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# Study of the gauge invariant, nonlocal mass operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ in Yang-Mills theories

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The nonlocal mass operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ is considered in Yang-Mills theories in Euclidean space-time. It is shown that the operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ can be cast in local form through the introduction of a set of additional fields. A local and polynomial action is thus identified. Its multiplicative renormalizability is proven by means of the algebraic renormalization in the class of linear covariant gauges. The anomalous dimensions of the fields and of the mass operator are computed at one-loop order. A few remarks on the possible role of this operator for the issue of the gauge invariance of the dimension two condensates are outlined.

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## I. INTRODUCTION

Dimension two condensates have received great attention in recent years. These condensates might play an important role for the infrared dynamics of Euclidean Yang-Mills theories, as supported by the considerable amount of results obtained through theoretical and phenomenological studies as well as from lattice simulations [1–26].

For instance, the gluon condensate $\langle A^a_\mu A^a_\mu\rangle$ has been largely investigated in the Landau gauge. As pointed out in [4], this condensate enters the operator product expansion (OPE) of the gluon propagator. Moreover, a combined OPE and lattice analysis has shown that this condensate can account for the $1/Q^2$ corrections which have been reported [18–21,24,25] in the running of the coupling constant and in the gluon correlation functions. An effective potential for $\langle A^a_\mu A^a_\mu\rangle$ has been obtained and evaluated in analytic form at two loop in [7,10,11,15,16], showing that a nonvanishing value of $\langle A^a_\mu A^a_\mu\rangle$ is favored as it lowers the vacuum energy. As a consequence, a dynamical gluon mass is generated. We also recall that, in the Landau gauge, the operator $A^a_\mu A^a_\mu$ is $BRST$ invariant on shell, a property which has allowed for an all orders proof of its multiplicative renormalizability. Its anomalous dimension is not an independent parameter, being expressed as a combination of the gauge $\beta$-function and of the anomalous dimension $\gamma_A$ of the gauge field $A^a_\mu$ [27]. This relation was conjectured and explicitly verified up to three-loop order in [28].

The dimension two operator $A^a_\mu A^a_\mu$ has been proven to be multiplicatively renormalizable to all orders in the more general class of linear covariant gauges [29]. An effective potential for the condensate $\langle A^a_\mu A^a_\mu\rangle$ in linear covariant gauges has been evaluated in [13], providing evidence for a nonvanishing value $\langle A^a_\mu A^a_\mu\rangle$ in these gauges.

A renormalizable mass dimension two operator can be introduced in other covariant renormalizable gauges, such as the Curci-Ferrari and the maximal Abelian gauge. In the Curci-Ferrari gauge the generalized gluon-ghost operator $(\frac{1}{2}A^a_\mu A^a_\mu + \alpha\bar{c}^a c^a)$ is $BRST$ invariant on shell, displaying multiplicative renormalizability to all orders [30]. The fields $\bar{c}^a$, $c^a$ stand for the Faddeev-Popov ghosts, while $\alpha$ denotes the gauge parameter. Evidence for a nonvanishing condensate $\langle\frac{1}{2}A^a_\mu A^a_\mu + \alpha\bar{c}^a c^a\rangle$ have been provided in [12]. Note that in the limit $\alpha \to 0$, corresponding to the Landau gauge, the operator $(\frac{1}{2}A^a_\mu A^a_\mu + \alpha\bar{c}^a c^a)$ reduces to $A^a_\mu A^a_\mu$. A mixed gluon-ghost operator, namely, $(\frac{1}{2}A^A_\mu A^A_\mu + \alpha\bar{c}^A c^A)$, can be introduced also in the maximal Abelian gauge [8,9,14]. Here the color index $A$ runs over the $N(N-1)$ off diagonal generators of the gauge group $SU(N)$, $A = 1, \ldots, N(N-1)$. As in the case of the Curci-Ferrari gauge, this operator is $BRST$ invariant on shell, being multiplicatively renormalizable to all orders [8,9,14,30,31]. Analytic evidence for a nonvanishing condensate $\langle\frac{1}{2}A^A_\mu A^A_\mu + \alpha\bar{c}^A c^A\rangle$ in the maximal Abelian gauge can be found in [14]. We underline that a nonvanishing condensate $\langle\frac{1}{2}A^A_\mu A^A_\mu + \alpha\bar{c}^A c^A\rangle$ gives rise to the dynamical mass generation for off diagonal gluons, a result of great relevance for the so-called Abelian dominance, supporting the dual superconductivity picture for color confinement.

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An off diagonal gluon mass has also been reported in lattice simulations [32,33].

Studies of the influence of these condensates on the gluon and ghost propagators when the nonperturbative effects of the Gribov copies are taken into account can be found in [34–37]. The output of this analysis is an infrared suppression of the components of the gluon propagator in the aforementioned gauges, a feature in agreement with the results available from lattice and Schwinger-Dyson studies [32,33,38–54].

Certainly, many aspects related to the dimension two condensates deserve a better understanding. This is the case, for example, of the gauge invariance, a central issue in order to give a precise physical meaning to these condensates. A recent study of this topic has been given in [55–57], where a set of conditions which should ensure the independence of the condensate $\langle A^a_\mu A^a_\mu \rangle$ from the gauge parameter in the class of linear covariant gauges has been proposed.

In this work we pursue the study on the aspects of the gauge invariance of the dimension two condensates. Our aim here is that of discussing the possibility of introducing a suitable colorless dimension two operator $\mathcal{O}(A)$ which preserves gauge invariance

$$\delta \mathcal{O}(A) = 0, \qquad \delta A^a_\mu = -D^{ab}_\mu \omega^b, \tag{1.1}$$

where $D^{ab}_\mu$ is the covariant derivative

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu. \tag{1.2}$$

This is a difficult task, due to the lack of a local gauge invariant mass term built up with gauge fields only. This problem could be overcome by looking at nonlocal operators. However, even if we allow for nonlocal operators, we cannot give up the requirement that a consistent computational framework, allowing to carry out higher loop calculations, has to be at our disposal. This is a strong requirement which, in practice, deeply constrains the type of nonlocality allowed for the dimension two operator. As a suitable proposition in order to obtain such a consistent framework, we could demand that the action to which the nonlocal gauge invariant operator $\mathcal{O}(A)$ is coupled, should have the property of being made local by the introduction of a suitable set of additional fields.

(I) Therefore, denoting by $S_{\mathcal{O}}$ the term which accounts for the introduction in the Yang-Mills action, $S_{YM}$, of the operator $\mathcal{O}(A)$ in its localized form, we require that $S_{\mathcal{O}}$ is gauge invariant.

(II) Also, on physical grounds, we demand that the introduction of the operator $\mathcal{O}(A)$ makes it possible to identify a quantized action which is multiplicatively renormalizable, a feature which should not be related to a specific choice of the gauge fixing term $S_{gf}$, of course on the condition that the usual Yang-Mills action $S$, quantized using the gauge fixing $S_{gf}$, thus $S = S_{YM} + S_{gf}$, is renormalizable.

As we shall see, these conditions will lead us to consider the nonlocal gauge invariant operator of mass dimension two

$$\mathcal{O}(A) = -\frac{1}{2}\int d^4x F^a_{\mu\nu}[(D^2)^{-1}]^{ab} F^b_{\mu\nu}. \tag{1.3}$$

Expression (1.3) can be made local by the introduction of a set of additional fields. Moreover, we will be able to prove that it is possible to identify a local and polynomial action which turns out to be multiplicatively renormalizable to all orders.

The identification of this action and the algebraic proof of its renormalizability, as explicitly checked through the evaluation of the one-loop anomalous dimensions, are the main results of the present investigation, signaling that the operator (1.3) could be relevant for a better understanding of the issue of the gauge invariance of the dimension two gluon condensate.

Besides the renormalizability, we should also provide a suitable framework to discuss the possible condensation of the operator (1.3), i.e. $\langle \mathcal{O}(A) \rangle \neq 0$, which would give rise to the dynamical gluon mass generation. Although being out of the aim of the present work, we remark that, in the Landau gauge, $\partial_\mu A^a_\mu = 0$, expression (1.3) reduces, to the first order, to the mass operator $\int d^4x A^a_\mu A^a_\mu$,

$$-\frac{1}{2}\int d^4x F^a_{\mu\nu}[(D^2)^{-1}]^{ab} F^b_{\mu\nu} = \int d^4x A^a_\mu A^a_\mu + \text{higher order terms}. \tag{1.4}$$

Thus, it is not inconceivable that a nonvanishing condensate $\langle A^a_\mu A^a_\mu \rangle \neq 0$ might provide a support in favor of a nonvanishing condensation of the operator (1.3), i.e. $\langle F \frac{1}{D^2} F \rangle \neq 0$.

The plan of the work is as follows. In Section II we give an account of a set of nonlocal and gauge invariant mass operators which can be introduced in the Abelian case. These include the Abelian version of the operator $\mathcal{O}(A)$ of Eq. (1.3), the operator $A^2_{\min}$ recently discussed in [5,6], the Stueckelberg term as well as the nonlocal mass operator $\int d^4x A^T_\mu A^T_\mu$, where $A^T_\mu$ stands for the transverse component of the gauge field $A_\mu$, $A^T_\mu = (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}) A_\nu$. Interestingly, in the Abelian case, it turns out that all these gauge invariant operators can be proven to be classically equivalent, i.e. they reduce to the same expression when the classical equations of motion are used. In Section III we present a detailed discussion of the non-Abelian generalization of these mass operators. We shall see that all operators introduced in the Abelian case possess a non-Abelian gauge invariant extension. However, the classical equivalence between them is now no longer valid. In particular, we point out that, in the non-Abelian case, the mass operator of Eq. (1.3) exhibits differences with respect to the operator $A^2_{\min}$. As we shall see, the latter can be expressed as an infinite sum of nonlocal terms, a feature which makes almost hopeless the possibility of achieving a consistent localization procedure for a generic choice of the gauge fixing condition. Section IV is devoted to the study of the localization procedure of the mass operator (1.3) and of the rich symmetry content of the resulting action. In Section V, the identification of a suitable local and polynomial action is provided. Its multiplicative renormalizability in the class of covariant linear gauges will be established by means of the algebraic renormalization. Having developed the general properties of the mass operator, we devote Section VI to the computation of its anomalous dimension at one loop. Our conclusions are presented in Section VII. For the benefit of the reader, we have found useful to collect in several appendices the explicit derivation of some relevant features of the various mass operators considered in this work.

## II. MASS OPERATORS IN THE ABELIAN CASE

In this section we shall discuss a set of nonlocal gauge invariant mass operators which can be added to the Maxwell action

$$\frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu}. \tag{2.1}$$

Perhaps, the simplest way of introducing a gauge invariant mass term is through the nonlocal gauge invariant variable $A^T_\mu$

$$A^T_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu. \tag{2.2}$$

Expression (2.2) is recognized to be the transverse component of the gauge field, $\partial_\mu A^T_\mu = 0$, and is invariant under the gauge transformations, i.e.

$$\delta A^T_\mu = 0, \tag{2.3}$$

with

$$\delta A_\mu = -\partial_\mu \omega. \tag{2.4}$$

Thus, for the gauge invariant mass term one writes

$$\mathcal{O}_1(A) = \int d^4x A^T_\mu A^T_\mu. \tag{2.5}$$

A second possibility of introducing an invariant mass term is provided by the operator $A^2_{\min}$, which has been recently analyzed in [5,6]. The operator $A^2_{\min}$ is obtained by minimizing the quantity $\int d^4x A_\mu A_\mu$ with respect to the gauge transformations, namely

$$\mathcal{O}_2(A) = A^2_{\min} = \min \int d^4x A_\mu A_\mu. \tag{2.6}$$

Making use of the decomposition of the gauge field $A_\mu$ into transverse and longitudinal parts

$$A_\mu = A^T_\mu + A^L_\mu, \qquad A^T_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu, \qquad A^L_\mu = \frac{\partial_\mu\partial_\nu}{\partial^2}A_\nu, \tag{2.7}$$

it follows that

$$\int d^4x A_\mu A_\mu = \int d^4x A^T_\mu A^T_\mu + \int d^4x A^L_\mu A^L_\mu. \tag{2.8}$$

Observe that both terms in Eq. (2.8) are positive definite. Moreover, as discussed in [5,6], the functional $\int d^4x A_\mu A_\mu$ achieves its minimum when $\partial_\mu A_\mu = 0$, i.e. $A^L_\mu = 0$, so that

$$\mathcal{O}_1(A) = \mathcal{O}_2(A), \tag{2.9}$$

which establishes the equivalence between expressions (2.5) and (2.6). It is worth mentioning that the gauge invariant functional $A^2_{\min}$ has been proven to be an order parameter for the study of the phase transition of compact three-dimensional QED [6].

A third possibility of introducing an invariant mass operator in the Abelian case is by means of the Stueckelberg term [58]

$$\mathcal{O}_3(A) = \int d^4x (A_\mu + \partial_\mu\phi)^2, \tag{2.10}$$

where $\phi$ is a dimensionless scalar field. Expression (2.10) is left invariant by the following transformations

$$\delta A_\mu = -\partial_\mu\omega, \qquad \delta\phi = \omega. \tag{2.11}$$

The mass term (2.10) can be rewritten in the form of a $U(1)$ gauged $\sigma$-model, by introducing the variable

$$U = e^{ie\phi}. \tag{2.12}$$

Thus

$$\mathcal{O}_3(A) = \int d^4x \left(A_\mu - \frac{i}{e}U^{-1}\partial_\mu U\right)^2. \tag{2.13}$$

Transformations (2.11) now read

$$A_\mu \to A_\mu + \frac{i}{e}V^{-1}\partial_\mu V, \qquad U \to UV, \tag{2.14}$$

with

$$V = e^{ie\omega}. \tag{2.15}$$

One checks that the quantity $(A_\mu - \frac{i}{e}U^{-1}\partial_\mu U)$ is left invariant by the transformations (2.14). Analogously to the operator $\mathcal{O}_2(A)$, expression (2.10) can be proven to be classically equivalent to the mass term of Eq. (2.5). This is easily seen by looking at the equations of motion which follow from the gauge invariant action

$$S = \frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x (A_\mu + \partial_\mu\phi)^2, \tag{2.16}$$

namely

$$\partial_\nu F_{\mu\nu} + m^2(A_\mu + \partial_\mu \phi) = 0, \qquad \partial^2\phi + \partial_\mu A_\mu = 0. \tag{2.17}$$

In particular, from the second equation of (2.17), we obtain

$$\phi = -\frac{1}{\partial^2}\partial A, \tag{2.18}$$

so that

$$\mathcal{O}_3(A) = \int d^4x\left(A_\mu - \frac{\partial_\mu\partial_\nu}{\partial^2}A_\nu\right)^2 = \int d^4x A^T_\mu A^T_\mu. \tag{2.19}$$

Thus

$$\mathcal{O}_3(A) = \mathcal{O}_1(A), \tag{2.20}$$

which establishes the classical equivalence between expressions (2.5) and (2.10). Also, from (2.18) one sees that the scalar field $\phi$ is related to the longitudinal mode of the gauge field $A_\mu$.

Finally, a fourth mass operator can be introduced by considering the nonlocal quantity

$$\mathcal{O}_4(A) = -\frac{1}{2}\int d^4x F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}. \tag{2.21}$$

Again, this term is seen to be equivalent to expression (2.5). In fact

$$\begin{aligned}\mathcal{O}_4(A) &= -\frac{1}{2}\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{1}{\partial^2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \frac{1}{2}\int d^4x\left[A_\nu\frac{1}{\partial^2}(\partial^2 A_\nu - \partial_\mu\partial_\nu A_\mu) \right. \\ &\quad \left. + A_\mu\frac{1}{\partial^2}(\partial^2 A_\mu - \partial_\nu\partial_\mu A_\nu)\right] \\ &= \int d^4x A_\nu\left(A_\nu - \frac{\partial_\mu\partial_\nu}{\partial^2}A_\mu\right) = \int d^4x A^T_\mu A^T_\mu,\end{aligned} \tag{2.22}$$

thus

$$\mathcal{O}_4(A) = \mathcal{O}_1(A). \tag{2.23}$$

Albeit nonlocal, the operator (2.21) can be made local through the introduction of suitable additional fields. More precisely, in the present case, one has

$$-\frac{1}{4}m^2\int d^4x F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu} \to \int d^4x\left(\frac{1}{4}\overline{B}_{\mu\nu}\partial^2 B_{\mu\nu} + \frac{im}{4} \times (F_{\mu\nu}B_{\mu\nu} - F_{\mu\nu}\overline{B}_{\mu\nu})\right), \tag{2.24}$$

where $\overline{B}_{\mu\nu}$ and $B_{\mu\nu}$ are a pair of antisymmetric complex fields and $m$ is a mass parameter. Eliminating $\overline{B}_{\mu\nu}$ and $B_{\mu\nu}$ by means of their equations of motion, one gets back the nonlocal action (2.21). One sees thus that, once cast in the local form, expression (2.21) looks renormalizable by power counting. It turns out in fact that, in the Abelian case, the localized term in the right-hand side (r.h.s.) of Eq. (2.24) can be added to the usual *QED* Lagrangian without destroying its renormalizability.

## III. MASS OPERATORS IN THE NON-ABELIAN CASE

As we have seen, there exist several ways of introducing nonlocal gauge invariant mass operators in the Abelian case. In particular, the four mass operators (2.5), (2.6), (2.10), and (2.21) turn out to be equivalent. Let us face now the more complex case of non-Abelian gauge theories. Let us start by considering the operator $A^2_{\min}$.

### A. Non-Abelian generalization of the operator $A^2_{\min}$

The operator $A^2_{\min}$ of expression (2.6) can be generalized to the non-Abelian case by minimizing the functional $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$ along the gauge orbit of $A_\mu$ [5,6,59–63], namely

$$\begin{aligned} A^2_{\min} &\equiv \min_{\{u\}}\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu, \\ A^u_\mu &= u^\dagger A_\mu u + \frac{i}{g}u^\dagger\partial_\mu u. \end{aligned} \tag{3.1}$$

A few remarks are in order. Although the minimization procedure along the gauge orbit of $A_\mu$ makes the operator $A^2_{\min}$ gauge invariant, it should be underlined that the explicit determination of the absolute minimum achieved by the functional $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$ is a highly nontrivial task which, in practice, requires the resolution of the issue of the Gribov copies. It has been proven that the operator $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$ achieves its absolute minimum along the gauge orbit of $A_\mu$ [59–63]. Moreover, it is also known that, in general, it possesses many relative minima along a given gauge orbit. Therefore, one has to be sure that the correct minimum has been selected. This requires a detailed knowledge of the so-called fundamental modular region, which is the set of all absolute minima in field space of the functional $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$. The fundamental modular region is contained in the Gribov region, which is defined as the set of all relative minima of $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$. While the Gribov region turns out to be still plagued by the presence of additional Gribov copies, the interior of the fundamental modular region is free from Gribov copies [5,6,59–63], a feature of primary importance for a correct quantization of Yang-Mills theories. However, a knowledge of the fundamental modular region of practical use in the Feynman path integral is not yet at our disposal. All this should give to the reader an idea of the real difficulty of obtaining an explicit expression for the absolute minimum configuration of the functional $\mathrm{Tr}\int d^4x A^u_\mu A^u_\mu$. A more modest program would be that of considering the Gribov region instead of the fundamental modular region, amounting to consider field configurations which are relative minima of $\mathrm{Tr}\int d^4xA^u_\mu A^u_\mu$. These configurations can be constructed in a relatively easy way as formal power series in the gauge field $A_\mu$. As discussed in Appendix A, a minimum configuration of $\mathrm{Tr}\int d^4xA^u_\mu A^u_\mu$ is attained when $u = h$ so that $A^h_\mu$ is a transverse field, $\partial_\mu A^h_\mu = 0$. The transversality condition can be solved order by order [64], allowing us to express $h$ as a formal power series in the gauge field $A_\mu$, i.e. $h = h(A)$. This gives

$$A^h_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\phi_\nu,$$
$$\phi_\nu = A_\nu - ig\left[\frac{1}{\partial^2}\partial A, A_\nu\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\nu\frac{1}{\partial^2}\partial A\right] + O(A^3). \tag{3.2}$$

In particular, the configuration $A^h_\mu$ turns out to be invariant under infinitesimal gauge transformations order by order in the gauge coupling $g$ [64], see also Appendix A, namely

$$\delta A^h_\mu = 0, \qquad \delta A_\mu = -\partial_\mu\omega + ig[A_\mu, \omega]. \tag{3.3}$$

Thus, from expression (3.2) it follows that

$$\begin{aligned} A^2_{\min} &= \mathrm{Tr}\int d^4xA^h_\mu A^h_\mu, \\ &= \frac{1}{2}\int d^4x\left[A^a_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A^a_\nu \right.\\ &\quad \left. - gf^{abc}\left(\frac{\partial_\nu}{\partial^2}\partial A^a\right)\left(\frac{1}{\partial^2}\partial A^b\right)A^c_\nu\right] + O(A^4). \end{aligned} \tag{3.4}$$

We see that the operator $A^2_{\min}$ can be expressed as an infinite sum of nonlocal terms. Such a nonlocal structure looks almost hopeless to be handled in a consistent way for a generic choice of the gauge fixing term. The only possibility here seems that of adopting the Landau gauge condition, $\partial_\mu A^a_\mu = 0$. In this case, all nonlocal terms in the r.h.s. of Eq. (3.4) drop out, so that

$$A^2_{\min} = \frac{1}{2}\int d^4xA^a_\mu A^a_\mu \text{ in the Landau gauge.} \tag{3.5}$$

It is worth remarking that, as proven in [27], the massive Yang-Mills action

$$S_m = \frac{1}{4}\int d^4xF^a_{\mu\nu}F^a_{\mu\nu} + \frac{m^2}{2}\int d^4xA^a_\mu A^a_\mu + \int d^4x(b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b), \tag{3.6}$$

where $b^a$ is the Lagrange multiplier enforcing the Landau condition, $\partial_\mu A^a_\mu = 0$, and $\bar{c}^a$, $c^a$ are the Faddeev-Popov ghosts, is multiplicatively renormalizable to all orders of perturbation theory.

In summary, we have seen that the operator $A^2_{\min}$ can be generalized to the non-Abelian case. In addition, when treated as a formal power series in the gauge field $A_\mu$, it has the pleasant property of reducing to the renormalizable operator $\int d^4xA^a_\mu A^a_\mu$ in the Landau gauge.

We also recall that the operator $\int d^4xA^a_\mu A^a_\mu$ turns out to be renormalizable to all orders of perturbation theory in the more general class of the linear covariant gauges [29], a fact which has made possible to give evidence of a nonvanishing condensate $\langle A^a_\mu A^a_\mu\rangle$ in these gauges [13]. However, outside of the Landau gauge, the relationship between $A^2_{\min}$ and $\int d^4xA^a_\mu A^a_\mu$ is lost, so that a study of the nonlocal operator $A^2_{\min}$ becomes difficult. The operator $A^2_{\min}$ lacks thus a simple computational framework outside of the Landau gauge.

### B. Non-Abelian generalization of the operator $\int d^4xA^T_\mu A^T_\mu$

The discussion of the previous section allows us to generalize the operator $\int d^4xA^T_\mu A^T_\mu$ in the non-Abelian case. In fact, according to expression (3.2) [64], see also Appendix A, it is possible to introduce a gauge invariant non-Abelian transverse field. It follows thus that the non-Abelian generalization of the mass operator $\int d^4xA^T_\mu A^T_\mu$ is provided by expression (3.4). This establishes the equivalence between the non-Abelian version of $\int d^4xA^T_\mu A^T_\mu$ and the functional $A^2_{\min}$ within the space of the formal power series. Moreover, the operator $\int d^4xA^T_\mu A^T_\mu$ is plagued by the same difficulties affecting $A^2_{\min}$.

### C. Non-Abelian generalization of the Stueckelberg term

The Stueckelberg term, Eq. (2.10), can be promoted to the non-Abelian case [58], namely

$$\mathcal{O}_S = \mathrm{Tr}\int d^4x\left(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right)^2, \tag{3.7}$$

with

$$U = e^{ig\phi^aT^a}, \tag{3.8}$$

where $\{T^a\}$, $a = 1, \ldots, N^2 - 1$, denote the Hermitian generators of the gauge group $SU(N)$, and where $\phi^a$ is a dimensionless scalar field in the adjoint representation. As shown in Appendix B, expression (3.7) is left invariant by the gauge transformations

$$A_\mu \to V^{-1}A_\mu V + \frac{i}{g}V^{-1}\partial_\mu V, \qquad U \to UV. \tag{3.9}$$

The resulting non-Abelian massive action

$$S_S = \frac{1}{2}\mathrm{Tr}\int d^4xF_{\mu\nu}F_{\mu\nu} + m^2\mathrm{Tr} \times \int d^4x\left(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right)^2, \tag{3.10}$$

looks local. However, it is not polynomial in the Stueckelberg field $\phi^a$. In fact, when expanded in a power series in the field $\phi^a$, the term $U^{-1}\partial_\mu U$ gives rise to an infinite number of vertices. This jeopardizes a consistent perturbative treatment of expression (3.10). To the best of our knowledge, the action (3.10) is not multiplicatively renormalizable [58], see also the recent discussion given in [65]. As done in the Abelian case, it is interesting to have a look at the classical equations of motion which follow from the action (3.10), i.e.

$$D_\mu\left(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right) = 0. \tag{3.11}$$

Equation (3.11) can be used to express the Stueckelberg field $\phi^a$ as a power series in the gauge field $A^a_\mu$ [66], see also Appendix B, yielding

$$\phi^a = -\frac{1}{\partial^2}\partial A^a + \frac{g}{\partial^2}\left(f^{abc}A^b_\mu\partial_\mu\frac{\partial A^c}{\partial^2} + \frac{1}{2}f^{abc}\partial A^b\frac{1}{\partial^2}\partial A^c\right) + O(A^3). \tag{3.12}$$

Therefore

$$\begin{aligned} A_\mu - \frac{i}{g}U^{-1}\partial_\mu U = T^a\bigg[ & A^a_\mu - \partial_\mu\frac{\partial A^a}{\partial^2} - \frac{g}{2}f^{abc}\frac{\partial_\mu}{\partial^2}\partial A^b\frac{1}{\partial^2}\partial A^c \\ & + \frac{g}{\partial^2}\partial_\mu\left(f^{abc}A^b_\nu\partial_\nu\frac{\partial A^c}{\partial^2} + \frac{1}{2}f^{abc}\partial A^b\frac{1}{\partial^2}\partial A^c\right)\bigg] + O(A^3). \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned} \mathcal{O}_S &= \mathrm{Tr}\int d^4x\left(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U\right)^2 \\ &= \frac{1}{2}\int d^4x\bigg[A^{aT}_\mu A^{aT}_\mu + 2gA^{aT}_\mu\frac{\partial_\mu}{\partial^2}\left(f^{abc}A^b_\nu\partial_\nu\frac{\partial A^c}{\partial^2} + \frac{1}{2}f^{abc}\partial A^b\frac{1}{\partial^2}\partial A^c\right) - gf^{abc}A^{aT}_\mu\left(\partial_\mu\frac{\partial A^b}{\partial^2}\right)\frac{\partial A^c}{\partial^2}\bigg] \\ &\quad + O(A^4). \end{aligned} \tag{3.14}$$

Moreover, taking into account that, due to the transversality of $A^{aT}_\mu$, the second term of the expression above vanishes by integration by parts, we obtain

$$\mathcal{O}_S = \frac{1}{2}\mathrm{Tr}\int d^4x\bigg[A^a_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A^a_\mu - gf^{abc}A^{aT}_\mu\left(\partial_\mu\frac{\partial A^b}{\partial^2}\right)\frac{\partial A^c}{\partial^2}\bigg] + O(A^4), \tag{3.15}$$

which coincides precisely with expression (3.4). This shows the classical equivalence, within the space of the formal power series, between the Stueckelberg mass operator and the functional $A^2_{\min}$ in the non-Abelian case.

### D. Non-Abelian generalization of the operator $\int d^4xF_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$

It remains now to discuss the non-Abelian generalization of the operator $\int d^4xF_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$, a task easily achieved by replacing the ordinary derivative $\partial$ by the covariant one $D$, namely

$$\mathrm{Tr}\int d^4xF_{\mu\nu}\frac{1}{D^2}F_{\mu\nu} \equiv \frac{1}{2}\int d^4xF^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}. \tag{3.16}$$

We remark that this term can be introduced in any gauge and, unlike the functional $A^2_{\min}$, does not require any specific knowledge of the properties of the Gribov region as well as of the fundamental modular region. It has already been considered in [67] in the case of the three-dimensional Yang-Mills theories, where the use of the operator (3.16) was based on its appearance in e.g. the two-dimensional Schwinger model. However, so far, it has not yet been analyzed in four dimensions. Although in the Abelian case the operator $\int d^4xF_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$ turns out to be equivalent to $A^2_{\min}$, this is no more true in the non-Abelian case. This can be understood by observing that, thanks to gauge invariance, the expression (3.4) for $A^2_{\min}$ can be rewritten directly in terms of the field strength $F_{\mu\nu}$. In fact, as proven in [60], it turns out that

$$\begin{aligned} A^2_{\min} = -\frac{1}{2}\mathrm{Tr}\int d^4x\bigg(& F_{\mu\nu}\frac{1}{D^2}F_{\mu\nu} + 2i\frac{1}{D^2}F_{\lambda\mu}\left[\frac{1}{D^2}D_\kappa F_{\kappa\lambda}, \frac{1}{D^2}D_\nu F_{\nu\mu}\right] \\ & - 2i\frac{1}{D^2}F_{\lambda\mu}\left[\frac{1}{D^2}D_\kappa F_{\kappa\nu}, \frac{1}{D^2}D_\nu F_{\lambda\mu}\right]\bigg) + O(F^4), \end{aligned} \tag{3.17}$$

from which the difference between the operator (3.16) and $A^2_{\min}$ becomes apparent. This interesting feature gives to the operator (3.16) a privileged role with respect to the localization procedure. In fact, while in the case of $A^2_{\min}$ one has to deal with an infinite number of nonlocal terms, expression (3.16) seems to be more manageable. In the next section the localization procedure of the operator (3.16) will be discussed.

## IV. LOCALIZING THE MASS OPERATOR $\mathrm{Tr}\int d^4xF_{\mu\nu}\frac{1}{D^2}F_{\mu\nu}$

The localization of the operator $\mathrm{Tr}\int d^4xF_{\mu\nu}\frac{1}{D^2}F_{\mu\nu}$ can be achieved by generalizing the procedure adopted in the localization of the Abelian operator $\int d^4xF_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$, Eq. (2.24). Let us start by considering the Yang-Mills action with the addition of the mass operator (3.16), i.e.

$$S_{YM} + S_{\mathcal{O}}, \tag{4.1}$$

where

$$S_{YM} = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}, \tag{4.2}$$

and

$$S_{\mathcal{O}} = -\frac{m^2}{4}\int d^4x F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}. \tag{4.3}$$

The term (4.3) can be localized by means of the introduction of a pair of complex bosonic antisymmetric tensor fields in the adjoint representation, $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu})$, according to

$$e^{-S_O} = \int D\bar{B}DB(\det D^2)^6 \exp\left[-\left(\frac{1}{4}\int d^4x \bar{B}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} + \frac{im}{4}\int d^4x (B-\bar{B})^a_{\mu\nu}F^a_{\mu\nu}\right)\right], \tag{4.4}$$

where the determinant, $(\det D^2)^6$, takes into account the Jacobian arising from the integration over the bosonic complex fields $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu})$. This term can also be localized by means of suitable anticommuting antisymmetric tensor fields $(\bar{G}^a_{\mu\nu}, G^a_{\mu\nu})$, namely

$$(\det D^2)^6 = \int D\bar{G}DG \exp\left(\frac{1}{4}\int d^4x \bar{G}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu}\right). \tag{4.5}$$

Therefore, we obtain a classical local action which reads

$$S_{YM} + S_{BG} + S_m, \tag{4.6}$$

where

$$S_{BG} = \frac{1}{4}\int d^4x(\bar{B}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} - \bar{G}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu}),$$
$$S_m = \frac{im}{4}\int d^4x(B-\bar{B})^a_{\mu\nu}F^a_{\mu\nu}. \tag{4.7}$$

The localization procedure does not destroy the gauge invariance of the resulting action. In fact, it is easily checked that expression (4.6) is left invariant by the gauge transformations

$$\delta A^a_\mu = -D^{ab}_\mu\omega^b, \qquad \delta B^a_{\mu\nu} = gf^{abc}\omega^b B^c_{\mu\nu},$$
$$\delta\bar{B}^a_{\mu\nu} = gf^{abc}\omega^b\bar{B}^c_{\mu\nu}, \qquad \delta G^a_{\mu\nu} = gf^{abc}\omega^b G^c_{\mu\nu}, \tag{4.8}$$
$$\delta\bar{G}^a_{\mu\nu} = gf^{abc}\omega^b\bar{G}^c_{\mu\nu},$$

$$\delta(S_{YM} + S_{BG} + S_m) = 0, \tag{4.9}$$

so that condition (I.) is fulfilled. Let us proceed thus with the identification of a suitable quantized action, associated to expression (4.6), which enjoys the property of being multiplicatively renormalizable. For that, we follow the setup successfully introduced by Zwanziger [68,69] in the localization of the nonlocal horizon function implementing the restriction to the Gribov region in the Landau gauge. In a series of papers, Zwanziger has been able to show that the restriction to the Gribov region can be implemented by adding to the Yang-Mills action a nonlocal term, known as the horizon function, which is given by

$$S_{\rm Horiz} = \gamma^4 g^2 \int d^4x f^{abc}A^b_\mu(\mathcal{M}^{-1})^{ad}f^{dec}A^e_\mu, \tag{4.10}$$

where $\gamma$ denotes the Gribov parameter [70] and $\mathcal{M}^{ab}$ is the Faddeev-Popov operator of the Landau gauge

$$\mathcal{M}^{ab} = -\partial_\mu(\partial_\mu\delta^{ab} + gf^{acb}A^c_\mu). \tag{4.11}$$

As proven in [68,69], the nonlocal horizon term (4.10) can be localized by means of a suitable set of additional fields, in a way analogous to that of Eq. (4.4). Remarkably, the resulting theory is renormalizable to all orders, obeying the renormalization group equations. Thus, it seems natural to us to adopt here the same procedure. According to [68,69], we treat the operators $B^a_{\mu\nu}F^a_{\mu\nu}$ and $\bar{B}^a_{\mu\nu}F^a_{\mu\nu}$, entering the expression for $S_m$ in Eq. (4.7), as composite operators coupled to suitable external sources $V_{\sigma\rho\mu\nu}(x)$, $\bar{V}_{\sigma\rho\mu\nu}(x)$. This amounts to replace the term $S_m$ by

$$\frac{1}{4}\int d^4x(V_{\sigma\rho\mu\nu}\bar{B}^a_{\sigma\rho}F^a_{\mu\nu} - \bar{V}_{\sigma\rho\mu\nu}B^a_{\sigma\rho}F^a_{\mu\nu}). \tag{4.12}$$

At the end, the sources $V_{\sigma\rho\mu\nu}(x)$, $\bar{V}_{\sigma\rho\mu\nu}(x)$ are required to attain their physical value, namely

$$\bar{V}_{\sigma\rho\mu\nu}|_{\rm phys} = V_{\sigma\rho\mu\nu}|_{\rm phys} = \frac{-im}{2}(\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\sigma\nu}\delta_{\rho\mu}), \tag{4.13}$$

so that expression (4.12) gives back the term $S_m$. As pointed out in [68,69], this procedure allows us to study the renormalization properties of the Green's functions obtained from the action $(S_{YM} + S_{BG})$ with the insertion of the composite operators $B^a_{\mu\nu}F^a_{\mu\nu}$ and $\bar{B}^a_{\mu\nu}F^a_{\mu\nu}$. Following [68,69], let us focus first on the properties of the action $(S_{YM} + S_{BG})$ which, as we shall see, displays a rich symmetry content.

### A. BRST invariance

In this section we shall discuss the symmetry content of the action $(S_{YM} + S_{BG})$, where $S_{YM}$ is the Yang-Mills action, Eq. (4.2), and $S_{BG}$ depends on the localizing fields $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, \bar{G}^a_{\mu\nu}, G^a_{\mu\nu})$, Eq. (4.7). Let us begin by introducing the gauge fixing term, chosen here to be that of the linear covariant gauges, namely

$$S = S_{YM} + S_{BG} + S_{gf}, \tag{4.14}$$

with

$$S_{gf} = \int d^4x\left(\frac{\alpha}{2}b^a b^a + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\right), \tag{4.15}$$

where $b^a$ is the Lagrange multiplier and $\bar{c}^a$, $c^a$ stand for the Faddeev-Popov ghosts. It turns out that the action $S$ is left invariant by the following $BRST$ transformation, i.e.

$$sA^a_\mu = -D^{ab}_\mu c^b, \qquad sc^a = \frac{g}{2} f^{abc} c^a c^b,$$
$$sB^a_{\mu\nu} = g f^{abc} c^b B^c_{\mu\nu} + G^a_{\mu\nu}, \qquad s\bar{B}^a_{\mu\nu} = g f^{abc} c^b \bar{B}^c_{\mu\nu},$$
$$sG^a_{\mu\nu} = g f^{abc} c^b G^c_{\mu\nu}, \qquad s\bar{G}^a_{\mu\nu} = g f^{abc} c^b \bar{G}^c_{\mu\nu} + \bar{B}^a_{\mu\nu},$$
$$s\bar{c}^a = b^a, \qquad sb^a = 0, \qquad s^2 = 0, \tag{4.16}$$

and

$$sS = 0. \tag{4.17}$$

This is easily verified by observing that the term $S_{BG}$ can be written as a pure $BRST$ variation, according to

$$S_{BG} = \frac{1}{4} s \int d^4x \bar{G}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu}. \tag{4.18}$$

Of course, the same property holds for the gauge fixing term

$$S_{gf} = s \int d^4x \left( \frac{\alpha}{2} \bar{c}^a b^a + \bar{c}^a \partial_\mu A^a_\mu \right). \tag{4.19}$$

In addition to the $BRST$ invariance, and in complete analogy with the Zwanziger action [68,69] implementing the restriction to the Gribov horizon, the model displays a global invariance $U(f)$, $f = 6$, expressed by

$$\mathcal{Q}_{\mu\nu\alpha\beta} S = 0, \tag{4.20}$$

where

$$\mathcal{Q}_{\mu\nu\alpha\beta} = \int d^4x \left( B^a_{\alpha\beta} \frac{\delta}{\delta B^a_{\mu\nu}} - \bar{B}^a_{\mu\nu} \frac{\delta}{\delta \bar{B}^a_{\alpha\beta}} + G^a_{\alpha\beta} \frac{\delta}{\delta G^a_{\mu\nu}} - \bar{G}^a_{\mu\nu} \frac{\delta}{\delta \bar{G}^a_{\alpha\beta}} \right). \tag{4.21}$$

The presence of the global invariance $U(f)$ , $f = 6$, means that one can make use of the composite index $i \equiv \{\mu\nu\}$, $i = (1, \ldots, 6)$. Therefore, setting

$$(B^a_i, \bar{B}^a_i, G^a_i, \bar{G}^a_i) = \frac{1}{2} (B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}), \tag{4.22}$$

we get

$$S_{BG} = \int d^4x (\bar{B}^a_i D^{ab}_\mu D^{bc}_\mu B^c_i - \bar{G}^a_i D^{ab}_\mu D^{bc}_\mu G^c_i), \tag{4.23}$$

and for the symmetry generator

$$\mathcal{Q}_{ij} = \int d^4x \left( B^a_i \frac{\delta}{\delta B^a_j} - \bar{B}^a_j \frac{\delta}{\delta \bar{B}^a_i} + G^a_i \frac{\delta}{\delta G^a_j} - \bar{G}^a_j \frac{\delta}{\delta \bar{G}^a_i} \right). \tag{4.24}$$

By means of the diagonal operator $\mathcal{Q}_f = \mathcal{Q}_{ii}$, the $i$-valued fields turn out to possess an additional quantum number, displayed in Table I, together with the dimension and the ghost number. Besides the global $U(f)$, $f = 6$, invariance, the action (4.14) possesses the following additional rigid symmetries

TABLE I. Dimension, ghost number, and $\mathcal{Q}_f$-charge of the fields.

| Fields | $A$ | $c$ | $\bar{c}$ | $b$ | $B$ | $\bar{B}$ | $G$ | $\bar{G}$ |
|---|---|---|---|---|---|---|---|---|
| Dimension | 1 | 0 | 2 | 2 | 1 | 1 | 1 | 1 |
| Ghost number | 0 | 1 | −1 | 0 | 0 | 0 | 1 | −1 |
| $\mathcal{Q}_f$-charge | 0 | 0 | 0 | 0 | 1 | −1 | 1 | −1 |

$$\mathcal{R}^{(A)}_{ij} S = 0, \tag{4.25}$$

where $A = \{1, 2, 3, 4\}$ and

$$\mathcal{R}^{(1)}_{ij} = \int d^4x \left( B^a_i \frac{\delta}{\delta G^a_j} - \bar{G}^a_j \frac{\delta}{\delta \bar{B}^a_i} \right),$$
$$\mathcal{R}^{(2)}_{ij} = \int d^4x \left( \bar{B}^a_i \frac{\delta}{\delta \bar{G}^a_j} + G^a_j \frac{\delta}{\delta B^a_i} \right), \tag{4.26}$$

$$\mathcal{R}^{(3)}_{ij} = \int d^4x \left( \bar{B}^a_i \frac{\delta}{\delta G^a_j} - \bar{G}^a_j \frac{\delta}{\delta B^a_i} \right),$$
$$\mathcal{R}^{(4)}_{ij} = \int d^4x \left( B^a_i \frac{\delta}{\delta \bar{G}^a_j} + G^a_j \frac{\delta}{\delta \bar{B}^a_i} \right). \tag{4.27}$$

Let us conclude this section by showing that also the source term (4.12) can be introduced in a $BRST$ invariant way. This is achieved by considering the following source term

$$S_{\rm aux} = s \int d^4x [(V_{i\mu\nu} \bar{G}^a_i - \bar{U}_{i\mu\nu} B^a_i) F^a_{\mu\nu} + \chi_1 \bar{U}_{i\mu\nu} \partial^2 V_{i\mu\nu} + \chi_2 \bar{U}_{i\mu\nu} \partial_\mu \partial_\alpha V_{i\nu\alpha} - \zeta (\bar{U}_{i\mu\nu} V_{i\mu\nu} \bar{V}_{j\alpha\beta} V_{j\alpha\beta} - \bar{U}_{i\mu\nu} V_{i\mu\nu} \bar{U}_{j\alpha\beta} U_{j\alpha\beta})], \tag{4.28}$$

with

$$sV_{i\mu\nu} = U_{i\mu\nu}, \qquad sU_{i\mu\nu} = 0, \qquad s\bar{U}_{i\mu\nu} = \bar{V}_{i\mu\nu},$$
$$s\bar{V}_{i\mu\nu} = 0, \qquad s^2 = 0. \tag{4.29}$$

The quantum numbers of the sources are displayed in Table II. Therefore, for $S_{\rm aux}$ one gets

TABLE II. Dimension, ghost number, and $\mathcal{Q}_f$-charge of the sources.

| Sources | $\bar{U}$ | $V$ | $U$ | $\bar{V}$ |
|---|---|---|---|---|
| Dimension | 1 | 1 | 1 | 1 |
| Ghost number | −1 | 0 | 1 | 0 |
| $\mathcal{Q}_f$-charge | −1 | 1 | 1 | −1 |

$$S_{\rm aux} = \int d^4x[\bar{U}_{i\mu\nu}G^a_i F^a_{\mu\nu} + V_{i\mu\nu}\bar{B}^a_i F^a_{\mu\nu} - \bar{V}_{i\mu\nu}B^a_i F^a_{\mu\nu} + U_{i\mu\nu}\bar{G}^a_i F^a_{\mu\nu} + \chi_1(\bar{V}_{i\mu\nu}\partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu}\partial^2 U_{i\mu\nu}) + \chi_2(\bar{V}_{i\mu\nu}\partial_\mu\partial_\alpha V_{i\nu\alpha} - \bar{U}_{i\mu\nu}\partial_\mu\partial_\alpha U_{i\nu\alpha}) - \zeta(\bar{U}_{i\mu\nu}U_{i\mu\nu}\bar{U}_{j\alpha\beta}U_{j\alpha\beta} + \bar{V}_{i\mu\nu}V_{i\mu\nu}\bar{V}_{j\alpha\beta}V_{j\alpha\beta} - 2\bar{U}_{i\mu\nu}U_{i\mu\nu}\bar{V}_{j\alpha\beta}V_{j\alpha\beta})]. \tag{4.30}$$

The parameters $\chi_1$, $\chi_2$, and $\zeta$ are free parameters, needed for renormalizability purposes. The action $S_{\rm aux}$ reduces to the term $S_m$ of Eq. (4.7) when the sources ($V_{i\mu\nu}$, $\bar{V}_{i\mu\nu}$, $U_{i\mu\nu}$, $\bar{U}_{i\mu\nu}$) attain their physical values, given now by

$$(V_{i\mu\nu}, \bar{V}_{i\mu\nu}, U_{i\mu\nu}, \bar{U}_{i\mu\nu}) = \frac{1}{2}(V_{\sigma\rho\mu\nu}, \bar{V}_{\sigma\rho\mu\nu}, U_{\sigma\rho\mu\nu}, \bar{U}_{\sigma\rho\mu\nu}), \tag{4.31}$$

$$\bar{V}_{\sigma\rho\mu\nu}|_{\rm phys} = V_{\sigma\rho\mu\nu}|_{\rm phys} = \frac{-im}{2}(\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\sigma\nu}\delta_{\rho\mu}), \quad U_{\sigma\rho\mu\nu} = \bar{U}_{\sigma\rho\mu\nu} = 0. \tag{4.32}$$

Thus

$$S_{\rm aux}|_{\rm phys} \to S_m - \frac{9}{4}\int d^4x \zeta m^4, \tag{4.33}$$

so that the term $S_m$ is recovered, modulo the constant quantity $\zeta m^4$. All ingredients needed to study the renormalizability of the action

$$S_{YM} + S_{BG} + S_{gf} + S_{\rm aux}, \tag{4.34}$$

are now at our disposal. This will be the task of the next section.

## V. IDENTIFICATION OF A MULTIPLICATIVELY RENORMALIZABLE ACTION

In order to discuss the renormalizability properties of our model, we have first to write down all possible Ward identities expressing the symmetry content of the starting classical action, Eq. (4.34). Let us begin by working out the Slavnov-Taylor identity. Following the algebraic renormalization procedure as described in [71], we need to introduce additional external sources ($\Omega^a_\mu$, $L^a$, $\bar{Y}^a_i$, $Y^a_i$, $\bar{X}^a_i$, $X^a_i$) in order to define at the quantum level the composite operators entering the nonlinear $BRST$ transformations of the fields ($A^a_\mu$, $c^a$, $B^a_i$, $\bar{B}^a_i$, $G^a_i$, $\bar{G}^a_i$), Eqs. (4.16). In the present case, this term reads

$$S_{ext} = s\int d^4x(-\Omega^a_\mu A^a_\mu + L^a c^a - \bar{Y}^a_i B^a_i - Y^a_i \bar{B}^a_i + \bar{X}^a_i G^a_i + X^a_i \bar{G}^a_i), \tag{5.1}$$

with

$$s\Omega^a_\mu = sL^a = 0, \tag{5.2}$$

and

$$sY^a_i = X^a_i, \qquad sX^a_i = 0, \tag{5.3}$$

$$s\bar{X}^a_i = -\bar{Y}^a_i, \tag{5.4}$$

$$s\bar{Y}^a_i = 0. \tag{5.5}$$

The quantum numbers of the external sources ($\Omega^a_\mu$, $L^a$, $\bar{Y}^a_i$, $Y^a_i$, $\bar{X}^a_i$, $X^a_i$) are displayed in Table III. For the complete action $\Sigma$

$$\Sigma = S_{YM} + S_{gf} + S_{BG} + S_{\rm aux} + S_{\rm ext}, \tag{5.6}$$

we obtain

$$\Sigma = S_{YM} + \int d^4x\left(\frac{\alpha}{2}b^a b^a + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\right) + \int d^4x(\bar{B}^a_i D^{ab}_\mu D^{bc}_\mu B^c_i - \bar{G}^a_i D^{ab}_\mu D^{bc}_\mu G^c_i) + \int d^4x((\bar{U}_{i\mu\nu}G^a_i + V_{i\mu\nu}\bar{B}^a_i - \bar{V}_{i\mu\nu}B^a_i + U_{i\mu\nu}\bar{G}^a_i)F^a_{\mu\nu} + \chi_1(\bar{V}_{i\mu\nu}\partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu}\partial^2 U_{i\mu\nu})) + \int d^4x\chi_2(\bar{V}_{i\mu\nu}\partial_\mu\partial_\alpha V_{i\nu\alpha} - \bar{U}_{i\mu\nu}\partial_\mu\partial_\alpha U_{i\nu\alpha}) - \int d^4x\zeta(\bar{U}_{i\mu\nu}U_{i\mu\nu}\bar{U}_{j\alpha\beta}U_{j\alpha\beta} + \bar{V}_{i\mu\nu}V_{i\mu\nu}\bar{V}_{j\alpha\beta}V_{j\alpha\beta} - 2\bar{U}_{i\mu\nu}U_{i\mu\nu}\bar{V}_{j\alpha\beta}V_{j\alpha\beta}) + \int d^4x\left(-\Omega^a_\mu D^{ab}_\mu c^b + \frac{g}{2}f^{abc}L^a c^b c^c + gf^{abc}\bar{Y}^a_i c^b B^c_i + gf^{abc}Y^a_i c^b\bar{B}^c_i + gf^{abc}\bar{X}^a_i c^b G^c_i + gf^{abc}X^a_i c^b\bar{G}^c_i\right). \tag{5.7}$$

TABLE III. Dimension, fermion number, and $\mathcal{Q}_f$-charge of the external sources.

| Sources | $\Omega$ | $L$ | $\bar{Y}$ | $Y$ | $\bar{X}$ | $X$ |
|---|---|---|---|---|---|---|
| Dimension | 3 | 4 | 3 | 3 | 3 | 3 |
| Ghost number | −1 | −2 | −1 | −1 | −2 | 0 |
| $\mathcal{Q}_f$-charge | 0 | 0 | −1 | 1 | −1 | 1 |

Expression (5.7) obeys several Ward identities, which we enlist below

(a) the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \tag{5.8}$$

$$S(\Sigma)=\int d^4x\Big[\frac{\delta\Sigma}{\delta\Omega^a_\mu}\frac{\delta\Sigma}{\delta A^a_\mu}+\frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a}+b^a\frac{\delta\Sigma}{\delta\bar{c}^a}+\Big(\frac{\delta\Sigma}{\delta\bar{Y}^a_i}+G^a_i\Big)\frac{\delta\Sigma}{\delta B^a_i}+\frac{\delta\Sigma}{\delta Y^a_i}\frac{\delta\Sigma}{\delta\bar{B}^a_i}+\frac{\delta\Sigma}{\delta\bar{X}^a_i}\frac{\delta\Sigma}{\delta G^a_i}+\Big(\frac{\delta\Sigma}{\delta X^a_i}+\bar{B}^a_i\Big)\frac{\delta\Sigma}{\delta\bar{G}^a_i}$$
$$+\bar{V}_{i\mu\nu}\frac{\delta\Sigma}{\delta\bar{U}_{i\mu\nu}}+U_{i\mu\nu}\frac{\delta\Sigma}{\delta V_{i\mu\nu}}-\bar{Y}^a_i\frac{\delta\Sigma}{\delta\bar{X}^a_i}+X^a_i\frac{\delta\Sigma}{\delta Y^a_i}\Big], \tag{5.9}$$

(b) the global $U(f)$ invariance , $f=6$, i.e.

$$\mathcal{Q}_{ij}\Sigma=0, \tag{5.10}$$

where

$$\mathcal{Q}_{ij}=\int d^4x\Big(B^a_i\frac{\delta}{\delta B^a_j}-\bar{B}^a_j\frac{\delta}{\delta\bar{B}^a_i}+G^a_i\frac{\delta}{\delta G^a_j}-\bar{G}^a_j\frac{\delta}{\delta\bar{G}^a_i}+U_{i\mu\nu}\frac{\delta}{\delta U_{j\mu\nu}}-\bar{U}_{j\mu\nu}\frac{\delta}{\delta\bar{U}_{i\mu\nu}}+V_{i\mu\nu}\frac{\delta}{\delta V_{j\mu\nu}}$$
$$-\bar{V}_{j\mu\nu}\frac{\delta}{\delta\bar{V}_{i\mu\nu}}+Y^a_i\frac{\delta}{\delta Y^a_j}-\bar{Y}^a_j\frac{\delta}{\delta\bar{Y}^a_i}+X^a_i\frac{\delta}{\delta X^a_j}-\bar{X}^a_j\frac{\delta}{\delta\bar{X}^a_i}\Big), \tag{5.11}$$

(c) the exact rigid symmetries

$$\mathcal{R}^{(A)}_{ij}\Sigma=0, \tag{5.12}$$

where $A=\{1,2,3,4\}$ and

$$\begin{aligned}
\mathcal{R}^{(1)}_{ij}&=\int d^4x\Big(B^a_i\frac{\delta}{\delta G^a_j}-\bar{G}^a_j\frac{\delta}{\delta\bar{B}^a_i}+V_{i\mu\nu}\frac{\delta}{\delta U_{j\mu\nu}}-\bar{U}_{j\mu\nu}\frac{\delta}{\delta\bar{V}_{i\mu\nu}}+Y^a_i\frac{\delta}{\delta X^a_j}+\bar{X}^a_j\frac{\delta}{\delta\bar{Y}^a_i}\Big),\\
\mathcal{R}^{(2)}_{ij}&=\int d^4x\Big(\bar{B}^a_i\frac{\delta}{\delta\bar{G}^a_j}+G^a_j\frac{\delta}{\delta B^a_i}+\bar{V}_{i\mu\nu}\frac{\delta}{\delta\bar{U}_{j\mu\nu}}+U_{j\mu\nu}\frac{\delta}{\delta V_{i\mu\nu}}-\bar{Y}^a_i\frac{\delta}{\delta\bar{X}^a_j}+X^a_j\frac{\delta}{\delta Y^a_i}\Big),\\
\mathcal{R}^{(3)}_{ij}&=\int d^4x\Big(\bar{B}^a_i\frac{\delta}{\delta G^a_j}-\bar{G}^a_j\frac{\delta}{\delta B^a_i}-\bar{V}_{i\mu\nu}\frac{\delta}{\delta U_{j\mu\nu}}+\bar{U}_{j\mu\nu}\frac{\delta}{\delta V_{i\mu\nu}}+\bar{Y}^a_i\frac{\delta}{\delta X^a_j}+\bar{X}^a_j\frac{\delta}{\delta Y^a_i}\Big),\\
\mathcal{R}^{(4)}_{ij}&=\int d^4x\Big(B^a_i\frac{\delta}{\delta\bar{G}^a_j}+G^a_j\frac{\delta}{\delta\bar{B}^a_i}-V_{i\mu\nu}\frac{\delta}{\delta\bar{U}_{j\mu\nu}}-U_{j\mu\nu}\frac{\delta}{\delta\bar{V}_{i\mu\nu}}-Y^a_i\frac{\delta}{\delta\bar{X}^a_j}+X^a_j\frac{\delta}{\delta\bar{Y}^a_i}\Big),
\end{aligned} \tag{5.13}$$

(d) the gauge fixing condition

$$\frac{\delta\Sigma}{\delta b^a}=\alpha b^a+\partial_\mu A^a_\mu \tag{5.14}$$

(e) the antighost Ward identity

$$\bar{\mathcal{G}}^a\Sigma=0, \tag{5.15}$$

where

$$\bar{\mathcal{G}}^a=\frac{\delta}{\delta\bar{c}^a}+\partial_\mu\frac{\delta}{\delta\Omega^a_\mu}. \tag{5.16}$$

### A. Determination of the most general local invariant counterterm

Having established all the Ward identities fulfilled by the complete action $\Sigma$, we can now turn to the characterization of the most general allowed counterterm $\Sigma^c$. Following the algebraic renormalization procedure [71], $\Sigma^c$ is an integrated local polynomial in the fields and sources with dimension bounded by four, with vanishing ghost number and $\mathcal{Q}_f$-charge, obeying the following constraints

$$\begin{aligned}
&\mathcal{Q}_{ij}\Sigma^c=0, \qquad \mathcal{R}^{(A)}_{ij}\Sigma^c=0,\\
&\frac{\delta\Sigma^c}{\delta b^a}=0, \qquad \bar{\mathcal{G}}^a\Sigma^c=0,
\end{aligned} \tag{5.17}$$

in addition to

$$\mathcal{B}_\Sigma\Sigma^c=0, \tag{5.18}$$

where $\mathcal{B}_\Sigma$ is the nilpotent linearized Slavnov-Taylor operator

$$\mathcal{B}_\Sigma=\int d^4x\Big[\frac{\delta\Sigma}{\delta\Omega^a_\mu}\frac{\delta}{\delta A^a_\mu}+\frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta\Omega^a_\mu}+\frac{\delta\Sigma}{\delta L^a}\frac{\delta}{\delta c^a}+\frac{\delta\Sigma}{\delta c^a}$$
$$\times\frac{\delta}{\delta L^a}+b^a\frac{\delta}{\delta\bar{c}^a}+\Big(\frac{\delta\Sigma}{\delta\bar{Y}^a_i}+G^a_i\Big)\frac{\delta}{\delta B^a_i}+\frac{\delta\Sigma}{\delta B^a_i}\frac{\delta}{\delta\bar{Y}^a_i}$$
$$+\frac{\delta\Sigma}{\delta Y^a_i}\frac{\delta}{\delta\bar{B}^a_i}+\Big(\frac{\delta\Sigma}{\delta\bar{B}^a_i}+X^a_i\Big)\frac{\delta}{\delta Y^a_i}+\frac{\delta\Sigma}{\delta\bar{X}^a_i}\frac{\delta}{\delta G^a_i}$$
$$+\Big(\frac{\delta\Sigma}{\delta G^a_i}-\bar{Y}^a_i\Big)\frac{\delta}{\delta\bar{X}^a_i}+\Big(\frac{\delta\Sigma}{\delta X^a_i}+\bar{B}^a_i\Big)\frac{\delta}{\delta\bar{G}^a_i}+\frac{\delta\Sigma}{\delta\bar{G}^a_i}\frac{\delta}{\delta X^a_i}$$
$$+\bar{V}_{i\mu\nu}\frac{\delta}{\delta\bar{U}_{i\mu\nu}}+U_{i\mu\nu}\frac{\delta}{\delta V_{i\mu\nu}}\Big], \tag{5.19}$$

$$\mathcal{B}_{\Sigma}\mathcal{B}_{\Sigma} = 0. \tag{5.20}$$

After a rather lengthy analysis, for the most general allowed counterterm we have found

$$\begin{aligned}
\Sigma^c =& a_0 S_{YM} + a_1\int d^4x A^a_\mu \frac{\delta S_{YM}}{\delta A^a_\mu} + \int d^4x\Big((a_1+a_2)(\Omega^a_\mu + \partial_\mu\bar{c}^a)\partial_\mu c^a + a_2 g f^{abc}(\Omega^a_\mu + \partial_\mu\bar{c}^a)A^b_\mu c^c - a_2\frac{g}{2}f^{abc}L^a c^b c^c\Big)\\
&+ \int d^4x\{(2a_3+a_4)\bar{B}^a_i\partial^2 B^a_i - (2a_3+a_4)\bar{G}^a_i\partial^2 G^a_i - (a_1+2a_3+a_4)gf^{abc}\bar{B}^a_i(\partial_\mu A^b_\mu + 2A^b_\mu\partial_\mu)B^c_i\\
&+ (2a_1+2a_3+a_4)g^2 f^{abd}f^{bce}\bar{B}^a_i A^d_\mu A^e_\mu B^c_i + (a_1+2a_3+a_4)gf^{abc}\bar{G}^a_i(\partial_\mu A^b_\mu + 2A^b_\mu\partial_\mu)G^c_i\\
&- (2a_1+2a_3+a_4)g^2 f^{abd}f^{bce}\bar{G}^a_i A^d_\mu A^e_\mu G^c_i - a_2 g f^{abc}c^a(\bar{Y}^b_i B^c_i + Y^b_i\bar{B}^c_i - \bar{X}^b_i G^c_i - X^b_i\bar{G}^c_i)\\
&+ [(a_1+a_3+a_5)2\partial_\mu A^a_\nu + (2a_1+a_3+a_5)gf^{abc}A^b_\mu A^c_\nu](\bar{U}_{i\mu\nu}G^a_i + V_{i\mu\nu}\bar{B}^a_i + U_{i\mu\nu}\bar{G}^a_i - \bar{V}_{i\mu\nu}B^a_i)\\
&+ \frac{\lambda^{abcd}}{16}(\bar{B}^a_i B^b_i - \bar{G}^a_i G^b_i)(\bar{B}^c_j B^d_j - \bar{G}^c_j G^d_j) + a_7(\bar{B}^a_i B^a_i - \bar{G}^a_i G^a_i)(\bar{V}_{i\mu\nu}V_{i\mu\nu} - \bar{U}_{i\mu\nu}U_{i\mu\nu}) + a_8\Big(\bar{B}^a_i G^a_j V_{i\mu\nu}\bar{U}_{j\mu\nu}\\
&+ \bar{G}^a_i G^a_j U_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{B}^a_i B^a_j V_{i\mu\nu}\bar{V}_{j\mu\nu} - \bar{G}^a_i B^a_j U_{i\mu\nu}\bar{V}_{j\mu\nu} - G^a_i B^a_j\bar{U}_{i\mu\nu}\bar{V}_{j\mu\nu} + \bar{G}^a_i\bar{B}^a_j U_{i\mu\nu}V_{j\mu\nu} - \frac{1}{2}B^a_i B^a_j\bar{V}_{i\mu\nu}\bar{V}_{j\mu\nu}\\
&+ \frac{1}{2}G^a_i G^a_j\bar{U}_{i\mu\nu}\bar{U}_{j\mu\nu} - \frac{1}{2}\bar{B}^a_i\bar{B}^a_j V_{i\mu\nu}V_{j\mu\nu} + \frac{1}{2}\bar{G}^a_i\bar{G}^a_j U_{i\mu\nu}U_{j\mu\nu}\Big) + a_9\zeta(\bar{V}_{i\mu\nu}V_{i\mu\nu} - \bar{U}_{i\mu\nu}U_{i\mu\nu})^2\\
&+ a_{10}\chi_1(\bar{V}_{i\mu\nu}\partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu}\partial^2 U_{i\mu\nu}) + a_{11}\chi_1(\bar{V}_{i\mu\nu}\partial_\mu\partial_\alpha V_{i\nu\alpha} - \bar{U}_{i\mu\nu}\partial_\mu\partial_\alpha U_{i\nu\alpha})\},
\end{aligned} \tag{5.21}$$

where ($a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_7$, $a_8$, $a_9$, $a_{10}$, $a_{11}$) are free parameters and $\lambda^{abcd}$ is an invariant tensor of rank four with indices in the adjoint representation and such that

$$\lambda^{abcd} = \lambda^{cdab}, \qquad \lambda^{abcd} = \lambda^{bacd}. \tag{5.22}$$

For a general discussion of the properties of higher rank invariant tensors we refer the reader to [72]. Let us only mention that an invariant rank 4 tensor like $\lambda^{abcd}$ obeys a generalized Jacobi identity

$$f^{man}\lambda^{mbcd} + f^{mbn}\lambda^{amcd} + f^{mcn}\lambda^{abmd} + f^{mdn}\lambda^{abcm} = 0. \tag{5.23}$$

These parameters $a_i$, $i = 0, \ldots, 11$, should correspond to a multiplicative renormalization of the fields, parameters, and sources of the starting classical action $\Sigma$. However, it turns out that the counterterm (5.21) cannot be reabsorbed through a renormalization of the parameters and fields of $\Sigma$. This means that the starting action $\Sigma$ is not stable against radiative corrections. Said otherwise, $\Sigma$ is not the most general local invariant action compatible with the Ward identities (5.8)–(5.15). In fact, from the expression (5.21) it follows that the term

$$\begin{aligned}
\int d^4x\Big[& a_7(\bar{B}^a_i B^a_i - \bar{G}^a_i G^a_i)(\bar{V}_{j\mu\nu}V_{j\mu\nu} - \bar{U}_{j\mu\nu}U_{j\mu\nu}) + \frac{\lambda^{abcd}}{16}(\bar{B}^a_i B^b_i - \bar{G}^a_i G^b_i)(\bar{B}^c_j B^d_j - \bar{G}^c_j G^d_j)\\
&+ a_8\Big(\bar{B}^a_i G^a_j V_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{G}^a_i G^a_j U_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{B}^a_i B^a_j V_{i\mu\nu}\bar{V}_{j\mu\nu} - \bar{G}^a_i B^a_j U_{i\mu\nu}\bar{V}_{j\mu\nu} - G^a_i B^a_j\bar{U}_{i\mu\nu}\bar{V}_{j\mu\nu}\\
&+ \bar{G}^a_i\bar{B}^a_j U_{i\mu\nu}V_{j\mu\nu} - \frac{1}{2}B^a_i B^a_j\bar{V}_{i\mu\nu}\bar{V}_{j\mu\nu} + \frac{1}{2}G^a_i G^a_j\bar{U}_{i\mu\nu}\bar{U}_{j\mu\nu} - \frac{1}{2}\bar{B}^a_i\bar{B}^a_j V_{i\mu\nu}V_{j\mu\nu} + \frac{1}{2}\bar{G}^a_i\bar{G}^a_j U_{i\mu\nu}U_{j\mu\nu}\Big)\Big],
\end{aligned} \tag{5.24}$$

fulfills all Ward identities. Moreover, this term does not correspond to a renormalization of the parameters and fields of $\Sigma$. This follows by noting that the counterterm (5.24) is in fact absent in the expression (5.7).

A stable action $\tilde{\Sigma}$ is thus obtained by adding to the action $\Sigma$ the following expression

$$\begin{aligned}
S_\lambda = \int d^4x\Big[& \lambda_1(\bar{B}^a_i B^a_i - \bar{G}^a_i G^a_i)(\bar{V}_{j\mu\nu}V_{j\mu\nu} - \bar{U}_{j\mu\nu}U_{j\mu\nu}) + \frac{\lambda^{abcd}}{16}(\bar{B}^a_i B^b_i - \bar{G}^a_i G^b_i)(\bar{B}^c_j B^d_j - \bar{G}^c_j G^d_j) + \lambda_3\Big(\bar{B}^a_i G^a_j V_{i\mu\nu}\bar{U}_{j\mu\nu}\\
&+ \bar{G}^a_i G^a_j U_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{B}^a_i B^a_j V_{i\mu\nu}\bar{V}_{j\mu\nu} - \bar{G}^a_i B^a_j U_{i\mu\nu}\bar{V}_{j\mu\nu} - G^a_i B^a_j\bar{U}_{i\mu\nu}\bar{V}_{j\mu\nu} + \bar{G}^a_i\bar{B}^a_j U_{i\mu\nu}V_{j\mu\nu} - \frac{1}{2}B^a_i B^a_j\bar{V}_{i\mu\nu}\bar{V}_{j\mu\nu}\\
&+ \frac{1}{2}G^a_i G^a_j\bar{U}_{i\mu\nu}\bar{U}_{j\mu\nu} - \frac{1}{2}\bar{B}^a_i\bar{B}^a_j V_{i\mu\nu}V_{j\mu\nu} + \frac{1}{2}\bar{G}^a_i\bar{G}^a_j U_{i\mu\nu}U_{j\mu\nu}\Big)\Big],
\end{aligned} \tag{5.25}$$

where $\lambda_1$, $\lambda_3$, are free parameters, namely, by taking as starting point the action

$$\tilde{\Sigma} = S_{YM} + S_{gf} + S_{BG} + S_{\rm aux} + S_\lambda + S_{\rm ext}. \tag{5.26}$$

The previous algebraic analysis can be repeated for the action $\tilde{\Sigma}$. For the most general allowed counterterm we find now

$$\begin{aligned}
\tilde{\Sigma}^c &= a_0 S_{YM} + a_1 \int d^4x A^a_\mu \frac{\delta S_{YM}}{\delta A^a_\mu} + \int d^4x\Big((a_1+a_2)(\Omega^a_\mu + \partial_\mu \bar{c}^a)\partial_\mu c^a + a_2 g f^{abc}(\Omega^a_\mu + \partial_\mu \bar{c}^a)A^b_\mu c^c - a_2 \frac{g}{2} f^{abc} L^a c^b c^c\Big) \\
&\quad + \int d^4x\{(2a_3+a_4)\bar{B}^a_i \partial^2 B^a_i - (2a_3+a_4)\bar{G}^a_i \partial^2 G^a_i - (a_1+2a_3+a_4) g f^{abc}\bar{B}^a_i(\partial_\mu A^b_\mu + 2A^b_\mu \partial_\mu)B^c_i \\
&\quad + (2a_1+2a_3+a_4)g^2 f^{abd}f^{bce}\bar{B}^a_i A^d_\mu A^e_\mu B^c_i + (a_1+2a_3+a_4) g f^{abc}\bar{G}^a_i(\partial_\mu A^b_\mu + 2A^b_\mu\partial_\mu)G^c_i \\
&\quad - (2a_1+2a_3+a_4)g^2 f^{abd}f^{bce}\bar{G}^a_i A^d_\mu A^e_\mu G^c_i - a_2 g f^{abc} c^a(\bar{Y}^b_i B^c_i + Y^b_i \bar{B}^c_i - \bar{X}^b_i G^c_i - X^b_i \bar{G}^c_i) \\
&\quad + [(a_1+a_3+a_5)2\partial_\mu A^a_\nu + (2a_1+a_3+a_5) g f^{abc} A^b_\mu A^c_\nu](\bar{U}_{i\mu\nu}G^a_i + V_{i\mu\nu}\bar{B}^a_i + U_{i\mu\nu}\bar{G}^a_i - \bar{V}_{i\mu\nu}B^a_i) \\
&\quad + (4a_3+\tilde{a}_6)\frac{\lambda^{abcd}}{16}(\bar{B}^a_i B^b_i - \bar{G}^a_i G^b_i)(\bar{B}^c_j B^d_j - \bar{G}^c_j G^d_j) + (2a_3+\tilde{a}_7)\lambda_1(\bar{B}^a_i B^a_i - \bar{G}^a_i G^a_i)(\bar{V}_{i\mu\nu}V_{i\mu\nu} - \bar{U}_{i\mu\nu}U_{i\mu\nu}) \\
&\quad + (2a_3+\tilde{a}_8)\lambda_3\Big(\bar{B}^a_i G^a_j V_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{G}^a_i G^a_j U_{i\mu\nu}\bar{U}_{j\mu\nu} + \bar{B}^a_i B^a_j V_{i\mu\nu}\bar{V}_{j\mu\nu} - \bar{G}^a_i B^a_j U_{i\mu\nu}\bar{V}_{j\mu\nu} - G^a_i B^a_j \bar{U}_{i\mu\nu}\bar{V}_{j\mu\nu} \\
&\quad + \bar{G}^a_i \bar{B}^a_j U_{i\mu\nu}V_{j\mu\nu} - \frac{1}{2}B^a_i B^a_j \bar{V}_{i\mu\nu}\bar{V}_{j\mu\nu} + \frac{1}{2}G^a_i G^a_j \bar{U}_{i\mu\nu}\bar{U}_{j\mu\nu} - \frac{1}{2}\bar{B}^a_i \bar{B}^a_j V_{i\mu\nu}V_{j\mu\nu} + \frac{1}{2}\bar{G}^a_i \bar{G}^a_j U_{i\mu\nu}U_{j\mu\nu}\Big) \\
&\quad + a_9\zeta(\bar{V}_{i\mu\nu}V_{i\mu\nu} - \bar{U}_{i\mu\nu}U_{i\mu\nu})^2 + a_{10}\chi_1(\bar{V}_{i\mu\nu}\partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu}\partial^2 U_{i\mu\nu}) + a_{11}\chi_1(\bar{V}_{i\mu\nu}\partial_\mu\partial_\alpha V_{i\nu\alpha} - \bar{U}_{i\mu\nu}\partial_\mu\partial_\alpha U_{i\nu\alpha})\},
\end{aligned} \tag{5.27}$$

As a useful check, let us show that $\tilde{\Sigma}^c$ can be reabsorbed by means of a multiplicative renormalization of the parameters, fields, and sources of $\tilde{\Sigma}$. Setting

$$\phi_0 = Z^{1/2}_\phi \phi, \qquad J_0 = Z_J J, \qquad \xi_0 = Z_\xi \xi, \tag{5.28}$$

where

$$\begin{aligned}
\phi &= \{A, b, c, \bar{c}, B, \bar{B}, G, \bar{G}\}, \\
J &= \{\Omega, L, \bar{U}, U, \bar{V}, V, X, \bar{X}, Y, \bar{Y}\}, \\
\xi &= \{g, \alpha, \chi_1, \chi_2, \zeta, \lambda_1, \lambda^{abcd}, \lambda_3\},
\end{aligned} \tag{5.29}$$

it follows

$$\tilde{\Sigma}(\phi_0, J_0, \xi_0) = \tilde{\Sigma}(\phi, J, \xi) + \eta\tilde{\Sigma}^c(\phi, J, \xi) + O(\eta^2). \tag{5.30}$$

In particular, the renormalization constants are found to be

$$Z^{1/2}_A = 1 + \eta\Big(\frac{a_0}{2} + a_1\Big), \tag{5.31}$$

$$Z^{1/2}_c = 1 - \eta\Big(\frac{a_1}{2} + \frac{a_2}{2}\Big), \tag{5.32}$$

$$Z^{1/2}_{\bar{c}} = Z^{1/2}_c, \tag{5.33}$$

$$Z^{1/2}_b = Z^{-1/2}_A, \tag{5.34}$$

$$Z_\Omega = Z^{1/2}_c, \tag{5.35}$$

$$Z_L = Z_A, \tag{5.36}$$

$$Z^{1/2}_B = Z^{1/2}_{\bar{B}} = Z^{1/2}_G = Z^{1/2}_{\bar{G}} = 1 + \eta\Big(a_3 + \frac{a_4}{2}\Big), \tag{5.37}$$

$$Z_V = Z_{\bar{V}} = Z_U = Z_{\bar{U}} = 1 - \eta\Big(\frac{a_0}{2} + \frac{a_4}{2} - a_5\Big), \tag{5.38}$$

$$Z_X = Z_{\bar{X}} = Z_Y = Z_{\bar{Y}} = Z^{1/2}_c Z^{1/2}_A Z^{-1/2}_B, \tag{5.39}$$

$$Z_g = 1 - \epsilon\frac{a_0}{2}, \tag{5.40}$$

$$Z_\alpha = Z_A, \tag{5.41}$$

$$Z_{\lambda_1} = 1 + \epsilon(a_0 - 2a_5 + \tilde{a}_7), \tag{5.42}$$

$$Z_{\lambda^{abcd}} = 1 - \epsilon(2a_4 - \tilde{a}_6), \tag{5.43}$$

$$Z_{\lambda_3} = 1 + \epsilon(a_0 - 2a_5 + \tilde{a}_8), \tag{5.44}$$

$$Z_{\chi_1} = 1 + \epsilon(a_0 + a_4 - 2a_5 + a_{10}), \tag{5.45}$$

$$Z_{\chi_2} = 1 + \epsilon(a_0 + a_4 - 2a_5 + a_{11}), \tag{5.46}$$

$$Z_\zeta = 1 + \epsilon(2a_0 + 2a_4 - 4a_5 - a_9). \tag{5.47}$$

### B. Summary

In summary, we have been able to identify a local and polynomial action, given in expression (5.26), which displays multiplicative renormalizability. This has been achieved by adding to the action $\Sigma$ the term $S_\lambda$, Eq. (5.25), which is compatible with the complete set of Ward identities. When the sources $(V_{i\mu\nu}, \bar{V}_{i\mu\nu}, U_{i\mu\nu}, \bar{U}_{i\mu\nu})$ attain their physical value, Eq. (4.32), $S_\lambda$ becomes

$$S_\lambda|_{\rm phys} = \int d^4x\bigg[-\frac{3}{8}m^2\lambda_1(\bar{B}^a_{\mu\nu}B^a_{\mu\nu} - \bar{G}^a_{\mu\nu}G^a_{\mu\nu}) + m^2\frac{\lambda_3}{32} \times (\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})^2 + \frac{\lambda^{abcd}}{16}(\bar{B}^a_{\mu\nu}B^b_{\mu\nu} - \bar{G}^a_{\mu\nu}G^b_{\mu\nu}) \times (\bar{B}^c_{\rho\sigma}B^d_{\rho\sigma} - \bar{G}^c_{\rho\sigma}G^d_{\rho\sigma})\bigg]. \tag{5.48}$$

This expression reminds us of a kind of Higgs term. There are, however, several differences. These are due to the antisymmetric character of the fields $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu})$ with respect to the Lorentz indices. Moreover, we remark that, while $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu})$ are bosonic, the fields $(G^a_{\mu\nu}, \bar{G}^a_{\mu\nu})$ are anticommuting. With the exception of the term containing the parameter $\lambda_3$, expression (5.48) displays thus a supersymmetric structure, a feature supported by the fact that, according to (4.16), the fields $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu})$ transform as *BRST* doublets. Therefore, a certain number of cancellations among the contributions arising from these fields might be expected in the evaluation of the Green's functions of the model. The possible use of this supersymmetric structure will be explored in the future, as well as its possible consequences for the Green's functions of the model.

To conclude, let us give explicitly the starting action when the sources $(V_{i\mu\nu}, \bar{V}_{i\mu\nu}, U_{i\mu\nu}, \bar{U}_{i\mu\nu})$ attain their physical value, Eq. (4.32), while the additional external sources $(\Omega^a_\mu, L^a, \bar{Y}^a_i, Y^a_i, \bar{X}^a_i, X^a_i)$ are put equal to zero.

$$\begin{aligned} S &= S_{YM} + S_{BG} + S_m + S_\lambda|_{\rm phys} + S_{gf} \\ &= \int d^4x\bigg[\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{im}{4}(B-\bar{B})^a_{\mu\nu}F^a_{\mu\nu} \\ &\quad + \frac{1}{4}(\bar{B}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} - \bar{G}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu}) \\ &\quad - \frac{3}{8}m^2\lambda_1(\bar{B}^a_{\mu\nu}B^a_{\mu\nu} - \bar{G}^a_{\mu\nu}G^a_{\mu\nu}) + m^2\frac{\lambda_3}{32}(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})^2 \\ &\quad + \frac{\lambda^{abcd}}{16}(\bar{B}^a_{\mu\nu}B^b_{\mu\nu} - \bar{G}^a_{\mu\nu}G^b_{\mu\nu})(\bar{B}^c_{\rho\sigma}B^d_{\rho\sigma} - \bar{G}^c_{\rho\sigma}G^d_{\rho\sigma}) \\ &\quad + \frac{\alpha}{2}b^ab^a + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\bigg]. \end{aligned} \tag{5.49}$$

Let us finally notice that each of the terms in Eq. (5.48) is invariant w.r.t. the gauge transformations (4.8). More precisely, one has

$$\delta(S_{YM} + S_{BG} + S_m + S_\lambda|_{\rm phys}) = 0. \tag{5.50}$$

## VI. ONE-LOOP RENORMALIZATION

We now turn to the details of the explicit one-loop renormalization of the Lagrangian (5.49) in the presence of the nonlocal operator. It is first worth noting some of the key features of (5.49) in relation to the extraction of the one-loop renormalization constants prior to discussing their calculation. First, considering the case when $m$ is zero then one has a gauge theory fixed in an arbitrary linear covariant gauge where in addition to the usual gluon and Faddeev-Popov ghost fields there are two additional auxiliary fields, $B^a_{\mu\nu}$ and $G^a_{\mu\nu}$, where the latter is anticommuting. Since these fields originate in localizing the nonlocal operator, when that operator is absent at $m = 0$, these new fields ought to play a completely passive role in the (one-loop) renormalization. In other words the gluon, Faddeev-Popov ghost and quark renormalization constants ought to be equivalent to those obtained when $B^a_{\mu\nu}$ and $G^a_{\mu\nu}$ are formally absent. However, when they are present the algebraic renormalization formalism has demonstrated that they generate a new quartic interaction through (one-loop) renormalization effects[1] which is indicated by the term with the independent coupling $\lambda^{abcd}$ in (5.49). In other words if one computes the $\bar{B}^aB^b\bar{B}^cB^d$ four-point function at one loop with $\lambda^{abcd}$ initially zero, there will be a divergent contribution at $O(g^2)$ which will be removed by the counterterm generated by the term involving $\lambda^{abcd}$. This is akin to the situation in $\lambda\phi^4$ theory where the Lagrangian is multiplicatively renormalizable in four dimensions. However, the interaction can be replaced by a cubic vertex involving an auxiliary scalar field. The renormalization of this version of the Lagrangian still proceeds as usual except that the Lagrangian ceases to be multiplicatively renormalizable since a $\phi^4$ vertex will naturally be generated from one-loop box diagrams. The standard $\lambda\phi^4$ $\beta$-function and renormalization group functions can still be extracted with the auxiliary field version but one has to take account of the effects of the generation of the extra interaction. Indeed a similar situation arises in two-dimensional four-Fermi theories where a formalism has been developed [73] and used to perform three-loop calculations. The situation for our current Lagrangian is the same. The quartic interaction is generated via loop interactions and will be $O(g^2)$. Thus it does not need to be taken into account for the extraction of the one-loop anomalous dimensions we are interested in. For the case when $m$ is nonzero, there is a similar situation. The algebraic renormalization demonstrates that the now localized mass operator $(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})F^a_{\mu\nu}$, which is dimension three, mixes into two gauge invariant dimension two operators being those associated with the couplings $\lambda_1$ and $\lambda_3$. In other words computing the renormalization of the operator with a massive gluon propagator will inevitably lead to the generation of these two additional operators. As such this is nothing new in that it follows the pattern already known

[1]It might be useful to remark here that, at one-loop order, the invariant rank four tensor which emerges from explicit calculations turns out to be proportional to $g^2(f^{eap}f^{ebq}f^{mcp}f^{mdq} + f^{eap}f^{ebq}f^{mdp}f^{mcq})$, which fulfills in fact the conditions (5.22).

for the renormalization of local composite operators (see, for example, [74]). Indeed it is reassuring that this property emerges in an elegant way from the algebraic renormalization formalism for a *localized* nonlocal operator. However, these two additional operators do not form the complete basis of the possible dimension two operators that higher dimensional operators can mix into when one uses the massive theory. Since each combination of pairs of the set $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\}$ are individually gauge invariant operators, to correctly treat the renormalization one would have to construct the full mixing matrix for this set. Though only those combinations with zero ghost number would be of importance. As we are primarily focused on extracting the anomalous dimension of the nonlocal operator itself, it will be apparent that this mixing matrix is not immediately required and we will defer its computation to a later article.

Having outlined the status of (5.49) it is now evident how one goes about extracting the renormalization constants which will lead to the anomalous dimension of $F^a_{\mu\nu}\frac{1}{D^2}F^a_{\mu\nu}$. Since we have localized this operator to $(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})F^a_{\mu\nu}$ then the anomalous dimension of $F^a_{\mu\nu}\frac{1}{D^2}F^a_{\mu\nu}$ is equivalent[2] to that of the gauge invariant operator $(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})F^a_{\mu\nu}$. Therefore, we can extract the anomalous dimension by inserting $(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})F^a_{\mu\nu}$ into a $B^a_{\mu\nu}A^b_\sigma$ two-point function and compute it using *massless* propagators. This is similar to how one determines the quark mass anomalous dimension by inserting the mass operator $\bar{\psi}\psi$ into a quark two-point function, [75,76]. For $(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu})F^a_{\mu\nu}$ we will need the $B^a_{\mu\nu}$ anomalous dimension. However, we have carried out the full renormalization of all the fields of (5.49) at one loop by making use of symbolic manipulation programmes. The Feynman diagrams for the relevant Green's functions are generated with the QGRAF package, [77], converted into FORM, [78], input notation before extracting the divergences with the MINCER package, [79]. This uses dimensional regularization in $d = 4 - 2\epsilon$ dimensions and we will remove the infinities with the (mass independent) $\overline{\text{MS}}$ renormalization scheme. If we define

$$\gamma_\phi(a) = \mu\frac{\partial}{\partial\mu}\ln Z_\phi, \tag{6.1}$$

for $\phi \in \{A^a_\mu, c^a, \psi, B^a_{\mu\nu}, G^a_{\mu\nu}\}$ where $a = g^2/(16\pi^2)$, then the renormalization constants give the explicit results

$$\begin{aligned}
\gamma_A(a) &= [(3\alpha - 13)C_A + 8T_F N_f]\frac{a}{6} + O(a^2),\\
\gamma_c(a) &= (\alpha - 3)C_A\frac{a}{4} + O(a^2),\\
\gamma_\psi(a) &= \alpha C_F a + O(a^2),\\
\gamma_B(a) &= \gamma_G(a) = (\alpha - 3)C_A a + O(a^2),
\end{aligned} \tag{6.2}$$

where $N_f$ is the number of quark flavours,[3] $T^aT^a = C_F I$, $f^{acd}f^{bcd} = C_A\delta^{ab}$, and $\mathrm{tr}(T^aT^b) = T_F\delta^{ab}$. For completeness we note that the massless momentum space propagators of the fields are

$$\begin{aligned}
\langle A^a_\mu(p)A^b_\nu(-p)\rangle &= -\frac{\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right],\\
\langle c^a(p)\bar{c}^b(-p)\rangle &= \frac{\delta^{ab}}{p^2}, \qquad \langle\psi(p)\bar{\psi}(-p)\rangle = \frac{\not{p}}{p^2},\\
\langle B^a_{\mu\nu}(p)\bar{B}^b_{\sigma\rho}(-p)\rangle &= -\frac{\delta^{ab}}{2p^2}[\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}],\\
\langle G^a_{\mu\nu}(p)\bar{G}^b_{\sigma\rho}(-p)\rangle &= -\frac{\delta^{ab}}{2p^2}[\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}],
\end{aligned} \tag{6.3}$$

where $p$ is the momentum. It is worth noting that the expressions for the gluon, Faddeev-Popov ghost, and quark are equivalent to those obtained in the absence of $B^a_{\mu\nu}$ and $G^a_{\mu\nu}$ as expected. Indeed from examining the contributions from the diagrams involving these fields it is evident that the anticommuting property of $G^a_{\mu\nu}$ introduces the necessary minus sign to exactly cancel the contribution from the graph involving a $B^a_{\mu\nu}$ loop. To verify that the $B^a_{\mu\nu}$ and $G^a_{\mu\nu}$ anomalous dimensions are correct, aside from correctly satisfying the equality demanded from the algebraic renormalization, Eq. (5.37), we have also renormalized both the gluon-$B$ and gluon-$G$ vertices at one loop and verified that the correct one-loop $\alpha$ independent coupling constant renormalization emerges as

$$\beta(a) = -\left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right]a^2 + O(a^3). \tag{6.4}$$

Hence the renormalization of the operator $B^a_{\mu\nu}F^a_{\mu\nu}$ proceeds by inserting $B^a_{\mu\nu}F^a_{\mu\nu}$ into a gluon-$B$ two-point function and extracting the divergence from the five one-loop diagrams. Although we are regarding $B^a_{\mu\nu}F^a_{\mu\nu}$ as multiplicatively renormalizable, since it is of dimension three it could in principle mix into the dimension three quark mass operator, $\bar{\psi}\psi$. However, at one loop there are no mixed diagrams of inserting $\bar{\psi}\psi$ into a gluon-$B$ Green's function or of inserting $B^a_{\mu\nu}F^a_{\mu\nu}$ into a quark two-point function. If we define

$$\mathcal{O}_{\mathrm{o}} = Z_{\mathcal{O}}\mathcal{O}, \tag{6.5}$$

where the subscript $_{\mathrm{o}}$ denotes the bare object, with

$$\mathcal{O} = B^a_{\mu\nu}F^a_{\mu\nu}, \tag{6.6}$$

[2]Up to an overall scaling factor.

[3]Although we did not consider matter fields in the previous analysis, it turns out that the multiplicative renormalizability of the action $\tilde{\Sigma}$, Eq. (5.26), can be extended to the case in which spinor fields are present.

then we find

$$Z_{\mathcal{O}} = 1 + \left(\frac{2}{3}T_F N_f - \frac{11}{6}C_A\right)\frac{a}{\epsilon} + O(a^2). \qquad (6.7)$$

Hence, with

$$\gamma_{\mathcal{O}}(a) = \mu \frac{\partial}{\partial \mu} \ln Z_{\mathcal{O}}, \qquad (6.8)$$

we deduce

$$\gamma_{\mathcal{O}}(a) = -\left[\frac{11}{6}C_A - \frac{2}{3}T_F N_f\right]a + O(a^2). \qquad (6.9)$$

As the original operator was gauge invariant it is reassuring to note that $\gamma_{\mathcal{O}}(a)$ is independent of $\alpha$. It is worth underlining here that the anomalous dimension $\gamma_{\mathcal{O}}(a)$ is equivalent to the one-loop $\beta$-function, where the overall factor of 2 is accounted for by noting that this is equivalent to the anomalous dimension of $m$ as opposed to that of $m^2$. This is interesting for various reasons. First in the one-loop renormalization of two-leg higher dimension operators in Yang-Mills theories the operators $F^a_{\mu\nu}F^a_{\mu\nu}$, $D_\mu F^a_{\nu\sigma}D_\mu F^a_{\nu\sigma}$, and $D_\mu D_\nu F^a_{\sigma\rho}D_\mu D_\nu F^a_{\sigma\rho}$ each have the same one-loop anomalous dimensions which is also the $\beta$-function,[4] [81,82]. What is intriguing in the present situation is that the *nonlocal* operator $F^a_{\mu\nu}\frac{1}{D^2}F^a_{\mu\nu}$, which has a similar Lorentz contraction as the higher dimension operators noted above, has an anomalous dimension which is the same at one loop. There would appear to be no *a priori* reason either from the algebraic renormalization or other methods to expect this. Obviously, having the two-loop correction to (6.9) would enhance our understanding of both the renormalization and significance of this nonlocal operator. With the exception of $F^a_{\mu\nu}F^a_{\mu\nu}$, the renormalization group behavior of the two-leg higher dimension operators is also unknown.[5] It would be interesting to pursue this study to find out if a gauge invariant *and* renormalization group invariant mass dimension two condensate could be found using $F^a_{\mu\nu}\frac{1}{D^2}F^a_{\mu\nu}$, provided the operator condenses. Since evidence for the existence of a nonzero dimension two condensate arises in the fitting of data for gauge variant objects [18,20,21,24,25], as a first step, it would seem natural in the light of (6.9) to find out whether one could extract an estimate for the one-loop renormalization group invariant condensate $\langle \alpha_s F^a_{\mu\nu}\frac{1}{D^2}F^a_{\mu\nu}\rangle$ by fitting for $1/Q^2$ power corrections in measurements of correlations of gauge *invariant* operators. We refer to [84–86] for a review of the role of such $1/Q^2$ corrections which go beyond the standard Shifman-Vainshtein-Zakharov (SVZ)-expansion [84,87,88].

## VII. CONCLUSIONS

In this work the properties of the nonlocal gauge invariant operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ of mass dimension two have been investigated. We started by looking at the Abelian case, where several nonlocal gauge invariant operators have been considered. Moreover, in this case, all operators turn out to reduce to the same expression when the classical equations of motion are employed. All Abelian operators generalize to the non-Abelian case. However, their classical equivalence does not hold anymore. In particular, the operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ exhibits differences with respect to the operator $A^2_{\mathrm{min}}$.

Albeit nonlocal, the operator $\mathrm{Tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ can be cast in local form by the introduction of a suitable set of additional fields, in contrast with the operator $A^2_{\mathrm{min}}$. A local and polynomial action has been identified, Eq. (5.26), and proven to be multiplicatively renormalizable to all orders in the class of linear covariant gauges by means of the algebraic renormalization. We point out that this action possesses a finite and relatively small number of parameters, a feature useful for higher order computations. We have calculated the one-loop renormalization group functions of the model. We have recovered the anomalous dimensions of the elementary fields, if already known. In the case of the nonlocal operator, we have found that the renormalization group behavior is dictated by the $\beta$-function at one-loop.

The possibility of having at our disposal a local and renormalizable action might provide us with a consistent framework for a future investigation of the possible existence of the condensate $\langle F\frac{1}{D^2}F\rangle$.

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## APPENDIX A: PROPERTIES OF THE FUNCTIONAL $f_A[u]$

In this appendix we recall some useful properties of the functional $f_A[u]$

[4]We have checked that the dimension ten operator $D_\mu D_\nu D_\sigma F^a_{\rho\theta}D_\mu D_\nu D_\sigma F^a_{\rho\theta}$ has the same one-loop anomalous dimension too [80].

[5]For details concerning the renormalization (group) properties of $F^a_{\mu\nu}F^a_{\mu\nu}$ with or without massless/massive quarks, we refer to [83].

$$f_A[u] \equiv \mathrm{Tr} \int d^4x A^u_\mu A^u_\mu = \mathrm{Tr} \int d^4x \left( u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right) \left( u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right). \tag{A1}$$

For a given gauge field configuration $A_\mu$, $f_A[u]$ is a functional defined on the gauge orbit of $A_\mu$. Let $\mathcal{A}$ be the space of connections $A^a_\mu$ with finite Hilbert norm $||A||$, i.e.

$$||A||^2 = \mathrm{Tr} \int d^4x A_\mu A_\mu = \frac{1}{2} \int d^4x A^a_\mu A^a_\mu < +\infty, \tag{A2}$$

and let $\mathcal{U}$ be the space of local gauge transformations $u$ such that the Hilbert norm $||u^\dagger \partial u||$ is finite too, namely

$$||u^\dagger \partial u||^2 = \mathrm{Tr} \int d^4x (u^\dagger \partial_\mu u)(u^\dagger \partial_\mu u) < +\infty. \tag{A3}$$

The following proposition holds [59–63]

(a) Proposition
The functional $f_A[u]$ achieves its absolute minimum on the gauge orbit of $A_\mu$.

This proposition means that there exists a $h \in \mathcal{U}$ such that

$$\delta f_A[h] = 0, \tag{A4}$$

$$\delta^2 f_A[h] \geq 0, \tag{A5}$$

$$f_A[h] \leq f_A[u], \qquad \forall u \in \mathcal{U}. \tag{A6}$$

The operator $A^2_{\min}$ is thus given by

$$A^2_{\min} = \min_{\{u\}} \mathrm{Tr} \int d^4x A^u_\mu A^u_\mu = f_A[h]. \tag{A7}$$

Let us give a look at the two conditions (A4) and (A5). To evaluate $\delta f_A[h]$ and $\delta^2 f_A[h]$ we set[6]

$$v = h e^{ig\omega} = h e^{ig\omega^a T^a}, \tag{A8}$$

$$[T^a, T^b] = i f^{abc}, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \tag{A9}$$

where $\omega$ is an infinitesimal Hermitian matrix and we compute the linear and quadratic terms of the expansion of the functional $f_A[v]$ in power series of $\omega$. Let us first obtain an expression for $A^v_\mu$

$$\begin{aligned} A^v_\mu &= v^\dagger A_\mu v + \frac{i}{g} v^\dagger \partial_\mu v \\ &= e^{-ig\omega} h^\dagger A_\mu h e^{ig\omega} + \frac{i}{g} e^{-ig\omega} (h^\dagger \partial_\mu h) e^{ig\omega} \\ &\quad + \frac{i}{g} e^{-ig\omega} \partial_\mu e^{ig\omega} \\ &= e^{-ig\omega} A^h_\mu e^{ig\omega} + \frac{i}{g} e^{-ig\omega} \partial_\mu e^{ig\omega}. \end{aligned} \tag{A10}$$

Expanding up to the order $\omega^2$, we get

$$\begin{aligned} A^v_\mu &= \left(1 - ig\omega - g^2 \frac{\omega^2}{2}\right) A^h_\mu \left(1 + ig\omega - g^2 \frac{\omega^2}{2}\right) + \frac{i}{g} \left(1 - ig\omega - g^2 \frac{\omega^2}{2}\right) \partial_\mu \left(1 + ig\omega - g^2 \frac{\omega^2}{2}\right) \\ &= \left(1 - ig\omega - g^2 \frac{\omega^2}{2}\right) \left(A^h_\mu + igA^h_\mu \omega - g^2 A^h_\mu \frac{\omega^2}{2}\right) + \frac{i}{g} \left(1 - ig\omega - g^2 \frac{\omega^2}{2}\right) \left(ig\partial_\mu \omega - \frac{g^2}{2} (\partial_\mu \omega)\omega - \frac{g^2}{2} \omega (\partial_\mu \omega)\right) \\ &= A^h_\mu + igA^h_\mu \omega - \frac{g^2}{2} A^h_\mu \omega^2 - ig\omega A^h_\mu + g^2 \omega A^h_\mu \omega - \frac{g^2}{2} \omega^2 A^h_\mu + \frac{i}{g} \Big( ig\partial_\mu \omega - \frac{g^2}{2} (\partial_\mu \omega)\omega - \frac{g^2}{2} \omega \partial_\mu \omega \\ &\quad + g^2 \omega \partial_\mu \omega \Big) + O(\omega^3), \end{aligned} \tag{A11}$$

from which it follows

$$A^v_\mu = A^h_\mu + ig[A^h_\mu, \omega] + \frac{g^2}{2} [[\omega, A^h_\mu], \omega] - \partial_\mu \omega + i\frac{g}{2} [\omega, \partial_\mu \omega] + O(\omega^3). \tag{A12}$$

We now evaluate

[6]The case of the gauge group $SU(N)$ is considered here.

$$
\begin{aligned}
f_A[v] &= \mathrm{Tr}\int d^4x A^u_\mu A^u_\mu \\
&= \mathrm{Tr}\int d^4x\bigg[\bigg(A^h_\mu + ig[A^h_\mu,\omega] + \frac{g^2}{2}[[\omega, A^h_\mu],\omega] - \partial_\mu\omega + i\frac{g}{2}[\omega,\partial_\mu\omega] + O(\omega^3)\bigg)\times\bigg(A^h_\mu + ig[A^h_\mu,\omega] \\
&\quad + \frac{g^2}{2}[[\omega, A^h_\mu],\omega] - \partial_\mu\omega + i\frac{g}{2}[\omega,\partial_\mu\omega] + O(\omega^3)\bigg)\bigg] \\
&= \mathrm{Tr}\int d^4x\bigg\{A^h_\mu A^h_\mu + igA^h_\mu[A^h_\mu,\omega] + g^2A^h_\mu\omega A^h_\mu\omega - \frac{g^2}{2}A^h_\mu A^h_\mu\omega^2 - \frac{g^2}{2}A^h_\mu\omega^2A^h_\mu - A^h_\mu\partial_\mu\omega + i\frac{g}{2}A^h_\mu[\omega,\partial_\mu\omega] \\
&\quad + ig[A^h_\mu,\omega]A^h_\mu - g^2[A^h_\mu,\omega][A^h_\mu,\omega] - ig[A^h_\mu,\omega]\partial_\mu\omega + g^2\omega A^h_\mu\omega A^h_\mu - \frac{g^2}{2}A^h_\mu\omega^2A^h_\mu - \frac{g^2}{2}\omega^2A^h_\mu A^h_\mu - \partial_\mu\omega A^h_\mu \\
&\quad - ig\partial_\mu\omega[A^h_\mu,\omega] + \partial_\mu\omega\partial_\mu\omega + i\frac{g}{2}[\omega,\partial_\mu\omega]A^h_\mu\bigg\} + O(\omega^3) \\
&= f_A[h] - \mathrm{Tr}\int d^4x\{A^h_\mu,\partial_\mu\omega\} + \mathrm{Tr}\int d^4x\bigg(g^2A^h_\mu\omega A^h_\mu\omega - \frac{g^2}{2}A^h_\mu A^h_\mu\omega^2 - \frac{g^2}{2}A^h_\mu\omega^2A^h_\mu - g^2[A^h_\mu,\omega][A^h_\mu,\omega] \\
&\quad + g^2\omega A^h_\mu\omega A^h_\mu - \frac{g^2}{2}A^h_\mu\omega^2A^h_\mu - \frac{g^2}{2}\omega^2A^h_\mu A^h_\mu\bigg) + \mathrm{Tr}\int d^4x\bigg(\partial_\mu\omega\partial_\mu\omega + i\frac{g}{2}[\omega,\partial_\mu\omega]A^h_\mu - ig\partial_\mu\omega[A^h_\mu,\omega] \\
&\quad - ig[A^h_\mu,\omega]\partial_\mu\omega + i\frac{g}{2}A^h_\mu[\omega,\partial_\mu\omega]\bigg) + O(\omega^3) \\
&= f_A[h] + 2\int d^4x tr(\omega\partial_\mu A^h_\mu) + \int d^4x tr\{2g^2\omega A^h_\mu\omega A^h_\mu - 2g^2A^h_\mu A^h_\mu\omega^2 - g^2(A^h_\mu\omega - \omega A^h_\mu)(A^h_\mu\omega - \omega A^h_\mu)\} \\
&\quad + \int d^4x tr\bigg(\partial_\mu\omega\partial_\mu\omega + i\frac{g}{2}\omega\partial_\mu\omega A^h_\mu - i\frac{g}{2}\partial_\mu\omega\omega A^h_\mu - ig\partial_\mu\omega A^h_\mu\omega + ig\partial_\mu\omega\omega A^h_\mu - igA^h_\mu\omega\partial_\mu\omega \\
&\quad + ig\omega A^h_\mu\partial_\mu\omega + i\frac{g}{2}A^h_\mu\omega\partial_\mu\omega - i\frac{g}{2}A^h_\mu\partial_\mu\omega\omega\bigg) + O(\omega^3) \\
&= f_A[h] + 2\mathrm{Tr}\int d^4x(\omega\partial_\mu A^h_\mu) + \mathrm{Tr}\int d^4x(\partial_\mu\omega\partial_\mu\omega + ig\omega\partial_\mu\omega A^h_\mu - ig\partial_\mu\omega\omega A^h_\mu - 2ig\partial_\mu\omega A^h_\mu\omega \\
&\quad + 2ig\partial_\mu\omega\omega A^h_\mu) + O(\omega^3).
\end{aligned} \tag{A13}
$$

Thus

$$
\begin{aligned}
f_A[v] &= f_A[h] + 2\mathrm{Tr}\int d^4x(\omega\partial_\mu A^h_\mu) + \mathrm{Tr}\int d^4x(\partial_\mu\omega\partial_\mu\omega \\
&\quad + ig\omega\partial_\mu\omega A^h_\mu - ig\partial_\mu\omega\omega A^h_\mu - ig(\partial_\mu\omega)A^h_\mu\omega \\
&\quad + ig(\partial_\mu\omega)\omega A^h_\mu) + O(\omega^3) \\
&= f_A[h] + 2\mathrm{Tr}\int d^4x(\omega\partial_\mu A^h_\mu) \\
&\quad + \mathrm{Tr}\int d^4x\{\partial_\mu\omega(\partial_\mu\omega - ig[A^h_\mu,\omega])\} \\
&\quad + O(\omega^3).
\end{aligned} \tag{A14}
$$

Finally

$$
\begin{aligned}
f_A[v] &= f_A[h] + 2\mathrm{Tr}\int d^4x(\omega\partial_\mu A^h_\mu) \\
&\quad - \mathrm{Tr}\int d^4x\omega\partial_\mu D_\mu(A^h)\omega + O(\omega^3),
\end{aligned} \tag{A15}
$$

so that

$$
\begin{aligned}
&\delta f_A[h] = 0 \Rightarrow \partial_\mu A^h_\mu = 0, \\
&\delta^2 f_A[h] > 0 \Rightarrow -\partial_\mu D_\mu(A^h) > 0.
\end{aligned} \tag{A16}
$$

We see therefore that the set of field configurations fulfilling conditions (A16), i.e. defining relative minima of the functional $f_A[u]$, belong to the so-called Gribov region $\Omega$, which is defined as

$$\Omega = \{A_\mu | \partial_\mu A_\mu = 0 \text{ and } -\partial_\mu D_\mu(A) > 0\}. \tag{A17}$$

Let us proceed now by showing that the transversality condition, $\partial_\mu A^h_\mu = 0$, can be solved for $h = h(A)$ as a power series in $A_\mu$. We start from

$$A^h_\mu = h^\dagger A_\mu h + \frac{i}{g}h^\dagger\partial_\mu h, \tag{A18}$$

with

$$h = e^{ig\phi} = e^{ig\phi^a T^a}. \tag{A19}$$

Let us expand $h$ in powers of $\phi$

$$h = 1 + ig\phi - \frac{g^2}{2}\phi^2 + O(\phi^3). \tag{A20}$$

From Eq. (A18) we have

$$A^h_\mu = A_\mu + ig[A_\mu, \phi] + g^2\phi A_\mu\phi - \frac{g^2}{2}A_\mu\phi^2 - \frac{g^2}{2}\phi^2 A_\mu - \partial_\mu\phi + i\frac{g}{2}[\phi, \partial_\mu] + O(\phi^3). \tag{A21}$$

Thus, condition $\partial_\mu A^h_\mu = 0$, gives

$$\begin{aligned}\partial^2\phi = {}& \partial_\mu A + ig[\partial_\mu A_\mu, \phi] + ig[A_\mu, \partial_\mu\phi] \\ &+ g^2\partial_\mu\phi A_\mu\phi + g^2\phi\partial_\mu A_\mu\phi + g^2\phi A_\mu\partial_\mu\phi \\ &- \frac{g^2}{2}\partial_\mu A_\mu\phi^2 - \frac{g^2}{2}A_\mu\partial_\mu\phi\phi - \frac{g^2}{2}A_\mu\phi\partial_\mu\phi \\ &- \frac{g^2}{2}\partial_\mu\phi\phi A_\mu - \frac{g^2}{2}\phi\partial_\mu\phi A_\mu - \frac{g^2}{2}\phi^2\partial_\mu A_\mu \\ &+ i\frac{g}{2}[\phi, \partial^2\phi] + O(\phi^3).\end{aligned} \tag{A22}$$

This equation can be solved iteratively for $\phi$ as a power series in $A_\mu$, namely

$$\begin{aligned}\phi = {}& \frac{1}{\partial^2}\partial_\mu A_\mu + i\frac{g}{\partial^2}\left[\partial A, \frac{\partial A}{\partial^2}\right] + i\frac{g}{\partial^2}\left[A_\mu, \partial_\mu\frac{\partial A}{\partial^2}\right] \\ &+ \frac{i}{2}\frac{g}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + O(A^3),\end{aligned} \tag{A23}$$

so that

$$\begin{aligned}A^h_\mu = {}& A_\mu - \frac{1}{\partial^2}\partial_\mu\partial A - ig\frac{\partial_\mu}{\partial^2}\left[A_\nu, \partial_\nu\frac{\partial A}{\partial^2}\right] \\ &- i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\left[\partial A, \frac{1}{\partial^2}\partial A\right] + ig\left[A_\mu, \frac{1}{\partial^2}\partial A\right] \\ &+ i\frac{g}{2}\left[\frac{1}{\partial^2}\partial A, \frac{\partial_\mu}{\partial^2}\partial A\right] + O(A^3).\end{aligned} \tag{A24}$$

Expression (A24) can be written in a more useful way, given in Eq. (3.2). In fact

$$\begin{aligned}A^h_\mu &= \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\left(A_\nu - ig\left[\frac{1}{\partial^2}\partial A, A_\nu\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\nu\frac{1}{\partial^2}\partial A\right]\right) + O(A^3) \\ &= A_\mu - ig\left[\frac{1}{\partial^2}\partial A, A_\mu\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu\frac{1}{\partial^2}\partial A\right] - \frac{\partial_\mu}{\partial^2}\partial A + ig\frac{\partial_\mu}{\partial^2}\partial_\nu\left[\frac{1}{\partial^2}\partial A, A_\nu\right] - i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\partial_\nu\left[\frac{\partial A}{\partial^2}, \frac{\partial_\nu}{\partial^2}\partial A\right] + O(A^3) \\ &= A_\mu - \frac{\partial_\mu}{\partial^2}\partial A + ig\left[A_\mu, \frac{1}{\partial^2}\partial A\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu\frac{1}{\partial^2}\partial A\right] + ig\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu}{\partial^2}\partial A, A_\nu\right] + i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + O(A^3)\end{aligned} \tag{A25}$$

which is precisely expression (A24). The transverse field given in Eq. (3.2) enjoys the property of being gauge invariant order by order in the coupling constant $g$. Let us work out the transformation properties of $\phi_\nu$ under a gauge transformation

$$\delta A_\mu = -\partial_\mu\omega + ig[A_\mu, \omega]. \tag{A26}$$

We have, up to the order $O(g^2)$,

$$\begin{aligned}\delta\phi_\nu &= -\partial_\nu\omega + ig\left[\frac{1}{\partial^2}\partial A, \partial_\nu\omega\right] - i\frac{g}{2}\left[\omega, \partial_\nu\frac{1}{\partial^2}\partial A\right] - i\frac{g}{2}\left[\frac{\partial A}{\partial^2}, \partial_\nu\omega\right] + O(g^2) \\ &= -\partial_\nu\omega + i\frac{g}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\nu\omega\right] + i\frac{g}{2}\left[\partial_\nu\frac{1}{\partial^2}\partial A, \omega\right] + O(g^2).\end{aligned} \tag{A27}$$

Therefore

$$\delta\phi_\nu = -\partial_\nu\left(\omega - i\frac{g}{2}\left[\frac{\partial A}{\partial^2}, \omega\right]\right) + O(g^2), \tag{A28}$$

from which the gauge invariance of $A^h_\mu$ is established.

Finally, let us work out the expression of $A^2_{\min}$ as a power series in $A_\mu$

$$
\begin{aligned}
A^2_{\min} &= \mathrm{Tr}\int d^4x A^h_\mu A^h_\mu \\
&= \mathrm{Tr}\int d^4x\left[\phi_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\phi_\nu\right] \\
&= \mathrm{Tr}\int d^4x\left[\left(A_\mu - ig\left[\frac{1}{\partial^2}\partial A, A_\mu\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu\frac{1}{\partial^2}\partial A\right]\right)\times\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\left(A_\nu - ig\left[\frac{1}{\partial^2}\partial A, A_\nu\right]\right.\right. \\
&\quad \left.\left. + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\nu\frac{1}{\partial^2}\partial A\right]\right)\right] \\
&= \mathrm{Tr}\int d^4x\left\{A_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu - 2ig\left(A_\nu - \partial_\nu\frac{\partial A}{\partial^2}\right)\left[\frac{\partial A}{\partial^2}, A_\nu\right] + ig\left(A_\nu - \partial_\nu\frac{\partial A}{\partial^2}\right)\left[\frac{\partial A}{\partial^2}, \partial_\nu\frac{\partial A}{\partial^2}\right]\right\} + O(A^4) \\
&= \mathrm{Tr}\int d^4x\left\{A_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu - 2igA_\nu\left[\frac{\partial A}{\partial^2}, A_\nu\right] + 2ig\frac{\partial_\nu\partial A}{\partial^2}\left[\frac{\partial A}{\partial^2}, A_\nu\right] + igA_\nu\left[\frac{\partial A}{\partial^2}, \partial_\nu\frac{\partial A}{\partial^2}\right]\right. \\
&\quad \left. - ig\frac{\partial_\nu\partial A}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial_\nu\frac{\partial A}{\partial^2}\right]\right\} + O(A^4) \\
&= \frac{1}{2}\int d^4x\left[A^a_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A^a_\nu - 2gf^{abc}\frac{\partial_\nu\partial A^a}{\partial^2}\frac{\partial A^b}{\partial^2}A^c_\nu - gf^{abc}A^a_\nu\frac{\partial A^b}{\partial^2}\frac{\partial_\nu\partial A^c}{\partial^2}\right] + O(A^4).
\end{aligned} \tag{A29}
$$

leading to the result quoted in Eq. (3.4).

We conclude this appendix by noting that, due to gauge invariance, $A^2_{\min}$ can be rewritten in a manifestly invariant way in terms of $F_{\mu\nu}$ and the covariant derivative $D_\mu$ [60], see Eq. (3.17).

## APPENDIX B: PROPERTIES OF THE STUECKELBERG TERM

In this appendix we derive some useful properties of the non-Abelian Stueckelberg term $\mathcal{O}_S$ [58], defined by the Eqs. (3.7) and (3.8). The expression (3.7) is left invariant by the gauge transformations given in Eq. (3.9). In fact

$$(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U) \to V^{-1}(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U)V. \tag{B1}$$

Thus

$$\mathrm{Tr}\,(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U)^2 \to \mathrm{Tr}(A_\mu - \frac{i}{g}U^{-1}\partial_\mu U)^2. \tag{B2}$$

Let us look now at the equations of motion of the Stueckelberg field $\phi^a$, as expressed in Eq. (3.11), from which

$$\partial_\mu A_\mu - \frac{i}{g}\partial_\mu(U^{-1}\partial_\mu U) - [A_\mu, U^{-1}\partial_\mu U] = 0. \tag{B3}$$

Expanding the term $U^{-1}\partial_\mu U$ in power series of $\phi^a$

$$
\begin{aligned}
U^{-1}\partial_\mu U = e^{-ig\phi^aT^a}\partial_\mu e^{ig\phi^aT^a} &= \left(1 - ig\phi^aT^a - \frac{g^2}{2}\phi^aT^a\phi^bT^b\right)\partial_\mu\left(1 + ig\phi^aT^a - \frac{g^2}{2}\phi^aT^a\phi^bT^b\right) + O(\phi^3) \\
&= \left(1 - ig\phi^aT^a - \frac{g^2}{2}\phi^aT^a\phi^bT^b\right)\left(ig\partial_\mu\phi^aT^a - \frac{g^2}{2}\partial_\mu\phi^a\phi^bT^aT^b - \frac{g^2}{2}\phi^a\partial_\mu\phi^bT^aT^b\right) \\
&= igT^a\partial_\mu\phi^a - \frac{g^2}{2}(\partial_\mu\phi^a)\phi^bT^aT^b - \frac{g^2}{2}\phi^a\partial_\mu\phi^bT^aT^b + g^2\phi^a\partial_\mu\phi^bT^aT^b + O(\phi^3) \\
&= igT^a\partial_\mu\phi^a - \frac{g^2}{2}(\partial_\mu\phi^a)\phi^b[T^a, T^b] + O(\phi^3),
\end{aligned} \tag{B4}
$$

yielding

$$U^{-1}\partial_\mu U = igT^a\partial_\mu\phi^a - \frac{g^2}{2}iT^cf^{abc}(\partial_\mu\phi^a)\phi^b + O(\phi^3). \tag{B5}$$

After substitution of expression (B5) in Eq. (B3), we have

$$0 = \partial_\mu A^a_\mu T^a + \partial_\mu\left(T^a \partial_\mu \phi^a - \frac{g}{2} T^c f^{abc} \partial_\mu \phi^a \phi^b\right) - igA^b_\mu \partial_\mu \phi^c [T^b, T^c] + \text{higher order terms}$$
$$= T^a\left(\partial A^a + \partial^2 \phi^a - \frac{g}{2} f^{abc} (\partial^2 \phi^b)\phi^c + g f^{abc} A^b_\mu \partial_\mu \phi^c\right) + \text{higher order terms}, \tag{B6}$$

from which

$$\partial^2 \phi^a = -\partial A^a - g f^{abc} A^b_\mu \partial_\mu \phi^c + \frac{g}{2} f^{abc} (\partial^2 \phi^b)\phi^c + \text{higher order terms}, \tag{B7}$$

and

$$\phi^a = -\frac{1}{\partial^2} \partial A^a - \frac{g}{\partial^2}\left(f^{abc} A^b_\mu \partial_\mu \phi^c - \frac{g}{2} f^{abc} (\partial^2 \phi^b)\phi^c\right) + \text{higher order terms}. \tag{B8}$$

Finally, substituting recursively for $\phi$, we obtain the expression (3.12).

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