathbf{o}) \geq SW_{\mathbf{f}}(\mathbf{O})$. Consider the fractional allocation $\mathbf{o}' = (o'_1, \ldots, o'_n)$, where $o'_i = O_i/m$, for every i. Then it is easy to see that for every i, $g_i(o'_i) = f_i(\lfloor O_i \rfloor) + (O_i - \lfloor O_i \rfloor) \cdot (f_i(\lfloor O_i \rfloor + 1) - f_i(\lfloor O_i \rfloor)) = f_i(O_i)$, since O_i is an integer. Therefore, $SW_{\mathbf{g}}(\mathbf{o}) \geq SW_{\mathbf{g}}(\mathbf{o}') = SW_{\mathbf{f}}(\mathbf{O})$, by the optimality of $\mathbf{o}$.

Now we show $SW_{\mathbf{f}}(\mathbf{O}) \geq SW_{\mathbf{g}}(\mathbf{o})$. Note that for any fractional allocation $\mathbf{x}$, such that $\sum_j x_j = m$, $\mathbb{E}_{\mathbf{X}(\mathbf{x})}[f_i(X_i(\mathbf{x}))] = f_i(\lfloor x_i \rfloor) + (x_i - \lfloor x_i \rfloor) \cdot (f_i(\lfloor x_i \rfloor + 1) - f_i(\lfloor x_i \rfloor)) = g_i(x_i/m)$, for every i. By the optimality of $\mathbf{O}$, $SW_{\mathbf{f}}(\mathbf{O}) \geq \mathbb{E}_{\mathbf{X}(m\cdot\mathbf{o})}[SW_{\mathbf{f}}(\mathbf{X}(m \cdot \mathbf{o}))] = SW_{\mathbf{g}}(\mathbf{o})$. □

E Missing Proofs of Section 5

Here we give the proofs of theorems and lemmas of Sect. 5. Throughout this section we assume that the players are ordered based on decreasing order of their valuations, i.e. $v_1 \geq v_2 \geq \ldots \geq v_n$. We also drop the word expected while referring to the revenue or to the maximum bid.

Proof of Theorem 14:

Upper bound: Based on the results of [1], inefficient Nash equilibria only exist when players' valuations are in the form $v_1 > v_2 = ... = v_k > v_{k+1} \geq ... \geq v_n$ (with $v_2 > 0$), where players $k+1$ through n bid zero with probability 1. W.l.o.g., we assume that $v_1 = 1$ and $v_i = v > 0$, for $2 \leq i \leq k$. Let P_1 be the probability that bidder 1 gets the item in any such mixed Nash equilibrium denoted by $\mathbf{B}$. Then the expected utility of bidder 1 in $\mathbf{b} \sim \mathbf{B}$ can be expressed by $\mathbb{E}[u_1(\mathbf{b})] = P_1 \cdot 1 - \mathbb{E}[b_1]$. Based on the characterization in [1], no player would bid above v in any Nash equilibrium and nobody bids exactly v with positive probability. Therefore, if player 1 deviates to v, she will gets the item with probability 1. By the definition of Nash equilibrium, we have $\mathbb{E}[u_1(\mathbf{b})] \geq \mathbb{E}[u_1(v, \mathbf{b}_{-i})] = 1 - v$, resulting in $P_1 \geq 1 - v + \mathbb{E}[b_1]$.

It has been shown in the proof of Theorem $2C$ in [1], that $\mathbb{E}[b_1]$ is minimized when players 2 through k play symmetric strategies. Following their results, we can extract the following equations (for a specific player i):

$$G_1(x) = \frac{x}{v \prod_{i' \neq 1, i} G_{i'}(x)}, \quad \forall x \in (0, v] \ ; \qquad \prod_{i' \neq 1} G_{i'}(x) = 1 - v + x, \quad \forall x \in (0, v] \ .$$

Recall that $G_{i'}(x)$ is the CDF according to which player i' bids in $\mathbf{B}$. Since players 2 through k play symmetric strategies, $G_{i'}(x)$ should be identical for $i' \neq 1$. Then, for some $i' \neq 1$,

$$G_1(x) = \frac{x}{v \cdot G_{i'}^{k-2}(x)} = \frac{x}{v \cdot (1 - v + x)^{\frac{k-2}{k-1}}} \ , \qquad \forall x \in (0, v] \ .$$

Note that $1 - v + x \leq 1$, and so we get $G_1(x) \leq \frac{x}{v(1-v+x)}$ (for two players, $G_1(x) = \frac{x}{v}$) and

$$\mathbb{E}[b_1] \geq \int_0^v \left(1 - \frac{x}{v\,(1 - v + x)}\right) dx = v - 1 - \frac{(1-v)\ln(1-v)}{v} \ .$$

Now we can derive that $P_1 \geq \frac{1-v}{v} \ln \frac{1}{1-v}$.
For two players, $\mathbb{E}[b_1] = \int_0^v (1 - x/v)\, dx = v/2$ and so $P_1 = 1 - v/2$.

The expected social welfare in $\mathbf{B}$ is $\mathbb{E}[SW(b)] \geq P_1 + (1-P_1)v \geq \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v$. The expression, $T(v) = \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v$, is minimized for $v \approx 0.5694$ and therefore, the price of anarchy is at most $T(0.5694) \approx 1.185$. Particularly, for two players, $\mathbb{E}[SW(b)] \geq 1 - v/2 + v^2/2$, which is minimized for $v = 1/2$ and therefore the price of anarchy for two players is at most $8/7$.

Lower bound: Consider n players, with valuations $v_1 = 1$ and $v_i = v > 0$, for $2 \leq i \leq n$. Let $\mathbf{B}$ be the Nash equilibrium, where bidders bid according to the following CDFs,

$$G_1(x) = \frac{x}{v\,(1-v+x)^{\frac{n-2}{n-1}}} \;\; x \in [0,v] \;\; ; \;\; G_i(x) = (1-v+x)^{\frac{1}{n-1}} \;\; x \in [0,v],\; i \neq 1 \;\; .$$

Note that $F_i(x) = \prod_{i' \neq i} G_{i'}(x)$ is the probability of bidder i getting the item when she bids x, for every bidder i.

$$F_1(x) = (1-v+x) \qquad x \in [0,v] \;\; ; \qquad\qquad F_i(x) = \frac{x}{v} \qquad x \in [0,v], \quad i \neq 1 \;\; .$$

If player 1 bids any value $x \in [0,v]$, her utility is $u_1 = F_1(x) \cdot 1 - x = 1 - v$. Bidding greater than v is dominated by bidding v. If any player $i \neq 1$ bids any value $x \in [0,v]$, her utility is $u_i = F_i(x) \cdot v - x = 0$. Bidding greater than v results in negative utility. Hence, $\mathbf{B}$ is a Nash equilibrium. Let P_1 be the probability that bidder 1 gets the item in $\mathbf{B}$, then

$$\mathbb{E}[SW(b)] = 1 \cdot P_1 + (1-P_1)v = v + (1-v)P_1 = v + (1-v)\int_0^v G_i^{n-1}(x) dG_1(x) \;\; .$$

When n goes to infinity, $\mathbb{E}[SW(b)]$ converges to $v + (1-v)\int_0^v \frac{1-v}{v(1-v+x)} dx = v + (1-v)\frac{1-v}{v} \ln \frac{1}{1-v} = \frac{(1-v)^2}{v} \ln \frac{1}{1-v} + v = T(v)$. If we set $v = 0.5694$, the price of anarchy is at least $T(v) \approx 1.185$.
For $n = 2$, $\mathbb{E}[SW(b)] = v + (1-v)\int_0^v \frac{1-v+x}{v} = v + (1-v)(1-v/2) = 1 - v/2 + v^2/2$, which for $v = 1/2$ results in price of anarchy at least $8/7$. □

Proof of Theorem 15: Let k be any integer greater or equal to 2, such that $v_1 \geq v_2 = \ldots = v_k \geq v_{k+1} \geq \ldots \geq v_n$. Let $F(x) = \prod_i G_i(x)$ be the CDF of the maximum bid h. By the characterization of [1], in any mixed Nash equilibrium, players with valuation less than v_2 do not participate (always bid zero) and there exist two players $1, i$ bidding continuously in the interval $[0, v_2]$. Then, by [1], $F_1 = (v_1 - v_2 + x)/v_1$ and $F_i(x) = x/v_2$, for any $x \in (0, v_2]$. Therefore, we get

$$F(x) = F_i(x) G_i(x) = \frac{x}{v_2} G_i(x) \;\; .$$

In the proof of Theorem 2C in [1], it is argued that $G_{i_1}(x)$ is maximized (and therefore the maximum bid is minimized) when all the k players play symmetrically (except for the first player, in the case that $v_1 > v_2$). So, $F(x)$ is maximized for $G_i = \left(\prod_{i' \neq 1} G_{i'}\right)^{\frac{1}{k-1}} = F_1^{\frac{1}{k-1}} = \left(\frac{v_1 - v_2 + x}{v_1}\right)^{\frac{1}{k-1}}$. Finally we get

$$
\begin{aligned}
E[h] &= \int_0^\infty (1 - F(x))dx \geq \int_0^{v_2} \left(1 - \frac{x}{v_2}\left(\frac{v_1 - v_2 + x}{v_1}\right)^{\frac{1}{k-1}}\right) dx \\
&\geq v_2 - \int_0^{v_2} \frac{x}{v_2} dx = \frac{1}{2} v_2 \ .
\end{aligned}
$$

The same lower bound also holds for the revenue, which is at least as high as the maximum bid. This lower bound is tight for the maximum bid, as indicated by our analysis, when k goes to infinity and for the symmetric mixed Nash equilibrium. In the next lemma, we show that this lower bound is also tight for the revenue. □

Proof of Lemma 16: In [1], the authors provide results for the revenue in all possible equilibria. For the case that $v_1 = v_2$, the revenue is always equal to v_2. To show a tight lower bound, we consider the case where $v_1 > v_2$ and there exist k players with valuation v_2 playing symmetrically in the equilibrium, by letting k go to infinity. For this case, based on [1], the revenue is equal to[3]

$$
\sum_i \mathbb{E}[b_i] = v^2 + (1 - v)\,\mathbb{E}[b_1] \ ,
$$

where $\mathbb{E}[b_1] = \int_0^v (1 - G_1(x))\,dx$. From the proof of Theorem 15 we can derive that $G_1(x) = F(x)/F_1(x) = \frac{x}{v}(1 - v + x)^{\frac{1}{k-1} - 1} = \frac{x}{v}(1 - v + x)^{-1}$, when k goes to infinity. By substituting we get,

$$
\begin{aligned}
\sum_i \mathbb{E}[b_i] &= v^2 + (1 - v) \int_0^v \left(1 - \frac{x}{v}(1 - v + x)^{-1}\right) dx \\
&= v^2 + (1 - v)\left(v - \frac{1}{v}\left(v + (1 - v)\ln(1 - v)\right)\right) \\
&= 2v - 1 - \frac{(1 - v)^2}{v} \ln(1 - v) \\
&= v - (1 - v)\left(1 + \frac{1 - v}{v}\ln(1 - v)\right) \ .
\end{aligned}
$$

By taking limits, we finally derive that $\lim_{v \to 0}\left(\frac{\sum_i \mathbb{E}[b_i]}{v}\right) = 1/2$. The same tightness result also holds for the maximum bid, which is at most the same as the revenue. □

Proof of Theorem 17: We will assert the statement of the theorem for the valuation profile $(1, v, 0, 0, \ldots, 0)$, where $v \in (0, 1)$ is the second highest value. It is safe to assume that $q_2 \in [0, q_1)$ [4]. We show that the following bidding profile is

[3] For simplicity we assume $v_1 = 1$ and $v_2 = v$.

[4] Otherwise, consider the tie-breaking rule that allocates the item equiprobably. Then for $q_2 \geq q_1$, the strategy profile where all players bid zero is strictly dominant.

a mixed Nash equilibrium. The first two bidders bid on the interval $[0, v(q_1 - q_2)]$ and the other bidders bid 0. The CDF of bidder 1's bid is $G_1(x) = \frac{x}{v(q_1-q_2)}$ and the CDF of bidder 2's bid is $G_2(x) = x/(q_1 - q_2) + 1 - v$. It can be checked that this is a mixed Nash equilibrium by the following calculations. For every bid $x \in [0, v(q_1 - q_2)]$,

$$u_1(x) = G_2(x) \cdot q_1 + (1 - G_2(x)) \cdot q_2 - x = q_1 - v(q_1 - q_2) \ ,$$

$$u_2(x) = G_1(x) \cdot q_1 v + (1 - G_1(x)) \cdot q_2 v - x = q_2 v \ .$$

The revenue is

$$\begin{aligned}
&\int_0^{v(q_1-q_2)} (1 - G_1(x))dx + \int_0^{v(q_1-q_2)} (1 - G_2(x))dx \\
=&\int_0^{v(q_1-q_2)} \left(1 - \frac{x}{v(q_1 - q_2)}\right) dx + \int_0^{v(q_1-q_2)} \left(1 - \left(\frac{x}{q_1 - q_2} + 1 - v\right)\right) dx \\
=&\frac{v(q_1 - q_2)}{2} + \frac{v^2(q_1 - q_2)}{2} \ .
\end{aligned}$$

When v goes to 0, the revenue go to $v(q_1 - q_2)/2 < v/2$ since $q_1 - q_2 < 1$. Obviously, the same happens with the maximum bid, which is at most the same as the revenue. □

F Conventional Procurement

In this section we give bounds on the expected revenue and maximum bid of the single-item first-price auction. In the following, we drop the word expected while referring to the revenue or to the maximum bid.

Theorem 37. *In any mixed Nash equilibrium, the revenue and the maximum bid lie between the two highest valuations. There further exists a tie-breaking rule, such that in the worst-case, these quantities match the second highest valuation (This can also be achieved, under the no-overbidding assumption).*

Lemma 38. *In any mixed Nash equilibrium, if the expected utility of any player i with valuation v_i is 0, then with probability 1 the maximum bid is at least v_i.*

Proof. Consider any mixed Nash equilibrium $\mathbf{b} \sim \mathbf{B}$ and let $h = \max_i\{b_i\}$ be the highest bid; h is a random variable induced by $\mathbf{B}$. For the sake of contradiction, assume that h is *strictly* less than v_i with probability $p > 0$. Then, there exists $\varepsilon > 0$ such that $h < v_i - \varepsilon$ with probability p. Consider now the deviation of player i to pure strategy $s_i = v_i - \varepsilon$. s_i would be the maximum bid with probability p and therefore the utility of player i would be at least $p(v_i - (v_i - \varepsilon)) = p \cdot \varepsilon > 0$. This contradicts the fact that $\mathbf{B}$ is an equilibrium and completes the proof of lemma. □

Lemma 39. *In any mixed Nash equilibrium, if v is the highest valuation, any player with valuation strictly less than v has expected utility equal to 0.*

Proof. In [8] (Theorem 5.4), they proved that the price of anarchy of mixed Nash equilibria, for the single-item first-price auction, is exactly 1. This means that the player(s) with the highest valuation gets the item with probability 1. Therefore, any player with valuation strictly less than v gets the item with zero probability and hence, her expected utility is 0. $\square$

Consider the players ordered based on their valuations so that $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$. In order to prove Theorem 37, we distinguish between two cases: i) $v_1 > v_2$ and ii) $v_1 = v_2$.

Lemma 40. *If $v_1 > v_2$, the maximum bid of any mixed Nash equilibrium, is at least v_2 and at most v_1. If we further assume no-overbidding, the maximum bid is exactly v_2.*

Proof. If $v_1 > v_2$, by Lemma 39, the expected utility of player 2 equals 0. From Lemma 38, the highest bid is at least v_2 with probability 1. Moreover, if there exist players bidding above v_1 with positive probability, then at least one of them (whoever gets the item with positive probability) would have negative utility for that bid and would prefer to deviate to 0; so, the bidding profile couldn't be an equilibrium. Therefore, the maximum bid lies between v_1 and v_2.

If we further assume no-overbidding, nobody, apart from player 1, would bid above v_2. So, the same hold for player 1, who has an incentive to bid arbitrarily close to v_2. $\square$

Corollary 41. *If $v_1 > v_2$, there exists a tie breaking rule, under which the maximum bid of the worst-case mixed Nash equilibrium is exactly v_2.*

Proof. Due to Lemma 40, it is sufficient to show a tie breaking rule, where there exists a mixed Nash equilibrium with highest bid equal to v_2. Consider the tie-breaking rule where, in a case of a tie with player 1 (the bidder of the highest valuation), the item is always allocated to player 1. Under this tie-breaking rule, the pure strategy profile, where everybody bids v_2 is obviously a pure Nash equilibrium, with v_2 being the maximum bid. $\square$

Lemma 42. *If $v_1 = v_2$, the maximum bid of any mixed Nash equilibrium, equals v_2.*

Proof. Consider a set S of $k \geq 2$ players having the same valuation $v_1 = v_2 = \ldots = v_k = v$ and the rest having a valuation strictly less than v. For any mixed Nash equilibrium $\mathbf{b} \sim \mathbf{B}$ and any player i, let G_i and F_i be the CDFs of b_i and $\max_{i' \neq i} b_{i'}$, respectively. We define $l_i = \inf\{x | G_i(x) > 0\}$ to be the infimum value of player's i support in $\mathbf{B}$. We would like to prove that $\max_i l_i = v$. For the sake of contradiction, assume that $\max_i l_i < v$ (Assumption 1).

We next prove that, under Assumption 1, $l_i = l$ for any player $i \in S$ and for some $0 \leq l < v$. We will assume that $l_j < l_i$ for some players $i, j \in S$ (Assumption 2) and we will show that Assumption 2 contradicts Assumption 1. There exists $\varepsilon > 0$ such that $l_j + \varepsilon < l_i$. Moreover, based on the definition of l_j, for any $\varepsilon' > 0$, $G_j(l_j + \varepsilon') > 0$ and so $G_j(l_j + \varepsilon) > 0$. When player's j bid is derived by the interval $[l_j, l_j + \varepsilon]$, she receives the item with zero probability, since $l_i > l_j + \varepsilon$. Therefore, for any bid of her support that is at most $l_j + \varepsilon$, her utility is zero ($G_j(l_j + \varepsilon) > 0$, so there should be such a bid). Since $\mathbf{B}$ is a mixed Nash equilibrium, her total expected utility should also be zero. In that case, Lemma 38 contradicts Assumption 1, and therefore Assumption 2 cannot be true (under Assumption 1). Thus, for any player $i \in S$, $l_i = l$ for some $0 \leq l < v$.

Moreover, Lemma 39 indicates that no player $i \notin S$ bids above l with positive probability, i.e. $G_i(l) = 1$ for all $i \notin S$. We now show that for any $i \in S$, G_i cannot have a mass point at l, i.e. $G_i(l) = 0$ for all $i \in S$.

Case 1. If $G_i(l) > 0$ for all i, then $p = \prod_i G_i(l) > 0$ is the probability that the highest bid is l, or more precisely, it is the probability that all players in S bid l and a tie occurs. Given that this event occurs, there exists a player $j \in S$ that gets the item with probability p_j strictly less than 1 (this is the conditional probability). Therefore, player j has an incentive to deviate from l to $l + \varepsilon$, for $\varepsilon < (1 - p_j)(v - l)$ (so that $p_j(v - l) < v - (l + \varepsilon)$); this contradicts the fact that $\mathbf{B}$ is an equilibrium.

Case 2. If $G_i(l) > 0$ and $G_j(l) = 0$ for some $i, j \in S$, then l is in the support of player i, but she does never receives the item when she bids l, since player j bids above l with probability 1. Therefore, the expected utility of player i is 0 and due to Lemma 38 this cannot happen under Assumption 1.

Overall, we have proved so far that, under Assumption 1 (that now has become $l < v$), $G_i(l) = 0$ for all $i \in S$ and $G_i(l) = 1$ for all $i \notin S$. Since $k \geq 2$, $F_i(l) = \prod_{i' \neq i} G_{i'}(l) = 0$ for all i. Consider any player $i \in S$ and let u_i be her expected utility. Based on the definition of l_i, for any $\varepsilon > 0$, there exists $x(\varepsilon) \in [l, l + \varepsilon]$, such that $x(\varepsilon)$ is in the support of player i. Therefore, $u_i \leq F_i(x(\varepsilon))(v - x(\varepsilon)) \leq F_i(l + \varepsilon)(v - l)$. As F_i is a CDF, it should be right-continuous and so for any $\delta > 0$, there exists some $\varepsilon > 0$, such that $F_i(l + \varepsilon)(v - l) < \delta$ and therefore, $u_i < \delta$. We can contradict Assumption 1, right away by using Lemma 38, but we give a bit more explanation. Assume that, in $\mathbf{B}$, the maximum bid h is strictly less than v with probability $p > 0$. Then, there exists some $\varepsilon' > 0$, such that $h < v - \varepsilon'$ with probability p. If we consider any $\delta < p(v - \varepsilon')$, it is straight forward to see that player i has an incentive to deviate to the pure strategy $v - \varepsilon'$. Therefore, we showed that Assumption 1 cannot hold and so the highest bid is at least v with probability 1. Similar to the proof of Lemma 40, nobody will bid above v in any mixed Nash equilibrium. $\square$