INVERSE MONOIDS AND IMMERSIONS OF 2-COMPLEXES

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Abstract. It is well known that under mild conditions on a connected topological space $\mathcal{X}$, connected covers of $\mathcal{X}$ may be classified via conjugacy classes of subgroups of the fundamental group of $\mathcal{X}$. In this paper, we extend these results to the study of *immersions* into 2-dimensional CW-complexes. An immersion $f : \mathcal{D} \to \mathcal{C}$ between CW-complexes is a cellular map such that each point $y \in \mathcal{D}$ has a neighborhood U that is mapped homeomorphically onto $f(U)$ by f. In order to classify immersions into a 2-dimensional CW-complex $\mathcal{C}$, we need to replace the fundamental group of $\mathcal{C}$ by an appropriate inverse monoid. We show how conjugacy classes of the closed inverse submonoids of this inverse monoid may be used to classify connected immersions into the complex.

Dedicated to Stuart Margolis, on the occasion of his 60th birthday.

1. Introduction

It is well known that under mild restrictions on a topological space $\mathcal{X}$, connected covers of $\mathcal{X}$ may be classified via conjugacy classes of subgroups of the fundamental group of $\mathcal{X}$. For this fact, and for general background in topology, we refer to the book by Munkres [6].

In this paper we study connected immersions between finite-dimensional CW-complexes. A CW-complex $\mathcal{C}$ is obtained from a discrete set $\mathcal{C}^0$ (the 0-skeleton of $\mathcal{C}$) by iteratively attaching cells of dimension n to the $(n-1)$-skeleton $\mathcal{C}^{n-1}$ of $\mathcal{C}$ for $n \geq 1$. We refer the reader to Hatcher's text [1], for the precise definition and basic properties of CW-complexes. In particular a continuous map between CW-complexes is homotopic to a *cellular map* ([1], Theorem 4.8), that is a continuous function that maps cells to cells of the same or lower dimension, so we will regard maps between CW-complexes as cellular maps. A subcomplex of a CW-complex is a closed subspace that is a union of cells.

An *immersion* of a CW-complex $\mathcal{D}$ into a CW-complex $\mathcal{C}$ is a cellular map $f : \mathcal{D} \to \mathcal{C}$ such that each point $y \in \mathcal{D}$ has a neighborhood U which

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is mapped homeomorphically onto $f(U)$ by f. So f maps n-cells to n-cells. Thus if $\mathcal{C}$ is an n-dimensional CW-complex, then $\mathcal{D}$ is an m-dimensional CW-complex with $m \leq n$. Every subcomplex of an n-dimensional CW-complex $\mathcal{C}$ immerses into $\mathcal{C}$. Every covering space of a CW-complex $\mathcal{C}$ has a CW-complex structure, and every covering map is in particular an immersion.

We classify connected immersions into a 2-dimensional CW-complex $\mathcal{C}$ via conjugacy classes of closed inverse submonoids of a certain inverse monoid associated with $\mathcal{C}$. The closed inverse submonoids of this inverse monoid enable us to keep track of the 1-cells and 2-cells of $\mathcal{C}$ that lift under the immersion, in much the same way as the subgroups of the fundamental group of $\mathcal{C}$ enable us to encode coverings of $\mathcal{C}$. We provide an iterative process for constructing the immersion associated with a closed inverse submonoid of this inverse monoid. In many cases this iterative procedure provides an algorithm for constructing the immersion, in particular if the closed inverse submonoid is finitely generated and $\mathcal{C}$ has finitely many 2-cells.

Section 2 of the paper outlines basic material on presentations of inverse monoids that we will need to build an inverse monoid associated with a 2-complex $\mathcal{C}$. Section 3 describes an iterative procedure for constructing closed inverse submonoids of an inverse monoid from generators for the submonoid. The main results of the paper linking immersions over a 2-complex $\mathcal{C}$ and closed inverse submonoids of an inverse monoid associated with $\mathcal{C}$ are described in detail in Section 4 of the paper (Theorem 4.9, Theorem 4.10 and Theorem 4.11). We close in Section 5 with several examples illustrating the connections between immersions over 2-complexes and the associated closed inverse submonoids.

These results extend some work of Margolis and Meakin [5] that classifies connected immersions over graphs (1-dimensional CW-complexes) via closed inverse submonoids of free inverse monoids. Some related work may be found in the thesis of Williamson [13]. However, the notion of immersion in this paper is considerably more general than the notion of immersion between 2-complexes in [13].

2. X-graphs and inverse monoids

Let X be a set and X^{-1} a disjoint set in one-one correspondence with X via a map $x \to x^{-1}$ and define $(x^{-1})^{-1} = x$. We extend this to a map on $(X \cup X^{-1})^*$ by defining $(x_1 x_2 \cdots x_n)^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}$, giving $(X \cup X^{-1})^*$ the structure of the free monoid with involution on X. Throughout this paper by an *X-graph* (or just an *edge-labeled graph* if the labeling set X is understood) we mean a strongly connected digraph Γ with edges labeled over the set $X \cup X^{-1}$ such that the labeling is consistent with an involution: that is, there is an edge labeled $x \in X \cup X^{-1}$ from vertex v_1 to vertex v_2 if and only if there is an inverse edge labeled x^{-1} from v_2 to v_1. The initial vertex of an edge e will be denoted by $\alpha(e)$ and the terminal vertex by $\omega(e)$. If $X = \emptyset$, then we view Γ as the graph with one vertex and no edges.

The label on an edge e is denoted by $l(e) \in X \cup X^{-1}$. There is an evident notion of *path* in an X-graph. A path p with initial vertex v_1 and terminal vertex v_2 will be called a (v_1, v_2) path. The initial (resp. terminal) vertex of a path p will be denoted by $\alpha(p)$ (resp. $\omega(p)$). The label on the path $p = e_1 e_2 \dots e_k$ is the word $l(p) = l(e_1)l(e_2)\dots l(e_k) \in (X \cup X^{-1})^*$.

It is customary when sketching diagrams of such graphs to include just the positively labeled edges (with labels from X) in the diagram.

X-graphs occur frequently in the literature. For example, the bouquet of $|X|$ circles is the X-graph B_X with one vertex and one positively labeled edge labeled by x for each $x \in X$. The Cayley graph $\Gamma(G, X)$ of a group G relative to a set X of generators is an X-graph: its vertices are the elements of G and it has an edge labeled by x from g to gx for each $x \in X \cup X^{-1}$.

If we designate an initial vertex (state) α and a terminal vertex (state) β of Γ, then the birooted X-graph $\mathcal{A} = (\alpha, \Gamma, \beta)$ may be viewed as an automaton. See for example the book of Hopcroft and Ullman [2] for basic information about automata theory. The language accepted by this automaton is the subset $L(\mathcal{A})$ of $(X \cup X^{-1})^*$ consisting of the words in $(X \cup X^{-1})^*$ that label paths in Γ starting at α and ending at β. This automaton is called an *inverse automaton* if it is deterministic (and hence injective), i.e. if for each vertex v of Γ there is at most one edge with a given label starting or ending at v. A deterministic X-graph Γ determines an immersion of Γ into B_X, obtained by mapping an edge labeled by $x \in X \cup X^{-1}$ onto the corresponding edge in B_X.

Recall that an *inverse monoid* is a monoid M with the property that for each $a \in M$ there exists a unique element $a^{-1} \in M$ (the inverse of a) such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. Every inverse monoid may be embedded in a suitable symmetric inverse monoid $\mathrm{SIM}(X)$. Here $\mathrm{SIM}(X)$ is the monoid of all partial injective functions from X to X (i.e. bijections between subsets of X) with respect to the usual composition of partial maps. If Γ is a deterministic X-graph, then each letter $x \in X \cup X^{-1}$ determines a partial injection of the set V of vertices of Γ that maps a vertex v_1 to a vertex v_2 if there is an edge labeled by x from v_1 to v_2. The submonoid of $\mathrm{SIM}(V)$ generated by these partial maps is an inverse monoid, called the transition monoid of the graph Γ.

We refer the reader to the books by Lawson [3] or Petrich [8] for the basic theory of inverse monoids. In particular, the *natural partial order* on an inverse monoid M is defined by $a \leq b$ iff $a = eb$ for some idempotent $e \in M$, or equivalently, if $a = aa^{-1}b$. This corresponds to restriction of partial injective maps when $M = \mathrm{SIM}(X)$. See [3] or [8] for the important role that the natural partial order plays in the structure of inverse monoids. If we factor an inverse monoid M by the congruence generated by pairs of the form $(aa^{-1}, 1)$, $a \in M$, we obtain a group. This congruence is denoted by σ, and M/σ is in fact the *greatest group homomorphic image of* M.

Since inverse monoids form a variety of algebras (in the sense of universal algebra - i.e. an equationally defined class of algebras), free inverse monoids exist. We will denote the free inverse monoid on a set X by FIM(X). This is the quotient of $(X \cup X^{-1})^*$, the free monoid with involution, by the congruence that identifies $ww^{-1}w$ with w and $ww^{-1}uu^{-1}$ with $uu^{-1}ww^{-1}$ for all words $u, w \in (X \cup X^{-1})^*$. See [8] or [3] for much information about FIM(X). In particular, [8] and [3] provide an exposition of Munn's solution [7] to the word problem for FIM(X) via birooted edge-labeled trees called *Munn trees.*

In his thesis [11] and paper [12], Stephen initiated the theory of presentations of inverse monoids by extending Munn's results about free inverse monoids to arbitrary presentations of inverse monoids. Here, a presentation of an inverse monoid M, denoted $M = Inv\langle X \mid u_i = v_i, i \in I\rangle$ (where the u_i and v_i are words in $(X \cup X^{-1})^*$) is the quotient of FIM(X) obtained by imposing the relations $u_i = v_i$ in the usual way. In order to study the word problem for such presentations, Stephen considers the *Schützenberger graph* $S\Gamma(M, X, w)$ (or simply $S\Gamma(w)$ if the presentation is understood) of each word $w \in (X \cup X^{-1})^*$. The Schützenberger graph of w is the restriction of the Cayley graph of M to the $\mathcal{R}$-class of w in M. That is, the vertices of $S\Gamma(w)$ are the elements $u \in M$ such that $uu^{-1} = ww^{-1}$ in M; there is an edge labeled by $x \in X \cup X^{-1}$ from u to v if $uu^{-1} = vv^{-1} = ww^{-1}$ and $ux = v$ in M. (Here, for simplicity of notation, we are using the same notation for a word $w \in (X \cup X^{-1})^*$ and its natural image in M; the context guarantees that no confusion should occur.)

The Schützenberger graphs of M are just the strongly connected components of the Cayley graph of M relative to the set X of generators for M. Of course, if G is a group, then it has just one Schützenberger graph, which is the Cayley graph $\Gamma(G, X)$. The *Schützenberger automaton* $S\mathcal{A}(w)$ of a word $w \in (X \cup X^{-1})^*$ is defined to be the birooted X-graph $S\mathcal{A}(w) = (ww^{-1}, S\Gamma(w), w)$. Thus $S\mathcal{A}(w)$ is an inverse automaton. In his paper [12], Stephen proves the following result.

Theorem 2.1. *Let $M = Inv\langle X \mid u_i = v_i, i \in I\rangle$ be a presentation of an inverse monoid. Then*

(a) For each word $u \in (X \cup X^{-1})^*$, the language accepted by the Schützenberger automaton $S\mathcal{A}(u)$ is the set of all words $w \in (X \cup X^{-1})^*$ such that $u \leq w$ in the natural partial order on M.

(b) $u = w$ in M iff $u \in L(S\mathcal{A}(w))$ and $w \in L(S\mathcal{A}(u))$.

(c) The word problem for M is decidable iff there is an algorithm for deciding membership in $L(S\mathcal{A}(w))$ for each word $w \in (X \cup X^{-1})^*$.

3. Closed inverse submonoids of inverse monoids

For each subset N of an inverse monoid M, we denote by N^ω the set of all elements $m \in M$ such that $m \geq n$ for some $n \in N$. The subset N of M is called *closed* if $N = N^\omega$. Thus the image in M of the language accepted by a Schützenberger automaton $S\mathcal{A}(u)$ of a word u relative to a presentation of M is a closed subset of M.

Closed inverse submonoids of an inverse monoid M arise naturally in the representation theory of M by partial injections on a set [9]. An inverse monoid M acts (on the right) by injective partial functions on a set Q if there is a homomorphism from M to $\mathrm{SIM}(Q)$. Denote by qm the image of q under the action of m if q is in the domain of the action by m. The following basic fact is well known (see [9]).

Proposition 3.1. *If M acts on Q by injective partial functions, then for every $q \in Q$, $Stab(q) = \{m \in M : qm = q\}$ is a closed inverse submonoid of M.*

Conversely, given a closed inverse submonoid H of M, we can construct a transitive representation of M as follows. A subset of M of the form $(Hm)^\omega$ where $mm^{-1} \in H$ is called a *right ω-coset* of H. Let X_H denote the set of right ω-cosets of H. If $m \in M$, define an action on X_H by $Y.m = (Ym)^\omega$ if $(Ym)^\omega \in X_H$ and undefined otherwise. This defines a transitive action of M on X_H. Conversely, if M acts transitively on Q, then this action is equivalent in the obvious sense to the action of M on the right ω-cosets of $Stab(q)$ in M for any $q \in Q$. See [9] or [8] for details.

The *ω-coset graph* $\Gamma_{(H,X)}$ (or just Γ_H if X is understood) of a closed inverse submonoid H of an X-generated inverse monoid M is constructed as follows. The set of vertices of Γ_H is X_H and there is an edge labeled by $x \in X \cup X^{-1}$ from $(Ha)^\omega$ to $(Hb)^\omega$ if $(Hb)^\omega = (Hax)^\omega$. Then Γ_H is a deterministic X-graph. The birooted X-graph (H, Γ_H, H) is called the *ω-coset automaton* of H. The language accepted by this automaton is H (or more precisely the set of words $w \in (X \cup X^{-1})^*$ whose natural image in M is in H). Clearly, if G is a group generated by X, then Γ_H coincides with the coset graph of the subgroup H of G.

Let M be an inverse monoid given by a presentation $M = Inv\langle X \mid u_i = v_i,\ i \in I\rangle$, and let Y be a subset of $(X \cup X^{-1})^*$. Let $\langle Y\rangle^\omega$ denote the closed inverse submonoid of M generated by the natural image of Y in M. We now provide an iterative construction of the ω-coset automaton of $\langle Y\rangle^\omega$. The construction extends the well-known construction of Stallings [10] of a finite graph associated with each finitely generated subgroup of a free group. See also [5] for the automata-theoretic point of view on Stallings' construction.

In [11], Stephen shows that the class of all birooted X-graphs forms a cocomplete category, and hence directed systems of birooted X-graphs have direct limits in this category. See Mac Lane [4] for background in category theory. Morphisms in this category are graph morphisms that take edges to edges and preserve edge labelings and initial (terminal) roots.

Given a finite presentation $M = Inv\langle X \mid u_i = v_i,\ i = 1, \ldots, n\rangle$ of an inverse monoid, we consider two types of operations on X-graphs (or birooted

X-graphs), namely edge foldings (in the sense of Stallings [10]) and expansions. If e_1 and e_2 are two edges with the same label and the same initial or terminal vertex, then an edge folding identifies these edges (an edge folding is called a "determination" in Stephen's terminology [12, 11]). Clearly, each edge folding of an X-graph results in another X-graph. If Γ is an X-graph with two vertices a and b and a path from a to b labeled by one side (say u_i) of one of the defining relations $u_i = v_i$ of the monoid M, but no path labeled by the other side, then we expand Γ to create another X-graph Δ by adding a new path from a to b labeled by the other side (v_i) of the relation. One of the results of Stephen [12] (Lemma 4.7) is that these processes are confluent.

The set of birooted X-graphs obtained by applying successive expansions and edge foldings to a birooted X-graph $\mathcal{A} = (\alpha, \Gamma, \beta)$ forms a directed system in the category of birooted X-graphs. The direct limit (colimit) of this system is an inverse automaton that we will denote by $\mathcal{A}^\omega$. This automaton is *complete*, in the sense that no edge foldings or expansions may be applied. Of course if finitely many applications of edge foldings and expansions transform $\mathcal{A}$ into a complete automaton $\mathcal{B}$, then $\mathcal{B} = \mathcal{A}^\omega$.

Any automaton $\mathcal{A}'$ obtained from $\mathcal{A}$ by applying successive expansions and edge foldings is called an *approximate automaton* of $\mathcal{A}^\omega$.

Theorem 3.2. *Let $M = Inv\langle X : u_i = v_i,\ i = 1, \dots n\rangle$ be a finitely presented inverse monoid. If $\mathcal{A}$ is a birooted X-graph (i.e. automaton) accepting the language $L \subseteq (X \cup X^{-1})^*$, then the language accepted by the direct limit automaton $\mathcal{A}^\omega$ is $L^\omega = \{w \in (X \cup X^{-1})^* : w \geq s \text{ in } M \text{ for some } s \in L\}$.*

Proof. The proof follows by a modification of the proof of Theorem 4.12 of Stephen [11], where it is proved that the Schützenberger automaton $S\mathcal{A}(s)$ of a word $s \in (X \cup X^{-1})^*$ is the colimit $Lin(s)^\omega$, where $Lin(s)$ is the "linear automaton" of s. See also Theorem 5.10 of [12] for a closely related result. The basic idea of the proof is that application of an expansion to some automaton $\mathcal{A}'$ just augments the language $L(\mathcal{A}')$ by words that are equal in M to words in $L(\mathcal{A}')$, while an edge folding augments this language by words that are greater than or equal in M to words in $L(\mathcal{A}')$. We provide some more detail below.

Let $\mathcal{A}^\omega = (\alpha^\omega, \Gamma^\omega, \beta^\omega)$. If $w \in L(\mathcal{A}^\omega)$, then the path labeled by w lifts to a path labeled by w from α' to β' in some approximate automaton $\mathcal{A}' = (\alpha', \Gamma', \beta')$ of $\mathcal{A}^\omega$ by Theorem 2.11 of [11]. This implies that $w \in L(\mathcal{A}')$. But it follows as in the proof of Theorem 5.5 and Lemma 5.6 of [12] that if $\mathcal{A}'$ is an approximate automaton of $\mathcal{A}^\omega$, then $L \subseteq L(\mathcal{A}') \subseteq L^\omega$. Hence $L(\mathcal{A}^\omega) \subseteq L^\omega$.

Conversely, if $w \geq s$ for some $s \in L$, then by Theorem 2.1 above, $w \in L(S\mathcal{A}(s))$. So w is in the language accepted by some approximate automaton $\mathcal{B}'$ of $S\mathcal{A}(s)$ by Theorem 5.12 of [12]. The automaton $\mathcal{B}'$ is obtained from the linear automaton of s by a finite number of edge foldings and expansions. Since $s \in L = L(\mathcal{A})$, we may apply the same sequence of edge foldings and expansions to $\mathcal{A}$ to obtain an approximate automaton $\mathcal{A}'$ of $\mathcal{A}^\omega$, and hence w is in the language accepted by this approximate automaton $\mathcal{A}'$. Since there is a morphism from $\mathcal{A}'$ to $\mathcal{A}^\omega$ by definition of the colimit, it follows from Lemma 2.4 of [12] that $w \in L(\mathcal{A}^\omega)$.

□

We now apply Stephen's iterative process as described above to construct the closed inverse submonoid of M generated by a subset Y of $(X \cup X^{-1})^*$. Start with the "flower automaton" $\mathcal{F}(Y)$. This is the birooted X-graph with one distinguished state 1 designated as initial and terminal state and a closed path based at 1 labeled by the word y for each $y \in Y$. (This is a finite automaton if Y is finite of course.) Now successively apply edge foldings and expansions to $\mathcal{F}(Y)$ to obtain the limit automaton $\mathcal{F}(Y)^\omega$.

Theorem 3.3. *Let $M = Inv\langle X \mid u_i = v_i,\ i = 1, \ldots, n\rangle$ be a finitely presented inverse monoid, let Y be a subset of $(X \cup X^{-1})^*$, and construct the inverse automaton $\mathcal{F}(Y)^\omega$ obtained from the flower automaton $\mathcal{F}(Y)$ by iteratively applying the processes of edge foldings and expansions as described above. Then the language $L(\mathcal{F}(Y)^\omega)$ accepted by this automaton is $\{w \in (X \cup X^{-1})^* : w \in \langle Y\rangle^\omega\}$, and $\mathcal{F}(Y)^\omega$ is the ω-coset automaton of the closed inverse submonoid $\langle Y\rangle^\omega$ of M. Thus the membership problem for the closed inverse submonoid $\langle Y\rangle^\omega$ is decidable if and only if there is an algorithm for deciding membership in the language $L(\mathcal{F}(Y)^\omega)$.*

Proof. The fact that $L(\mathcal{F}(Y)^\omega) = \{w \in (X \cup X^{-1})^* : w \in \langle Y\rangle^\omega\}$ is immediate from Theorem 3.2 above. Hence the automaton $\mathcal{F}(Y)^\omega$ and the ω-coset automaton of the closed inverse submonoid $\langle Y\rangle^\omega$ are birooted deterministic X-graphs that accept the same language. But it is routine to see that any two birooted (connected) deterministic X-graphs that accept the same language are isomorphic as birooted X-graphs.

□

This theorem shows in particular that the membership problem for the finitely generated closed inverse submonoid $\langle Y\rangle^\omega$ of M is decidable if the iterative procedure described above for constructing $\mathcal{F}(Y)^\omega$ terminates after a finite number of edge foldings and expansions, since in that case $\mathcal{F}(Y)^\omega$ is a finite inverse automaton.

We remark that if M is the free group $FG(X)$, viewed as an inverse monoid with presentation $FG(X) = Inv\langle X \mid xx^{-1} = x^{-1}x = 1\rangle$, then finitely generated closed inverse submonoids of M coincide with finitely generated subgroups of $FG(X)$, and the construction of $\mathcal{F}(Y)^\omega$ from a finite set Y of words produces the coset graph of the subgroup. The core of this graph is, of course, the Stallings graph (automaton) of the corresponding subgroup [10], obtained by pruning all trees off the coset graph; the reduced words accepted by the coset automaton (or by the Stallings automaton) coincide with the reduced words in the subgroup.

4. Immersions of 2-complexes

Recall the following definition [1] of a finite dimensional CW-complex $\mathcal{C}$:

(1) Start with a discrete set $\mathcal{C}^0$, the 0-cells of $\mathcal{C}$.
(2) Inductively, form the n-skeleton $\mathcal{C}^n$ from $\mathcal{C}^{n-1}$ by attaching n-cells C^n_α via maps $\varphi_\alpha \colon S^{n-1} \to \mathcal{C}^{n-1}$. This means that $\mathcal{C}^n$ is the quotient space of $\mathcal{C}^{n-1} \dot\cup_\alpha D^n_\alpha$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. The cell C^n_α is a homeomorphic image of $D^n_\alpha - \partial D^n_\alpha$ under the quotient map.
(3) Stop the inductive process after a finite number of steps to obtain a finite dimensional CW-complex $\mathcal{C}$.

The dimension of the complex is the largest dimension of one of its cells. We denote the set of n-cells of $\mathcal{C}$ by $\mathcal{C}^{(n)}$. Throughout the remainder of this paper, by a 2-*complex* we mean a connected CW-complex of dimension less than or equal to 2. The 1-skeleton of a 2-complex is an undirected graph, but it is more convenient for our purposes to regard it as a digraph, with two oppositely directed edges for each undirected edge.

An immersion between CW-complexes always maps n-cells to n-cells, and the restriction of an immersion to a subcomplex is also an immersion. It is easy to see that a cellular map $f \colon \mathcal{C} \to \mathcal{D}$ is an immersion if and only if it is locally injective at the 0-cells, that is, each 0-cell $v \in \mathcal{C}^{(0)}$ has a neighborhood that is homeomorphic to its image under f. For graphs, this definiton of immersions is equivalent to Stallings' definition in [10].

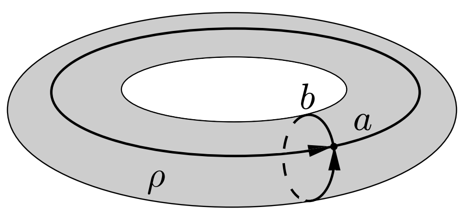
In this section, we classify immersions over 2-complexes using inverse monoids. Our results extend the results of [5], where the authors classify immersions over graphs by keeping track of which closed paths lift to closed paths. This is essentially what we do in this paper, with the added information about when 2-cells lift. It will be convenient to label the 1-cells over some set $X \cup X^{-1}$ and the 2-cells over some disjoint set P as described below. With every 2-cell, we associate a distinguished vertex (root) and walk on its boundary, consistent with the labeling. We first describe the process of choosing a root and boundary walk for 2-cells.

Let $\mathcal{C}$ be a 2-complex and let C be a 2-cell of $\mathcal{C}$ with the attaching map $\varphi_C : S^1 \to \mathcal{C}^1$. Choose a point x_0 on the circle S^1 in such a way that φ_C maps x_0 to a 0-cell of $\mathcal{C}$. Consider S^1 as the interval $[0,1]$ with its endpoints glued together and identified with x_0. Consider the closed path (in the topological sense) $p_C \colon [0,1] \to \mathcal{C}$, with $p_C|_{(0,1)} = \varphi_C|_{(0,1)}$, $p_C(0) = p_C(1) = \varphi_C(x_0)$. Since the closure of every 2-cell meets only finitely many 0-cells or 1-cells ([1], Proposition A.1), the image of this path corresponds to a closed path in $\mathcal{C}^1$ (in the graph theoretic sense) that we call the boundary walk of C: we denote it by $bw(C)$. We allow for the possibility that $bw(C)$ might have no edges. We call the 0-cell $\varphi_C(x_0)$ the *base* or *root* of the 2-cell C and of the closed path $bw(C)$ and denote it by $\alpha(C)$.

Let B_X be the bouquet of $|X|$ circles. We build a 2-complex $B_{X,P}$ by attaching labeled 2-cells to B_X with labels coming from a set P (which we assume to be disjoint from $X \cup X^{-1}$), and with a specified boundary walk for each 2-cell, as described above. The labeling is chosen so that different 2-cells in $B_{X,P}$ have different labels (even if they have the same boundary in B_X). We allow for the possibility that $P = \emptyset$ or that $X = \emptyset$. Denote the label of a 2-cell C in $B_{X,P}$ by $l(C) \in P$.

Every 2-complex $\mathcal{C}$ admits an immersion $f\colon \mathcal{C} \to B_{X,P}$ for some sets X and P: one could choose X as an index set for the (undirected) edges of $\mathcal{C}$ and P as an index set for the 2-cells for example, but we would normally choose smaller sets X and P if possible. This mapping f induces a labeling on $\mathcal{C}$ by giving each 1-cell or 2-cell in $\mathcal{C}$ the label of its image in $B_{X,P}$ under f. From now on, by a *labeled 2-complex*, we mean a labeling induced by an immersion into some complex $B_{X,P}$. The 1-skeleton of a 2-complex $\mathcal{C}$ labeled this way is a deterministic X-graph that immerses via the restriction of f into B_X; 2-cells of $\mathcal{C}$ have the same label in P if they map to the same 2-cell in $B_{X,P}$.

Example 4.1. Let $X = \{a, b\}$, $P = \{\rho\}$, and let $B_{X,P}$ be the 2-complex with one 2-cell C (labeled by ρ) corresponding to the attaching map that takes S^1 to the closed path labeled by $aba^{-1}b^{-1}$. Then $l(bw(C)) = aba^{-1}b^{-1}$, and $B_{X,P}$ is the presentation complex of the free abelian group of rank 2, and is homeomorphic to the torus. We could have chosen any cyclic conjugate of $aba^{-1}b^{-1}$ or its inverse and obtained the same 2-complex, but with a different boundary walk.



If $\mathcal{C}, \mathcal{D}$ are 2-complexes and $f\colon \mathcal{C} \to \mathcal{D}$ an immersion, and $\mathcal{D}$ is labeled by an immersion $g\colon \mathcal{D} \to B_{X,P}$, then $g \circ f\colon \mathcal{C} \to B_{X,P}$ is an immersion, and it induces a labeling on $\mathcal{C}$ that is respected by f; that is, $l(C) = l(f(C))$ and $l(e) = l(f(e))$ for all 2-cells C and 1-cells e in $\mathcal{C}$. Therefore we may, without loss of generality, assume that immersions respect the labeling.

Lemma 4.2. *Let $\mathcal{C}, \mathcal{D}$ be labeled 2-complexes and let $f\colon \mathcal{C} \to \mathcal{D}$ be an immersion that respects the labeling. For an arbitrary 2-cell C of $\mathcal{C}$, $f(\alpha(C)) = \alpha(f(C))$ and $f(bw(C)) = bw(f(C))$. Furthermore, $bw(C)$ is uniquely determined by f and $bw(f(C))$.*

Proof. Let $\varphi_C\colon S^1 \to \mathcal{C}$ and $\varphi_{f(C)}\colon S^1 \to \mathcal{D}$ be the attaching maps corresponding to C and $f(C)$. If $\varphi_{f(C)}$ maps the circle to a point, then so does φ_C, and our statement trivially holds. For the remainder of the proof, we suppose that is not the case.

We first prove that $f \circ \varphi_C = \varphi_{f(C)}$. Consider $\mathcal{C}$ as $\mathcal{C}^1 \mathbin{\dot\cup} D^2_\alpha$ with identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^2_\alpha$. Thus the closure of our 2-cell C is a quotient of $\varphi_C(S^1) \dot\cup D^2$ by identifying the points $x \sim \varphi_C(x)$ for $x \in \partial D^2$. Since f is an immersion, then $f|_C$ is a homeomorphism, and so $f(\overline{C})$ is $f(\varphi_C(S^1)) \dot\cup D^2$ with the identifications $x \sim f(\varphi_C(x))$ for $x \in \partial D^2$. But $f(\overline{C})$ is the closure of the 2-cell $f(C)$ in $\mathcal{D}$, so it is also $\varphi_{f(C)}(S^1) \dot\cup D^2$ with identifications $x \sim \varphi_{f(C)}(x)$ for $x \in \partial D^2$. That is, the points $x \in \partial D^2$ and $y \in \mathcal{D}^1$ are identified on one hand if and only if $y = f(\varphi_C(x))$, on the other hand, if and only if $y = \varphi_{f(C)}(x)$, which yields that $f(\varphi_C(x)) = \varphi_{f(C)}(x)$ for all $x \in S^1$.

Regard S^1 as $[0,1]$ with its endpoints glued together to x_0 in such a way that $\varphi_{f(C)}(x_0) = \alpha(f(C)) \in \mathcal{D}^0$. Then for the paths corresponding to the attaching maps, we have $f \circ p_C = p_{f(C)}$, that is, $f(bw(C)) = bw(f(C))$. In particular, $\alpha(f(C)) = f(\alpha(C))$. Since f respects the labeling, this also yields $l(bw(C)) = l(f(bw(C))) = l(bw(f(C)))$.

To prove the uniqueness of $bw(C)$, all we need to prove is that $\varphi_C(x_0)$ is uniquely determined, as the label of the boundary walk of C and the root $\alpha(C) = \varphi_C(x_0)$ determine $bw(C)$ uniquely. Take a neighborhood N of x_0 in the disk D^2. Denote the images of N in $\mathcal{C}$ and $\mathcal{D}$ by $N_\mathcal{C}$ and $N_\mathcal{D}$ respectively after the identifications $x \sim \varphi_C(x)$ and $x \sim \varphi_{f(C)}(x)$ for $x \in \partial D^2$. Naturally, $\varphi_C(x_0) \in N_\mathcal{C}$ and $\varphi_{f(C)}(x_0) \in N_\mathcal{D}$. Since $f|_C$ is a homeomorphism, it takes $int(N_\mathcal{C})$ to $int(N_\mathcal{D})$ homeomorphically, and therefore takes $N_\mathcal{C}$ to $N_\mathcal{D}$. If N is small enough, there is only one preimage of $\varphi_{f(C)}(x_0)$ in $N_\mathcal{D}$, and that is $\varphi_C(x_0) = \alpha(f(C))$.

□

We point out that the second part of the theorem is non-trivial when $l(bw(C)) = x^n$ for some word x, in which case there may be more than one vertex on $bw(C)$ from which $l(bw(C))$ can be read.

We have just seen that for an immersion $f\colon \mathcal{C} \to \mathcal{D}$ and for any 2-cell $C \in \mathcal{C}^2$, we have $l(bw(C)) = l(bw(f(C)))$. In particular, when $\mathcal{D} = B_{X,P}$, then for any 2-cells $C_1, C_2 \in \mathcal{C}$ with $l(C_1) = l(C_2) = \rho$, we have $l(bw(C_1)) = l(bw(C_2))$: this common label (called the "boundary label" of ρ) will often be denoted by $bl(\rho)$. Thus $bl(\rho) \in (X \cup X^{-1})^*$.

As in covering space theory, paths of a 2-complex $\mathcal{C}$ are our tools to classify immersions over $\mathcal{C}$. The point of the following construction is to generalize the notion of graph-theoretic paths to 2-complexes.

We associate an edge-labeled graph $\Gamma_\mathcal{C}$ with the 2-complex $\mathcal{C}$ as follows:

$$V(\Gamma_\mathcal{C}) = \mathcal{C}^{(0)}$$

$$E(\Gamma_\mathcal{C}) = \mathcal{C}^{(1)} \cup \{e_C : C \in \mathcal{C}^{(2)}\},$$

where e_C denotes a loop based at $\alpha(C)$ and labeled by $l(C)$. Thus the edges in $\mathcal{C}^{(1)}$ are labeled over $X \cup X^{-1}$ and the edges of the form e_C (for C a 2-cell) are labeled over P. Since an edge labeled by $\rho \in P$ is always a loop, we may identify P with P^{-1} and regard $\Gamma_\mathcal{C}$ as an $X \cup P$-graph in the sense of section 2 of the paper.

Lemma 4.3. *For any labeled complex $\mathcal{C}$, the labeled graph $\Gamma_{\mathcal{C}}$ is deterministic.*

Proof. Let $f\colon \mathcal{C} \to B_{X,P}$ be the immersion inducing the labeling on $\mathcal{C}$. The subgraph corresponding to the 1-skeleton of $\mathcal{C}$ is deterministic, as its labeling is induced by the immersion $f|_{\mathcal{C}^1}$ over B_X (see [5]). Therefore we only need to check if different edges labeled by ρ are based at different vertices, that is, if different 2-cells in $\mathcal{C}$ labeled by ρ have different roots. Denote the set of ρ-labeled 2-cells of $\mathcal{C}$ by $\{C_\alpha : \alpha \in A\}$, and the corresponding attaching maps $\varphi_\alpha\colon S^1 \to \mathcal{C}$ for $\alpha \in A$. Again, regard S^1 as the unit interval with its endpoints identified with x_0, and let N be a neighborhood of x_0 in the disk D^2. Let N_α denote the image of N induced by the attaching map φ_α. Since f maps all ρ-labeled 2-cells to one cell, $f(N_\alpha) = f(N_{\alpha'})$ for all $\alpha, \alpha' \in A$ and for any neighborhood N. Since f is locally injective, this implies that the $f(N_\alpha)$ $(\alpha \in A)$ are pairwise disjoint, therefore the roots $\varphi_\alpha(x_0)$ of the 2-cells are all different. □

The paths in the graph $\Gamma_{\mathcal{C}}$ will play the role of paths in $\mathcal{C}$ in our paper. One can think of these paths as paths in $\mathcal{C}^1$ (in the graph-theoretic sense) extended with the possibility of "stepping" on a 2-cell at its basepoint, thus including it in the path.

Lemma 4.4. *For two labeled 2-complexes $\mathcal{C}$ and $\mathcal{D}$ there exists an immersion $\mathcal{C} \to \mathcal{D}$ (that respects the labeling) if and only if there is an immersion $\Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{D}}$ (that respects the labeling).*

Proof. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an immersion that respects the labeling. Regarding $\mathcal{C}^1$ as a subgraph of $\Gamma_{\mathcal{C}}$, we define $g\colon \Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{D}}$ to be f on $\mathcal{C}^1$, and for an edge e_C corresponding to a 2-cell C, let $g(e_C) = e_{f(C)}$. It is easy to see that if f is locally injective at the vertices, so is g, hence an immersion. For the converse, suppose $g\colon \Gamma_{\mathcal{C}} \to \Gamma_{\mathcal{D}}$ is an immersion that respects the labeling. Define $f\colon \mathcal{C} \to \mathcal{D}$ to be g on $\mathcal{C}^1$, and for a 2-cell C of $\mathcal{C}$, let $f(C)$ be the 2-cell for which $g(e_C) = e_{f(C)}$ holds. Note that if g is an immersion, then so is $f|_{\mathcal{C}^1}$. Suppose that $f|_{\mathcal{C}^1}$ is an immersion, but f is not. Then there is a vertex v with two 2-cells C_1 and C_2 with $v \in \partial C_1 \cap \partial C_2$ that f identifies around v, that is, for any neighborhood N of v, $f(C_1 \cap N) = f(C_2 \cap N)$. Since f is locally injective on to the 1-skeleton — in paticular, on $bw(C_1)$ and $bw(C_2)$ —, this can only happen if C_1 and C_2 have the same boundary walk, so e_{C_1} and e_{C_2} are based at the same vertex. But since $g(e_{C_1}) = g(e_{C_2})$, that contradicts our assumption. Hence f is an immersion, and it respects the labeling.

□

We are now ready to define the inverse monoid which will play the role of the fundamental group. Let $\mathcal{C}$ be a labeled 2-complex with the edges (1-cells) labeled over the set $X \cup X^{-1}$ and the 2-cells labeled over the set P, consistent with an immersion over some complex $B_{X,P}$. We define a partial action of the inverse monoid

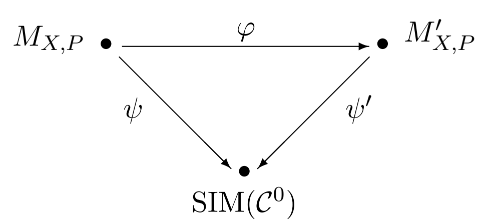
$$M_{X,P} = Inv \left\langle X \cup P \mid \rho^2 = \rho,\ \rho \leq bl(\rho) : \rho \in P \right\rangle$$

on the vertices (0-cells) of $\mathcal{C}$. For $x \in X \cup X^{-1}$, let $vx = w$ if there is an edge labeled x from v to w, and vx is undefined otherwise. For $\rho \in P$, let $v\rho = v$ if there is a 2-cell labeled ρ based at v, and $v\rho$ is undefined otherwise. This action extends to an action of $\mathrm{FIM}(X)$ in a natural way. Since the action of ρ is always idempotent, and is always a restriction of the action of $bl(\rho)$, it also extends to an action of $M_{X,P}$. Note that the action of $M_{X,P}$ on the vertices of $\mathcal{C}$ corresponds to the usual partial action induced by edges in $\Gamma_{\mathcal{C}}$. We will denote the stabilizer of a vertex $v \in \mathcal{C}^0$ under this action by $M_{X,P}$ by $Stab(\mathcal{C}, v)$.

Proposition 4.5. *The inverse monoid $M_{X,P}$ and its action on $\mathcal{C}^0$ do not depend on the boundary walks and roots chosen for the 2-cells.*

Proof. Suppose we chose different roots and boundary walks for the 2-cells of $\mathcal{C}$, and let $bl'(\rho)$ denote the new boundary label corresponding to the 2-cells labeleld by ρ. The inverse monoid corresponding to these boundary walks is $M'_{X,P} = \langle X, P \mid \rho^2 = \rho,\ \rho \leq bl'(\rho)\rangle$. The word $bl'(\rho)$ is a cyclic conjugate of $bl(\rho)$ or $(bl(\rho))^{-1}$. Since $\rho \leq bl(\rho)$ holds if and only if $\rho \leq (bl(\rho))^{-1}$, reversing the boundary walk does not effect $M_{X,P}$, so we may assume that $bl'(\rho)$ is a cyclic conjugate of $bl(\rho)$. Suppose $bl(\rho) = p_\rho q_\rho$, $bl'(\rho) = q_\rho p_\rho$. Note that $p_\rho \rho p_\rho^{-1}$ is an idempotent of $M'_{X,P}$, since ρ is an idempotent of $M'_{X,P}$. Also, since $\rho \leq q_\rho p_\rho$ in $M'_{X,P}$, it follows that $p_\rho \rho p_\rho^{-1} = p_\rho \rho q_\rho p_\rho p_\rho^{-1} \leq p_\rho \rho q_\rho \leq p_\rho q_\rho$ in $M'_{X,P}$. Hence the map $x \mapsto x, \rho \mapsto p_\rho \rho p_\rho^{-1}$, where $x \in X$, $\rho \in P$, extends to a well-defined morphism $\varphi\colon M_{X,P} \to M'_{X,P}$. Also, for $\rho \in M'_{X,P}$, $p_\rho^{-1}(p_\rho \rho p_\rho^{-1})p_\rho = \rho$, so φ is surjective; and it is injective since it is injective on the generators of $M_{X,P}$, so it is an isomorphism.

Moreover, denoting the maps from $M_{X,P}$ and $M'_{X,P}$ to $\mathrm{SIM}(\mathcal{C}^0)$ corresponding to their actions on the vertices by ψ and ψ' respectively, the following diagram commutes:



The commutativity of the diagram follows directly from the facts that φ is the identity on X, and that for $\rho \in M_{X,P}$, the action of $\varphi(\rho)$ on the vertices is the same as that of ρ. $\square$

We now define an inverse category of paths on $\Gamma_{\mathcal{C}}$. A category C is called *inverse* if for every morphism p in C there is a unique inverse morphism p^{-1}

such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$. The loop monoids $L(C, v)$ of an inverse category, that is, the set of all morphisms from v to v, where v is an arbitrary vertex, form an inverse monoid. The free inverse category $\mathrm{FIC}(\Gamma)$ on a graph Γ is the free category on Γ factored by the congruence induced by relations of the form $p = pp^{-1}p$, $p^{-1} = p^{-1}pp^{-1}$, and $pp^{-1}qq^{-1} = qq^{-1}pp^{-1}$ for all paths p, q in Γ with $\alpha(p) = \alpha(q)$.

Now let $\sim$ be the congruence on the free category on $\Gamma_{\mathcal{C}}$ generated by the relations defining $\mathrm{FIC}(\Gamma_{\mathcal{C}})$ and the ones of the form $p^2 = p$ and $p = pq$, where p, q are coterminal paths with $l(p) \in P$ and $l(q) = bl(l(p))$. The inverse category $\mathrm{IC}(\mathcal{C})$ corresponding to the 2-complex $\mathcal{C}$ is obtained by factoring the free category on $\Gamma_{\mathcal{C}}$ by $\sim$. The loop monoids $L(\mathrm{IC}(\mathcal{C}), v)$ consist of $\sim$-classes of (v, v)-paths, these monoids play the role of the fundamental group, and $\mathrm{IC}(\mathcal{C})$ plays the role of the fundamental groupoid in the classification of immersions. We will denote $L(\mathrm{IC}(\mathcal{C}), v)$ by $L(\mathcal{C}, v)$ for brevity.

Proposition 4.6. *For any vertex v in a connected 2-complex $\mathcal{C}$, the greatest group homomorphic image of $L(\mathcal{C}, v)$ is the fundamental group of $\mathcal{C}$.*

Proof. The proof follows from the fact that the fundamental groupoid of $\mathcal{C}$ is $\mathrm{IC}(\mathcal{C})$ factored by the congruence generated by relations of the form $xx^{-1} = \mathrm{id}_{\alpha(x)}$ for any morphism x (which implies $bw(C) = \mathrm{id}_{\alpha(C)}$ for any 2-cell C). Hence $L(\mathcal{C}, v)/\sigma = \pi_1(\mathcal{C})$. □

Note that the relations of $\sim$ are closely related to the ones defining $M_{X,P}$, that is, two coterminal paths p, q are in the same $\sim$-class if and only if $l(p) = l(q)$ in $M_{X,P}$. This enables us to identify morphisms from some vertex v with their (common) label in $M_{X,P}$. Using this identification, we have $L(\mathcal{C}, v) = \mathrm{Stab}(\mathcal{C}, v)$ for any vertex v. The following proposition is a direct consequence of our previous observation and Proposition 3.1.

Proposition 4.7. *Each loop monoid of $\mathrm{IC}(\mathcal{C})$ is a closed inverse submonoid of $M_{X,P}$.*

Given a closed inverse submonoid H of $M_{X,P}$, we construct a complex with H as a loop monoid using the ω-coset graph Γ_H of H. First note that the action of $M_{X,P}$ by right multiplication on the right ω-cosets of H is by definition the same as the action on the vertices of Γ_H induced by the edges. Suppose there is a closed path based at H labeled by $x\rho y$, where $\rho \in P,\ x, y \in (X \cup X^{-1} \cup P)^*$. Then $x\rho y \in H$, and since H is closed and $x\rho y \leq xy$ in $M_{X,P}$, we also have $xy \in H$, hence xy also labels a closed path based at H. This implies that ρ always labels a loop in the coset graph. Similarly, $x\rho y \leq x(bl(\rho))y$, so $x(bl(\rho))y$ labels a closed path based at H. Therefore whenever there is a loop in the coset graph labeled ρ based at v, there is a closed path labeled $bl(\rho)$ based at v.

The labeled *coset complex* $\mathcal{C}_H$ of H is defined the following way:

$$\mathcal{C}_H^{(0)} = V(\Gamma_H),$$

$$\mathcal{C}_H^{(1)} = \{e \in E(\Gamma_H) : l(e) \in X \cup X^{-1}\},$$

$$\mathcal{C}_H^{(2)} = \{C_e \in E(\Gamma_H) : l(e) \in P\},$$

where the boundary walk of a 2-cell C_e is the closed path rooted at $\alpha(e)$ and labeled by $bl(\rho)$ where $\rho = l(e)$. In short, we take the graph Γ_H, and substitute edges labeled by P with 2-cells in the natural way. Note that the labeling of C_H corresponds to the immersion over the 2-complex $B_{X,P}$, in which the attaching map of a 2-cell labeled by ρ is given by $bl(\rho)$.

The following proposition gives the relationships between the complexes associated with the coset graphs and graphs associated with complexes.

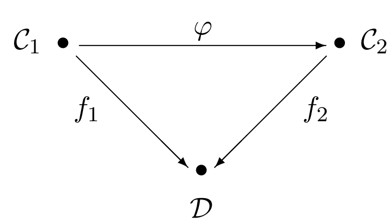
Proposition 4.8. *Let $\mathcal{C}$ be a labeled 2-complex. If H is a closed inverse submonoid of $M_{X,P}$ for which $\Gamma_H \cong \Gamma_{\mathcal{C}}$, then $\mathcal{C}_H \cong \mathcal{C}$. There is an isomorphism $\varphi\colon \Gamma_H \to \Gamma_{\mathcal{C}}$ if and only if $H = \mathrm{Stab}(\mathcal{C}, \varphi(H))$.*

Proof. The first statement follows directly from the definitions of $\mathcal{C}_H$ and $\Gamma_{\mathcal{C}}$. For the second statement, suppose $H = \mathrm{Stab}(\mathcal{C}, v)$ for some $v \in \mathcal{C}^0$. First we observe that the set of words labeling closed paths from $\mathrm{Stab}(\mathcal{C}, v)$ to $\mathrm{Stab}(\mathcal{C}, v)$ in $\Gamma_{\mathrm{Stab}(\mathcal{C},v)}$ is the same as the set of words labeling closed paths from v to v in $\Gamma_{\mathcal{C}}$. Indeed, p is a closed (v, v)-path in $\Gamma_{\mathcal{C}}$ if and only if $l(p) \in \mathrm{Stab}(\mathcal{C}, v)$, which is if and only if p is a closed path from $\mathrm{Stab}(\mathcal{C}, v)$ to $\mathrm{Stab}(\mathcal{C}, v)$ in $\Gamma_{\mathrm{Stab}(\mathcal{C},v)}$. We now define an isomorphism $\varphi\colon \Gamma_{\mathrm{Stab}(\mathcal{C},v)} \to \Gamma_{\mathcal{C}}$ by $\mathrm{Stab}(\mathcal{C}, v) \mapsto v$, and all $(\mathrm{Stab}(\mathcal{C}, v), \mathrm{Stab}(\mathcal{C}, v))$-paths map to the (unique) (v, v)-path with the same label. It is routine to verify that this is a graph isomorphism.

Now for the converse, suppose $H \neq \mathrm{Stab}(\mathcal{C}, v)$ for any vertex v. Then the set of labels of closed (H, H)-paths in Γ_H and the ones of closed (v, v) paths in $\Gamma_{\mathcal{C}}$ are different, for all $v \in V(\Gamma_{\mathcal{C}})$, hence the two graphs cannot be isomorphic. $\square$

Let H, K be two closed inverse submonoids of $M_{X,P}$. Define H to be *conjugate* to K, denoted by $H \approx K$, if there exists $m \in M_{X,P}$ such that $m^{-1}Hm \subseteq K$ and $mKm^{-1} \subseteq H$. It is easy to see that $\approx$ is an equivalence relation (called "conjugation") on the set of closed inverse submonoids of $M_{X,P}$. The equivalence classes of $\approx$ are called *conjugacy classes*. We remark that conjugate closed inverse submonoids of $M_{X,P}$ are not necessarily isomorphic (see [5]).

We call the two (labeled) immersions $f_1\colon \mathcal{C}_1 \to \mathcal{D}$ and $f_2\colon \mathcal{C}_2 \to \mathcal{D}$ *equivalent* if there is a labeled isomorphism $\varphi\colon \mathcal{C}_1 \to \mathcal{C}_2$ which makes the following diagram commute:



The following two theorems state the main result of the paper. They are generalizations of Theorem 4.4 and 4.5 in [5], and most of the proofs are analogous to those. When the 2-complexes contain no 2-cells (that is, they are graphs), these theorems reduce to Theorem 4.4 and 4.5 in [5].

Theorem 4.9. *Let $\mathcal{C}$ be a 2-complex, with edges labeled over the set $X \cup X^{-1}$, 2-cells labeled over the set P, consistent with an immersion over some complex $B_{X,P}$. Then each loop monoid is a closed inverse submonoid of $M_{X,P}$, and the set of all loop monoids $L(\mathcal{C}, v)$ for $v \in \mathcal{C}^0$ forms a conjugacy class of the set of closed inverse submonoids of $M_{X,P}$. Conversely, if H is a closed inverse submonoid of $M_{X,P}$, then there is a 2-complex $\mathcal{C}$ and an immersion $f\colon \mathcal{C} \to B_{X,P}$ such that H is a loop monoid of* IC($\mathcal{C}$)*, furthermore, $\mathcal{C}$ is unique (up to isomorphism), and f is unique (up to equivalence).*

Proof. We saw in Proposition 4.7 that loop monoids are closed. Take two loop monoids $L(\mathcal{C}, v_1)$ and $L(\mathcal{C}, v_2)$, and let $m \in (X \cup X^{-1} \cup P)^*$ label a (v_1, v_2)-path in $\mathcal{C}$. If $n \in L(\mathcal{C}, v_2)$, then n labels a (v_2, v_2)-path, and mnm^{-1} labels a (v_1, v_1)-path, so $mL(\mathcal{C}, v_2)m^{-1} \subseteq L(\mathcal{C}, v_1)$. Since m^{-1} labels a (v_2, v_1)-path, we get $m^{-1}L(\mathcal{C}, v_1)m \subseteq L(\mathcal{C}, v_2)$ similarly. Now suppose $H \approx L(\mathcal{C}, v_1)$. Then there exists some $m \in M_{X,P}$ such that $m^{-1}L(\mathcal{C}, v_1)m = H$ and $mHm^{-1} = L(\mathcal{C}, v_1)$, in particular, $mm^{-1} \in L(\mathcal{C}, v_1)$. Therefore, regarding m as an element of $(X \cup X^{-1} \cup P)^*$, it labels a path from v_1 to some vertex v_2. If $h \in H$ (and again regard h as an element of $(X \cup X^{-1} \cup P)^*$), then mhm^{-1} labels a (v_1, v_1)-path, hence h labels a path form v_2 to v_2. Therefore $H \subseteq L(\mathcal{C}, v_2)$. On the other hand, if $n \in L(\mathcal{C}, v_2)$, then $mnm^{-1} \in L(\mathcal{C}, v_1)$, and $m^{-1}mnm^{-1}m \subseteq H$. Since H is closed and $m^{-1}mnm^{-1}m \leq n$, this yields $n \in H$, therefore $H = L(\mathcal{C}, v_2)$. This proves that the set of all loop monoids $L(\mathcal{C}, v)$ for $v \in \mathcal{C}^0$ form a conjugacy class of the set of closed inverse submonoids of $M_{X,P}$.

Now suppose that H is a closed inverse submonoid of $M_{X,P}$, and build the coset complex $\mathcal{C}_H$. There is a natural immersion $f\colon \mathcal{C}_H \to B_{X,P}$, namely the one sending all edges and 2-cells to the ones corresponding to their labels.

It follows from Proposition 4.8 that the graph $\Gamma_{\mathcal{C}}$ is unique, and it uniquely determines $\mathcal{C}$. The uniqueness of f follows from the fact that f respects the labeling.

□

Theorem 4.10. *Let $f\colon \mathcal{C}_2 \to \mathcal{C}_1$ be an immersion over $\mathcal{C}_1$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are 2-complexes with edges labeled over the set $X \cup X^{-1}$, 2-cells labeled over the set P consistent with an immersion over some complex $B_{X,P}$, and f respects the labeling. If $v_i \in \mathcal{C}_i^0$, $i = 1, 2$, such that $f(v_2) = v_1$, then f induces an embedding of $L(\mathcal{C}_2, v_2)$ into $L(\mathcal{C}_1, v_1)$. Conversely, let $\mathcal{C}_1$ be a labeled 2-complex and let H be a closed inverse submonoid of $M_{X,P}$ such that $H \subseteq L(\mathcal{C}_1, v_1)$ for some $v_1 \in \mathcal{C}_1^0$. Then there exists a 2-complex $\mathcal{C}_2$ and an immersion $f\colon \mathcal{C}_2 \to \mathcal{C}_1$ and a vertex $v_2 \in \mathcal{C}_2^0$ such that $f(v_2) = v_1$ and $L(\mathcal{C}_2, v_2) = H$. Furthermore, $\mathcal{C}_2$ is unique (up to isomorphism), and f*

is unique (up to equivalence). If H, K are two closed inverse submonoids of $M_{X,P}$ with $H, K \subseteq L(\mathcal{C}_1, v_1)$, then the corresponding immersions are equivalent if and only if $H \approx K$ in $M_{X,P}$.

Proof. Suppose first that $f(v_2) = v_1$. The assertion that $L(\mathcal{C}_2, v_2) \subseteq L(\mathcal{C}_1, v_1)$ follows easily from the fact that if p is a closed path in $\mathcal{C}_2$ based at v_2, then $f(p)$ is a closed path in $\mathcal{C}_1$ based at $f(v_2) = v_1$ and $l(p) = l(f(p))$. For the converse, suppose H is a closed inverse submonoid of $M_{X,P}$ such that $H \subseteq L(\mathcal{C}_1, v_1)$, and construct the coset complex $\mathcal{C}_H$ and the coset graph Γ_H, and let Γ_1 denote $\Gamma_{\mathcal{C}_1}$. Put $\mathcal{C}_2 = \mathcal{C}_H$, and $v_2 = H$ We saw in Proposition 4.8 that $H = L(\mathcal{C}_H, H)$. We construct an immersion $g \colon \Gamma_H \to \Gamma_1$ that respects the labeling. Let $f(H) = v_1$, and note that if $(Hm)^\omega$ is a right ω-coset, then $mm^{-1} \in H \subseteq L(\mathcal{C}_1, v_1)$, so m labels a path starting at v_1 in Γ_1. Now we define g to take all paths starting at H to the (unique) path with the same label, starting at v_1. Then g is locally injective at the vertices, hence it is an immersion, and it respects the labeling by definition. By Lemma 4.4, g yields an immersion $f \colon \mathcal{C}_H \to \mathcal{C}_1$ that commutes with the labeling.

The uniqueness of f and $\mathcal{C}_2$ again follow from the uniqueness of $\Gamma_{\mathcal{C}_2}$ by Proposition 4.8, and from the fact that f respects the labeling. For the last statement, recall that according to Lemma 4.2, the immersion f and the complex $\mathcal{C}_2$ determine the boundary walks and therefore the graph $\Gamma_{\mathcal{C}_2}$ uniquely, that is, $\Gamma_{\mathcal{C}_2}$ and the pair $(f, \mathcal{C}_2)$ are in one-one correspondence. The fact $\Gamma_H \cong \Gamma_{H'}$ if and only if H and H' are conjugate completes our proof. □

We close this section with some observations about the inverse monoids $M_{X,P}$ and their closed inverse submonoids. In particular, we give an algorithm to construct $\mathcal{C}_H$ for a finitely generated closed inverse submonoid H of $M_{X,P}$ if X and P are finite.

Theorem 4.11. *(a) If X and P are finite sets, then the Schützenberger graphs of $M_{X,P}$ are finite (and effectively constructible) and so the word problem for $M_{X,P}$ is decidable.*

(b) If X and P are finite sets and H is a finitely generated closed inverse submonoid of $M_{X,P}$, then the associated 2-complex $\mathcal{C}$ is finite and effectively constructible.

Proof. (a) If w is a word in $(X \cup X^{-1})^*$ then no defining relation for $M_{X,P}$ applies, so the corresponding Schützenberger graph $S\Gamma(w)$ is the Munn tree of w (see [7, 3]), so it is finite and effectively constructible. On the other hand, if w is a word in $(X \cup X^{-1} \cup P)^*$ that does contain some letter $\rho \in P$, then any application of the relation $\rho^2 = \rho$ turns the edge labeled by ρ into a loop. Any application of the relation $\rho \leq bl(\rho)$ (i.e. $\rho = \rho bl(\rho)$) just introduces a new path labeled by $bl(\rho)$ to the approximate automaton. Once the relations $\rho = \rho^2$ and $\rho \leq bl(\rho)$ have been applied, this occurrence of ρ is not involved in any further application of relations involved in iteratively constructing $S\Gamma(w)$. As the automaton we started out with was finite, this iterative process (as outlined in Section 2 above - Theorem 4.12 of [12]) must terminate in a finite number of steps and the Schützenberger automaton $S\mathcal{A}(w)$ is finite and effectively constructible.

(b) The proof of part (b) of the theorem is similar. If we start with the flower automaton $\mathcal{F}(Y)$ of a finite subset $Y \subset (X \cup X^{-1} \cup P)^*$ and iteratively apply edge foldings and expansions corresponding to the defining relations of $M_{X,P}$, this process terminates in a finite number of steps, providing an effective construction of the ω-coset automaton of the corresponding closed inverse submonoid $\langle Y \rangle^\omega$ of $M_{X,P}$ by Theorem 3.3. The result then follows from Theorem 4.8 (and the fact that the associated complex $\mathcal{C}$ is the coset complex of $\langle Y \rangle^\omega$).

□

5. Examples and special cases

Recall that a *covering space* of a space X is a space $\tilde{X}$ together with a map $f\colon \tilde{X} \to X$ called a covering map, satisfying the following condition: there exists an open cover U_α of X such that for each α, $f^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α by f. It is easy to see that a cellular map $f\colon \mathcal{C} \to \mathcal{D}$ between CW-complexes is a covering map if and only if each 0-cell $v \in C^0$ has a neighborhood U_v that is homeomorphic to a neighborhood $U_{f(v)}$ of $f(v)$. This happens if and only if f is an immersion for which the neighborhoods of 0-cells "lift completely", that is, whenever v is on the boundary of a cell C in $\mathcal{D}$, then each 0-cell in $f^{-1}(v)$ is on the boundary of a cell in $f^{-1}(C)$.

The following theorem characterizes those immersions between 2-complexes which are also covering maps, in the sense of the previous theorem.

Theorem 5.1. *Let $\mathcal{C}, \mathcal{D}$ be 2-complexes labeled by an immersion over some complex $B_{X,P}$, let $f\colon \mathcal{C} \to \mathcal{D}$ be an immersion that respects the labeling, and let $v \in \mathcal{C}^0$ be an arbitrary 0-cell. Then f is a covering map if and only if $L(\mathcal{C}, v)$ is a full closed inverse submonoid of $L(\mathcal{D}, f(v))$, that is, it contains all idempotents of $L(\mathcal{D}, f(v))$.*

Proof. First, suppose that f is a covering, and suppose there is an idempotent $e \in L(\mathcal{D}, f(v))$. Regarding e as an element of $(X \cup X^{-1} \cup P)^*$, the closed path in $\mathcal{D}$ labeled by e, starting at $f(v)$ lifts to a path labeled by e, starting at v in $\mathcal{C}$, because f is a covering. Since e is idempotent, the action of any path labeled by e on $\mathcal{C}^0$ is the restriction of the identity, therefore a path labeled by e must always be closed. This yields $e \in L(\mathcal{C}, v)$.

For the converse, suppose $L(\mathcal{C}, v)$ is a full closed inverse submonoid of $L(\mathcal{D}, f(v))$. Suppose there is an edge starting at $f(v)$, labeled by s in $\mathcal{D}$. Then $ss^{-1} \in L(\mathcal{D}, f(v))$, and since ss^{-1} is idempotent, that implies $ss^{-1} \in L(\mathcal{C}, v)$. Which yields that there is an edge labeled by s, starting form v in $L(\mathcal{C}, v)$, that is, the neighborhood of $f(v)$ lifts completely. By induction on distance from v, we obtain that all 0-cells are in the image of f, therefore their neighborhoods lift completely.

□

It is easy to see that $L(\mathcal{C}, v)$ is a full closed inverse submonoid of $L(\mathcal{D}, f(v))$ if and only if whenever $m \in L(\mathcal{C}, v)$ and $n \in L(\mathcal{D}, f(v))$ such that $m \geq n$ holds in $L(\mathcal{D}, f(v))$, then $n \in L(\mathcal{C}, v)$. Therefore combining the result above with Theorem 4.10, we obtain that an immersion $f\colon \mathcal{C} \to \mathcal{D}$ is a covering if and only if whenever an element $m \in L(\mathcal{D}, f(v))$ is comparable with $n \in L(\mathcal{C}, v)$ in the natural partial order, we have $n \in L(\mathcal{D}, f(v))$.

We briefly compare our results with the theorem classifying covers via subgroups of the fundamental group when applied to 2-complexes. Recall (Proposition 4.6) that the fundamental group $\pi_1(\mathcal{C})$ of a (connected) 2-complex $\mathcal{C}$ is the greatest group homomorphic image of any loop monoid of $\mathcal{C}$, denoted by $L(\mathcal{C}, v)/\sigma$. The greatest group homomorphic image of $M_{X,P}$, denoted by $G_{X,P}$, is the group with the same presentation as $M_{X,P}$. Since in groups, $\rho = \rho^2$ implies $\rho = 1$, and $\rho \leq bl(\rho)$ implies $\rho = bl(\rho) = 1$, that is just

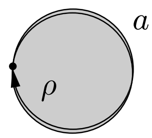
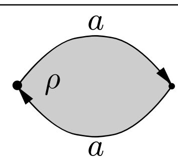
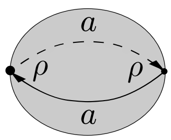
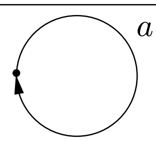
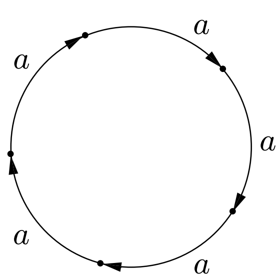
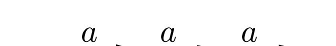
$$G_{X,P} = Gp\langle X \mid bl(\rho) = 1\rangle.$$

This is the fundamental group of the corresponding complex $B_{X,P}$, and the fundamental group of a complex immersing into $B_{X,P}$ is a subgroup of $G_{X,P}$.

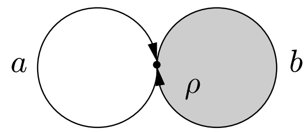
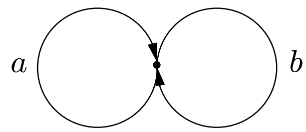
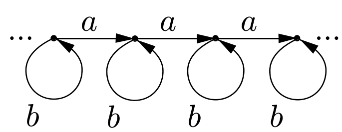
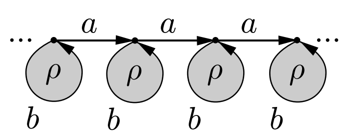
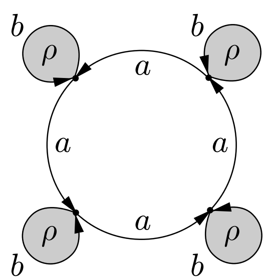
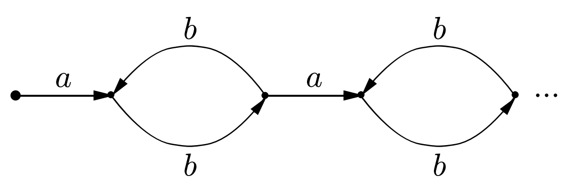
Naturally, the fundamental groups of 2-complexes immersing into a 2-complex $\mathcal{C}$ are always subgroups of $\pi_1(\mathcal{C})$, but distinct immersing 2-complexes may give rise to the same subgroup of $\pi_1(\mathcal{C})$ — for example, any immersing tree has the trivial group as its fundamental group. When restricting to covers, however, it is well-known that the fundamental groups of the covering spaces are in one-to-one correspondence with the conjugacy classes of subgroups of the fundamental group of the base space. Therefore the loop monoids of different covering spaces all have different greatest group homomorphic images. Suppose $f\colon \mathcal{C} \to \mathcal{D}$ is a covering that respects the labeling, and let $v \in \mathcal{C}^0$. Let $\sigma^\natural\colon L(\mathcal{D}, f(v)) \to \pi_1(\mathcal{D})$ be the natural homomorphism corresponding to the congruence σ. Recall ([3]) that σ is generated by pairs (m, n) such that $m \leq n$. Therefore by Theorem 5.1 (and the observation that followed), it is clear that $L(\mathcal{C}, v)$ is the union of some σ-classes of $L(\mathcal{D}, f(v))$, namely it is the full inverse image of $\pi_1(\mathcal{C})$ under $\sigma^\natural$.

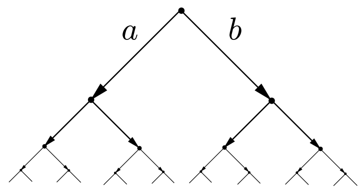
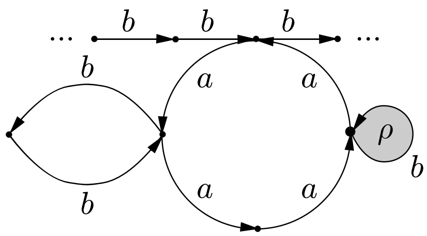
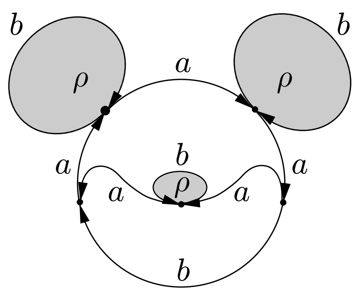
In [13], Williamson uses similar methods to classify immersions over a slightly restricted class of complexes with one 0-cell. The notion of immersion $f : \mathcal{C} \to \mathcal{D}$ in [13] has the additional property that every 0-cell in the fiber $f^{-1}(v_0)$ of a 0-cell v_0 on the boundary of a 2-cell of $\mathcal{D}$ is required to be part of the boundary of some 2-cell of $\mathcal{C}$.

Example 5.2. Let $X = \{a\}$, $P = \{\rho\}$, and $\mathcal{C}$ be the labeled 2-complex with one loop labeled by a and one 2-cell attached to the path a^2. (This complex is homeomorphic to the projective plane.) Then its loop monoid is $M_{X,P} = Inv\langle a, \rho \mid \rho^2 = \rho,\ \rho \leq a^2\rangle$. Here is a list of all 2-complexes immersing into $\mathcal{C}$, and a representative from the corresponding conjugacy class of closed inverse submonoids of $M_{X,P}$. (The basepoint of the representative is denoted by a larger dot when necessary.) The complex that immerses into the projective plane uniquely determines the immersion (up to equivalence).

	$\langle a, \rho\rangle^\omega$, the projective plane
	$\langle \rho\rangle^\omega$
	$\langle \rho,\ a\rho a\rangle^\omega$, the universal cover
	$\langle a\rangle^\omega$
	$\langle a^n\rangle^\omega$, $n \in \mathbb{N}$, $(n = 5)$
•	$\langle 1\rangle^\omega$
	$\langle a^n a^{-n}\rangle^\omega$, $n \in \mathbb{N}$, $(n = 4)$
	$\langle a^n a^{-n} : n \in \mathbb{N}\rangle^\omega$
	$\langle a^{-n} a^n : n \in \mathbb{N}\rangle^\omega$
	$\langle a^n a^{-n} : n \in \mathbb{Z}\rangle^\omega$

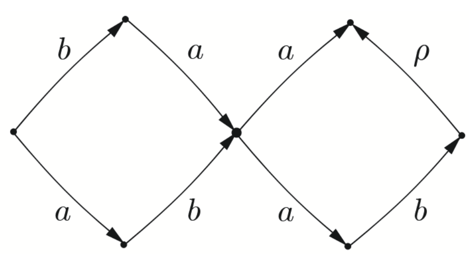
Example 5.3. Let $X = \{a, b\}$, $P = \{\rho\}$, and let $\mathcal{D}$ be the labeled 2-complex with two loops labeled by a and b, and one 2-cell attached to the path b. Then its loop monoid is $M_{X,P} = Inv\langle a, b, \rho \mid \rho^2 = \rho,\ \rho \leq b\rangle$. Here are some examples of 2-complexes immersing into $\mathcal{D}$, and a representative from the corresponding conjugacy class of closed inverse submonoids of $M_{X,P}$.

	$\langle a, b, \rho\rangle^\omega$
	$\langle a, b\rangle^\omega$
	$\langle a^n b a^{-n} : n \in \mathbb{Z}\rangle^\omega$
	$\langle a^n \rho a^{-n} : n \in \mathbb{Z}\rangle^\omega$, the universal cover
	$\langle a^k, a^n \rho a^{-n} : n \in \{1, \ldots, k\}\rangle^\omega$ $k \in \mathbb{N}$, $(k = 4)$
	$\langle (ab)^n ab^2 a^{-1} (ab)^{-n} : n \in \mathbb{N}\rangle^\omega$

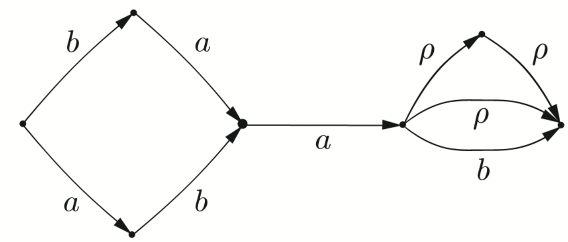
	$\{ww^{-1} : w \in \{a,b\}^*\}$
	$\langle a^4,\ \rho,\ a^2b^2a^{-2},\ ab^nb^{-n}a^{-1} : n \in \mathbb{Z}\rangle^\omega$
	$\langle \rho,\ a\rho a^{-1},\ a^2ba,\ a^3\rho a^2\rangle^\omega$

Example 5.4. Regard the torus as the 2-complex seen in Example 4.1. Its loop monoid is $M_{X,P} = \langle a, b, \rho \mid \rho^2 = \rho, \rho \leq aba^{-1}b^{-1}\rangle$. We construct the unique complex $\mathcal{C} = \mathcal{C}_H$ with a loop monoid $H = \langle a^{-1}b^{-1}ab,\ ab\rho a^{-1}\rangle^\omega \leq M_{X,P}$ using the method described in Theorem 3.3. Recall that the inequality $\rho \leq aba^{-1}b^{-1}$ can be written as $\rho = \rho aba^{-1}b^{-1}$.

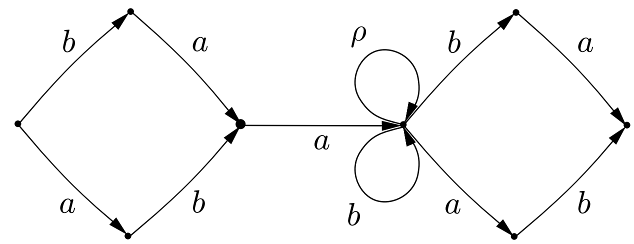
The flower automaton:



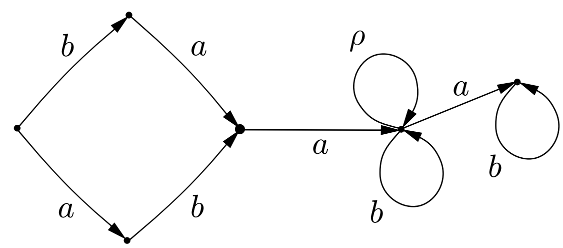
Folding a, then expanding by ρ^2:



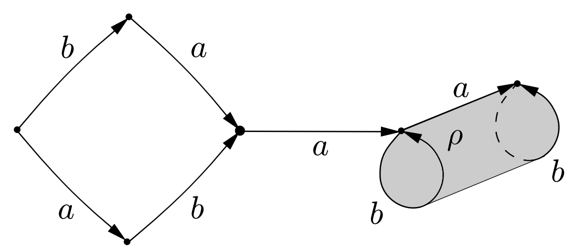
Folding ρ, then expanding by $\rho aba^{-1}b^{-1}$, and folding ρ right away:



Folding b and then a, the resulting graph is complete, thus it is Γ_H:



The coset complex $\mathcal{C}_H$:



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References

[1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press (2001).
[2] J.E. Hopcroft and J.D. Ullman, *Introduction to Automata Theory, Languages and Computation*, Addison-Wesley, New York (1979).
[3] M.V. Lawson, *Inverse Semigroups: the Theory of Partial Symmetries*, World Scientific (1998).
[4] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, New York (1977).
[5] S. Margolis and J. Meakin, *Free inverse monoids and graph immersions*, Int. J. Algebra and Computation, **3**, No. 1 (1993) 79-99.
[6] James R. Munkres, *Topology*, Second Edition, Prentice Hall (2000).

[7] W.D. Munn, *Free inverse semigroups*, Proc. London Math. Soc. **30** (1974), 384-404.
[8] M. Petrich, *Inverse semigroups*, Wiley (1984).
[9] B.M. Schein, *Representations of generalized groups*, Izv. Vyss. Ucebn. Zav. Mat. **3** (1962) 164-176 (in Russian).
[10] J.R. Stallings, *Topology of finite graphs*, Invent. Math. **71**, No. 3, (1983), 551-565.
[11] J.B. Stephen, *Applications of automata theory to presentations of monoids and inverse monoids*, Ph. D Thesis, Univ. of Nebraska-Lincoln, (1987).
[12] J.B. Stephen, *Presentations of inverse monoids*, J. Pure and Appl. Algebra **63** (1990) 81-112.
[13] Helen Williamson, *Immersions of* 2*-complexes and inverse monoids*, PhD Thesis, University of York (1995).

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