Author(s): Sihvola, Ari

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Polarizability of a tri-isotropic sphere

Ari Sihvola*
Electromagnetics and Acoustics Laboratory, Swiss Federal Institute of Technology, DE-EPFL, CH-1015 Lausanne, Switzerland and Electromagnetics Laboratory, Helsinki University of Technology, P.O. Box 3000, FIN-02015 HUT, Finland
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This report presents the polarizability matrix of a tri-isotropic sphere. A tri-isotropic material exhibits coupling between electric, magnetic, and in addition, a third kind of material response. Each of these three fields creates three different kind of polarizations. The paper derives the polarizability components of such a sphere in vacuum and makes physical interpretations of the results for the copolarizabilities and cross polarizabilities.

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I. INTRODUCTION

In the electromagnetics literature of the late 20th century, materials with magnetoelectric coupling often carry the name bi-isotropic or bi-anisotropic media [1]. The term *bi-anisotropy* generalizes ordinary anisotropy to the case that the electric and magnetic material constitutive relations are not independent of each other. This term was coined by Kong in his doctoral dissertation in 1968 [2]: ''A bianisotropic medium is defined as one in which the field vectors $\overline{D}$ and $\overline{H}$ depend upon both $\overline{E}$ and $\overline{B}$ but may not be parallel to either.'' Then also, *bi-isotropic* is a natural label for materials that retain the magnetoelectric coupling but are isotropic.

The behavior of electromagnetic fields in bi-isotropic and bi-anisotropic materials have been amply analyzed and many important results from these are available in the literature. The aim of the present paper is to generalize some results into still more complex class of materials: those of *tri-isotropic* materials.

Let us define here tri-isotropic materials in the following way. First, they are isotropic media. Second, there are three independent fields to which the material responds with polarization, e.g., both electric and magnetic fields cause both electric and magnetic polarization but in addition, there is a third field with analogous flux response, and this third field causes both electric and magnetic polarization, as well as the electric and magnetic fields cause polarization of this third type.

What could be such a third force and response? It has to be a field mechanism of the Poisson-type with the field deriving from a potential and the responding flux or flow proportional to the characteristic isotropic material parameter. And then, there needs to be a coupling to the electric and magnetic regimes. One example of such equilibrium process would be a response to gravitational field. If the medium possesses nonzero gravoelectric and gravomagnetic couplings (meaning, for example, that gravity would cause ion separation and induce electric and magnetic polarization), such a multifield material would be tri-isotropic.

But one might also think of tri-isotropy in connection with dissipative, transport phenomena. The potential could be electrical, chemical, or due to temperature, with fluxes involving charge, particles, or heat [3,4]. Materials exist that exhibit thermoelectric, thermomagnetic, or other diffusion effects with coupled flows between different species of particles. In addition to flow phenomena, one might probably make use of the results of this paper in the description of coupled phenomena involving elastic properties of materials, piezoelectricity, and piezomagnetism, as the mathematical description there is analogous to electrostatics.

Anyway, regardless of the character of this third force, the additional coupling that it creates in the response of matter to fields may give rise to interesting possibilities in the design of complex composite materials. With the following analysis, hopefully such possibilities get more justification. In the remaining sections of this paper, the polarizability matrix of a homogeneous tri-isotropic sphere is derived and the polarizability components are interpreted physically.

II. TRI-ISOTROPIC MATERIALS AND NOTATION

As was mentioned above, materials that display magnetoelectric coupling can be called bi-anisotropic. Bi-isotropic materials [1] also exhibit magnetoelectric coupling but their response is independent of the direction of the exciting field (not anisotropic), and hence the material parameters are equivalent to scalars. Bi-isotropic constitutive relations between the electric and magnetic fields **E**, **H**, and displacements **D**, **B** are the following:

$$\mathbf{D}=\epsilon\epsilon_0\mathbf{E}+\sqrt{\mu_0\epsilon_0}\xi\mathbf{H}, \tag{1}$$

$$\mathbf{B}=\sqrt{\mu_0\epsilon_0}\zeta\mathbf{E}+\mu\mu_0\mathbf{H}, \tag{2}$$

where ϵ is the relative permittivity, μ the relative permeability, and ξ and ζ the two magnetoelectric parameters. The free-space parameters are ϵ_0 and μ_0.

The different electric and magnetic units call for a renormalization of the quantities when there is magnetoelectric coupling. Also, to achieve a compact notation for the material response analysis, a good technique is the six-vector notation [5] with which the constitutive parameters are contained in a material matrix **M**.

*FAX: +358-9-4512267. Email address: ari.sihvola@hut.fi

$$\begin{pmatrix} c\eta\mathbf{D} \\ c\mathbf{B} \end{pmatrix} = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \end{pmatrix}, \tag{3}$$

where the fields and displacements now carry the same dimensions (V/m), and the material matrix components are dimensionless.[1]

The topic of the present paper is to include into the analysis a coupling of electric and magnetic fields with another force field **F**. In the absence of **E** and **H**, the material responds to this field in a similar way as to electrostatic or magnetostatic potential fields:

$$\mathbf{G} = \lambda \mathbf{F}, \tag{4}$$

where λ is the permissivity.[2] Now, assuming that there exists cross coupling between this third field and with electric and magnetic components, the constitutive relation of the material is given in the nine-vector form by the following matrix **M**:

$$\begin{pmatrix} c\eta\mathbf{D} \\ c\mathbf{B} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \epsilon & \xi & \sigma \\ \zeta & \mu & \tau \\ \psi & \theta & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \\ \mathbf{F} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \\ \mathbf{F} \end{pmatrix}. \tag{5}$$

Note that because of the assumed isotropy of the material, there are only nine components in the material matrix. If anisotropy were allowed, each of these would in the most general form have nine components. Hence the number of degrees of freedom in the tri-anisotropic materials description is 81.

In the analysis we also need the (trivial) response of free space. The tri-isotropic material matrix of free space, $\mathbf{M}_0$, is a 3×3 identity matrix:

$$\begin{pmatrix} c\eta\mathbf{D} \\ c\mathbf{B} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \\ \mathbf{F} \end{pmatrix} = \mathbf{I} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \\ \mathbf{F} \end{pmatrix}. \tag{6}$$

III. POLARIZABILITY

The ordinary dielectric polarizability of an inclusion, α, is the ratio between the dipole moment induced in the inclusion by an incident field and the field [6]: $\mathbf{p} = \alpha \mathbf{E}$. The polarizability can be often a dyadic (tensor of second rank), due to the shape of the scatterer or its composition and structure. For spherical, homogeneous, and isotropic scatterers, however, the polarizability is a scalar. For example, the (electric) polarizability of a small spherical scatterer with relative permittivity ϵ and volume V is

$$\frac{\alpha}{\epsilon_0} = 3V \frac{\epsilon - 1}{\epsilon + 2}. \tag{7}$$

Analogously with the analysis of polarizabilities of bi-isotropic scatterers [7], we can write for the polarizability matrix of a tri-isotropic sphere:

$$\begin{pmatrix} c\eta\mathbf{p}_e \\ c\mathbf{p}_m \\ \mathbf{p}_f \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{E} \\ \eta\mathbf{H} \\ \mathbf{F} \end{pmatrix}, \tag{8}$$

with the electric, magnetic, and third-type dipole moments $\mathbf{p}_e, \mathbf{p}_m, \mathbf{p}_f$ normalized in such a way that the polarizability matrix **A** has dimensions of volume.

The polarizability matrix can be calculated like in the six-vector notation [5]:

$$\begin{aligned} A &= 3V(\mathbf{M} - \mathbf{M}_0) \cdot (\mathbf{M} + 2\mathbf{M}_0)^{-1} \cdot \mathbf{M}_0 \\ &= 3V(\mathbf{M} - \mathbf{I}) \cdot (\mathbf{M} + 2\mathbf{I})^{-1}. \end{aligned} \tag{9}$$

Enumeration of the components of the polarizability matrix is a straightforward algebraic operation, with the result

$$\mathbf{A} = \begin{pmatrix} A_{\mathrm{ee}} & A_{\mathrm{em}} & A_{\mathrm{ef}} \\ A_{\mathrm{me}} & A_{\mathrm{mm}} & A_{\mathrm{mf}} \\ A_{\mathrm{fe}} & A_{\mathrm{fm}} & A_{\mathrm{ff}} \end{pmatrix}, \tag{10}$$

where the copolarizability components are

$$\begin{aligned} \frac{A_{\mathrm{ee}}}{3V/D} &= (\epsilon - 1)(\mu + 2)(\lambda + 2) - (\epsilon - 1)\tau\theta - (\mu + 2)\psi\sigma \\ &\quad - (\lambda + 2)\xi\zeta + \xi\psi\tau + \zeta\sigma\theta, \end{aligned}$$

$$\begin{aligned} \frac{A_{\mathrm{mm}}}{3V/D} &= (\epsilon + 2)(\mu - 1)(\lambda + 2) - (\epsilon + 2)\tau\theta - (\mu - 1)\psi\sigma \\ &\quad - (\lambda + 2)\xi\zeta + \xi\psi\tau + \zeta\sigma\theta, \end{aligned}$$

$$\begin{aligned} \frac{A_{\mathrm{ff}}}{3V/D} &= (\epsilon + 2)(\mu + 2)(\lambda - 1) - (\epsilon + 2)\tau\theta - (\mu + 2)\psi\sigma \\ &\quad - (\lambda - 1)\xi\zeta + \xi\psi\tau + \zeta\sigma\theta, \end{aligned}$$

and the cross polarizabilities

$$A_{\mathrm{em}} = 9V[(\lambda + 2)\xi - \sigma\theta]/D,$$

$$A_{\mathrm{ef}} = 9V[(\mu + 2)\sigma - \tau\xi]/D,$$

$$A_{\mathrm{me}} = 9V[(\lambda + 2)\zeta - \psi\tau]/D,$$

$$A_{\mathrm{mf}} = 9V[(\epsilon + 2)\tau - \zeta\sigma]/D,$$

$$A_{\mathrm{fe}} = 9V[(\mu + 2)\psi - \theta\zeta]/D,$$

$$A_{\mathrm{fm}} = 9V[(\epsilon + 2)\theta - \xi\psi]/D,$$

with the denominator

[1] The coefficients in the renormalization are the vacuum constants $c = 1/\sqrt{\epsilon_0\mu_0}$ and $\eta = \sqrt{\mu_0/\epsilon_0}$.

[2] Let us call this third type of susceptivity ''permissivity'' to distinguish it from the electric permittivity and magnetic permeability. Also, choose the dimensions for **F** and **G** to be those of the electric field. With such a choice we are left with a dimensionless permissivity λ.

$$D=(\epsilon+2)(\mu+2)(\lambda+2)-(\epsilon+2)\tau\theta-(\mu+2)\psi\sigma$$
$$-(\lambda+2)\xi\zeta+\xi\psi\tau+\zeta\sigma\theta.$$

IV. DISCUSSION

The nine components of the isotropic polarizability matrix contain all the nine material parameters of the inclusion, and they are coupled in a complicated manner. This is obvious from the definition that essentially involves an inversion of the material matrix. It is easy to check that the earlier known polarizabilities of dielectric and magnetic spheres, as well as the polarizability matrix of a bi-isotropic sphere [7] are recovered from the formulas above.

A. Weak cross coupling

Often it happens that the magnetoelectric coupling is weak. This means that the amplitudes of the cross terms ξ and ζ are much smaller than ϵ and μ. It is also fair to assume that the cross couplings to the third response are weak, meaning that also σ, τ, ψ, and θ are small in comparison with ϵ, μ, and λ.[3] Then the second- and higher-order products of the small parameters can be neglected, giving the approximative polarizabilities

$$A_{ee}=3V\frac{\epsilon-1}{\epsilon+2},\quad A_{mm}=3V\frac{\mu-1}{\mu+2},\quad A_{ff}=3V\frac{\lambda-1}{\lambda+2} \tag{11}$$

and

$$A_{em}=\frac{9V\xi}{(\epsilon+2)(\mu+2)},\quad A_{ef}=\frac{9V\sigma}{(\epsilon+2)(\lambda+2)},$$
$$A_{mf}=\frac{9V\tau}{(\mu+2)(\lambda+2)}, \tag{12}$$
$$A_{me}=\frac{9V\zeta}{(\epsilon+2)(\mu+2)},\quad A_{fe}=\frac{9V\psi}{(\epsilon+2)(\lambda+2)},$$
$$A_{fm}=\frac{9V\theta}{(\mu+2)(\lambda+2)}. \tag{13}$$

These results show that in the copolarizabilities, there is no first-order effect of the cross terms of the material matrix. The copolarizabilities are just like those of an inclusion of a noncoupled medium (i.e., a material with $\xi=\zeta=\tau=\sigma=\psi=\theta=0$). Also each cross-polarizability coefficient is directly proportional to the corresponding cross term in the material matrix. Knowing that the ratio of the electric field inside a spherical inclusion with relative permittivity ϵ to the incident field is $3/(\epsilon+2)$, we can see that the proportionality factor is the product of the field ratios in the two associated domains (for example, the electric and magnetic field ratios for the magnetoelectric coefficients ξ and ζ).

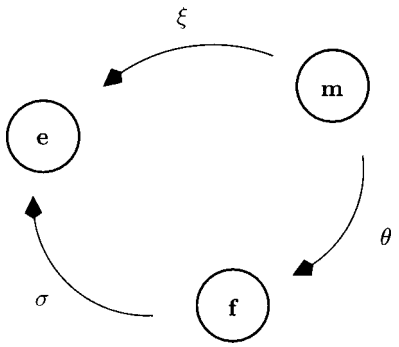

FIG. 1. The direct way of the magnetoelectric coupling is through the parameter ξ: the indirect one is through the third force and its response (by θ and σ).

B. Direct and indirect cross coupling

All cross-polarizability components in the matrix (10) are of the similar form: they contain the ''direct'' cross term, proportional to the corresponding component in the constitutive material dyadic **M**, and an ''indirect'' one that is proportional to the product between two other cross terms of the material parameters. For instance, the numerator of the magnetically caused electric polarizability A_{em} is

$$(\lambda+2)\xi-\sigma\theta,$$

and here the direct effect is of course ξ. But if there is coupling between both electric and magnetic domains with the third force, the second, indirect, component here is nonzero. A magnetic field causes response in the third force through θ, and electric response results from the third force **F** with the coupling amplitude σ, also shown in Fig. 1.

This means that a tri-isotropic sphere can exhibit magnetoelectric coupling even if the direct coefficients vanish ($\xi=\zeta=0$), if only the indirect route through the third force is open. An interesting detail is the opposite sign between the direct and indirect routes. This is probably due to the depolarizing character of the field due to internal dipoles. The depolarized field is an oppositely directed field, created by the induced polarization [8]. Therefore in the indirect route, the force field giving rise to the second-order polarization is negative. So is, subsequently, the second-order polarization itself.

Note also that the two effects can cancel each other [given the condition $(\lambda+2)\xi=\theta\sigma$]. Sometimes such a property could be useful in composite materials design when a magnetoelectric effect, that is present in the components (or any other of the cross effects in tri-isotropic media) is undesirable and one would need to ''wash-it-out'' in the macroscopic medium.

Figure 2 illustrates the quantitative behavior of the cross polarizability. There the magnetoelectric cross polarizability, which depends on all nine material parameters, is shown as a function of the most important ones, the cross terms ξ and σ. To simplify the illustration, a symmetrical matrix is chosen

[3]Note that λ itself is not small: even in the limit of vanishing susceptibility in this third dimension, the relative permissivity is $\lambda=1$.

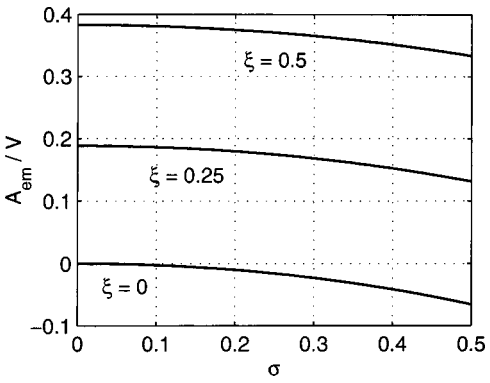

FIG. 2. The magnetoelectric cross-polarizability coefficient A_{em} for three values of the magnetoelectric parameter ξ, as a function of the other cross-material parameter $\sigma=\theta$. The other values assumed here are $\epsilon=2$, $\mu=1$, $\lambda=1$, $\xi=\zeta$, $\sigma=\psi$, and $\tau=\theta$.

($\xi=\zeta$, $\sigma=\psi$, and $\tau=\theta$). Furthermore, the coupling strengths of the third force to the electric and magnetic ones are chosen equal ($\sigma=\theta$). Indeed, with increasing σ and θ (provided that they are of the same sign as in this example), the magnetoelectric polarizability decreases.

C. Coupling mechanisms

In bi-isotropic magnetoelectric materials, the two effects responsible for the cross-coupling terms are chirality (with amplitude κ) and nonreciprocity (χ) [1]:

$$\xi=\chi-j\kappa, \tag{14}$$

$$\zeta=\chi+j\kappa. \tag{15}$$

The nonreciprocal in-phase coupling χ was suggested by Tellegen in 1948 [9], and the reciprocal chiral effect κ is present in handed materials. In chiral media, there is characteristic the 90° phase shift between the force field and the response flux.[4]

The Tellegen model for the magnetoelectric coupling is conceptually very simple. The medium possesses permanent dielectric and magnetic dipole moments that orient themselves according to the exciting field in the Debye-Langevin manner [10]. But the moments are coupled pairwise, ''glued together'' so that when the electric field gives a torque to the electric dipole, also the magnetic dipole turns, and vice versa. Figure 3 contains such an element.

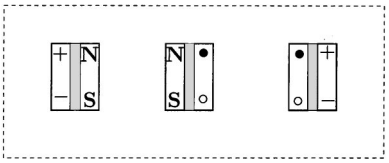

FIG. 3. The coupling elements, which, if mixed with large numbers and random orientation to a neutral background medium, make the material tri-isotropic. Left: a Tellegen element where a permanent dipole moment (poles + and −) is glued together with a permanent magnetic moment (poles **N** and **S**). Middle: magnetic moment tied together with the moment representing the third force (poles: positive ●, and negative ○). Right: third-force moment and electric moment. The gray color represents the glue that ties the pairs together.

In Fig. 3, the Tellegen elements are generalized to tri-isotropic regime. There are three types of dipole pairs, in addition to the magnetoelectric element also the third-force-type dipole coupled with electric and also with magnetic dipole. In addition, here the positive ends of each of the element pairs are tied together. If a large number of this type of elements are mixed with random orientation to a matrix, the mixture is tri-isotropic, with the cross-coupling amplitudes proportional to the density of these elements.[5] The coupling in the figure corresponds to positive values of all cross parameters in the matrix **M**. Furthermore, because of the symmetry of the pairs, also the matrix **M** is symmetric for the material with this ''Tellegen-type'' geometry, in other words, $\xi-\zeta$, $\sigma=\psi$, and $\tau=\theta$.

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[4]This is the reason for the imaginary unit j, which is *not* here implying losses. For a lossless materials, both χ and κ are real.

[5]Another possibility to achieve tri-isotropic coupling *à la Tellegen* would be to use only one type of element: dipole triplets where the three different type of permanent dipoles are all glued together. This would lead to a very symmetric coupling where all the off-diagonal components of the **M** matrix are equal.

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