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# A genus six cyclic tetragonal reduction of the Benney equations

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**Abstract.** A reduction of Benney's equations is constructed corresponding to Schwartz-Christoffel maps associated with a family of genus six cyclic tetragonal curves. The mapping function, a second kind Abelian integral on the associated Riemann surface, is constructed explicitly as a rational expression in derivatives of the Kleinian  $\sigma$ -function of the curve.

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## 1. Introduction

In [1, 2] it was shown that Benney's equations

$$A_t^n = A_x^{n+1} + nA^{n-1}A_x^0, \quad n \geq 0,$$

admit *reductions* in which only finitely many  $N$  of the moments  $A^n$  are independent, and that a large class of such reductions may be parametrised by conformal maps from the upper half  $p$ -plane to a slit domain - the upper half  $\lambda$ -plane, cut along  $N$  nonintersecting Jordan arcs, which have one fixed end point on  $\Im(\lambda) = 0$ , and whose other 'free' end is a Riemann invariant of the reduced equations.

A natural subclass of these occurs where these Jordan arcs are straight lines, leading to a polygonal domain and hence an  $N$ -parameter Schwartz-Christoffel map; an important and tractable subfamily of these is the case in which the angles are all rational multiples of  $\pi$ , and in this case the mapping is given by an integral of a second kind Abelian differential [3] on an algebraic curve. Such examples have been worked out explicitly, in [3], [4], [5] and [6]. These have looked at elliptic and hyperelliptic curves as well as a cyclic trigonal example. It is thus worthwhile to generalise this to other algebraic curves.

In all these examples, reductions have been constructed explicitly - both the integrand and its integral were evaluated using quotients of derivatives of the  $\sigma$ -function associated with the respective curves. These curves are all specific examples from the wider class of cyclic  $(n, s)$  curves which have equations of the form:

$$y^n = x^s + \mu_{s-1}x^{s-1} + \dots + \mu_1x + \mu_0. \quad (1)$$

We suppose that  $(n, s)$  are coprime with  $n < s$ , in which case the curves have genus  $g = \frac{1}{2}(n-1)(s-1)$ , and a unique branch point  $\infty$  at infinity. In this paper we consider a reduction associated with a cyclic tetragonal curve, that is, from the class (1) with  $n = 4$ . For simplicity we look at the case with  $s = 5$  here.

## 2. Benney's Equations

In 1973, Benney considered an approximation for the two-dimensional equations of motion of an incompressible perfect fluid under a gravitational force [7]. He showed that if moments are defined by:

$$A_n(x, t) = \int_0^h u^n dy,$$

where  $u(x, y, t)$  is the horizontal fluid velocity and  $h(x, t)$  the height of the free surface, the moments  $A_n(x, t)$  satisfy an infinite set of hydrodynamic type equations

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + n A_{n-1} \frac{\partial A_0}{\partial x} = 0 \quad (n = 1, 2, \dots), \quad (2)$$

now called the Benney moment equations.

Identical moment equations can alternatively be derived from a Vlasov equation [8], [9]:

$$\frac{\partial f}{\partial t_2} + p \frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x} \frac{\partial f}{\partial p} = 0. \quad (3)$$

Here  $f = f(x, p, t)$  is a distribution function and the moments are defined instead by

$$A_n = \int_{-\infty}^{\infty} p^n f \, dp.$$

We assume throughout that  $f$  is such that all these moments exist. The equation of motion (3) has the Lie-Poisson structure :

$$\frac{\partial f}{\partial t} + \left\{ f, \frac{\delta H}{\delta f} \right\}_{p,x} = 0, \quad (4)$$

where  $\{\cdot, \cdot\}_{p,x}$  is the canonical Poisson bracket. Kupershmidt and Manin showed directly that the moment equations are Hamiltonian [10], [11]. If we set  $H = \frac{1}{2}H_2 = \frac{1}{2}(A_2 + A_0^2)$ ,  $\mathbf{A} = (A_0, A_1, \dots)$ , then

$$\frac{\partial \mathbf{A}}{\partial t} = B \frac{\partial H}{\partial \mathbf{A}} \quad (5)$$

where the matrix operator  $B$  is given by

$$B_{n,m} = nA_{n+m-1} \frac{\partial}{\partial x} + m \frac{\partial}{\partial x} \cdot A_{n+m-1}.$$

This is consistent with (4) in the sense that if  $H$  is some function only of the moments, the moment equations resulting from (4) and (5) are identical.

Benney showed in [7] that system (2) has infinitely many conserved densities, polynomial in the  $A_n$ . One of the most direct ways to calculate these is to use generating functions [10]. Let  $\lambda(x, p, t)$ , a formal series in  $p$ , be the generating function of the moments

$$\lambda(x, p, t) = p + \sum_{n=0}^{\infty} \frac{A_n}{p^{n+1}} \quad (6)$$

and let  $p(x, \lambda, t)$  be the inverse series

$$p(x, \lambda, t) = \lambda - \sum_{m=0}^{\infty} \frac{H_m}{\lambda^{m+1}}.$$

We note here that if  $A_n = \int_{-\infty}^{\infty} p^n f \, dp$  is substituted into (6), then this can be understood as the asymptotic series, as  $p \rightarrow \infty$ , of an integral

$$\lambda = p + \int_{-\infty}^{\infty} \frac{f(x, p', t)}{(p - p')} \, dp'. \quad (7)$$

Here  $p'$  runs along the real axis, and we take  $\Im(p) > 0$ . It follows that  $\lambda(p)$  is analytic in its domain of definition. If  $f(p)$  is Hölder continuous, the boundary value, on the real  $p$ -axis, of  $\lambda(p)$  will itself be Hölder continuous.

Comparing the first derivatives of  $\lambda(x, p, t)$ , we obtain the PDE

$$\frac{\partial \lambda}{\partial t} + p \frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial p} \left( \frac{\partial p}{\partial t} + p \frac{\partial p}{\partial x} + \frac{\partial A_0}{\partial x} \right). \quad (8)$$

If we now hold  $p$  constant, this gives

$$\frac{\partial \lambda}{\partial t} + p \frac{\partial \lambda}{\partial x} - \frac{\partial A_0}{\partial x} \frac{\partial \lambda}{\partial p} = 0 \quad (9)$$

which is a Vlasov equation of the same form as (3). Thus (3) and (9) have the same characteristics. Any function of  $\lambda$  and  $f$  must satisfy the same equation.

Alternatively, if we hold  $\lambda$  constant in (8), then we obtain the conservation equation

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} p^2 + A_0 \right) = 0. \quad (10)$$

Substituting the formal series of  $p(x, \lambda, t)$  into (10), we see that each  $H_n$  is polynomial in the  $A_n$  and is a conserved density. Any of the  $H_n$  could be used as the Hamiltonian in (4), and the resulting flows all commute. From this we define the Benney hierarchy to be the family of evolution equations

$$\frac{\partial f}{\partial t_n} + \left\{ f, \frac{1}{n} \frac{\delta H_n}{\delta f} \right\} = 0.$$

Again,  $\lambda$  satisfies an equation analogous to this,

$$\frac{\partial \lambda}{\partial t_n} + \left\{ \lambda, \frac{1}{n} \frac{\delta H_n}{\delta f} \right\} = 0,$$

so that both  $f$  and  $\lambda$  are advected along the same characteristics. These characteristics are flows of a Hamiltonian vector field, with Hamiltonians  $\frac{\delta H_n}{\delta f}$  given by the relation:

$$\left( \frac{1}{n} \frac{\delta H_n}{\delta f} \right) = \left( \frac{\lambda^n}{n} \right)_+$$

where  $(\cdot)_+$  denotes the polynomial part of the Laurent expansion.

### 2.1. Reductions of the moment equations

Suppose that for some family of points,  $p = \hat{p}_i(x, t)$ ,  $\lambda(\hat{p}_i) = \hat{\lambda}_i(x, t)$ , we have

$$\frac{\partial \lambda}{\partial p} \Big|_{p=\hat{p}_i} = 0.$$

Then (8) reduces to:

$$\frac{\partial \hat{\lambda}_i}{\partial t} + \hat{p}_i \frac{\partial \hat{\lambda}_i}{\partial x} = 0$$

where  $\frac{\partial \hat{\lambda}_i}{\partial t} = \frac{\partial \lambda}{\partial t} \Big|_{p=\hat{p}_i}$  and  $\frac{\partial \hat{\lambda}_i}{\partial x} = \frac{\partial \lambda}{\partial x} \Big|_{p=\hat{p}_i}$ . We say that  $\hat{\lambda}_i$  is a Riemann invariant with characteristic speed  $\hat{p}_i$ . We will see that there are families of functions  $\lambda(p)$  which are invariant under the Benney dynamics, and are parametrised by  $N$  Riemann invariants  $\lambda_i$ .

A hydrodynamic type system with  $N \geq 3$ , independent variables can not in general be expressed in terms of Riemann invariants. If such a system does have  $N$  Riemann invariants, it is called diagonalisable. Tsarev showed in [12] that if a diagonal hydrodynamic-type system

$$\frac{\partial \hat{\lambda}_i}{\partial t} + v_i(\hat{\lambda}) \frac{\partial \hat{\lambda}_i}{\partial x} = 0 \quad (i = 1, 2, \dots, N). \quad (11)$$

is semi-Hamiltonian, that is if

$$\partial_j \left( \frac{\partial_i v_k}{v_i - v_k} \right) = \partial_i \left( \frac{\partial_j v_k}{v_j - v_k} \right), \quad i \neq j \neq k,$$

for  $i, j, k$  distinct, where

$$\partial_k = \frac{\partial}{\partial \hat{\lambda}_k},$$

then it can be solved by the hodograph transformation. Any Hamiltonian system of hydrodynamic type is semi-Hamiltonian. Given a second equation of type (11)

$$\frac{\partial \hat{\lambda}_i}{\partial \tau} + w_i(\hat{\lambda}) \frac{\partial \hat{\lambda}_i}{\partial x} = 0 \quad (i = 1, 2, \dots, N), \quad (12)$$

and requiring it to be consistent with (11), we find that the  $w_i(\hat{\lambda})$  must satisfy the over-determined linear system

$$\frac{\partial_k w_i}{w_i - w_k} = \frac{\partial_k v_i}{v_i - v_k}, \quad i \neq k. \quad (13)$$

These equations are consistent provided (11) is semi-Hamiltonian. If the condition (13) holds, we say that (11) and (12) commute. In this case a set of equations for the unknowns  $\hat{\lambda}_i(x, t)$  is given by :

$$w_i(\hat{\lambda}) = v_i(\hat{\lambda}) t + x, \quad (i = 1, 2, \dots, N)$$

where  $t$  and  $x$  are the independent variables. Thus any reduction of this type can be solved in principle.

This generalized hodograph construction cannot easily be applied directly to the Benney equations however, as these have infinitely many dependent variables. Instead we will now consider families of distribution functions  $f$ , which are parameterised by finitely many  $N$  Riemann invariants  $\hat{\lambda}_i(x, t)$ . We are interested in the case [1], [2] where the function  $\lambda(p, x, t)$  is such that only  $N$  of the moments are independent. Then there are  $N$  characteristic speeds, assumed real and distinct, and  $N$  corresponding Riemann invariants  $(\hat{p}_i, \hat{\lambda}_i)$ , so Benney's equations reduce to a diagonal system of hydrodynamic type with finitely many dependent variables  $\hat{\lambda}_i$ ,

$$\frac{\partial \hat{\lambda}_i}{\partial t} + \hat{p}_i(\hat{\lambda}) \frac{\partial \hat{\lambda}_i}{\partial x} = 0 \quad (i = 1, 2, \dots, N). \quad (14)$$

Such a system is called a reduction of Benney's equations.

The construction of a more general family of solutions for equations of this type was outlined in [1] and [2]. An elementary example is the case where the map  $\lambda_+$  takes the upper half  $p$ -plane to the upper half  $\lambda$ -plane with a vertical slit as follows. This is a Schwarz-Christoffel map:

$$\lambda_+(x, p, t) = p + \int_{\infty}^p \frac{p' - \hat{p}_1}{\sqrt{(p' - p_1)(p' - p_2)}} dp'.$$

If the residue at infinity is set to be zero, then this imposes the condition  $\hat{p}_1 = \frac{1}{2}(p_1 + p_2)$  and we get the solution

$$\begin{aligned}\lambda_+(x, p, t) &= \hat{p}_1 + \sqrt{p^2 - (p_1 + p_2)p + p_1 p_2} \\ &= \hat{p}_1 + \sqrt{(p - \hat{p}_1)^2 + 2A_0}\end{aligned}$$

(from the expansion as  $p \rightarrow \infty$ ). This gives a steadily translating solution of Benney's equations (4)

$$\frac{\partial f}{\partial t} + \left\{ f, \frac{1}{2}p^2 + A_0 \right\}_{p,x} = 0.$$

The two parameters  $p_1$  and  $p_2$  are not independent, as for consistency their sum must be a constant. Hence only the end point of the slit in the  $\lambda$ -plane is variable. This is the Riemann invariant.

This construction can be generalized, [2], mapping the upper half  $p$ -plane to the upper half  $\lambda$ -plane with  $N$  curvilinear slits. However, in this paper we are specifically interested in *straight* slits - these mappings are all of Schwartz-Christoffel type.

## 2.2. Schwartz-Christoffel reductions

The case of a polygonal  $N$ -slit domain is of particular interest. The real  $p$ -axis has  $M$  vertices  $u_j$  marked on it; the preimage in the  $p$ -plane of each slit runs from a vertex  $\hat{p}_j$ , to a point  $\hat{v}_i$ , the preimage of the end of the slit, and then to another vertex  $\hat{p}_{j+1}$ . The angle  $\pi$  in the  $p$ -plane at  $\hat{p}_j$  is mapped to an angle  $\alpha_j \pi$  at the image point. The internal angle at the end of each slit is  $2\pi$ .

The mapping function is then given, up to a constant of integration, by

$$\lambda = \int^p \left[ \frac{\prod_{i=1}^N (p - \hat{v}_i)}{\prod_{j=1}^{2N} (p - \hat{p}_j)^{1-\alpha_j}} \right] dp.$$

If the integrand is to converge to 1 as  $p \rightarrow \infty$ , we require  $\sum_{j=1}^{2N} \alpha_j = N$ , while to avoid a logarithmic singularity, we further impose  $\sum_{j=1}^M \alpha_j \hat{p}_j = \sum_{i=1}^N \hat{v}_i$ . We then define  $\lambda$  more precisely as

$$\lambda = p + \int_{-\infty}^p \left[ \frac{\prod_{i=1}^N (p - \hat{v}_i)}{\prod_{j=1}^M (p - \hat{p}_j)^{1-\alpha_j}} - 1 \right] dp.$$

Other constraints are imposed by requiring the vertices  $\hat{p}_j$  to map to points  $\lambda_i^0$ , the fixed base points of the slits; there remain  $N$  independent parameters, which can be taken to be the movable end points of the slits  $\lambda_i(x, t)$ . These satisfy the equations of motion

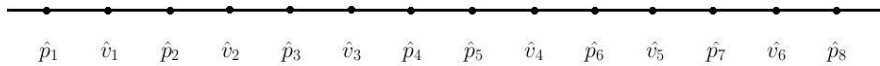
$$\frac{\partial \lambda_i}{\partial t} + \hat{v}_i \frac{\partial \lambda_i}{\partial x} = 0.$$

To understand and to solve these equations, it is necessary to understand the dependence of the  $\hat{v}_i$  on the Riemann invariants  $(\lambda_1, \dots, \lambda_N)$  - we thus need to evaluate the Schwartz-Christoffel integral explicitly.

The most tractable cases are where all the  $\alpha_j$  are rational, so that the integrand becomes a meromorphic second kind differential on some algebraic curve. In this case the only singularity is as  $p \rightarrow \infty$ , where the integrand has a double pole with no residue, and the integral thus has a simple pole. For specific families of curves, as in [3], [4], [5] and [6], this integral has been worked out explicitly. In each case, these mappings were found as rational functions of derivatives of the Kleinian  $\sigma$ -function of the associated curve.

### 3. A tetragonal reduction

We will consider reductions that allow us to work on a tetragonal surface; these have not been considered before, in the context of this application. Such reductions will require two or more sets of straight slits, making angles of  $\pi/4, \pi/2, 3\pi/4$  to the horizontal. Define  $\mathcal{P}$  to be the upper half  $p$ -plane with 14 points marked on the real axis, as in Figure 1.



**Figure 1.** The domain  $\mathcal{P}$  within the  $p$ -plane.

These points satisfy

$$\hat{p}_1 < \hat{v}_1 < \hat{p}_2 < \hat{v}_2 < \hat{p}_3 < \hat{v}_3 < \hat{p}_4 < \hat{p}_5 < \hat{v}_4 < \hat{p}_6 < \hat{v}_5 < \hat{p}_7 < \hat{v}_6 < \hat{p}_8.$$

Then define the domain  $\mathcal{L}'$  as the upper-half  $\lambda$ -plane with two triplets of slits, as described above. We let the first trio of slits radiate from the fixed point  $p_1$ , with the end points of these three slits labelled  $v_1, v_2$  and  $v_3$  respectively. Similarly, let the second trio of slits radiate from  $p_5$  and have end points  $v_4, v_5, v_6$ . Finally impose the conditions that

$$\begin{aligned} \lambda(\hat{p}) &= p, & p_1 &= p_2 = p_3 = p_4, \\ \lambda(\hat{v}) &= v, & p_5 &= p_6 = p_7 = p_8. \end{aligned}$$

We then see that  $\mathcal{L}'$  is the slit domain as shown in Figure 2, and the mapping  $\lambda : \mathcal{P} \rightarrow \mathcal{L}'$  can be given in Schwartz Christoffel form by

$$\lambda(p) = p + \int_{\infty}^p [\varphi(p') - 1] dp' \quad (15)$$

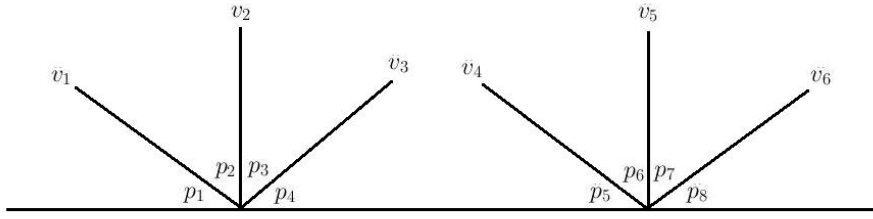
where

$$\varphi(p) = \frac{\prod_{i=1}^6 (p - \hat{v}_i)}{[\prod_{i=1}^8 (p - \hat{p}_i)]^{\frac{3}{4}}} = \frac{\prod_{i=1}^6 (p - \hat{v}_i)}{y^3} \quad (16)$$

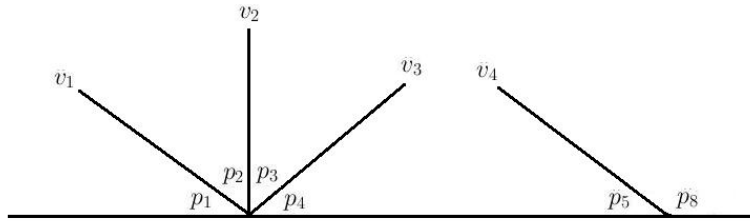
where

$$y^4 = \prod_{i=1}^8 (p - \hat{p}_i). \quad (17)$$





**Figure 2.** The domain  $\mathcal{L}'$  within the  $\lambda$ -plane.



**Figure 3.** The domain  $\mathcal{L}$  within the  $\lambda$ -plane.

Note that we require the following zero residue property:

$$\lim_{p \rightarrow \infty} \varphi(p) \sim 1 + O\left(\frac{1}{p^2}\right). \quad (18)$$

This mapping would lead us to consider the Riemann surface given by points  $(p, y)$  that satisfy (17). However, we wish to consider the simplest possible tetragonal surface (one with only six branch points) and so we collapse two of the slits, (the final two by choice). This simplifies our  $\lambda$ -plane to  $\mathcal{L}$ , given in Figure 3.

Further, as in the trigonal case, the analysis of this surface is eased if we put it into canonical form, by mapping one of the branch points ( $p_8$  by choice), to infinity. So we use the following invertible rational map to perform these simplifications on our curve and integrand.

$$\begin{aligned} \hat{p}_6 &= \hat{p}_8, & \hat{p}_7 &= \hat{p}_8, & \hat{v}_5 &= \hat{p}_8, & \hat{v}_6 &= \hat{p}_8, \\ p &= \hat{p}_8 - (1/t), & \hat{p}_i &= \hat{p}_8 - \frac{1}{T_i}, & i &= 1, \dots, 5 \end{aligned} \quad (19)$$

$$y = \frac{sk}{t^2} \quad \text{where} \quad k^4 = -\prod_{i=1}^5 (\hat{p}_8 - \hat{p}_i) = -\prod_{i=1}^5 \frac{1}{T_i}$$

If we perform the mapping (19) on the curve (17) we obtain

$$\begin{aligned} \frac{s^4 k^4}{t^8} &= \left[ \prod_{i=1}^5 \left( \hat{p}_8 - \frac{1}{t} - \hat{p}_8 + \frac{1}{T_i} \right) \right] \left( \hat{p}_8 - \frac{1}{t} - \hat{p}_8 \right)^3 = -\frac{1}{t^3} \prod_{i=1}^5 \left( \frac{1}{T_i} - \frac{1}{t} \right) \\ &= -\frac{1}{t^3} \left[ \prod_{i=1}^5 (t - T_i) \frac{1}{t T_i} \right] = \left[ \prod_{i=1}^5 (t - T_i) \right] \cdot \left( \frac{1}{t^8} \right) \cdot (-1) \left[ \prod_{i=1}^5 \frac{1}{T_i} \right]. \end{aligned}$$

This simplifies to give the following canonical form of (17).

$$s^4 = \prod_{i=1}^5 (t - T_i) = t^5 + \mu_4 t^4 + \mu_3 t^3 + \mu_2 t^2 + \mu_1 t + \mu_0, \quad (20)$$

for constants  $\mu_0, \dots, \mu_4$ . Let  $C$  denote the Riemann surface defined by (20).

We now consider  $\lambda(p)$  as mapping  $\mathcal{P} \rightarrow \mathcal{L}$  by performing (19) on the integrand (16).

$$\begin{aligned} \varphi(p)dp &= \left( \frac{t^6}{s^3 k^3} \right) \left[ \prod_{i=1}^4 \left( \hat{p}_8 - \frac{1}{t} - \hat{v}_i \right) \right] \left( \hat{p}_8 - \frac{1}{t} - \hat{p}_8 \right)^2 \left( -\frac{1}{t^2} dt \right) \\ &= \left( \frac{t^6}{s^3 k^3} \right) \left[ \prod_{i=1}^4 ((\hat{p}_8 - \hat{v}_i)t - 1) \right] \left[ \frac{1}{t^4} \right] \left( \frac{1}{t} \right)^2 \left( -\frac{1}{t^2} dt \right) \\ &= K [A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + 1] \frac{1}{t^2} \frac{dt}{4s^3} \equiv \varphi(t)dt, \end{aligned} \quad (21)$$

where  $K = -4/k^3$  and  $A_1, \dots, A_4$  are constants. We will evaluate this integrand using Kleinian functions defined upon  $C$ .

#### 4. Properties of the tetragonal surface $C$

The Riemann surface  $C$  is defined by (20), the cyclic tetragonal curve of genus six. This is also referred to as the *cyclic (4,5)-curve*. The surface is constructed from four sheets of the complex plane, with branch points of order 4 at  $T_1, \dots, T_5, T_6 = \infty$ , a local coordinate at  $t = \infty$  given by  $\xi = t^{-1/4}$  and branch cuts along the intervals

$$[T_1, T_2], \quad [T_2, T_3], \quad [T_3, T_4], \quad [T_5, \infty].$$

This surface was recently considered in [13], where the aim was to generalise the theory of the Weierstrass  $\wp$ -function to Abelian functions associated with (20). We will give the essential properties of the surface here, but refer the reader to [13] for some of the details and proofs.

We start by noting that there are a set of *Sato weights* associated to the surface, which render all equations within the theory homogeneous. The weights of the variables  $t, s$  and the curve constants can be determined up to a constant factor, from the curve equation. We may set the weights as below.

	$t$	$s$	$\mu_4$	$\mu_3$	$\mu_2$	$\mu_1$	$\mu_0$
<b>Weight</b>	-4	-5	-4	-8	-12	-16	-20

The weights of other variables and function in the theory are derived uniquely from these, with all other constants are assigned zero weight.

We define a basis of holomorphic differentials upon  $C$  by

$$\begin{aligned} \mathbf{du} &= (du_1, \dots, du_6), \quad du_i(t, s) = \frac{g_i(t, s)}{4s^3} dt, \\ \text{where} \quad &g_1(t, s) = 1, \quad g_2(t, s) = t, \quad g_3(t, s) = s, \\ &g_4(t, s) = t^2, \quad g_5(t, s) = ts, \quad g_6(t, s) = s^2. \end{aligned} \quad (22)$$

We can use the local parameter  $\xi$  to express these as series,

$$\begin{aligned} du_1 &= -\xi^{10} + O(\xi^{11})d\xi & du_4 &= -\xi^2 + \frac{3}{4}\mu_4\xi^6 + O(\xi^7)d\xi \\ du_2 &= -\xi^6 + O(\xi^7)d\xi & du_5 &= -\xi^1 + \frac{1}{2}\mu_4\xi^5 + O(\xi^6)d\xi \\ du_3 &= -\xi^5 + O(\xi^6)d\xi & du_6 &= -1 + \frac{1}{4}\mu_4\xi^4 + O(\xi^5)d\xi. \end{aligned} \quad (23)$$

We know from the general theory that any point  $\mathbf{u} \in \mathbb{C}^6$  can be expressed as

$$\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6) = \sum_{i=1}^6 \int_{\infty}^{P_i} d\mathbf{u},$$

where the  $P_i$  are six variable points upon  $C$ . Integrating (23) gives

$$\begin{aligned} u_1 &= -\frac{1}{11}\xi^{11} + O(\xi^{15}) & u_3 &= -\frac{1}{6}\xi^6 + O(\xi^{10}) & u_5 &= -\frac{1}{2}\xi^2 + O(\xi^6) \\ u_2 &= -\frac{1}{7}\xi^7 + O(\xi^{11}) & u_4 &= -\frac{1}{3}\xi^3 + O(\xi^7) & u_6 &= -\xi + O(\xi^5). \end{aligned} \quad (24)$$

from which we can conclude that the weights of  $\mathbf{u}$  are:

	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$	$u_6$
<b>Weight</b>	+11	+7	+6	+3	+2	+1

Next we choose a basis of cycles (closed paths) upon the surface defined by  $C$ . We denote them

$$\alpha_i, \beta_j, \quad 1 \leq i, j \leq 6,$$

and ensure they have intersection numbers

$$\alpha_i \cdot \alpha_j = 0, \quad \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Let  $\Lambda$  denote the lattice generated by the integrals of the basis of holomorphic differentials around this basis of cycles in  $C$ . Then the manifold  $\mathbb{C}^6/\Lambda$  is the Jacobian variety of  $C$ , denoted by  $J$ . Next, for  $k = 1, 2, \dots$  define  $\mathfrak{A}$ , the *Abel map* from the  $k$ th symmetric product  $\text{Sym}^k(C)$  to  $J$ .

$$\begin{aligned} \mathfrak{A} : \text{Sym}^k(C) &\rightarrow J \\ (P_1, \dots, P_k) &\mapsto \left( \int_{\infty}^{P_1} d\mathbf{u} + \dots + \int_{\infty}^{P_k} d\mathbf{u} \right) \pmod{\Lambda}, \end{aligned} \quad (25)$$

where the  $P_i$  are again points upon  $C$ . Denote the image of the  $k$ th Abel map by  $W^{[k]}$ , and let  $[-1](u_1, \dots, u_6) = (-u_1, \dots, -u_6)$ . We then define the  *$k$ th standard theta subset* (also referred to as the  *$k$ th stratum*) by

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}.$$

When  $k = 1$  the Abel map gives a one dimensional image of the curve  $C$ . Since our mapping was given by a single integral with respect to one parameter, it will make sense to rewrite this as an integral on the one-dimensional stratum,  $\Theta^{[1]}$ . In addition to [4],[5] and [6], similar problems of inverting meromorphic differentials on lower dimensional strata of the Jacobian have been studied, in the case of hyperelliptic surfaces, in [14], [15], [16] and [17] for example.

Let us also define a basis of second kind meromorphic differentials,  $\mathbf{dr}$ , for the surface; these have their only pole at  $\infty$ . These are determined modulo the space spanned by the  $\mathbf{du}$  and can be expressed as

$$\mathbf{dr} = (dr_1, \dots, dr_6), \quad \text{where} \quad dr_j(t, s) = \frac{h_j(t, s)}{4s^3} dx.$$

A specific set was derived in [13] in order to construct Klein's explicit realisation of the fundamental differential of the second kind. This set was given as

$$\begin{aligned} h_1 &= -s^2(11t^3 + 8t^2\mu_4 + 5t\mu_3 + 2\mu_2), & h_2 &= -s^2(7t^2 + 4t\mu_4 + \mu_3), \\ h_3 &= -2ts(3t^2 + 2t\mu_4 + \mu_3), & h_4 &= -3ts^2, & h_5 &= -2t^2s, & h_6 &= -t^3. \end{aligned}$$

We then define the period matrices  $\omega', \omega'', \eta'$  and  $\eta''$  by

$$\begin{aligned} 2\omega' &= \left( \oint_{\alpha_k} du_\ell \right)_{k,\ell=1,\dots,6} & 2\omega'' &= \left( \oint_{\beta_k} du_\ell \right)_{k,\ell=1,\dots,6} \\ 2\eta' &= \left( \oint_{\alpha_k} dr_\ell \right)_{k,\ell=1,\dots,6} & 2\eta'' &= \left( \oint_{\beta_k} dr_\ell \right)_{k,\ell=1,\dots,6} \end{aligned}.$$

We can combine these into

$$M = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix},$$

which satisfies the generalised Legendre relation,

$$M \begin{pmatrix} 0 & -1_6 \\ 1_g & 0 \end{pmatrix} M^T = -\frac{i\pi}{2} \begin{pmatrix} 0 & -1_6 \\ 1_g & 0 \end{pmatrix}.$$

## 5. The Kleinian $\sigma$ -function associated with $C$

We will now define the multivariate  $\sigma$ -function associated with  $C$ , which is constructed from the  $\theta$ -function, (see for example, [18]). This function is a generalisation of the classical Weierstrass elliptic  $\sigma$ -function, and as in the elliptic case, it can be used to construct Abelian functions on  $J$ .

**Definition 5.1.** *The Kleinian  $\sigma$ -function associated with  $C$  is*

$$\begin{aligned} \sigma(\mathbf{u}) &= \sigma(\mathbf{u}; M) = c \exp\left(-\frac{1}{2}\mathbf{u}\eta'(\omega')^{-1}\mathbf{u}^T\right) \times \theta[\delta]((\omega')^{-1}\mathbf{u}^T \mid (\omega')^{-1}\omega'') \\ &= c \exp\left(-\frac{1}{2}\mathbf{u}\eta'(\omega')^{-1}\mathbf{u}^T\right) \times \sum_{m \in \mathbb{Z}^6} \exp\left[ \right. \\ &\quad \left. 2\pi i \left\{ \frac{1}{2}(m + \delta')^T (\omega')^{-1} \omega'' (m + \delta') + (m + \delta')^T ((\omega')^{-1} \mathbf{u}^T + \delta'') \right\} \right]. \end{aligned}$$

Here  $c$  is a constant dependent upon the curve parameters,  $(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ . The results of this paper are independent of this constant, so we do not discuss its value here.

The matrix  $\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix}$  is the theta function characteristic which gives the Riemann constant for  $C$  with respect to the base point  $\infty$  and the period matrix  $[\omega', \omega'']$ , (see [19] p23-24).

We will evaluate the integrand using derivatives of  $\sigma(\mathbf{u})$ , with respect to the variables  $\mathbf{u}$ . We denote these  $\sigma$ -derivatives by adding subscripts. For example we write  $\frac{\partial \sigma}{\partial u_i}$  as  $\sigma_i$ .

**Lemma 5.2.** *We summarise the fundamental properties of the  $\sigma$ -function in this lemma. Further details and proofs are available in [13]. For a detailed study of the general multivariate  $\sigma$ -function, we refer the reader to [19].*

- Given  $\mathbf{u} \in \mathbb{C}^6$ , denote by  $\mathbf{u}'$  and  $\mathbf{u}''$  the unique elements in  $\mathbb{R}^6$  such that  $\mathbf{u} = \mathbf{u}'\omega' + \mathbf{u}''\omega''$ . Let  $\ell$  represent a point on the period lattice

$$\ell = \ell'\omega' + \ell''\omega'' \in \Lambda.$$

For  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^6$  and  $\ell \in \Lambda$ , define  $L(\mathbf{u}, \mathbf{v})$  and  $\chi(\ell)$  as follows:

$$\begin{aligned} L(\mathbf{u}, \mathbf{v}) &= \mathbf{u}^T(\eta'\mathbf{v}' + \eta''\mathbf{v}''), \\ \chi(\ell) &= \exp[\pi i(2(\ell'^T\delta'' - \ell''^T\delta') + \ell'^T\ell'')]. \end{aligned}$$

Then, for all  $\mathbf{u} \in \mathbb{C}^6, \ell \in \Lambda$  the function  $\sigma(\mathbf{u})$  is quasi-periodic.

$$\sigma(\mathbf{u} + \ell) = \chi(\ell) \exp\left[L\left(\mathbf{u} + \frac{\ell}{2}, \ell\right)\right] \cdot \sigma(\mathbf{u}). \quad (26)$$

- For  $\gamma \in Sp(12, \mathbb{Z})$  we have

$$\sigma(\mathbf{u}; \gamma M) = \sigma(\mathbf{u}; M). \quad (27)$$

- $\sigma(\mathbf{u})$  has zeroes of order 1 when  $\mathbf{u} \in \Theta^{[5]}$ , and is non-zero elsewhere.
- In the case when all the curve parameters are set to zero, the function  $\sigma(\mathbf{u})$  is equal to a constant  $\mathcal{K}$  times the Schur-Weierstrass polynomial,

$$\begin{aligned} SW_{4,5} &= \frac{1}{8382528}u_6^{15} + \frac{1}{336}u_6^8u_5^2u_4 - \frac{1}{12}u_6^4u_1 - \frac{1}{126}u_6^7u_3u_5 - \frac{1}{6}u_4u_3u_5u_6^4 \\ &\quad - \frac{1}{72}u_4^3u_6^6 - \frac{1}{33264}u_6^{11}u_5^2 + \frac{1}{27}u_5^6u_6^3 + \frac{2}{3}u_4u_5^3u_3 - 2u_4^2u_6u_3u_5 - u_2^2u_6 \\ &\quad - \frac{2}{9}u_5^3u_3u_6^3 - u_4u_3^2 + \frac{1}{12}u_4^4u_6^3 - \frac{1}{3024}u_6^9u_4^2 - \frac{1}{756}u_6^7u_5^4 + \frac{1}{1008}u_6^8u_2 \\ &\quad + \frac{1}{3}u_5^4u_2 + \frac{1}{3}u_6^3u_3^2 - \frac{1}{9}u_4u_5^6 + \frac{1}{399168}u_6^{12}u_4 + u_4u_6u_5^2u_2 + \frac{1}{4}u_4^5 \\ &\quad + 2u_5u_3u_2 + \frac{1}{6}u_5^2u_6^4u_2 + \frac{1}{12}u_6^5u_2u_4 - \frac{1}{2}u_4^2u_6^2u_2 + \frac{1}{2}u_4^3u_6^2u_5^2 \\ &\quad - \frac{1}{3}u_4^2u_6u_5^4 - \frac{1}{36}u_5^4u_4u_6^4 + u_4u_6u_1 - u_5^2u_1. \end{aligned} \quad (28)$$

- The sigma function must have definite parity and weight. From  $SW_{4,5}$  above we can conclude that  $\sigma(\mathbf{u})$  is odd with weight +15.

One of the key results in [13] was a Taylor series expansion for  $\sigma(\mathbf{u})$  about the origin, used to derive relations between the Abelian functions associated with  $C$ . We will use this expansion in the evaluation of the integrand given in Section 7. In [13] it was shown that this expansion could be constructed in the form

$$\sigma(\mathbf{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6) = C_{15}(\mathbf{u}) + C_{19}(\mathbf{u}) + \dots + C_{15+4n}(\mathbf{u}) + \dots$$

where each  $C_k$  was a finite, odd isobaric polynomial composed of sums of monomials in  $u_i$  of total weight  $+k$ , each multiplied by a monomial in  $\mu_j$  of total weight  $15 - k$ . From Lemma 5.2 we can conclude that  $C_{15} = SW_{4,5}$ , while the other  $C_k$  were found in turn, up to  $C_{59}$  by considering the possible terms, and ensuring the expansion satisfied known properties of  $\sigma(\mathbf{u})$ . (See [13] for full details of the construction, and [20] for a link to the expansion.)

## 6. Deriving relations between the $\sigma$ -function and its derivatives

We will evaluate the integrand (21) as a function of  $\sigma$ -derivatives restricted to  $\Theta^{[1]}$ . In order to achieve this we will need to derive equations that hold between the various  $\sigma$ -derivatives. In [13], sets of relations between the Abelian functions associated with  $C$  were calculated. However, these were the relations that held everywhere on  $J$ , and do not give us sufficient information for the behaviour of  $\sigma(\mathbf{u})$  on the strata. To derive such relations we start by considering a theorem of Jorgenson [21]:

**Theorem 6.1.** *Let  $\mathbf{u} \in \Theta^{[k]}$  for some  $k < g$ . Then for a set of  $k$  points  $P_i = (t_i, s_i)$  on  $C$  we have*

$$\mathbf{u} = \sum_{i=1}^k \int_{\infty}^{P_i} d\mathbf{u},$$

and the following statement holds for vectors  $\mathbf{a}, \mathbf{b}$  of arbitrary constants.

$$\frac{\sum_{j=1}^g a_j \sigma_j(\mathbf{u})}{\sum_{j=1}^g b_j \sigma_j(\mathbf{u})} = \frac{\det[\mathbf{a} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_k) | d\mathbf{u}(P_k)^{(g-k-1)} | \cdots | d\mathbf{u}(P_k)^{(1)}]}{\det[\mathbf{b} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_k) | d\mathbf{u}(P_k)^{(g-k-1)} | \cdots | d\mathbf{u}(P_k)^{(1)}]}$$

Here,  $d\mathbf{u}^{(i)}$  denotes the column of  $i$ th derivatives of the holomorphic differentials  $d\mathbf{u}$ , and should be ignored if  $i < 1$ .

Below we state that the strata of  $C$  can be defined by the zeros of the  $\sigma$ -function, and of its derivatives. The definition of  $\Theta^{[5]}$  is a classical result (in Lemma 5.2) while the others can be derived from the theorem above. (See Appendix A for full details.)

$$\begin{aligned} \Theta^{[5]} &= \{\mathbf{u} \mid \sigma(\mathbf{u}) = 0\} \\ \Theta^{[4]} &= \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = 0\} \\ \Theta^{[3]} &= \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = 0\} \\ \Theta^{[2]} &= \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = \sigma_4(\mathbf{u}) = 0\} \\ \Theta^{[1]} &= \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = \sigma_4(\mathbf{u}) = \sigma_3(\mathbf{u}) = 0\} \end{aligned} \tag{29}$$

We use these defining relations to generate further relations between the  $\sigma$ -derivatives, holding on each stratum. We use a systematic method, implemented in Maple, to achieve this.

Start with the relation  $\sigma(\mathbf{u}) = 0$  valid for  $\mathbf{u} \in \Theta^{[5]}$ . Consider  $\mathbf{u}$  as it descends to  $\Theta^{[4]}$ . We write  $\mathbf{u}$  as  $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_\xi$  where  $\hat{\mathbf{u}}$  is an arbitrary point on  $\Theta^{[4]}$  and  $\mathbf{u}_\xi$  is a vector containing the series expansions, (24). We can calculate the Taylor series expansion in  $\xi$  for  $\sigma(\hat{\mathbf{u}} + \mathbf{u}_\xi) = 0$  as

$$\begin{aligned} 0 &= \sigma(\hat{\mathbf{u}} + \mathbf{u}_\xi) = \sigma(\hat{\mathbf{u}}) - \sigma_6(\hat{\mathbf{u}})\xi + \frac{1}{2}[\sigma_{66}(\hat{\mathbf{u}}) - \sigma_5(\hat{\mathbf{u}})]\xi^2 + [\frac{1}{2}\sigma_{56}(\hat{\mathbf{u}}) \\ &\quad - \frac{1}{3}\sigma_4(\hat{\mathbf{u}}) - \frac{1}{6}\sigma_{666}(\hat{\mathbf{u}})]\xi^3 + [\frac{1}{8}\sigma_{55}(\hat{\mathbf{u}}) + \frac{1}{3}\sigma_{46}(\hat{\mathbf{u}}) - \frac{1}{4}\sigma_{566}(\hat{\mathbf{u}}) \\ &\quad + \frac{1}{24}\sigma_{6666}(\hat{\mathbf{u}})]\xi^4 + [\frac{1}{12}\sigma_{5666}(\hat{\mathbf{u}}) + \frac{1}{6}\sigma_{45}(\hat{\mathbf{u}}) - \frac{1}{120}\sigma_{66666}(\hat{\mathbf{u}}) - \frac{1}{6}\sigma_{466}(\hat{\mathbf{u}}) \\ &\quad - \frac{1}{8}\sigma_{556}(\hat{\mathbf{u}}) + \frac{1}{20}\sigma_6(\hat{\mathbf{u}})\mu_4]\xi^5 + O(\xi^6) \end{aligned}$$

Setting the coefficients of  $\xi$  to zero gives us a set of relations for  $\mathbf{u} \in \Theta^{[4]}$ :

$$\begin{aligned}
\sigma_6(\mathbf{u}) &= 0 \\
\sigma_{66}(\mathbf{u}) &= \sigma_5(\mathbf{u}) \\
\sigma_{666}(\mathbf{u}) &= 3\sigma_{56}(\mathbf{u}) - 2\sigma_4(\mathbf{u}) \\
\sigma_{6666}(\mathbf{u}) &= 6\sigma_{566}(\mathbf{u}) - 8\sigma_{46}(\mathbf{u}) - 3\sigma_{55}(\mathbf{u}) \\
\sigma_{66666}(\mathbf{u}) &= 10\sigma_{5666}(\mathbf{u}) - 20\sigma_{466}(\mathbf{u}) - 15\sigma_{556}(\mathbf{u}) + 20\sigma_{45}(\mathbf{u}) + 6\mu_4\sigma_6(\mathbf{u}) \\
&\vdots
\end{aligned} \tag{30}$$

If we calculate the expansion to a higher order of  $\xi$  then more relations can be obtained. Note however, that the expansion for  $\mathbf{u}_\xi$  must first be calculated to a sufficiently high order first. We have calculated an expansion for  $\sigma(\hat{\mathbf{u}} + \mathbf{u}_\xi)$  up to  $O(\xi^{29})$ , using the weight properties of  $\sigma(\mathbf{u})$  to simplify the calculation. This expansion can be found on-line at [20].

The next step in this process will be to find the relations valid for  $\mathbf{u} \in \Theta^{[3]}$ . Since  $\Theta^{[3]} \subset \Theta^{[4]}$  we can conclude that the relations (30) are valid here also. However, we can derive a larger set of relations for  $\mathbf{u} \in \Theta^{[3]}$  by repeating the descent procedure for those relations that are valid on  $\Theta^{[4]}$ .

We do not need to consider the relation  $\sigma(\mathbf{u}) = 0$  since that will only give us the same relations as above. Instead choose the second defining relation  $\sigma_6(\mathbf{u}) = 0$ . We again write  $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_\xi$  where  $\mathbf{u}_\xi$  is the vector of expansions as before and  $\hat{\mathbf{u}}$  is now an arbitrary point on  $\Theta^{[3]}$ . We do not need to calculate the Taylor series expansion in  $\xi$  for  $\sigma_6(\hat{\mathbf{u}} + \mathbf{u}_\xi)$  as before. Instead we can take the previous expansion and simply add 6 to each index:

$$\begin{aligned}
0 &= \sigma_6(\hat{\mathbf{u}}) - \sigma_{66}(\hat{\mathbf{u}})\xi + \frac{1}{2}[\sigma_{666}(\hat{\mathbf{u}}) - \sigma_{56}(\hat{\mathbf{u}})]\xi^2 + [\frac{1}{2}\sigma_{566}(\hat{\mathbf{u}}) - \frac{1}{3}\sigma_{46}(\hat{\mathbf{u}}) \\
&\quad - \frac{1}{6}\sigma_{6666}(\hat{\mathbf{u}})]\xi^3 + \dots
\end{aligned}$$

Setting the coefficients of  $\xi$  to zero gives us more relations valid for  $\mathbf{u} \in \Theta^{[3]}$ , starting with  $\sigma_{66}(\mathbf{u}) = 0$ . We can obtain further relations for  $\mathbf{u} \in \Theta^{[3]}$  by descending all of (30). We automate this process in Maple as follows:

1. Take a relation valid for  $\mathbf{u} \in \Theta^{[4]}$  and expand as a Taylor series in  $\xi$ . To do this we replace each  $\sigma$ -derivative by the Taylor series expansion for  $\sigma(\hat{\mathbf{u}} + \mathbf{u}_\xi)$ , adding the relevant index to each  $\sigma$ -derivative in the expansion.
2. Set each coefficient with respect to  $\xi$  to zero, and save the resulting equations.
3. Repeat steps 1 and 2 for all known relations valid for  $\mathbf{u} \in \Theta^{[4]}$ .
4. Use the set of equations we have obtained, to express the higher index  $\sigma$ -derivatives using lower-index derivatives. If we have  $\sigma$ -derivatives with the same number of indices, solve for those with the higher indices first.

Once we have finished this process we will have a set of relations valid for  $\mathbf{u} \in \Theta^{[3]}$ . We can repeat the process by descending each of these to  $\Theta^{[2]}$  creating another set of relations which we can finally descend to  $\Theta^{[1]}$ . We end up with a set of relations valid

for  $\mathbf{u} \in \Theta^{[1]}$ , some of which are contained in Appendix B, with the full set we have derived available online at [20].

The surprising result of these calculations was that on  $\Theta^{[1]}$ , we have  $\sigma_1(\mathbf{u}) = \sigma_2(\mathbf{u}) = 0$  along with the other first derivatives of  $\sigma(\mathbf{u})$ , concluded to be zero using Theorem 6.1. These calculations were computationally much more difficult than in [6]. The latter stages were performed in parallel on a small cluster of machines, using Distributed Maple, (see [22] and [23]).

## 7. Evaluating the integrand

Recall our integrand, (21)

$$\varphi(t)dt = K \left( A_4 t^2 + A_3 t + A_2 + \frac{A_1}{t} + \frac{1}{t^2} \right) \left( \frac{dt}{4s^3} \right).$$

Now,  $\lambda(t)$  was given by a single integral with respect to one parameter, the point  $(t, s)$  on  $C$ . So we rewrite this as an integral on the one-dimensional stratum  $\Theta^{[1]}$  of  $J$ , which we will parametrise by  $u_1$ . We will then evaluate it using  $\sigma$ -derivatives restricted to  $\Theta^{[1]}$ . In [6] Jorgenson's Theorem was used to express  $t$  in terms of  $\sigma$ -derivatives. However, if we solved (A.3) naively for  $\mathbf{u} \in \Theta^{[1]}$  we would find

$$t = -\frac{\sigma_1(\mathbf{u})}{\sigma_2(\mathbf{u})}$$

which makes no sense given that  $\sigma_1(\mathbf{u}) = 0$  for  $\mathbf{u} \in \Theta^{[1]}$ . Instead let us take equation (A.2) which was also derived from Jorgenson's Theorem (in Appendix A) and which holds for  $\mathbf{u}$  on  $\Theta^{[2]}$ . We consider what happens to this as  $\mathbf{u}$  descends to  $\Theta^{[1]}$ . We replace  $(t_2, s_2)$  with the expansions in the parameter  $\xi$  and replace the  $\sigma$ -derivatives by their Taylor series in  $\xi$ . If we then take series expansion of this in  $\xi$  and set  $\xi = 0$  we find that for  $\mathbf{u} \in \Theta^{[1]}$  we have

$$\frac{a_1 \sigma_{23}(\mathbf{u}) + a_2 \sigma_{34}(\mathbf{u})}{b_1 \sigma_{23}(\mathbf{u}) + b_2 \sigma_{34}(\mathbf{u})} = \frac{a_1 t - a_2}{b_1 t - b_2}.$$

Solving this for  $t$  gives:

$$t = -\frac{\sigma_{23}(\mathbf{u})}{\sigma_{34}(\mathbf{u})} \tag{31}$$

for  $\mathbf{u} \in \Theta^{[1]}$ . Therefore, using the basis of differentials (22) and equation (31) we can rewrite our integrand as:

$$\varphi(t)dt = K[\varphi_1(t)dt + \varphi_2(t)dt] \tag{32}$$

where

$$\varphi_1(t)dt = A_2 du_1 + A_3 du_2 + A_4 du_4, \tag{33}$$

$$\varphi_2(t)dt = \left( \left( \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} \right)^2 - A_1 \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} \right) du_1 \equiv \varphi_2(\mathbf{u})du_1. \tag{34}$$



Thus  $\varphi_1$  is a sum of holomorphic differentials on  $C$ , and  $\varphi_2$  is a second kind meromorphic differential. As in the previous cases we will need to find a suitable function  $\Psi(\mathbf{u})$  such that

$$\frac{d}{du_1}\Psi(\mathbf{u}) = \varphi_2(\mathbf{u}), \quad \mathbf{u} \in \Theta^{[1]}. \quad (35)$$

We will identify such a function  $\Psi(\mathbf{u})$  as follows. First we must derive the expansions for  $\varphi_2(\mathbf{u})$  at its poles. We will then find a function  $\Psi(\mathbf{u})$ , which has simple poles at the same points as the double poles of  $\varphi_2(\mathbf{u})$ , and which varies by at worst an additive constant as  $\mathbf{u}$  moves round the  $\alpha$  and  $\beta$ -cycles of  $\Theta_1$ . The function will be chosen so that  $\frac{d}{du_1}\Psi(\mathbf{u})$ , has the same expansion at the poles as  $\varphi_2$  and is regular elsewhere. It then follows that the difference  $\frac{d}{du_1}\Psi(\mathbf{u}) - \varphi_2(\mathbf{u})$  is holomorphic and Abelian; by Liouville's theorem we conclude that this difference is a constant. This constant may be evaluated at any convenient point.

### 7.1. The expansion of $\varphi_2(\mathbf{u})$ at the poles

Recall that  $\sigma(\mathbf{u})$  was an entire function, and so  $\varphi_2(\mathbf{u})$  will have poles only when  $\sigma_{23}(\mathbf{u}) = 0$ . Since we are working with  $\mathbf{u} \in \Theta^{[1]}$ , by (31) this will occur at the points, one on each sheet, where  $t = 0$ . The cyclic symmetry  $[i]$ , relating the different sheets of the curve, acts on  $(t, s)$  by  $[i](t, s) \mapsto (t, is)$ ; hence, it will act on  $\mathbf{u}$  as follows:

$$\begin{aligned} u_1 &\mapsto iu_1 & u_2 &\mapsto iu_2 & u_3 &\mapsto -u_3 \\ u_4 &\mapsto iu_4 & u_5 &\mapsto -u_5 & u_6 &\mapsto -iu_6. \end{aligned} \quad (36)$$

Let  $\mathbf{u}_0$  be the Abel image of the point on the principal sheet where  $\sigma_{23} = 0$ . This is the point where  $t = 0$  and  $s = (\mu_0)^{1/4}$ . Then the full set of zeros of  $\sigma_{23}$  are given by  $\mathbf{u}_{0,N} = [i]^N \mathbf{u}_0$ ,  $N = 0, 1, 2, 3$ . We will require the poles to match at all four of these points.

We need to find an expansion for  $\varphi_2(\mathbf{u})$  at these points. To start, we consider  $\mathbf{u} \in \Theta^{[1]}$  and calculate the Taylor series of  $\sigma(\mathbf{u})$  around the point  $\mathbf{u} = \mathbf{u}_0 = (u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}, u_{0,5}, u_{0,6})$ . Writing  $w_i = (u_i - u_{0,i})$  we have

$$\begin{aligned} \sigma(\mathbf{u}) &= \sigma(\mathbf{u}_0) + [\sigma_1(\mathbf{u}_0)w_1 + \sigma_2(\mathbf{u}_0)w_2 + \sigma_3w_3(\mathbf{u}_0) + \sigma_4(\mathbf{u}_0)w_4 \\ &\quad + \sigma_5(\mathbf{u}_0)w_5 + \sigma_6(\mathbf{u}_0)w_6] + [\frac{1}{2}\sigma_{11}(\mathbf{u}_0)w_1^2 + \sigma_{12}(\mathbf{u}_0)w_1w_2 + \sigma_{13}(\mathbf{u}_0)w_1w_3 \\ &\quad + \sigma_{14}(\mathbf{u}_0)w_1w_4 + \sigma_{15}(\mathbf{u}_0)w_1w_5 + \sigma_{16}(\mathbf{u}_0)w_1w_6 + \frac{1}{2}\sigma_{22}(\mathbf{u}_0)w_2^2 + \dots \end{aligned}$$

We will have similar expansions around the other  $\mathbf{u}_{0,N}$ , and we can also use this expansion to easily compute the expansions for the  $\sigma$ -derivatives, (by simply adding the relevant indices). Note, that since  $\mathbf{u}_{0,N}$  are the points where  $t = 0$ , we can write their components as

$$u_{0,i}^{[N]} = \int_{\infty}^0 du_i, \quad i = 1, \dots, 6.$$

evaluated on the sheet where  $s = [i]^N(\mu_0)^{1/4}$ . Therefore

$$w_{i,N} := (u_i - u_{0,i}^{[N]}) = \int_{\infty}^t du_i - \int_{\infty}^0 du_i = \int_0^t du_i,$$

evaluated on this sheet. Using (22), our basis of holomorphic differentials, we can find expansions for  $w_{1,N}, \dots, w_{6,N}$  in the parameter  $t$ .

$$\begin{aligned}
 w_{1,N} &= \frac{1}{4} \frac{i^N}{\mu_0^{3/4}} t - \frac{3}{32} \frac{i^{3N} \mu_1}{\mu_0^{7/4}} t^2 - \frac{i^{3N}}{128} \frac{8\mu_2\mu_0 - 7\mu_1^2}{\mu_0^{11/4}} t^3 \\
 &\quad - \frac{i^N}{2048} \frac{(96\mu_3\mu_0^2 - 168\mu_1\mu_2\mu_0 + 77\mu_1^3)}{\mu_0^{15/4}} t^4 + O(t^5) \\
 w_{2,N} &= \frac{1}{8} \frac{i^N}{\mu_0^{3/4}} t^2 - \frac{1}{16} \frac{i^N \mu_1}{\mu_0^{7/4}} t^3 - \frac{3i^N}{512} \frac{8\mu_2\mu_0 - 7\mu_1^2}{\mu_0^{11/4}} t^4 + O(t^5) \\
 w_{3,N} &= \frac{1}{4} \frac{i^{2N}}{\mu_0^{1/2}} t - \frac{1}{16} \frac{i^{2N} \mu_1}{\mu_0^{3/2}} t^2 - \frac{i^{2N}}{96} \frac{4\mu_2\mu_0 - 3\mu_1^2}{\mu_0^{5/2}} t^3 \\
 &\quad - \frac{i^{2N}}{256} \frac{8\mu_3\mu_0^2 - 12\mu_1\mu_2\mu_0 + 5\mu_1^3}{\mu_0^{7/2}} t^4 + O(t^5) \\
 w_{4,N} &= \frac{1}{12} \frac{i^N}{\mu_0^{3/4}} t^3 - \frac{3}{64} \frac{i^N \mu_1}{\mu_0^{7/4}} t^4 + O(t^5) \\
 w_{5,N} &= \frac{1}{8} \frac{i^{2N}}{\mu_0^{1/2}} t^2 - \frac{1}{24} \frac{i^{2N} \mu_1}{\mu_0^{3/2}} t^3 - \frac{i^{2N}}{128} \frac{4\mu_2\mu_0 - 3\mu_1^2}{\mu_0^{5/2}} t^4 + O(t^5) \\
 w_{6,N} &= \frac{1}{4} \frac{i^{3N}}{\mu_0^{1/4}} t - \frac{1}{32} \frac{i^{3N} \mu_1}{\mu_0^{5/4}} t^2 - \frac{i^{3N}}{384} \frac{8\mu_2\mu_0 - 5\mu_1^2}{\mu_0^{9/4}} t^3 \\
 &\quad - \frac{i^{3N}}{2048} \frac{(32\mu_3\mu_0^2 - 40\mu_1\mu_2\mu_0 + 15\mu_1^3)}{\mu_0^{13/4}} t^4 + O(t^5)
 \end{aligned} \tag{37}$$

Note that all these expansions are given for the general sheet since we need to check the behaviour at all the poles. We can move between the sheets by selecting the appropriate value of  $N$ . We can invert (37) on the  $N$ -th sheet to give an expansion for  $t$  in  $w_{1,N}$ , allowing us to use  $w_{1,N}$  as a local parameter near  $\mathbf{u}_{0,N}$ .

$$t = 4 i^{3N} \mu_0^{3/4} w_{1,N} + 6\mu_1 i^{6N} \mu_0^{1/2} w_{1,N}^2 + O(w_{1,N}^3).$$

We start by substituting for  $t$  to give the expansions of  $w_{2,N}, \dots, w_{6,N}$  with respect to  $w_{1,N}$ .

$$\begin{aligned}
 w_{2,N} &= 2i^{3N} \mu_0^{3/4} w_{1,N}^2 + O(w_{1,N}^3) \\
 &\quad \vdots \\
 w_{6,N} &= i^{2N} \mu_0^{1/2} w_{1,N} + \mu_0^{1/4} \mu_1 i^{5N} w_{1,N}^2 + O(w_{1,N}^3).
 \end{aligned} \tag{38}$$

We use these in turn to give the  $\sigma$ -derivative expansions at  $\mathbf{u}_{0,N}$  as series in  $w_{1,N}$ . For example we have,

$$\begin{aligned}
 \sigma_{23}(\mathbf{u}) &= \sigma_{23}(\mathbf{u}_{0,N}) + \left( i^{2N} \mu_0^{1/2} \sigma_{236} + \sigma_{123} + i^N \mu_0^{1/4} \sigma_{233} \right) (\mathbf{u}_{0,N}) w_{1,N} \\
 &\quad + O(w_{1,N}^2).
 \end{aligned}$$

We substitute these into (34) to obtain an expansion of  $\varphi_2(\mathbf{u})$  at  $\mathbf{u} = \mathbf{u}_{0,N}$ , as a series in  $w_{1,N}$ .

$$\varphi_2(\mathbf{u}) = \frac{1}{w_{1,N}^2} \left( \frac{\sigma_{34}}{i^N \mu_0^{1/4} \sigma_{233} + i^{2N} \mu_0^{1/2} \sigma_{236} + \sigma_{123}} \right)^2 (\mathbf{u}_{0,N}) + \tag{39}$$

$$+ \frac{\mathcal{C}(\mathbf{u}_{0,N})}{w_{1,N}} + O(w_{1,N}^0) \quad (40)$$

where  $\mathcal{C}(\mathbf{u}_{0,N})$  is a polynomial in the  $\sigma$ -derivatives, which we need to evaluate to ensure that  $\varphi_2 dt$  has zero residue.

In the previous section we derived a set of relations for  $\mathbf{u} \in \Theta^{[1]}$ , but these are not sufficient to simplify  $\mathcal{C}(\mathbf{u}_{0,N})$ . We need to generate a further set of relations which are valid only at  $\mathbf{u} = \mathbf{u}_{0,N}$ . We do this using a similar approach to the previous section. We take a relation valid on  $\Theta^{[1]}$  and calculate its expansion around  $\mathbf{u} = \mathbf{u}_{0,N}$  as a series in  $w_{1,N}$ , using the series derived above. We then set to zero the coefficients of  $w_{1,N}$ . We do this for each relation valid on  $\Theta^{[1]}$  and obtain a set of equations between  $\sigma$ -derivatives at the points  $\mathbf{u} = \mathbf{u}_{0,N}$ . The first few such relations are given in Appendix C, with a fuller set available online at [20].

If we substitute these into the expansion of  $\varphi_2(\mathbf{u})$  we obtain

$$\varphi_2(\mathbf{u}) = \left[ \frac{i^{2N}}{16} \frac{1}{\mu_0^{3/2}} \right] \frac{1}{w_{1,N}^2} + \left[ \frac{i^N}{16} (4\mu_0 A_1 - 3\mu_1) \right] \frac{1}{w_{1,N}} + O(w_{1,N}^0) \quad (41)$$

Recall equation (18) which stated that  $\varphi(p)$  has zero residue at  $p = \infty$  on all sheets. Since residues are invariant under conformal maps, we can conclude that  $\varphi_2(\mathbf{u})$  must also have zero residue, and so the constant  $A_1$  must be equal to

$$A_1 = \frac{3\mu_1}{4\mu_0}, \quad \text{giving} \quad \varphi_2(\mathbf{u}) = \left[ \frac{i^{2N}}{16} \frac{1}{\mu_0^{3/2}} \right] \frac{1}{w_{1,N}^2} + O(w_{1,N}^0). \quad (42)$$

## 7.2. Finding a suitable function $\Psi(\mathbf{u})$

We need to derive a function  $\Psi(\mathbf{u})$  such that the Laurent expansion of  $\frac{d\Psi(\mathbf{u})}{du_1}$  has the same principal part at the poles as  $\varphi_2(\mathbf{u})$ , so we will restrict our search to linear expressions in  $\sigma$ -derivatives, divided by  $\sigma_{23}(\mathbf{u})$ . For these functions we will derive expansions in  $w_{1,N}$  at  $\mathbf{u} = \mathbf{u}_{0,N}$  using the techniques described above. Let us take the function

$$\Psi(\mathbf{u}) = \sum_{1 \leq i \leq 6} \eta_i \frac{\sigma_i(\mathbf{u})}{\sigma_{23}(\mathbf{u})} + \sum_{\substack{1 \leq i < j \leq 6 \\ [i,j] \neq [2,3]}} \eta_{ij} \frac{\sigma_{ij}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} + \sum_{1 \leq i < j < k \leq 6} \eta_{ijk} \frac{\sigma_{ijk}(\mathbf{u})}{\sigma_{23}(\mathbf{u})},$$

where the  $\eta_{ij}$  and  $\eta_{ijk}$  are undetermined constants. (Through trial and error we found that 3-index  $\sigma$ -derivatives were required in the numerator). Now, since we are working on  $\Theta^{[1]}$  we will find that many of these  $\sigma$ -derivatives are equal to zero, or can be expressed as a linear combination of other such functions using the equations in Appendix B. Let us set the coefficients of these functions to zero, leaving us with

$$\begin{aligned} \Psi(\mathbf{u}) = & [\eta_{22}\sigma_{22} + \eta_{34}\sigma_{34} + \eta_{111}\sigma_{111} + \eta_{122}\sigma_{122} + \eta_{123}\sigma_{123} + \eta_{134}\sigma_{134} \\ & + \eta_{222}\sigma_{222} + \eta_{223}\sigma_{223} + \eta_{224}\sigma_{224} + \eta_{225}\sigma_{225} + \eta_{226}\sigma_{226} + \eta_{233}\sigma_{233} \\ & + \eta_{234}\sigma_{234} + \eta_{235}\sigma_{235} + \eta_{236}\sigma_{236} + \eta_{334}\sigma_{334} + \eta_{344}\sigma_{344} + \eta_{345}\sigma_{345} \\ & + \eta_{346}\sigma_{346}] (\mathbf{u}) \cdot \frac{1}{\sigma_{23}(\mathbf{u})}. \end{aligned}$$

We emphasise that we need to work with the *total*, not the partial, derivative of  $\Psi(\mathbf{u})$  with respect to  $u_1$ ; in practice the other  $u_i$  are expressed in terms of  $w_{1,N}$  in the vicinity of  $\mathbf{u}_{0,N}$  so there is no ambiguity. Note from (22) that

$$\frac{\partial}{\partial u_2} = t \frac{\partial}{\partial u_1}, \quad \frac{\partial}{\partial u_3} = s \frac{\partial}{\partial u_1}, \quad \frac{\partial}{\partial u_4} = t^2 \frac{\partial}{\partial u_1}, \quad \frac{\partial}{\partial u_5} = st \frac{\partial}{\partial u_1}, \quad \frac{\partial}{\partial u_6} = s^2 \frac{\partial}{\partial u_1}.$$

Therefore

$$\begin{aligned} D_1 &:= \left. \frac{d}{du_1} \right|_{\Theta^{[1]}} = \frac{\partial}{\partial u_1} + t \frac{\partial}{\partial u_2} + s \frac{\partial}{\partial u_3} + t^2 \frac{\partial}{\partial u_4} + st \frac{\partial}{\partial u_5} + s^2 \frac{\partial}{\partial u_6} \\ &= \frac{\partial}{\partial u_1} - \frac{\sigma_{23}(\mathbf{u})}{\sigma_{34}(\mathbf{u})} \frac{\partial}{\partial u_2} + s \frac{\partial}{\partial u_3} + \left( \frac{\sigma_{23}(\mathbf{u})}{\sigma_{34}(\mathbf{u})} \right)^2 \frac{\partial}{\partial u_4} - s \frac{\sigma_{23}}{\sigma_{34}} \frac{\partial}{\partial u_5} + s^2 \frac{\partial}{\partial u_6} \end{aligned}$$

We can now evaluate  $\frac{d}{du_1} \Psi(\mathbf{u})$  as a sum of quotients of  $\sigma$ -derivatives. For example

$$\begin{aligned} D_1 \left( \frac{\sigma_{236}}{\sigma_{23}} \right) &= \left[ \frac{\sigma_{1236}}{\sigma_{23}} - \frac{\sigma_{236}\sigma_{123}}{\sigma_{23}^2} - \frac{\sigma_{2236}}{\sigma_{34}} + \frac{\sigma_{236}\sigma_{223}}{\sigma_{23}\sigma_{34}} + s \frac{\sigma_{2336}}{\sigma_{23}} - s \frac{\sigma_{236}\sigma_{233}}{\sigma_{23}^2} \right. \\ &\quad \left. + \frac{\sigma_{23}\sigma_{2346}}{\sigma_{34}^2} - \frac{\sigma_{236}\sigma_{234}}{\sigma_{34}^2} - s \frac{\sigma_{2356}}{\sigma_{34}^2} + s \frac{\sigma_{236}\sigma_{235}}{\sigma_{23}\sigma_{34}} + s^2 \frac{\sigma_{2366}}{\sigma_{23}} - s^2 \frac{\sigma_{236}^2}{\sigma_{23}^2} \right]. \end{aligned}$$

Now let us consider the expansion of  $D_1(\Psi(\mathbf{u}))$  at  $\mathbf{u} = \mathbf{u}_{0,N}$ . We generate series expansions in  $w_{1,N}$  for the relevant  $\sigma$ -derivatives using the method described in the previous subsection. We can use the relations in Appendix B and Appendix C to simplify these expansions, and so obtain a series in  $w_{1,N}$  for  $D_1(\Psi(\mathbf{u}))$ . We find,

$$\left. \frac{d}{du_1} \Psi(\mathbf{u}) \right|_{\mathbf{u}=\mathbf{u}_{0,N}} = \frac{N(\mathbf{u}_{0,N})}{\sigma_{22}(\mathbf{u}_{0,N})} \left[ \frac{1}{w_{1,N}^2} \right] + O(w_{1,N}^0),$$

where  $N(\mathbf{u}_{0,N})$  is a linear polynomial in  $\{\sigma_{22}, \sigma_{122}, \sigma_{222}, \sigma_{223}, \sigma_{224}, \sigma_{225}, \sigma_{226}\}$ . This set of  $\sigma$ -derivatives can be used to express all other 2 and 3-index  $\sigma$ -derivatives when  $\mathbf{u} = \mathbf{u}_{0,N}$ , (as in Appendix C). We find the coefficients of  $N(\mathbf{u}_{0,N})$  with respect to each of these seven  $\sigma$ -derivatives and determine conditions on the constants  $\eta_{ij}, \eta_{ijk}$  that set all the coefficients, except that of  $\sigma_{22}$ , to zero. We then obtain further conditions by ensuring the expansion we are left with (now independent of any  $\sigma$ -derivatives) is equal to (41) on the four sheets. Imposing these conditions on  $\Psi(\mathbf{u})$  leaves us with

$$\Psi(\mathbf{u}) = \left[ \eta_{22} \frac{\sigma_{22}}{\sigma_{23}} + \eta_{111} \frac{\sigma_{111}}{\sigma_{23}} + 2\eta_{334} \frac{\sigma_{235}}{\sigma_{23}} + \frac{8\eta_{22}\mu_0 - 1}{4\mu_0} \frac{\sigma_{236}}{\sigma_{23}} + \eta_{334} \frac{\sigma_{334}}{\sigma_{23}} \right](\mathbf{u}).$$

Note from Appendix C that the terms containing  $\sigma_{111}, \sigma_{235}, \sigma_{334}$  all vanish at the points  $\mathbf{u} = \mathbf{u}_{0,N}$  and so have no effect on the expansion here. Let us discard these to leave,

$$\Psi(\mathbf{u}) = \eta_{22} \frac{\sigma_{22}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} + \frac{1}{4} \frac{(8\eta_{22}\mu_0 - 1)}{\mu_0} \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}$$

We now have two functions,  $\varphi_2(\mathbf{u})$  and  $D_1(\Psi(\mathbf{u}))$ , which both have poles at  $\mathbf{u}_{0,N}$ . We have derived expansions at these points, given in the local parameter  $w_{1,N}$ , and ensured that they match. However, we should also separately check, explicitly, what happens at the point  $\mathbf{u} = \mathbf{0}$ , since  $w_{1,N}$  is not a suitable local parameter here. We can instead use the Taylor series expansion of  $\sigma(\mathbf{u})$  presented in [13] and described in Section 5. We differentiate this to give expansions for the  $\sigma$ -derivatives and then, since we are

at the origin, replace the variables  $u_1, \dots, u_6$  with their expansions, (24), in the local parameter  $\xi$ .

Now, the sigma expansion was given as a sum of polynomials with increasing weight in  $\mathbf{u}$  and hence the expansions will have increasing order of  $\xi$ . Since the functions we consider all contain ratios of  $\sigma$ -derivatives we will only need the leading terms from each expansion, in order to check regularity. Hence we only require a minimum amount of the sigma expansion, sufficient to give non-zero expansions for the derivatives we consider. We find that for the functions used here, we can truncate the expansion after  $C_{35}$ .

Substituting these expansions into  $\varphi_2(\mathbf{u})$ , we find

$$\lim_{\mathbf{u} \rightarrow \mathbf{0}} \varphi_2(\mathbf{u}) = \lim_{\xi \rightarrow 0} \left[ \frac{3\mu_1}{4\mu_0} \xi^4 + \xi^8 + O(\xi^{36}) \right] = 0. \quad (43)$$

So  $\varphi_2(\mathbf{u})$  is regular at the origin, and hence we must ensure that  $\Psi(\mathbf{u})$  is as well. Upon substitution into  $\Psi(\mathbf{u})$ , we find that we must set  $\eta_{22} = 0$  for the expansion to be regular. This leaves us with

$$\Psi(\mathbf{u}) = -\frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})},$$

with

$$\lim_{\mathbf{u} \rightarrow \mathbf{0}} \Psi(\mathbf{u}) = \lim_{\xi \rightarrow 0} \left[ -\frac{1}{28} \frac{\mu_3}{\mu_0} \xi^7 + \frac{1}{176} \frac{(-8\mu_2 + 3\mu_4\mu_3)}{\mu_0} \xi^{11} + O(\xi^{15}) \right] = 0. \quad (44)$$

Now all that remains is to check the periodicity properties of the functions. Recall equation (26) which gave the quasi-periodicity property of  $\sigma(\mathbf{u})$ . We can differentiate this, and use the relations in Appendix B to show that, first, for any  $\ell \in \Lambda$ , the lattice of half-periods,

$$\begin{aligned} \sigma_{23}(\mathbf{u} + \ell) &= \chi(\ell) \exp(L(\mathbf{u} + \frac{\ell}{2}, \ell)) \sigma_{23}(\mathbf{u}), \\ \sigma_{34}(\mathbf{u} + \ell) &= \chi(\ell) \exp(L(\mathbf{u} + \frac{\ell}{2}, \ell)) \sigma_{34}(\mathbf{u}), \end{aligned}$$

for  $\sigma$  and its first derivatives vanish on  $\Theta^{[1]}$ ; hence from equation (34) we can see that  $\varphi_2(\mathbf{u} + \ell) = \varphi_2(\mathbf{u})$  and so is indeed Abelian. Further, we find that:

$$\sigma_{236}(\mathbf{u} + \ell) = \chi(\ell) \exp(L(\mathbf{u} + \frac{\ell}{2}, \ell)) \cdot \left[ \frac{\partial}{\partial u_6} L(\mathbf{u} + \frac{\ell}{2}, \ell) \cdot \sigma_{23} + \sigma_{236} \right]. \quad (45)$$

It then follows that:

$$\Psi(\mathbf{u} + \ell) = -\frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{236}(\mathbf{u} + \ell)}{\sigma_{23}(\mathbf{u} + \ell)} = \Psi(\mathbf{u}) - \frac{1}{4} \frac{1}{\mu_0} \frac{\partial}{\partial u_6} (L(\mathbf{u} + \frac{\ell}{2}, \ell))$$

Hence  $D_1(\Psi(\mathbf{u}))$  is Abelian, though  $\Psi$  itself is not, and we may now write our integrand  $\varphi(t)dt$  as

$$\varphi_2(\mathbf{u}) du_1 + A_2 du_1 + A_3 du_2 + A_4 du_4 = D_1(\Psi(\mathbf{u})) + \mathbf{B}^T d\mathbf{u}, \quad (46)$$

for some vector of constants  $\mathbf{B}^T = (B_1, B_2, B_3, B_4, B_5, B_6)$ .

### 7.3. Evaluating the vector $\mathbf{B}$

We can evaluate the vector  $\mathbf{B}$  by considering the integral of equation (46) at the point  $\mathbf{u} = \mathbf{0}$ . We can use the expansions in (23), (43) and (44) to obtain the following series.

$$\begin{aligned}
 0 = & B_6\xi + \frac{B_5}{2}\xi^2 + \left(-\frac{A_4}{3} + \frac{B_4}{3}\right)\xi^3 - \frac{B_6}{20}\mu_4\xi^5 + \left(-\frac{B_5}{12}\mu_4 + \frac{B_3}{6}\right)\xi^6 \\
 & + \frac{1}{28}\frac{1}{\mu_0}(3\mu_4\mu_0A_4 - \mu_3 - 3\mu_4\mu_0B_4 - 4A_3\mu_0 + 4B_2\mu_0)\xi^7 \\
 & + \frac{B_6}{288}(5\mu_4^2 - 8\mu_3)\xi^9 + \left(-\frac{B_3}{20}\mu_4 - \frac{B_5}{20}\mu_3 + \frac{3B_5}{80}\mu_4^2\right)\xi^{10} \\
 & - \frac{1}{352}\frac{1}{\mu_0}(32A_2\mu_0 - 32B_1\mu_0 + 16\mu_2 - 6\mu_4\mu_3 - 24\mu_0A_4\mu_3 + 21\mu_0A_4\mu_4^2 \\
 & + 24\mu_0B_4\mu_3 - 21\mu_0B_4\mu_4^2 - 24\mu_4\mu_0A_3 + 24\mu_4\mu_0B_2)\xi^{11} + O(\xi^{13})
 \end{aligned}$$

Setting each coefficient of  $\xi$  to zero, we find

$$\begin{aligned}
 B_1 = \frac{1}{2}\frac{(\mu_2 + 2A_2\mu_0)}{\mu_0}, & \quad B_2 = \frac{1}{4}\frac{(4A_3\mu_0 + \mu_3)}{\mu_0}, & \quad B_3 = 0, \\
 B_4 = A_4, & \quad B_5 = 0, & \quad B_6 = 0.
 \end{aligned}$$

## 8. Obtaining an explicit formula for $\lambda(p)$

We now use the results of Section 7 to derive an explicit formula for the mapping  $\lambda(p)$ . Start by applying the change of coordinates given in (19) to  $\lambda(p)$  as given in (15).

$$\begin{aligned}
 \lambda(p) &= p + \int_{\infty}^p [\varphi(p') - 1] dp' = \left(\hat{p}_8 - \frac{1}{t}\right) + \int_0^{\frac{1}{\hat{p}_8 - p}} \left(\varphi(t) - \frac{1}{t^2}\right) dt \\
 &= \left(\hat{p}_8 - \frac{1}{t}\right) - \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{t^2}\right] dt \\
 &\quad + \int_0^{\frac{1}{\hat{p}_8 - p}} \left(K[A_4t^4 + A_3t^3 + A_2t^2 + A_1t + 1]\frac{1}{t^2}\frac{dt}{4s^3}\right),
 \end{aligned}$$

where the constants  $A_1, A_2, A_3, A_4$  and  $K$  were defined by equation(21). Note from (31) that

$$p = \hat{p}_8 + \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}, \quad \mathbf{u} \in \Theta^{[1]}. \quad (47)$$

So let us take  $\mathbf{u} \in \Theta^{[1]}$ , and use (47) and the evaluation of  $\varphi(t)dt$  from the previous section to write  $\lambda(p)$  as

$$\begin{aligned}
 \lambda(p) &= \left(\hat{p}_8 + \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}\right) - \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{t^2}\right] dt + K \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{2}\frac{(\mu_2 + 2A_2\mu_0)}{\mu_0} du_1 \right. \\
 &\quad \left. + \frac{1}{4}\frac{(4A_3\mu_0 + \mu_3)}{\mu_0} du_2 + A_4 du_4\right] + K \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{d}{du_1} \left(-\frac{1}{4}\frac{1}{\mu_0}\frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}\right)\right] du_1.
 \end{aligned}$$

Integrating, gives

$$\lambda(p) = \left(\hat{p}_8 + \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}\right) - \left[\frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})}\right] + K \left[\frac{1}{2}\frac{(\mu_2 + 2A_2\mu_0)}{\mu_0} u_1\right]$$

$$\begin{aligned}
 & + \frac{1}{4} \frac{(4A_3\mu_0 + \mu_3)}{\mu_0} u_2 + A_4 u_4 \Big] + K \left[ -\frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} \right] + \hat{C} \\
 & = \hat{p}_8 + K \left[ -\frac{1}{4\mu_0} \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} + \frac{\mu_2 + 2A_2\mu_0}{2\mu_0} u_1 + \frac{4A_3\mu_0 + \mu_3}{4\mu_0} u_2 + A_4 u_4 \right] + \hat{C}.
 \end{aligned}$$

for some constant  $\hat{C}$ . We can determine  $\hat{C}$  by ensuring that the following condition on the mapping is satisfied.

$$\lim_{p \rightarrow \infty} \lambda(p) = p + O\left(\frac{1}{p}\right)$$

Note from (47) that  $p \rightarrow \infty$  implies  $\sigma_{23}(\mathbf{u}) \rightarrow 0$  and therefore  $\mathbf{u} \rightarrow \mathbf{u}_{0,N}$ .

$$\begin{aligned}
 \lim_{p \rightarrow \infty} [\lambda(p) - p] &= \lim_{\mathbf{u} \rightarrow \mathbf{u}_{0,N}} \left[ \hat{C} - \frac{\sigma_{34}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} \right. \\
 & \left. + K \left( -\frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} + \frac{\mu_2 + 2A_2\mu_0}{2\mu_0} u_1 + \frac{4A_3\mu_0 + \mu_3}{4\mu_0} u_2 + A_4 u_4 \right) \right]
 \end{aligned}$$

Let us ensure that the condition is met on the sheet of the surface  $C$  associated with  $\lim_{t \rightarrow 0}(s) = \mu_0^{1/4}$ . We can write this as a series expansion in the local parameter  $w_1$ , (as described in Section 7.1). Recall that  $u_i = w_i + u_{0,i}$ , and use the expansions (38) and the existing expansions for the  $\sigma$ -derivatives to obtain

$$\begin{aligned}
 \lim_{p \rightarrow \infty} [\lambda(p) - p] &= \left[ \frac{1}{4} \frac{1}{\mu_0^{3/4}} - \frac{K}{16} \frac{1}{\mu_0^{3/2}} \right] \frac{1}{w_1} + \left[ \hat{C} - \frac{3}{8} \frac{\mu_1}{\mu_0} + K \left( -\frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{226}(\mathbf{u}_0)}{\sigma_{22}(\mathbf{u}_0)} \right. \right. \\
 & \left. \left. + \frac{1}{32} \frac{\mu_1}{\mu_0^{7/4}} + \frac{(\mu_2 + 2A_2\mu_0)}{2\mu_0} u_{0,1} + \frac{(4A_3\mu_0 + \mu_3)}{4\mu_0} u_{0,2} + A_4 u_{0,4} \right) \right] + O(w_1).
 \end{aligned}$$

Therefore, we must set the constants of integration,  $\hat{C}$ , to be

$$\begin{aligned}
 \hat{C} &= \frac{3}{8} \frac{\mu_1}{\mu_0} + K \left( + \frac{1}{4} \frac{1}{\mu_0} \frac{\sigma_{226}(\mathbf{u}_0)}{\sigma_{22}(\mathbf{u}_0)} - \frac{1}{32} \frac{\mu_1}{\mu_0^{7/4}} - \frac{1}{2} \frac{(\mu_2 + 2A_2\mu_0)}{\mu_0} u_{0,1} \right. \\
 & \left. - \frac{1}{4} \frac{(4A_3\mu_0 + \mu_3)}{\mu_0} u_{0,2} - A_4 u_{0,4} \right)
 \end{aligned}$$

giving us the following explicit formula for the mapping  $\lambda(p)$ .

$$\begin{aligned}
 \lambda(p) &= \hat{p}_8 + \frac{3}{8} \frac{\mu_1}{\mu_0} + K \left[ -\frac{1}{4} \frac{1}{\mu_0} \left[ \frac{\sigma_{236}(\mathbf{u})}{\sigma_{23}(\mathbf{u})} - \frac{\sigma_{226}(\mathbf{u}_0)}{\sigma_{22}(\mathbf{u}_0)} \right] - \frac{1}{32} \frac{\mu_1}{\mu_0^{7/4}} \right. \\
 & \left. + \frac{\mu_2 + 2A_2\mu_0}{2\mu_0} (u_1 - u_{1,0}) + \frac{4A_3\mu_0 + \mu_3}{4\mu_0} (u_2 - u_{2,0}) + A_4 (u_4 - u_{4,0}) \right],
 \end{aligned}$$

where  $\mathbf{u} \in \Theta^{[1]}$  and  $\mathbf{u}_0$  is the point on the principal sheet of the surface  $C$  where  $t = 0$ .

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## Appendix A. Deriving defining relations for the strata from Jorgenson's Theorem

Consider Theorem 6.1 in the case when  $k = 5$ . Then for arbitrary  $\mathbf{a}, \mathbf{b}$

$$\frac{\sum_{j=1}^6 a_j \sigma_j(\mathbf{u})}{\sum_{j=1}^6 b_j \sigma_j(\mathbf{u})} = \frac{\det[\mathbf{a} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_5)]}{\det[\mathbf{b} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_5)]}.$$

Now, as  $\mathbf{u} \in \Theta^{[5]}$  approaches  $\Theta^{[4]}$  we have the point  $P_5 = (t_5, s_5)$  approaching  $\infty$ . We can use the local coordinate  $\xi$  here and hence replace the final column of the determinants by the expansions (23). When  $\mathbf{u}$  arrives at  $\Theta^{[4]}$  we will have  $\xi = 0$  and hence the determinant in the numerator becomes

$$\begin{vmatrix} a_1 & \frac{dt_1}{4s_1^3} & \cdots & \frac{dt_4}{4s_4^3} & 0 \\ a_2 & \frac{t_1 dt_1}{4s_1^3} & \cdots & \frac{t_4 dt_4}{4s_4^3} & 0 \\ a_3 & \frac{s_1 dt_1}{4s_1^3} & \cdots & \frac{s_4 dt_4}{4s_4^3} & 0 \\ a_4 & \frac{t_1^2 dt_1}{4s_1^3} & \cdots & \frac{t_4^2 dt_4}{4s_4^3} & 0 \\ a_5 & \frac{t_1 s_1 dt_1}{4s_1^3} & \cdots & \frac{t_4 s_4 dt_4}{4s_4^3} & 0 \\ a_6 & \frac{s_1^2 dt_1}{4s_1^3} & \cdots & \frac{s_4^2 dt_4}{4s_4^3} & -1 \end{vmatrix} = \frac{dt_1}{3s_1^3} \cdots \frac{dt_4}{3s_4^3} \begin{vmatrix} a_1 & 1 & \cdots & 1 & 0 \\ a_2 & t_1 & \cdots & t_4 & 0 \\ a_3 & s_1 & \cdots & s_4 & 0 \\ a_4 & t_1^2 & \cdots & t_4^2 & 0 \\ a_5 & t_1 s_1 & \cdots & t_4 s_4 & 0 \\ a_6 & s_1^2 & \cdots & s_4^2 & -1 \end{vmatrix}$$

The determinant in the denominator will be identical, except with the entries of  $\mathbf{a}$  replaced by the entries of  $\mathbf{b}$ . Hence the factored terms will cancel, leaving us with the simpler determinants. It is clear from the final column, that when we expand the determinants the resulting quotient of polynomials will not vary with the arbitrary constant  $a_6$ . Hence we must conclude that for  $\mathbf{u} \in \Theta^{[4]}$ ,  $\sigma_6(\mathbf{u}) = 0$ . The same conclusion could have been drawn from considering  $b_6$ .

$$\Theta^{[4]} = \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = 0\}.$$

We repeat this process by considering Theorem 6.1 in the case when  $k = 4$ .

$$\frac{\sum_{j=1}^6 a_j \sigma_j(\mathbf{u})}{\sum_{j=1}^6 b_j \sigma_j(\mathbf{u})} = \frac{\det[\mathbf{a} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_4) | d\mathbf{u}(P_4)^{(1)}]}{\det[\mathbf{b} | d\mathbf{u}(P_1) | \cdots | d\mathbf{u}(P_4) | d\mathbf{u}(P_4)^{(1)}]}.$$

This time we consider  $\mathbf{u}$  descending to  $\Theta^{[3]}$ , by letting the fourth point move towards infinity. The penultimate column in each determinant can be given with the expansions (23) as before. For the final column we will need to determine the derivative of these expansions:

$$\begin{aligned} \frac{d^2 u_1}{d\xi^2} &= -10\xi^9 + O(\xi^{10}) & \frac{d^2 u_4^2}{d\xi} &= -2\xi + \frac{9}{2}\mu_4 \xi^5 + O(\xi^6) \\ \frac{d^2 u_2}{d\xi^2} &= -6\xi^5 + O(\xi^6) & \frac{d^2 u_5}{d\xi^2} &= -1 + \frac{5}{2}\mu_4 \xi^4 + O(\xi^5) \\ \frac{d^2 u_3}{d\xi^2} &= -5\xi^4 + O(\xi^5) & \frac{d^2 u_6}{d\xi^2} &= \mu_4 \xi^3 + O(\xi^4). \end{aligned} \tag{A.1}$$



When  $\mathbf{u}$  arrives at  $\Theta^{[3]}$  we will have  $\xi = 0$ . Our determinants will again factor and cancel to leave the numerator as

$$\begin{vmatrix} a_1 & 1 & \cdots & 1 & 0 & 0 \\ a_2 & t_1 & \cdots & t_3 & 0 & 0 \\ a_3 & s_1 & \cdots & s_3 & 0 & 0 \\ a_4 & t_1^2 & \cdots & t_3^2 & 0 & 0 \\ a_5 & t_1 s_1 & \cdots & t_3 s_3 & 0 & -1 \\ a_6 & s_1^2 & \cdots & s_3^2 & -1 & 0 \end{vmatrix}$$

with the denominator identical with  $\mathbf{b}$  instead of  $\mathbf{a}$ . From the final two columns it is clear that the resulting quotient of polynomials will not vary with the arbitrary constant  $a_6$  and  $a_5$ . Hence we must conclude that

$$\Theta^{[3]} = \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = 0\}.$$

We repeat the procedure once more for  $k = 3$ . We let  $\mathbf{u}$  descend to  $\Theta^{[2]}$  and use the previous expansions along with the derivatives of (A.1) for the final three columns. We let  $\xi = 0$ , cancel the common factors and expand the determinants to reduce the statement to

$$\frac{\sum_{j=1}^6 a_j \sigma_j(\mathbf{u})}{\sum_{j=1}^6 b_j \sigma_j(\mathbf{u})} = \frac{a_1 t_1 s_2 - a_1 s_1 t_2 + a_2 s_1 - a_2 s_2 - a_3 t_1 + a_3 t_2}{b_1 t_1 s_2 - b_1 s_1 t_2 + b_2 s_1 - b_2 s_2 - b_3 t_1 + b_3 t_2} \quad (\text{A.2})$$

for  $\mathbf{u} \in \Theta^{[2]}$ . We therefore conclude that

$$\Theta^{[2]} = \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = \sigma_4(\mathbf{u}) = 0\}.$$

Finally we consider Theorem 6.1 in the case when  $k = 2$ . Here, when we let  $\mathbf{u}$  descend to  $\Theta^{[1]}$ , we find that the statement of the Theorem involves singular matrices, (resulting from the final set of series for the derivatives all equalling zero when  $\xi = 0$ ), and hence gives us no information.

Instead we can consider equation (A.2) which held for  $\mathbf{u} \in \Theta^{[2]}$ . Let  $\mathbf{u}$  descend to  $\Theta^{[1]}$  here, by using the expansions in  $\xi$  for  $(t_2, s_2)$ . We find that

$$\frac{\sum_{j=1}^6 a_j \sigma_j(\mathbf{u})}{\sum_{j=1}^6 b_j \sigma_j(\mathbf{u})} = \frac{a_1 t_1 - a_2}{b_1 t_1 - b_2} + O(\xi) \quad (\text{A.3})$$

and so for  $u \in \Theta^{[1]}$  we can see there is no dependence on  $a_3, a_4, a_5$  or  $a_6$ . Hence

$$\Theta^{[1]} = \{\mathbf{u} \mid \sigma(\mathbf{u}) = \sigma_6(\mathbf{u}) = \sigma_5(\mathbf{u}) = \sigma_4(\mathbf{u}) = \sigma_3(\mathbf{u}) = 0\}.$$

## Appendix B. Relations between $\sigma$ -derivatives on $\Theta^{[1]}$

The following list of equations are valid for  $\mathbf{u} \in \Theta^{[1]}$ . This set contains all those relations we have obtained that express  $n$ -index  $\sigma$ -functions for  $n \leq 4$ . A larger set that includes relations for  $n > 4$  is available online at [20].

$$\begin{aligned}
 \sigma_1 &= 0 & \sigma_{11} &= 0, & \sigma_{24} &= 0 & \sigma_{44} &= 0 \\
 \sigma_2 &= 0 & \sigma_{12} &= 0, & \sigma_{25} &= -\sigma_{34} & \sigma_{45} &= 0 \\
 \sigma_3 &= 0 & \sigma_{13} &= 0, & \sigma_{26} &= 0 & \sigma_{46} &= 0 \\
 \sigma_4 &= 0 & \sigma_{14} &= -\frac{1}{2}\sigma_{22}, & \sigma_{33} &= 0 & \sigma_{55} &= 0 \\
 \sigma_5 &= 0 & \sigma_{15} &= -\sigma_{23}, & \sigma_{35} &= 0 & \sigma_{56} &= 0 \\
 \sigma_6 &= 0 & \sigma_{16} &= 0, & \sigma_{36} &= 0 & \sigma_{66} &= 0 \\
 \\ 
 \sigma_{112} &= \mu_0\sigma_{34} + \mu_1\sigma_{23} & \sigma_{156} &= -\sigma_{236} & \sigma_{445} &= 0 \\
 \sigma_{113} &= 0 & \sigma_{166} &= \sigma_{23} & \sigma_{446} &= 0 \\
 \sigma_{114} &= -\sigma_{122} + \mu_1\sigma_{34} + \mu_2\sigma_{23} & \sigma_{244} &= \sigma_{23} + \mu_4\sigma_{34} & \sigma_{455} &= 0 \\
 \sigma_{115} &= -2\sigma_{123} & \sigma_{245} &= -\frac{1}{2}\sigma_{344} & \sigma_{456} &= 0 \\
 \sigma_{116} &= 0 & \sigma_{246} &= 0 & \sigma_{466} &= 0 \\
 \sigma_{124} &= -\frac{1}{6}\sigma_{222} + \frac{1}{2}\mu_2\sigma_{34} + \frac{1}{2}\mu_3\sigma_{23} & \sigma_{255} &= -2\sigma_{345} & \sigma_{555} &= 0 \\
 \sigma_{125} &= -\frac{1}{2}\sigma_{223} - \sigma_{134} & \sigma_{256} &= -\sigma_{346} & \sigma_{556} &= 0 \\
 \sigma_{126} &= 0 & \sigma_{266} &= \sigma_{34} & \sigma_{566} &= 0 \\
 \sigma_{133} &= 0 & \sigma_{333} &= 0 & \sigma_{666} &= 0 \\
 \sigma_{135} &= -\frac{1}{2}\sigma_{233} & \sigma_{335} &= 0 \\
 \sigma_{136} &= 0 & \sigma_{336} &= -2\sigma_{23} \\
 \sigma_{144} &= -\sigma_{224} + \mu_4\sigma_{23} + \mu_3\sigma_{34} & \sigma_{355} &= 0 \\
 \sigma_{145} &= -\sigma_{234} - \frac{1}{2}\sigma_{225} & \sigma_{356} &= -\sigma_{34} \\
 \sigma_{146} &= -\frac{1}{2}\sigma_{226} & \sigma_{366} &= 0 \\
 \sigma_{155} &= -\sigma_{334} - 2\sigma_{235} & \sigma_{444} &= 3\sigma_{34} \\
 \\ 
 \sigma_{1136} &= -\mu_0\sigma_{34} \\
 \sigma_{1144} &= \mu_3\sigma_{223} - 2\sigma_{1224} - \frac{1}{6}\sigma_{2222} + 2\mu_4\sigma_{123} + 2\mu_2\sigma_{234} + 2\mu_3\sigma_{134} + \mu_1\sigma_{344} \\
 \sigma_{1145} &= \mu_2\sigma_{235} - 2\sigma_{1234} - \sigma_{1225} - \frac{1}{3}\sigma_{2223} + \frac{1}{2}\mu_3\sigma_{233} + \mu_2\sigma_{334} + \mu_1\sigma_{345} \\
 \sigma_{1146} &= -\sigma_{1226} + \mu_2\sigma_{22} + \mu_2\sigma_{236} + \mu_1\sigma_{346} \\
 \sigma_{1155} &= -2\sigma_{1334} - 4\sigma_{1235} - \sigma_{2233} + 2\mu_2\sigma_{22} \\
 \sigma_{1156} &= -\mu_1\sigma_{34} - 2\sigma_{1236} \\
 \sigma_{1166} &= 2\sigma_{123} \\
 \sigma_{1244} &= -\frac{1}{3}\sigma_{2224} + \sigma_{123} + \frac{1}{2}\mu_4\sigma_{223} + \mu_4\sigma_{134} + \frac{1}{2}\mu_2\sigma_{344} + \mu_3\sigma_{234} \\
 \sigma_{1245} &= \frac{1}{4}\mu_4\sigma_{233} - \frac{1}{6}\sigma_{2225} - \frac{1}{2}\sigma_{2234} - \frac{1}{2}\sigma_{1344} + \frac{1}{2}\mu_3\sigma_{334} \\
 &\quad + \frac{1}{2}\mu_2\sigma_{345} + \frac{1}{2}\mu_3\sigma_{235} \\
 \sigma_{1246} &= \frac{1}{2}\mu_3\sigma_{22} - \frac{1}{6}\sigma_{2226} + \frac{1}{2}\mu_3\sigma_{236} + \frac{1}{2}\mu_2\sigma_{346} \\
 \sigma_{1255} &= \mu_3\sigma_{22} - \sigma_{2235} - 2\sigma_{1345} - \sigma_{2334} \\
 \sigma_{1256} &= -\frac{1}{2}\sigma_{2236} - \frac{1}{2}\mu_2\sigma_{34} - \sigma_{1346} \\
 \sigma_{1266} &= \frac{1}{2}\sigma_{223} + \sigma_{134} \\
 \sigma_{1333} &= -6\mu_0\sigma_{34} + 2\mu_1\sigma_{23}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{1335} &= \frac{4}{3}\mu_2\sigma_{23} - \frac{4}{3}\mu_1\sigma_{34} - \frac{1}{3}\sigma_{2333}\sigma_{1336} = -2\sigma_{123} \\
 \sigma_{1355} &= -\sigma_{2335} - \frac{1}{3}\sigma_{3334} - \frac{2}{3}\mu_2\sigma_{34} + 2\mu_3\sigma_{23} \\
 \sigma_{1356} &= -\frac{1}{2}\sigma_{223} - \sigma_{134} - \frac{1}{2}\sigma_{2336} \\
 \sigma_{1366} &= \frac{1}{2}\sigma_{233} \\
 \sigma_{1444} &= \frac{3}{2}\sigma_{223} + 3\sigma_{134} - \frac{3}{2}\sigma_{2244} + 3\mu_4\sigma_{234} + \frac{3}{2}\mu_3\sigma_{344} \\
 \sigma_{1445} &= \mu_3\sigma_{345} + \frac{1}{2}\sigma_{233} - \sigma_{2245} - \sigma_{2344} + \mu_4\sigma_{334} + \mu_4\sigma_{235} \\
 \sigma_{1446} &= \mu_4\sigma_{22} - \sigma_{2246} + \mu_3\sigma_{346} + \mu_4\sigma_{236} \\
 \sigma_{1455} &= -\frac{1}{2}\sigma_{3344} - 2\sigma_{2345} - \frac{1}{2}\sigma_{2255} + \mu_4\sigma_{22} \\
 \sigma_{1456} &= -\frac{1}{2}\sigma_{2256} - \frac{1}{2}\mu_3\sigma_{34} - \sigma_{2346} \\
 \sigma_{1466} &= \sigma_{234} - \frac{1}{2}\sigma_{2266} \\
 \sigma_{1555} &= -3\sigma_{2355} + 8\mu_4\sigma_{23} - 3\sigma_{3345} \\
 \sigma_{1556} &= -2\sigma_{234} - \sigma_{3346} - 2\sigma_{2356} \\
 \\
 \sigma_{1566} &= \sigma_{334} - \sigma_{2366} + \sigma_{235} & \sigma_{3556} &= -2\sigma_{345} \\
 \sigma_{1666} &= \sigma_{22} + 3\sigma_{236} & \sigma_{3566} &= -2\sigma_{346} \\
 \sigma_{2444} &= 3\sigma_{234} + \frac{3}{2}\mu_4\sigma_{344} & \sigma_{3666} &= \sigma_{34} \\
 \sigma_{2445} &= \sigma_{334} + \sigma_{235} - \frac{1}{3}\sigma_{3444} + \mu_4\sigma_{345} & \sigma_{4444} &= 6\sigma_{344} \\
 \sigma_{2446} &= \sigma_{22} + \sigma_{236} + \mu_4\sigma_{346} & \sigma_{4445} &= 3\sigma_{345} \\
 \sigma_{2455} &= \sigma_{22} - \sigma_{3445} & \sigma_{4446} &= 3\sigma_{346} \\
 \sigma_{2456} &= -\frac{1}{2}\mu_4\sigma_{34} - \frac{1}{2}\sigma_{3446} & \sigma_{4455} &= 0 \\
 \sigma_{2466} &= \frac{1}{2}\sigma_{344} & \sigma_{4456} &= -\sigma_{34} \\
 \sigma_{2555} &= 10\sigma_{23} + 2\mu_4\sigma_{34} - 3\sigma_{3455} & \sigma_{4466} &= 0 \\
 \sigma_{2556} &= -\sigma_{344} - 2\sigma_{3456} & \sigma_{4556} &= 0 \\
 \sigma_{2666} &= 3\sigma_{346} & \sigma_{4566} &= 0 \\
 \sigma_{3333} &= 0 & \sigma_{4666} &= 0 \\
 \sigma_{3335} &= 0 & \sigma_{5555} &= 0 \\
 \sigma_{3336} &= -3\sigma_{233} & \sigma_{5556} &= 0 \\
 \sigma_{3355} &= 0 & \sigma_{4555} &= 4\sigma_{34} \\
 \sigma_{3356} &= -2\sigma_{334} - 2\sigma_{235} & \sigma_{5566} &= 0 \\
 \sigma_{3366} &= -2\sigma_{22} - 4\sigma_{236} & \sigma_{5666} &= 0 \\
 \sigma_{3466} &= \sigma_{345} - \sigma_{2566} & \sigma_{6666} &= 0 \\
 \sigma_{3555} &= 0
 \end{aligned}$$

**Appendix C. Relations between  $\sigma$ -derivatives at  $\mathbf{u}_{0,N}$** 

The following list of equations were not valid in general for  $\mathbf{u} \in \Theta^{[1]}$ , but are true at  $\mathbf{u} = \mathbf{u}_{0,N}$ . This set contains all those relations we have obtained that express  $n$ -index  $\sigma$ -functions for  $n \leq 4$ . A larger set that includes relations for  $n > 4$  is available online at [20].

$$\begin{aligned}
 \sigma_{34} &= \frac{1}{2} \frac{\sigma_{22}}{i^N \mu_0^{1/4}} & \sigma_{233} &= -i^N \mu_0^{1/4} \sigma_{22} \\
 \sigma_{111} &= 0 & \sigma_{234} &= \frac{1}{6} \frac{\sigma_{222}}{i^N \mu_0^{1/4}} - \frac{1}{4} \frac{\mu_2 \sigma_{22}}{i^{2N} \sqrt{\mu_0}} \\
 \sigma_{112} &= \frac{1}{2} \sigma_{22} i^{3N} \mu_0^{3/4} & \sigma_{235} &= -\frac{1}{2} \frac{\sigma_{223}}{i^N \mu_0^{1/4}} \\
 \sigma_{113} &= 0 & \sigma_{236} &= -\frac{1}{2} \sigma_{22} \\
 \sigma_{123} &= -\frac{1}{2} i^{2N} \sqrt{\mu_0} \sigma_{22} & \sigma_{334} &= +\frac{\sigma_{223}}{i^N \mu_0^{1/4}} \\
 \sigma_{134} &= \frac{i^{3N} \sigma_{122}}{2 \mu_0^{1/4}} - \frac{i^{2N} \sigma_{22} \mu_1}{2 \sqrt{\mu_0}} - \frac{\sigma_{223}}{2} & \sigma_{344} &= -\frac{1}{2} \frac{\sigma_{22} \mu_3}{i^{2N} \sqrt{\mu_0}} + \frac{\sigma_{224}}{i^N \mu_0^{1/4}} \\
 \\
 \sigma_{345} &= \frac{i^{3N} \sigma_{225}}{2 \mu_0^{1/4}} + \frac{i^{2N} \sigma_{222}}{6 \sqrt{\mu_0}} - \frac{i^N \mu_2 \sigma_{22}}{4 \mu_0^{3/4}} \\
 \sigma_{346} &= \frac{1}{2} \frac{\sigma_{226}}{i^N \mu_0^{1/4}} \\
 \sigma_{1111} &= -6 \mu_0^{\frac{3}{2}} \sigma_{22} i^{2N} \\
 \sigma_{1112} &= -3 \mu_1 i^{2N} \sqrt{\mu_0} \sigma_{22} + \frac{3}{2} \mu_0^{\frac{3}{4}} \sigma_{122} i^{3N} \\
 \sigma_{1113} &= -\frac{3}{2} \mu_0^{5/4} \sigma_{22} i^N \\
 \sigma_{1114} &= -\frac{9}{4} i^{2N} \sqrt{\mu_0} \mu_2 \sigma_{22} - \frac{3}{2} \frac{i^{2N} \mu_1^2 \sigma_{22}}{\sqrt{\mu_0}} \\
 &\quad - \frac{3}{2} \sigma_{1122} + \frac{3}{2} \frac{\mu_1 \sigma_{122}}{i^N \mu_0^{1/4}} + \frac{1}{2} \mu_0^{\frac{3}{4}} \sigma_{222} i^{3N} \\
 \sigma_{1115} &= -\frac{3}{2} i^N \sigma_{22} \mu_1 \mu_0^{\frac{1}{4}} + 3 \sqrt{\mu_0} \sigma_{122} i^{2N} \\
 \sigma_{1116} &= \frac{3}{2} \sigma_{22} \mu_0 \\
 \sigma_{1123} &= \frac{1}{2} i^{3N} \mu_0^{\frac{3}{4}} \sigma_{223} - \sqrt{\mu_0} i^{2N} \sigma_{122} \\
 \sigma_{1124} &= -\frac{3}{4} \frac{i^{2N} \mu_2 \sigma_{22} \mu_1}{\sqrt{\mu_0}} - \frac{3}{4} i^{2N} \sqrt{\mu_0} \sigma_{22} \mu_3 - \frac{1}{3} \sigma_{1222} + \frac{1}{2} \mu_0^{\frac{3}{4}} \sigma_{224} i^{3N} \\
 &\quad + \frac{1}{6} \frac{\mu_1 \sigma_{222}}{i^N \mu_0^{1/4}} + \frac{1}{2} \frac{\mu_2 \sigma_{122}}{i^N \mu_0^{1/4}} \\
 \sigma_{1125} &= -\frac{1}{2} \mu_2 \sigma_{22} i^N \mu_0^{\frac{1}{4}} - \frac{i^N \mu_1^2 \sigma_{22}}{\mu_0^{\frac{3}{4}}} - \frac{1}{2} \frac{\sigma_{1122}}{i^N \mu_0^{1/4}} + \frac{1}{2} \mu_0^{\frac{3}{4}} \sigma_{225} i^{3N} \\
 &\quad + \frac{\mu_1 \sigma_{122}}{i^{2N} \sqrt{\mu_0}} + \frac{2}{3} \frac{\sqrt{\mu_0} \sigma_{222}}{i^{2N}}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{1126} &= \frac{1}{2}\mu_1\sigma_{22} + \frac{1}{2}\sigma_{226}i^{3N}\mu_0^{\frac{3}{4}} \\
 \sigma_{1133} &= 2\sigma_{22}\mu_0 \\
 \sigma_{1134} &= \frac{i^N\mu_1^2\sigma_{22}}{\mu_0^{\frac{3}{4}}} - \frac{1}{4}\mu_2\sigma_{22}i^N\mu_0^{\frac{1}{4}} - \sigma_{1223} + \frac{1}{2}\frac{\sigma_{223}\mu_1}{i^N\mu_0^{1/4}} + \frac{1}{2}\frac{\sigma_{1122}}{i^N\mu_0^{1/4}} \\
 &\quad - \frac{1}{2}\frac{\sqrt{\mu_0}\sigma_{222}}{i^{2N}} - \frac{\mu_1\sigma_{122}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{1135} &= i^{2N}\sqrt{\mu_0}\sigma_{223} + i^N\mu_0^{\frac{1}{4}}\sigma_{122} \\
 \sigma_{1233} &= \sigma_{22}\mu_1 - \frac{\sqrt{\mu_0}\sigma_{223}}{i^{2N}} - \frac{\sigma_{122}\mu_0^{\frac{1}{4}}}{i^{3N}} \\
 \sigma_{1234} &= -\frac{1}{2}i^N\mu_0^{\frac{1}{4}}\sigma_{22}\mu_3 + \frac{1}{2}\frac{\mu_2\sigma_{22}i^N\mu_1}{\mu_0^{\frac{3}{4}}} - \frac{1}{6}\sigma_{2223} + \frac{1}{4}\frac{\sigma_{223}\mu_2}{i^N\mu_0^{1/4}} + \frac{1}{6}\frac{\sigma_{1222}}{i^N\mu_0^{1/4}} \\
 &\quad - \frac{1}{2}\frac{\sqrt{\mu_0}\sigma_{224}}{i^{2N}} - \frac{1}{6}\frac{\mu_1\sigma_{222}}{i^{2N}\sqrt{\mu_0}} - \frac{1}{4}\frac{\mu_2\sigma_{122}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{1235} &= \frac{1}{2}\frac{i^{2N}\sigma_{223}\mu_1}{\sqrt{\mu_0}} - \frac{1}{2}\sqrt{\mu_0}i^{2N}\sigma_{225} - \frac{1}{6}\mu_0^{\frac{1}{4}}i^N\sigma_{222} + \frac{1}{2}\mu_2\sigma_{22} - \frac{1}{2}\frac{\sigma_{1223}}{i^N\mu_0^{1/4}} \\
 \sigma_{1236} &= -\frac{1}{2}i^{2N}\sqrt{\mu_0}\sigma_{226} - \frac{1}{2}\sigma_{122} \\
 \sigma_{1334} &= -\frac{i^{2N}\sigma_{223}\mu_1}{\sqrt{\mu_0}} + \frac{1}{2}\mu_2\sigma_{22} - \frac{1}{2}\sigma_{2233} + \frac{\sigma_{1223}}{i^N\mu_0^{1/4}} \\
 \sigma_{1344} &= \frac{i^N\mu_3\sigma_{22}\mu_1}{\mu_0^{\frac{3}{4}}} - \frac{3}{2}i^N\mu_0^{\frac{1}{4}}\sigma_{22}\mu_4 + \frac{1}{2}\frac{i^N\mu_2^2\sigma_{22}}{\mu_0^{\frac{3}{4}}} - \sigma_{2234} + \frac{1}{6}\frac{\sigma_{2222}}{i^N\mu_0^{1/4}} \\
 &\quad + \frac{1}{2}\frac{\sigma_{223}\mu_3}{i^N\mu_0^{1/4}} + \frac{\sigma_{1224}}{i^N\mu_0^{1/4}} - \frac{1}{2}\frac{\mu_3\sigma_{122}}{i^{2N}\sqrt{\mu_0}} - \frac{\mu_1\sigma_{224}}{i^{2N}\sqrt{\mu_0}} - \frac{1}{3}\frac{\mu_2\sigma_{222}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{1345} &= -\frac{1}{2}\sigma_{2235} - \frac{1}{4}\sigma_{22}\mu_3 + \frac{3}{4}\frac{\mu_1\mu_2\sigma_{22}}{\mu_0} + \frac{1}{2}\frac{\sigma_{1225}}{i^N\mu_0^{1/4}} - \frac{1}{6}\frac{\sigma_{2223}}{i^N\mu_0^{1/4}} + \frac{1}{2}\frac{\mu_0^{\frac{1}{4}}\sigma_{224}}{i^{3N}} \\
 &\quad - \frac{1}{4}\frac{\mu_2\sigma_{122}}{i^{3N}\mu_0^{3/4}} - \frac{1}{2}\frac{\mu_1\sigma_{225}}{i^{2N}\sqrt{\mu_0}} + \frac{1}{4}\frac{\sigma_{223}\mu_2}{i^{2N}\sqrt{\mu_0}} + \frac{1}{6}\frac{\sigma_{1222}}{i^{2N}\sqrt{\mu_0}} - \frac{1}{3}\frac{\mu_1\sigma_{222}}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{1346} &= -\frac{1}{6}\sigma_{222} - \frac{1}{2}\sigma_{2236} - \frac{1}{4}\frac{\mu_2\sigma_{22}}{i^N\mu_0^{1/4}} + \frac{1}{2}\frac{\sigma_{1226}}{i^N\mu_0^{1/4}} - \frac{1}{2}\frac{\mu_1\sigma_{226}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{2333} &= -2\frac{\mu_1\sigma_{22}i^{3N}}{\mu_0^{1/4}} - 3i^N\mu_0^{1/4}\sigma_{223} \\
 \sigma_{2334} &= \frac{1}{2}\sigma_{22}\mu_3 + \frac{1}{3}\frac{\sigma_{2223}}{i^N\mu_0^{1/4}} - \frac{1}{2}\frac{\sigma_{223}\mu_2}{i^{2N}\sqrt{\mu_0}} - \frac{\mu_0^{1/4}\sigma_{224}}{i^{3N}} \\
 \sigma_{2335} &= -i^N\mu_0^{1/4}\sigma_{225} - \frac{2}{3}\sigma_{222} - \frac{1}{2}\frac{\sigma_{2233}}{i^N\mu_0^{1/4}} + \frac{\mu_2\sigma_{22}}{i^N\mu_0^{1/4}} \\
 \sigma_{2336} &= -i^N\mu_0^{1/4}\sigma_{226} - \sigma_{223} \\
 \sigma_{2344} &= \frac{1}{2}\frac{\mu_2\sigma_{22}i^N\mu_3}{\mu_0^{3/4}} - 2i^N\mu_0^{1/4}\sigma_{22} + \frac{1}{2}\frac{\sigma_{223}\mu_4}{i^N\mu_0^{1/4}} + \frac{1}{3}\frac{\sigma_{2224}}{i^N\mu_0^{1/4}}
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{6} \frac{\sigma_{222}\mu_3}{i^{2N}\sqrt{\mu_0}} - \frac{1}{2} \frac{\sigma_{224}\mu_2}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{2345} = & -\frac{1}{2}\sigma_{22}\mu_4 + \frac{3}{8} \frac{\mu_2^2\sigma_{22}}{\mu_0} + \frac{1}{6} \frac{\sigma_{2225}}{i^N\mu_0^{1/4}} - \frac{1}{2} \frac{\sigma_{2234}}{i^N\mu_0^{1/4}} + \frac{1}{12} \frac{\sigma_{2222}}{i^{2N}\sqrt{\mu_0}} \\
 & - \frac{1}{4} \frac{\sigma_{225}\mu_2}{i^{2N}\sqrt{\mu_0}} + \frac{1}{4} \frac{\sigma_{223}\mu_3}{i^{2N}\sqrt{\mu_0}} - \frac{1}{4} \frac{\mu_2\sigma_{222}}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{2346} = & -\frac{1}{2}\sigma_{224} + \frac{1}{6} \frac{\sigma_{2226}}{i^N\mu_0^{1/4}} - \frac{1}{4} \frac{\sigma_{226}\mu_2}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{2355} = & -\frac{\sigma_{2235}}{i^N\mu_0^{1/4}} + \frac{\sigma_{22}\mu_3}{i^N\mu_0^{1/4}} - \frac{1}{3} \frac{\sigma_{2223}}{i^{2N}\sqrt{\mu_0}} + \frac{1}{2} \frac{\sigma_{223}\mu_2}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{2356} = & -\frac{1}{2}\sigma_{225} - \frac{1}{2} \frac{\sigma_{222}}{i^N\mu_0^{1/4}} - \frac{1}{2} \frac{\sigma_{2236}}{i^N\mu_0^{1/4}} + \frac{1}{2} \frac{\mu_2\sigma_{22}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{2366} = & -\sigma_{226} + \frac{1}{2} \frac{\sigma_{223}}{i^N\mu_0^{1/4}} \\
 \sigma_{2566} = & -\frac{1}{2} \frac{\mu_2\sigma_{22}i^N}{\mu_0^{3/4}} + \frac{1}{2} \frac{\sigma_{225}}{i^N\mu_0^{1/4}} - \frac{1}{2} \frac{\sigma_{2266}}{i^N\mu_0^{1/4}} + \frac{1}{3} \frac{\sigma_{222}}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{3334} = & \sigma_{222} + \frac{3}{2} \frac{\sigma_{2233}}{i^N\mu_0^{1/4}} - \frac{5}{2} \frac{\mu_2\sigma_{22}}{i^N\mu_0^{1/4}} \\
 \sigma_{3344} = & 2\sigma_{22}\mu_4 - \frac{\mu_2^2\sigma_{22}}{2\mu_0} + 2 \frac{\sigma_{2234}}{i^N\mu_0^{1/4}} - \frac{\sigma_{223}\mu_3}{i^{2N}\sqrt{\mu_0}} - \frac{1}{6} \frac{\sigma_{2222}}{i^{2N}\sqrt{\mu_0}} + \frac{1}{3} \frac{\mu_2\sigma_{222}}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{3345} = & \frac{\sigma_{2235}}{i^N\mu_0^{1/4}} - \frac{\sigma_{22}\mu_3}{i^N\mu_0^{1/4}} + \frac{1}{3} \frac{\sigma_{2223}}{i^{2N}\sqrt{\mu_0}} - \frac{1}{2} \frac{\sigma_{223}\mu_2}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{3346} = & \frac{1}{3} \frac{\sigma_{222}}{i^N\mu_0^{1/4}} + \frac{\sigma_{2236}}{i^N\mu_0^{1/4}} \\
 \sigma_{3444} = & \frac{3}{4} \frac{i^N\sigma_{22}\mu_3^2}{\mu_0^{3/4}} + \frac{3}{4} \frac{i^N\mu_4\mu_2\sigma_{22}}{\mu_0^{3/4}} + \frac{3}{2} \frac{\sigma_{223}}{i^N\mu_0^{1/4}} + \frac{3}{2} \frac{\sigma_{2244}}{i^N\mu_0^{1/4}} \\
 & - \frac{3}{2} \frac{\sigma_{224}\mu_3}{i^{2N}\sqrt{\mu_0}} - \frac{1}{2} \frac{\sigma_{222}\mu_4}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{3445} = & -\sigma_{22} + \frac{3}{4} \frac{\mu_2\sigma_{22}\mu_3}{\mu_0} + \frac{\sigma_{2245}}{i^N\mu_0^{1/4}} - \frac{1}{2} \frac{\sigma_{225}\mu_3}{i^{2N}\sqrt{\mu_0}} + \frac{1}{3} \frac{\sigma_{2224}}{i^{2N}\sqrt{\mu_0}} \\
 & - \frac{1}{3} \frac{\sigma_{222}\mu_3}{i^{3N}\mu_0^{3/4}} - \frac{1}{2} \frac{\sigma_{224}\mu_2}{i^{3N}\mu_0^{3/4}} \\
 \sigma_{3446} = & -\frac{1}{2} \frac{\sigma_{22}\mu_4}{i^N\mu_0^{1/4}} + \frac{\sigma_{2246}}{i^N\mu_0^{1/4}} - \frac{1}{2} \frac{\sigma_{226}\mu_3}{i^{2N}\sqrt{\mu_0}} \\
 \sigma_{3455} = & -\frac{\sigma_{22}\mu_4}{i^N\mu_0^{1/4}} + \frac{1}{2} \frac{\sigma_{2255}}{i^N\mu_0^{1/4}} + \frac{1}{2} \frac{\mu_2^2\sigma_{22}}{i^N\mu_0^{5/4}} + \frac{1}{3} \frac{\sigma_{2225}}{i^{2N}\sqrt{\mu_0}} - \frac{1}{2} \frac{\sigma_{225}\mu_2}{i^{3N}\mu_0^{3/4}} \\
 & + \frac{1}{12} \frac{\sigma_{2222}}{i^{3N}\mu_0^{3/4}} - \frac{1}{3} \frac{\mu_2\sigma_{222}}{\mu_0} \\
 \sigma_{3456} = & -\frac{1}{2} \frac{\sigma_{224}}{i^N\mu_0^{1/4}} + \frac{1}{2} \frac{\sigma_{2256}}{i^N\mu_0^{1/4}} + \frac{1}{6} \frac{\sigma_{2226}}{i^{2N}\sqrt{\mu_0}} + \frac{1}{4} \frac{\sigma_{22}\mu_3}{i^{2N}\sqrt{\mu_0}} - \frac{1}{4} \frac{\sigma_{226}\mu_2}{i^{3N}\mu_0^{3/4}}
 \end{aligned}$$

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