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Vacuum polarisation and the muon $g - 2$ in lattice QCD[*]

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We measure the hadronic contribution to the vacuum polarisation tensor, and use it to estimate the hadronic contribution to $(g - 2)_\mu$, the muon anomalous magnetic moment.

1. Introduction

The vacuum polarisation $\Pi(q^2)$ is defined by

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= \mathrm{i} \int \mathrm{d}^4 x \, e^{\mathrm{i}qx} \langle 0| T \, J_\mu(x) J_\nu(0)|0\rangle \\
&\equiv (q_\mu q_\nu - q^2 g_{\mu\nu}) \, \Pi(Q^2).
\end{aligned}
\tag{1}
$$

J_μ is the electromagnetic current, $Q^2 = -q \cdot q$.

The 1-loop diagram is logarithmically divergent, Π is additively renormalised. The value of Π can be shifted up and down by a constant depending on scheme and scale without any physical effects, but the Q^2 dependence of $\Pi(Q^2)$ is physically meaningful, and it must be independent of scheme or regularisation.

The vacuum polarisation tensor enters physics in several important ways. It is responsible for the running of α_{em}, which must be known very precisely for high-precision electro-magnetic calculations (for example of $g - 2$). Secondly, the optical theorem implies that the cross-section for $e^+ e^- \to hadrons$ is proportional to the imaginary part of Π at timelike q. This cross-section is usually given in terms of the ratio R,

$$
R(s) = \frac{\sigma_{e^+ e^- \to \mathrm{hadrons}}(s)}{\sigma_{e^+ e^- \to \mu^+ \mu^-}(s)}.
\tag{2}
$$

[*]Talk presented by P. Rakow at Lattice 2003

There is an important set of dispersion relations connecting R and Π

$$
\begin{aligned}
&12\pi^2 Q^2 \frac{(-1)^n}{n!} \left(\frac{d}{dQ^2}\right)^n \Pi(Q^2) \\
&= Q^2 \int_{4m_\pi^2}^\infty ds \frac{R(s)}{(s + Q^2)^{n+1}}.
\end{aligned}
\tag{3}
$$

2. The lattice calculation

Our present calculation is carried out in the quenched approximation, using clover fermions and an $O(a)$ improved electromagnetic current. We have used three β values (6.0, 6.2 and 6.4) with several quark masses in each case. Full details of the calculation can be found in [1].

Π can be split into two parts, a fermion line connected contribution C_Π, and a disconnected contribution A_Π, Fig. 1.

$$
\begin{aligned}
-12\pi^2 \Pi(Q^2) &= \sum_f e_f^2 \, C_\Pi(Q^2, m_f) \\
&+ \sum_{f,f'} e_f e_{f'} \, A_\Pi(Q^2, m_f, m_{f'}).
\end{aligned}
\tag{4}
$$

We only calculate C_Π, but the term A_Π (which violates Zweig's rule) is expected to be very small.

In Fig. 2 we compare our lattice data with continuum perturbation theory. Firstly, we see very
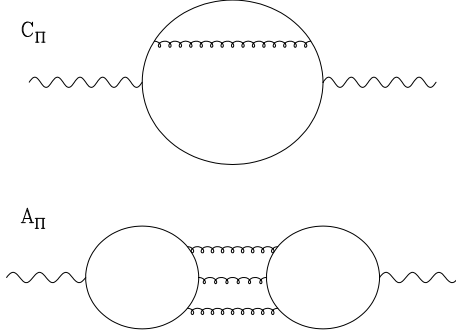
C_Π

A_Π

Figure 1. *Examples of fermion-line-connected and disconnected contributions to the vacuum polarisation.*

good agreement between the two lattice sizes, showing that finite size effects are not a major problem. Perturbation theory works very well except at the largest Q^2 values, where discretisation errors start to show up, and at low Q^2 (below $\sim 2\,\mathrm{GeV}^2$). Both lattice sizes agree, so this is not a finite size effect. Can we understand this in terms of non-perturbative physics?
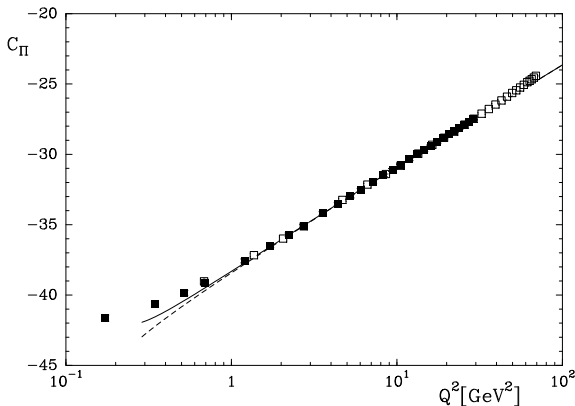
Figure 2. *A check for finite size effects, comparing a 16^4 lattice (white) and a 32^4 lattice (black) at $\beta = 6.0$, $\kappa = 0.1345$. The curves show continuum perturbation theory. The dashed curve is at zero quark mass, the solid curve includes mass effects.*

One approach, introduced in [2], uses the dis-

persion relations eq.(3). These relations connecting $R(s)$ to $\Pi(Q^2)$ are very stable, so even a crude model of R is likely to give a good estimate of Π. The simplest model of R consists of δ-function contributions from the vector mesons (ρ, ω, ϕ) and a flat continuum beginning at a threshold s_0,

$$R(s) = \sum_f e_f^2 \left(A\delta(s - m_V^2) + B\Theta(s - s_0) \right). \quad (5)$$

Using the dispersion relation gives

$$C_\Pi(Q^2) = B \ln[a^2(Q^2 + s_0)] - \frac{A}{Q^2 + m_V^2} + K. (6)$$

We have already measured m_V and f_V (the decay constant) so the weight and position of the δ-function are already known. B and s_0 are found by fitting. As seen in Fig. 3, eq.(6) gives an excellent fit, and can be used to extrapolate our data to lower Q^2.
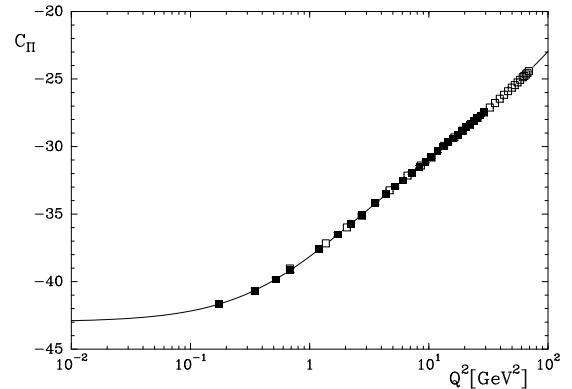
Figure 3. *Eq.(6) compared with the lattice data.*

3. The muon anomalous magnetic moment

The anomalous magnetic moment of the muon can be calculated to very high order in QED (5 loop), and measured very precisely. $(g-2)_\mu$ is more sensitive to high-energy physics than $(g-2)_e$, by a factor m_μ^2/m_e^2, so it is a more promising place to look for signs of new physics, but to identify new physics we need to know the

conventional contributions very accurately. QED perturbative calculations take good account of muon and electron loops, but at the two-loop level quarks can be produced, which in turn will produce gluons. The dominant contribution comes from photons with virtualities $\sim m_\mu^2$, which is a region where QCD perturbation theory will not work well.

The traditional route for estimating these hadronic contributions is by using a dispersion relation involving the cross-section $R(s)$.

$$a_\mu^{had} = \frac{\alpha_{em}^2}{3\pi^2} \int_{4m_\pi^2}^\infty \frac{ds}{s} K\left(\frac{s}{m_\mu^2}\right) R(s) . \qquad (7)$$

By distorting the contour of integration this can be rewritten as

$$a_\mu^{had} = \frac{\alpha_{em}^2}{3\pi^2} \sum_f e_f^2 I^f \qquad (8)$$

where

$$I^f(m_f) = \int_0^\infty \frac{dQ^2}{Q^2} F\left(\frac{Q^2}{m_\mu^2}\right) \left[C_\Pi(Q^2) - C_\Pi(0)\right] \quad (9)$$

The kernel in this relation is

$$F\left(\frac{Q^2}{m_\mu^2}\right) = \frac{(4m_\mu^2/Q^2)^2}{\left(1 + \sqrt{1 + \frac{4m_\mu^2}{Q^2}}\right)^4 \sqrt{1 + \frac{4m_\mu^2}{Q^2}}} . \quad (10)$$

For each data set we calculate the integral I^f. The integral is dominated by $Q^2 \sim 3m_\mu^2$, so some Q^2 extrapolation is needed.

In Fig. 4 we extrapolate the integral I^f to the continuum limit and to the physical quark masses ($m_{ps}^2 = m_\pi^2$, $2m_K^2 - m_\pi^2$) using the ansatz

$$I^f = (A_1 + A_2 a^2) + (B_1 + B_2 a^2)m_{ps}^2. \qquad (11)$$

Our values are

$$I^u = I^d = 0.0389(21); \ I^s = 0.0287(9) . \qquad (12)$$

Adding these values, we get our final answer

$$a_\mu^{had} = \frac{\alpha_{em}^2}{3\pi^2} \frac{4I^u + I^d + I^s}{9} = 446(23) \times 10^{-10} (13)$$

This value agrees with the pioneering lattice calculation of [3], $a_\mu^{had} = 460(78) \times 10^{-10}$. However both lattice values lie lower than the experimental value $683.6(8.6) \times 10^{-10}$ [4]. Possibly this shortfall is due to the absence of two-pion states in the quenched calculation.
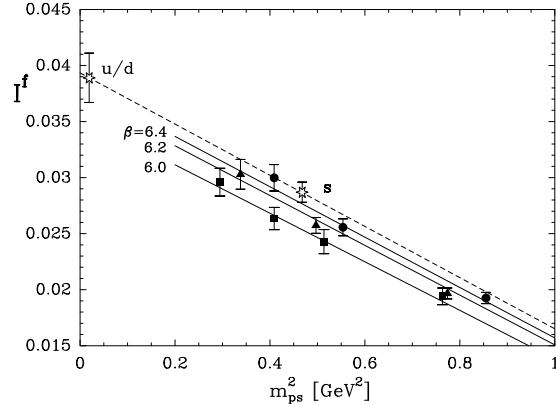

Figure 4. *The results for the integral I^f, and our extrapolations to the continuum limit.*

4. Conclusions

We have seen that perturbation theory works above $Q^2 \sim 2\,\text{GeV}^2$. We can describe the entire Q^2 region very well with a dispersion relation using a simple model of $R(s)$ [2].

From the low Q^2 region of the vacuum polarisation we can extract a lattice value for a_μ^{had}, the hadronic contribution to the muon's anomalous magnetic moment, which is of the right order of magnitude. Our estimate could be improved by using a larger lattice size, enabling us to reach lower Q^2 which would reduce uncertainties from extrapolation. Naturally, dynamical calculations would be very interesting.

Acknowledgements

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