LTH 561
hep-th/0209111

General classical solutions in the noncommutative CP^{N-1} model

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We give an explicit construction of general classical solutions for the noncommutative CP^{N-1} model in two dimensions, showing that they correspond to integer values for the action and topological charge. We also give explicit solutions for the Dirac equation in the background of these general solutions and show that the index theorem is satisfied.

September 2002

The two-dimensional CP^{N-1} model[1][2] has long been considered as a useful test-bed for four-dimensional gauge theories, since it possesses many of the same features, such as conformal invariance, asymptotic freedom and a topological charge taking integer values. There is also the possibility of a $\frac{1}{N}$ expansion giving useful information about confinement[3][4]. The action is minimised for self-dual and anti-self-dual instanton configurations[2][3] for which the action is proportional to the topological charge. The instanton contribution to the functional integral defining the quantum field theory was evaluated in Refs. [5][6]. However, the instanton and anti-instanton solutions do not exhaust the solutions to the classical field equations which should be used in this stationary phase approximation; it turns out that the general solution can be given by a very elegant construction[7] based on a similar analysis for the $O(2k+1)$ σ-model in Ref. [8]. These solutions are in general saddle points of the action, which still takes well-defined integer values as of course does the topological charge[9]. Moreover the solution to the Dirac equation in the background of a general classical solution can be given in an equally elegant fashion[10].

Quantum field theory on noncommutative spacetime has received much attention recently, largely due to its emergence in M-theory (for reviews see Refs. [11][12]). In particular a good deal of work has been devoted to instanton solutions in noncommutative gauge theory[13]. As in the commutative case, this has also naturally led to interest in the CP^{N-1} model[14]. It was shown that the instanton and anti-instanton solutions could straightforwardly be generalised to the noncommutative case. In this paper we shall show that all the work on the general classical solutions in Refs. [7]–[10] generalises equally naturally.

We consider the CP^{N-1} model defined on the noncommutative complex plane[14]. The co-ordinates x_1, x_2 satisfy

$$[x_1, x_2] = i\theta, \tag{1}$$

with $\theta > 0$, but as usual it will prove useful to introduce complex co-ordinates

$$x_+ = \frac{x_1 + ix_2}{\sqrt{2}}, \qquad x_- = \frac{x_1 - ix_2}{\sqrt{2}} \tag{2}$$

which satisfy

$$[x_+, x_-] = \theta. \tag{3}$$

We can represent $x_\pm$ by creation and annihilation operators, acting on the harmonic oscillator Hilbert space with ground state $|0>$ satisfying $x_+|0>= 0$ and $|n>= \frac{1}{\sqrt{\theta^n n!}}(x_-)^n|0>$, so that

$$x_+|n>= \sqrt{\theta n}|n-1>, \quad x_-|n>= \sqrt{\theta(n+1)}|n+1> . \tag{4}$$

Then the derivatives $\partial_\pm = \frac{\partial}{\partial x_\pm}$ can be represented as

$$\partial_+ z = -\theta^{-1}[x_-, z], \quad \partial_- z = \theta^{-1}[x_+, z]. \tag{5}$$

Correspondingly, the integration over the noncommutative plane may be represented by the trace over the harmonic oscillator Hilbert space. The Lagrangian is given by

$$L = \partial_\mu \overline{z}.\partial^\mu z + (\overline{z}.\partial_\mu z)(\overline{z}.\partial^\mu z) = \overline{D_\mu z}.D_\mu z, \tag{6}$$

where z is an N-dimensional vector over the noncommutative complex plane, subject to the constraint

$$\overline{z}.z = 1, \tag{7}$$

where for any N-vector X

$$D_\mu X = \partial_\mu X - X\overline{z}.\partial_\mu z, \tag{8}$$

and

$$|X|^2 = \overline{X}.X \tag{9}$$

(Strictly speaking it is an abuse of terminology to call z a vector since vector spaces are defined over a field, in which by definition multiplication is commutative. However, all the standard theorems and properties of vector spaces with regard to bases, dimensionality and orthogonality will remain valid in the noncommutative case, as long as we adhere to the convention that multiplication of a vector by a scalar is on the right.) Note that the complex conjugate satisfies $\overline{fg} = \overline{g}\overline{f}$.

The corresponding action is

$$S = \mathrm{Tr} L = 2\pi\theta \sum_{n\geq 0} <n|L|n>, \tag{10}$$

where $|n>$, $n = 0, 1, 2\ldots$ are the usual normalised harmonic oscillator energy eigenstates. The model has a local $U(1)$ symmetry

$$z \to zg(x), \quad g(x) \in U(1), \tag{11}$$

under which

$$D_\mu X \to D_\mu X g(x). \tag{12}$$

Note that here as always the ordering is crucial. On introducing covariant derivatives $D_\pm$ corresponding to $x_\pm$, we can rewrite L as

$$L = |D_+ z|^2 + |D_- z|^2, \tag{13}$$

and the field equation can be written in the equivalent forms

$$\begin{aligned} D_+ D_- z + z|D_- z|^2 &= 0, \\ \text{or alternatively} \quad D_- D_+ z + z|D_+ z|^2 &= 0. \end{aligned} \tag{14}$$

The commutative CP^{N-1} model has well-known instanton and anti-instanton solutions satisfying

$$D_\pm z = 0, \tag{15}$$

given by

$$z = f(x_\mp) \frac{1}{|f(x_\mp)|}. \tag{16}$$

Here due to gauge invariance f may be assumed without loss of generality to be a polynomial N-vector. It can be shown that (with this ordering) these remain solutions in the noncommutative case. A feature which appears here and elsewhere is that awkward derivatives such as that of $\frac{1}{|f(x_\mp)|}$ get projected out.

It was shown in Ref. [7] that the commutative CP^{N-1} model has additional classical solutions which are neither instantons nor anti-instantons, and these were presented explicitly using a simple, elegant construction. We shall show here that these solutions (correctly ordered) remain valid in the noncommutative case. We claim that a general solution is given by

$$z^{(k)} = \hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|}, \tag{17}$$

where

$$\hat{z}^{(k)} = \partial_+^k f - \sum_{l,m=0}^{k-1} \partial_+^l f M^{-1}_{l,m} \partial_+ M_{m,k-1}, \tag{18}$$

where $f(x_+)$ is a polynomial N-vector[1] and the matrix M has entries

$$M_{l,m} = \overline{\partial_+^l f}.\partial_+^m f, \qquad l,m = 0, \dots k-1. \tag{19}$$

[1] As noted in Ref. [14], the use of functions with singularities presents difficulties in the noncommutative case

(We assume that $f, \partial_+ f, \ldots \partial_+^{N-1} f$ are linearly independent.) We start by noting the following identities which follow immediately from Eq. (18):

$$\overline{\hat{z}^{(k)}}.\partial_- \hat{z}^{(k)} = 0, \tag{20a}$$

$$\overline{\partial_+^i f}.\hat{z}^{(k)} = \delta^{ik} |\hat{z}^{(k)}|^2, \quad i = 0, 1, \ldots k, \tag{20b}$$

$$\overline{\hat{z}^{(k)}}.\partial_+ \hat{z}^{(i-1)} = \delta^{ik} |\hat{z}^{(k)}|^2, \quad i = 0, 1, \ldots k. \tag{20c}$$

It is then easy to establish the following results, using Eq. (20a):

$$D_+ X = \partial_+ \left(X \frac{1}{|\hat{z}^{(k)}|} \right) |\hat{z}^{(k)}|, \tag{21a}$$

$$D_- X = \partial_- \left(X |\hat{z}^{(k)}| \right) \frac{1}{|\hat{z}^{(k)}|}. \tag{21b}$$

Two additional useful identities are as follows:

$$\partial_+ \left(\hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} \right) = \hat{z}^{(k+1)} \frac{1}{|\hat{z}^{(k)}|^2}, \tag{22a}$$

$$\partial_- \hat{z}^{(k+1)} = -\hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} |\hat{z}^{(k+1)}|^2. \tag{22b}$$

To prove Eqs. (22), we start by defining

$$L_i = \{f, \partial_+ f, \ldots \partial_+^{i-1} f\} = \{\hat{z}^{(0)}, \ldots \hat{z}^{(i-1)}\}, \tag{23}$$

where by this notation we mean that L_i is the subspace whose basis is as shown (the second equality holds since it is clear from Eqs. (18), (20b) that $\overline{\hat{z}^{(i)}}.\hat{z}^{(i')} = 0$ for $i \neq i'$). Then Eqs. (22) are easily proved, by first noting that the LHS of Eq. (22a) is clearly in L_{k+2} and that of Eq. (22b) is clearly in L_{k+1}. Then expanding the LHS of each identity in terms of the basis vectors $\hat{z}^{(i)}$, we may readily establish the coefficients. For instance, we may write (using Eq. (20a))

$$\partial_+ \left(\hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} \right) = \sum_{j=0}^{k+1} \hat{z}^{(j)} \alpha^{(j)} = \partial_+ \hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} - \hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} \overline{\hat{z}^{(k)}}.\partial_+ \hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2}, \tag{24}$$

and then take the scalar product with $\hat{z}^{(i)}$, $i = 0, \ldots k+1$ in turn. Then for $i = 0, 1, \ldots k-1$,

$$|\hat{z}^{(i)}|^2 \alpha^{(i)} = \overline{\hat{z}^{(i)}}.\partial_+ \hat{z}^{(k)} \frac{1}{|\hat{z}^{(k)}|^2} = \left[\partial_+ (\overline{\hat{z}^{(i)}}.\hat{z}^{(k)}) - \overline{\partial_- \hat{z}^{(i)}}.\hat{z}^{(k)} \right] \frac{1}{|\hat{z}^{(k)}|^2} = 0 \tag{25}$$

(since $\partial_-\hat{z}^{(i)} \in L_i$). For $i = k$, $|\hat{z}^{(k)}|^2\alpha^{(k)} = 0$ follows immediately from Eq. (24). Finally, using Eq. (20c) we find $\alpha^{(k+1)} = \frac{1}{|\hat{z}^{(k)}|^2}$, thus completing the proof of Eq. (22a). Eq. (22b) is proved in similar fashion.

We can now write, using Eqs. (17), (21), (22),

$$\begin{aligned} D_+D_-z^{(k)} =&\partial_+\left(\partial_-\hat{z}^{(k)}\frac{1}{|\hat{z}^{(k)}|^2}\right)|\hat{z}^{(k)}| \\ =&-z^{(k)}|\hat{z}^{(k)}|\frac{1}{|\hat{z}^{(k-1)}|^2}|\hat{z}^{(k)}| \\ =&-z^{(k)}|D_-z^{(k)}|^2, \end{aligned} \tag{26}$$

showing that $z^{(k)}$ is a solution of Eq. (14). A useful representation of these solutions[9] is derived by defining the operator P_+ as

$$P_+g = \partial_+g - g\frac{1}{|g|^2}(\overline{g}.\partial_+g). \tag{27}$$

We then have

$$\hat{z}^{(k)} = P_+^k f. \tag{28}$$

Eq. (28) is easily proved using induction. Assuming it true for $k = 0, 1, \ldots l$, we have

$$P_+^{l+1}f = \partial_+\hat{z}^{(l)} - \hat{z}^{(l)}\frac{1}{|\hat{z}^{(l)}|^2}\overline{\hat{z}^{(l)}}.\partial_+\hat{z}^{(l)}. \tag{29}$$

Since $\hat{z}^{(l)} \in L_{l+1}$, $P_+^{l+1}f \in L_{l+2}$. Since $\hat{z}^{(l)}$ is orthogonal to L_l, we have for $i \leq l-1$

$$\overline{\partial_+^i f}.P_+^{l+1}f = \partial_+\left[\overline{\partial_+^i f}.\hat{z}^{(l)}\right] = 0, \tag{30}$$

and also

$$\begin{aligned} \overline{\partial_+^l f}.P_+^{l+1}f =&\partial_+[\overline{\partial_+^l f}.\hat{z}^{(l)}] - \overline{\partial_+^l f}.\hat{z}^{(l)}\frac{1}{|\hat{z}^{(l)}|^2}\overline{\hat{z}^{(l)}}.\partial_+\hat{z}^{(l)} \\ =&\left[\partial_+(|\hat{z}^{(l)}|^2) - \overline{\hat{z}^{(l)}}.\partial_+\hat{z}^{(l)}\right] = 0, \end{aligned} \tag{31}$$

using Eq. (20). Hence $P_+^{l+1}f$ is in L_{l+2} and orthogonal to L_{l+1}. So we must have $P_+^{l+1}f = \hat{z}^{(l+1)}\mu$ for some μ, and using Eq. (20c) we find $\mu = 1$. Moreover the result is trivially true for $k = 0$, completing the inductive proof.

We shall now show that the topological charge is given in the same way as for the commutative case. The action $S^{(k)}$ corresponding to a solution $z^{(k)}$ may be written

$$S^{(k)} = 2\pi\tilde{Q}^{(k)} + 2I^{(k)}, \tag{32}$$

where the topological charge $\tilde{Q}^{(k)}$ is given by

$$\tilde{Q}^{(k)} = \frac{1}{2\pi}\mathrm{Tr}[Q^{(k)}], \tag{33}$$

with the topological charge density $Q^{(k)}$ defined as

$$Q^{(k)} = |D_+ z^{(k)}|^2 - |D_- z^{(k)}|^2, \tag{34}$$

and where

$$I^{(k)} = \mathrm{Tr}|D_- z^{(k)}|^2. \tag{35}$$

It can easily be shown using Eqs. (20a), (21) that

$$Q^{(k)} = |\hat{z}^{(k)}|\partial_- \mathcal{O}^{(i)} \frac{1}{|\hat{z}^{(k)}|}, \tag{36}$$

where

$$\mathcal{O}^{(i)} = \frac{1}{|\hat{z}^{(i)}|^2}\partial_+ |\hat{z}^{(i)}|^2, \tag{37}$$

and moreover (as can be established using induction combined with Eq. (22))

$$|D_- z^{(k)}|^2 = |\hat{z}^{(k)}|\partial_- \sum_{i=0}^{k-1} \mathcal{O}^{(i)} \frac{1}{|\hat{z}^{(k)}|}. \tag{38}$$

Inside the traces in Eqs. (33), (35), the factors of $|\hat{z}^{(k)}|$ and $\frac{1}{|\hat{z}^{(k)}|}$ cancel. We therefore find ourselves interested in computing

$$X^{(i)} = \mathrm{Tr}[\partial_- \mathcal{O}^{(i)}] = 2\pi\theta \sum_{n=0}^{\infty} <n|\partial_- \mathcal{O}^{(i)}|n> . \tag{39}$$

Following Ref. [14], and using Eqs. (4), (5), we write

$$\begin{aligned} X^{(i)} = 2\pi \sum_{n=0}^{\infty} \Big[<n+1|\mathcal{O}^{(i)} x_+|n+1> - <n|\mathcal{O}^{(i)} x_+|n>\Big] \\ =& 2\pi \lim_{N\to\infty} <N|\mathcal{O}^{(i)} x_+|N> = -2\pi\theta^{-1} \lim_{N\to\infty} <N|\frac{1}{|\hat{z}^{(i)}|^2}[x_-, |\hat{z}^{(i)}|^2]x_+|N> . \end{aligned} \tag{40}$$

After some use of Eq. (3), together with[14]

$$\begin{aligned} x_+ g(x_- x_+) =& g(x_- x_+ + \theta)x_+, \\ x_- g(x_- x_+) =& g(x_- x_+ - \theta)x_-, \end{aligned} \tag{41}$$

we can write $|\hat{z}^{(i)}|^2 = h^{(i)}(x_-x_+, \theta)$, where $h^{(i)}(x_-x_+, \theta)$ is a homogeneous rational polynomial in x_-x_+ and θ. We find

$$
\begin{aligned}
X^{(i)} =& -2\pi\theta^{-1} \lim_{N\to\infty} <N|\frac{1}{h^{(i)}(x_-x_+,\theta)}[h^{(i)}(x_-x_+ - \theta, \theta) - h^{(i)}(x_-x_+,\theta)]x_-x_+|N> \\
=& -2\pi\theta^{-1} \lim_{N\to\infty} \frac{1}{h^{(i)}(\theta N,\theta)}[h^{(i)}(\theta N - \theta, \theta) - h^{(i)}(\theta N,\theta)]\theta N \\
=& 2\pi \lim_{x\to\infty} \frac{x H^{(i)\prime}(x)}{H^{(i)}(x)},
\end{aligned}
\tag{42}
$$

where $H^{(i)}(x) = h^{(i)}(x, 0)$, corresponding to the commutative result for $|\hat{z}^{(i)}|^2$. In other words

$$X^{(i)} = 2\pi\gamma^{(i)} \tag{43}$$

where the commutative $|\hat{z}^{(i)}|^2 \sim (x_-x_+)^{\gamma^{(i)}}$ for large x_-, x_+. It is shown in Ref. [9] that for the case where f is a polynomial with degree β we have

$$\gamma^{(i)} = \beta - 2i, \tag{44}$$

leading to

$$
\begin{aligned}
\tilde{Q}^{(k)} =& \beta - 2k \\
I^{(k)} =& 2\pi k(\beta - k + 1) \\
S^{(k)} =& 2\pi[(2k+1)\beta - 2k^2].
\end{aligned}
\tag{45}
$$

As advertised, these are precisely the same results as obtained in the commutative case[9].

We now discuss the generalisation of these solutions to the supersymmetric CP^{N-1} model, with Lagrangian

$$
\begin{aligned}
L = & \overline{D_\mu z} D_\mu z - i\overline{\psi}\not{D}\psi \\
& + \tfrac{1}{4}\left[(\overline{\psi}\psi)^2 + (\overline{\psi}\gamma_5\psi)^2 - (\overline{\psi}\gamma_\mu\psi)^2\right],
\end{aligned}
\tag{46}
$$

where the fermion field is subject to

$$\overline{z}.\psi = 0. \tag{47}$$

The solution of the full set of coupled equations for z and ψ was discussed in Ref. [15], using superfields. There seems no obstacle in principle to generalising these solutions to the noncommutative case, but the formalism is somewhat complex. Here instead we consider the simpler problem of a fermion in the fixed background of a bosonic solution, as in Ref. [10]. This can be considered[15] as the first-order term in a Grassmann expansion of the full solution. The Dirac equation becomes

$$\not{D}\psi - z\,(\overline{z}.\not{D}\psi) = 0, \tag{48}$$

with the constraint Eq. (47). Decomposing ψ in terms of eigenstates of γ_5,

$$\psi = \psi^+ \begin{pmatrix} 1 \\ -i \end{pmatrix} + \psi^- \begin{pmatrix} 1 \\ i \end{pmatrix}, \tag{49}$$

we have

$$D_\pm \psi^\pm = z\lambda_\pm, \quad \overline{z}.\psi^\pm = 0, \tag{50}$$

where $\lambda_\pm$ are functions of $x_\pm$.

It is now easy to show using Eqs. (21a), (22) that a positive helicity solution to Eq. (50) is given by

$$\psi^{(k)+} = \sum_{i \neq k} \hat{z}^{(i)} \frac{1}{|\hat{z}^{(i)}|^2} \overline{\hat{z}^{(i)}}.g^+(x_-)|\hat{z}^{(k)}|, \tag{51}$$

(where following Ref. [10] we take g^+ to be a polynomial) provided $\overline{\hat{z}^{(k+1)}}.g^+$ is a function of x_- alone and hence is also polynomial (any denominator is inevitably a function of x_-x_+). The form of the solution can be further restricted by requiring it to be normalizable on a sphere. Here we face the problem of defining the normalisation condition in the noncommutative case; a symmetric possibility is to impose

$$\mathrm{Tr}\left[\frac{1}{1+\frac{1}{2}\{x_+, x_-\}}|\psi^\pm|^2\right] < \infty. \tag{52}$$

(In fact any ordering will lead to the same conclusions.) Following a similar procedure as in the discussion of topological charge, we can write $|\psi^{(k)+}|^2$ as a homogeneous function of x_-x_+ and θ. We then find

$$\mathrm{Tr}\left[\frac{1}{1+\frac{1}{2}\{x_+, x_-\}}|\psi^{(k)+}|^2\right] = \sum_{n=0}^{\infty} \frac{1}{1+n+\frac{1}{2}\theta} \sum_{i \neq k} G^{(i)}(n, \theta), \tag{53}$$

where (since $\overline{\hat{z}^{(k+1)}}.g^+$ is polynomial) $G^{(k+1)}(n, \theta) = O(n^{\tilde{Q}^{(k)} - \tilde{Q}^{(k+1)} + D})$, where D is the degree of $\overline{\hat{z}^{(k+1)}}.g^+$. Since $\tilde{Q}^k > \tilde{Q}^{k+1}$, for normalisability (i.e. convergence of Eq. (53)) we must have $\overline{\hat{z}^{(k+1)}}.g^+ = 0$. Repeating the argument, we deduce in turn that $\overline{\hat{z}^{(i)}}.g^+$

is polynomial and thence zero for $i = k+2, \dots N-1$. We therefore find the general normalisable solution to be

$$\psi^{(k)+} = \sum_{i=0}^{k-1} \hat{z}^{(i)} \frac{1}{|\hat{z}^{(i)}|^2} \overline{\hat{z}^{(i)}}.g^{+}(x_{-})|\hat{z}^{(k)}|. \tag{54}$$

Similarly a general negative helicity solution is given by

$$\psi^{(k)-} = \sum_{i \neq k} \hat{z}^{(i)} \frac{1}{|\hat{z}^{(i)}|^2} \overline{\hat{z}^{(i)}}.g^{-}(x_{+}) \frac{1}{|\hat{z}^{(k)}|}, \tag{55}$$

where now $\frac{1}{|\hat{z}^{(k-1)}|^2} \overline{\hat{z}^{(k-1)}}.g^{-}(x_{+})$ is a function of x_{+} alone and hence polynomial. Proceeding as for $\psi^{(k)+}$, we find the general normalisable solution is

$$\psi^{(k)-} = \sum_{i=k+1}^{N-1} \hat{z}^{(i)} \frac{1}{|\hat{z}^{(i)}|^2} \overline{\hat{z}^{(i)}}.g^{-}(x_{+}) \frac{1}{|\hat{z}^{(k)}|}. \tag{56}$$

The counting of the number $\xi_{\pm}$ of independent solutions $\psi^{\pm}$ then proceeds just as in the commutative case[10] with the result that $\xi_{\pm}$ satisfy the index theorem

$$\xi_{+} - \xi_{-} = -NQ^{(k)}. \tag{57}$$

We have seen that most of the results of Refs. [7]–[10] are unaffected by the generalisation to the noncommutative case. However, it still remains to establish whether Eqs. (17), (18) exhaust the set of classical solutions in the noncommutative as well as in the commutative[7] case. The answer may have to await an extension of complex analysis to the noncommutative context. Further work could include the construction of Green functions as was done in the commutative case in Ref. [16]. Moreover, it seems likely that the solutions found for Grassmannian σ-models (where z becomes an $N \times p$ matrix and the model has a local $U(p)$ invariance) in Refs. [17] will straightforwardly extend to the noncommutative case; indeed this has already been shown for the pure instanton case in Refs. [18].

Acknowledgements

IJ thanks the Dept. of Mathematics and Statistics at the University of Melbourne for its hospitality, and the Australian Research Council for financial support, while part of this work was carried out. DRTJ was supported by a PPARC Senior Fellowship.

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