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GLOBAL STABILITY OF A PRICE MODEL WITH MULTIPLE DELAYS

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Abstract. Consider the delay differential equation

$$\dot{x}(t) = a\left(\sum_{i=1}^{n} b_i\big[x(t-s_i) - x(t-r_i)\big]\right) - g(x(t)),$$

where $a > 0$, $b_i > 0$ and $0 \le s_i < r_i$ $(i \in \{1,\dots,n\})$ are parameters, $g\colon \mathbb{R} \to \mathbb{R}$ is an odd C^1 function with $g'(0) = 0$, the map $(0,\infty) \ni \xi \mapsto g(\xi)/\xi \in \mathbb{R}$ is strictly increasing and $\sup_{\xi>0} g(\xi)/\xi > 2a$. This equation can be interpreted as a price model, where $x(t)$ represents the price of an asset (e.g. price of share or commodity, currency exchange rate etc.) at time t. The first term on the right-hand side represents the positive response for the recent tendencies of the price and $-g(x(t))$ is responsible for the instantaneous negative feedback to the deviation from the equilibrium price.

We study the local and global stability of the unique, non-hyperbolic equilibrium point. The main result gives a sufficient condition for global asymptotic stability of the equilibrium. The region of attractivity is also estimated in case of local asymptotic stability.

1. Introduction.

In this paper we consider the delay differential equation

$$\dot{x}(t) = a\left(\sum_{i=1}^{n} b_i\big[x(t-s_i) - x(t-r_i)\big]\right) - g(x(t)), \tag{1.1}$$

where n is a positive integer, $a > 0$, $b_i > 0$ and $0 \le s_i < r_i$ $(i \in \{1,\dots,n\})$ are parameters such that $\max_{1\le i\le n} r_i = 1$, $\sum_{i=1}^{n} b_i = 1$, and $g\colon \mathbb{R} \to \mathbb{R}$ is an odd C^1 function with $g'(0) = 0$; moreover we assume that the map $(0,\infty) \ni \xi \mapsto g(\xi)/\xi \in \mathbb{R}$

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is strictly increasing and $\sup_{\xi>0} g(\xi)/\xi > 2a$. It is important to note, that this also implies that g is strictly increasing. Assumption $\max_{1\le i\le n} r_i = 1$ does not restrict generality as it may be achieved by rescaling time. The assumption that g is odd could be avoided, but it would make the proofs technically more involved.

Erdélyi, Brunovský and Walther [4, 5, 21] studied the following special case of equation (1.1):

$$\dot{x}(t) = a\big[x(t) - x(t-1)\big] - \beta|x(t)|x(t), \tag{1.2}$$

where β is a positive parameter. Erdélyi [7] gave a detailed interpretation of equation (1.2) according to which $x(t)$ represents the price of an asset at time t (e.g. price of share or commodity, currency exchange rate etc.). The positive response to the recent tendency of the price is represented by $a[x(t)-x(t-1)]$, while $-\beta|x(t)|x(t)$ is responsible for the negative feedback to the deviation from the unique equilibrium, which is the origin.

From the modelling point of view it is natural to assume that when we are trying to figure out the tendencies of the price, we are more likely to think of it as a weighted sum of recent changes of the price (i.e. $\sum_{i=1}^n b_i\big[x(t-s_i) - x(t-r_i)\big]$, $0 \le s_i < r_i$, presumably with smaller weights on less recent values of the price), rather than to compare the current price only to one previous value of it, $x(t-1)$. It is also natural to allow more general functions for the instantaneous feedback than $x \mapsto -\beta|x|x$. Both these possibilities are incorporated in equation (1.1).

Numerical simulations provided by Erdélyi [7] suggested the existence of a stable (slowly oscillating) periodic solution of equation (1.2) for $a > 1$, which was established in [4, 5]. This result has recently been generalized by Stumpf [20] for a state-dependent delay version of equation (1.2). Walther analyzed further the slowly oscillating periodic solution of equation (1.2) and showed that it converges to a square-wave solution as a tends to infinity [21], and that the period tends to infinity as $a \to 1^+$ [22].

It is shown in [7] that the unique equilibrium of (1.2) is unstable for $a > 1$ and locally asymptotically stable for $a < 1$. Since the equilibrium is non-hyperbolic, the latter was carried out by a center manifold reduction. Numerical observations in [7] also indicated global attractivity of the unique equilibrium of (1.2) for $a < 1$.

In this paper we prove that the unique, non-hyperbolic equilibrium of (1.1) (the origin) is unstable if $a\sum_{i=1}^n b_i(r_i - s_i) > 1$, locally asymptotically stable if $a\sum_{i=1}^n b_i(r_i - s_i) < 1$, and we provide a lower bound on the domain of attraction of the equilibrium. Furthermore, sufficient conditions for the global asymptotic stability of the origin are given. More precisely, we show that all solutions of equation (1.1) converge to 0 as $t \to \infty$ if

$$a^2 \sum_{i=1}^n b_i\big(r_i^2 - s_i^2\big) < \left(1 - a\sum_{i=1}^n b_i(r_i - s_i)\right)^2.$$

In particular, the result yields that the zero solution of equation (1.2) is globally asymptotically stable if $a < \frac{1}{2}$.

In order to show local stability and to estimate the region of attraction of a non-hyperbolic equilibrium, the application of center manifolds seems natural. It works here as well. However, taking advantage of the particular structure of the equation, we use another technique. This technique gives global results in addition to local ones. Moreover, we believe that the estimation for the region of attractivity are better than those could be obtained via the center manifold reduction. The main idea is that equation (1.1) is considered in a neutral equation form and its solutions are transformed (in an invertible way) to solutions of a linear, infinite delay equation. Then stability and convergence are guaranteed by the 3/2-type stability results due to Krisztin [14]. These are stated and proved in Section 3. Similar ideas were applied in [3, 9].

The constant 3/2 as a sharp upper bound first appeared in the book of Myškis [18] for the linear equation $\dot{x}(t) = -px(t-\tau(t))$ showing that $pr \leq 3/2$ implies stability of the zero solution where $p \geq 0$ and τ is continuous with $\tau(t) \in [0, r]$. It is remarkable that the same 3/2 can be obtained for a large class of nonlinear problems, see e.g. Wright [23], Yorke [24], Barnea [2], Kato [13], Lillo [15], Hale [10], Liz, Tkachenko and Trofimchuk [16], Ivanov, Liz and Trofimchuk [11]. Krisztin [14] gives an extension to the case of distributed and infinite delays which naturally arises here.

Section 4 is devoted to the single delay case, i.e. $n = 1$ with $s_1 = 0$. In this special case one can use the Poincaré–Bendixson-type theorem and some monotonicity properties of (possible) periodic solutions by Mallet-Paret and Sell [17] to improve the condition $a < \frac{1}{2}$.

In Section 5, some relevant examples are given to illustrate the results, and we also show some directions on possible further studies in the topic.

In the next section we introduce the notations and recall some preliminary results that will be used in subsequent sections.

2. Preliminaries.

Let $\mathbb{N}_k$ denote the set of positive integers not greater than k, and let the m-fold Cartesian product $\mathbb{N}_k \times \cdots \times \mathbb{N}_k$ be denoted by $\mathbb{N}_k^m$. We say that a continuous function $x\colon [-1, \infty) \to \mathbb{R}$ is a *solution* of equation (1.1) if it is differentiable for $t > 0$ and satisfies equation (1.1) for $t > 0$. Let $C = C([-1, 0], \mathbb{R})$ denote the Banach space of continuous real functions on the interval $[-1, 0]$, endowed with the maximum norm: $\|\varphi\|_C = \max_{-1 \leq t \leq 0} |\varphi(t)|$, for $\varphi \in C$. For a given continuous map $\psi\colon I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, and $t \in \mathbb{R}$ with $[t-1, t] \subseteq I$, let the segment $\psi_t \in C$ be defined by $\psi_t(s) = \psi(t+s)$ for $-1 \leq s \leq 0$. By the method of steps it can be shown that for every $\varphi \in C$ there exists a unique solution $x^\varphi\colon [-1, \infty) \to \mathbb{R}$ of equation (1.1), for which $x_0^\varphi = \varphi$.

Theorem 2.1 is one of the key tools in the proof of our main results. In order to formulate it, we need to introduce several definitions and notions (see also [14]). Let BC denote the set of bounded, continuous functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and for $\varphi \in BC$ let $\|\varphi\|_{BC} = \sup_{s \leq 0} |\varphi(s)|$. For $\alpha \in \mathbb{R}$, $\psi \in C((-\infty, \alpha], \mathbb{R})$ and $t \leq \alpha$, let $\psi_t \in BC$ be defined by $\psi_t(s) = \psi(t+s),\ s \leq 0$. As ψ_t may now denote two similar, but different maps, we will always make it unambiguous by writing either $\psi_t \in C$ or $\psi_t \in BC$.

Consider the functional differential equation

$$\dot{x}(t) = F(t, x_t), \tag{2.1}$$

where $F\colon [0, \infty) \times BC \to \mathbb{R}$, $F(\,\cdot\,, 0) \equiv 0$, and for any $\psi \in C(\mathbb{R}, \mathbb{R})$ with $\psi_t \in BC$ the function $t \mapsto F(t, \psi_t)$ is continuous on $[0, \infty)$. The function $x(\,\cdot\,) = x(\,\cdot\,; t_0, \varphi) \in C((-\infty, t_0 + \omega), \mathbb{R})$ is a *solution* of equation (2.1) through a given pair $(t_0, \varphi) \in [0, \infty) \times BC$ on $[t_0, t_0 + \omega)$, $\omega > 0$, if $x_{t_0} = \varphi$ and equation (2.1) holds on $(t_0, t_0 + \omega)$. We also assume that some additional conditions are satisfied for F guaranteeing that a unique solution exists on $[t_0, \infty)$ for all $t_0 \geq 0$ and $\varphi \in BC$ (see [6, 12]).

The zero solution of (2.1) is said to be

(i) *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that

$$t \geq t_0 \geq 0, \quad \|\varphi\| < \delta \quad \text{imply} \quad |x(t; t_0, \varphi)| < \varepsilon;$$

(ii) *uniformly asymptotically stable* if it is uniformly stable and there exist $\delta_0 > 0$ and a function $T = T(\varepsilon)$ such that, for any $\varepsilon > 0$

$$\|\varphi\| < \delta_0, \quad t_0 \geq 0, \quad t \geq t_0 + T \quad \text{imply} \quad |x(t; t_0, \varphi)| < \varepsilon.$$

Let $\mathcal{M}$ denote the space of functions $\mu\colon [0,\infty) \to [0,\infty)$ that are bounded, non-decreasing, continuous from the left and not identically constant. For $\mu \in \mathcal{M}$ let

$$\begin{aligned} \mu_0 &= \int_0^\infty d\mu(s), \qquad \mu_1 = \int_0^\infty s\, d\mu(s) \quad \text{and} \\ \mu_2 &= \mu_1 + \frac{\mu_0}{2} \int_0^{1/\mu_0} \left(\frac{1}{\mu_0} - s\right)^2 d\mu(s). \end{aligned} \tag{2.2}$$

Now we are in a position to state the following generalization of Yorke's theorem [24].

Theorem 2.1 ([14, Theorem 1.2])**.** *Assume that there exists $\mu \in \mathcal{M}$ so that for all $t \geq 0$ and $\varphi \in BC$ it satisfies the condition*

$$-\int_0^\infty M_u(\varphi)\, d\mu(u) \leq F(t,\varphi) \leq \int_0^\infty M_u(-\varphi)\, d\mu(u), \tag{2.3}$$

where M_u is defined by

$$M_u(\varphi) = \max\left\{0, \max_{s\in[-u,0]} \varphi(s)\right\}$$

for $\varphi \in BC$ and $u \geq 0$. Then the following statements hold.

(i) If $\mu_2 \leq 3/2$, then the zero solution of equation (2.1) is uniformly stable.

(ii) If $\mu_2 < 3/2$, and condition

$$\left\{ \begin{matrix} t_n \to \infty,\ \varphi_n \in BC,\ c \in \mathbb{R},\ c \neq 0,\ \varphi_n(s) \to c \text{ uniformly on} \\ \text{compact subsets of } (-\infty, 0] \text{ imply that } F(t_n, \varphi_n) \not\to 0 \text{ as } n \to \infty \end{matrix} \right\} \tag{2.4}$$

is satisfied, then the zero solution of equation (2.1) is uniformly asymptotically stable.

(iii) If $\mu_2 \leq 3/2$ and condition (2.3) is only assumed to hold for all $t \in [0, t_0]$ and $\varphi \in BC$, then for any $\varepsilon > 0$ and $\psi \in BC$ fixed

$$\|\psi\|_{BC} \leq \varepsilon e^{-5/2} \quad \text{implies} \quad \|x_t(\,\cdot\,; t_0, \psi)\|_{BC} \leq \varepsilon \quad \text{for all} \quad t \leq t_0.$$

Proof. The first two statements and their proofs can be found in [14]. Statement (iii) can also be proved by arguing the same way (with straightforward modifications) as in the proof of statement (i), thus the proof is omitted here. ☐

It is easy to check that $\mu_2 - \mu_1 \leq 1/2$, therefore the following corollary holds.

Corollary 2.2 ([14, Corollary 1.3])**.** *If the conditions $\mu_2 \leq 3/2$ and $\mu_2 < 3/2$ in Theorem 2.1 are replaced by $\mu_1 \leq 1$ and $\mu_1 < 1$, respectively, then the statements of the theorem remain true.*

3. **Results.** Let us turn our attention to equation (1.1). Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$h(\xi) = \begin{cases} \dfrac{g(\xi)}{\xi}, & \text{for } \xi \neq 0, \\ 0, & \text{for } \xi = 0. \end{cases} \tag{3.1}$$

It is clear from the assumptions on g (given in equation (1.1)), that function h is even, continuous and it is positive and strictly increasing on $(0,\infty)$. Let h^{-1} denote the inverse of the restricted map $h|_{[0,\infty)}$ and note that the domain of h^{-1} contains the interval $[0, 2a]$. We have the following boundedness result for solutions of (1.1).

Lemma 3.1. *For every $\varphi \in C$, the following statement holds:*

$$-h^{-1}(2a) \leq \liminf_{t\to\infty} x^{\varphi}(t) \leq \limsup_{t\to\infty} x^{\varphi}(t) \leq h^{-1}(2a).$$

Proof. Note that this lemma is proved in [4, Proposition 2.4] (see also [5]) for equation (1.2).

First note that the definition of the map h combined with the assumption that $g(\xi)/\xi$ is strictly increasing yields that the following hold for all $\xi > 0$:

$$\left.\begin{aligned} \xi < h^{-1}(2a) &\iff g(\xi) < 2a\xi, \\ \xi = h^{-1}(2a) &\iff g(\xi) = 2a\xi, \\ \xi > h^{-1}(2a) &\iff g(\xi) > 2a\xi. \end{aligned}\right\} \tag{3.2}$$

It is easy to see that for any $M \geq h^{-1}(2a)$, if $\|\varphi\|_C < M$, then $\|x_t^{\varphi}\|_C < M$ for all $t > 0$. To see this, assume to the contrary that there exists $t_0 > 0$ such that $|x(t_0)| = M$, and $|x(t)| < M$ for all $t \in [-1, t_0)$. Let us consider the case $x(t_0) = M$, as the case of $x(t_0) = -M$ is analogous. Then we have

$$0 \leq \dot{x}(t_0) = a\left(\sum_{i=1}^{n} b_i\big[x(t_0 - s_i) - x(t_0 - r_i)\big]\right) - g(x(t_0)) < 2aM - g(M) \leq 0,$$

a contradiction proving the claim.

This implies that for any $\varphi \in C$, $x(t) := x^{\varphi}(t)$ is bounded on $t \in [0,\infty)$. Thus there exists $M := \limsup_{t\to\infty} x(t)$ and $m := \liminf_{t\to\infty} x(t)$, both finite. We may suppose that $|M| \geq |m|$, as the case $|M| \leq |m|$ can be handled similarly.

By way of contradiction, suppose that $M > h^{-1}(2a)$. Properties (3.2) and continuity of g guarantee that there exists $\varepsilon > 0$ and $\delta > 0$ small enough such that

$$g(M - \varepsilon) \geq 2a(M + \varepsilon) + \delta > 0. \tag{3.3}$$

Let ε and δ be fixed this way. By the definition of M, there exists $T = T(\varepsilon) \geq 0$ such that $|x(t)| < M + \varepsilon$ for $t > T - 1$. Now, if an arbitrary $t_0 \geq T$ is such that $x(t_0) \geq M - \varepsilon$ holds, then we infer

$$\begin{aligned} \dot{x}(t_0) &= a\left(\sum_{i=1}^{n} b_i\big[x(t_0 - s_i) - x(t_0 - r_i)\big]\right) - g(x(t_0)) \\ &\leq 2a(M + \varepsilon) - g(M - \varepsilon) \\ &\leq -\delta < 0. \end{aligned} \tag{3.4}$$

This means that either $x(T) \leq M - \varepsilon$ and then $x(t) \leq M - \varepsilon$ holds for all $t > T$ as well, or $x(T) > M - \varepsilon$. In the latter case, inequality (3.4) implies that there exists $T' > T$ such that $x(t) < M - \varepsilon$ for all $t > T'$. Both cases contradict to the assumption that $M = \limsup_{t\to\infty} x(t)$, which proves our claim. □

We will transform equation (1.1) to an infinite delay differential equation, for which we need some further notations. For a given $\alpha \in \mathbb{R}$ we introduce the weighted space of continuous functions

$$C_\alpha = \Big\{ \varphi \in C((-\infty, 0], \mathbb{R}) : \lim_{t \to -\infty} e^{\alpha t} \varphi(t) = 0 \Big\}$$

endowed with the norm

$$\|\varphi\|_\alpha = \sup_{t \le 0} e^{\alpha t} |\varphi(t)|.$$

Note that for any $\varphi \in BC$, $\varphi \in C_\alpha$ also holds for any $\alpha > 0$. For $\alpha \in \mathbb{R}$ let BV_α be the set of functions $\mu\colon [0,\infty) \to \mathbb{R}$ of bounded variation satisfying

$$\|\mu\|_\alpha \coloneqq \int_0^\infty e^{\alpha t} \, d|\mu|(t) < \infty,$$

where $|\mu|$ denotes the total variation function of μ. For $\mu \in BV_\alpha$ let the convolution operator L_μ, mapping C_α to itself, be defined by

$$(L_\mu \varphi)(t) = \int_0^\infty \varphi(t-s) \, d\mu(s) \qquad (t \le 0). \tag{3.5}$$

Now, let $\delta \in BV_\alpha$ and $\nu_i \in BV_\alpha$ for all $i \in \mathbb{N}_n$ be defined by

$$\delta(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0, \end{cases} \quad \text{and} \quad \nu_i(t) = \begin{cases} ab_i s_i & \text{if } 0 \le t \le s_i \\ ab_i t & \text{if } s_i < t \le r_i \\ ab_i r_i & \text{if } t > r_i, \end{cases} \tag{3.6}$$

and let

$$\nu = \sum_{i=1}^n \nu_i, \qquad \eta = \delta - \nu. \tag{3.7}$$

It is clear that $\eta \in BV_\alpha$ for any $\alpha \in \mathbb{R}$. Then equation (1.1) can be written in the following neutral equation form:

$$\frac{d}{dt} \left[x(t) - \sum_{i=1}^n \int_0^\infty x(t-s) \, d\nu_i(s) \right] = -g(x(t)) \qquad (t > 0),$$

or equivalently

$$\frac{d}{dt} \big[(L_\eta x_t)(0) \big] = -g(x(t)) \qquad (t > 0). \tag{3.8}$$

Here function x can be extended to the interval $(-\infty, -1)$ for example by letting $x(t) = x(-1)$ for all $t < -1$. However, as η is constant on $[1, \infty)$, the value of $L_\eta x_t$ does not depend on the extension.

For brevity we will frequently use the following notations:

$$A \coloneqq a \sum_{i=1}^n b_i (r_i - s_i) \quad \text{and} \quad B \coloneqq \frac{a}{2} \sum_{i=1}^n b_i \big(r_i^2 - s_i^2 \big). \tag{3.9}$$

It is clear that the total variation of ν is A. We claim that if $A < 1$, then $\operatorname{Re} \lambda \le \log A < 0$ holds for all roots $\lambda \in \mathbb{C}$ of the (characteristic) equation

$$1 - \int_0^1 e^{-\lambda s} \, d\nu(s) = 0.$$

This is indeed true, since for $\operatorname{Re} \lambda > \log A$ we have the following estimates

$$\left| \int_0^1 e^{-\lambda s} \, d\nu(s) \right| \le \int_0^1 \max_{s \in [0,1]} \left| e^{-\lambda s} \right| d|\nu|(s) < e^{-\log A} A = 1.$$

Now, by [10, Theorem XII.4.1], the operator $D\colon C \to \mathbb{R}$, defined by $D\varphi = (L_\eta \varphi^e)(0)$, is stable, where $\varphi^e \in BC$ denotes an extension of $\varphi \in C$. For definiteness, one may take $\varphi^e(t) = \varphi(-1)$ for $t < -1$, however, the value of $D\varphi$ does not depend on the extension itself.

This ensures that the following lemma by Staffans [19] holds. See also Lemma 2.2 and Remark 2.1 of [8].

Lemma 3.2. *If $a\sum_{i=1}^n b_i(r_i - s_i) < 1$, then for $\alpha > 0$ small enough, the operator L_η defined by (3.5)–(3.7) maps C_α into itself continuously, it has a continuous inverse L_η^{-1}, and there exists a function $\tilde\eta \in BV_\alpha$, such that the inverse operator is the convolution operator $L_{\tilde\eta}$, i.e. $L_{\tilde\eta} L_\eta = L_\eta L_{\tilde\eta} = L_\delta = \mathrm{id}$. Moreover $L_{\tilde\eta} = (L_\delta - L_\nu)^{-1}$ can be expressed by the convergent power series $L_{\tilde\eta} = \sum_{k=0}^\infty L_\nu^k$.*

Remark 3.3. Note that from the power series expansion $L_{\tilde\eta} = \sum_{k=0}^\infty L_\nu^k$ and from the monotonicity of ν it follows that function $\tilde\eta$ is also monotonic. Moreover, as we will only integrate continuous functions with respect to $\tilde\eta$, we may assume without loss of generality that $\tilde\eta$ is continuous from the left. Consequently $\tilde\eta \in \mathcal{M}$ can be assumed.

Now we are ready to state our main theorem.

Theorem 3.4. *The zero solution of (1.1) is*

(i) unstable if $A > 1$;

(ii) locally asymptotically stable if $A < 1$; moreover $x^\varphi(t) \to 0$ provided that

$$\|\varphi\|_C < \frac{(1-A)e^{-5/2}}{1+A} h^{-1}\left(\frac{(1-A)^2}{B}\right);$$

(iii) globally asymptotically stable (i.e. locally stable and globally attractive) if

$$A < 1 \quad \text{and} \quad 2aB < (1-A)^2.$$

Proof. To prove statement (i), assume that $A > 1$ and consider the characteristic equation of the linearization of equation (1.1):

$$\Delta(\lambda) := a\sum_{i=1}^n b_i\big(e^{-\lambda s_i} - e^{-\lambda r_i}\big) - \lambda = 0. \tag{3.10}$$

Observe that $\Delta(\lambda) \to -\infty$ as $\lambda \to \infty$. Then continuity of the map Δ combined with $\Delta(0) = 0$ and $\frac{d}{d\lambda}\Delta(0) = A - 1 > 0$ yields that there exists at least one positive real characteristic root, proving statement (i).

Since 0 is always a characteristic root, a linearized stability theorem cannot be applied here. A different approach is necessary to prove local stability. We will transform our equation to a non-autonomous infinite delay equation of the form (3.8) and apply Corollary 2.2.

For the rest of the proof, let us assume that $A < 1$. For fixed $\varphi \in C$ extend the solution $x = x^\varphi : [-1, \infty) \to \mathbb{R}$ to a map $\mathbb{R} \to \mathbb{R}$ by $x(t) = x(-1)$ for $t < -1$. We denote the extension also by x. Then $x_t \in BC$ for all $t \in \mathbb{R}$. Using notations (3.5)–(3.7) and letting

$$\begin{aligned} y(t) = (L_\eta x_t)(0) &= x(t) - \int_0^\infty x(t-s)\, d\nu(s) \\ &= x(t) - a\sum_{i=1}^n \left(b_i \int_{s_i}^{r_i} x(t-s)\, ds\right) \end{aligned} \tag{3.11}$$

for all $t \in \mathbb{R}$, one obtains that $y_t \in BC$ for all $t \in \mathbb{R}$, and y satisfies

$$\dot{y}(t) = -g(x(t)) = -h(x(t))x(t) \tag{3.12}$$

for all $t > 0$, where the map h is defined by (3.1). On the other hand, Lemma 3.2 and Remark 3.3 guarantee that there exists $\tilde{\eta} \in \mathcal{M}$ such that

$$x(t) = (L_{\tilde{\eta}} y_t)(0) \qquad (t \in \mathbb{R}). \tag{3.13}$$

Using the above notations and $\beta(t) := -h(x(t))$ we have that $z = y$ is a solution of the non-autonomous, linear differential equation with infinite delay

$$\dot{z}(t) = \beta(t) \int_0^\infty z(t-s)\, d\tilde{\eta}(s) \qquad (t > 0). \tag{3.14}$$

Our aim is to apply Corollary 2.2 for equation (3.14). For this reason we need to calculate $\tilde{\eta}_0$ and $\tilde{\eta}_1$ defined by (2.2). Let $\varphi_0(t) \equiv 1$ and $\varphi_1(t) \equiv -t$ for all $t \in \mathbb{R}$. Then one easily gets that

$$\tilde{\eta}_0 = \int_0^\infty d\tilde{\eta}(s) = (L_{\tilde{\eta}}\varphi_0)(0) = \sum_{k=0}^\infty (L_\nu^k \varphi_0)(0) = \sum_{k=0}^\infty A^k = \frac{1}{1-A}. \tag{3.15}$$

Similarly, one obtains the following:

$$\begin{aligned} \tilde{\eta}_1 &= \int_0^\infty s\, d\tilde{\eta}(s) = (L_{\tilde{\eta}}\varphi_1)(0) \\ &= \sum_{k=0}^\infty (L_\nu^k \varphi_1)(0) \\ &= (L_\delta \varphi_1)(0) + \sum_{k=1}^\infty \sum_{I \in \mathbb{N}_n^k} \left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(0) \\ &= \sum_{k=1}^\infty \sum_{I \in \mathbb{N}_n^k} \left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(0). \end{aligned} \tag{3.16}$$

Let us further examine this last product of operators. For simplicity, fix $I = \mathbb{N}_k$ for now. By the definition of operator L_{ν_i} we have

$$\begin{aligned} \left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(t) &= \int_0^\infty \cdots \int_0^\infty \varphi_1\big(t - (u_1 + \cdots + u_k)\big) d\nu_1(u_1) \cdots d\nu_k(u_k) \\ &= \left(\prod_{i=1}^k ab_i \right) \int_{s_k}^{r_k} \cdots \int_{s_1}^{r_1} (u_1 + \cdots + u_k - t)\, du_1 \cdots du_k. \end{aligned}$$

Thus

$$\left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(0) = \left(\prod_{i=1}^k ab_i \right) \int_{s_k}^{r_k} \cdots \int_{s_1}^{r_1} (u_1 + \cdots + u_k)\, du_1 \cdots du_k$$

holds. We claim that

$$\int_{s_k}^{r_k} \cdots \int_{s_1}^{r_1} (u_1 + \cdots + u_k)\, du_1 \cdots du_k = \frac{1}{2} \left(\sum_{j=1}^k (r_j + s_j) \right) \left(\prod_{i=1}^k (r_i - s_i) \right).$$

This trivially holds for $k = 1$. Assume that the claim holds for some $k \geq 1$. Then one easily gets that

$$\begin{aligned}
&\int_{s_{k+1}}^{r_{k+1}} \cdots \int_{s_1}^{r_1} (u_1 + \cdots + u_{k+1})\, du_1 \cdots du_{k+1} \\
&\quad = \int_{s_{k+1}}^{r_{k+1}} \left[\int_{s_k}^{r_k} \cdots \int_{s_1}^{r_1} (u_1 + \cdots + u_k)\, du_1 \cdots du_k \right] du_{k+1} \\
&\qquad + \int_{s_{k+1}}^{r_{k+1}} \cdots \int_{s_1}^{r_1} u_{k+1}\, du_1 \cdots du_{k+1} \\
&\quad = \frac{1}{2} \left(\sum_{j=1}^{k} (r_j + s_j) \right) \left(\prod_{i=1}^{k} (r_i - s_i) \right) \int_{s_{k+1}}^{r_{k+1}} du_{k+1} \\
&\qquad + \left(\prod_{i=1}^{k} (r_i - s_i) \right) \int_{s_{k+1}}^{r_{k+1}} u_{k+1}\, du_{k+1} \\
&\quad = \frac{1}{2} \left(\sum_{j=1}^{k+1} (r_j + s_j) \right) \left(\prod_{i=1}^{k+1} (r_i - s_i) \right),
\end{aligned}$$

proving our claim. This yields that

$$\left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(0) = \frac{1}{2} \left(\sum_{j=1}^{k} (r_j + s_j) \right) \left(\prod_{i=1}^{k} a b_i (r_i - s_i) \right).$$

In a similar fashion one obtains that

$$\left(\left(\prod_{i \in I} L_{\nu_i} \right) \varphi_1 \right)(0) = \frac{1}{2} \left(\sum_{j \in I} (r_j + s_j) \right) \left(\prod_{i \in I} a b_i (r_i - s_i) \right) \tag{3.17}$$

holds for any set of indices I.

Now, observe that for any positive integer k we have

$$\begin{aligned}
&\sum_{I \in \mathbb{N}_n^k} \left[\left(\prod_{i \in I} a b_i (r_i - s_i) \right) \left(\sum_{j \in I} (r_j + s_j) \right) \right] \\
&\qquad = k \left(a \sum_{j=1}^{n} b_j \left(r_j^2 - s_j^2 \right) \right) \left(a \sum_{i=1}^{n} b_i (r_i - s_i) \right)^{k-1}.
\end{aligned} \tag{3.18}$$

Substituting formulas (3.17) and (3.18) into (3.16) and using notations (3.9) we obtain

$$\tilde{\eta}_1 = B \left[\sum_{k=0}^{\infty} (k+1) A^k \right] = \frac{B}{(1-A)^2}. \tag{3.19}$$

Now we are in a position to prove statement (ii). Let $\varepsilon > 0$ be fixed arbitrarily. We will give $\delta = \delta(\varepsilon)$ such that $\|x_t\|_C < \varepsilon$ holds for all $t > 0$ provided that $\|x_0\|_C < \delta$.

To prove this, let $\beta_0 > 0$ and $\varepsilon_0 > 0$ be fixed such that

$$\beta_0 < \frac{(1-A)^2}{B} = \frac{1}{\tilde{\eta}_1} \tag{3.20}$$

$$\varepsilon_0 := \min \left\{ (1-A)\varepsilon,\ \frac{h^{-1}(\beta_0)}{\tilde{\eta}_0} \right\} = (1-A) \min\{\varepsilon,\ h^{-1}(\beta_0)\} \tag{3.21}$$

hold. Note that if for some $t_0 > 0$, $|y(t)| \leq \varepsilon_0$ holds for all $t \in (-\infty, t_0]$, then, by formula (3.15),

$$|\beta(t)| = |h(x(t))| = \left|h\Big(\textstyle\int_0^\infty y(t-s)\, d\tilde{\eta}(s)\Big)\right| \leq h\Big(\varepsilon_0 \textstyle\int_0^\infty d\tilde{\eta}(s)\Big) \leq \beta_0 \tag{3.22}$$

also holds for all $t \in (-\infty, t_0]$. Finally, let $\varepsilon_1 \in (0, \varepsilon_0)$,

$$\delta = \delta(\varepsilon) = \frac{\varepsilon_1 e^{-5/2}}{1+A}, \tag{3.23}$$

and $\|x_0\|_C < \delta$. Then for $t \leq 0$,

$$|y(t)| = \big|(L_\eta x_t)(0)\big| = \left|x(t) - \int_0^\infty x(t-s)\, d\nu(s)\right| \leq \delta + \delta \int_0^\infty d\nu(s) = \delta(1+A)$$

holds, yielding $\|y_0\|_{BC} < \varepsilon_1 e^{-5/2}$. We claim that $\|y_t\|_{BC} < \varepsilon_0$ holds for all $t > 0$. Assume to the contrary that this is not the case. Since $\|y_0\|_{BC} \leq \varepsilon_1 e^{-5/2} < \varepsilon_1 < \varepsilon_0$, thus there must exist $t_0 > 0$ such that $|y(t_0)| = \varepsilon_0$ and $|y(t)| < \varepsilon_0$ for all $t \in (-\infty, t_0)$. From (3.22) it follows that $|\beta(t)| \leq \beta_0$ for $t \in (-\infty, t_0]$. Now recall that

$$\dot{y}(t) = \beta(t) \int_0^\infty y(t-s)\, d\tilde{\eta}(s)$$

holds for all $t > 0$ and note that for $\mu := \beta_0 \tilde{\eta}$ one has $\mu_1 < 1$, from which $\mu_2 < 3/2$ also follows (see (2.2) for the definition of μ_1 and μ_2). Use notation $F(t, \varphi) := \beta(t) \int_0^\infty \varphi(t-s)\, d\tilde{\eta}(s)$ and observe that (2.4) can only be violated if $\beta(t) \to 0$ as $t \to \infty$, but in that case we readily have $x(t) \to 0$. It is easy to verify that all conditions on F, required by Theorem 2.1, are satisfied. Thus Theorem 2.1 (iii) can be applied to conclude that $|y(t)| < \varepsilon_1 < \varepsilon_0$ holds for all $t \in [0, t_0]$, which is a contradiction. Therefore $\|y_t\|_{BC} < \varepsilon_0$ for all $t > 0$.

Now, observe that by formulas (3.15) and (3.21) and $\tilde{\eta} \in \mathcal{M}$

$$|x(t)| = \left|\int_0^\infty y(t-s)\, d\tilde{\eta}(s)\right| \leq \varepsilon_0 \tilde{\eta}_0 = \frac{\varepsilon_0}{1-A} \leq \varepsilon$$

holds for all $t > 0$, which proves that the zero equilibrium of equation (1.1) is locally stable.

To prove asymptotic stability, first we claim that $y(t) \to 0$ implies $x(t) \to 0$ (as $t \to \infty$). To see this, let $\varepsilon > 0$ be fixed arbitrarily and assume that $y(t) \to 0$ as $t \to \infty$. Let t_0 be such that $|y(t)| < \varepsilon_2 := \frac{\varepsilon}{2}(1-A)$ for all $t \geq t_0$. Now let N be a positive integer such that $KA^N < \varepsilon_2$, where $K = \max\{\varepsilon_2, \|y_{t_0}\|_{BC}\}$. Then using $\max_{i \in \mathbb{N}_n} r_i = 1$ and that ν is monotone non-decreasing we obtain for all $t > t_0 + N - 1$ the following estimates:

$$\begin{aligned}
|x(t)| &= \left|y(t) + \sum_{k=1}^\infty (L_\nu^k y_t)(0)\right| \\
&\leq |y(t)| + \sum_{k=1}^\infty \left(\int_0^1 \cdots \int_0^1 |y(t-u_1-\cdots-u_k)|\, d\nu(u_1)\cdots d\nu(u_k)\right) \\
&\leq \varepsilon_2 + \sum_{k=1}^\infty \max_{s \in [t-k,t]} |y(s)| A^k \leq \varepsilon_2 \sum_{k=0}^{N-1} A^k + K \sum_{k=N}^\infty A^k < \frac{\varepsilon_2 + KA^N}{1-A} < \varepsilon,
\end{aligned}$$

which proves the claim.

Now, fix $\varepsilon > 0$ arbitrarily so that

$$\varepsilon < h^{-1}\left(\tfrac{(1-A)^2}{B}\right)$$

holds. Note that during the proof of local stability we also showed that if

$$\delta < \frac{(1-A)e^{-5/2}\varepsilon}{1+A}$$

and $\|x_0\|_C < \delta$, then $|x(t)| < \varepsilon$ for all $t \geq -1$. Consequently $|\beta(t)| < h(\varepsilon) < (1-A)^2/B$ holds for all $t \geq -1$. Finally, applying Theorem 2.1 (ii) and Corollary 2.2 for equation (3.14) with $\mu := h(\varepsilon)\tilde{\eta}$ and $F(t,\varphi) := \beta(t)\int_0^\infty \varphi(t-s)\,\tilde{\eta}(s)$, we obtain that the zero solution of equation (3.14) is uniformly asymptotically stable. As equation (3.14) is linear, this means that every solution of (3.14) converges to zero, and in particular, $y(t) \to 0$ as $t \to \infty$, from which $x(t) \to 0$ follows. This completes the proof of statement (ii).

To prove assumption (iii), note that $2aB < (1-A)^2$ guarantees that δ can be chosen small enough so that $(2a+\delta)B < (1-A)^2$ still holds. Then Lemma 3.1 yields that for t large enough $|\beta(t)| < 2a+\delta$. Finally, applying Theorem 2.1 (ii) and Corollary 2.2 for equation (3.14), $\mu := (2a+\delta)\tilde{\eta}$ and $F(t,\varphi) := \beta(t)\int_0^\infty \varphi(t-s)\,\tilde{\eta}(s)$, and using again the linearity of equation (3.14), we obtain that every solution of (3.14) converges to zero, and thus $y(t) \to 0$ as $t \to \infty$ from which $x(t) \to 0$ also follows. □

The third statement of the above theorem can be slightly amended by using criterion $\mu_2 < 3/2$ instead of $\mu_1 < 1$. This result is formulated in the next theorem.

Theorem 3.5. *The zero solution of* (1.1) *is globally asymptotically stable if*

$$A < 1 \quad \textit{and} \quad \frac{2aB}{(1-A)^2} + \min\left\{\frac{1}{2}, \frac{7(1-A)}{12}\right\} < \frac{3}{2},$$

where A and B are defined by (3.9).

Proof. We may assume that

$$\frac{7(1-A)}{12} < \frac{1}{2},$$

since otherwise the statement of the theorem coincides with Theorem 3.4 (iii). Then by the conditions of the theorem there exists $\delta > 0$ small enough such that for $M := (2a+\delta)$ the inequality

$$\frac{MB}{(1-A)^2} + \frac{1-A}{2} + \frac{a(1-A)}{6M} < \frac{3}{2}$$

holds. Then Lemma 3.1 guarantees that $M \geq |\beta(t)|$ holds for t large enough.

Condition $A < 1$ ensures that equation (1.1) can be transformed to equation

$$\dot{y}(t) = F(t, y_t),$$

with $F(t,\varphi) := \beta(t)\int_0^\infty \varphi(t-s)\,d\tilde{\eta}(s)$. We will apply Theorem 2.1 to this equation with $\mu := M\tilde{\eta}$. Therefore we need to estimate μ_2 defined by (2.2). By formula (3.19) we have

$$\mu_1 = \frac{MB}{(1-A)^2}, \tag{3.24}$$

so we only need an estimate on $\mu_2 - \mu_1$, which reads as

$$\mu_2 - \mu_1 = \frac{\mu_0}{2}\int_0^{1/\mu_0}\left(\frac{1}{\mu_0} - s\right)^2 d\mu(s) = \frac{M\mu_0}{2}\int_0^{1/\mu_0}\left(\frac{1}{\mu_0} - s\right)^2 d\tilde{\eta}(s).$$

Using the notation

$$\varphi_2(t) = \begin{cases} \left(\frac{1}{\mu_0} + t\right)^2 & \text{for } t \in \left[-\frac{1}{\mu_0}, 0\right], \\ 0 & \text{otherwise,} \end{cases}$$

we obtain similarly as in formula (3.16) that

$$\int_0^{1/\mu_0} \left(\frac{1}{\mu_0} - s\right)^2 d\tilde{\eta}(s) = (L_{\tilde{\eta}}\varphi_2)(0) = \frac{1}{\mu_0^2} + \sum_{k=1}^{\infty} (L_\nu^k \varphi_2)(0).$$

We claim that

$$(L_\nu^k \varphi_2)(0) \le \frac{aA^{k-1}}{3\mu_0^3}$$

holds for all $k \ge 1$. Indeed, using the definition of φ_2 one has the estimates

$$\begin{aligned}(L_\nu^k \varphi_2)(0) &= \int_0^\infty \cdots \int_0^\infty \varphi_2(-u_1 - \cdots - u_k)\, d\nu(u_1) \cdots d\nu(u_k) \\ &\le \int_0^\infty \cdots \int_0^\infty \varphi_2(-u_1)\, d\nu(u_1) \cdots d\nu(u_k) \\ &\le A^{k-1} \left[\sum_{i=1}^n ab_i \int_0^{1/\mu_0} \left(\frac{1}{\mu_0} - u_1\right)^2 du_1 \right] \\ &= \frac{aA^{k-1}}{3\mu_0^3}.\end{aligned}$$

In the light of the above formulas we have obtained that

$$\mu_2 - \mu_1 \le \frac{M}{2}\left(\frac{1}{\mu_0} + \frac{a}{3(1-A)\mu_0^2}\right).$$

Using $\mu_0 = M\tilde{\eta}_0 = M/(1-A)$ one infers the inequality

$$\mu_2 - \mu_1 \le \frac{1-A}{2} + \frac{a(1-A)}{6M}. \tag{3.25}$$

Finally, application of Theorem 2.1 (ii) and a similar argument to that presented in the proof of Theorem 3.4 (ii)–(iii) completes the proof. □

4. **The single delay case.** In this section we improve the results of Theorems 3.4 (iii) and 3.5 for the case when $n = 1$ and $s_1 = 0$ by applying the Poincaré–Bendixson-type theorem and some monotonicity properties of (possible) periodic solutions by Mallet-Paret and Sell [17].

In this case our equation reads as

$$\dot{x}(t) = a\big[x(t) - x(t-1)\big] - g(x(t)), \tag{4.1}$$

with $a > 0$ and with the same assumptions on the feedback function g as before. Theorem 3.4 (iii) implies that the zero solution of equation (4.1) is globally asymptotically stable if $a < 1/2$. This can be slightly improved by applying Theorem 3.5 to obtain that a less than approximately 0.525 is sufficient for global stability.

We will shortly show that the region of global stability with respect to a can be risen up to at least $a < 0.61$. To prove this, we will need Lemma 4.1.

Define

$$m(a) = \begin{cases} a & \text{for } 0 < a \le \log 2, \\ a(e^a - 1) & \text{for } \log 2 < a \le \log 3, \\ 2a & \text{for } a > \log 3. \end{cases}$$

Lemma 4.1. *For all periodic solutions x of equation (4.1) the inequality*

$$\max_{t\in\mathbb{R}} |x(t)| \le h^{-1}(m(a))$$

holds.

Proof. Let x be a non-constant periodic solution with minimal period $T>0$. From Theorems 7.1 and 7.2 of [17] we know that x has the special symmetry $x(t+T/2) = -x(t)$, $t\in\mathbb{R}$. Without loss of generality we may assume $M = \max_{t\in[0,T]} x(t) = x(0) = x(T)$. By the special symmetry of x, $\max_{t\in\mathbb{R}} |x(t)| = M$ follows. The statement of the lemma is that

$$h(M) = \frac{g(M)}{M} \le m(a).$$

Lemma 3.1 gives the result for $a \ge \log 3$. Thus, in the sequel, we consider only the case $0 < a < \log 3$.

As x has a maximum at T, from equation (4.1) one obtains that

$$x(T-1) = M - \frac{g(M)}{a}.$$

In case $x(T-1)\ge 0$, this equality implies $h(M) \le a \le m(a)$.

Assume $x(T-1) < 0$. Let $t_0 \in (T-1,T)$ be minimal with $x(t_0)=0$, and let $t_1 \in (T-1,T)$ be maximal with $x(t_1)=0$. From equation (4.1) we obtain the inequality

$$\dot{x}(t) \le ax(t) + C$$

for the intervals $(T-1,t_0)$ and (t_1,T) with $C = aM + g(M)$ and $C = aM$, respectively. On these intervals

$$\frac{d}{dt}\left[\left(x(t) + \frac{C}{a}\right)e^{-at}\right] \le 0$$

is satisfied. Using $x(t_0) = x(t_1) = 0$ and $x(T-1) = M - g(M)/a$, integrations on the respective intervals yield

$$e^{a(t_0-(T-1))} \ge \frac{1}{2} + \frac{h(M)}{2a}$$

and

$$e^{a(T-t_1)} \ge 2.$$

It follows that

$$1 \ge (t_0 - (T-1)) + (T-t_1) \ge \frac{1}{a}\log\frac{a+h(M)}{2a} + \frac{1}{a}\log 2 = \frac{1}{a}\log\left(1 + \frac{h(M)}{a}\right).$$

Hence, using $a < \log 3$ as well, we obtain

$$h(M) \le a(e^a - 1) \le m(a).$$

This completes the proof. ☐

Remark 4.2. We note that Lemma 3.1 holds for any solution and gives an upper bound $|x(t)| \le h^{-1}(2a)$ as a special case for periodic solutions. However $m(a)$ is smaller than $2a$ if and only if $a < \log 3$, thus Lemma 4.1 yields a better upper bound for possible periodic solutions in that case. These lemmas are used to show that there exist no non-constant periodic solutions of equation (4.1).

Theorem 4.3. *If $a \in (0,1)$ satisfies*

$$\frac{m(a)a}{2(1-a)^2} + \frac{1-a}{2} + \frac{a(1-a)}{6m(a)} \le \frac{3}{2},$$

then the zero solution of equation (4.1) is globally asymptotically stable. In particular, it is globally asymptotically stable for $a \in (0, 0.61)$.

Proof. First we claim that to prove the convergence of all solutions of (4.1) to zero, it is sufficient to exclude non-constant periodic solutions.

Since the origin is the only equilibrium, its local stability (shown in Theorem 3.4) excludes homoclinic solutions. Finally, the Poincaré–Bendixson theorem by Mallet-Paret and Sell [17, Theorem 2.1] infers that the ω-limit set of a bounded solution of equation (4.1) is either a single non-constant periodic orbit, or else it is the unique equilibrium. Since all solutions are bounded as $t \to \infty$ by Lemma 3.1, the claim is proved.

We will use the notations introduced in the proof of Theorem 3.4. Assume that for some $\varphi \in C$, the solution x^φ is non-constant and periodic, and fix $x := x^\varphi$. Lemma 4.1 yields that $|\beta(t)| \le m(a)$ holds. By letting $\mu = m(a)\tilde{\eta}$, one obtains analogously to formulas (3.24) and (3.25) that

$$\mu_2 \le \frac{m(a)a}{2(1-a)^2} + \frac{1-a}{2} + \frac{a(1-a)}{6m(a)}. \tag{4.2}$$

Whence, applying Theorem 2.1 we may infer that $y(t) \to 0$ and consequently $x(t) \to 0$ as $t \to \infty$ provided that

$$\frac{m(a)a}{2(1-a)^2} + \frac{1-a}{2} + \frac{a(1-a)}{6m(a)} < \frac{3}{2}.$$

For $a < 0.61 < \log 2$ we have $m(a) = a$, thus the above inequality reduces to

$$\frac{a^2}{2(1-a)^2} + \frac{2(1-a)}{3} < \frac{3}{2}.$$

It is elementary to check that this holds for $a \in (0, 0.61)$, which completes the proof. ☐

5. **Discussion and examples.** We note that if feedback function g is fixed, and possibly some further restrictions (e.g. $n = 1$) are made, one may use the domain of attraction – obtained in Theorem 3.4 (ii) – to improve the results on global stability. This could be carried out for certain parameters by showing that, for a conveniently chosen ε, and for all $\varphi \in C$ with $\|\varphi\|_C < h^{-1}(m(a)) + \varepsilon$, solution x_t^φ gets inside of the above mentioned domain, which would imply global stability. This could presumably be done for $n = 1$, $s_1 = 0$ and for approximately $A \le 0.999$ by a computer aided proof similarly to that applied in [1] for the Wright equation.

In the following examples we will apply Theorem 3.4 to two special cases. Both of them seem relevant from the modelling point of view. In both cases $s_i = 0$ is assumed, that is, we compare previous values of the price only to the current price.

Example 5.1 (The case of "uniformly weighted memory")**.** Let

$$b_i = \frac{1}{n} \qquad r_i = \frac{i}{n}, \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \in \mathbb{N}_n. \tag{5.1}$$

Then Theorem 3.4 implies that the origin is locally stable if

$$L_0(n) := \frac{2n}{1+n} > a$$

and unstable if $a > L_0(n)$. For $n > 1$, the zero solution is globally asymptotically stable if

$$G_0(n) := 2\sqrt{6} \cdot \sqrt{\frac{2n^3 + n^2}{(n-1)^2(n+1)}} - \frac{6n}{n-1} > a.$$

Considering the fraction of the measures of the regions of global stability and local stability $G_0(n)/L_0(n)$, it can be easily proved that $G_0(n)/L_0(n)$ is strictly decreasing in n and converges to $-3 + 2\sqrt{3} \approx 0.464$, as n tends to infinity.

Example 5.2 (The case of "linearly fading memory")**.** In this case, the weights corresponding to the delayed terms decrease linearly with respect to the delay. From the modelling point of view it means that when considering the tendency of the price, i.e. when one compares the current price to previous ones, the more recent the price, the more impact it has on our feedback.

Accordingly, let

$$b_i = \frac{n+1-i}{\sum_{j=1}^{n} j} = \frac{2(n+1-i)}{n(n+1)}, \qquad r_i = \frac{i}{n} \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \in \mathbb{N}_n.$$

For $n > 1$, Theorem 3.4 yields that the origin is locally stable if

$$L_1(n) := \frac{3n}{2+n} > a,$$

unstable if $a > L_1(n)$, and globally asymptotically stable if

$$G_1(n) := 3\sqrt{6} \cdot \sqrt{\frac{n^3 + n^2}{(n-1)^2(n+2)}} - \frac{6n}{n-1} > a.$$

Applying an analogous argument to that presented in the previous example, one obtains that $G_1(n)/L_1(n)$ is strictly decreasing in n and converges to $-2 + 2\sqrt{6} \approx 0.449$, as n tends to infinity.

Let us modify the above example by letting $s_i > 0$.

Example 5.3. Let $n = 4$ and

$$b_i = \frac{5-i}{10}, \qquad r_i = \frac{i}{n}, \qquad s_i = r_i - \frac{1}{8} \quad \text{for} \quad i \in \mathbb{N}_4.$$

Then Theorem 3.4 yields that the zero solution is locally asymptotically stable if $a < 8$, unstable if $a > 8$, and globally asymptotically stable if $a < 4(\sqrt{7} - 1)/3 \approx 2.19$.

Remark 5.4. If we choose s_i close to r_i (e.g. by letting $r_i - s_i = \delta(n)$ for all $i \in \mathbb{N}_n$), let n tend to infinity and assume $\delta(n) \to 0$, then equation (1.1) approaches a neutral differential equation of the form

$$\frac{d}{dt}\left(x(t) - a\int_0^1 x(t-s)b(s)\,ds\right) = -g(x(t)), \qquad b(s) \geq 0 \quad \text{for} \quad s \in [0,1]. \tag{5.2}$$

The neutral equation with single delay

$$\frac{d}{dt}\big[x(t) - ax(t-1)\big] = -g(x(t)) \tag{5.3}$$

also appears naturally. The stability questions and the description of the dynamics in case zero is unstable are interesting open problems.

Remark 5.5. To prove analogous global stability results for equation (5.2) and (5.3), we only miss a boundedness result analogous to Lemma 3.1. All other steps of the proof can be carried out to get sufficient conditions for global stability.

However, if $s_i \approx r_i$, then unfortunately our theorems on global stability do not seem to be efficient. To see this, fix n, b_i, r_i and $s_i = r_i - \delta$ for all $i \in \mathbb{N}_n$. Then we get that the zero solution of equation (1.1) is locally asymptotically stable if $a < 1/\delta$. From Theorem 3.4 we can easily derive a formula $G(\delta)$, such that the zero solution is globally asymptotically stable if $a < G(\delta)$ and it is not hard to prove that $\sqrt{\delta}G(\delta) \to 1$ as $\delta \to 0$ (even if we let $n \to \infty$), meaning in particular that the fraction of the length of the regions (obtained by Theorem 3.4) of global and local stability, respectively tends to zero as $\delta \to 0$. Nevertheless, numerical simulations suggest that global stability is implied by local stability in this case, as well.

The reason for this ineffectiveness is that Lemma 3.1 is insensitive to the values $\delta = r_i - s_i$. To demonstrate this, let us consider the equation

$$\dot{x}(t) = a\left(\sum_{i=1}^{n} \frac{1}{n}\left[x\left(t - \tfrac{i-1}{n}\right) - x\left(t - \tfrac{i}{n}\right)\right]\right) - g(x(t)). \tag{5.4}$$

Indeed, this equation can be written much more simply, as follows:

$$\dot{x}(t) = \frac{a}{n}\left[x(t) - x(t-1)\right] - g(x(t)) \tag{5.5}$$

However, for $\limsup_{t\to\infty} |h(x(t))|$, Lemma 3.1 gives upper bounds $2a$ and $2a/n$, respectively. Similarly, if we apply Theorem 4.3, we need

$$n > \frac{1}{2}\left(a^2 + 2a\right) + \frac{1}{2}\sqrt{a^4 + 4a^3}$$

to prove global stability for equation (5.4), which implies that $a < \sqrt{n}$ is required. For equation (5.5), global stability is granted by $a < n/2$. Note that using the upper bound $2a/n$ one would get, by the argument presented in the proof of Theorem 3.4 (iii), that $2aB/n < (1-A)^2$ implies global stability (instead of requiring condition $2aB < (1-A)^2$), which reduces to condition $a < n/2$, as well.

We also note that the zero solution is locally asymptotically stable if $a < n$.

In order to handle this issue, one would need effective boundedness results on the derivative of the solutions.

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