Author(s): Nukala, Phani K. V. V. & Zapperi, Stefano & Alava, Mikko J. & Simunovic, Srdan

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Effect of disorder and notches on crack roughness

Phani K. V. V. Nukala
Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831-6164, USA

Stefano Zapperi
CNR–INFM, SMC, Dipartimento di Fisica, Sapienza–Università di Roma, P. le A. Moro 2, 00185 Roma, Italy and ISI Foundation, Viale S. Severo 65, 10133 Torino, Italy

Mikko J. Alava
Laboratory of Physics, Helsinki University of Technology, FIN-02015 HUT, Finland

Srđan Šimunović
Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831-6164, USA
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We analyze the effect of disorder and notches on crack roughness in two dimensions. Our simulation results based on large system sizes and extensive statistical sampling indicate that the crack surface exhibits a universal local roughness of $\zeta_{loc}=0.71$ and is independent of the initial notch size and disorder in breaking thresholds. The global roughness exponent scales as $\zeta=0.87$ and is also independent of material disorder. Furthermore, we note that the statistical distribution of crack profile height fluctuations is also independent of material disorder and is described by a Gaussian distribution, albeit deviations are observed in the tails.

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I. INTRODUCTION

The statistical properties of fracture in disordered media are interesting for theoretical reasons and practical applications [1,2]. An important theoretical issue is represented by the scaling of crack surfaces. Experiments on several materials under different loading conditions have shown that the fracture surface is self-affine [3] and, in three dimensions, the out-of-plane roughness exponent displays a universal value of $\zeta \simeq 0.8$ irrespective of the material studied [4]. The scaling regime is sometimes quite impressive, spanning five decades in metallic alloys [4]. In particular, experiments have been done in metals [5], glass [6], rocks [7], and ceramics [8], covering both ductile and brittle materials. Later on, a smaller exponent $\zeta=0.4-0.6$ was observed at smaller length scales. It was conjectured that crack roughness displays a universal value of $\zeta \simeq 0.8$ only at larger scales and at higher crack speeds, whereas another roughness exponent in the range of 0.4–0.6 is observed at smaller length scales under quasistatic or slow crack propagation [4]. It was recently shown that the short-scale value is not present in silica glass, even when cracks move at extremely low velocities [9]. In addition, in sandstone and glass ceramics, one only measures a value of 0.45 even at high velocities [10–12]. The current interpretation associates the value $\zeta \simeq 0.8$ with rupture processes occurring inside the fracture process zone (FPZ), where elastic interactions would be screened, and the value $\zeta \simeq 0.45$ with large-scale elastic fracture [9,13]. In two dimensions, the available experimental results, mainly obtained for paper samples, indicate a roughness exponent in the range $\zeta \simeq 0.6-0.7$ [14–17].

In this work, we investigate the influence of fracture process zone on crack roughness in two dimensions through two key variables: material disorder, expressed as a distribution in breaking thresholds, and preexisting notches. Material disorder and the size of preexisting notches play a significant role in determining the size of the FPZ ahead of the crack tip. When the disorder is weak, the size of the FPZ is small and the material fracture response is dictated by the stress concentrations around the notches. On the other hand, when the disorder is strong, a relatively large fracture process zone is generated ahead of the crack tips. Similarly, the influence of preexisting notches on the FPZ in the presence of disorder is nontrivial. This is especially the case when the initial notch size is small and disorder is sufficiently strong to allow for significant damage accumulation. As the damage starts evolving, multiple cracks develop, which in turn influence the stress concentration around the initial preexisting notch. Even in the simplest case of noninteracting cracks, the stress fields become additive and hence the proportionality with respect to inverse of square root of the initial notch size is lost. The presence of interacting cracks further complicates this scenario and the stress concentration around notches depends in a nontrivial fashion on the initial notch size. For large notches, the effect of disorder should be weaker since the fracture process is dominated by a single crack.

The question we would like to address is how the roughness of the fracture surfaces depends on the material disorder and the relative sizes of the preexisting notches, given their influence on the fracture process zone. Studies on the random fuse model with uniform and power law disorder have indicated that spatial correlations in the damage accumulated prior to the peak load (the maximum load before catastrophic failure) are negligible, and that the damage is accumulated more or less uniformly up to the peak load [18]. This suggests that the origin of self-affine roughness in the random fuse model should not depend on whether there is strong or weak disorder, since the spatial correlations are built in the system only at the final stage of macroscopic failure. Earlier studies that investigated the effect of disorder on crack roughness are controversial: based on two-dimensional disordered beam lattice simulations, Ref. [19] suggested a universal roughness exponent of $\simeq 0.86$, whereas using two-dimensional disordered fuse lattice simulations, Ref. [20] argued against the universality of the roughness exponent. Clearly, the situation warrants further investigation especially in light of the role played by the FPZ in the current interpretation for different values of the roughness exponents.

II. MODEL

In this paper, we study the effect of disorder and notches on the crack roughness, by numerical simulations of the two-dimensional random fuse model (RFM), where a lattice of fuses with random threshold is subject to an increasing external voltage [21,22]. The results show that the roughness exponent does not depend on the breaking threshold's disorder strength or on the presence of a notch. We consider a triangular lattice of linear size L with a central notch of length a_0. All of the lattice bonds have the same conductance, but the bond breaking thresholds t are randomly distributed based on the threshold probability distribution $p(t)$. The burning of a fuse occurs irreversibly, whenever the electrical current in the fuse exceeds the breaking threshold current value t of the fuse. Periodic boundary conditions are imposed in the horizontal directions (x direction) to simulate an infinite system, and a constant voltage difference V is applied between the top and the bottom of the lattice system bus bars.

A power-law threshold distribution $p(t)$ is used by assigning $t=X^D$, where $X\in[0,1]$ is a uniform random variable with density $p_X(X)=1$ and D represents a quantitative measure of disorder. The larger D is, the stronger the disorder. This results in t values between 0 and 1, with a cumulative distribution $P(t)=t^{1/D}$. The average breaking threshold is $\langle t\rangle=1/(D+1)$, and the probability that a fuse will have a breaking threshold less than the average breaking threshold $\langle t\rangle$ is $P(\langle t\rangle)=[1/(D+1)]^{1/D}$. That is, the larger D is, the smaller the average breaking threshold and the larger the probability that a randomly chosen bond will have breaking threshold smaller than the average breaking threshold.

Numerically, a unit voltage difference, $V=1$, is set between the bus bars (in the y direction), and the Kirchhoff equations are solved to determine the current flowing in each of the fuses. Subsequently, for each fuse j, the ratio between the current i_j and the breaking threshold t_j is evaluated, and the bond j_c having the largest value, $\max_j(i_j/t_j)$, is irreversibly removed (burnt). The current is redistributed instantaneously after a fuse is burnt, implying that the current relaxation in the lattice system is much faster than the breaking of a fuse. Each time a fuse is burnt, it is necessary to recalculate the current redistribution in the lattice to determine the subsequent breaking of a bond. The process of breaking bonds, one at a time, is repeated until the lattice system falls apart.

Using the algorithm proposed in Ref. [23], we have performed numerical simulation of fracture up to system sizes $L=320$. Our simulations cover an extensive parametric space of (L, D, and a_0) given by $L=\{64,128,192,256,320\}$; $D=\{0.3,0.4,0.5,0.6,0.75,1.0\}$; and $a_0/L=\{1/32,1/16,3/32,1/8,3/16,1/4,5/16,3/8\}$. A minimum of 200 realizations has been performed for each case, but for many cases 2000 realizations have been used to reduce the statistical error.

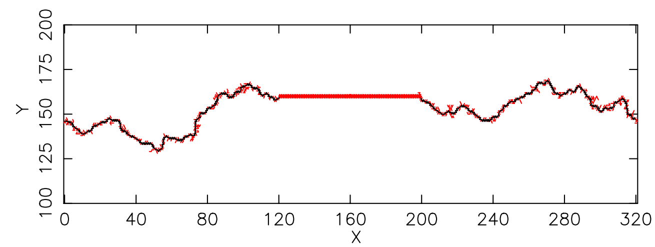

FIG. 1. (Color online) A typical final crack in a system of size $L\times L$ with $L=320$ and initial notch size $a_0=80$. Note that the crack shows dangling ends and overhangs, which are removed to obtain a single-valued crack line. The initial central notch is not considered in the roughness calculations.

III. CRACK ROUGHNESS

Once the sample has failed, we identify the final crack, which typically displays dangling ends and overhangs (see Fig. 1). We remove them and obtain a single-valued crack line h_x, where the values of $x\in[0,L]$. For self-affine cracks, the local width $w(l)\equiv\langle\Sigma_x[h_x-(1/l)\Sigma_X h_X]^2\rangle^{1/2}$, where the sums are restricted to regions of length l and the average is over different realizations, scales as $w(l)\sim l^{\zeta}$ for $l\ll L$ and saturates to a value $W=w(L)\sim L^{\zeta}$ corresponding to the global width. The power spectrum $S(k)\equiv\langle\hat{h}_k\hat{h}_{-k}\rangle/L$, where $\hat{h}_k\equiv\Sigma_x h_x \exp i(2\pi xk/L)$, decays as $S(k)\sim k^{-(2\zeta+1)}$. When anomalous scaling is present [24–26], the exponent describing the system size dependence of the surface *differs* from the local exponent measured for a fixed system size L. In particular, the local width scales as $w(\ell)\sim\ell^{\zeta_{loc}}L^{\zeta-\zeta_{loc}}$, so that the global roughness W scales as L^{ζ} with $\zeta>\zeta_{loc}$. Consequently, the power spectrum scales as $S(k)\sim k^{-(2\zeta_{loc}+1)}L^{2(\zeta-\zeta_{loc})}$.

In the following we investigate the influence of disorder D and initial notch size a_0 on crack roughness. Figure 2(a) presents the scaling of local and global crack widths in systems with different disorder values and an initial relative notch size of $a_0/L=1/16$. The slopes of the curves presented in Fig. 2(a) suggest that the local roughness exponent $\zeta_{loc}=0.71$ and is independent of the disorder. The global roughness exponent is estimated to be $\zeta=0.87$, and differs considerably from the local roughness exponent ζ_{loc}. The collapse of the data in Fig. 2(b) clearly demonstrates that crack widths follow an anomalous scaling law. The inset in Fig. 2(b) reports the data collapse of the power spectra based on anomalous scaling for different disorder values. This collapse of the data once again suggests that local roughness is independent of disorder. A fit of the power-law decay of the spectrum yields a local roughness exponent of $\zeta_{loc}=0.74$. This result is in close agreement with the real space estimate, and we can attribute the differences to the bias associated with the methods employed [27].

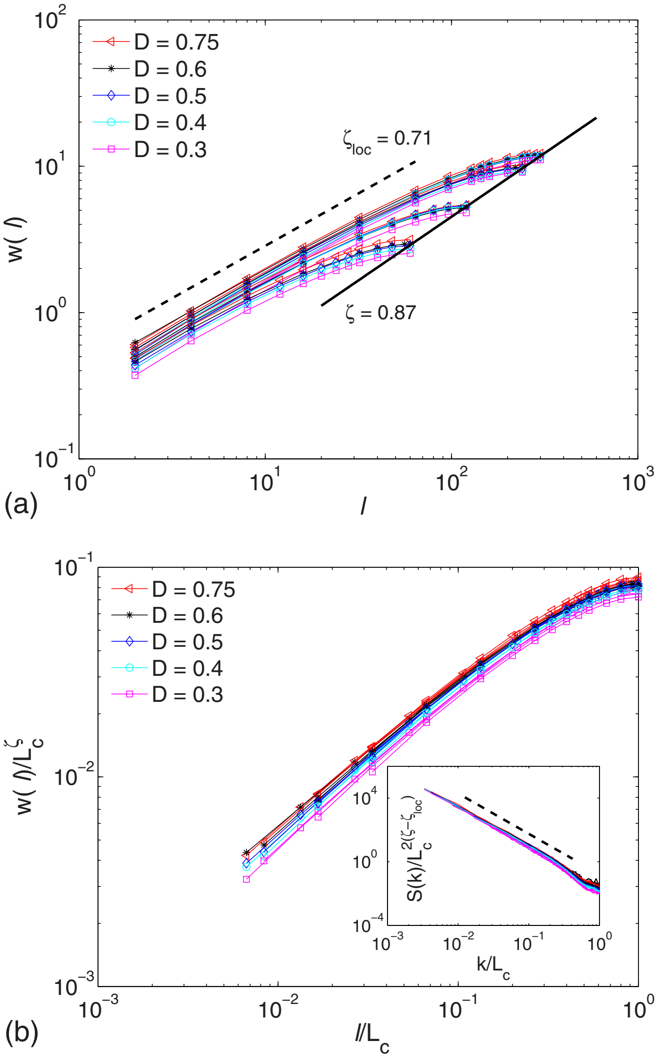

FIG. 2. (Color online) (a) Scaling of local and global widths $w(l)$ and W of the crack for different system sizes $L=\{64,128,256,320\}$, disorder values D, and a fixed $a_0/L=1/16$ value (top). The local crack width exponent $\zeta_{loc}=0.71$ is independent of disorder and differs considerably from the global crack width exponent $\zeta=0.87$. (b) Collapse of the crack width data using the anomalous scaling law (bottom). $L_c=(L-a_0)$ is the effective length of the crack profile. Collapse of the data for a given disorder value implies that local and global roughness exponents are independent of disorder. The inset shows collapse of the power spectrum $S(k)$ using the anomalous scaling law with $\zeta_{loc}=0.71$ and $\zeta=0.87$. The slope in the inset defines the local exponent via $-(2\zeta_{loc}+1)=-2.48$. (a) and (b) present a total of 20 data sets.

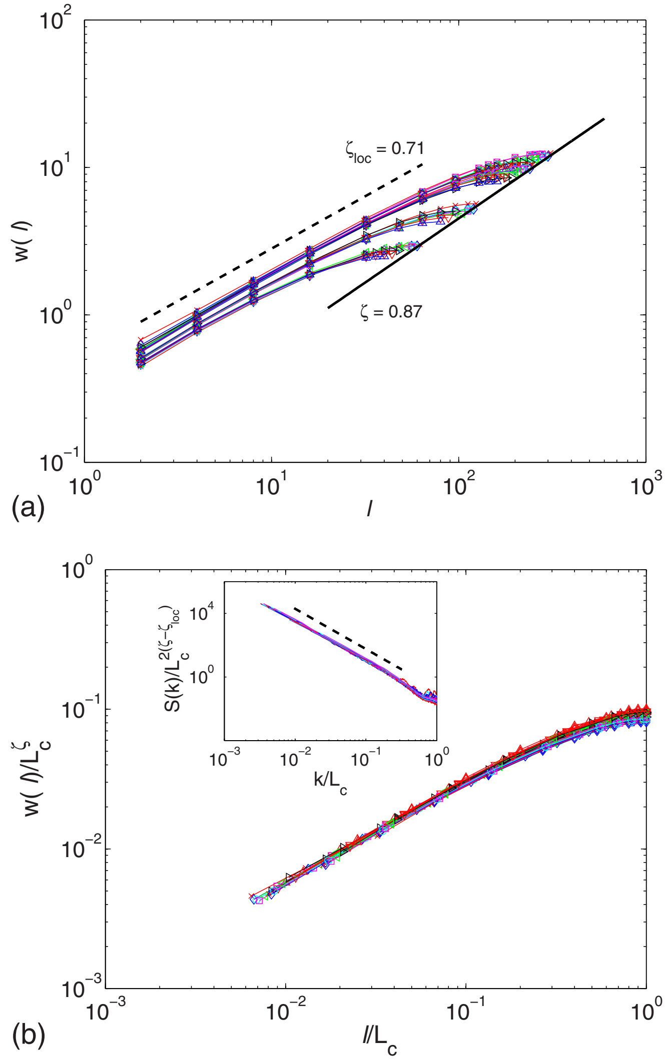

FIG. 3. (Color online) (a) Scaling of local and global widths $w(l)$ and W of the crack for different system sizes $L=\{64,128,256,320\}$, notch sizes $a_0/L=\{1/32,1/16,3/32,1/8,3/16,1/4,5/16,3/8\}$, and a constant disorder of $D=0.6$ (top). Once again, the local crack width exponent $\zeta_{loc}=0.71$ is independent of notch size a_0 and differs considerably from the global crack width exponent $\zeta=0.87$. (b) Collapse of the crack width data using the anomalous scaling law (bottom). $L_c=(L-a_0)$ is the effective length of the crack profile. The inset shows collapse of the power spectrum $S(k)$ using the anomalous scaling law with $\zeta_{loc}=0.71$ and $\zeta=0.87$. The slope in the inset defines the local exponent via $-(2\zeta_{loc}+1)=-2.53$. (a) and (b) present a total of 32 data sets.

The influence of initial notch size on crack roughness is presented in Fig. 3(a). The curves presented in Fig. 3(a) represent the scaling of local and global widths for various notch sizes a_0. Once again, the local roughness exponent is estimated to be $\zeta_{loc}=0.71$ and is independent of the initial notch size, whereas the global roughness exponent $\zeta=0.87$. Figure 3(b) presents the data collapse of crack widths based on the anomalous scaling law, which once again confirms that crack roughness follows anomalous scaling. The collapse of the power spectra in the inset of Fig. 3(b) for different notch sizes confirms that the local roughness is independent of the initial notch size. A fit of the power-law decay of the spectrum yields a local roughness exponent value of $\zeta_{loc}=0.77$. The close agreement of these results with the $\zeta_{loc}=0.72$ obtained for the unnotched, strong disorder case [28] indicates that the crack roughness is universal and is independent of disorder and initial notch size. The global crack width W, however, scales as $W\equiv[\langle(h_b-\overline{h_b})^2\rangle]^{1/2}\sim L^{\zeta}$ with $\zeta=0.87\pm0.03$, and is also independent of disorder and crack size.

The scaling properties of the crack profiles $h(x)$ can also be studied using the probability density distribution $p[\Delta h(\ell)]$ of the height differences $\Delta h(\ell)=[h(x+\ell)-h(x)]$ of the crack

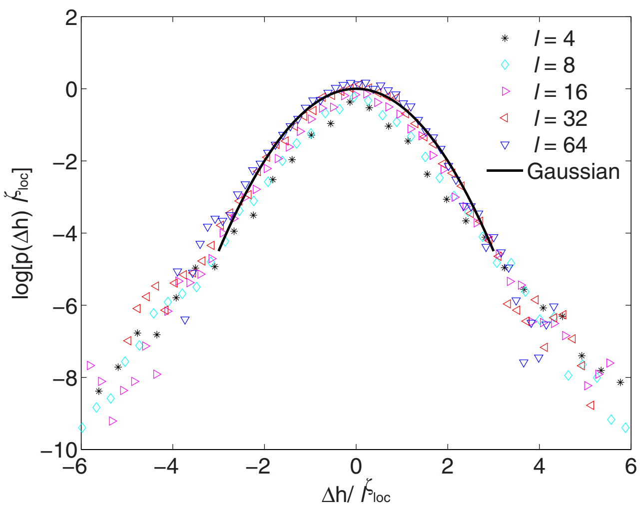

FIG. 4. (Color online) Logarithm (ln) of probability distributions $p[\Delta h(\ell)]$ of the height differences $\Delta h(\ell)=[h(x+\ell)-h(x)]$ of the crack profile $h(x)$ for various bin sizes $\ell=4,8,16,32,64$. The results are for crack profiles obtained using a system size of $L=320$, a disorder of $D=0.75$, and a relative crack size of $a_0/L=1/16$. As a guide for the eye, we present a Gaussian fit for $\ell=32$. It can be seen that the central parts of the distributions are Gaussian, but the tails do not follow a Gaussian.

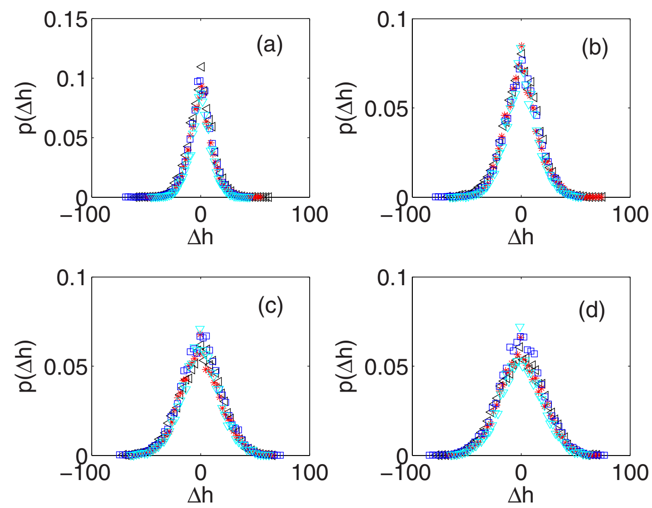

FIG. 5. (Color online) Collapse of the probability density distributions $p[\Delta h(\ell)]$ of the crack profiles for various disorder values $D=0.3$ (▽), $D=0.4$ (□), $D=0.5$ (*), $D=0.75$ (◁). $\ell=$(a) 32, (b) 64, (c) 96, and (d) 128. The results are obtained for a system size of $L=320$.

profile between any two points on the reference line (x axis) separated by a distance ℓ. Recently, there has been a debate over the scaling of this $p[\Delta h(\ell)]$ distribution [29–31], i.e., whether the scaling properties of $p[\Delta h(\ell)]$ can be described by a single scaling exponent ζ or multiple scaling exponents are required to describe the scaling of $p[\Delta h(\ell)]$. The self-affine property of the crack profiles implies that the probability density distribution $p[\Delta h(\ell)]$ follows the relation

$$p(\Delta h(\ell)) \sim \ell^{-\zeta_{loc}} f\left(\frac{\Delta h(\ell)}{\ell^{\zeta_{loc}}}\right), \tag{1}$$

whereas multiscaling of fracture surface implies that Eq. (1) is not valid. Although multiscaling of $p[\Delta h(\ell)]$ was argued in Ref. [29], it has been shown in Refs. [30,31] that $p[\Delta h(\ell)]$ follows a self-affine monoscaling relation given by Eq. (1) and that multiscaling is an artifact that results at small scales due to the removal of crack profile overhangs. In the following, we investigate whether disorder has any influence on the scaling of $p[\Delta h(\ell)]$.

First, we present the probability distributions $p[\Delta h(\ell)]$ of the height differences $\Delta h(\ell)=[h(x+\ell)-h(x)]$ for various bin sizes $\ell=4,8,16,32,64$ for a disorder of $D=0.75$, a system size of $L=320$, and a relative crack size of $a_0/L=1/16$. Figure 4 shows the collapse of the central parts of the probability distributions $p[\Delta h(\ell)]$ for larger ℓ values, but still within the local width scaling regime. The deviation for smaller ℓ values may be attributed to steps in the single-valued crack height profiles, which inevitably arise due to the removal of overhangs on the crack surface. For larger ℓ values, the central parts of these distributions approach Gaussian, but clear deviations can be observed in the tails of the distribution from a Gaussian distribution. Similar deviations in the tails were observed for the uniform disorder case as well [30,31].

Second, we present the collapse of $p[\Delta h(\ell)]$ distributions in Figs. 5(a)–5(d) for various bin sizes ℓ and disorders. The collapse of these $p[\Delta h(\ell)]$ distributions for various disorders D at each bin size ℓ indicates that $p[\Delta h(\ell)]$ distributions and the roughness exponent ζ_{loc} are unaffected by the material disorder.

IV. DISCUSSION

In summary, the evidence presented in this paper indicates that the crack surface roughness is unaffected by the material disorder and the presence of preexisting notches. This can be inferred from the fact that material disorder, whether strong or weak, has a significant influence on the amount of damage accumulated prior to the peak load; however, the spatial correlations in the damage accumulated prior to the peak load are negligible [18]. Indeed, Figs. 6(a)–6(d) show the snapshots of damage and crack propagation at peak load in typical fracture simulations of system size $L=320$ having weak to strong disorder and small to large preexisting notches. By following the damage growth process, one can easily see from these figures that there is very little crack extension at peak load whether the material is strongly disordered or weakly disordered. In addition, Fig. 6 shows that the FPZ cannot be defined from a single damage snapshot, but it is necessary to average the damage over many realizations of the disorder. When this is done, we find that ξ_{FPZ} depends strongly on disorder D (ranging from one to 12 lattice units for the values considered) but only weakly on a_0 and L [32]. The independence of the roughness exponent on D suggests that self-affinity in the random fuse model is not related to the FPZ.

This is further corroborated by our numerical simulations on a simplified random fuse model, in which failure events

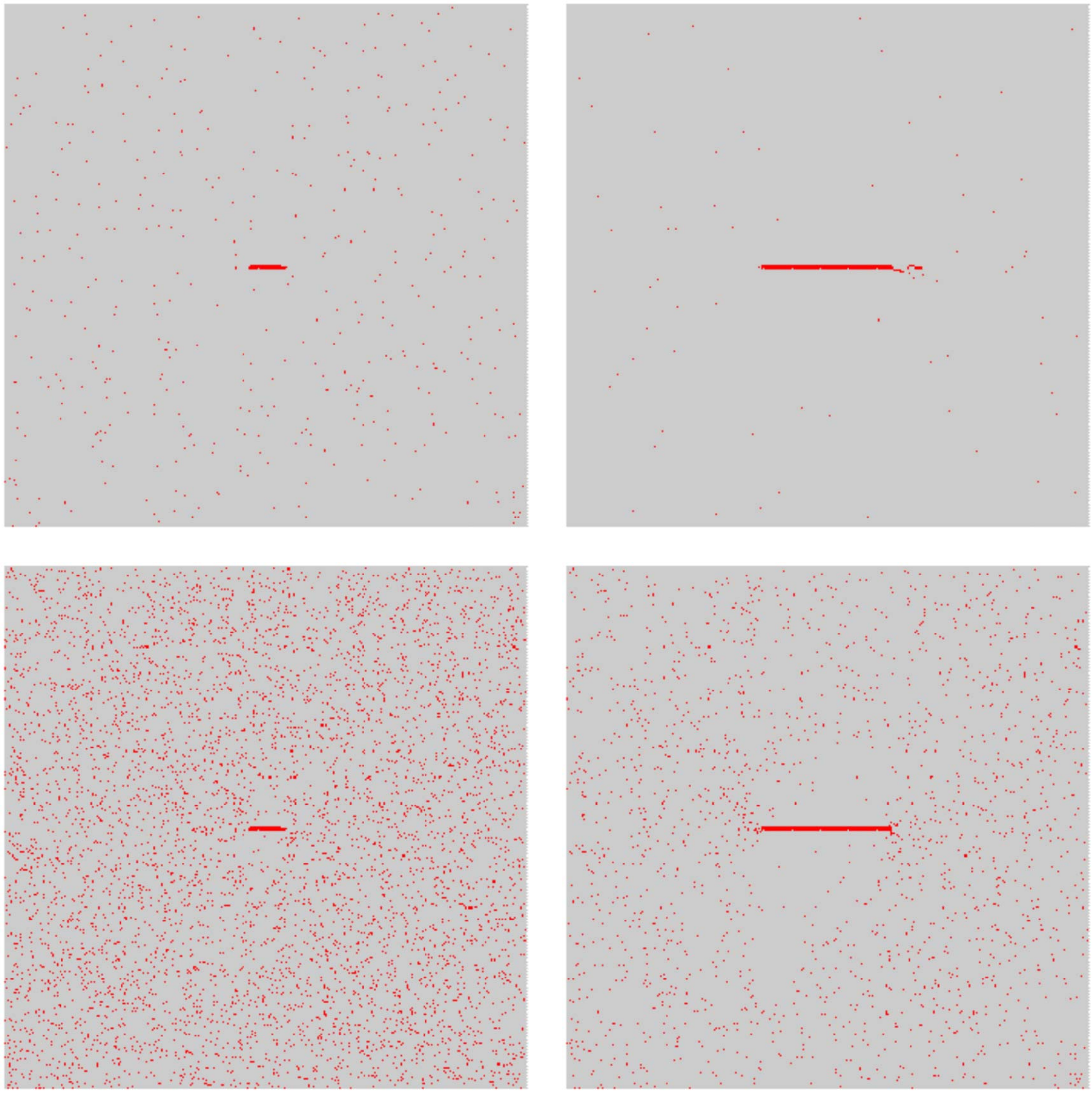

FIG. 6. (Color online) Snapshots of damage and crack evolution in disordered notched specimens of size $L=320$. (a) $D=0.3$ and $a_0=20$; (b) $D=0.6$ and $a_0=20$; (c) $D=0.3$ and $a_0=80$; (d) $D=0.6$ and $a_0=80$.

form a connected crack thereby excluding damage nucleation in the bulk [30]. In this model, after breaking the weakest fuse, successive failure events are only allowed on fuses that are connected to the crack. Otherwise, the rules of this simplified model strictly follow those of the usual RFM. Consequently, this model tracks only the connected crack along with its dangling ends in a disordered medium, and hence forms the most simplified model to study the effect of disorder on crack roughness. As was shown in Ref. [30], this simplified model exhibits the same roughness and height-height correlation characteristics as that of a conventional RFM. Even for this simplified model, the collapse of the power spectrum and local width of crack profiles for different disorders ($D=\{0.25,0.4,0.5,0.6,0.75,1.0\}$) results in a roughness exponent value of $\zeta_{loc}=0.7$, which clearly demonstrates once again that disorder is irrelevant for identifying the crack roughness exponent.

As also shown in Ref. [30], when the branching of the cracks or damage within the fracture process zone is not allowed, thereby limiting the crack extension to only the crack tips, a local roughness exponent of 0.5 is obtained. But, as soon as branching was present, the value of the roughness exponent was increased from $\zeta_{loc}=0.5$ to $\zeta_{loc}\simeq 0.7$. This may also explain why there is no transition in the value of the roughness exponent from strong to weak disorder in the presented simulations, since damage is always present even for the lowest values of disorders considered. A similar conclusion was reached by Bouchbiner *et al.* studying a model for crack growth with damage nucleation [33]. According to this work, as soon as a FPZ was introduced in the model the roughness exponent increased from $\zeta_{loc}=0.5$ to a higher value. Thus it appears that the roughness exponent in two dimensions does not depend on the size of the FPZ, but only on whether or not a FPZ is present. In three dimensions the situation should be different, since experiments suggest that the roughness exponent displays a crossover precisely at the FPZ size. It would be interesting to investigate this issue using three-dimensional simulations.

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