

Mixing of scalar glueballs and flavor-singlet scalar mesons

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We discuss in detail the extraction of hadronic mixing strengths from lattice studies. We apply this to the mixing of a scalar glueball and a scalar meson in the quenched approximation. We also measure correlations appropriate for flavor-singlet scalar mesons using dynamical quark configurations from UKQCD. This enables us to compare the results from the quenched study of the mixing with the direct determination of the mixed spectrum. Improved methods of evaluating the disconnected quark diagrams are also presented.

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I. INTRODUCTION

Lattice techniques are well developed to describe mass spectra. What is much less well studied are hadronic transitions. Here we shall concentrate on purely hadronic transitions such as glueball mixing with scalar mesons, string breaking, flavor-singlet pseudoscalar mass generation, etc. In full QCD studies on a lattice, much as in experiment, one will obtain the mass values of the resulting mixed states. By varying quark masses and the number of quark flavors, one may be able to go beyond experiment and so help to substantiate or vitiate phenomenological models.

Within a quenched or partially quenched lattice approach, one can in principle learn much more: obtaining estimates of the mixing strengths themselves. This is the approach that we analyze in detail. We then apply it to the mixing of a glueball with a scalar meson. This is of considerable phenomenological interest: the fate of the glueball is widely debated [1,2].

As a counterpoint to our quenched study of this mixing, we also determine the mixed spectrum directly for two flavors of degenerate sea quark. This provides a check on our approach and, incidentally, indicates evidence for a surprisingly light scalar state at the lattice spacing we employ.

We include an Appendix giving details of the variance reduction technique relevant to determining disconnected fermionic loops which are needed in our study of flavor-singlet mesons.

II. LATTICE ANALYSIS

Here we discuss the formalism on a lattice to extract hadronic mixing. To set the scene, the variational approach is first summarized and the simpler case of weak or electromagnetic matrix elements is reviewed. Then we discuss hadronic mixing matrix elements.

A. Variational methods

Consider a hadronic correlator $C_{ij}(t)$ where t is the lattice separation in the time direction and i, j label the type of operator used to create or destroy the hadron (e.g., whether local, fuzzed, etc). We assume there are N types of operator.

In an ideal world of infinite statistics the matrix $C_{ij}(t)$ of correlations can be written in terms of the eigenstates of the transfer matrix, assuming these eigenstates to be discrete and non-degenerate, as

$$C(t) = A^T e^{-mt} A \quad (1)$$

where the intermediate state sums are over all ‘‘particles’’ allowed. Here A is a rectangular matrix.

However in practice, because of noise there has to be a truncation of the sum over intermediate states in the above equation.

One standard approach to circumvent these problems is the variational method. This can be motivated by maximizing

$$u_i C_{ij}(t) u_j \quad (2)$$

subject to constant $u_i C_{ij}(t-1) u_j$ which leads to the generalized eigenvalue equation

$$C_{ij}(t) u_j^\alpha = \lambda^\alpha C_{ij}(t-1) u_j^\alpha \quad (3)$$

with $\alpha = 0, \dots, N-1$ and where the ground state with $\alpha = 0$ has the largest eigenvalue λ^0 . It is usual to relate these eigenvalues to masses (energies in general) by $\lambda^\alpha = e^{-m\alpha}$. Given exact data, N eigenvalues of Eq. (3) can be determined. Note that these eigenvalues will only correspond to the true eigenvalues [of Eq. (1)] in the limit $N \rightarrow \infty$. In practice, one finds that a good choice of operators to create or destroy the hadron will yield a close approximation to the masses even with $N=2$ or 3. Bounds can also be derived, such that the variational estimate of the ground state mass is an upper bound.

In order to isolate a particular state, usually the ground state, one can use these variational eigenvectors to form a new basis. This is

$$\bar{C}_{\alpha\beta}(t) = u_i^\alpha C_{ij}(t) u_j^\beta \quad (4)$$

Then at times t and $t-1$, $\bar{C}_{\alpha\beta}$ will be diagonal and the diagonal elements will decrease like λ^α (i.e. as $e^{-m\alpha}$) as time increases from $t-1$ to t . It is convenient to normalize u^α such that $\bar{C}_{\alpha\beta}(t) = \delta_{\alpha\beta} (\lambda^\alpha)^t$.

A typical use of the variational method to extract the ground state signal is to form this variational basis using a low value of t where statistical errors are relatively small and then to explore the t dependence of $\bar{C}_{00}(t)$ at larger t values to extract the asymptotic behavior. Note that the finite (N state) variational basis derived at finite t will not match exactly to the true spectrum of excited states and hence \bar{C}_{00} will have some small remaining contamination of excited states. Indeed it is possible to use the variational estimate of the mass gap to the first excited state ($m_1 - m_0$) to control this extrapolation to large t to determine m_0 .

In order to make contact with the alternative procedure of fitting C directly to M states over some range of t :

$$C_{ij}(t) = \sum_{\alpha=0}^{M-1} c_i^\alpha c_j^\alpha e^{-m_\alpha t} \quad (5)$$

we note that u corresponds to the right eigenvectors of the nonsymmetric matrix $C^{-1}(t-1)C(t)$ and one can introduce left eigenvectors v , suitably normalized, which satisfy

$$v_i^\alpha = (\lambda^\alpha)^{(1-t)} C_{ij}(t-1) u_j^\alpha \quad (6)$$

and are orthogonal to the right eigenvectors

$$v_i^\alpha u_i^\beta = \delta_{\alpha\beta}. \quad (7)$$

In terms of these eigenvectors, we have

$$C_{ij}(t) = \sum_{\alpha=0}^{N-1} v_i^\alpha v_j^\alpha (\lambda^\alpha)^t. \quad (8)$$

and

$$C_{ij}(t-1) = \sum_{\alpha=0}^{N-1} v_i^\alpha v_j^\alpha (\lambda^\alpha)^{(t-1)}. \quad (9)$$

Thus we see that the variational method corresponds to making an exact fit to the data at t and $t-1$ with N states with the eigenvalues giving the masses and the left eigenvectors v are the couplings c .

B. Matrix elements: Operator insertions

Here we consider first the simpler case where a three point function is evaluated with an explicit operator (current) at an intermediate time t_1 . This is the case of weak and electromagnetic current insertions and also for some hadronic studies. One example is the study of semileptonic decays [3]. In general the quantum numbers of the states propagating before and after the insertion may be different. We shall assume that the spectrum of states is discrete. In a finite spatial volume, the two particle spectrum will indeed be discrete and it is possible to make use of this to explore relevant matrix elements [4] for two particle systems.

We study the general behavior of the three point function with insertion at t_1 where t is the lattice separation in the time direction and i, j label the type of operator used to create or destroy the hadronic state at times 0 and t where there

are N, N' types of operator, respectively. We include M states with masses (energies) m_α from time 0 to t_1 and M' states with masses (energies) m'_β from t_1 to t :

$$X_{ij}(t_1, t) = \sum_{\alpha=0}^{M-1} \sum_{\beta=0}^{M'-1} c_i^\alpha e^{-m_\alpha t_1} x_{\alpha\beta} e^{-m'_\beta(t-t_1)} d_j^\beta. \quad (10)$$

The task is usually to determine the matrix element x_{00} corresponding to the ground state hadrons. One can employ a fit to the above expression along with fitting the two point functions (C with couplings c and masses m and C' with couplings d and masses m'). Care should be taken to include sufficient states M and M' . A sensible criterion is that M should be chosen so that a good fit is obtained to the two point correlator over time interval t_1 for the appropriate hadron with creation or destruction operators as used in the three point function for that hadron and M' should be chosen likewise so that a good fit was obtained for the appropriate two point correlator over time interval $t - t_1$.

Another way to approach this analysis is to use of the variational technique discussed above. Then with a variational basis for the appropriate two point functions (of size N, N' respectively) one can form

$$\bar{X}_{\alpha\beta}(t_1, t) = u_i^\alpha X_{ij}(t_1, t) u_j'^\beta. \quad (11)$$

This expression will have non-zero off-diagonal elements in general since the operator insertion need not be diagonal in the spectrum basis. In this variational approach, one can extract x_{00} by taking a ratio to the variational ground state two point functions determined as above in Eq. (4),

$$x_{00} = \bar{X}_{00}(t_1, t) / [\bar{C}_{00}(t_1) \bar{C}'_{00}(t - t_1)] \quad (12)$$

where this ratio should be independent of t_1 provided t_1 and $t - t_1$ are not too small.

C. Matrix elements: Hadronic mixing

The problem of determining hadronic matrix elements involved in mixing—for example in mixing of glueball and scalar meson, in string breaking or in flavor-singlet meson studies—is much less straightforward. In full QCD, mass eigenstates can be determined directly and one is able, much as in experiment, to determine the masses of the resulting mixed states. In quenched, or partially quenched, studies it is possible to study mixing more directly by evaluating correlators between the different states involved. This is the area we explore here.

In lattice QCD with Euclidean time, the main factor is that the lightest state allowed will dominate the correlation at large time separation t . Thus in a study of glueball decay to two pions, for example, the two pion state will be lightest and dominate. In general this makes hadronic decays difficult to study [5]. As we shall see, the way forward is to restrict study to transitions in which the energies are similar: on-shell transitions. In the case of glueball decay to two pseudoscalar mesons, this would imply varying the quark mass until the two pseudoscalar mesons had comparable energy to

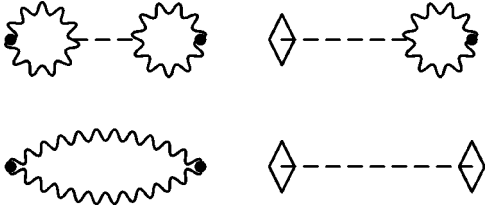


FIG. 1. We illustrate correlations among scalar glueballs (created by a closed Wilson loop) and scalar mesons made from quark-antiquark. Clockwise from the top left: the disconnected fermionic correlation (D^{ss}), the cross correlation of a fermion loop with a Wilson loop (C^{ms} or H), the correlation of Wilson loops (C^{mm}) and the connected fermionic correlation (C^{ss}).

the glueball, as was arranged in a pioneering study [6].

Here we have in mind several situations where energies are comparable: (i) scalar glueball mixing with scalar mesons, (ii) $B \bar{B}$ mixing with a static $Q\bar{Q}$ at separation R , (iii) mixing of $s\bar{s}$ pseudoscalar mesons with singlet $q\bar{q}$ pseudoscalar mesons. In each case one system is fully treated (i.e. the gluonic interactions in the scalar glueball and the static $Q\bar{Q}$ or the $q\bar{q}$ state with sea quarks q in the vacuum) while the other has valence quarks which are treated in a quenched (or partially quenched) approach.

Our notation is that we consider M states with masses m_α to be fully described (that is to say they are either fermionic with those quarks present in the vacuum or they are fully described in a quenched theory as are glueballs or potentials). We also consider M' states with masses s_β which are quenched or partially quenched (i.e. fermionic with valence quarks which do not participate in the sea). Here we assume that the spectrum of states is discrete in each case. To be more specific, in one application one can think of the m states as glueballs and the s states as scalar mesons. Some relevant correlations are illustrated in Fig. 1.

More generally, the s states do not contribute to correlators unless an explicit operator creates or destroys them whereas the m states can occur as intermediate states in any correlator with the correct quantum numbers. We shall treat the s states as those given by the relevant connected fermionic correlator. The disconnected contribution to any correlator will be included explicitly. The model used for these disconnected contributions may depend on the application.

Assuming that the disconnected fermionic loops in the s state to s state correlator are joined by gauge links, as in the example of Fig. 1 where a glueball intermediate state is appropriate, one would expect transitions between these s states and m states to be the relevant description and so would not allow direct transitions from s state to s state. In some applications, however, such as pseudoscalar mesons, the gluonic links between the disconnected loops are expected to be short ranged (effectively a contact interaction) and so are treated as local. In this case an explicit mixing coefficient between disconnected s states should be introduced.

We will assume that transitions between states of mass m_α and s_β are local and have strength $x_{\alpha\beta}$. The goal is usually to determine the transition strength between ground

states: x_{00} . Because the transition is hadronic, there is no explicit matrix element insertion. One must deduce the strength of the transition from a study of the two point correlators alone.

Consider now the two point correlators at separation t between operators creating either of these types of state. For each type of creation or annihilation operator, we consider several different operators labeled $i, j = 1, N$ and $k, l = 1, N'$ respectively (here we have in mind different smearing or fuzzing prescriptions). Then we have a description in terms of transfer matrix eigenstates as

$$C_{ij}^{mm}(t) = \sum_{\alpha=0}^{M-1} c_i^\alpha e^{-m_\alpha t} c_j^\alpha \quad (13)$$

$$C_{ik}^{ms}(t) = \sum_{t_1=0}^t \sum_{\alpha=0}^{M-1} \sum_{\beta=0}^{M'-1} c_i^\alpha e^{-m_\alpha t_1} x_{\alpha\beta} e^{-s_\beta(t-t_1)} d_k^\beta. \quad (14)$$

There will also be additional terms with $t \rightarrow T-t$ for a lattice periodic in time with extent T —these we do not write explicitly. Now for the quenched s correlation one can separate it into connected and disconnected fermion contributions

$$C_{kl}^{ss}(t) = C_{kl}^{ss}(t) + D_{kl}^{ss}(t) \quad (15)$$

with

$$C_{kl}^{ss}(t) = \sum_{\beta=0}^{M'-1} d_k^\beta e^{-s_\beta t} d_l^\beta \quad (16)$$

and

$$D_{kl}^{ss}(t) = \sum_{t_1=0}^t \sum_{\beta=0}^{M'-1} \sum_{\beta'=0}^{M'-1} \sum_{t_2=t_1}^t \sum_{\alpha=0}^{M-1} d_k^\beta e^{-s_\beta t_1} x_{\alpha\beta'} e^{-m_\alpha(t_2-t_1)} x_{\alpha\beta'} e^{-s_{\beta'}(t-t_2)} d_l^{\beta'}. \quad (17)$$

One might also include contributions to D^{ss} coming from a direct (i.e., not via an m state) transition at t_1 from state s_β to $s_{\beta'}$ with mixing strength $y_{\beta\beta'}$. As discussed above, this is appropriate for a discussion of pseudoscalar meson mixing, for example. In general, if propagation of the states of mass m_α over zero time steps is included, then the above formula does include contributions giving the effect of such direct transitions.

The main problem with extracting the mixing matrix elements x_{00} is in removing the excited state contributions. Here we assume that there is a finite mass gap between the excited state and ground state. Unlike in the case of the explicit insertion at t_1 , here we have no dependence on t_1 (and t_2) in the observables. This implies that t_1 values from 0 to t will be allowed so that there will be excited state contributions to the correlator which are not suppressed (since $e^{-(m_1-m_0)t_1}$ with $t_1=0$ is large, for example).

The cleanest way to circumvent this is in the case when the masses are equal, i.e., $m_0=s_0$. In this case, called the on-shell case, the mixing observable $C_{ik}^{ms}(t)$ will have a con-

tribution from the sum over t_1 of $c_i^0 x_{00} t e^{-m_0 t} d_k^0$ from the ground states whereas the dominant excited state contributions behave as $c_i^0 x_{01} e^{-m_0 t} d_k^1$ and will be suppressed by a relative factor of $1/t$. This is a much smaller relative suppression than the factor of $e^{-(m_1 - m_0)t}$ which applies to two point correlators, but it is sufficient to remove excited state contributions. This implies that x_{00} can in principle be extracted unambiguously. Likewise $D_{kl}^{ss}(t)$ has a contribution of $d_k^0 x_{00} t^2 e^{-m_0 t} x_{00} d_l^0 / 2$ from the ground states which also dominates, by a factor of t , any excited state contributions. This gives a further cross-check since x_{00} can be determined in these two independent ways.

The analysis when $m_0 \neq s_0$ is trickier. The only t dependences that can be observed will be of the form $e^{-m_\alpha t}$ and $e^{-s_\beta t}$ [and also $t e^{-s_\beta t}$ for $D^{ss}(t)$]. Then assuming that the M, M' state fits to $C_{ij}^{mm}(t)$ and $C_{kl}^{ss}(t)$ yield the masses m_α , s_β and couplings c_i^α and d_k^β of both ground states and excited states, the t dependence of C_{ik}^{ms} and D_{kl}^{ss} are available to determine $x_{\alpha\beta}$. In principle there are enough such independent t dependences to determine the mixing parameters, given sufficiently precise data.

One way to see this is to use the variational formalism with eigenvectors u_i^α obtained from $C_{ij}^{mm}(t)$ and w_i^β obtained from $C_{kl}^{ss}(t)$ using time values of t and $t-1$. Then we can project into this variational basis

$$H_{\alpha\beta}(t) = u_i^\alpha C_{ik}^{ms}(t) w_k^\beta. \quad (18)$$

If the variational basis corresponds to the exact spectrum then we would have, using continuum evaluation of the sum over t_1 as would be appropriate for a very small lattice spacing,

$$H_{\alpha\beta}(t) = x_{\alpha\beta} \frac{e^{-m_\alpha t} - e^{-s_\beta t}}{s_\beta - m_\alpha} \quad (19)$$

from which $x_{\alpha\beta}$ can be extracted. In general, however, the variational basis does not correspond to the exact spectrum. A fit to the t dependence then is needed. Provided enough operators N and N' are used (namely $N > M, N' > M'$), there is sufficient information to extract the parameters $x_{\alpha\beta}$ in principle. A similar variational analysis of $D_{kl}^{ss}(t)$ is also possible and this gives another way to determine constraints on the parameters $x_{\alpha\beta}$. If $\beta = \beta'$ then, using continuum evaluation of the sums over t_1 and t_2 ,

$$D_{\beta\beta'}(t) = \sum_\alpha x_{\alpha\beta} x_{\alpha\beta'} \frac{e^{-m_\alpha t} - e^{-s_\beta t} [1 + (s_\beta - m_\alpha)t]}{(s_\beta - m_\alpha)^2} \quad (20)$$

while if $\beta \neq \beta'$, then

$$D_{\beta\beta'}(t) = \sum_\alpha x_{\alpha\beta} x_{\alpha\beta'} \left(\frac{e^{-m_\alpha t}}{(s_\beta - m_\alpha)(s_{\beta'} - m_\alpha)} + \frac{e^{-s_\beta t}}{(s_\beta - m_\alpha)(s_\beta - s_{\beta'})} + \frac{e^{-s_{\beta'} t}}{(s_{\beta'} - m_\alpha)(s_{\beta'} - s_\beta)} \right). \quad (21)$$

Note that this approach assumes that an accurate description of the diagonal two-point correlators exists which is valid down to $t=0$. This is in practice hard to achieve. In particular the $t=0$ correlator is often quite different (sometimes even in sign) from the $t>0$ correlators. We now address the possibility of excited states that contribute only to $t=0$ since these are needed to cope with this data.

Let us explore this situation with $m_0 \neq s_0$ in a simple example. We will assume precise data are available for the correlations at all t values and that, by an optimal choice of operators, the diagonal correlations C^{mm} and C^{ss} are described exactly by one state for $t \geq 1$ and so have an additional excited state contribution of the form

$$C^{mm}(t) = c^0 e^{-m_0 t} c^0 + c^1 \delta_{t0} c^1 \quad (22)$$

$$C^{ss}(t) = d^0 e^{-s_0 t} d^0 + d^1 \delta_{t0} d^1. \quad (23)$$

Here we suppress the operator labels (i, j, k , etc.) since we are considering the case that an optimum combination of them has already been taken to isolate the ground state (i.e., just taking $\alpha=0$ and $\beta=0$ above). Then the cross correlation has the form for $t>0$, including the excited state contributions from $t_1=0$ and $t_1=t$,

$$H(t) = \sum_{t_1} c^0 e^{-m_0 t_1} x_{00} e^{-s_0(t-t_1)} d^0 + c^1 x_{10} e^{-s_0 t} d^0 + c^0 e^{-m_0 t} x_{01} d^1. \quad (24)$$

Now by completing the sum over t_1 as a discrete sum, $H(t)$ can be expressed as

$$H = c^0 d^0 e^{-m_0 t} (A + B \lambda^t) \quad (25)$$

with $\lambda = e^{-(s_0 - m_0)}$,

$$A = \frac{x_{00}}{1 - \lambda} + \frac{d^1 x_{01}}{d^0} \quad (26)$$

and

$$B = -\frac{\lambda x_{00}}{1 - \lambda} + \frac{c^1 x_{10}}{c^0}. \quad (27)$$

Here d^0, d^1, c^0, c^1 and λ are known in principle but the mixing parameters x_{00} , x_{01} and x_{10} are to be determined. With perfectly precise data for H , only the coefficients A and B can be determined; hence the three mixing parameters cannot be independently determined. Thus x_{00} cannot be determined, even in principle. The exception to this is when $\lambda \rightarrow 1$, since

the contribution of x_{00} is then greatly enhanced. In fact in this case x_{00} can be read off from the coefficient of the linear term in t in H as discussed above.

Now when data are available for the correlation D then additional constraints exist. For our example this is

$$D(t) = d^0 d^0 e^{-m_0 t} (X + Y\lambda^t + Zt\lambda^t) \quad (28)$$

with

$$X = \left(\frac{x_{00}}{1-\lambda} \right)^2 + (d^1 x_{01} / d^0)^2 + \frac{2x_{00}x_{01}d^1}{(1-\lambda)d^0} \quad (29)$$

$$Y = -\lambda(2-\lambda) \left(\frac{x_{00}}{1-\lambda} \right)^2 - \frac{2x_{00}x_{01}d_1\lambda}{(1-\lambda)d^0} + 2x_{10}x_{11}d_1/d^0 + x_{10}^2 \quad (30)$$

and

$$Z = -\lambda x_{00}^2 / (1-\lambda) + x_{10}^2. \quad (31)$$

Here x_{11} is a new mixing: between the excited states in both sectors. We now have three additional constraints (X , Y and Z) given accurate data for $D(t)$, with only one additional parameter. Thus in the case of this simple model, the measurable quantities overdetermine the mixing parameters. Again as $\lambda \rightarrow 1$, this expression simplifies and the coefficient of t^2 in $D(t)$ gives x_{00}^2 directly.

One way to check that ground state contributions dominate is to extract x_{00} from H and D at several t values neglecting excited states and check for consistency. So the relevant expressions will be

$$x_H(t) = \frac{H(t)}{\sqrt{C^{mm}(t)C^{ss}(t)}} \frac{\lambda^{t/2}}{1+\lambda+\dots+\lambda^t} \quad (32)$$

and

$$x_D(t) = \sqrt{\frac{D(t)}{C^{ss}(t)}} \frac{\lambda^{t/2}}{\sqrt{1+2\lambda+\dots+(t+1)\lambda^t}}. \quad (33)$$

In our application below $\lambda=0.64$ for strange quarks, so the enhancement of the ground state mixing (x_{00}) by factors of $1/(1-\lambda)$ is not very big. However, we use information from both H and D which does provide a cross check in principle of our assumption that the excited state contributions (such as x_{01} and x_{10} in the above example) are negligible.

In summary, when we have an on-shell hadronic transition we can extract the mixing matrix element with a power suppression in t of excited states compared to the exponential suppression which applies to extracting masses. When the masses are not equal, excited states can still be removed in principle if a precise study is made of both correlations of type H and D . Note that in practice, if the difference in masses is significant, the extraction of the mixing strength will be very difficult.

TABLE I. Properties of the $N_f=2$ lattices from UKQCD [7] with $\beta=5.2$ and $C_{SW}=1.76$ and (last two rows) $N_f=0$ lattices [8] with $\beta=5.7$ and $C_{SW}=1.57$.

L_s	κ	r_0/a	$m_\pi a$	m_π/m_ρ
12	0.1390	3.05	0.707(5)	0.78
16	0.1390	3.03	0.701(6)	0.78
12	0.1395	3.44	0.558(8)	0.71
16	0.1395	3.44	0.564(4)	0.72
12	0.1398	3.65	0.476(16)	0.67
16	0.1398	3.65	0.468(5)	0.67
12	0.13843	2.94	0.736(2)	0.78
12	0.14077	2.94	0.529(2)	0.65

III. LATTICES

We choose to study sea quark effects using the configurations with $N_f=2$ at $\beta=5.2$ with $C_{SW}=1.76$ from UKQCD [7]. Two spatial lattice sizes are available (12^3 and 16^3) so that finite size effects can be explored. We use the three lightest sea quark masses available and we use valence quark masses equal to the sea-quark mass. The lattice information is summarized in Table I. Local and non-local meson operators were used with fuzzing radius 2 with 5 iterative levels with coefficient 2.5.

We also use for comparison a quenched lattice at $\beta=5.7$ of size $12^3 24$ with valence fermions of $\kappa=0.14077$ (approximately strange mass) and $\kappa=0.13843$ (approximately twice strange mass) with $C_{SW}=1.57$, as studied previously [8].

In order to improve the statistics we measure the disconnected diagrams on configurations separated by less trajectories than for the connected correlators as shown in Table II. Even though there may be some autocorrelation among these measurements separated by less trajectories, we find that this approach does allow the statistical error to be reduced. Indeed this is the approach that was used in glueball studies, where the measurement time is very small so one might as well measure almost every configuration—indeed we follow this approach here when considering glueballs, as we will discuss later.

TABLE II. Statistics of connected, disconnected and glueball calculations. For the fermionic disconnected correlation, the variance reduction methods with N_S samples were different for $L_s=12$ and 16 as described in the Appendix.

L_s	κ	N_S	Connected configs	Disconnected configs	Glueball configs
12	0.1390	24	151	301	301
16	0.1390	200	90	94	390
12	0.1395	24	121	253	505
16	0.1395	48	100	106	424
12	0.1398	24	98	169	170
16	0.1398	48	69	75	298
12	0.13843	24	482	100	100
12	0.14077	24	482	100	100

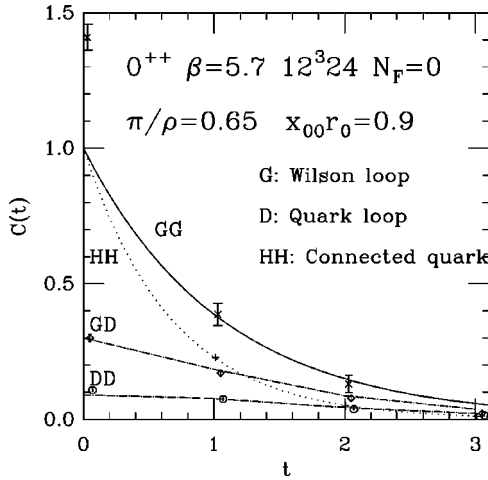


FIG. 2. Quenched scalar correlations with quark masses approximately strange. Here HH is the connected mesonic correlation (C^{ss}), GG is the glueball correlation (C^{mm}), DD is the disconnected mesonic correlation (D) and GD is the cross correlation between glueball and meson operators (H). Lattice results are illustrated for one glueball operator and one (local) meson operator. The curves are a simple mixing model, as described in the text.

IV. SCALAR MESONS

Within the quenched approximation, there will be two distinct types of scalar meson: $q\bar{q}$ mesons and scalar glueballs. The flavor singlet $q\bar{q}$ scalar meson will have a pole and a double pole contribution, as in the well known flavor singlet pseudoscalar case, with the pole mass being the same as the flavor non-singlet scalar meson mass.

In full QCD, these two types of state will mix, resulting in the observed experimental spectrum of scalar mesons. As an aid to disentangling this experimental situation, we here explore the lattice predictions for scalar mesons. As a first step we evaluate the mixing matrix elements in the quenched approximation. This has been explored [9] previously and here we discuss the problems associated with determining such hadronic mixing on the lattice.

A. Quenched lattice results

We explored this mixing in the quenched approximation (see Tables I and II) using SW-clover valence quarks of two different masses [8]. The zero-momentum glueball operators were measured at every time slice in the usual way [10] and the disconnected quark loops are measured as described in the appendix, namely with sufficient stochastic samples that no significant error arises from the stochastic algorithm. The connected quark correlators were taken from previous measurements [8]. Since the scalar meson or glueball has vacuum quantum numbers, we subtract the vacuum contribution in the other types of correlation we measure. Our results for all of these types of correlation are illustrated in Fig. 2 for the case of one choice of glueball operator and one (local) mesonic operator at our lighter quark mass.

The connected quark propagator is the same for degenerate quark masses for the flavor singlet and non-singlet me-

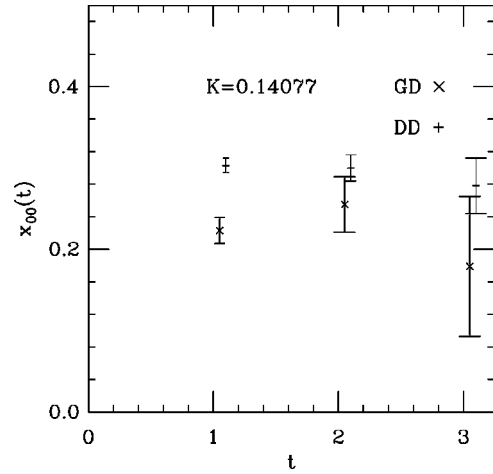


FIG. 3. The mixing coefficient x_{00} is determined in lattice units with $a^{-1} \approx 1$ GeV, at each t value, from quenched scalar correlations with quark masses approximately strange: DD is the disconnected mesonic correlation (D) and GD is the cross correlation between glueball and meson operators (H).

son. From the connected quark correlator C^{ss} , at hopping parameters $\kappa=0.14077$ and 0.13843 , we find scalar $q\bar{q}$ masses of $1.39(5)$, $1.36(2)$ respectively in lattice units, fitting local and two types of smeared operator to one state in the t range 2 to 8. Note that the mass ordering is not as expected (namely meson with lighter quarks being lighter) but the errors are large enough to cover near equality. This meson mass value is somewhat larger than that reported [9] at the same β value but using Wilson quarks of mass corresponding to strange (i.e., our $\kappa=0.14077$), namely $1.29(2)$. This discrepancy is not surprising since the SW-clover formalism we use has improved control of order a effects compared to the Wilson discretization.

For the glueball mass, which is of course independent of fermion formalism in quenched studies, we use the higher statistics result [9] of $0.95(2)$ in lattice units obtained for $t \geq 2$. Our result for the glueball correlator [$C^{mm}(t)$] is consistent with a single exponential with this mass for $t \geq 1$. Since our glueball correlator has large errors for $t > 1$, in evaluating the expressions shown in Figs. 3 and 4 we use our measured glueball correlations at $t=1$ but for $t > 1$ we assume the glueball correlation has the mass dependence given by $ma=0.95$ as found in higher statistics studies. As shown in Fig. 5, this glueball mass lies below the continuum extrapolation because of order a^2 lattice artifacts. To convert to physical units, we use $r_0/a=2.94$ and then conventionally $r_0 \approx 0.5$ fm, so $a^{-1} \approx 1.1$ GeV.

Then given these mass values, one can attempt to describe the disconnected and cross correlations (D and H) in terms of the one free parameter, the mixing strength x_{00} . The errors on our determinations of these correlations are quite large: 25% for D and 50% for H at $t=3$. We use local fermionic operators in these comparisons since the smearings used in the determination of the connected and disconnected fermionic correlator were different for historical reasons.

The curves shown in Fig. 2 are from the lattice model described above with ground state contribution only and with

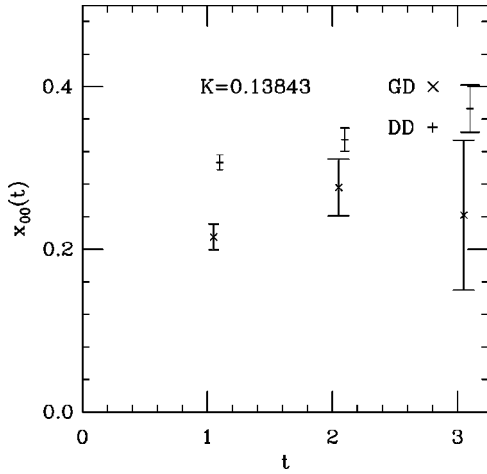


FIG. 4. As Fig. 3 but with quenched scalar correlations with quark masses approximately twice strange.

$x_{00}a = 0.3$. This is seen to give a reasonable overall description for $1 \geq t \geq 3$. Alternatively, in Figs. 3, 4 we give the value of the mixing parameter x_{00} for each t -value obtained from taking the data on H and D respectively and assuming the lattice mixing expressions of the previous section [Eqs. (32), (33)] with no excited state contributions. We find for each quark mass that $x_{00}a \approx 0.3$.

Since our results for the diagonal correlations of glueball and meson operators (C^{mm} and C^{ss}) are only reasonably approximated by the ground state for $t_g \geq 1$ and $t_q \geq 1$, as discussed above, the mixing should be studied from correlations with $t \geq 2$. As shown in Fig. 3, we find that the signal becomes very noisy by $t=3$ already. With 100 times the statistics we would be able to determine the $t=4$ mixing correlations (D and H) to 10%. This implies that much larger data sets (number of gauge configurations about 10000) are

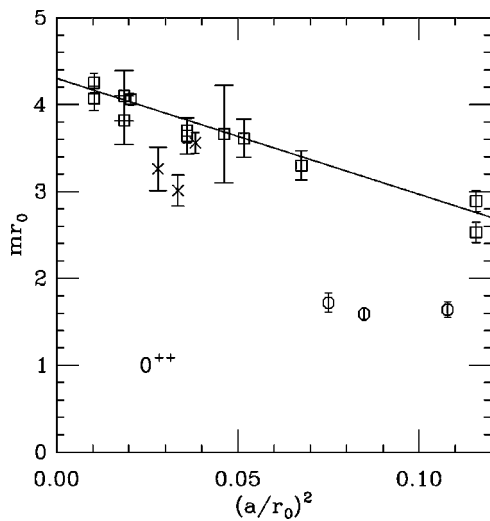


FIG. 5. The scalar mass versus a^2 . The quenched results [12,10,13,14] are for the scalar glueball and are shown by boxes. The results from $N_f=2$ flavors of sea quark are from glueballs [15] (crosses from SESAM) and the lightest flavor singlet scalar we find here (circles).

needed to give a more definitive answer to the mixing in this case. Even then since there is only a power suppression of excited states, one would need precise data to large t to have small systematic errors from excited state contributions.

The consistency of the determinations of x_{00} from different t values and different quantities does, however, act as a cross check that our results are consistent with the assumption that a single ground state dominates. Because of the lack of control of excited state contributions, we can only quote the statistical error on the mixing x_{00} . Assuming no excited state contamination and taking $t \geq 2$, we obtain $x_{00}a = 0.26(4)$ for strange quarks, corresponding to $x_{00}r_0 = 0.76(12)$. This is the quenched result for one flavor and we see no sign of any significant difference in x_{00} as we vary the quark mass since we have $x_{00}a = 0.32(4)$ for heavier quarks.

A previous work [9] has studied glueball mixing with a scalar meson in the quenched approximation. They used several β values, Wilson fermions and concentrated on the cross correlation H to determine x_{00} . At $\beta=5.7$ and for quark mass near strange, we can make a direct comparison, bearing in mind that the order a corrections are significant at such a coarse lattice spacing and will be different for Wilson and SW-clover fermion formalisms, indeed the SW-clover formalism we use is focused on removing these order a effects.

At $\beta=5.7$ they have $t_q=2$ and $t_g=2$ and they determine their mixing coefficient from data for H with $t \geq 2$ by assuming no excited state contributions to H . They do not consider data on D . Note that, as discussed above, from measurement of H alone, it is impossible in principle to confirm that excited state contributions are absent. Their quoted result for strange quarks is $x_{00}a = 0.211(16)$. This is broadly compatible with our estimate of $x_{00}a = 0.26(4)$ bearing in mind that different fermion discretizations were used.

At larger β (up to 6.2) their results for the mixing are that, at the strange quark mass, the mixing tends to a very small value in the continuum limit. The situation concerning excited state contamination is even worse at larger β since they find $t_q=6$ and $t_g=4$ at $\beta=6.2$, whereas they still fit H for $t \geq 2$. Thus their mixing estimates at larger β are even more susceptible to excited state contamination. Moreover, they do not make use of the disconnected correlator D to constrain their assumptions further.

We find a significant mixing at coarse lattice spacings using a fermion formalism that has been shown to have reduced order a corrections. This implies that we would expect a substantial mixing in the continuum if order a^2 corrections were also to be small. This is in contrast to the conclusion [9] that the a dependence of the mixing (in physical units) is such that the continuum mixing is very small. We conclude that we find evidence for a mixing strength $x_{00}r_0 = 0.76(12)$ with one flavor of quarks of strange mass at our lattice spacing.

B. Full QCD scalar mesons

To compare with our results using dynamical sea quarks with $N_f=2$, we estimate the effects of our quenched determination of the mixing if applied to that case. For this estimate we take $x_{00}a = 0.3$ for one quark flavor. Then for quarks

of strange mass our unmixed states with $m_0a=0.95$ and $s_0a=1.39$ will be mixed by off diagonal element $x_{00}a=0.3\sqrt{N_f}=0.42$ giving a mass mixing matrix:

$$\begin{pmatrix} am_0 & \sqrt{2}ax_{00} \\ \sqrt{2}ax_{00} & as_0 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.42 \\ 0.42 & 1.39 \end{pmatrix}$$

which gives mass eigenstates pushed apart to $ma=0.69$ and 1.65 (and for the heavier quarks with the same mixing strength then 0.95 and 1.36 will be mixed to 0.68 and 1.63). Thus for strange quarks, the lightest scalar meson would be reduced in mass by approximately 0.24 in lattice units. If our quenched mixing strength were to be applied to the scalar mass matrix, it results in a downward shift for $N_f=2$ of the lattice glueball mass by 25% .

We now consider the sea-quark case explicitly, where the mixing will be observed directly from the resulting mass values. What we can measure in that case is the non-singlet mass and the ground and excited states in the flavor singlet sector. Based on the results from the quenched study, we would expect in the flavor singlet sector that the ground state mass lies considerable below the flavor non-singlet mass and the first excited state mass is slightly above the flavor non-singlet mass.

We use 4 scalar meson operators (i) closed Wilson loops (glueball operators) of two different sizes (Teper-smear [10]) and (ii) $q\bar{q}$ operators which are local and separated by fuzzed links. For the fermionic correlations, we include the connected and disconnected contributions as given by the Wick formalism. We evaluate the 4×4 matrix of correlations. Since different numbers of gauge configurations are analyzed for different operators, where necessary, we average results from nearby gauge configurations to have a consistent bootstrap sample for error analysis.

We should also consider two particle states (two pseudo-scalar mesons with momentum $\pm 2\pi n/L_s$, for example) which can mix with scalar mesons and glueballs. On a lattice the lightest such state with overall momentum zero will have energy $2a(m_\pi^2 + 4\pi^2 n^2/L_s^2)^{1/2}$ which is at 1.12 for the lightest case of $n=0$ and with $\kappa=0.1395$ —this will apply to the flavor singlet case. For the flavor non-singlet scalar meson then the $\pi\eta$ mode will be the lightest and this will be even heavier. These two particle energies are sufficiently heavy that we shall ignore these states in our present work. They will, however, become important as the quark mass is reduced.

The glueball and fermionic singlet correlations have a vacuum contribution. For the glueball fits we deal with this by using 2 state fits with one state constrained to have mass=0 (namely the vacuum). In such fits it is important to use correlated fits to have a meaningful expression for χ^2 . Since we are fitting to many (up to 60) different types of data (different operators at source-sink and different t values), we use the technique of retaining exactly the N_e largest eigenvalues of the correlation among the data set and setting the remaining eigenvalues equal [11]. This avoids spurious correlations being induced because of our limited sample size. We use $N_e=10,8$ for $L_s=12,16$ respectively. For our fits to

TABLE III. Ground state scalar masses from fits to the glueball sector (GB), the whole flavor singlet sector (FS: glueball and fermionic operators) and to the fermionic non-singlet sector (NS). The errors quoted are statistical only.

L_s	κ	$m_{GB}a$	$m_{FS}a$	$m_{NS}a$
12	0.1390	0.40(6)	0.54(3)	1.23(4)
16	0.1390	0.53(7)	0.47(3)	1.19(5)
12	0.1395	0.49(4)	0.46(2)	1.23(4)
16	0.1395	0.70(9)	0.75(4)	1.18(8)
12	0.1398	0.48(10)	0.47(3)	1.00(5)
16	0.1398	0.58(8)	0.66(4)	0.99(6)

the full 4×4 matrix of singlet correlations, since the number of gauge samples is different for different observables which makes the vacuum contribution depend on the observable, it is preferable to subtract the vacuum contribution and fit the resulting connected correlation. The errors quoted on the fits are statistical from bootstrap analysis and do not include systematic errors from varying the fit range or fit function: these are at least comparable in size.

We can now measure directly the scalar spectrum for $N_f=2$ to explore this. Results are shown in Table III. We obtain the flavor non-singlet mass (m_{NS}) from a two state fit from $t=2$ to 7 to the 2×2 matrix of connected fermionic correlators. For the singlet, we now use both glueball and $q\bar{q}$ operators (vacuum subtracted) and find an acceptable fit with 1 state to the 4×4 matrix of correlations for t from 2 to 7 : the results are shown as $m_{FS}a$ in Table III. We find that the mass obtained from fitting only the 2×2 matrix of glueball correlations (m_{GB}) is consistent with the full fit, as it should be. Moreover, we see a surprisingly low scalar mass—as emphasized in Fig. 5 which compares with quenched results and the SESAM $N_f=2$ values [15].

It would be interesting, as discussed above, to obtain the excited state mass in the flavor singlet sector. We expect this above m_{NS} at around $am'=1.4$. We have used the variational method for $t=1,2$ to extract the two lightest mass eigenstates from our 4×4 matrix of vacuum-subtracted correlators. This variational ground state mass ($0.44(1)$ for $\kappa=0.1395$) agrees quite well with the fitted value shown in Table III, as expected. The next state is poorly determined although for the case with best statistics ($L_s=12$ and $\kappa=0.1395$) we find $am'=1.27$ which is close to am_{NS} . It will be interesting to explore this further with higher statistics. Note that it is difficult to determine this mass since the signal is swamped by that of two lighter states (the vacuum at $m=0$ and the ground state at $am\approx 0.5$).

We do expect a relatively light flavor-singlet scalar mass because of mixing effects as described above which would reduce the mass by 25% . This could explain in part our low scalar mass but other explanations are also worth exploring. For example the order a^2 corrections might be anomalously large for our lattice implementation (e.g. twice as large as in the quenched Wilson case).

Another possible explanation of the light flavor-singlet scalar mass we find would be a partial restoration of chiral

symmetry in a finite volume, however, this would be expected only for very light quark masses. We do find that our flavor-singlet scalar meson masses are *lighter* than the pseudoscalar non-singlet (pion) mass for $L_s=12$ for the range of sea quark masses considered here. The spatial size with $L_s=12$ is 1.7 fm and no evidence of finite size effects was seen here in a study of flavor non-singlet correlators [7]. Indeed we see no significant sign of any spatial size dependence in the non-singlet scalar masses reported in Table III.

To explore this further, we have made a study of flavor singlet correlators on 16^3 spatial lattices to check directly for finite size effects and the results are presented in Table III. The signal to noise from zero momentum correlations is worse for the larger volume for glueball and disconnected correlations. Also we find that the excited state contributions are relatively stronger for our operators. Thus the systematic fit errors for $L_s=16$ are also considerably larger than those for the smaller volume. Even though the signal from the larger spatial volume is relatively poorly determined, we do see some evidence (at the 2σ level if the systematic errors are taken as comparable to the statistical ones) of a higher scalar mass (for m_{GB} and m_{FS}) on the larger spatial lattice. In order to explore larger L_s values, the signal to noise can be improved by considering non-zero momentum correlators, as is the case for glueball studies [10].

One conclusion is that it would be valuable to use a finer lattice spacing or an improved gauge discretization so that any suppression of the glueball mass by order a^2 effects would be reduced. This would increase the glueball mass and hence reduce the magnitude of the signal we see, but it would move the parameters into a region closer to experiment.

V. CONCLUSION

Hadronic mixing as exemplified by the glueball mixing with a scalar meson can be explored using lattice methods. In the quenched approximation, one can determine the mixing strengths although the systematic errors in this determination are large as we have discussed. In studies with sea quarks, the mixed spectrum itself is obtained which gives complementary information.

In a preliminary study of this glueball mixing, we find a large mixing in a quenched study and, consistently, a large suppression of the mass of the lightest scalar meson when sea quarks are included. These studies are computationally difficult and we have used a coarse lattice spacing, albeit with an improved (SW-clover) fermion formalism. It will be necessary to extend these studies to smaller lattice spacing in order to have more confidence in their relevance to the phenomenological situation.

APPENDIX: DISCONNECTED FERMION LOOP EVALUATION

To evaluate disconnected quark loops with zero momentum, we need to sum over propagators from sources at each spatial location at a given time slice. This problem has been approached using stochastic source methods [18–21]. Here

we describe in more detail the variance reduction techniques described previously [16]. For a similar method see Ref. [22].

To study flavor-singlet mesons, we need to consider quark loops which are disconnected (often called hairpins), namely evaluate $\text{Tr}\Gamma M^{-1}$ where the sum (trace) is over all space (for zero momentum) at a given time value and all colors and spins. Here M is the lattice fermion matrix, Γ is a combination of the appropriate γ -matrix and a product of spatial gauge links if a non-local (fuzzed) operator is used for the meson.

Using a random volume source ξ (where $\langle \xi^* \xi \rangle = 1$ for the same color, spin and space-time component and zero otherwise) then solving $M\phi = \xi$, one can evaluate unbiased estimates of the propagator M_{xy}^{-1} from $\langle \xi_y^* \phi_x \rangle$ where the average is over the N_S samples of the stochastic source. The computational overhead of this method lies entirely in the inversion of M to obtain ϕ from ξ for each of the N_S samples.

The drawback of this approach is that the variance on these estimates can be very large, so that typically hundreds of samples are needed. Here we present a method which succeeds in reducing this variance substantially at rather small computational expense.

The variance reduction is based on expressing the fermion matrix M as

$$M = C - D = C(1 - C^{-1}D) = (1 - DC^{-1})C \quad (\text{A1})$$

where C is easy to invert, for example the SW-clover term which is local in space or in the Wilson case where one can chose $C=1$. Then we have the exact identity

$$M^{-1} = C^{-1} + C^{-1}DC^{-1} + \dots + (C^{-1}D)^m C^{-1} + (C^{-1}D)^{n_1} M^{-1} (DC^{-1})^{n_2} \quad (\text{A2})$$

with $n_1 + n_2 = n = m + 1$.

Our strategy will be to evaluate the last term in this expression stochastically and to evaluate the preceding terms as exactly as possible. We will refer to these terms in our following discussion as the stochastic and exact terms. If these exact terms can be evaluated precisely, then it is plausible that the stochastic term will contribute less variance to the overall estimate of M^{-1} than in the $m=0$ case where there would be no such exact terms. These exact terms can be evaluated either directly (for example terms with odd powers of D vanish in the evaluation of a local trace) or as a subsidiary stochastic calculation with more samples to achieve good precision and at relatively small computational overhead since no inversion is required.

Since this approach is a variant of the hopping parameter expansion, it might be suspected that the convergence was poor since at each higher order 8 extra terms are present with coefficients which are of order $\kappa \approx 1/8$. In our application, however, these 8 terms contribute with random strengths—like a random walk. So they have an effective weight which is more like $\sqrt{8}$ which is smaller. So we do find a reduced

variance including more terms, but at the cost of some extra computation in evaluating the additional exact terms on the right hand side of Eq. (A2).

A special case of this ($n_1=n_2=2$) with Wilson fermions (for which $C=1$ and the terms with up to 3 powers of D vanish for $\text{Tr}M^{-1}$) employing Gaussian noise was used by the bermion group [17] previously.

Using the stochastic volume source, the variance reduced expression can be rewritten (assuming C is Hermitian) as

$$\sum_x \Gamma_{xy} M_{yx}^{-1} = \dots + \langle [(C^{-1}D^\dagger)^{n_2} \xi]_x^* \Gamma_{xy} [(C^{-1}D)^{n_1} \phi]_y \rangle \quad (\text{A3})$$

so the stochastic term may be evaluated as an average over stochastic samples ξ after inversion to obtain $\phi = M^{-1} \xi$. In the application of this paper we take Γ to have the Dirac structure I whereas in other applications [16,23] we consider γ_5 and $\gamma_5 \gamma_4$ also.

Only the even exact terms in the series when Γ is local are non-zero and we calculate the $m=0$ and 2 cases explicitly. For hadron operators with fuzzed paths of length n_F the series starts at $m=n_F$ (this we calculate explicitly) and then has alternate terms zero. The explicit calculations referred to are rather cumbersome for clover fermions, so in some cases we actually evaluate the simpler Wilson expression and then evaluate the difference stochastically. The generic non-zero terms (A) in the series were calculated stochastically using $4N_S$ Z2 noise samples ξ using

$$\sum_x \Gamma_{xy} A_{yx} = \Gamma_{xy} \langle \xi_x^* A_{yz} \xi_z \rangle. \quad (\text{A4})$$

Taking $n_1 \approx n_2$ gives an averaging over a smaller volume than taking an asymmetric choice. We find that an asymmetric choice gives a smaller variance, presumably because it does involve averaging over a larger volume. For different disconnected observables, the optimum strategy is not necessarily the same. In this work, we find an overall good

choice to be $n_1=0, n_2=16$ with Gaussian noise. This results in a reduction by a factor of 0.2 to 0.3 of the standard deviation of the samples.

Using larger values of n_1 and n_2 implies that the estimate of M^{-1} is very non-local, involving ξ values up to n_1 and n_2 away. To evaluate correlators between traces at t_1 and t_2 , one must require that the samples of stochastic volume source used in the two cases are different so that there is no bias. We use $N_S=24$ stochastic samples and this condition is readily implemented with essentially no loss of statistics. This number of samples was chosen to make the stochastic sampling error smaller than the intrinsic variance from one time slice (for example at $L_s=12$ with $\kappa=0.1395$, the ratio of the stochastic sampling error to the standard deviation over time slices was 0.5, 1.0, 0.2 for $\Gamma = \gamma_5, \gamma_5 \gamma_4, I$ respectively for local operators and the ratio was about 50% bigger for fuzzed hadronic operators). The computational effort in terms of the number of inversions is equivalent to that in obtaining two conventional propagators (from two sources of all color spins). We also make use of the fact [20] that one can truncate the inversion at a larger residual without measurable bias, using 10^{-9} in an MR inverter as residual or source.

In conclusion, using $N_S=24$ samples for each gauge configuration corresponds to 2 conventional propagator determinations (from all 12 color-spin combinations) and so is not a particularly big computational challenge and yet the resulting measurement of the disconnected fermion loop has a stochastic error which is unbiased and less than its intrinsic error.

For spatial size 16, we made use of existing solver codes and chose not to use the variance reduction method described above, using instead $n_1=n_2=0$. To achieve some variance reduction, we used the method [24] of using pairs of sources with the same random Z2 numbers but with the second of the pair multiplied by $\gamma_i \gamma_5$ where i is chosen randomly from 1...3. This has the effect of reducing the standard error by a factor of two for $\Gamma = \gamma_5$ for only a doubling of CPU effort. In this case we used either $N_S=48$ or 200, the larger value being motivated by the need to get a more precise estimate for $\Gamma = \gamma_5 \gamma_4$.

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