A basis for the Birman-Wenzl algebra

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Abstract

This paper provides an explicit isomorphism between the *Birman-Wenzl algebra* BW_n, constructed by J. Birman and H. Wenzl, and the *Kauffman algebra* MT_n, subsequently constructed by H.R. Morton and P. Traczyk. The Birman-Wenzl algebra is defined algebraically using generators and relations while the Kauffman algebra has a geometric formulation in terms of tangles. The isomorphism is obtained by constructing an explicit basis in BW_n, analogous to a basis previously constructed for MT_n using 'Brauer connectors'. The geometric isotopy arguments used for MT_n are systematically replaced by algebraic versions using the Birman-Wenzl relations. This not only gives a direct way of determining the dimension of the Birman-Wenzl algebra, but also clarifies the role played by the ring of coefficients, Λ, and its specialisations.

Foreword

This paper is a very lightly edited version of an article originally written in 1989 but never fully completed. It was intended to be a joint paper with A.J.Wassermann. He had planned to write a final section to make use of the ability to change at will between the Birman-Wenzl algebra, as given by generators and relations, and the geometric framework of the tangles, so as to look in more detail at the representation theory.

One goal of our original approach was to make sure that specialisations of the coefficient ring could be handled confidently, and that the translations to and from the tangle context were on a sound footing.

Subsequently others have made progress in this way, in works such as [6], but we had a number of requests for our earlier account, and so I put it into this more accessible form on the Liverpool knot theory pages in 2000.

In order to have a more permanent place for it I have now put it on ArXiv. I had hoped to make this a joint submission, but I have been unable to re-establish contact with Wassermann to get his formal agreement. The present paper is largely the result of our joint discussions, although I take responsibility for the eventual content and exposition.

1 Introduction

In recent years there has been considerable interest in deformations of the classical 'centraliser algebras' of Schur, Weyl and Brauer. These play an important role in several areas, including exactly solvable models in statistical mechanics, quantum groups, von Neumann algebras and knot theory. It has long been recognized that these links are more than tenuous and if properly exploited lead to fruitful interchanges between the different disciplines. The first and most spectacular instance of this was of course Vaughan Jones' pioneering work on subfactors, which led to his discovery of new link invariants. Subsequently these invariants were understood in terms of solutions of the quantum Yang-Baxter equation and vertex models. The central thread running through all these topics is the quantum group obtained by deforming the universal enveloping algebra of the unitary group. One has also to deform the centraliser algebras, which amounts to replacing the group algebra of the symmetric group by the Hecke algebra (of type A).

After these discoveries, somewhat curiously history took a reverse turn. Kauffman discovered new link invariants, based on a purely skein theoretic characterisation of Jones' original invariants. It was natural to ask whether Kauffman's invariants could be obtained by algebraic means. This led Birman and Wenzl to introduce a deformation of an abstract algebra first introduced by R. Brauer. An alternative knot-theoretic approach to deforming Brauer's algebra was later given in terms of tangles by Morton-Traczyk and by Kauffman himself. The original algebra of Brauer bore the same relation to the orthogonal group as the group algebra of the symmetric group did to the unitary group.

By exploiting the new insights provided by Vaughan Jones' work on subfactors (in particular his 'basic construction'), Wenzl was able to acquire a fuller understanding of Brauer's algebra, and resolve some old questions on semisimplicity raised by Brauer and Weyl. Subsequent studies have shown that the Birman-Wenzl algebra does indeed provide the correct analogue of the Hecke algebra for the quantum group corresponding to the orthogonal group. Most recently Wenzl has been able to construct new examples of subfactors using these algebras as a substitute for the Hecke algebras.

In this paper we provide an explicit isomorphism between the *Birman-*

Wenzl algebra BW_n, constructed by J. Birman and H. Wenzl in [1], and the *Kauffman algebra* MT_n, subsequently constructed by the author and P. Traczyk in [11]. The Birman-Wenzl algebra is defined algebraically using generators and relations while the Kauffman algebra has a geometric formulation in terms of tangles. We obtain this isomorphism by constructing an explicit basis in BW_n, analogous to a basis previously constructed for MT_n using 'Brauer connectors'. The geometric isotopy arguments used in [11] are systematically replaced by algebraic versions using the Birman-Wenzl relations. This not only gives a direct way of determining the dimension of the Birman-Wenzl algebra, but also clarifies the role played by the ring of coefficients, Λ, an integral domain.

In fact, in [1] the authors prefer to consider the algebra $BW_n \otimes_\Lambda k$, where k is the field of fractions of Λ. This enables them to imitate V. Jones' basic construction and thus determine the structure of the algebra. At a crucial point in proof of their main result (theorem 3.7) they need to use a specialisation of Λ. Since the algebra is defined by generators and relations over Λ, any such specialisation automatically extends to BW_n although not necessarily to the algebra $BW_n \otimes_\Lambda k$. This difficulty can be overcome by observing that the existence of a basis implies that BW_n is free as a module over Λ. The arguments presented in [2] p.55 to prove that the Hecke algebra is generically semisimple may then be adapted to prove the same result for BW_n, i.e. the specialisations of BW_n are semisimple for a Zariski open subset of the parameter space $\mathrm{Spec}(\Lambda)$. Wenzl has carried out a more detailed analysis, based on Jones' basic construction, in order to determine precisely when the algebras fail to be semisimple.

This paper is divided into six sections. In section 2 we review the definitions of the algebras to be studied, with some historical comments. In section 3 we use the basic solution of the Yang-Baxter equation for the orthogonal group together with a simple skein-theoretic argument to provide a short self-contained definition of Kauffman's two-variable link invariant. We also briefly discuss the duality between the quantum orthogonal group and the Birman-Wenzl algebra. In section 4 we give an inductive definition of a basis for the Birman-Wenzl algebra and outline the more formal aspects of the proof. The inductive procedure relies on a natural filtration analogous to the one extensively used by Hanlon and Wales in their studies of Brauer's algebra [7]. Effectively the proof that the proposed basis is a spanning set is achieved by a double induction, which from the point of view of tangles depends both on the number of strings and then on the number of 'through' strings. The remaining two sections are devoted to various stages of the inductive argument showing that the natural surjective maps from the Birman-Wenzl algebras to the tangle algebras are isomorphisms. In section 5 we treat the case in which there are no 'through' strings: a complete understanding of this case is crucial for the subsequent reasoning since it allows us to use geometry in place of algebra in a controlled way. Finally in section 6 we perform the main step of the induction.

2 Three algebras

2.1 The Birman-Wenzl algebra

We start by recalling the definition of the Birman-Wenzl algebra. We have made a slight change by the introduction of some minus signs, in accordance with Kauffman's 'Dubrovnik' version of his link invariant. As explained in [11] and below, this makes it much easier to see the connection with Brauer's centraliser algebras.

Let Λ be the quotient ring $\mathbf{Z}[\lambda^{\pm 1}, z, \delta]/ < \lambda^{-1} - \lambda - z(\delta - 1) >$. Thus Λ (or more accurately its complexification) is the coordinate ring of the irreducible quasiprojective variety defined by $\lambda \neq 0, \lambda^{-1} - \lambda = z(\delta - 1)$ in $\mathbf{A}^3$.

Definition. The *Birman-Wenzl algebra* BW_n is the quotient of the free algebra over Λ with generators $g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_{n-1}^{\pm 1}$ and $e_1, e_2, \ldots, e_{n-1}$ modulo the ideal generated by the relations:
(1) (Kauffman skein relation) $g_i - g_i^{-1} = z(1 - e_i)$.
(2) (Idempotent relation) $e_i^2 = \delta e_i$.
(3) (Braid relations) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| > 1$.
(4) (Tangle relations) $e_i e_{i \pm 1} e_i = e_i$ and $g_i g_{i \pm 1} e_i = e_{i \pm 1} e_i$.
(5) (Delooping relations) $g_i e_i = e_i g_i = \lambda e_i$ and $e_i g_{i \pm 1} e_i = \lambda^{-1} e_i$.

Remark. If z is taken to be invertible then the idempotent relation follows from the delooping and skein relations.

In Birman and Wenzl's original version several of their relations could be omitted without loss, given invertibility of z. They use v in place of λ in the coefficient ring.

The presentation given here is intended to be sufficiently symmetric to allow for easy comparison with the tangle algebra, while maintaining the coefficient ring Λ as in [11].

2.2 Kauffman's tangle algebra

Definition. An (m, n)-tangle is a piece of knot diagram in a rectangle R in the plane, consisting of arcs and closed curves, so that the end points of the arcs consist of m points at the top of the rectangle and n points at the bottom, in some standard position.

An example of a $(4, 2)$-tangle is shown in figure 2.1.

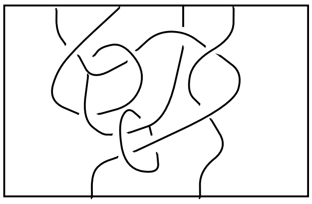

Figure 2.1

Definition. Two tangles are *ambient isotopic* if they are related by a sequence of Reidemeister's moves I, II and III, (see figure 2.2), together with isotopies of R fixing its boundary.

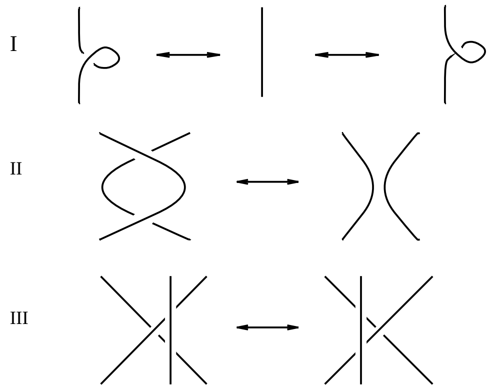

Figure 2.2

They are *regularly isotopic* if Reidemeister move I is not used.

Notation. Write $\mathcal{U}_n^m$ for the set of (m, n)-tangles up to regular isotopy.

The set $\mathcal{U}_n^n$ admits an *associative multiplication*, defined by placing representative tangles one below the other.

A well-known subset B_n consists of geometric braids, in this context represented by tangles (necessarily without closed components) where the height coordinate in R increases monotonically on each component. It can be shown that B_n is the full group of units in $\mathcal{U}_n^n$ under the multiplication.

The *closure*, $\hat{T}$, of an (n,n)-tangle T, is defined, by analogy with the closure of a braid, to be the link diagram (or $(0,0)$-tangle) given from T by joining the points on the top of R to those on the bottom by arcs lying outside R with no further crossings.

We define a closure map $\varepsilon : \mathcal{U}_n^n \to \mathcal{U}_0^0$, by $\varepsilon(T) = \hat{T}$.

From $\mathcal{U}_n^n$ we construct the algebra MT_n, which we call Kauffman's tangle algebra, as an algebra over a ring Λ, as in [11]. We shall take Λ to be the ring

$$\Lambda = \mathbf{Z}[\lambda^{\pm 1}, z, \delta]/ < \lambda^{-1} - \lambda = z(\delta - 1) > .$$

Then Λ is isomorphic to a subring of $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$, by taking $\delta = 1 + (\lambda^{-1} - \lambda)/z$. It admits a homomorphism $e : \Lambda \to \mathbf{Z}[\delta]$ with $e(z) = 0$, $e(\lambda) = 1$ and $e(\delta) = \delta$. The main aim of this paper is to show that the Birman-Wenzl algebra BW_n is isomorphic to MT_n, on specialisation of coefficients.

Certain features of MT_n, for example its dimension, and its relation to Brauer's algebra [3], appear here very simply, using the homomorphism e and the Dubrovnik invariant $\mathcal{D}$. These features of BW_n, not proved directly in the original approach, then follow at once.

Definition. *Kauffman's tangle algebra*, MT_n, is the Λ-module, constructed from $\Lambda[\mathcal{U}_n^n]$ by factoring out three sets of relations:

$$T^+ - T^- = z(T^0 - T^\infty), \tag{1}$$

where $T^\pm, T^0, T^\infty$ are represented by tangles differing only as in figure 2.3,

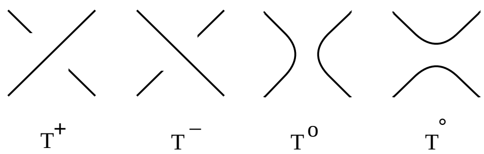

Figure 2.3

$$T^{\text{right}} = \lambda^{-1} T,\ T^{\text{left}} = \lambda T, \tag{2}$$

where T^{right} and T^{left} are given from T by adding a left or right hand curl as in figure 2.4,

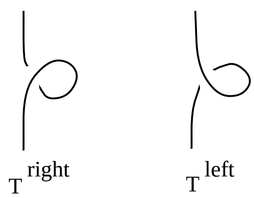

Figure 2.4

$$T \amalg O = \delta T, \tag{3}$$

where $T \amalg O$ is the union of T and a circle having no crossings with T or itself.

Proposition 2.1 *Composition of tangles induces a Λ-bilinear multiplication on MT_n making MT_n an algebra over Λ.*

Proof : The relations (1)-(3) carry down under the multiplication in $\Lambda[\mathcal{U}_n^n]$. □

Proposition 2.2 *The map ε induces a Λ-linear map $\varepsilon : MT_n \to MT_0$.*

We now give the homomorphism $\varphi : BW_n \to MT_n$ which provided the intuition behind Birman and Wenzl's description of BW_n.

Definition. Write G_i, E_i respectively for the tangles in $\mathcal{U}_n^n$ illustrated in figure 2.5. Use the same letters for the elements represented by these tangles in MT_n, called s_i, h_i in [11].

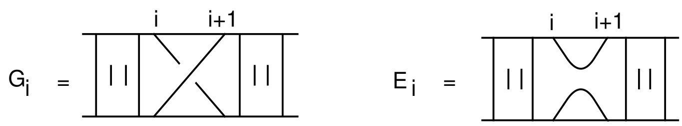

Figure 2.5

Then

$$G_i - G_i^{-1} = z(1 - E_i)$$

in MT_n, from relation (1) applied to the only crossing in G_i.

Similarly, relation (2) shows that

$$\begin{aligned} G_i E_i &= E_i G_i = \lambda^{-1} E_i \\ G_i^{-1} E_i &= E_i G_i^{-1} = \lambda E_i \end{aligned}$$

and relation (3) that

$$E_i^2 = \delta E_i.$$

Theorem 2.3 *A homomorphism $\varphi : BW_n \to MT_n$ may be defined by $\varphi(g_i) = G_i,\ \varphi(e_i) = E_i$.*

Proof : The relations in BW_n are respected. We have already noted that the skein relation and delooping relations are satisfied by E_i, G_i in MT_n. The other relations hold even at the level of the tangle semigroup $\mathcal{U}_n^n$. □

Our goal is to prove that φ is an isomorphism for all n. In this section we find explicit spanning sets for MT_n, and show that φ is surjective.

In section 4 we give the proof from [11] that the chosen spanning sets are a free basis for MT_n, using the existence of Kauffman's invariant.

The proof that φ is injective will subsequently be built up in stages, with the recurring pattern of taking spanning sets for selected subspaces of BW_n and proving that they map to independent sets in MT_n.

To save later effort we note here some symmetry of BW_n, which carries over by φ to two natural operations in MT_n.

Definition. (1) Write $\rho_n : BW_n \to BW_n$ for the automorphism defined by

$$\rho_n(g_i) = g_{n-i},\ \rho_n(e_i) = e_{n-i}.$$

(2) Write $\alpha : BW_n \to BW_n$ for the reversing *anti* automorphism defined by

$$\alpha(g_i) = g_i,\ \alpha(e_i) = e_i.$$

Remark. The symmetry of the relations in BW_n ensures that ρ_n, α are well-defined.

Proposition 2.4 *There is an automorphism ρ_n of MT_n, and an antiautomorphism α, with $\varphi \circ \alpha = \alpha \circ \varphi$ and $\varphi \circ \rho_n = \rho_n \circ \varphi$.*

Proof : Write $\rho_n, \alpha : \mathcal{U}_n^n \to \mathcal{U}_n^n$ for the natural symmetries given by rotating a tangle T through π about one of the two axes shown in figure 2.6.

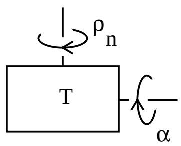

Figure 2.6

Clearly $\alpha(G_i) = G_i,\ \rho_n(G_i) = G_{n-i}$, and similarly for E_i. The skein relations are preserved by ρ_n and α so that they induce $\rho_n, \alpha : MT_n \to MT_n$. Since $\rho_n(ST) = \rho_n(S)\rho_n(T)$ and $\alpha(ST) = \alpha(T)\alpha(S)$ these are respectively an automorphism and an antiautomorphism, satisfying the stated relations on the generators of BW_n. □

We now continue with the proof that MT_n has a finite spanning set, and at the same time we develop the notation to relate these algebras readily with Brauer's centraliser algebras.

2.3 Connectors and Brauer's algebras

An (n,n)-tangle T consists of n arcs and a number, $|T|$, of closed curves. If each arc joins a point at the top to a point at the bottom then the tangle determines a permutation in S_n.

Definition. For a general tangle we extend the idea of a permutation to that of an *n-connector*, defined to be a pairing of $2n$ points into n pairs.

The set C_n of n-connectors has $(2n)!/2^n n!$ elements, the product of the first n odd integers.

Take the set of $2n$ points to be the end points of (n,n)-tangles. The arcs of any $T \in \mathcal{U}_n^n$ pair these end points to give a connector, which we write as $\text{conn}(T) \in C_n$.

Remark. (Brauer's algebra) Brauer [3] uses C_n as the basis for an algebra over $\mathbf{Z}[\delta]$, (writing n in place of δ and f in place of n). He divides the $2n$ points to be connected into two subsets $t_1, \ldots, t_n$ and $b_1, \ldots, b_n$, arranged along the top and bottom of a rectangle, and views a connector c as a set of n intervals with these $2n$ points as endpoints, which join the points paired by c. Two connectors c_1 and c_2 are composed by placing one rectangle above the other, giving n arcs whose endpoints are the new top and bottom points, together with some number $r \geq 0$ of closed curves.

Brauer sets $c_1 c_2 = \delta^r d$, where d is the connector defined by the new arcs. This defines an associative multiplication on $\mathbf{Z}[\delta][C_n] = A_n$ making it an algebra over $\mathbf{Z}[\delta]$, called *Brauer's algebra.*

Having divided the $2n$ points in this way there is a natural embedding $S_n \subset C_n$.

We can modify the map $\text{conn} : \mathcal{U}_n^n \to C_n$ to give a multiplicative homomorphism $c : \mathcal{U}_n^n \to A_n$, which extends to $c : MT_n \to A_n$ as follows.

For $T \in \mathcal{U}_n^n$ set $c(T) = \delta^{|T|}\text{conn}(T) \in A_n$. This can be extended to $c : \Lambda[\mathcal{U}_n^n] \to A_n$ by setting $c(\Sigma\, \lambda_i T_i) = \Sigma\, e(\lambda_i) c(T_i)$, using the ring homomorphism $e : \Lambda \to \mathbf{Z}[\delta]$.

Theorem 2.5 *There is an induced homomorphism $c : MT_n \to A_n$.*

Proof : The relations (1)-(3) defining MT_n are respected. □

Remark. We show later that A_n is isomorphic to the $\mathbf{Z}[\delta]$ algebra $MT_n \otimes_\Lambda \mathbf{Z}[\delta]$ given from MT_n by replacing the coefficients Λ with $\mathbf{Z}[\delta]$, using the homomorphism e.

The existence of $c : MT_n \to A_n$ can be viewed as the consequence of specialising the coefficients so that the relations no longer distinguish under- from over-crossings. Then tangles pass to their projections, retaining only the information of their connectors. The crucial technical feature here is that we can specialise Λ so as to retain δ, while fixing $\lambda = 1$ and $z = 0$. Complications arise if we try to do this while working in the ring $\mathbf{Z}[\lambda^{\pm 1}, z^{\pm 1}]$.

Definition. Given a tangle T, choose a sequence of base-points, consisting firstly of one end point of each arc, and then one point on each closed component. Say that T is *totally descending* (with this choice of base points) if on traversing all the strands of T, starting from the base point of each component in order, each crossing is first met as an overcrossing.

Remark. We shall assume that for each connector $c \in C_n$ a choice of ordering of base-points for the arcs has been made, and we use this same choice for all tangles T with $c = \operatorname{conn} T$. Note that there are $n!\,2^n$ potentially different choices possible for each connector. The precise choice is not material, and we shall have occasion to vary the choice in the course of later proofs. The result will be simply to alter the choice of linear basis in MT_n.

An example of a totally descending $(3, 3)$-tangle is shown in figure 2.7, with base-points numbered according to a choice of order.

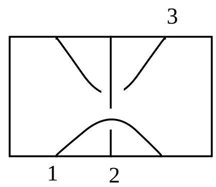

Figure 2.7

Theorem 2.6 *MT_n is spanned by totally descending tangles.*

Proof : Let T be a tangle representing an element of MT_n. Choose base points for T according to the choice for $\operatorname{conn}(T)$. Traverse the arcs of T in order. At the first non-descending crossing use relation (1) with $T = T_\pm$. Note that $\operatorname{conn}(T_+) = \operatorname{conn}(T_-)$, so that $T_\mp$, resulting from T with the crossing switched, has fewer non-descending crossings. Then T is a linear combination of three tangles, two with fewer crossings and one with fewer non-descending crossings. The theorem follows by induction, firstly on the number of crossings, then on the number of non-descending crossings. □

Corollary 2.7 *MT_n is spanned by totally descending tangles without closed components.*

Proof : If T is totally descending, with r closed components, then these components are unknotted curves stacked below the arcs of T. The tangle can then be altered by regular isotopy so that the unknotted components lie well away from the arcs. By using Reidemeister move I as well they can be changed to have no self-crossings. Then by (2) and (3), $T = \lambda^k \delta^r T'$ in MT_n, where T' consists simply of the arcs of T. □

Remark. This result holds as stated for $n = 0$, provided that we admit the 'empty tangle' as an element of $\mathcal{U}_0^0$. In any event MT_0 is spanned by a single element.

Corollary 2.8 *MT_0 is cyclic.*

Theorem 2.9 *Let S and T be totally descending (n, n)-tangles, without closed components, such that conn(S) = conn(T). Then S and T are ambient isotopic, and so $S = \lambda^k T$ in MT_n, for some k.*

Proof : Number the arcs of S and T according to the order of their base points. Since conn(S) = conn(T), the ith arc in each tangle joins the same pair of end points. The arcs can be arranged to lie in disjoint levels 1 to n above the plane of R, since arc i lies above arc j at every crossing when $i < j$. Each individual arc is unknotted, because the tangle is descending, so it can be changed by ambiemt isotopy to an arc without self-crossings in its level. The resulting tangles are then ambient isotopic by level-preserving isotopy. □

Remark. If the arcs of S and T have no self-crossings initially then S and T are regularly isotopic.

Remark. (Construction) For each connector $c \in C_n$, choose an order for the arcs. With this order construct a totally descending tangle with connector c such that any two arcs cross at most once. (Start for example from a diagram of the connector in which any two arcs cross at most once, and make it descending, by choosing the sense of each crossing according to the order of the arcs.) The element $T_c \in \mathcal{U}_n^n$ represented by this tangle then depends only on c and the chosen order, by Theorem 2.9.

Remark. For $c \in S_n$ and a natural choice of order the resulting tangles T_c have been studied, [4, 5], under the name 'positive permutation braids'.

They can be represented by a braid in B_n with positive crossings and permutation c in which any two strings cross at most once.

These braids have also been used in [9, 10], to give easily handled generators for the Hecke algebra H_n.

Theorem 2.10 *MT_n is spanned, for every choice of order, by the finite set $\{T_c\},\ c \in C_n$.*

Proof : By theorem 2.6 and its corollary, MT_n is spanned by tangles which are ambient isotopic to T_c, for various c. By use of relation (2), any tangle ambient isotopic to T_c represents $\lambda^k T_c$ in M_n, for some k. $\Box$

Remark. The number of crossings in a totally descending tangle T_c depends on the connector c, not on the order of arcs used. It is simply the number of pairs of arcs which cross in c, as dictated by whether or not their endpoints interlock on the boundary rectangle.

Clearly any tangle with k crossings can always be written in MT_n as a linear combination of totally descending tangles with at most k crossings, by induction on k, using the procedure of theorem 2.6. It follows that if T'_c, T_c are totally descending tangles with the same connector c, arising from different choices of the order of arcs then

$$T'_c = T_c + \sum_d \lambda_d T_d,$$

where d runs over connectors with fewer crossings than c.

We finish this section by proving:

Theorem 2.11 *The map $\varphi : BW_n \to MT_n$ is surjective.*

Proof : We must show that MT_n is generated by E_i, G_i, $1 \le i \le n-1$. It is enough to show that each totally descending tangle T_c is a monomial in $\{E_i\}$ and $\{G_i^{\pm 1}\}$.

Assuming that the connector c pairs r points at the top with r at the bottom, and connects the remaining $2k = n - r$ points as k pairs, leaving $2k$ points at the bottom connected as k pairs.

We can then draw the tangle T_c (for any order of the arcs) so that there are r arcs running monotonically from top to bottom, k arcs running with a single local minimum from top to top, and k arcs from bottom to bottom with a single local maximum. We can further assume, since the arcs never cross twice, that all the local minima on the top arcs are higher up than the local maxima, so that there are only r arcs passing through the middle part of the rectangle.

Now pair arbitrarily the local maxima and minima, and isotop the tangle so that each local minimum moves down to lie directly above its corresponding maximum. We can now decompose the tangle level by level into a composite of simple tangles in each of which there are n strings all running vertically, except for one pair, which either cross simply, giving $G_i^{\pm 1}$, or form a paired minimum and maximum, giving E_i. An example is shown in figure 2.8.

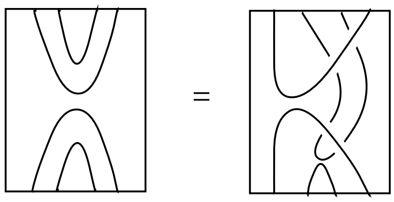

Figure 2.8

□

Remark. It is useful to regard the tangle T_c with r through strings as a composite of an (n, r)-tangle and an (r, n)-tangle, and it suggests that a counterpart of (n, r)-tangles might helpfully be studied in relation to BW_n.

3 Kauffman's link polynomial

In this section we discuss Kauffman's Dubrovnik invariant of links, and its relation to the solutions of the Yang-Baxter equation for the orthogonal group.

Kauffman's polynomial, in its Dubrovnik form, is a non-zero function $\mathcal{D} : \mathcal{U}_0^0 \to \Lambda$, i.e. a function on knot diagrams which is unaltered by *regular* isotopy.

This function $\mathcal{D}$ has the basic properties:

(1) $\mathcal{D}(K^+) - \mathcal{D}(K^-) = z(\mathcal{D}(K^0) - \mathcal{D}(K^\infty))$ (skein relation)

where the diagrams $K^\pm$, K^0 and K^∞ differ only as in figure 2.3, and

(2) $\mathcal{D}(K^{\text{left}}) = \lambda \mathcal{D}(K)$, $\mathcal{D}(K^{\text{right}}) = \lambda^{-1} \mathcal{D}(K)$,

where K^{left} and K^{right} are given from K as in figure 2.4.

It also satisfies

(3) $\mathcal{D}(K \amalg O) = \delta \mathcal{D}(K)$,

where $K \amalg O$ is the union of K and a circle having no crossings with K or with itself, and $\delta \in \Lambda$ satisfies $\lambda^{-1} - \lambda = z(\delta - 1)$.

Proposition 3.1 *Kauffman's invariant exists if and only if the cyclic module MT_o is free.*

Proof : We have shown already that MT_0 is cyclic, so MT_0 is free if and only if there is a non-zero Λ-homomorphism $\varphi : MT_0 \to \Lambda$.

If MT_0 is free then we may define $\mathcal{D}$ on any diagram K by $\mathcal{D}(K) = \varphi(K)$. Conversely, if $\mathcal{D}$ satisfies (1)-(3) then it defines a non-zero Λ-homomorphism $\mathcal{D} : MT_0 \to \Lambda$. □

Remark. (Uniqueness of Kauffman's invariant) It follows simply from section 2 that Kauffman's invariant is unique, because MT_0 is cyclic. It is determined uniquely by its value on O, the diagram of the unknot without any crossings. $\mathcal{D}$ was originally normalised so that $\mathcal{D}(O) = 1$. It now appears more natural to assign the value 1 to the 'empty knot', so that $\mathcal{D}(O) = \delta$.

Kauffman's original proof of the existence of $\mathcal{D}$, [8], requires a considerable amount of combinatorial argument to show that the elements of Λ reached by different routes from a given diagram K are independent of any intermediate choices.

We note here an alternative existence proof, using the Yang-Baxter orthogonal invariants.

Proposition 3.2 *There exists a regular isotopy invariant of knot diagrams in $\mathbf{Z}[s^{\pm 1}]$ which satisfies relations (1)-(3) with $z = s - s^{-1}$, $\lambda = s^{2n-1}$, $\delta = 1 + (\lambda - \lambda^{-1})/z$, and takes the value 1 on the empty knot.*

Proof (Turaev): The invariant is constructed from the q-analogue of the fundamental representation of the Lie algebra of $SO(2n)$. □

For each n we have a ring homomorphism $e_n : \Lambda \to \mathbf{Z}[s^{\pm 1}]$ defined by $e_n(\lambda) = s^{2n-1}$, $e_n(z) = s - s^{-1}$. Turaev's invariant then defines a map $\varphi_n : MT_0 \to \mathbf{Z}[s^{\pm 1}]$ with $\varphi_n(aK) = e_n(a)\varphi_n(K)$ for $a \in \Lambda$.

Proposition 3.3 *MT_0 is a free Λ-module.*

Proof : Suppose not. Then there exists $a \in \Lambda,\ a \neq 0$ such that $aK = 0$, where K is the empty diagram. Now $\varphi_n(K) = 1$ so $0 = \varphi_n(aK) = e_n(a)$ for all n. This is impossible, since for any given $a \neq 0$ there exists n with $e_n(a) \neq 0$. □

This proves the existence of $\mathcal{D}$, given Turaev's invariants. In principle $\mathcal{D}(K)$ could be calculated explicitly for a given link diagram K from knowledge of the invariants $\varphi_n(K)$ for sufficiently many n, as follows:

Proof : We know that any element a of Λ can be written as a polynomial in $\lambda^{\pm 1}$, z and δ. Now $z\delta = \lambda^{-1} - \lambda + z$, so $z^k a$ can be rewritten as a polynomial in $\lambda^{\pm 1}$ and z alone, for large enough k.

A simple induction, as in theorem 2.6, shows that $z^{|K|}\mathcal{D}(K) \in \Lambda$ can always be written as a polynomial in $\lambda^{\pm 1}$ and z; say

$$\begin{aligned} z^{|K|}\mathcal{D}(K) &= \sum_{r=m}^{M} \lambda^r P_r(z) \\ &= \sum_{r=m}^{M} \lambda^r Q_r(s), \end{aligned}$$

where $Q_r(s) = P_r(s - s^{-1})$.

It is then enough to find $Q_r(s)$, $m \leq r \leq M$.

Now for each n,

$$\begin{aligned} \sum_{r=m}^{M} s^{r(2n-1)} Q_r(s) &= e_n(z^{|K|}\mathcal{D}(K)) \\ &= (s - s^{-1})^{|K|}\varphi_n(K). \end{aligned}$$

Write V for the $k \times k$ Vandermonde matrix with entries

$$s^{(2n-1)r}, \; 1 \leq n \leq k, \; m \leq r \leq M, \quad \text{with } k = M - m + 1.$$

Then

$$V \begin{pmatrix} Q_m \\ Q_{m+1} \\ \vdots \\ Q_M \end{pmatrix} = (s - s^{-1})^{|K|} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_k \end{pmatrix}.$$

Since V is invertible, we have $Q_m, \ldots, Q_M$, and hence $\mathcal{D}(K)$ in terms of $\varphi_1, \ldots, \varphi_k$. □

In order to make these calculations explicitly we need bounds for m and M, in terms of K. It is certainly sufficient to note that $|m|, M \leq |K| + c(K)$, where $c(K)$ is the number of crossings in the diagram, although these bounds may turn out to be quite generous.

4 A basis for the tangle algebra

In this section we set out the induction to be used in proving that the algebra BW_n defined by generators and relations is isomorphic to the Kauffman algebra, defined by tangles. We start by reviewing the position for MT_n.

The algebra MT_n is shown in [11] to be free over Λ, of the same dimension, $|C_n|$, as Brauer's algebra A_n. The proof, which we give here, is an easy consequence of the existence, however established, of Kauffman's Dubrovnik invariant $\mathcal{D} : MT_0 \to \Lambda$. We make use of the homomorphism $e : \Lambda \to \mathbf{Z}[\delta]$.

Proposition 4.1 $e(\mathcal{D}(K)) = \delta^{|K|}$.

Proof : It follows from condition (1) that $e(\mathcal{D}(K))$ is unaltered when any crossing in a diagram is switched, and from (2) that it is unaltered by Reidemeister move I. Now any diagram can be changed to any other with the same number of components by a sequence of crossing switches and Reidemeister moves, so $e(\mathcal{D}(K)) = e(\mathcal{D}(K'))$, where K' is the disjoint union of $|K|$ simple closed curves, giving the result by (3). □

Theorem 4.2 *Any set of tangles* $\{T_c\}$, $c \in C_n$, *without closed components, having* $c = conn(T_c)$ *and spanning* MT_n *forms a free* Λ*-basis for* MT_n.

Proof : Define a bilinear map $b : MT_n \times MT_n \to \Lambda$ by $b(S, T) = \mathcal{D}(\varepsilon(ST))$. Write A for the $|C_n| \times |C_n|$ matrix with entries $a_{cd} = b(T_c, T_d)$.

Suppose that $\Sigma\lambda_i T_i = 0$, $\lambda_i \in \Lambda$. We want to show that $\lambda_i = 0$ for all i. For each $c \in C_n$ replace the cth column of A by the linear combination of the columns of A with coefficients λ_i. The new matrix then has determinant $\lambda_c \det A$ and a zero column. The required result follows by proving that $\det A \neq 0$, since Λ has no zero-divisors.

Now $\varepsilon(T_c T_d) \in MT_0$ is represented by a link with r components, say. Each component contains at least one arc from T_c and one from T_d, so $r \leq n$. When $r = n$ each component must have exactly one arc from each, so that the connector d is the 'mirror image' of c, given by interchanging the roles of the top and bottom points. Set $\overline{c} = d$ in this case, so that we have $r = n$ if and only if $d = \overline{c}$.

Now apply the homomorphism e to the entries in A. Then, by proposition 4.1, $e(a_{cd}) = \delta^r$, $r \leq n$, and $r = n$ if and only if $d = \overline{c}$. The matrix $e(A)$ has then one entry δ^n in each row and column, so $e(\det A) = \det(e(A)) \in \mathbf{Z}[\delta]$ has a non-zero coefficient for δ^{n^2}. Thus $e(\det A) \neq 0$, so $\det A \neq 0$. □

This shows that MT_n is a deformation of Brauer's algebra A_n, in the following sense.

Theorem 4.3 *There is an isomorphism of* $\mathbf{Z}[\delta]$*-algebras induced by* c *between* $MT_n \otimes_\Lambda \mathbf{Z}[\delta]$ *and* A_n.

Proof : The map $c : MT_n \to A_n$, defined in section 2, factors through a $\mathbf{Z}[\delta]$-homomorphism $MT_n \otimes_\Lambda \mathbf{Z}[\delta] \to \mathbf{A_n}$. Since $MT_n \otimes_\Lambda \mathbf{Z}[\delta]$ is spanned over $\mathbf{Z}[\delta]$ by $\{T_c\}$ which maps onto a *basis* of A_n of the same cardinality, this set must be a $\mathbf{Z}[\delta]$-basis in the specialisation, and the map is hence an isomorphism. □

Corollary 4.4 (to theorem 4.2) *Any set of tangles with distinct connectors forms an independent set in MT_n.*

Proof : We have shown that $c : MT_n \to A_n$ carries a free Λ-basis to a free $\mathbf{Z}[\delta]$-basis. It follows, using determinantal criteria for independence as in theorem 4.2, that k elements of MT_nwhose images are independent in A_n must themselves be independent. □

We shall prove by induction on n that the homomorphism $\varphi : BW_n \to MT_n$ is an isomorphism. In the course of the proof we shall construct explicit bases $\varphi^{-1}\{T_c\}$ in BW_n. As part of the induction we shall use natural filtrations $BW_n^{(r)}$ and $MT_n^{(r)}$ by 2-sided ideals, analogous to the filtration of A_n used by Hanlon and Wales, [7]. In the case of MT_n this filtration arises from the geometric viewpoint, as in [11], when we consider tangles of rank $\leq r$.

Definition. A tangle $T \in \mathcal{U}_n^n$ has *rank* $\leq r$ if it is the composite $T = AB$ of an (n, r) and an (r, n) tangle.

Remark. Then conn(T) has at most r arcs connecting top to bottom. However this is not sufficient for T to have rank r. For example, the tangle T in figure 4.1 has rank 2, although conn(T) has no connecting arcs from top to bottom.

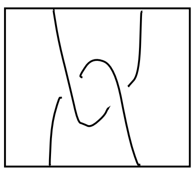

Figure 4.1

Write $MT_n^{(r)}$ for the subspace of MT_n spanned by tangles of rank $\leq r$. Clearly $MT_n^{(r)}$ is a 2-sided ideal, with

$$MT_n = MT_n^{(n)} \supset MT_n^{(n-2)} \supset \dots .$$

Proposition 4.5 *$MT_n^{(r)}$ is generated, as an ideal, by the element $E_1E_3\ldots E_{2k-1}$, where $2k = n - r$.*

Proof : For $r > 0$ we can write the identity tangle in $\mathcal{U}_r^r$ as

$$I = C(E_1E_3\ldots E_{2k-1})D,$$

where C is an (r, n) tangle and D is an (n, r) tangle, as in figure 4.2.

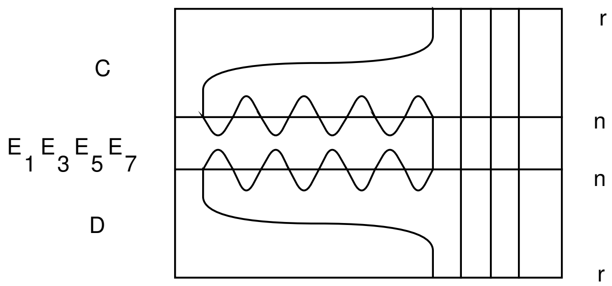

Figure 4.2

Then any tangle $T = AB$ of rank $\leq r$ can be written as $T = AC(E_1E_3\ldots E_{2k-1})BD$ with $AC, BD \in \mathcal{U}_n^n$.

The case $r = 0$ can be handled similarly, by first writing a tangle T of rank 0 as $T = AE_1B$ where A is an $(n, 2)$ tangle and B is a $(2, n)$ tangle. □

Definition. For $r = n - 2k$ write $BW_n^{(r)}$ for the 2-sided ideal of BW_n generated by $e_1e_3\ldots e_{2k-1}$.

Then

$$BW_n = BW_n^{(n)} \supset BW_n^{(n-2)} \supset \ldots .$$

Clearly $\varphi : BW_n \to MT_n$ restricts to $\varphi : BW_n^{(r)} \to MT_n^{(r)}$.

Our main result, that φ is an isomorphism, follows from

Theorem 4.6 *$\varphi : BW_n^{(r)} \to MT_n^{(r)}$ is injective for all n, r.*

Proof : The detailed lemmas needed appear in later sections. The scheme of the proof follows here.

For fixed n we prove the result for $r = 0, 1$ in section 5 from the injectivity of φ on BW_{n-1} ($= BW_{n-1}^{(n-1)}$) using induction on n.

The proof then continues by induction on r.

For this induction we construct a linear subspace $V_n^{(r)} \subset BW_n^{(r)}$, complementing $BW_n^{(r-2)}$. The induction step for injectivity of φ follows by establishing:

(1) $V_n^{(r)} + BW_n^{(r-2)}$ is a 2-sided ideal in $BW_n^{(r)}$,
(2) $\varphi|V_n^{(r)} \to MT_n$ is injective,
(3) $e_1 e_3 \ldots e_{2k-1} \in V_n^{(r)}$.

In the construction, given later in this section, we exhibit an explicit spanning set for $V_n^{(r)}$ whose image in MT_n is an independent set of totally descending tangles. This establishes property (2).

Property (3) is immediate from the construction, and property (1) is proved in section 6. □

To describe certain elements in BW_n we now draw on Artin's braid group.

The braid group on n strings, defined by geometric braids, (particular types of (n, n) tangles), is known to have the presentation with generators $\sigma_i,\ i \le n$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

There is then a homomorphism $\psi : B_n \to BW_n$ defined by $\sigma_i \mapsto g_i$. Any two monomials in BW_n in $g_i^{\pm 1}$ which arise from the same geometric braid β will then be equal, and we shall use β to picture the element $\psi(\beta)$. We shall also refer to monomials in $g^{\pm 1}$ as braids in BW_n.

There is an antihomomorphism $\mathrm{perm} : B_n \to S_n$ defined by $\mathrm{perm}(\sigma_i) = \tau_i = (i\ i+1)$. With our convention of composition of geometric braids, the strings in a braid β then join the point i at the top to the point $\pi(i)$ at the bottom, with $\pi = \mathrm{perm}(\beta)$.

Among the elements of B_n we shall use particularly the *positive permutation braids* and, as special cases, the *Lorenz braids* .

Definition. A braid in B_n in which all crossings are positive and every pair of strings crosses at most once is called a *positive permutation braid* .

Theorem 4.7 *A positive permutation braid β is determined by the permutation $\pi = perm(\beta)$ induced by its strings.*

Proof : Such braids are examples of 'totally descending tangles', as defined in section 2, in which the arcs of the connector all join top to bottom and are ordered by the order of their initial points. □

We shall write β_π for the positive permutation braid with permutation $\pi = \mathrm{perm}(\beta_\pi)$, whose strings join the points i at the top with $\pi(i)$ at the bottom. The element $b_\pi = \psi(\beta_\pi) \in BW_n$, which we shall also call a positive permutation braid, can be conveniently referred to by the permutation π, rather than choosing one of the many ways of writing it as a monomial in g_i. For example, the permutation $\pi = (14)(23) \in S_4$ gives $b_\pi = g_1g_2g_3g_1g_2g_1 = g_2g_1g_2g_3g_2g_1 = \ldots$.

Definition. A *Lorenz braid* of type (ℓ, r) is a braid β_π where $\pi \in S_n$, $n = \ell+r$, does not permute the first ℓ 'left-hand' strings, or the last r 'right-hand' strings among themselves.

For fixed (ℓ, r) there are $\binom{n}{r}$ Lorenz braids, as a Lorenz permutation π is determined simply by the free choice of endpoints for the right-hand strings. Note that π is an (ℓ, r) Lorenz permutation if and only if $\pi(i) < \pi(j)$ for $1 \le i < j \le \ell$ and for $\ell + 1 \le i < j \le n$. An example of a $(3, 4)$ Lorenz braid is shown in figure 4.3.

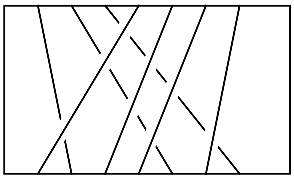

Figure 4.3

Where π^{-1} is a Lorenz permutation the braid $\beta_\pi = \alpha(\beta_{\pi^{-1}})$ can be viewed as a Lorenz braid $\beta_{\pi^{-1}}$ turned upside down. Call $\alpha(\beta_{\pi^{-1}})$ a *reverse Lorenz braid* . Note that $(\beta_\pi)^{-1}$ is not the same braid as $\beta_{\pi^{-1}}$ but has all the crossings switched.

Definition. For each n and $r = n - 2k$ write $V_n^{(r)}$ for the linear subspace of $BW_n^{(r)}$ spanned by elements $b_\pi w_{2k} b_\tau b_\mu$, where π, μ are $(2k, r)$ Lorenz permutations, τ is a permutation of the last r strings only, and $w_{2k} \in BW_{2k}^{(0)}$.

Proposition 4.8 *Given that $\varphi|BW_{2k}^{(0)}$ is injective for $n \ge 2k$ then $\varphi|V_n^{(r)} \to MT_n$ is injective.*

Proof : We know that $MT_{2k}^{(0)}$ is spanned by $|C_k|^2$ totally descending tangles, one for each k-connector of rank 0, and that $\varphi|BW_{2k}^{(0)} \to MT_{2k}^{(0)}$ is surjective. By hypothesis we can choose a spanning set of $|C_k|^2$ elements for $BW_{2k}^{(0)}$ with this set of tangles as image.

Then $V_n^{(r)}$ is spanned by the $\binom{n}{r}^2 |C_k|^2 r!$ elements $b_\pi w_{2k} b_\tau b_\mu$, where π^{-1}, μ are drawn independently from $(2k, r)$ Lorenz permutations, τ from permutations in S_r and w_{2k} from the spanning set for $BW_{2k}^{(0)}$.

The elements $\varphi(b_\pi w_{2k} b_\tau b_\mu)$ are represented by tangles in $MT_n^{(r)}$ each with exactly r through strings, and all having different connectors. A typical such tangle with $k = 2$, $r = 4$ is illustrated in figure 4.4. It follows by the corollary to theorem 4.2 that these tangles are independent in MT_n, and hence that $\varphi|V_n^{(r)}$ is injective. □

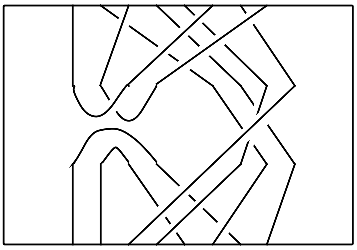
Figure 4.4

This establishes property (2) of theorem 4.6 under its induction hypothesis.

From theorem 4.6 we eventually build a basis for BW_n as a union of spanning sets for each $V_n^{(r)}$. The image of this basis in MT_n can be represented by a set of tangles each with a different connector, and each totally descending, for some ordering of the arcs.

We note that this gives a complicated check that the dimension of BW_n is

$$|C_n| = \sum_{k=0}^{[n/2]} \tbinom{n}{r}^2 \, |C_k|^2 \, r!,$$

where we write $r = n - 2k$.

5 Generators and relations for the tangle algebra: the base for induction

In this section we prove injectivity of φ on $BW_n^{(0)}$ or $BW_n^{(1)}$, depending on the parity of n, given injectivity of $\varphi|BW_{n-1}$. The corresponding sets of tangles in MT_n are those with at most one through string.

We start with some results in BW_n which use only the regular isotopy relations.

Definition. The shift map $S : MT_n \to MT_{n+1}$ is a homomorphism defined on an n-tangle T as shown in figure 5.1.

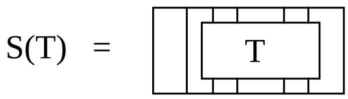

Figure 5.1

Thus $S(G_i) = G_{i+1},\ S(E_i) = E_{i+1}$.

It is clear, from the behaviour on tangles, as shown in figure 5.2, that $WA_m = A_m S(W)$ for $W \in MT_m$, where $A_m = G_m G_{m-1} \dots G_1$.

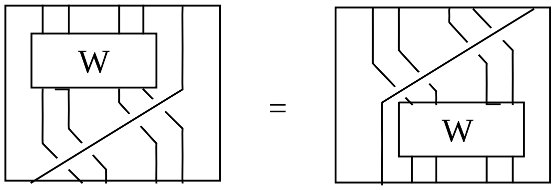

Figure 5.2

We can define a shift map with similar properties in BW as follows.

Definition. The shift map $S : BW_n \to BW_{n+1}$ is defined as a homomorphism by

$$S(g_i) = g_{i+1},\ S(e_i) = e_{i+1},$$

extended linearly.

It is simply necessary to check that the relations are respected by S.

Proposition 5.1 *The homomorphism S satisfies*

$$wa_m = a_m S(w), \quad wb_m = b_m S(w)$$

for any $w \in BW_m$, where

$$a_m = g_m g_{m-1} \dots g_1,\ b_m = g_m^{-1} g_{m-1}^{-1} \dots g_1^{-1}.$$

Proof : When $w = g_i^{\pm 1}$ or $w = e_i$ the result is an immediate consequence of the relations, and it follows for monomials w by induction on their length.□

We now define $F_k \in MT_n,\ 2k \leq n$, to be the element represented by the tangle shown in figure 5.3.

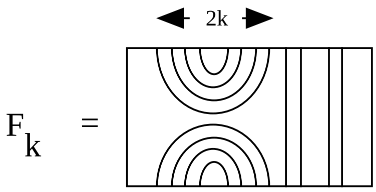

Figure 5.3

The following equations in MT_n are clear from inspection of representative tangles.

Proposition 5.2 *For all $i < k$*
(1) $G_iF_k = G_{2k-i}F_k$,
(2) $E_iF_k = E_{2k-i}F_k$,
(3) $F_kG_i = F_kG_{2k-i}$,
(4) $F_kE_i = F_kE_{2k-i}$.

An example of equation (1) is illustrated in figure 5.4, with $i = 1$ and $k = 3$.

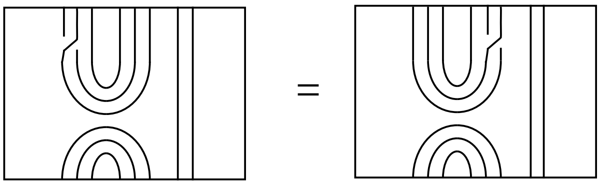

Figure 5.4

Again it is clear from inspection of the tangles, as shown in figure 5.5, that

$$F_k = \alpha(A_{2k-2})F_{k-1}E_{2k-1}A_{2k-2}.$$

By analogy we define $f_k \in BW_n$ inductively, setting $f_0 =$ identity, and

$$f_k = \alpha(a_{2k-2})f_{k-1}e_{2k-1}a_{2k-2}.$$

We then have $F_k = \varphi(f_k)$, and $\alpha(f_k) = f_k$, since f_{k-1} and e_{2k-1} commute.

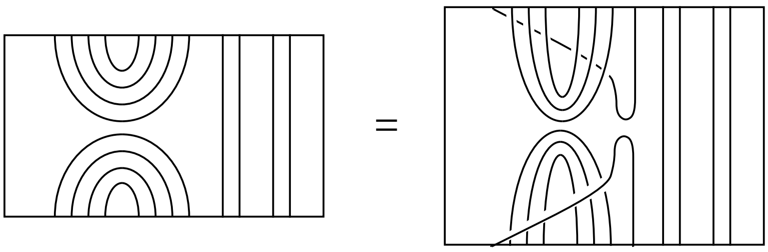

Figure 5.5

Remark. While it is clear that $\rho_{2k}(F_k) = F_k$ in MT_n, it is difficult to prove directly from the definition and relations in BW_n that $\rho_{2k}(f_k) = f_k$.

Proposition 5.3 $BW_n^{(n-2k)} \subset BW_n$ *is the 2-sided ideal generated by* f_k.

Proof : By definition $BW_n^{(n-2k)}$ is the 2-sided ideal generated by $e_1e_3 \ldots e_{2k-1}$. By induction on k we can write $f_k = \alpha(b_k)e_1e_3 \ldots e_{2k-1}b_k$ for some invertible element $b_k \in BW_{2k}$, in fact b_k can be chosen to be a braid. □

We now make use of the relations in BW_n to prove the analogous results to proposition 5.2.

Proposition 5.4 *For all $i < k$*
(1) $g_i f_k = g_{2k-i} f_k$,
(2) $e_i f_k = e_{2k-i} f_k$,
(3) $f_k g_i = f_k g_{2k-i}$,
(4) $f_k e_i = f_k e_{2k-i}$.
The same results hold with $\rho_{2k}(f_k)$ in place of f_k.

Proof : Cases (3) and (4) follow from (1) and (2) by applying α. Applying ρ_{2k} gives the results for $\rho_{2k}(f_k)$. The result is immediate for $k = 1$. For $i > 1$ the result follows from proposition 5.1 by induction on k.

For example, in case (1),

$$\begin{aligned} g_i f_k &= g_i \alpha(a_{2k-2}) f_{k-1} e_{2k-1} a_{2k-2} \\ &= \alpha(a_{2k-2}) g_{i-1} f_{k-1} e_{2k-1} a_{2k-2}, \text{ by applying } \alpha \text{ to 5.1} \\ &= \alpha(a_{2k-2}) g_{2k-i-1} f_{k-1} e_{2k-1} a_{2k-2}, \text{ by induction} \\ &= g_{2k-i} \alpha(a_{2k-2}) f_{k-1} e_{2k-1} a_{2k-2}, \ (i \geq 2) \\ &= g_{2k-i} f_k. \end{aligned}$$

To prove 5.4 when i=1 we set $h_j = \alpha(a_j)\alpha(a_{j-2}) e_{j+1} e_{j-1}$.

Since $f_k = h_{2k-2} f_{k-2} a_{2k-4} a_{2k-2}$, the result for cases (1) and (2) will follow by showing that
$(1')$ $g_1 h_j = g_{j+1} h_j$
$(2')$ $e_1 h_j = e_{j+1} h_j$,
for all j.

We prove $(1')$ and $(2')$ by induction on j, starting with $j = 2$. For $j = 2$ we have

$$h_2 = g_1 g_2 e_1 e_3 = e_2 e_1 e_3 = e_2 e_3 e_1 = g_3 g_2 e_3 e_1.$$

Then $g_3 h_2 = g_3 g_1 g_2 e_1 e_3 = g_1 h_2$ and $e_1 h_2 = e_1 e_2 e_1 e_3 = e_1 e_3 = e_3 h_2$.

For the induction step, use the braid relations to write

$$\alpha(a_j)\alpha(a_{j-2}) = g_2 g_1 S(\alpha(a_{j-1})\alpha(a_{j-3})).$$

(Compare the two braids illustrated in figure 5.6.)

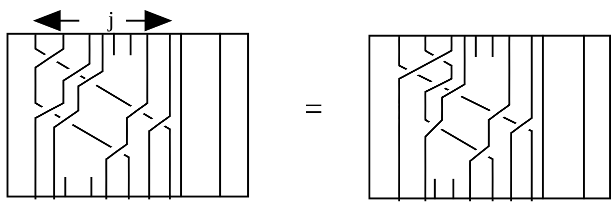

Figure 5.6

Then $h_j = g_2 g_1 S(h_{j-1})$. So

$$\begin{aligned} g_1 h_j &= g_1 g_2 g_1 S(h_{j-1}) = g_2 g_1 g_2 S(h_{j-1}) = g_2 g_1 S(g_1 h_{j-1}) \\ &= g_2 g_1 S(g_j h_{j-1}), \text{ by induction on } j, \\ &= g_2 g_1 g_{j+1} S(h_{j-1}) = g_{j+1} h_j, \text{ for } j > 2. \end{aligned}$$

Similarly $e_1 h_j = e_{j+1} h_j$, using the relation in BW_n that $e_1 g_2 g_1 = g_2 g_1 e_2$. □

Lemma 5.5 *Suppose that $\varphi : BW_{m+1} \to MT_{m+1}$ is injective. Then $BW_{m+1} e_m = BW_m e_m$.*

Proof : By hypothesis it is enough to prove the corresponding result

$$MT_{m+1} E_m = MT_m E_m.$$

For an $(m+1, m+1)$ tangle T define $\varepsilon_m(T)$ to be the (m, m) tangle shown in figure 5.7.

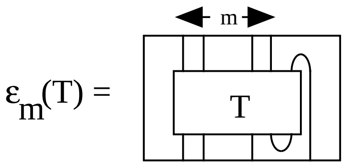

Figure 5.7

Using the standard interpretation of $\varepsilon_m(T)$ as an $(m+1, m+1)$ tangle it is clear that $\varepsilon_m(T) E_m = T E_m$. Extend the definition of ε_m to linear combinations of tangles to define a linear map $\varepsilon_m : MT_{m+1} \to MT_m$, (the relations are respected). Then any element $X E_m$ with $X \in MT_{m+1}$ can be rewritten as $X E_m = \varepsilon_m(X) E_m \in MT_m E_m$. □

Corollary 5.6 *Under the same conditions, $BW_{m+1} e_1 = S(BW_m) e_1$.*

Proof : Apply the automorphism ρ_{m+1}. □

Proposition 5.7 *Suppose that $\varphi | BW_{n-1} \to MT_{n-1}$ is injective. Then*
(1) $BW_{2k} f_k = BW_k f_k$, *for all k with $2k \le n$,*
(2) $BW_{2k+1} S(f_k) = BW_{k+1} S(f_k)$, *for all k with $2k+1 \le n$.*

Proof :

(1) The case $k = 1$ is immediate, since $g_1^{\pm 1} e_1$ and $e_1 e_1$ are multiples of e_1.

For $k \geq 2$ we have $k + 1 \leq n - 1$ so that $BW_{k+1} e_k = BW_k e_k$ by lemma 5.5. It is enough to show that

$$\begin{aligned} g_i BW_k f_k &\subset BW_k f_k \\ e_i BW_k f_k &\subset BW_k f_k, \text{ for all } i < 2k. \end{aligned}$$

This is immediate for $i < k$. For $i > k$ it follows from 5.4, and the fact that BW_k then commutes with e_i and g_i.

Write $f_k = e_k r_k$ for some $r_k \in BW_{2k}$, by induction on k. The remaining cases with $i = k$ follow by noting that

$$\begin{aligned} g_k BW_k e_k &\subset BW_{k+1} e_k = BW_k e_k \\ \text{and } e_k BW_k e_k &\subset BW_{k+1} e_k = BW_k e_k. \end{aligned}$$

(2) The case $k = 1$ will be proved directly.

For $k \geq 2$ we have $k+2 \leq n-1$ so that $BW_{k+2} e_{k+1} = BW_{k+1} e_{k+1}$ from lemma 5.5. We must show that

$$\begin{aligned} g_i BW_{k+1} S(f_k) &\subset BW_{k+1} S(f_k), \\ e_i BW_{k+1} S(f_k) &\subset BW_{k+1} S(f_k), \text{ for } i \leq 2k. \end{aligned}$$

This is immediate for $i < k + 1$. For $i > k + 1$ it follows from proposition 5.4, since BW_{k+1} commutes with g_i and e_i. The remaining cases follow as in (1), since $S(f_k) = e_{k+1} S(r_k)$.

We finish the proof of (2) by showing that $BW_3 e_2 = BW_2 e_2$. Now $BW_2 e_2$ is spanned by e_2, $e_1 e_2$ and $g_1 e_2$, so we must show that products of these elements with g_2 or e_2 on the left still lie in $BW_2 e_2$. It is a matter of a quick check from the relations in BW_3, to see that $e_2^2 = \delta e_2$, $e_2 e_1 e_2 = e_2$, $e_2 g_1 e_2 = \lambda^{-1} e_2$, $g_2 e_2 = \lambda e_2$, $g_2 e_1 e_2 = g_1^{-1} e_2$ and $g_2 g_1 e_2 = e_1 e_2$. □

Corollary 5.8 *Suppose that $\varphi | BW_{n-1}$ is injective. Then the ideals generated by f_k in BW_n, with $k = [n/2]$, can be written as:*

(1) $BW_{2k}^{(0)} = BW_k f_k BW_k$ when $n = 2k$, and

(2) $BW_{2k+1}^{(1)} = BW_{k+1} S(f_k) BW_{k+1}$ when $n = 2k + 1$.

Proof (1):

$$\begin{aligned} BW_{2k}^{(0)} &= BW_{2k} f_k BW_{2k} \\ &= BW_k f_k BW_{2k} \text{ by 5.7,} \\ &= BW_k f_k BW_k \text{ applying } \alpha \text{ to 5.7.} \end{aligned}$$

□

Proof (2): The ideal $BW^{(1)}_{2k+1}$ generated by f_k is equally generated by $S(f_k) = a^{-1}_{2k} f_k a_{2k}$ so the result follows using 5.7 (2) exactly as in (1). □

We complete this section by showing the injectivity of φ on the 2-sided ideals generated by f_k in $BW_n,\ k = [n/2]$, given injectivity on BW_{n-1}.

Theorem 5.9 *Suppose that $\varphi|BW_{n-1} \to MT_{n-1}$ is injective. Then $\varphi|BW^{(0)}_{2k} \to MT^{(0)}_{2k}$ is injective, when $n = 2k$, and $\varphi|BW^{(1)}_{2k+1} \to MT^{(1)}_{2k+1}$ is injective, when $n = 2k+1$.*

Proof : In the case $n = 2k$ we know that $\varphi|BW_k$ is an isomorphism to MT_k. We may then choose elements $t_c \in BW_k, c \in C_k$, spanning BW_k, with $\varphi(t_c)$ represented by a totally descending tangle T_c say, having connector c.

By corollary 5.8 we have $BW^{(0)}_{2k} = BW_k f_k BW_k$. This is spanned by $|C_k|^2$ elements $t_c f_k t_d$, $c, d \in C_k$. It is enough to prove that the images of these elements are independent in MT_{2k}.

Now these images are represented by the tangles $T_c F_k T_d$. Different pairs of connectors (c, d) give tangles $T_c F_k T_d$ with different connectors in C_{2k}, since the tangles consist of a top and a bottom half, each with k arcs, affected independently by the connectors c and d. The tangles then represent independent elements in MT_{2k}, by corollary 4.4.

Similarly when $n = 2k+1$ we know that $\varphi|BW_{k+1}$ is an isomorphism to MT_{k+1}. We may then choose spanning elements $t_c \in BW_{k+1}, c \in C_k$, with $\varphi(t_c)$ represented by a totally descending tangle T_c say, having connector c. Again, by corollary 5.8, we have a spanning set $\{t_c S(f_k) t_d\}$, $c, d \in C_{k+1}$ with $|C_{k+1}|^2$ elements, for the ideal $BW^{(1)}_{2k+1}$.

The images of these elements are represented by tangles $T_c S(F_k) T_d$. Once more we can see that different pairs of connectors (c, d) give tangles with different connectors in C_{2k+1} because all but one of the arcs stays either in the top or in the bottom of the tangle. This guarantees independence in MT_{2k+1}, as before. □

Remark. We could in fact show that the composite tangles used in this proof are themselves totally descending, for some suitable ordering of their arcs.

We continue in the next section to examine $BW^{(r)}_n$ for larger r having established here the start of our induction on r. Note that we could prove similarly that $BW^{(r)}_{2k+r} = BW_{k+r} S^r(f_k) BW_{k+r}$ and find a spanning set of $|C_{k+r}|^2$ elements. However, a similar attempt to prove the (false) result for

$r > 1$ that these are independent would fail, because some different pairs of connectors in C_{k+r} can yield the same connector in C_{2k+r}.

6 Isomorphism between Kauffman's tangle algebras and the Birman-Wenzl algebras

We finish the proof of injectivity of $\varphi : BW_n \to MT_n$ by proving the remaining induction step, namely that if $\varphi|BW_{n-1}$ is injective, and $\varphi|BW_n^{(r-2)}$ is injective then $\varphi|BW_n^{(r)}$ is injective. We do this by finding a complementary subspace $V_n^{(r)}$ to $BW_n^{(r-2)}$ in $BW_n^{(r)}$ on which φ is injective.

We recall the definition of $V_n^{(r)}$ given in section 4 as the subspace spanned by $\{b_\pi BW_{2k}^{(0)} b_\tau b_\mu\}$, where $n = 2k + r$, $\alpha(b_\pi)$, b_μ are $(2k, r)$ Lorenz braids in B_n and b_τ is a positive permutation braid on the last r strings in $S^{2k}(B_r)$. Following the scheme of proof in theorem 4.6 we already know, by induction on n, that $\varphi|V_n^{(r)}$ is injective.

It remains to show that $V_n^{(r)} + BW_n^{(r-2)} = BW_n^{(r)}$. Since $BW_n^{(r)}$ is the 2-sided ideal generated by f_k, and $f_k \in V_n^{(r)}$ we need only show that $V_n^{(r)} + BW_n^{(r-2)}$ is a 2-sided ideal. Now $\alpha(V_n^{(r)}) = V_n^{(r)}$, since the elements b_τ in $S^{2k}(BW_r)$ commute with BW_{2k}. Hence it is enough to show that $V_n^{(r)} + BW_n^{(r-2)}$ is a left ideal.

Proposition 6.1 *Let $n = r+2k$ and let $X_n^{(r)}$ be the subspace spanned by the set $\{b_\pi b_\tau BW_{2k} f_k\}$, where $\alpha(b_\pi)$ is a $(2k, r)$ Lorenz braid and b_τ is a positive permutation braid in $S^{2k}(B_r)$. Suppose also that $\varphi|BW_{n-1}$ is injective and that $r \geq 2$. Then*

$$L_n^{(r)} = X_n^{(r)} + BW_n^{(r-2)}$$

is a left ideal.

Corollary 6.2 $V_n^{(r)} + BW_n^{(r-2)}$ *is a left ideal, under the hypotheses of proposition 6.1, and hence theorem 4.6 is established.*

Proof : Since $L_n^{(r)}$ is a left ideal, by 6.1, it follows that $V_n^{(r)} + BW_n^{(r-2)}$ is a left ideal, by noting that $BW_{2k}^{(0)} = BW_{2k} f_k BW_{2k}$. □

The proof of proposition 6.1 occupies the rest of this section. The principal ingredient is an analysis of the elements $g_i b_\pi$ and $e_i b_\pi$ for positive permutation braids b_π. The following two lemmas are a consequence primarily of the braid relations.

Lemma 6.3 *Let ρ be any permutation, and let ρ_1 be the permutation $\rho \circ (i\, i+1)$. Then the positive permutation braid b_{ρ_1} satisfies the equation*

$$\begin{aligned} b_{\rho_1} &= g_i b_\rho && \text{if } \rho(i) < \rho(i+1), \\ b_\rho &= g_i b_{\rho_1} && \text{if } \rho(i) > \rho(i+1). \end{aligned}$$

Proof : If $\rho(i) < \rho(i+1)$ then each pair of strings in the braid $g_i b_\rho$ crosses at most once, so it is a positive permutation braid. Its permutation is ρ_1, so $g_i b_\rho = b_{\rho_1}$.

If $\rho(i) > \rho(i+1)$ then $\rho_1(i) < \rho_1(i+1)$ and the same argument holds with ρ_1 in place of ρ. □

Corollary 6.4 *Any positive permutation braid b_ρ can be written as the product of a word in $\{g_i\}$, $i \neq \ell$, and an (ℓ, r) Lorenz braid.*

Proof : By induction on the length of b_ρ, using 6.3 to write $b_\rho = g_i b_{\rho_1}$ for some $i \neq \ell$ if b_ρ is not already an (ℓ, r) Lorenz braid. □

Lemma 6.5 *Let ρ be any permutation with $\rho(i+1) = \rho(i) + 1$. Then $g_i b_\rho = b_\rho g_{\rho(i)}$ and $e_i b_\rho = b_\rho e_{\rho(i)}$.*

Proof : This can be viewed as allowing us to pass a simple crossing along two parallel strings from top to bottom of a braid. By the hypothesis on ρ, both $g_i b_\rho$ and $b_\rho g_{\rho(i)}$ are positive permutation braids, and both have the same permutation. Hence they are equal, using only the braid relations, by the fundamental theorem on positive permutation braids. It follows that $g_i^{-1} b_\rho = b_\rho g_{\rho(i)}^{-1}$ and hence, by the skein relation, that $z\, e_i b_\rho = z\, b_\rho e_{\rho(i)}$.

The lemma follows, if we assume that z is invertible in Λ. Without inverting z the result follows by induction on the length of b_ρ, together with the relation $e_i g_{i+1} g_i = g_{i+1} g_i e_{i+1}$ and its reverse in BW_n. For we can write $b_\rho = g_j b_{\rho_1}$ for some j. Then $j \neq i$, since the strings i and $i+1$ do not cross under ρ.

If $j = i+1$ then $\rho(i+2) < \rho(i+1) = \rho(i) + 1$, so $\rho(i+2) < \rho(i)$. We can then, by lemma 6.3, write $b_\rho = g_{i+1} g_i b_{\rho_2}$, and then $e_i b_\rho = g_{i+1} g_i e_{i+1} b_{\rho_2}$. Now $\rho_2(i+2) = \rho(i+1) = \rho_2(i+1) + 1$ and b_{ρ_2} is shorter than b_ρ, so that we can use induction.

A similar argument can be used when $j = i-1$, while otherwise $|i-j| > 2$, and $e_i g_j = g_j e_i$, giving an immediate inductive proof. □

Lemma 6.6 $X_n^{(r)} S^{2k}(BW_r) \subset L_n^{(r)}$.

Proof : $X_n^{(r)} e_j \subset BW_n^{(r-2)} \subset L_n^{(r)}$ for $j > 2k$, since $f_k e_j \in BW_n^{(r-2)}$ for $j > 2k$.

Let b_τ be any positive permutation braid in $S^{2k} B_r$ and let $j > 2k$. Then by 6.3, either

$$\begin{aligned} b_\tau g_j &= b_{\tau'} \\ \text{or } b_\tau g_j &= b_{\tau'} g_j^2 = b_{\tau'} + z b_\tau - z b_\tau e_j. \end{aligned}$$

Hence $x g_j \in L_n^{(r)}$ for any spanning element $x = b_\pi w_{2k} f_k b_\tau \in X_n^{(r)}$.

Thus $X_n^{(r)} g_j \subset L_n^{(r)}$ for $j > 2k$. □

We now continue the proof of 6.1, to show that $L_n^{(r)}$ is a left ideal. Lemma 6.6 shows in particular that $L_n^{(r)} b_\tau \subset L_n^{(r)}$ for $b_\tau \in S^{2k}(B_r)$. It is enough to show that $e_i x$, $g_i x \in L_n^{(r)}$ for each $x = b_\pi w_{2k} f_k \in X_n^{(r)}$ and each i, where $\alpha(b_\pi)$ is a $(2k, r)$ Lorenz braid and $w_{2k} \in BW_{2k}$.

Suppose then that x and i are given. We may further suppose that $\pi(i+1) > \pi(i)$, otherwise $b_\pi = g_i b_{\pi_1}$ with $\pi_1(i+1) > \pi_1(i)$. We then need only prove that $e_i x' \in L_n^{(r)}$ where $x' = b_{\pi_1} w_{2k} f_k$, since $g_i x = g_i^2 x' = x' + z g_i x' - z g_i e_i x' = x' + z x - \lambda z e_i x'$ and $e_i x = e_i g_i x' = \lambda e_i x'$ from the skein and delooping relations.

Since π is a reverse $(2k, r)$ Lorenz permutation then $\pi(i+1) = \pi(i) + 1$ if either $\pi(i+1) \le 2k$ or $\pi(i) > 2k$. By 6.5 $e_i x = b_\pi e_{\pi(i)} w_{2k} f_k$ in either case. This lies in $X_n^{(r)}$ if $\pi(i) < 2k$ and in $BW_n^{(r-2)}$ if $\pi(i) > 2k$, and similarly $g_i x \in L_n^{(r)}$. It remains to deal with $e_i x$ and $g_i x$ when $\pi(i+1) > 2k$ and $\pi(i) \le 2k$. In this case $g_i b_\pi$ is a reverse $(2k, r)$ Lorenz braid, by 6.3, so that $g_i x \in X_n^{(r)}$ and we are left to consider $e_i x$.

Given π and i, let ρ be the permutation given by

$$\rho(j) = \begin{cases} 2k & j = \pi(i), \\ j - 1, & \pi(i) < j \le 2k, \\ j + 1, & 2k + 1 \le j < \pi(i+1), \\ 2k + 1, & j = \pi(i+1), \\ j, & \text{otherwise.} \end{cases}$$

Now ρ only makes pairs of strings cross which have not already been made to cross by the reverse Lorenz braid b_π, so that $b_\pi b_\rho$ is also a positive permutation braid. Then $b_\pi b_\rho = b_{\pi_1}$, where $\pi_1 = \rho \circ \pi$. Now ρ permutes the first $2k$ strings and the last r strings among themselves, moving $\pi(i)$ to $2k$ and $\pi(i+1)$ to $2k+1$, so $x = b_{\pi_1} (b_\rho)^{-1} w_{2k} f_k$ with $b_\rho \in BW_{2k} S^{2k}(B_r)$. Note that $\pi_1^{-1}(i) < \pi_1^{-1}(j)$ for $i < j \le 2k - 1$.

It is enough, by lemma 6.6, to show that $e_i x' \in L_n^{(r)}$, for $x' = b_{\pi_1} w'_{2k} f_k$. Now $\pi_1(i+1) = 2k+1 = \pi_1(i)+1$ so, by 6.3, $e_i x' = b_{\pi_1} e_{2k} w'_{2k} f_k$. This does not finish the proof, since the element e_{2k} is stuck between BW_{2k} and $S^{2k}(BW_r)$ and we have to use our inductive knowledge of $w'_{2k} f_k \in BW_{2k}^{(0)}$ to free it.

Lemma 6.7 *Suppose that $\varphi | BW_{2k}^{(0)}$ is injective. Then every element in $BW_{2k} f_k$ is a linear combination of elements in the sets*

$$g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} BW_{2k} f_k, m = 1, \ldots, 2k-2, \textit{ and } e_{2k-1} BW_{2k} f_k.$$

Lemma 6.8 *For each $m = 1, \ldots, 2k-2$ and each positive permutation braid b_ρ with $\rho^{-1}(i) < \rho^{-1}(j)$ for $i < j \le 2k-1$ and $\rho^{-1}(2k+1) = \rho^{-1}(2k)+1$ we have*

$$b_\rho e_{2k} g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} = b_{\rho'} e_{2k-1}$$

for some positive permutation braid $b_{\rho'}$.

Proposition 6.1 then follows from 6.7 and 6.8, since we can write the element $e_i x'$ as a linear combination of elements of the form $b_{\rho'} BW_{2k} f_k$. All of these lie in $L_n^{(r)}$, since any positive permutation braid $b_{\rho'}$ can be written as the product $b_\pi b_{\rho''}$ of a reverse $(2k, r)$ Lorenz braid b_π with a positive braid which does not involve the generator g_{2k}, by the corollary to lemma 6.3, applied to the reverse braids.

Proof of lemma 6.7: By hypothesis, φ gives an isomorphism from $BW_{2k} f_k \subset BW_{2k}^{(0)}$ to $MT_{2k} F_k$. Now every element of $MT_{2k} F_k$ can be written as a linear combination of totally descending tangles T_c, where the connectors c join points of the top to the top in some way, and join the bottom points as for F_k. We may choose the order of strings for each connector c as we wish, so let us assume that in each tangle T_c the string whose end point is at position $2k$ on the top lies above all the others. By isotopy of the strings we may then write each of these tangles T_c as

$$G_m G_{m+1} \ldots G_{2k-2} E_{2k-1} T F_k$$

for some $m = 1, \ldots, 2k-2$ and some $T \in MT_{2k}$ as illustrated in figure 6.1. The isomorphism φ then gives a spanning set for $BW_{2k} f_k$ as stated. $\Box$

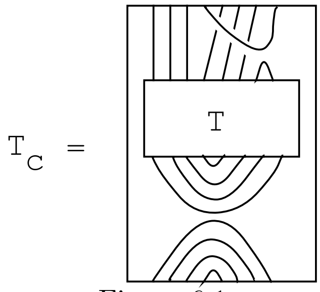

Figure 6.1

Proof of lemma 6.8: We have

$$b_\rho e_{2k} g_m g_{m+1} \dots g_{2k-2} e_{2k-1} = b_\rho g_m g_{m+1} \dots g_{2k-2} e_{2k} e_{2k-1}.$$

Now the reverse braid $g_{2k-2} \dots g_{m+1} g_m \alpha(b_\rho)$ is a positive permutation braid, b_{ρ_1} say, since $g_{2k-2} \dots g_m$ is a positive permutation braid on the first $2k-1$ strings only, while $\alpha(b_\rho) = b_{\rho^{-1}}$ does not make these strings cross. Now $\rho_1(2k+1) = \rho_1(2k)+1$, so either $g_{2k} g_{2k-1} b_{\rho_1}$ or $g_{2k}^{-1} g_{2k-1}^{-1} b_{\rho_1}$ is a positive permutation braid, b_{ρ_2}, say, depending on whether $\rho_1(2k-1) < \rho_1(2k)$ or $\rho_1(2k-1) > \rho_1(2k)$, by 6.3.

We can write $e_{2k-1} e_{2k} = e_{2k-1} g_{2k} g_{2k-1} = e_{2k-1} g_{2k}^{-1} g_{2k-1}^{-1}$ by the relations in BW_n. Then $e_{2k-1} e_{2k} b_{\rho_1} = e_{2k-1} b_{\rho_2}$. Apply the reversing map to give

$$\begin{aligned} b_\rho e_{2k} g_m g_{m+1} \dots g_{2k-2} e_{2k-1} &= \alpha(e_{2k-1} e_{2k} b_{\rho_1}) \\ &= \alpha(e_{2k-1} b_{\rho_2}) \\ &= b_{\rho''} e_{2k-1}, \end{aligned}$$

where $\rho'' = \rho_2^{-1}$. □

This concludes the proof of proposition 6.1, and the inductive proof of theorem 4.6. We have now established that φ is an isomorphism from BW_n to MT_n for all n, so that we are able to use tangle based arguments in dealing with the algebra BW_n. We have established its dimension over Λ and also the geometric description of the natural chain of ideals generated by the elements f_k, so we can also study the composition series of this chain by using the corresponding ideals in MT_n generated by F_k.

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