

Evaluation of Executive Stock Options in

Continuous and Discrete Time



Antti Pirjetä

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Abstract

What follows is a treatise on executive stock options (ESOs), employing both theoretical and empirical methods. The title speaks of evaluation, instead of ‘plain’ valuation, because subjective ESO values depend on the employee’s private data including risk aversion, initial wealth, and labor income.

Economic theory assumes that utility is drawn from consumption, and consumers are impatient. It follows that the decision to exercise or sell ESOs deals with consumption smoothing. Assuming that the ESOs have vested, the option grantee asks: Is it better to sell now and consume the proceeds, or to wait and possibly enjoy higher consumption later on? Obviously, the answer depends (among other things) on individual risk preferences.

ESOs are options in the sense that they provide the call option payoff. However, since the option grantee cannot usually hedge his position, usual arguments leading to risk-neutral valuation are not applicable, suggesting that ESO pricing is different from standard contracts. This claim can be tested, since ESOs are traded in the Helsinki stock exchange.

Now let us consider a generalized Black-Scholes model with (implied) volatility as free parameter. If ESO pricing differs from standard options, it should be seen in their volatilities. It is shown in *Market pricing of ESOs* that volatilities (i.e. relative prices) of ESOs are lower compared to standard options. Given the prominence of illiquidity and hedging under uncertainty in mathematical finance, we ask if these factors could explain the discount in ESO prices.

In the case of Nokia ESOs, illiquidity is not the story, since the underlying stock is fiercely traded. However, hedging is complicated by ESOs having longer maturities than standard contracts. The average maturity of ESOs is 3.1 years, compared to few months for liquid standard options. Hence, hedging ESOs by taking opposite position in standard options is subject to volatility risk. Obviously, hedging is also possibly by shorting the stock and using cash account, but this involves frequent trading and hedging error. Nonetheless, volatility risk and hedging error can explain the discount only to an extent. Finally, we conclude that the employees benefit from public trading of ESOs. It allows the employees to receive some time value¹, which they would lose if ESOs were not traded and had to be exercised.

Keywords: Executive stock options, risk aversion, empirical option pricing

¹Time value is defined as option price less intrinsic value, i.e. $C - (S - K)$ where option, stock and strike prices are denoted by C , S , and K .

Evaluation of Executive Stock Options in Continuous and Discrete Time

*Doctoral Dissertation in
Quantitative Methods of Economics and Management Science*

Antti Pirjetä¹
Helsinki School of Economics
2009

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²Each essay contains a detailed table of contents.

Alkusanat (Foreword in Finnish)

Tämä kirja sisältää johdannon ja neljä esseetä, joissa tarkastellaan johdannais-sopimusten hinnoittelua riskiä karttavan yksilön näkökulmasta. *Introduction* on johdanto moderniin rahoitusteoriaan, ja esittelee optioiden hinnoittelua ja kulutus-sijoitusongelman ratkaisua jatkuvan ajan mallissa, olettaen joko täydelliset tai epätäydelliset markkinat. Koska johdon optiot muodostavat epätäydellisten markkinoiden mallin erikoistapauksen, *Introduction* sivuuttaa ne lyhyesti. Ydinasiat *Introssa* ovat riskineutraali hinnoittelu, riskin markkinahintojen vektori, rajahyötyyn perustuva hinnoittelu ja konvekssi optimointi.

ESO valuation under IFRS 2 on väitöskirjaprojektini ensimmäinen tutkimus, ja perustuu havaintoon, että johdon optioiden subjektiivinen arvo on alhaisempi kuin niiden tuloslaskennassa käytettävä arvo, joka yleensä perustuu joko Black-Scholes-malliin tai ns. binomimalliin (Cox-Ross-Rubinstein 1979). Matemaattinen perustelu on varsin yksinkertainen. Olettaen, että optionsaaja on riskinkarttaja (konkaavi hyötyfunktio), option päättymispäivän rahavirran varmuusekvivalenttihinta on alhaisempi kuin sen riskineutraali odotusarvo. Näiden erotus kasvaa riskin karttamisen myötä.

Computational methods for incentive option valuation yleistää edellisen esseen varmuusekvivalentin indifferenssihinnoitteluksi. Siinä tarkastellaan yhden periodin mallin asemesta usean periodin kulutus- ja sijoituspäätöksiä. Lähtökohdiana on yksilö, jolla on sekä palkkatuloja että sijoitusvarallisuutta. Molempiin liittyy riskejä, ja mallin erityispiirre on, että työmarkkinoihin ja sijoitukseen liittyvät riskit voivat olla korreloituneita. Tämä on täysin realistinen oletus, sillä usein johdon palkat ovat sidottuja yrityksen tulokseen, joka osaltaan vaikuttaa pörssikurssiin ja optioiden arvoon. Tutkimuksessa johdetaan realistinen arvostusmalli työsuhdeoptioille. Numeerisesti ratkaistava optimointimalli tuottaa herkkyyksianalyyskejä koskien sijoitus- ja kulutuspäätöksiä sijoitustuottojen ja palkkatulojen jakauman vaihdellessa.

Market pricing of executive stock options tarkastelee johdon optioiden hinnoittelua empiirisesti perustuen Helsingin pörssissä noteerattuihin Nokian optioihin. Eri malleja käyttäen todetaan, että henkilöstöoptioiden suhteelliset hinnat (mittarina implisiittinen volatilitiitti) ovat alhaisempia verrattuna tavanomaisiin Nokian optioihin. Tulokset kertovat, että suhteellinen alennus option hinnassa kasvaa osakekurssin myötä. Lisäksi tutkimuksessa lasketaan optionsaajien riskinkarttamiskerroin vertaamalla optiomarkkinoiden riskineutraalia ja osakemarkkinoiden objektiivista tuottojakaamaa. Tulosten perusteella keskimääräisen optionsaajan preferenssejä kuvaa logaritminen hyötyfunktio eli suhteellinen riskinkarttamisker-

roin on hiukan yli yhden.

Empirical option pricing with copula-based correlation tarkastelee optioiden hinnoittelua Hull-White-mallissa, jossa sekä osakekurssi että tuoton varianssi noudattavat geometrista Brownin liikettä, eli volatilitteetti onkin satunnaismuuttuja eikä vakio, kuten Black-Scholes-mallissa. Edelleen em. prosessit voivat olla korreloituneita, ja tutkimus keskittyy tuon korrelaation robustiin laskentaan. Osake-tuoton ja volatilitteetin korrelaatio on avainmuuttuja, koska se vaikuttaa riskineutraalin todennäköisyysjakauman muotoon. Taustalla on idea, että korrelaation avulla voidaan hallita osakkeen tuottojakauman kolmatta ja neljättä momenttia, eli kaltevuutta ja huipukkuutta. Tutkimuksessa mallinnetaan tuoton ja volatilitteetin riippuvuutta käyttäen copuloita eli kertymäfunktioiden riippuvuusmalleja. Niiden tärkeimmät matemaattiset ominaisuudet ovat ns. Sklarin teoria (Abe Sklar 1959) ja invarianssiominaisuus.

Kiitokset

Haluan ilmaista kiitollisuuteni perheenjäsenille, ystäville ja työtovereille, joiden tuki on mahdollistanut tämän hankkeen valmistumisen. Tosiasiallisesti aloitin jatko-opinnot toimiessani Jyväskylän yliopistossa vs. yliassistenttina lukuvuonna 2003-04. Se oli mukava työpaikka, ilmapiiriltään yksi parhaista kokemistani. Kiitän erityisesti Antti Rautiaista ja Risto Tammelaä antoisista keskusteluista liittyen johdon optioihin ja niiden kohteluun tilinpäätöksessä. Nykyisin Antti työskentelee lehtorina yliopistossa ja Risto riskienhallintapäällikkönä Aktia Pankissa; toivon molemmille onnea ja menestystä! En myöskään unohda professori Esko Leskisen tilastotieteen luentoja, joista on ollut runsaasti hyötyä myöhemmin.

Jyväskylästä palasin Helsingin kauppakorkeakouluun assistentiksi 2004-08 kvantitatiivisten menetelmien aineeseen. Haluan lämpimästi kiittää professori Antti Kantoa (nykyisin Tampereen yliopistossa), joka rekrytoi minut assistentin virkaan ja ohjasi tutkimustyötäni parin ensimmäisen vuoden ajan. Hän oli positiivinen ja kannustava ohjaaja. Tahdon kiittää koko aineen henkilökuntaa: professori Pekka Korhonen, lehtori Tapani Lehtonen, Helena Knuuttila, Pekka Malo ja Jan-Erik Antipin ovat työtovereina (ja Pekka K. esimiehenä) tukeneet minua monin tavoin näiden vuosien aikana.

Toimin kuluneen lukuvuoden 2008-09 rahoituksen yliassistentin viransijaisena. Kiitän kaikkia työtovereita antoisista keskusteluista ja professori Matti Keloharjua esimiehenä joustavasta suhtautumisesta tutkimukseni viimeistelyyn, joka on väistämättä vienyt aikaa opetustyöltä.

Olen saanut työskennellä professoritason tutkijoiden kanssa. Markku Kalliolta olen toivoakseni oppinut jotain optimoinnista ja matemaattisen tekstin kirjoit-

tamisesta. Seppo Ikäheimo ja Vesa Puttonen, joiden kanssa olen tutkinut johdon optioita, ovat jaksaneet kannustaa ja olla optimistisia julkaisuprosessimme pitkällä ja kivisellä tiellä. Toivottavasti heidän pitkäjänteisyydestään on tarttunut jotain minuun.

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Työn loppumetreillä olen hyötynyt esitarkastajieni, professorien Seppo Pynnönen (Vaasan yliopisto) ja Tomas Björk (Handelshögskolan i Stockholm) kommentteista. On ilo ja kunnia saada Seppo Pynnönen vastaväittäjäksi ja Pekka Korhonen kustokseksi.

Helsingissä 7.7.2009

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Introduction to
*Evaluation of executive stock options using
continuous and discrete time methods*

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Abstract

Consistent with the title, this work shows how the pricing of ESOs, or other equity-based compensation, falls in modern arbitrage theory. Since the following essays are practically orientated, this work assumes a theoretical perspective for completeness. In particular, we treat ESO pricing in a Merton problem framework, since most 'executives' invest in other securities, consume and receive labor income. Therefore, we consider the Merton problem using a martingale method. Indeed, the results are applicable for pricing of any incentive that is a contingent claim.

Further, the 'executive' faces an incomplete market, if she is constrained from trading in the options or underlying stock. It is shown that the main tool in incomplete market pricing is the state price density. It is used for preference-dependent (i.e. utility-based) pricing in cases, where exact replication is impossible.

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1 Introduction

1.1 Fundamentals of arbitrage pricing

This chapter provides an overview of modern arbitrage theory and utility-based option pricing. The connection to ESO pricing is found by applications of certainty equivalent (CE) principle and indifference pricing. The CE concept appears already in Pratt's [40] classic paper on risk aversion and has recently been developed to indifference pricing¹ (surveys are given by Broadie and Detemple [5] or Hobson [19]). The 'archaic' form of CE pricing considers only the option cash flow, and disregards others, such as investment returns and labor income.

For mathematics, our main reference is the monograph of Karatzas and Shreve [30], building on convex analysis and optimization. Their book covers both complete and incomplete markets. In the latter case, relevant portfolio constraints (e.g. on short sales) can be formulated as convex cones. Further, to make the model more realistic, we add labor income following the tracks of Cuoco [10].

A natural starting point for expanding CE valuation to full managerial portfolio problem is the Merton [34], [35] problem. Instead of isolated valuation of an option, the Merton problem optimizes consumption and investments dynamically and allows for correlated risks. This approach has gained popularity after Cox and Huang [8] proposed to solve the problem by a martingale method. It replaces Merton's dynamic budget constraint by a static one. The martingale method leads to "feedback solutions" (due to Karatzas and Shreve) given by inverse of marginal utility function [cf. Eqs. (4.17)-(4.18)]. Working in continuous time, Lemma 2.5 of Cox and Huang shows that Eq. (1.1) must hold when there is no arbitrage.

Let us introduce some notation. E^Q and E are expectation operators using risk-neutral and objective probabilities. Moreover, r_t is the risk-free rate, $g(t)$ is an adapted process (e.g. consumption or a derivative payoff), and $\Lambda(t)$ is the state price density (SPD) defined below² by Eq. (3.5).

$$E^Q \left[e^{-\int r(s)ds} \int_0^T g(s)ds \right] = E \left[\int_0^T \Lambda(s)g(s)ds \right] \quad (1.1)$$

Originally, the SPD was defined in this manner by Harrison and Kreps [12]. In essence, the relation (1.1) shows how to incorporate preferences in contingent claim valuation using the state price density. Moreover, this procedure is consistent with no arbitrage.

More concretely, consider an agent who writes the contingent claim X and so has to cover the liability $X(T)$. Assuming no arbitrage, the product process

¹By definition, the manager is *indifferent* between receiving the CE price or risky payoff (i.e. contingent claim), leading to the notion of indifference price.

²This chapter contains some forward references. We apologize for any inconvenience.

of SPD $\Lambda(t)$ and hedging portfolio $X(t)$ is in general a supermartingale as in Eq. (1.2).

$$\Lambda(s)X(s) \geq E[\Lambda(t)X(t) \mid \mathcal{F}(s)] \quad \text{for } 0 \leq s < t \leq T \quad (1.2)$$

For an optimal choice of Λ and X the process becomes a martingale, and (1.2) holds as equality (Kramkov and Schachermayer [32, Th. 2]). In words, optimum means finding the cheapest hedge for the claim. Once we move on to consider a consumption-investment problem, it represents the best available tradeoff. If the agent invests 1€ now ($X(0) = 1$), her *expected* time- T consumption is given by (1.3), using the SPD definition (3.5).

$$E[X(T)] = 1 \cdot \frac{\Lambda(0)}{E[\Lambda(T)]} = 1 \cdot \exp\left(\int_0^T r(s) ds\right) \quad (1.3)$$

However, the *actual* time- T consumption, affected by market uncertainty, equals (1.4). In this formula $\lambda = (\mu - r)/\sigma$ denotes the price of risk and W a Wiener process³. Note that λ determines the compensation for letting time- T consumption to depend on market return.

$$X(T) = 1 \cdot \exp\left[\int_0^T \lambda dW + \int_0^T \left(r(s) + \frac{1}{2}\lambda^2\right) ds\right] \quad (1.4)$$

Our next task is to infer an economic interpretation for the SPD. This involves two steps. The first one links SPD and marginal utility, and the second one shows the relation to *marginal rate of substitution*. For the first step, recall Eq. (4.17) from Section 4.3, saying that optimal consumption is given by $c^*(t) = I_1(t, \mathcal{Y}(x)\Lambda(t))$ with $I = (U')^{-1}$. This implies directly $U'(c_t^*) = \mathcal{Y}(x)\Lambda(t)$. Further, using the notation $\mathcal{Y}(x) = y$ we deduce that $U'(c_t^*) = y\Lambda(t)$, which completes the first step⁴. The second step combines the previous result with Eq. (1.2), which leads to the classical *Euler equation* (1.5). Again, for optimally chosen $c(t)$ and $X(t)$ this relation becomes an equality.

$$E\left[\frac{U'(c_t)}{U'(c_s)} \frac{X(t)}{X(s)} \mid \mathcal{F}(s)\right] \leq 1 \quad \text{for } s < t. \quad (1.5)$$

In Eq. (1.5) the first term inside brackets is equal to marginal rate of substitution. This specification is the foundation of *asset pricing theory*, summarized by Campbell and Viceira [6].

So far we have been silent on how to identify the SPD (or related objects) from practical data. Assuming this objective, Aït-Sahalia and Lo [1] list three fundamental objects:

³A Wiener process, or equivalently Brownian motion, is a random process with normally distributed increments: $W(t) - W(s) \sim N(0, t - s)$ for $s < t$. See Björk [3] for exact definition.

⁴In derivatives pricing context, the result that SPD is proportional to marginal utility is proved by Kramkov and Schachermayer [32].

1. *Preferences of the representative agent (implying risk aversion)*
2. *Asset price dynamics*
3. *State price density*

If one can measure two items on the list, economic theory implies the third one. In particular, since economists disagree widely on risk aversion, why not estimate it using data on (2) and (3)? They are directly measurable from stock and option prices. In this dissertation, Pirjetä et al. [38] perform a similar exercise, and infer risk aversion from trading prices of ESOs.

Valuation methods based on state price density are topical in mathematical finance. In fact, we can evaluate *any* contingent claim within the consumption-investment model, including an ESO. This is discussed in Section 5.6 in the light of *utility-based price* of Hugonnier et al. [22] and *utility indifference pricing* of Hobson [19]. Regularity conditions for SPD-based valuation have been recently derived by Kramkov and Schachermayer [32], and extended by Hugonnier and Kramkov [21] to handle nontraded contingent claims. These papers verify a solution in terms of the SPD subject to reasonable conditions, only one of which is critical. It requires that some hedging portfolio dominates the derivative payoffs at expiration (cf. [21, Lemma 1]). When this condition fails, the upper hedging price (5.9) is infinite (Karatzas and Shreve [30, Prop. 5.8.6]).

In the area of hedging, Kramkov and Sirbu [33] have recently introduced *utility-based hedging*. This method yields a marginal hedging strategy for cases, where perfect hedging is impossible. The setup applies well to a typical manager receiving options, and has indeed been investigated by Henderson [15]. She shows how to hedge non-traded assets using correlated traded assets. It would be interesting to add consumption and labor income to her model.

1.2 Summary of main results

Main results of the four essays are briefly summarized in the following paragraphs. Each one was written from a different perspective given by the heading. Let us remark that each essay provides a review of area-specific literature.

Essay 1: ESO valuation from an accounting perspective

This paper discusses the implications and valuation of employee stock options under IFRS 2, i.e. the accounting standard that listed companies have to comply with. Executive stock options (ESOs) are analyzed in an agency model. Risk-neutral option values are calculated using the Cox, Ross and Rubinstein [9] binomial model. The employee calculated ESOs using the certainty equivalent principle, which leads to a discount to risk-neutral value. Hence, the fair option value stated as an expense in the profit and loss statement should be lower than the value suggested by risk-neutral option pricing models. Further, the gap

between employer's and employee's valuation (*deadweight loss*) grows with the volatility and employee risk aversion. It is found that the ESO risk premium is time dependent, and it decreases going towards expiration. Finally, we discuss the effects of ESOs on managerial behavior using the framework of Ross [41].

Essay 2: Variations on a managerial portfolio problem

Here we ask: *How does inclusion of market frictions and labor income impact ESO values in a Merton problem?* The answer is provided using a discrete-time model using double binomial tree. While the Cox, Ross and Rubinstein [9] tree is a discretization of one-dimensional Brownian motion (BM), our model discretizes a two-dimensional BM representing market and stock-specific risk factors.

The paper employs stochastic programming on solving option values in the double binomial model. The ESOs considered may have European or American-style exercise. These options are valued considering hedging restrictions and other market frictions, like transaction and shorting costs. The model also allows for different interest rates for borrowing and lending.

Perhaps the most original feature is that model incorporates risky labor income. It turns out that correlation of income and investment risks decreases the value of ESOs. However, if income is riskless or the risk is independent, ESO values may increase.

Another innovation in this paper is that it considers *ask price functions*, which model the option price depending on the sold amount. There are only a few papers in the incomplete markets literature modelling the dependence of price and quantity⁵. One such model is given by Cetin et al. [7], where the supply curve (analogous to our ask price function) is caused by liquidity effects. Recently, Pennanen [36] has proposed a model, where (possibly discontinuous) demand and supply curves integrate to a total cost function. Building on convex analysis, he derives conditions for no arbitrage in marginal or scalable form.

Essay 3: Empirical view on pricing and risk preferences

In spite of their popularity in equity-based compensation, executive options are not publicly traded outside Finland to our knowledge. (Indeed, below we give reasons why they *should* be.) This study uses a unique dataset of Nokia ESO prices to shed light on two issues: pricing of ESOs vis-à-vis standard options and managerial risk preferences.

When managers get to trade in options received as compensation, their trading prices reveal several aspects of subjective option pricing and risk preferences. Two subjective pricing models are fitted to show that executive stock option prices incorporate a subjective discount. It depends positively on implied volatility and negatively on option moneyness. Further, risk preferences are estimated using the semiparametric model of Aït-Sahalia and Lo [1]. The results suggest

⁵This is probably due to the fact that in a complete market the option price is independent of quantity. However, this is not the case for SPD-based valuation models.

that relative risk aversion is just above 1 for a certain stock price range. This level of risk aversion is low but reasonable, and it may be explained by the typical manager being wealthy and having low marginal utility. Related to risk aversion, it is found that marginal rate of substitution increases considerably in states with low stock prices.

Essay 4: Correlation of ESO volatility and stock returns

Correlation of stock returns and volatility is a key parameter in stochastic volatility models. An original method for computing this correlation is developed below. Under this method, a copula is fitted in pairwise data of option delta and volatility, and linear correlation is inferred from rank correlation that is measured by Kendall's tau. Fitness of the estimate can be evaluated using goodness of fit tests for copulas, based on bootstrap techniques. Also, an ANOVA procedure based on prediction error table is proposed for the purpose. The method is demonstrated using the Hull-White option pricing model and a smooth volatility function. In an empirical application it turns out that volatility and stock returns are negatively dependent, and both Gauss and Student t copulas provide acceptable dependence models.

1.3 Recommendation based on the results

To our knowledge, ESOs are not publicly traded outside Finland. *Based on empirical results in [38], we find that public trading of ESOs is beneficial to both the employees and the owners who grant the options.* When ESOs are traded in a stock exchange, their trading prices include non-negative time value⁶. It is particularly significant when the ESOs have long maturity. In contrast, if employees could only exercise (and not sell) the options, all time value would be lost. It can be argued that positive time value cuts *deadweight loss*, that is the gap between objective (risk-neutral) and subjective values (see [28]). In addition, there are tax-based reasons for public trading of ESOs (see [38]). Finally, it adds transparency in financial reporting.

2 First look at the Merton problem

In this section we introduce the Merton portfolio problem [34], [35], by adopting the setup of Ch. 5 of Campbell and Viceira [6]. The aim is to provide maximal intuition at a minimum level of formality, and characterise the solution in a way that is consistent with the generalized analysis of Chapters 3-5. The main difference concerns the budget constraint. Here it only balances the cash flows of a consumer. Later on (in Section 3.3) we formulate a budget constraint that closes out arbitrage possibilities.

⁶Define time value as option price less intrinsic value, i.e. $C - (S - K)$ where option, stock and strike prices are denoted by C , S , and K .

We start formally by establishing the asset price dynamics. Write S for a stock (not necessarily traded), B for a riskless bond, and M for a state variable interpreted as the stock market risk. It is natural to consider S as the employer's equity in a managerial compensation context.

$$\begin{aligned}\frac{dS}{S} &= \mu_S(S, t)dt + \sigma_S(S, t)dW^{(1)} \\ \frac{dB}{B} &= r(t)dt \\ dM &= \mu_M(M, t)dt + \sigma_M(M, t)dW^{(2)}\end{aligned}$$

In the above display, $W^{(1)}$ and $W^{(2)}$ are Wiener processes; $W_{T-t} \sim N(0, T-t)$. They are allowed to be correlated: $dW^{(1)}dW^{(2)} = \rho dt$. Also, a *diffusion process* is defined as an adapted stochastic process satisfying two conditions; the uncertainty is generated by Wiener functionals, and the process allows a martingale representation (cf. Björk [3, Th. 11.2]).

The consumption-investment problem calls for additional notation. Denote consumption by C and wealth by X . Portfolio weights of the stock and bond are α and $1-\alpha$. Preferences are set by a nondecreasing and concave utility function $U(C)$. Then $U'(C)$ is decreasing and so is the inverse $(U')^{-1}(C)$. The problem is to maximize the utility of consumption (2.1), subject to budget constraint (2.2). It follows from the requirement that the growth rate of wealth equals return on investments less the consumption rate, i.e.

$$\frac{dX}{X} = \alpha \frac{dS}{S} + (1-\alpha) \frac{dB}{B} - \frac{C}{X} dt.$$

Both consumption and wealth are required to stay positive with zero wealth being an absorbing state. Infinite horizon is chosen to simplify the problem. It is shown below that choosing a finite horizon leads to qualitatively similar results.

$$\max_{C, \alpha} E \left[\int_0^\infty U(C, t) dt \right] \tag{2.1}$$

$$\text{s.t. } dX = \{ \alpha [(\mu_P - r) + r] X - C \} dt + \alpha X \sigma_P dW^{(1)} \tag{2.2}$$

From optimal control theory (see Björk [3, Ch. 19]) we know that the optimal controls (C and α) are found by solving the Hamilton-Jacobi-Bellman equation (2.3). The value function that gives expected utility is denoted $V(X, S, t)$. It is presumed that $\lim_{t \rightarrow \infty} V(X, S, t) = 0$. Heuristically, in a dynamic problem, $V(X, S, t)$ is the equivalent of utility function.

$$\max_{C, \alpha} \left[U(C, t) + \frac{1}{dt} E(V(X, S, t)) \right] = 0 \tag{2.3}$$

The HJB equation says that in the optimum, instantaneous utility from consuming one euro equals the utility of investing it and consuming the proceeds

later. The next step is to write the stochastic differential (2.4) using Ito's lemma. Subscripts denote partial derivatives, e.g. $V_t = \partial V / \partial t$.

$$dV(X, S, t) = V_X dX + V_S dS + V_t dt \quad (2.4)$$

$$+ \frac{1}{2} \left[V_{XX} (dX)^2 + 2V_{XS} dX dS + V_{SS} (dS)^2 \right]$$

Continue by combining Eq. (2.4) and the HJB equation (2.3). Lengthy but straightforward calculations lead to the following statement.

$$\max_{C, \alpha} \left\{ \begin{aligned} &U(C, t) + V_X [\alpha [(\mu_S - r) + r] X - C] + V_M \mu_M + V_t \\ &+ \frac{1}{2} V_{XX} \alpha^2 X^2 \sigma_S^2 + V_{XS} X \alpha \rho \sigma_S \sigma_M + \frac{1}{2} V_{MM} \sigma_M^2 \end{aligned} \right\} = 0 \quad (2.5)$$

Differentiation of Eq. (2.5) with respect to consumption (C) and stock weight (α) results in first-order conditions (2.6)-(2.7) of the Merton problem. Their interpretations turn out to be useful.

$$U_C - V_X = 0 \quad (2.6)$$

$$V_X (\mu_S - r) X + V_{XX} \alpha X^2 \sigma_S^2 + V_{XS} X \rho \sigma_S \sigma_M = 0$$

After reordering terms and denoting relative risk aversion by $\text{RRA} = -\frac{V_{XX} \cdot X}{V_X}$, the second equation yields the stock weight (α). Consequently the allocation to risk-free asset is $1 - \alpha$.

$$\alpha = \underbrace{\frac{1}{\text{RRA}} \left(\frac{\mu_S - r}{\sigma_S^2} \right)}_{\text{Merton constant}} - \underbrace{\frac{V_{SX}}{V_{XX} \cdot X} \frac{\sigma_M}{\sigma_S} \rho}_{\text{hedging demand}} \quad (2.7)$$

The envelope condition (2.6) implies that marginal utility and marginal value functions are both decreasing. It is important, because optimal consumption can be calculated using the inverse of marginal utility, that is $C = (U')^{-1}(V_X)$. Obviously, such calculations go smoothly using log utility. This relation proves useful in the generalized discussion of the Merton problem in complete and incomplete markets.

Moving forward, the optimal stock weight (2.7) contains two terms open to interesting interpretations. The first term, or *Merton constant*, shows that the optimal stock weight is increasing in market price of risk, and decreasing in relative risk aversion. The second term, or *hedging demand*, gives the response of an optimal investor to changes in the investment opportunity set. Note how it depends on the ratio of standard deviations $\frac{\sigma_M}{\sigma_S}$ and the correlation of stock and market risk processes.

3 Arbitrage pricing of assets driven by diffusion processes

The simple Merton problem of Ch. 2 leads to the result that optimal portfolio weights are proportional to price of risk defined by Eq. (3.4). However, no

assumptions were made as to how it is determined. This section takes up the issue, and relates the price of risk to asset prices in an arbitrage-free market, where returns are driven by Wiener processes (i.e. Brownian motions). This leads to definition of *state price density*, which is our most important pricing tool. Here and below arbitrage-free prices are given by variations of the risk-neutral pricing formula (3.12). This chapter is mainly based on Chapters 10, 11 and 14 of Björk [3].

3.1 Absence of arbitrage and change of measure

This section recalls some results on arbitrage in a complete market. It also treats the Girsanov theorem that relates statistical and risk-neutral Wiener processes. Let us start by stating the First Fundamental Theorem of finance. It says that absence of arbitrage follows from the existence of an equivalent⁷ martingale measure. It generates a "risk-neutral" distribution of Q -probabilities, rendering asset prices martingales when discounted by risk-free rate. In general, this is not true for the observed (or "objective") distribution of P -probabilities. Further, the Second Fundamental Theorem says that the market is complete, if the Q -probabilities are unique.

In fact, the above theorems may not apply in pricing incentive options. The first theorem becomes less powerful (though not redundant), if the option or the underlying stock are not traded, or selling short the stock is prohibited. The second theorem does not apply, when one or fewer risk factors cannot be hedged, and the market becomes incomplete and risk-neutral probabilities are no more unique.

Unfortunately, the risk-neutral distribution cannot be observed, however it is related to the statistical one by the *Girsanov theorem*. It was originally Harrison and Kreps [12] who suggested using Girsanov for transforming the actual P -probabilities into risk-neutral Q -probabilities. This operation changes the drift but leaves the diffusion parameter untouched. Harrison and Kreps did also prove that the Girsanov kernel, i.e. φ in Eq. (3.1), equals the market price of risk with different sign, i.e. $\varphi = -\lambda$.

In order to formalize the above discussion, let L_t be the Radon-Nikodym derivative, or *likelihood process*

$$L_t = \frac{dQ}{dP}.$$

$L(t)$ is a non-negative P -martingale and it has expected value of one. Moreover, it is the *likelihood ratio* of P and Q , which entails that $E^Q(X) = E^P(L \cdot X)$ for some random variable X . Presuming that the market is driven by a k -dimensional Wiener functional, $L(t)$ has the same dimension. Based on its properties, $L(t)$ has a martingale representation of the form (3.1), where W_t is a P -Wiener process.

$$dL_t(\varphi) = \varphi'_t L_t dW_t \quad L_0 = 1 \quad (3.1)$$

⁷'Equivalent' means that the martingale measure and statistical measure have the same null sets, i.e. sets of zero probability events.

In Eq. (3.1) φ_t is the *Girsanov kernel*, in general being an adapted process. In particular, φ_t equals the market price of risk after a change of sign. An application of Ito's lemma results in the formula (3.2) for the likelihood process.

$$L_t(\varphi) = \exp\left(\int_0^t \varphi'_s dW_s - \frac{1}{2} \int_0^t \|\varphi_s\|^2 ds\right) \quad (3.2)$$

In practice, relevance of the likelihood process resides in that it changes the drift of a Wiener process. Let W_t^* denote the Q -Wiener process that generates the risk-neutral distribution. Then the statistical and risk-neutral processes are related as

$$W_t = \int_0^t \varphi_s ds + W_t^*. \quad (3.3)$$

3.2 Price of risk

Consider a financial market, where asset prices are driven by a k -dimensional Wiener functional. For the purpose of measuring expected compensation from holding risky assets, Eq. (3.4) defines the (market) *price of risk* (λ).

$$\lambda = \sigma^{-1}(\mu - r) \quad (3.4)$$

The definition (3.4) assumes n risky assets and k risk factors, hence λ is an n -vector. Moreover, σ is an $n \times k$ diffusion matrix, wherefore the covariance matrix (assumed invertible) is given by $\Sigma = \sigma\sigma'$. Going forward, μ and r are n -vectors of expected returns and the risk-free rate. Of course, all elements of r are equal.

It is clear by now that the risk-neutral drift must be equal to the risk-free rate. As a result the price of risk is the negative of Girsanov kernel, put formally as $\lambda_t = -\varphi_t$. To see this, consider an arbitrary asset S that follows a diffusion process; $dS = \mu_t S dt + \sigma_t S dW_t$. Application of Eqs. (3.3) and (3.4) results in $dS = rS dt + \sigma_t S dW_t^*$, provided that we choose $\lambda_t = -\varphi_t$.

For introduction of the risk-neutral valuation formula in Ch. 2.4, it is necessary to define the state price density (SPD), denoted Λ_t . (Björk [3] calls this concept the stochastic discount factor.) The intuition is that it gives the present value of an asset or option payoff determined by the risk-neutral distribution. Therefore, the SPD is calculated as the likelihood transformation $L_t(-\lambda)$ times a risk-free discount factor, which motivates the following equation.

$$\begin{aligned} \Lambda_t &= e^{-\int_0^s r(s) ds} L_t(-\lambda) \\ &= \exp\left[-\int_0^t \lambda'_s dW_s - \int_0^t \left(r(s) + \frac{1}{2} \|\lambda_s\|^2\right) ds\right] \end{aligned} \quad (3.5)$$

The above formula implies that the SPD is an exponential Brownian motion, and the observed values follow a lognormal distribution. This is a natural outcome, and supported by the fact the Black-Scholes model gives rise to lognormal asset price distribution. Obviously the current setup is more general, since it allows for time-varying drift and diffusion parameters. Not surprisingly, drift of

the SPD process equals the negative of risk-free rate. To see this, note that Eq. (3.5) solves the stochastic differential equation (3.6). It follows from combining (3.1) and (3.5) and writing $dB_t/B_t = r(t)dt$ for the risk-free bond.

$$d\Lambda_t = -r\Lambda_t dt + \frac{1}{B_t} dL_t \quad (3.6)$$

The preceding analysis of the SPD is a mathematical one, but the concept also allows an economic interpretation. It turns out that the SPD equals the (discounted) ratio of future and current marginal utilities, a quantity known as *marginal rate of substitution (MRS)* in what is known as "asset pricing" literature (see the collection of papers by Campbell and Viceira [6]).

The definition of SPD implies that it takes large values when returns are negative (i.e. $W_t < 0$) and marginal utility of consumption is large, provided the utility function is concave. What this means is that the SPD weighs risk-neutral prices by their marginal rates of substitution, as shown by Eq. (3.8) below. Doing so assigns large weights to low consumption states, where marginal utility is high. This result is analyzed by Campbell and Viceira [6, Ch. 2] in a discrete time setup.

Let us now demonstrate how these concepts apply to portfolio problem with ESOs. The indifference valuation framework of Kallio and Pirjetä [28] fits well for the purpose. For notation, write \mathcal{C}_k for option price at node k , and $\mathcal{V}(\cdot)$ for indifference valuation operator. The manager buys a small share ϵ of option \mathcal{Z} with payoff function $\mathcal{Z}(S_k)$. Indifference means that utility given up now, or the left-hand side of (3.7), equals expected utility from future consumption, given by the right-hand side of (3.7).

$$\epsilon \mathcal{V}(\mathcal{C}_k) \eta_0 = \epsilon \sum_k \mathcal{Z}(S_k) \eta_k \quad (3.7)$$

In Eq. (3.7) η_0 and η_k are Lagrange multipliers of a budget constraint. They depend on marginal utilities in the root node and node k ; $\eta_0 = u'(c_0)$ and $\eta_k = \pi_k u'(c_k)$. Rewriting Eq. (3.7) yields the indifference value of the option $\mathcal{V}(\mathcal{C}_k)$ with cash flow stream $h(S_k)$.

$$\mathcal{V}(\mathcal{C}_k) = \sum_k \frac{\eta_k}{\eta_0} \mathcal{Z}(S_k) = \sum_k \frac{\pi_k u'(c_k)}{u'(c_0)} \mathcal{Z}(S_k) \quad (3.8)$$

In Eq. (3.8) π_k is the probability of node k , $u'(\cdot)$ is the marginal utility and $\mathcal{Z}(S_k)$ is the cash flow from the option in node k . Indifference valuation is consistent with the SPD approach in the sense that the SPD equals the marginal rate of substitution. Let us add that the indifference valuation is free of arbitrage, as verified in Kallio and Pirjetä [28].

3.3 State price density implied by option prices

Theoretically speaking, the state price density (SPD) is a mathematical device that enables a mapping from risk-neutral to risk-averse prices. This gives rise

to the following practical idea. Once we observe risk-neutral option prices and risk-averse stock prices, perhaps it is possible to infer the SPD, and even say something about preferences?

The answer is positive, because options are state-contingent claims and their expiration dates are known. This brilliant idea goes back to Breeden and Litzenberger [4], and it is the basis for option-implied risk aversion estimators. Aït-Sahalia and Lo [1] have suggested a prominent method for estimating relative risk aversion as function of the underlying asset price. It is applied in Pirjetä et al. [38], part of this dissertation, to estimate managerial risk aversion from Nokia ESO data.

Besides having fixed expiration dates, options bring other benefits. They are short-term claims, and it is easier to forecast the economy in the short run, which adds to the robustness of option-implied SPDs. Another factor favouring options is market efficiency, enhanced by the easy formation of arbitrage portfolios. It should be added that the last argument does not apply to the ESO market investigated by Ikäheimo et al. [24] and Pirjetä et al. [38].

Based on its importance, the main result of Breeden and Litzenberger [4] is illustrated here. Let \mathcal{S} denote a stock that has time- T price distribution given by $\mathcal{S}_T = 1, 2, \dots, \bar{\mathcal{S}}_T < \infty$; increasing prices compare to economic states of increasing wealth. For the distribution it suffices to assume finite expected return and variance. the interest rate is set to zero for simplicity. Traded assets include three call options written on the stock. Write $\mathcal{C}(1)$, $\mathcal{C}(2)$, and $\mathcal{C}(3)$ for call options with strike prices of 1, 2, and 3. Fix T as the expiration date. The following table summarizes option values at expiration.

\mathcal{S}_T	$\mathcal{C}(1)$	$\mathcal{C}(2)$	$\mathcal{C}(3)$	$\mathcal{C}(1) - \mathcal{C}(2)$	$\mathcal{C}(2) - \mathcal{C}(3)$
1	0	0	0	0	0
2	1	0	0	1	0
3	2	1	0	1	1
4	3	2	1	1	1
...
$\bar{\mathcal{S}}_T$	$\bar{\mathcal{S}}_T - 1$	$\bar{\mathcal{S}}_T - 2$	$\bar{\mathcal{S}}_T - 3$	1	1

What the table demonstrates is that a certain option portfolio provides a unit payoff if $\mathcal{S}_T = 2$ and zero payoff otherwise, justifying the notion of *state-contingent claim*. The desired portfolio buys call options with strike prices of 1 and 3, and shorts two calls with strike price 2. This setup is easily generalized by using step size Δ instead of 1. Mathematically Δ is considered as the difference operator. We proceed by forming a portfolio that buys call options with strikes $x + \Delta x$ and $x - \Delta x$, and shorts two calls with strike x . It is easy to verify (left to the reader) that the portfolio pays off Δx if time- T stock price turns out to be x ; otherwise it pays off nothing. In order to scale the payoff to 1, we buy $1/\Delta x$ shares at the cost $\mathcal{P}(x, \Delta)$.

$$\mathcal{P}(x; \Delta) = \frac{[\mathcal{C}(x + \Delta x) - \mathcal{C}(x)] - [\mathcal{C}(x) - \mathcal{C}(x - \Delta x)]}{\Delta x}$$

Note that the numerator above is equal to the second difference of option price, i.e. $\Delta^2 \mathcal{C}(x + \Delta x)$. In economic terms the distribution of state prices is given by $\mathcal{P}(\mathcal{S}_T, \Delta)$. To get the state price density $\mathcal{P}'(\mathcal{S}_T)$, we differentiate the above expression, and replace the difference operator $1/\Delta x$ by the differential operator $\partial/\partial x$, which yields Eq. (3.9). Integral equation (3.10) is added to illustrate that the discounted price of a claim that pays off 1 in all states must be equal to 1.

$$\mathcal{P}'(\mathcal{S}_T) = \left(\frac{\partial}{\partial x} \mathcal{P}(x) \right)_{x=\mathcal{S}_T} = \left(\frac{\partial^2 \mathcal{C}}{\partial x^2} \right)_{x=\mathcal{S}_T} \quad (3.9)$$

$$\int_{\mathcal{S}_T} \mathcal{P}'(x) dx = 1 \quad (3.10)$$

The above formulas require that the option pricing function is twice differentiable. Moreover, since density functions are positive, Eq. (3.9) implies that the \mathcal{C} is convex in x . Finally, returning to the previous notation, it holds for stock \mathcal{S} that

$$\mathcal{P}'(\mathcal{S}_T) = \Lambda_T.$$

3.4 Risk-neutral pricing formula

Let us now proceed to the calculation of risk-neutral asset prices consistent with no arbitrage. For option pricing purposes, consider the price (at time t) of a European-style contingent claim \mathcal{Z} that has payoff function $h(\mathcal{S}_T)$ and expires at time T . Clearly, the risk-neutral price, based on Q -probabilities, equals the discounted value of expected cash flows. Unfortunately, Eq. (3.11) has little practical meaning because the Q -probabilities cannot be observed.

$$\mathcal{Z}_t = E^Q \left[e^{-\int r^{(s)} ds} h(\mathcal{S}_T) \mid \mathcal{F}_t \right] \quad (3.11)$$

For practical purposes it is more relevant to use the observed P -probabilities for calculations. Here we need the state price density to execute the probability transformation. Our risk-neutral pricing formula is given by (3.12), based on the definition (3.5). For simplicity we write $E(\cdot)$ for expectation under P .

$$\mathcal{Z}_t = E \left[\frac{\Lambda(T)}{\Lambda(t)} h(\mathcal{S}_T) \mid \mathcal{F}_t \right] \quad (3.12)$$

4 Another look at the Merton problem

The standard way of solving a Merton problem is to find a utility-maximizing consumption pattern and portfolio weights, subject to a budget constraint. Mathematically speaking, this is the *primal problem*, exemplified by Eq. (2.1). However, sometimes the optimization is simplified by considering the *dual problem* that asks: *If we insist on following a fixed consumption pattern, what is the minimum utility that allows it?*

This chapter illustrates how to solve for optimal consumption and investment working with the dual utility function. As shown by Theorem 1, this gives the benefit that the original dynamic problem becomes a static one. The analysis takes place in (x, y) -plane, where x and y are given in units of consumption and utility. As usual, utility function is an increasing and concave map from x to y . In contrast, the dual is a decreasing and convex map from y to x .

While our analysis of duality is based on Karatzas and Shreve [30], already Bismut [2] applied this principle to a Merton problem and proved that optimal consumption depends on marginal value function. Indeed, recent years have seen numerous applications of the dual approach. In continuous time it has been used in the incomplete market models of He and Pearson [14], Cuoco [10], and Henderson [15], [16], to name a few. An equivalent discrete time approach is used for ESO valuation in Kallio and Pirjetä [28] and for equity valuation in Kallio [27]. We will work mainly in continuous time to formulate the theory. Needless to say, the duality theory is applicable in discrete time models, as shown in Section 3.6.

Let us now provide some intuition for the dual approach. Consider a dynamic problem, where consumption and investments are financed by initial wealth X_0 . Write $(X_t)_{t=0}^T$ for a martingale process⁸, and f for a concave value function. Then the *value function process* $f(X)$ becomes a supermartingale and thereby bounded. This follows from Jensen's inequality as shown next.

$$\begin{aligned} E[f(X_t | \mathcal{F}_s)] &\leq f \circ E[X_t | \mathcal{F}_s] \quad \text{for } 0 < s < t \\ &\leq f \circ E[X_s | \mathcal{F}_0] \\ &\leq f(X_0) \end{aligned}$$

Hence, initial wealth sets an upper bound on value function. This leads to the result that the Lagrange multiplier of budget constraint, or 'shadow price of consumption,' relates optimal consumption and marginal value function, cf. Eqs. (4.17)-(4.18) below. Finally, the notation of this chapter is summarized in the Appendix for the reader's convenience.

4.1 Stock price dynamics

We will work with a model that generalizes Ch. 1 and is close enough to Ch. 14 of Björk [3] for citing his arbitrage conditions. The financial market consists of N stocks indexed by $(S_n)_{n=1}^N$ and riskfree asset B . Below we write S_t for the N -vector of stock prices stopped at time t . All uncertainty is generated by K -dimensional P -Wiener process, where P refers to objective probability and $N \leq K$.

Stock diffusions depend on a drift vector $\mu_t = (\mu_1, \dots, \mu_N)'$ and a $N \times K$ diffusion matrix $[\sigma_{nk}]$, rows of which are given by $\sigma_{n\bullet} = (\sigma_{n1}, \dots, \sigma_{nK})$. Drift and diffusion parameters and the bond process B_t are adapted random variables,

⁸A martingale measure is assumed to exist, however it may not be unique. This will happen in the context of incomplete market.

formally stated as $(r, \mu, \sigma_{nk}) \in \mathcal{F}_t^W$. (Time subscripts are omitted to simplify the notation.) Consequently, the following price processes are observed in this market.

$$\frac{dS_n}{S_n} = \mu_n dt + \sum_{k=1}^K \sigma_{nk} dW_k \quad \text{for } n = 1, \dots, N \quad (4.1)$$

$$\frac{dB_t}{B_t} = r dt \quad (4.2)$$

$$S_t = (S_1, \dots, S_N)' \quad (4.3)$$

For the risk-neutral Q -Wiener process we write W^* , and assume that the likelihood process $L_t = \frac{dQ}{dP}$ is a P -martingale. Completeness of this market depends on the order of numbers N and K , as explained below. We remark that the duality theory of Section 3.2 applies to both complete and incomplete markets, assuming that the state price density can be defined in a sensible way. Now consider the order of N and K .

- $N > K$: The market is complete assuming $\text{rank}(\sigma) \geq K$. There are (at most) $N - K$ redundant assets, given by linear combinations of K assets. This is equivalent to the complete market case, and therefore not treated separately.
- $N = K$: The market is complete assuming $\text{rank}(\sigma) = N$. This is the complete market case treated in Sections 3.3-3.5.
- $N < K$: The market is incomplete. There are unhedgeable risks, and in general the martingale measure is non-unique. This case is discussed in Chapter 4.

Hence, we can set $N \leq K$ and $\text{rank}(\sigma) = N$ without a loss of generality. Thereby, in a complete (*resp.* an *incomplete*) market, σ (*resp.* $\sigma\sigma'$) is an invertible $N \times N$ matrix. This has implications on how the price of risk is stated. In particular, suppressing time indices,

$$\begin{aligned} \lambda &= (\sigma')^{-1} (\mu - r) \quad \text{complete market} \\ \lambda &= \sigma' \left(\sigma\sigma' \right)^{-1} (\mu - r) \quad \text{incomplete market.} \end{aligned}$$

When we need risk-neutral probabilities, the likelihood process L_t of Eq. (3.1) will be used for change of measure. Let us remark that the following martingale measure equation has a solution that is unique if the market is complete. Dimensions are given to clarify the situation. Note that r denotes an n -vector with each element equal to the risk-free rate.

$$\begin{matrix} \sigma & \lambda & = & \mu & - & r \\ (N \times K) & (K \times 1) & & (N \times 1) & & (N \times 1) \end{matrix}$$

In the context of incomplete market it pays off to consider the diffusion matrix as a linear map $\sigma : \mathcal{R}^K \rightarrow \mathcal{R}^N$, and recall the definitions of kernel and

image, as well a result from linear algebra (see e.g. Hirsch and Smale [18]). Our working assumption will be $\text{rank}(\sigma) = \dim(\text{Im}(\sigma)) = N$.

$$\begin{aligned} \text{Ker}(\sigma) &= \{x \in \mathcal{R}^K \mid \sigma x = 0\} \\ \text{Im}(\sigma) &= \{y \in \mathcal{R}^N \mid \sigma x = y \text{ for some } x\} \\ \dim(\text{Im}(\sigma)) + \dim(\text{Ker}(\sigma)) &= \dim(\mathcal{R}^K) \end{aligned}$$

Now we're in a position to give an arbitrage condition. *According to Proposition 14.4 of Björk [3], the current model is generically arbitrage free if and only if $\text{rank}(\sigma) = \dim(\text{Im}(\sigma)) = N$.* Further details are given in Ch. 14 of Björk [3]. In theory, an incomplete market may be completed by adding $\dim(\text{Ker}(\sigma)) = K - N$ 'stocks' that are orthogonal to the traded stocks. This will be refined in Section 4.2.

4.2 A dual approach

Here we develop the notion of dual utility function. It is tightly connected to the inverse of marginal utility, our first object of interest. Let us commence with some definitions. Write U for an increasing and concave utility function. It has a lower bound $\underline{x} := \inf \{x \in \mathcal{R}; U(x) > -\infty\}$. Let $U'(x)$ be a marginal utility function that is decreasing in x and has an upper limit of $U'(\underline{x}^+) := \lim_{x \downarrow \underline{x}} U'(x)$.

It comes with the inverse $I(y) := (U')^{-1}(y)$ that is decreasing in y and characterized by the properties (4.4)-(4.5) given below. When these functions are time-dependent, we write $U(t, x)$ and $I(t, y)$. The notation $f \circ g$ stands for a composite function like $f(g(\cdot))$.

$$I \circ U'(x) = x \quad x > \underline{x} \tag{4.4}$$

$$U' \circ I(y) = y \quad 0 < y < U'(\underline{x}^+) \tag{4.5}$$

Heuristically, the above equations say that a change of variable $x \rightleftharpoons y$ is possible in a certain domain. Thus, we can fix either x or y a priori, and then solve the problem in terms of the other variable. Assuming that we start by fixing y , the solution is given by the *convex dual* $\tilde{U}(y)$ defined as

$$\tilde{U}(y) = \sup_{x \in \mathcal{R}} \{U(x) - xy\}. \tag{4.6}$$

The mathematics of $\tilde{U}(y)$ are described in the Appendix. Economically, $\tilde{U}(y)$ maps the optimal consumption-investment tradeoff. For taking the alternative route, Eq. (4.7) restates the convex dual (4.6) in terms of $U(x)$. It clarifies the previous idea of finding the minimum utility (y) to attain a desired level of consumption (x). This form will be used in the incomplete market context.

$$U(x) = \inf_{y \in \mathcal{R}} \{\tilde{U}(y) + xy\}. \tag{4.7}$$

The next step is to rewrite $\tilde{U}(y)$ in a more operational form (4.8) using the properties (4.4)-(4.5). Also the derivative (4.9) is provided for future use. It proves that $\tilde{U}(y)$ is a decreasing function⁹.

$$\tilde{U}(y) = U \circ I(y) - yI(y) \tag{4.8}$$

$$\tilde{U}'(y) = -I(y) \tag{4.9}$$

It is clarified in the Appendix that Eq. (4.8) is applicable in optimizing consumption-investment trade-off, with y interpreted as shadow price of consumption. We proceed by verifying that both (4.6) and (4.7) are satisfied by choosing $y = U'(x)$. For (4.6) this is true, because the function $y \mapsto \tilde{U}(y) + xy$ has y -derivative $-I(y) + x$. Therefore, (4.7) is minimized by $x = I(y)$. On the other hand, $x \mapsto U(x) - xy$ has x -derivative $U'(x) - y$, and thus $y = U'(x)$ maximizes (4.6). This is applied in the next section, where the consumption-investment problem is solved using the convex dual.

4.3 Problem formulation

Let us return to the consumption-investment problem by adopting the setup of Karatzas and Shreve [30, Ch. 3]. The dual approach for a complete market was introduced by Cox and Huang [8]. Formally, the agent wishes to maximize Eq. (4.10), i.e. the sum of utilities from consumption C_t and terminal wealth ξ . The set of feasible solutions $\mathcal{A}(x)$ is given by admissible pairs (C, ξ) satisfying the budget constraint. State price density (SPD) and initial wealth are denoted by Λ_0 and x .

$$\max_{(c, \xi) \in \mathcal{A}(x)} E \left[\int_0^T U_1(C_t) dt + U_2(\xi) \right] \tag{4.10}$$

$$\text{s.t. } E \left[\int_0^T \Lambda(s) c(s) ds + \Lambda(T) \xi \right] \leq x \tag{4.11}$$

The main difference between the current formulation and earlier problem (2.1)-(2.2) is that the budget constraint (4.11) enforces no arbitrage in the sense that the present value of consumption cannot exceed initial resources. State price density $\Lambda(t)$, being uniquely defined by Eq. (3.5) in a complete market, assumes the role of a discount factor. The following result from Cox and Huang [8, Lemma 2.5] shows the fundamental economic role of the SPD.

$$E^Q \left[\int_0^T \frac{c(s)}{B(s)} ds + \frac{\xi}{B(T)} \right] = E \left[\int_0^T \Lambda(s) c(s) ds + \Lambda(T) \xi \right] \tag{4.12}$$

The next theorem illustrates that an application of the convex dual solves the problem (4.10)-(4.11) in a clever way. Using the jargon of Karatzas and

⁹Since $I(y)$ takes values on $(0, U'(\underline{x}^+))$, $\tilde{U}'(y) = -I(y)$ takes values on $(-U'(\underline{x}^+), 0)$ hence $\tilde{U}(y)$ is nowhere increasing.

Shreve [30], the solution takes a *feedback form*, with inverse of marginal utility $I(y)$ playing an instrumental part.

Theorem 1 (Duality in a complete market) *Consider the dynamic consumption-investment problem (4.10)-(4.11). Assume that the market is complete in the sense that $N = K$. In this setup the dynamic problem can be transformed to static one by using the convex dual (4.6). The problem has a unique solution, characterized by consumption and terminal given by the function $I(y)$ defined above.*

Proof. *If we denote by y for Lagrange multiplier that arises in the constrained optimisation, the Lagrangian is written as*

$$\begin{aligned} E \left[\int_0^T U_1(C_t) + U_2(\xi) \right] + y \left\{ x - E \left[\int_0^T \Lambda(s) c(s) ds + \Lambda(T) \xi \right] \right\} = \\ xy + E \left[\int_0^T (U_1(C_s) - y\Lambda(s) c(s)) ds + U_2(\xi) - \Lambda(T) \xi \right] \\ \leq xy + E \left[\int_0^T \tilde{U}_1(y\Lambda(s)) ds + \tilde{U}_2(\xi\Lambda(T)) \right]. \end{aligned}$$

The third line follows from the definition of convex dual (4.6). For the budget constraint to be binding we require that the Lagrange multiplier $y > 0$. It follows from Eqs. (4.6) and (4.8) that the inequality " \leq " above becomes an equality " $=$ " if we choose

$$c(t) = I_1(t, y\Lambda(t)) \tag{4.13}$$

$$\xi = I_2(y\Lambda(T)) \tag{4.14}$$

Moreover, convexity of \tilde{U} renders the solution unique. ■

Knowing the form of the optimal controls (c, ξ) allows us define the value function, i.e. maximum utility consistent initial wealth x and no arbitrage as

$$V(x) = E \left[\int_0^T U_1(c(t)) dt + U_2(\xi) \right]. \tag{4.15}$$

Optimal consumption and terminal wealth are determined by Eq. (4.13)-(4.14) once we find the right y , which is the purpose of the following lemma.

Lemma 1 *In the dual problem context, one-to-one correspondence of initial wealth (x) and utility (y) is established by functions $\mathcal{X}(y)$ and $\mathcal{Y}(x)$ defined below. In particular, $\mathcal{Y}(x)$ is the marginal value function of the problem (4.10)-(4.11). Moreover, assuming power utility, $\mathcal{X}(1)$ is interpreted as the minimum initial wealth that allows consumption at unit rate. This will be useful in the context of Eqs. (4.21)-(4.22) below.*

Proof. Let us begin by defining the function $\mathcal{X} : (0, \infty) \rightarrow (\mathcal{X}(\infty), \infty)$ as

$$\mathcal{X}(y) = E \left[\int_0^T \Lambda(t) I_1(t, y\Lambda(t)) dt + \Lambda(T) I_2(y\Lambda(T)) \right]. \quad (4.16)$$

Above results on $I(\cdot)$ and $\Lambda(\cdot)$ imply that $\mathcal{X}(y)$ is decreasing and the inverse of marginal value function, that is $\mathcal{X}(y) = (V')^{-1}(y)$. We will proceed by interpreting $\mathcal{X}(1)$. Consider the power utility $U(x) = \frac{1}{p}x^p$ with $p < 1$ and $p \neq 0$. It satisfies $I \circ U'(1) = I(1) = 1$. Given that $\Lambda(t)$ takes the role of discount factor, the interpretation holds since $\mathcal{X}(1) = E \left[\int_0^T \Lambda(t) \cdot 1 dt + \Lambda(T) \cdot 1 \right]$. Now define the inverse of \mathcal{X} as $\mathcal{Y}(x) : (\mathcal{X}(\infty), \infty) \rightarrow (0, \bar{y})$ where \bar{y} is an upper bound. Therefore, $\mathcal{Y}(x)$ is the marginal value function evaluated at x , i.e. $\mathcal{Y}(x) = V'(x)$. Moreover, it is decreasing and satisfies $\mathcal{X} \circ \mathcal{Y}(x) = x$ so there will be a one-to-one correspondence $x \rightleftharpoons y$. ■

Utilizing the concepts just defined, we can write the optimality conditions for consumption process and terminal wealth as

$$c^*(t) = I_1(t, \mathcal{Y}(x)\Lambda(t)) \quad (4.17)$$

$$\xi^* = I_2(\mathcal{Y}(x)\Lambda(T)). \quad (4.18)$$

By definition, $c^*(t)$ and ξ^* are the maximal quantities consistent with initial wealth x and absence of arbitrage. When the above conditions (4.17)-(4.18) hold, the optimal Lagrange multiplier equals $\mathcal{Y}(x)$. Also, as stated by the next equation, the budget constraint holds as equality in the optimum.

$$\mathcal{X}(\mathcal{Y}(x)) = E \left[\int_0^T \Lambda(t) c^*(t) dt + \Lambda(T) \xi^* \right] = x \quad (4.19)$$

Let us now look at the optimal wealth $X(t)$. Provided that we choose $(c^*(t), \xi^*)$, it equals the present value of consumption for the remaining period $(t, T]$ and terminal wealth. $X(t)$ satisfies the initial and terminal conditions $X(0) = x$ and $X(T) = \xi^*$ almost surely. Note that $X(t)$ is analogous to Eq. (4.12).

$$X(t) = \frac{1}{\Lambda(t)} E \left[\int_t^T \Lambda(s) c^*(s) ds + \Lambda(T) \xi^* \right] \quad \forall t \in [0, T] \quad (4.20)$$

Let us now turn to the issue of optimal portfolio weights. All we have to say is that they are chosen proportionally to the Merton constant, defined as $\Sigma^{-1}(\mu - r)$, where Σ is a covariance matrix. Karatzas and Shreve [30, Remark 3.8.9] recover the mutual fund theorem of Merton [35] for the current model. The theorem says that regardless of preferences, the optimal investor allocates her funds between two funds. They consists of a stock fund with weights given by $\Sigma^{-1}(\mu - r)$ and a money market fund that yields the risk-free rate.

4.4 Solution in feedback form

The aim of this section is to make the previous results more tangible by demonstrating how Eqs. (4.17)-(4.18) can be used to solve a consumption-investment problem. The solution has a reasonable interpretation despite not being explicit. For preferences, assume power utility for both consumption and terminal wealth, i.e. $U_1(x) = U_2(x) = \frac{1}{p}x^p$. This entails that relative risk aversion equals $\text{RRA} = 1 - p$. Let us now find out the optimal pair $(C^*(t), \xi^*)$. The first step is to compute $I(y) = y^{1/(p-1)}$. Using this equality Eq. (4.16) becomes

$$\mathcal{X}(y) = E \left[\int_0^T (\Lambda(t))^{p/(p-1)} y^{1/(p-1)} dt + (\Lambda(T))^{p/(p-1)} y^{1/(p-1)} \right].$$

But the factor $y^{1/(p-1)}$ does not depend on time, allowing us to take it outside the integral. This results in the second line below, once we recall that $I(1) = 1$.

$$\begin{aligned} \mathcal{X}(y) &= y^{1/(p-1)} E \left[\int_0^T (\Lambda(t))^{p/(p-1)} dt + (\Lambda(T))^{p/(p-1)} \right] \\ &= y^{1/(p-1)} \mathcal{X}(1) \end{aligned}$$

The inverse of $\mathcal{X}(y)$ is easily calculated to be $\mathcal{Y}(x) = \left[\frac{x}{\mathcal{X}(1)} \right]^{p-1}$. Combining this with the optimum (4.17)-(4.18) gives the following *feedback form solutions*.

$$c^*(t) = \frac{x}{\mathcal{X}(1)} [\Lambda(t)]^{1/(p-1)} = \frac{x}{\mathcal{X}(1)} \frac{1}{[\Lambda(t)]^{1/\text{RRA}}} \quad (4.21)$$

$$\xi^* = \frac{x}{\mathcal{X}(1)} [\Lambda(T)]^{1/(p-1)} = \frac{x}{\mathcal{X}(1)} \frac{1}{[\Lambda(T)]^{1/\text{RRA}}} \quad (4.22)$$

What can we make of the above solutions? The first term shows how consumption depends on initial wealth. But how does consumption develop with economic states? Holding $\mathcal{X}(1)$ fixed, the second term shows how consumption and terminal wealth respond to variance in the SPD. Note from Eq. (3.5) that $\Lambda(t)$ takes large values when 'returns', i.e. the realisations of process W_t , are negative¹⁰. Further, smoothness of the consumption path increases with risk aversion, because $\text{RRA} > 0$.

4.5 Excursion to discrete time

Discrete-time modeling is often beneficial for valuing contingent claims with complicated exercise properties. An executive option with vesting period provides a good example. Such option cannot be exercised until it vests, but thereafter it is exercisable at any time, up to the expiration. Under these circumstances neither contingent claim values nor optimal consumption can be solved in closed form.

¹⁰To be exact, $\mathcal{X}(1)$ is also affected by changes in time- t SPD. However, this effect is likely very small as $\mathcal{X}(1)$ is determined by the average SPD; cf. Eq. (4.16).

Specifically, calculation of the state price density (3.5) is easier in discrete time, thanks to a factor form representation. Also, the stock price distribution, parametric form of which may be unknown, can be inferred from discrete outcomes. Thus, discrete time models are advantageous in practice, whereas continuous time models are better aligned for theoretical analysis. This raises the question: *How to discretize the problem in a way that ensures a solution that converges to an optimum?*

Let us first elaborate on the notions of optimum and convergence. The former refers to a no-arbitrage solution based on optimality conditions (4.17)-(4.18). The latter refers to *weak convergence* (or equivalently, convergence in distribution). It is denoted $X_n \rightsquigarrow X$, and stands for convergence in mean, i.e. $E[f(X_n)] \rightarrow E[f(X)]$ as $n \rightarrow \infty$, where X_n refers to a sequence and X to the law of a random variable. This notion is weaker than pointwise convergence and does not imply it. We also recall the *continuous mapping theorem*, saying that $X_n \rightsquigarrow X$ implies $g_n(X_n) \rightsquigarrow g_0(X)$, where g is a right-continuous function with limits on the left, in keeping with discrete changes in time. Mathematical details are given in Ch. 18 of van der Vaart [42].

In fact, few authors have paid attention to the convergence issue in the Merton problem context. One of them is He [13], who formulates three convergence theorems on consumption, portfolio weights and value function. Below we apply He's Theorem 1 to show the convergence of consumption policies and value function. What motivates this proof is that it applies directly to the model of Kallio and Pirjetä [28], assuming the market is complete. For the case of incomplete market we have not seen an applicable convergence theorem.

We begin formally by taking finite differences of the state price density (3.5), and writing the outcome in a 'factorized' form (4.23). Discretized processes are denoted by tilde, as in $\tilde{\Lambda}_t$. Indeed, it justifies the equivalent notion of 'stochastic discount factor'. The market is assumed complete, so $N = K$ in the notation of Section 3.1, and the price of risk (λ) is a K -vector. Recall that the definition (3.1) entails $\Lambda_0 = 1$.

$$\begin{aligned} \tilde{\Lambda}_t &= \frac{\tilde{L}_t}{\tilde{B}_t} = \left[\frac{1 + \lambda'_t \sqrt{\Delta t} Z}{1 + r_t \Delta t} \right] \frac{L_{t-1}}{B_{t-1}} \\ &= \dots = \prod_{\tau=1}^t \left[\frac{1 + \lambda'_\tau \sqrt{\Delta t} Z}{1 + r_\tau \Delta t} \right] \end{aligned} \tag{4.23}$$

In the above display Z denotes a K -vector of independent standard normal variates. Hence, $\sqrt{\Delta t} Z$ is distributed as $N(0, \text{diag}(\Delta t))$, and thus approximates the (vector) Wiener process $W_{t+\Delta t} - W_t$. Naturally, expected value of Eq. (4.23) reduces to the risk-free discount factor¹¹, i.e. $E[\Lambda_t | \mathcal{F}_0] = 1/[1 + r_0]^t$. We will now give a theorem with proof.

¹¹Formally, we have only assumed the risk-free rate to be an adapted process, rendering the bond locally riskless. However, by rational expectations we can write $E[r_t | \mathcal{F}_0] = r_0$ for any $t \in [0, T]$.

Theorem 2 (Optimal consumption) *Consider the consumption-investment problem (4.10)-(4.11), to be solved in discrete time. The discrete-time optimum is characterized by conditions equivalent to (4.17)-(4.18), using the discrete state price density (4.23). Moreover, optimal consumption, terminal wealth, and the value function will converge weakly to their continuous-time optima.*

Proof. *For notation, a tilde is used for discretized processes and a star for optimal quantities. Based on optimality conditions (4.17)-(4.18), the optimum is characterized by a sequence of Lagrange multipliers $y^* > 0$, and present value of consumption that equals initial wealth. Otherwise, if $y < y^*$ ($y > y^*$), the consumer attempts to spend more (less) than her initial wealth. Similar to Eq. (4.16), utility value of y requires an initial wealth of*

$$\mathcal{X}(y) = E \left[\int_0^T \tilde{\Lambda}(t) I_1(t, y\tilde{\Lambda}(t)) dt + \tilde{\Lambda}(T) I_2(y\tilde{\Lambda}(T)) \right].$$

We assume that $\mathcal{X}(y)$ satisfies integrability conditions (cf. He [13]). For optimization, the choice variable is y , not the state price density $\tilde{\Lambda}(t)$. Recall that optimality conditions for consumption and terminal wealth are given by

$$\tilde{c}_t^* = I_1(t, y^*\tilde{\Lambda}(t)); \quad \tilde{\xi}^* = I_2(y^*\tilde{\Lambda}(T)).$$

These conditions imply that an optimal multiplier $y^ > 0$ exists. Moreover, the budget constraint holds as equality and discrete processes converge weakly to their continuous-time optima, i.e.*

$$E \left[\int_0^T \tilde{\Lambda}(t) \tilde{c}^*(t) dt + \tilde{\Lambda}(T) \tilde{\xi}^* \right] \rightsquigarrow E \left[\int_0^T \Lambda(t) c^*(t) dt + \Lambda(T) \xi^* \right] = x.$$

For brevity, we will not write the convergence " \rightsquigarrow " explicitly below. Now suppose that $y < y^$. Because $I_1(\cdot)$ and $I_2(\cdot)$ are decreasing in y , the budget constraint is exceeded, which is not permitted. Formally,*

$$E \left[\int_0^T \tilde{\Lambda}(t) I_1(t, y\tilde{\Lambda}(t)) dt + \tilde{\Lambda}(T) I_2(y\tilde{\Lambda}(T)) \right] > x.$$

Alternatively, suppose that $y > y^$. In this case the consumer can improve her utility by consuming more, since the budget constraint is loose, stated formally as*

$$E \left[\int_0^T \tilde{\Lambda}(t) I_1(t, y\tilde{\Lambda}(t)) dt + \tilde{\Lambda}(T) I_2(y\tilde{\Lambda}(T)) \right] < x.$$

The outcome is that y must converge to y^ at the optimum. Moreover, convergence of the value function follows from continuous mapping theorem (van der Vaart [42, Th. 18.11]). ■*

4.6 An explicit solution in continuous time

While the feedback form solutions are valid under general conditions, they suffer from not being explicit. In this section we derive more explicit solutions for

consumption and portfolio weights, allowed by some restrictive assumptions. Our objective is to have them explicit enough for simulating the model. In such use parameter values can be validated by sensitivity analysis and 'sanity checks' of the results.

A prerequisite for the above objective is the employment of the dual value function. This will be rewarded by a simplified HJB equation. It becomes a linear second-order partial differential equation. Subject to certain assumptions, this PDE can be solved by the method of undetermined coefficients, as shown below. We step on the route by defining the *convex dual* \tilde{V} of value function V by Eq. (4.24).

$$\tilde{V}(y) = \sup_{x \in \mathcal{R}} \{V(x) - xy\} \quad (4.24)$$

We first consider the case of deterministic coefficients on diffusion parameters. This implies that stock prices obey the Markov property. Further, at time $t \in [0, T]$, the expected value of optimally invested wealth is known based on risk-neutral pricing formula (3.12). To quote Karatzas and Shreve [30, p.119], "in the case of deterministic coefficients, current level of wealth is a sufficient statistic for utility maximization". It follows that the first order conditions simplify somewhat. To see this, consider the discounted utility value of optimal wealth $\mathcal{Y}(x)\Lambda(t)$. Below, we write $\mathcal{Y}(x) = y$ to emphasize this quantity is deterministic. Expressions for $c^*(t)$ and ξ^* use Eqs. (4.17) and (4.18).

$$\begin{aligned} \mathcal{Y}(x)\Lambda(t) &= y \exp \left[- \int_0^t \lambda'_s dW_s - \int_0^t \left(r(s) + \frac{1}{2} \|\lambda_s\|^2 \right) ds \right] \\ &= \mathcal{Y}(t, x) \\ \Rightarrow c^*(t) &= I_1(t, \mathcal{Y}(t, x)); \quad \xi^* = I_2(\mathcal{Y}(T, x)) \end{aligned}$$

In the interest of completeness, and without fear for excessive notation, we remark that the dual value function can be represented in terms of dual utility functions. Note the arguments of \tilde{U}_1 and \tilde{U}_2 follow from above.

$$\tilde{V}(t, y) = E \left[\int_0^t \tilde{U}_1(\mathcal{Y}(t, x)) ds + \tilde{U}_2(\mathcal{Y}(T, x)) \right] \quad \text{for } y > 0.$$

Computation of $\tilde{V}(y)$ is based on the budget constraint (4.11) that sets an upper limit, in terms of initial wealth (x), for total consumption and terminal wealth. However, the dual value function (4.24) is defined in terms of y . This requires a change of variable, implemented by function $\mathcal{X}(y)$ that returns the maximum x corresponding to a fixed y . Therefore, maximum of the value function $V(x)$ is equal to $V(\mathcal{X}(y))$ as stated by Eq. (4.25). It is followed by the y -derivative $\tilde{V}'(y) > 0$. In economic terms, $\tilde{V}'(\Delta y)$ represents the change in initial wealth that causes the dual value function to change by Δy . Analysis of $\tilde{V}(y)$ is completed by the following display, where the second line is based on

$V' \circ \mathcal{X}(y) = \mathcal{Y} \circ \mathcal{X}(y) = y$, where \mathcal{Y} is the inverse of \mathcal{X} .

$$\tilde{V}(y) = V \circ \mathcal{X}(y) - y\mathcal{X}(y) \quad (4.25)$$

$$\begin{aligned} \tilde{V}'(y) &= [V' \circ \mathcal{X}(y)] \mathcal{X}'(y) - [\mathcal{X}(y) - y\mathcal{X}'(y)] \\ &= -\mathcal{X}(y). \end{aligned} \quad (4.26)$$

Actual solution of the consumption-investment problem will be obtained by making an ansatz on the dual value function that solves a partial differential equation. Then we use the method of undetermined coefficients to derive the exact solution. The PDE to be solved follows from Theorem 3.8.12 of Karatzas and Shreve [30]. They prove that, subject to regularity conditions, the Cauchy problem (4.27)-(4.28) provides necessary and sufficient conditions for Eq. (4.24) to hold. In the following lines, subscripts of \tilde{V} denote partial derivatives. \tilde{U}_1 and \tilde{U}_2 are dual utility functions determined by Eq. (4.8).

$$\tilde{V}_t(t, y) + \frac{1}{2} \|\lambda\|^2 y^2 \tilde{V}_{yy}(t, y) - ry\tilde{V}_y(t, y) = -\tilde{U}_1(y) \quad (4.27)$$

$$\tilde{V}(T, y) = \tilde{U}_2(y) \quad (4.28)$$

For preferences, we continue to assume power utility with the added twist of *subsistence consumption rate* \underline{c} . Also, for terminal wealth we set the lower bound \underline{x} , interpreted as a *retirement fund*. This is implemented by the power utility functions $U_1(c - \underline{c}) = \frac{1}{p}(c - \underline{c})^p$ and $U_2(x - \underline{x}) = \frac{1}{p}(x - \underline{x})^p$. The following display is based on application of Eq. (4.8). It presents the inverse marginal utility functions first, because \tilde{U}_1 and \tilde{U}_2 depend on I_1 and I_2 .

$$\begin{aligned} I_1(y) &= y^{1/(p-1)} + \underline{c} \\ \tilde{U}_1(y) &= \frac{1-p}{p} y^{1/(p-1)} - \underline{c}y \\ I_2(y) &= y^{1/(p-1)} + \underline{x} \\ \tilde{U}_2(y) &= \frac{1-p}{p} y^{1/(p-1)} - \underline{x}y \end{aligned}$$

Coming to the ansatz, based on the above forms of \tilde{U}_1 and \tilde{U}_2 and following Karatzas and Shreve [30], we try

$$\tilde{v}(t, y) = \frac{1-p}{p} k(t) y^{1/(p-1)} - l(t)y. \quad (4.29)$$

It is shown below that the functions $k(t)$ and $l(t)$, of which only $l(t)$ will have a direct interpretation, need to satisfy the following conditions. α is an auxiliary parameter (or a function) to be specified below.

$$\begin{aligned} \alpha k(t) - k'(t) &= -1 \\ l'(t) - rl(t) &= -\underline{c} \end{aligned}$$

Before elaborating on functions $k(t)$ and $l(t)$, we show that the ansatz (4.29) in fact solves the PDE (4.27). It implies that

$$\begin{aligned} \frac{1-p}{p} k' y y^{1/(p-1)} - l' y + \frac{1}{2} \left(\frac{1}{1-p} \right) \|\lambda\|^2 k y y^{1/(p-1)} - r y (-k y^{1/(p-1)} - l) \\ = -\frac{1-p}{p} y y^{1/(p-1)} + \underline{c} y \end{aligned}$$

To simplify the above expression, divide by y , recall that $l' - r l = -\underline{c}$, and finally divide by $y^{1/(p-1)}$. This yields a representation for $k(t)$.

$$k' + \frac{1}{2} \|\lambda\|^2 k \frac{p}{(1-p)^2} + r k \frac{p}{1-p} = -1.$$

Based on the previous equation, the condition $\alpha k(t) - k'(t) = -1$ is satisfied by the following choice for α . It may be constant or a function, depending on the setup. Solution of the PDE is completed by explicit forms for functions $k(t)$ and $l(t)$. Remark that the boundary condition (4.28) holds as $l(T) = \underline{x}$.

$$\alpha = \frac{p}{(1-p)^2} \left(\frac{1}{2} \|\lambda\|^2 + r(1-p) \right) \quad (4.30)$$

$$k(t) = e^{\alpha(T-t)} \left[1 + \frac{1}{\alpha} \right] - \frac{1}{\alpha} \quad (4.31)$$

$$l(t) = \underline{x} e^{-r(T-t)} + \frac{\underline{c}}{r} \left[1 - e^{-r(T-t)} \right] \quad (4.32)$$

As promised above, $l(t)$ has a natural interpretation. The first term equals present value of retirement fund \underline{x} , and the second term gives present value of steady consumption at rate \underline{c} during the period $(0, T]$.

Now we turn to the value function. Knowing that \tilde{v} solves the PDE we can write $\tilde{v}(t, y) = \tilde{V}(t, y)$, to be evaluated using Eq. (4.29). Marginal value function \mathcal{Y} and its inverse \mathcal{X} are calculated using $\mathcal{X}(t, y) = -\tilde{V}'(y)$, and the actual value function by integration as $V(x) = \int \mathcal{Y}(t, x) dx$. This yields the following equalities. It is required that $x > l(t)$.

$$\mathcal{X}(t, y) = k(t) y^{1/(p-1)} + l(t) \quad (4.33)$$

$$\mathcal{Y}(t, x) = \left(\frac{x - l(t)}{k(t)} \right)^{p-1} \quad (4.34)$$

$$V(x) = \frac{k(t)}{p} \left(\frac{x - l(t)}{k(t)} \right)^p \quad (4.35)$$

Finally, optimal consumption and portfolio weights are obtained using the feedback form solutions provided by Theorem 3.8.8 of Karatzas and Shreve [30]. It puts the feedback solutions in a simplified form (cf. Eqs. (4.17) and (??)).

$$\begin{aligned} c^*(t, x) &= I_1 \circ \mathcal{Y}(t, x) \\ \pi^*(t, x) &= -(\sigma')^{-1} \lambda(t) \frac{\mathcal{Y}(t, x)}{\mathcal{Y}_x(t, x)} \end{aligned}$$

Now we can state optimal consumption and investments in closed form, which was the objective set above. Note that both are proportional to $x - l(t)$, which equals wealth in excess of subsistence consumption and retirement fund. Also, the amount invested in stocks is proportional to market price of risk and inversely proportional to relative risk aversion. Of course, these results are not new, but they show how the feedback form solutions reduce to well-known results under certain simplifying assumptions.

$$\begin{aligned} c^*(t, x) &= \underline{c} + \frac{x - l(t)}{k(t)} \\ \pi^*(t, x) &= (\sigma')^{-1} \lambda(t) \left(\frac{x - l(t)}{1 - p} \right) \end{aligned}$$

Figure 4.1 illustrates key properties of the solution, with parameter values given in the figure caption. Here we only discuss some qualitative aspects. Panel A verifies that the value function $V(x)$ of Eq. (4.35) is nondecreasing and concave, and therefore it is a utility function. Panel B plots the marginal value function $V'(x) = \mathcal{Y}(x)$ of Eq. (4.34) that has the properties of marginal utility function. Panel C shows that the dual value function $\tilde{V}(y)$ is nonincreasing and convex. In fact, its derivative can be read from Panel B because $\tilde{V}'(y) = -\mathcal{X}(y)$ and $\mathcal{X}(y)$ is the inverse of $\mathcal{Y}(x)$, cf. Eqs. (4.33)-(4.34). In panel D, smooth line is time zero optimal consumption c^* , and dashed line is subsistence consumption, both given as function of initial wealth.

5 Extension to incomplete markets

So far the working assumption has been a complete market, where the number of stocks is at least equal to the dimension of the Wiener process, i.e. $N \geq K$ in Eq. (4.1). Here we relax this assumption, and also allow for unhedgeable risks such as stochastic volatility or labor income. In particular, we're interested in how incompleteness affects the market price of risk. Let us quantify this by rewriting the stock dynamics (4.1). Write σ_n for the n th row of a $N \times K$ diffusion matrix and W^* for the risk-neutral Wiener process. The price of risk is λ , being a K -vector.

$$\frac{dS}{S} = (\mu - \sigma\lambda) dt + \sigma dW^* \tag{5.1}$$

Incomplete market models are characterised by ambiguous prices, because arbitrage conditions only give rise to upper and lower hedging prices (5.9)-(5.10). Unique prices require additional assumptions on market equilibrium. To this end, Sections 5.2-5.3 discuss the minimal (or minimax) martingale measure of He and Pearson [14] and Hofmann et al. [20]. Sections 5.4 and 5.5 introduce portfolio constraints and labor income to the Merton problem (4.10)-(4.11) using the methods of Karatzas and Shreve [30, Ch. 5] and Cuoco [10]. There we change the market price of risk to $\lambda_\nu \geq \lambda$. Section 5.6 discusses certainty

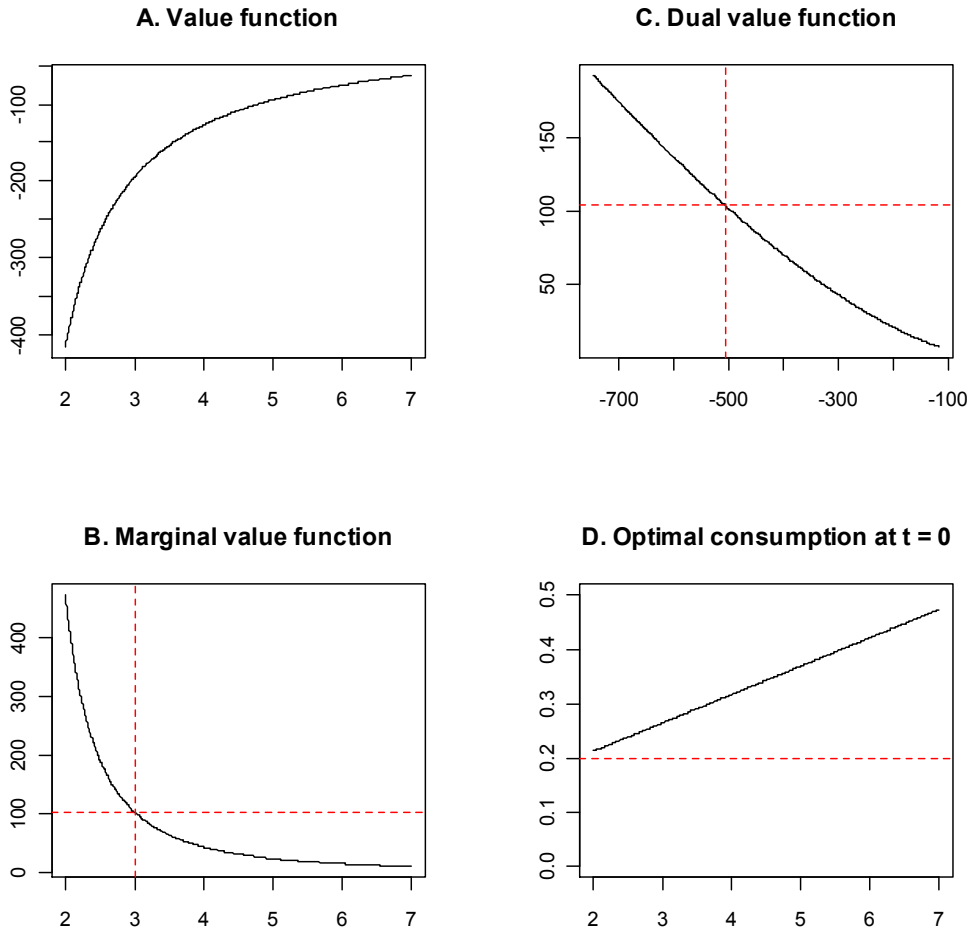


Figure 4.1: Solution of the consumption-investment problem, obtained using the following parameters. In Panels A-C initial wealth is fixed at $x = 3$, and in D it is the x -variable. Other parameters (in all panels): subsistence consumption $\underline{c} = 0.2$, retirement fund $\underline{x} = 0.3$, relative risk aversion $1 - p = 2$, expected stock return $\mu = 0.12$, stock volatility $\sigma = 0.15$ and risk-free rate $r = 0.05$. The problem has horizon $T = 5$, and the plotted functions are calculated at time zero. All plots are drawn in (x, y) -coordinates, where x and y are given in units of consumption and utility

equivalent pricing that is directly applicable to incentive options. However, we would like to begin with two results on hedging.

5.1 Two representation results

Here we aim to clarify the notions of contingent claim, martingale measure and hedging strategies, building on existence results from Jacka [26] and Kramkov [31]. For notation, write $\mathcal{M}(X)$ for a family of martingale measures, which means that the value process becomes a martingale under each $Q \in \mathcal{M}(X)$. Following Jacka [26], a contingent claim \mathcal{Z} with value process X_t has the following properties¹². Below S_t denotes a vector of traded stocks as in Eqs. (4.1) and (4.3).

- $X_t = x_0 + \int_0^t \theta_t dS_t$ for an adapted portfolio process θ_t
- X_t is a martingale under any $Q \in \mathcal{M}(X)$
- No-arbitrage price of the claim equals $\mathcal{Z}_t = E^Q[\beta_T X_T | \mathcal{F}_t]$, where β_T is a discount factor

What Jacka [26] proves is that a contingent claim has the same price under any $Q \in \mathcal{M}(X)$. For example, in the context of stochastic volatility this means choosing a price of risk that sets the stock price drift equal to risk free rate. Obviously, this entails finding a set of risk-neutral probabilities that make the discounted process $\beta \cdot X$ a martingale.

Presuming that a family of martingale measures exists, we're interested in the cheapest hedge of the claim, corresponding to a minimal process X_t . It is characterised by Kramkov [31], whose results apply to both the pricing of contingent claims and Merton problems. In the latter case, finding the cheapest hedge allows for maximal consumption, as shown by Eq. (5.2). In a Merton problem context, the triplet (v_0, θ_t, C_t) is interpreted as initial wealth, investment portfolio and consumption.

$$X_t = x_0 + \int_0^t \theta_t dS_t - C_t \quad (5.2)$$

By Kramkov's Theorem 3.1, X_t is a supermartingale for all $Q \in \mathcal{M}(X)$. This is exactly what is needed for a duality solution, as shown in Section 4.2. For derivatives pricing, X_t represents a hedging portfolio that is self-financing in the case $C = 0$. The same theorem says X_t is a supermartingale in general, and a martingale if it is self-financing. Moreover, Kramkov characterises the minimal hedging strategy as the smallest X_t . Mathematically this corresponds to a Doob-Meyer decomposition of supermartingale X_t with maximal C_t . In addition, we know that the minimal hedging strategy is a linear combination of traded assets as in Eq. (5.2). While this looks like a simple result, the proof is very complicated, because $\mathcal{M}(X)$ is not a countable set.

¹²Even if the claim is a linear combination of traded stocks, it cannot be hedged, because the number of risk factor exceeds the number of assets, i.e. $N < K$ in Eq. (4.1).

5.2 Pricing of contingent claims

Citing general equilibrium theory, He and Pearson [14] start from the idea that demand and supply must meet in a balanced market. Thereby, a marginal investor is indifferent between buying and selling any security, whether or not replicable, at prevailing prices. Their approach, that has become standard in the literature, is to solve the Merton problem by an extension of Eq. (4.7). Hofmann et al. [20] show that in option valuation this leads to a minimal martingale measure.

Henderson [15], [16] applies the above ideas to price and hedge non-traded assets in the presence of correlated traded asset. Her results apply to the case, where a manager cannot trade in his stock options, but is allowed to trade in stock market index. She finds that under such conditions the manager wants to maximize the systematic risk part of stock variance.

Let us now look at contingent claim valuation using the minimal martingale measure (MMM) of Hofmann et al. [20]. By definition, it forces expected returns of *traded assets* equal to the risk-free rate. In contrast, expected returns of *nontraded assets* are left unchanged, so their valuation remains dependent on subjective views. Equation (5.3) defines the MMM through the price of risk vector $\lambda^* \in \mathcal{R}^K$. Dimensions are added for clarification. Note that σ is not a square matrix but $\sigma\sigma'$ is such and also invertible.

$$\lambda^*_{(K \times 1)} = \sigma'_{(K \times N)} (\sigma\sigma')_{(N \times N)}^{-1} (\mu - r)_{(N \times 1)} \quad (5.3)$$

Given an incomplete market ($N < K$), there are generally infinitely many martingale measures. Thus we ask: *How does the MMM relate to a generic martingale measure, associated with price of risk vector λ_{gen} ?* The latter complies with a standard definition, i.e.

$$\sigma_{(N \times K)} \lambda_{gen}_{(K \times 1)} = (\mu - r)_{(N \times 1)}.$$

It is straightforward to show that the MMM and generic measure are related by (5.4), where ν is a K -vector. Because we require $\sigma\lambda_{gen} = \sigma\lambda^*$, it satisfies $\sigma\psi = 0$, so we write $\psi \in \text{Ker}(\sigma)$, and the next equality holds.

$$\lambda_{gen} = \lambda^* + \psi \quad (5.4)$$

In fact, finding a qualifying vector ν is easy, because $\dim(\text{Ker}(\sigma)) = K - N$ (cf. Sect. 3.1). The system $\sigma\psi = 0$ consists of N linear homogenous equations in K unknowns, and it has infinitely many solutions with $K > N$. Under the circumstances the market may be theoretically completed by adding $K - N$ nontraded assets. They can be subjectively priced and expected returns on these assets need not equal the risk-free rate (Hofmann et al. [20]).

5.3 Example: a stochastic volatility model

What follows is an application of the previous results in a stochastic volatility model. It is originally due to Hull and White [23], and applied to executive

option pricing in Pirjetä [37], part of this dissertation. Consider the following P -dynamics of stock S and squared volatility Y . (Diffusion parameters r, γ, η, μ may be time-varying.) Naturally, volatility is not a traded asset. Correlation of the S and Y processes is ρ , and we assume $\rho \neq 0$ to avoid a technical problem below¹³. Other diffusion parameters in (5.5) have their usual interpretations.

$$\begin{pmatrix} dS \\ dY \end{pmatrix} = \begin{pmatrix} \mu S \\ \eta Y \end{pmatrix} dt + \begin{bmatrix} \sqrt{Y}\sqrt{1-\rho^2}S & \sqrt{Y}\rho S \\ 0 & \gamma Y \end{bmatrix} \begin{pmatrix} dW_S \\ dW_Y \end{pmatrix} \quad (5.5)$$

In the notation of this section, we have an incomplete market with $N = 1$ and $K = 2$. Moreover, the diffusion matrix (in fact, a vector) is $\sigma_{1 \times 2} = \sqrt{Y} \begin{pmatrix} \sqrt{1-\rho^2}, \rho \end{pmatrix}$. We take as objective to determine the MMM and show that expected return on S equals the risk-free rate after measure change. Application of Eq. (5.3) gives the result

$$\hat{\lambda} = \frac{1}{\sqrt{Y}} \begin{pmatrix} \sqrt{1-\rho^2} \\ \rho \end{pmatrix} (\mu - r).$$

The desired result follows immediately from an application of the Girsanov theorem (3.3). Once we set $\varphi = -\hat{\lambda}$, the risk-neutral stock dynamics are given by the following display, and expected stock return is equal to the risk-free rate.

$$dS = rSdt + S\sqrt{Y} \left(\sqrt{1-\rho^2}dW_S^* + \rho dW_Y^* \right)$$

In this case, vector $\nu = (\nu_1, \nu_2)' \in \text{Ker}(\sigma)$ satisfies the equality $\sqrt{1-\rho^2}\nu_1 + \rho\nu_2 = 0$. This implies that we can choose an arbitrary ν_1 and then fix ν_2 so that the vector becomes $\nu = \nu_1 \begin{pmatrix} 1, -\frac{\sqrt{1-\rho^2}}{\rho} \end{pmatrix}'$. In general, the drift of this non-tradable asset will not be the risk-free rate.

5.4 Portfolio constraints

In this section we deal with constrained portfolios, and ask how the cost of hedging is affected by prohibition of short sales, non-traded stocks or minimum capital requirements. Such constraints are often faced by managers who receive equity-based compensation. The presentation is based on the results of Karatzas and Shreve [30, Ch. 5], Karatzas and Kou [29] and Cuoco [10]. The outcome will be that adding constraints leads to lower and upper hedging prices. They form an arbitrage-free interval that includes the unconstrained price. Unfortunately, this interval may be quite wide.

For a formal start we formulate the constraints. Define \mathcal{K} as the set of feasible portfolios, and an $(N+1)$ -vector $\nu = (\nu_0, \nu_-)'$. It is seen below in Eq. (5.8) that ν maps the effect of constraints into the price of risk. In order to add

¹³There is no problem in applying the MMM with $\rho = 0$, but the market cannot be completed since it requires $\sqrt{1-\rho^2}\nu_1 + \rho\nu_2 = 0$.

the constraints we need the convex function $\delta(\nu)$ given by (5.6), which is finite on the effective domain $\tilde{\mathcal{K}}$ defined by (5.7). Amounts invested in riskless bond and stocks are denoted by scalar α and N -vector θ .

$$\delta(\nu) = \sup_{(\alpha, \theta) \in \mathcal{K}} -(\alpha \nu_0 + \theta' \nu_-) \quad (5.6)$$

$$\tilde{\mathcal{K}} = \{\nu \in \mathcal{R}^{N+1} : \delta(\nu) < \infty\} \quad (5.7)$$

Intuitively, $\delta(\nu)$ measures the increase in cost of hedging caused by trading constraints, an idea clarified by the budget constraint (5.14) below. In particular, we have $\delta(\nu) \geq 0$ whenever (α, ν) belongs to \mathcal{K} , and $\delta(\nu) = 0$ on $\tilde{\mathcal{K}}$ (with one exception). Further, if borrowing is allowed in the money market, we have $\nu_0 \geq 0$ on $\tilde{\mathcal{K}}$ (see Cuoco [10, Remark on p. 40]).

Enforcing the constraints is done by modifying the market price of risk. First recall the stock and bond dynamics (4.1)-(4.2) and fix $N = K$. We wish to set up an artificial market \mathcal{M}_ν , where the market price of risk is given by

$$\lambda_\nu = \sigma^{-1} [\mu + \nu_- - (r + \nu_0) \mathbf{1}]. \quad (5.8)$$

Note how vector ν enters Eq. (5.8). *Assuming $\nu \geq 0$ it implies an increase in market price of risk to $\lambda_\nu \geq \lambda$. Because the objective drift is unchanged (μ), true risk-neutral drift decreases to $\mu - \sigma \lambda_\nu$ in Eq. (5.1).*

Adjusted price of risk (5.8) has an impact on the likelihood process and state price density. The market \mathcal{M}_ν and associated martingale measure $Q_\nu \in \mathcal{M}_\nu$ are characterised by the risk-neutral drift of $r + \nu_0$. By analogy to Eq. (3.5), the likelihood process (i.e. change of measure) is given by

$$L(-\lambda_\nu) = \frac{dQ_\nu}{dP} = \exp \left(- \int_0^t \lambda'_\nu dW_s - \frac{1}{2} \int_0^t \|\lambda_\nu\|^2 ds \right).$$

Let us further characterise the market \mathcal{M}_ν by fixing the risk-neutral discount factor B_ν and state price density Λ_ν as

$$\begin{aligned} B_\nu(t) &= \exp \left[- \int_0^t (r + \nu_0) ds \right] \\ \Lambda_\nu(t) &= \exp \left\{ - \int_0^t \lambda'_\nu dW - \int_0^t \left[(r + \nu_0) + \frac{1}{2} \|\lambda_\nu\|^2 \right] ds \right\}. \end{aligned}$$

Modification of the price risk implies that $W_\nu^*(t) = W(t) + \int_0^t \lambda_\nu ds$. Now we can proceed to formulate constraints on short sales, non-tradability and minimum capital (i.e. collateral requirement).

- *Short sales constraint* on stock 1. Set $\mathcal{K} = \{(\alpha, \theta) \in \mathcal{R}^{N+1} : \theta_1 > 0\}$ and $\tilde{\mathcal{K}} = \{\nu_- \in \mathcal{R}^{N+1} : \nu_1 > 0 \wedge \nu_2 = \dots = \nu_N = 0\}$. This implies $\delta(\nu) = 0$ on $\tilde{\mathcal{K}}$. Based on price of risk (5.8), the discounted process of stock 1 becomes a supermartingale¹⁴.

¹⁴This is the case, because risk-neutral drift is $r + \nu_0$ and the drift of stock 1 becomes $r + \nu_0 - \nu_1$.

- *Non-tradability constraint* on stock 1. Fix $\mathcal{K} = \{(\alpha, \theta) \in \mathcal{R}^{N+1} : \theta_1 = 0\}$ and $\tilde{\mathcal{K}} = \{\nu_- \in \mathcal{R}^{N+1} : \nu_2 = \dots = \nu_N = 0\}$. We still have $\delta(\nu) = 0$ on $\tilde{\mathcal{K}}$.
- *Minimum capital requirement* written as $\alpha + \theta' \underline{1} \geq M$. This is equivalent to *collateral requirement* of $\alpha \geq M - \nu_0^{-1} \sum_{n=1}^N \theta_n \nu_n$. Now the support function is $\delta(\nu) = -M\nu_0$ on $\tilde{\mathcal{K}}$. One can think of $\nu_n > 0$ as collateral values or risk weights.

Note that the collateral requirement α increases with short sales ($\theta < 0$). When constraints are present, the range of no-arbitrage prices is bounded by upper and lower hedging prices. First consider a trader who has written the option and has to pay \mathcal{Z} at expiration (T). Based on Th. 5.6.2 Karatzas and Shreve [30], his hedging price will be p_ν^{up} (possibly infinite). Alternatively, according to Th. 5.9.10 of *op.cit.*, if the trader is long the option and receives (non-negative) payoff \mathcal{Z} at expiration, his hedging price equals p_ν^{low} (with lower bound zero).

$$p_\nu^{up} = \sup_{\nu \in \tilde{\mathcal{K}}} E[\Lambda_\nu(T) \mathcal{Z}] \quad (5.9)$$

$$p_\nu^{low} = \inf_{\nu \in \tilde{\mathcal{K}}} E[\Lambda_\nu(T) \mathcal{Z}] \quad (5.10)$$

In line with intuition, the process $\Lambda_\nu(t) X(t)$, i.e. SPD times hedging portfolio, is a supermartingale in general, and a martingale for optimal ν . This result is made rigorous in Th. 5.8.1 of Karatzas and Shreve [30].

5.5 Labor income

Following Cuoco [10], the previous model is extended here to include labor income. Again we work with an artificial economy, where the constrained portfolios become optimal. It will be seen that adding constraints decreases terminal consumption and utility. Let us remark that Kallio and Pirjetä [28] solve a similar problem in discrete time and look at the effect of labor income on ESO valuation. Using stochastic optimization, they find that inclusion of labor income may increase or decrease ESO values, depending on the circumstances.

In addition to considering labor income, Cuoco's model contributes by allowing for portfolio constraints and borrowing against expected income flow. The market is incomplete, because labor income risk cannot be hedged. By analogy to complete market case, the solution will be given in feedback form similar to Eqs. (4.17)-(4.18). In particular, optimal consumption is still determined by marginal utility.

We will now provide a demonstration. The utility function is assumed to be in HARA class¹⁵, which includes both power and log utilities. Wealth dynamics are governed by Eq. (5.11), holding onto notation explained before (5.6). Diffusion parameters α, r, μ may be time-varying even though this is not

¹⁵HARA = hyperbolic absolute risk aversion

made explicit. For brevity, time indices are also dropped on consumption (C) and labor income¹⁶ (Y). The latter is broadly defined as an adapted random process, with details given in Cuoco [10].

$$X(t) = x_0 + \int_0^t [\alpha r + \theta' \mu] dt + \int_0^t \theta' \sigma dW - \int_0^t [C - Y] dt \quad (5.11)$$

In the above equation W is the Wiener process that drives stock prices as in Eq. (4.1), and σ is a $N \times N$ diffusion matrix. Clearly, labor income and stock market risks may be correlated if labor income is a diffusion process. Based on the modified market price of risk (5.8), we have the following SDE for risk-neutral dynamics using Q_ν -probabilities.

$$\begin{aligned} X(t) &= x_0 + \int_0^t [(\alpha + \theta' \underline{1})(r + \nu_0) - (\alpha \nu_0 + \theta' \nu_-)] dt + \int_0^t \theta' \sigma dW_\nu^* - \int_0^t [C - Y] dt \\ &\leq x_0 + \int_0^t [(\alpha + \theta' \underline{1})(r + \nu_0) + \delta(\nu)] dt + \int_0^t \theta' \sigma dW_\nu^* - \int_0^t [C - Y] dt. \end{aligned} \quad (5.12)$$

The second line implies that optimal wealth decreases if there are any constraints. The third line follows from Eq. (5.6). Remark that short sales and tradeability constraints imply $\delta(\nu) = 0$. Now we can state the consumption-investment problem, still using the Q_ν -probabilities, by Eqs. (5.13)-(5.14). The budget constraint is derived from Eq. (5.12).

$$\max_C \int_0^T U(C, t) dt \quad (5.13)$$

$$\text{s.t. } E^{Q_\nu} \left[\int_0^T B_\nu(C - Y - \delta(\nu)) dt \right] \leq x_0 \quad (5.14)$$

The budget constraint says that the present value of consumption less labor income cannot exceed initial wealth. Note the presence of support function $\delta(\nu) \geq 0$, which implies that a fixed consumption flow requires higher initial wealth. As expected, without constraints and labor income Eq. (5.14) reduces to the complete market case. Similar to Section 4.3, the dynamic problem can be transformed to a static one by using duality. For this purpose, write the Lagrangian, with multiplier ϕ , as

$$\mathcal{L} = \int_0^T U(C, t) dt + \phi \left[x_0 - E^{Q_\nu} \left(\int_0^T B_\nu(C - Y - \delta(\nu)) dt \right) \right].$$

The next item is to move from Q_ν - to P -probabilities. Changing the expectation operator $E^{Q_\nu} \Rightarrow E$ induces a change of discount factor $B_\nu \Rightarrow \Lambda_\nu$. Now the above Lagrangian becomes

$$\mathcal{L} = \int_0^T U(C, t) dt - \phi \left[\Lambda_\nu C + x_0 + \int_0^T \Lambda_\nu (Y - \delta(\nu)) dt \right]$$

Differentiating the above expression with respect to C yields the first-order condition (5.15) for consumption using the notation $I = (U')^{-1}$. Note the analogy to Eqs. (4.17)-(4.18).

$$C^* = I(\phi \Lambda_\nu) \quad (5.15)$$

¹⁶We apologize for using Y to denote squared volatility above. This is done to comply with the cited paper being part of this dissertation.

Using the definition (4.6) of convex dual, the above Lagrangian simplifies to

$$\mathcal{L} = \int_0^T \tilde{U}(\phi\Lambda_\nu, t) dt + \phi \left[x_0 + \int_0^T \Lambda_\nu (Y - \delta(\nu)) dt \right]. \quad (5.16)$$

This suggests that we can complete the solution using similar reasoning as in Ch. 4, that is by minimizing the above function with respect to ϕ and ν . Below we summarize the solution in two bullet points. These conditions are formalized by Proposition 1 of Cuoco [10].

- Optimal consumption is given by the first-order condition $c_t^* = I(\phi\Lambda_\nu)$, where I is the inverse function of marginal utility, Λ_ν is the state price density, and (ν, ϕ) are optimal Lagrangian multipliers specified below.
- (ν, ϕ) follow from solving the dual problem, which amounts to minimizing the function (5.16).

We remark that the discrete-time optimization of Kallio and Pirjetä [28] is based on equivalent arguments. Consumption follows from first order conditions, and optimal multipliers are calculated subject to portfolio restrictions. Discretizing the problem allows for numerical solutions. Closed-form solutions for problems this complex are hard to find, with the notable exception of logarithmic utility.

The current model also relates to the Ingersoll [25] model fitted in ESO data by Pirjetä et al. [38]. Ingersoll's model can be seen as a special case of Cuoco [10] with labor income excluded. Specifically, setting a holding constraint on underlying stock changes the market price of risk as in Eq. (5.8), and applicable drift in option pricing will be less than risk-free rate. A similar result is derived in Th. 5.7.1 of Karatzas and Shreve [30].

5.6 Utility-based option pricing

While utility-based pricing, or indifference pricing, is topical in mathematical finance, in economics the idea goes back to (at least) 1960s. Already Pratt [40] introduced the concept of *certainty equivalent (CE)*, typically defined as (5.17), where V^{CE} denotes the CE price and X an uncertain cash flow.

$$U(V^{CE}) = E[U(X)] \iff V^{CE} = U^{-1}\{E[U(X)]\}. \quad (5.17)$$

Eq. (5.17) applies directly to ESO pricing when the only cash flow is produced by the option. This is the case in Pirjetä and Rautiainen [39], who set to find out how severely incentive options convexify the utility function. While the paper provides a literature review, we would like to point out Hall and Murphy [11] as a useful summary of ESO pricing using the CE approach.

Mathematical finance literature has advanced CE pricing in (at least) two ways. Instead of a single cash flow, it considers a hedging portfolio $X_t = x_0 + \int_0^t \theta_t dS_t$ that is a semimartingale. Arbitrage possibilities are excluded by a budget constraint of the type $E[\Lambda_T X_T] \leq x_0$, saying that discounted value of

cash flows to come cannot exceed initial wealth. Under an incomplete market this gives rise to an optimization problem, where one wants to minimize the state price density. This leads to Theorem 3 defining the (marginal) utility-based price, or indifference price in line with Hobson [19] and Hugonnier et al. [22, Th. 3.1]. At this price the agent is indifferent between buying and selling the claim, i.e. his optimal demand is zero.

Let us highlight a technical issue before giving the main result. Existence of utility-based price requires that $|k\mathcal{Z}| \leq X_T$ for some hedging portfolio X and constant k . If this is satisfied, Hugonnier and Kramkov [21] show that a state price density exists, and the problem can be solved using duality¹⁷. Obviously, this is equivalent to the upper hedging price (5.9) being finite (Karatzas and Shreve [30, Th. 5.8.9]).

Theorem 3 (Utility-based price) *Assuming no arbitrage, the utility-based price of claim \mathcal{Z} for k contracts, being a dynamic counterpart of the certainty equivalent price of Pratt [40], is given by*

$$p(\mathcal{Z}) = \frac{p(k\mathcal{Z})}{k} = E[\Lambda_T^0 \mathcal{Z}] \quad (5.18)$$

where Λ_T^0 is the zero-level state price density (clarified below). Note that in general p depends on k , the number of claims.

Proof. *We will work out a utility maximization problem along the lines of Hobson [19], but remark that the solution by duality is due to He and Pearson [14]. Throughout the proof, X_T denotes a portfolio process like (5.2) that involves consumption. Now proceed by writing the actual problem as*

$$\begin{aligned} & \sup_{X_T} E[U(X_T)] \\ \text{s.t. } & E[\Lambda_T X_T] \leq x_0 \end{aligned}$$

and denote the optimum as $u(x_0, k) := \max E[U(X_T + k\mathcal{Z})]$. First-order condition of this problem yields with $k = 0$ the zero-level SPD $\Lambda_T^0 = U'(X_T)/\phi$, where ϕ is a Lagrangian multiplier. Moreover, we have $X_T = I(\phi\Lambda_T^0)$ where $I = (U')^{-1}$. Since Λ_T^0 is proportional to marginal utility, $p(\mathcal{Z})$ can be called marginal utility-based price. Let us now write the Lagrangian as (note that we add and subtract $\phi\Lambda_T k\mathcal{Z}$ and leave out the expectation operator)

$$\mathcal{L} = U(X_T + k\mathcal{Z}) - \phi\Lambda_T(X_T + k\mathcal{Z}) + \phi(x_0 + \Lambda_T k\mathcal{Z}).$$

Next, employing compact notation $x = X_T + k\mathcal{Z}$ and $y = \phi\Lambda_T$, we will use the duality results of Hugonnier and Kramkov [21, Th. 1] below. They require that $\tilde{u}(y) < \infty$ and $|\mathcal{Z}| < cX_T$. Then a state price density and that the process

¹⁷Because Hugonnier and Kramkov [21] allow the number of claims to be random, exact replication with $|\mathcal{Z}| = cX_T$ is not possible. Nonetheless, their Lemma 1 assumes existence of a portfolio that dominates time T payoffs, i.e. $|\mathcal{Z}| < cX_T$.

$\Lambda_t(X_T + kZ)$ is a supermartingale, which ensures no arbitrage. Moreover, the following functions are conjugates.

$$\begin{aligned} \tilde{u}(y) = \sup_{x>0} \{u(x) - xy\} &\iff u(x) = \inf_{y>0} \{\tilde{u}(y) + xy\} \\ \tilde{u}(y) &= u(I(y)) - yI(y) \end{aligned}$$

Now the above Lagrangian becomes $\mathcal{L} = \tilde{u}(y) + xy$. Therefore, the maximum utilities without and with the claim are given by

$$\begin{aligned} u(x_0, 0) &\leq \inf_{\psi} \inf_{\Lambda} E [\tilde{u}(\psi\Lambda_T^0) + \psi x_0] \\ u(x_0 - p(kZ), k) &\leq \inf_{\psi} \inf_{\Lambda} E [\tilde{u}(\psi\Lambda_T^0) + \psi(x_0 - p(kZ) + k\Lambda_T^0 Z)] \end{aligned}$$

In compliance with Hobson [19] and Hugonnier et al. [22], we compute utility-based price as $u(x_0 - p(kZ), k) = u(x_0, 0)$, which leads to the desired result that $p(kZ) = E[k\Lambda_T^0 Z]$ and price per unit equals $p(Z) = E[\Lambda_T^0 Z]$. ■

Remark 1 If we take the limit $k \rightarrow 0$, the utility-based price becomes $\lim_{k \rightarrow 0} p(Z) = E[\Lambda_T^0 Z]$, which is equivalent to the fair price of Karatzas and Kou [29] and marginal indifference price of Kallio and Pirjetä [28].

Proof. Karatzas and Kou [29] consider the following setup. Suppose that the agent optimizes (5.13)-(5.14), and he buys δ/p shares of the option $Z(T)$ at price p . In this case his value function is $W(\delta, p, x)$ given below. $X(x - \delta; T)$ refers to optimal wealth with an initial wealth of $x - \delta$.

$$W(\delta, p, x) = \sup_{(c, \xi) \in \mathcal{A}(x-\delta)} EU \left(X(x - \delta; T) + \frac{\delta}{p} Z(T) \right)$$

Karatzas and Kou [29] define the fair price p by Eq. (5.19) below and show that it falls in an arbitrage-free interval. Going back to Section 3.2, this definition agrees with the indifference price (3.7) of Kallio and Pirjetä [28].

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0 \tag{5.19}$$

■

Indeed, the above results relate executive option pricing to the theory of incomplete markets. Theorem 3 gives the CE price using the state price density. The latter is proportional to marginal utility, and it emphasizes the value of payoffs occurring in "poor times".

While advances have been made on valuation under incompleteness, unsolved problems remain. A key issue concerns hedging. In particular, when perfect hedges are not available, a decision rule is needed to evaluate the shortfalls. This task is assumed by Kramkov and Sirbu [33] using utility-based hedging.

Their definition of utility-based price is equivalent to (5.18). Further, they define a wealth process of utility-based hedge as

$$G(x, k) = X(x_0, p(\mathcal{Z})) - X(x_0, \mathcal{Z})$$

where $X(x_0, p(\mathcal{Z}))$ is optimal wealth by investing the utility-based price $p(\mathcal{Z})$, and $X(x_0, \mathcal{Z})$ is generated by hedging the claim \mathcal{Z} . Kramkov and Sirbu [33] characterize preference-dependent hedges that minimize $G(x, k)$. Their results open the door for many applications. For instance, how does the utility-based hedge of a manager depend on his option position and investable wealth? Investigation of this issue would certainly be of interest.

A Conjugacy in optimization

The intention here is to relate the convex dual of Eq. (4.6) to convex analysis and conjugacy. In particular, the latter plays a role in optimization. The presentation follows Section E of Hiriart-Urruty and Lemaréchal [17].

Let us start with a convex function $f : \mathcal{R}^n \rightarrow \mathcal{R}$. For a moment, assume that it is *smooth*, so that the gradient $\nabla f(x) = s(x)$ exists. (f is said to be smooth when all its partial derivatives are continuous.) Our focal point will be the inverse mapping of $\nabla f(x)$, denoted by $x = (\nabla f)^{-1}(s)$. In our applications $x \in \mathcal{R}_+$ corresponds to the smallest initial wealth that allows certain consumption flow, excluding arbitrage opportunities. More generally, in optimization $x \in \mathcal{R}^n$ refers to a stationary point of f .

Interestingly, x itself is the gradient of function h given by¹⁸ $h = \langle s, x \rangle - f(x(s))$. The notation $\langle s, x \rangle$ refers to scalar product of two vectors, i.e. $\langle x, y \rangle = x'y$ where x' is the transpose of x . Now define the *conjugate* of f by Eq. (A.1). It is also known as the Legendre-Fenchel transform.

$$f^*(s) = \sup_x \{\langle s, x \rangle - f(x)\} \tag{A.1}$$

This supremum is related to optimal f below. As noted above, existence of $\nabla f(x)$ requires f to be smooth, which may not hold in all cases. In order to deal with non-smooth f , the gradient is replaced by *subdifferential* of f (taken at x), defined by Eq. (A.2).

$$\partial f(x) = \{s \in \mathcal{R}^n : f(y) \geq f(x) + \langle s, y - x \rangle \text{ for any } y \text{ in } \mathcal{R}^n\} \tag{A.2}$$

The subdifferential is set-valued, and its elements are called *subgradients*. In words, they represent slopes of affine functions (lines in \mathcal{R}^2) that minorize f and coincide with f at x . The subdifferential is related to constrained optimization. In particular, x^* maximizes $\langle s, x \rangle - f(x)$ over \mathcal{R}^n when $0 \in \partial f(x) - \{s\}$ by Theorem E.1.4.1 in [17]. This theorem also verifies that Eq. (A.1) holds as equality

¹⁸This can be verified by taking total differential; $dh = \langle ds, x \rangle + \langle s, dx \rangle - \langle \nabla f, dx \rangle$. However, $\nabla f = s$, which implies $dh = \langle s, dx \rangle$ so we can write $\nabla h(x) = s$.

in optimum so that the supremum exists. In fact, one can show using the definition of $\partial f(x)$ that the optimum is characterized by Eq. (A.3). Moreover, there is primal-dual correspondence, i.e. $s \in \partial f(x)$ implies that $x \in \partial f^*(s)$.

$$f^*(s) + f(x) - \langle s, x \rangle \leq 0. \quad (\text{A.3})$$

But how are these results related to the consumption-income problem of Section 4? The connection is established by simply by changing the sign of utility function, or setting

$$f(x) = -U(x).$$

This way the convexity of f is guaranteed by the concavity of U . Moreover, the conjugate is given by

$$f^*(s) = \tilde{U}(-y).$$

Combining the above relation with Eq. (A.1) gives the Karatzas-Shreve definition of convex dual, i.e. $\tilde{U}(y) = \sup_x \{U(x) - xy\}$. Further, since the utility function is smooth, $U'(x)$ and $I(y)$ play the roles of gradient and its inverse. (Of course, the subdifferential reduces to gradient in this case.)

$$\tilde{U}(y) = U(I(y)) - yI(y).$$

In economic terms, the convex dual represents the optimal consumption-investment trade-off. At the margin, a small increase in current consumption increases current utility by $U(I(y))$, however it reduces future utility by $yI(y)$. Note how this representation highlights the roles of $I(y)$ and y , equal to inverse marginal utility and shadow price of consumption.

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ESO valuation under IFRS 2 – considerations of agency theory, risk aversion and the binomial model

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Abstract

This paper discusses the implications and valuation of employee stock options under IFRS 2. We analyze ESOs using agency theory. The employee calculates ESO value using the certainty equivalent principle, which leads to a discount to risk-neutral value. Hence, the fair option value stated as an expense in the profit and loss statement should be lower than the value suggested by risk-neutral option pricing models. Further, the gap between employer's and employee's valuation grows with the volatility and employee risk aversion. It is found that the ESO risk premium is time dependent, and it decreases going towards expiration. Finally, we discuss the effects of ESOs on managerial behavior using the framework of Ross (2004).

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1 Introduction

In an employee stock option (or executive stock option, in short ESO) plan the employee gets the right to subscribe shares of the employer after a vesting period, in order to align the interests of the owner and the employee. However, the real motivational and committing effects of an ESO contract are ambiguous (Hull and White 2004; Tian 2004; Ikäheimo et al. 2004). The recent IFRS 2 about share-based payment (with effective date of 1.1.2005) requires the recognition of the value of ESO plan as an expense in the company's profit and loss statement. However, the adoption of IFRS 2 may increase the amount of judgmental valuations in the profit and loss statement. Therefore, the implications and valuation problems involved in an ESO contract under IFRS 2 are discussed in this paper.

An employee stock option is basically a contract between the agent (employee) and the principal (employer). Both parties try to benefit from the contract, although they may have dissimilar power over the design and outcome of the contract and varying risk preferences. In this paper, the implications of the ESO contracts and IFRS 2 are discussed in the light of agency theory (Harris and Raviv 1979) and Pratt's (1964) risk premium.

In the agency theory framework incentive schemes of companies are designed so that the manager's behavior according to his/her self-interest also benefits the owner. Further, the lack of congruence between the agent's and principal's interests is thought to diminish with proper contract design and accounting disclosure (e.g. Jensen and Meckling 1976; Macintosh 1994, 29–37). Thus, the agency theory provides some support for granting options to corporate managers. In an influential paper, Harris and Raviv (1979) prove that if the agent is risk averse and his action is observable, optimal contract always depends on the agent's action. However, we argue that an unbounded linear contract is not feasible in reality, since the contract function (agent's compensation) becomes negative if the payoff is negative. In contrast, the agent will accept a linear contract bounded to positive outcomes. This contract is equivalent to call option on the payoff combined with a fixed payoff. Hence, agency theory implies that both the employer and the employee are better off, when compensation is tied to payoff, equal to change in market value. This result is of course conditional to the assumption that change in market value is an unbiased measure of the employee's effort.

Further, accounting disclosure is an ex post control device of the principal about the agent's action informing the owner's capital has been maintained. Function of the financial statements is to convey the true and fair view of the financial position, performance and changes in financial position of an entity. Fair presentation of financial statements is usually expected to be the result from applying generally accepted accounting principles (see IFRS Framework, paragraph 46). However, the agent prepares the statement and may use methods convenient for his or her purposes which may lead to a distorted view of the operations. Hence, in order to facilitate agency theoretical considerations, we define here the true and fair profit and loss statement as a reasoned and

materially accurate calculation of the change in the owners' wealth associated with, and caused by, the operations of the reporting entity during a specified period.

2 Expensing ESOs under IFRS 2

In share-based payment the reporting entity receives goods or services as a consideration for its equity instruments, such as ESOs. As the issuance of shares or rights to shares is recognized as equity, the offsetting debit entry (the grant date fair value of the share-based payment) is recognized as an expense. However, calculating the fair value of an American option with a vesting period and a non-listed or non-liquid underlying asset becomes complicated with existing option pricing methodology. Appendix B of the IFRS 2 requires that factors such as early exercise and changes in the expected volatility are considered in valuation, since they affect the fair value. However, vesting conditions are not to be considered in calculating the fair value according to IFRS 2.

According to appendix B of the IFRS 2, these considerations may sometimes preclude the use of "Black-Scholes-Merton (1973) formula". We argue that some of the complications are easier to deal with in the binomial model and hence we use it below. Common vesting conditions concern restrictions in selling, exercising or transferring the option. These conditions complicate the valuation of options and diminish the manager's perceived value of the option, at least in case of a risk-averse manager. The gap between the perceived value of an ESO to the company and to the manager is called deadweight loss. It is caused among other things by vesting conditions, trading restrictions and lack of diversification (Meulbroek 2001). Moreover, the accuracy of financial statements is impaired, if the value stated in the profit and loss statement differs substantially from the fair market value.

After the grant date fair value of an ESO is determined, this amount is expensed over the vesting period of the option plan. Corrections to the annual expense figure may normally be caused by a change in the number of options, but not by a change in the market value of the options after the grant date. However, if an employee leaves the company during the vesting period and thus forfeits the right to options, the expense is corrected. Hence, the number of options expensed is the number of options that actually vest.

An option plan decreases shareholder wealth by the dilution effect and by the opportunity cost of issuing shares below market price, provided that the options are exercised. Because of the dilution effect, in order to benefit from the ESO plan the owners should witness a share price growth above the market growth in the industry (as would have witnessed without the option plan). The adoption of IFRS 2 takes steps to disclose the costs of ESOs explicitly in the profit and loss statement. Therefore, earnings per share will be lower than without the recognition of options as expense. However, the market value of an ESO does not usually equal the prediction of a pricing model (e.g. Ikäheimo et al. 2004); nor will the grant date value of an option equal the cost of option plan to the

owners, or to the company, if the price of stock goes down and the options will not be exercised. Recall that later (i.e. post-issue) share price fluctuations have no effect on the stated expense.

Further, the cost of an option plan may exceed the benefits to the owner for many reasons, but especially if the employee is not committed to common goals. The difference in the perceived value of ESOs to the agent and to the principal (the deadweight loss) causes also a threat to the motivational effects intended. Motivational effects seem low if the exercise price is too low or too high (Tian 2004). Thus, according to Hall and Murphy (2000), more than 90 % of S&P 500 companies set the exercise price of ESOs at-the-money. We agree with Hall and Murphy in that it is difficult to say why most options are issued at-the-money, but it seems likely that the risk-averse manager's valuation of out-of-the money calls would be much below the market, and the owners reject the idea of issuing in-the-money calls.

When financial statements comply with IFRS 2, they reveal the burden of management incentives. It is entirely possible that option-related expenses turn profit into loss. Hence, the recognition of ESO costs facilitates the owners' judgement of whether the management is performing properly. This justifies the idea of reporting the cost of an ESO plan from the principal's point of view. However, the choice of the option valuation method is in the hands of the agent, and thus the reported value may not equal fair value of the options nor the cost to owners. Further, the value of the option plan may vanish if the market value of the underlying asset falls. Thus, the company may benefit from the incentive scheme in better motivation and records the corresponding expense, but nobody – not even the owner – has to compensate this in reality if the option plan has become worthless.

From an agency theory point of view, the agent is more exposed to the option value than the principal if the option generates a significant addition of the agent's wealth or is expected to do so. The principal is usually less exposed, because the opportunity cost of options becomes high only with excellent price performance. In this case, the principal has realized material capital gains, relieving the pain of issuing cheap shares. Next, the valuation problems of ESO plans, contract design features and the effects of risk-aversion are discussed and illustrated in more detail.

3 An agency model for executive stock options

Here we will refer to agency theory results to show that in the presence of asymmetric information and risk-averse agent, Pareto-optimal contract involves the agent's action. Further, the optimal contract links compensation to realized payoff, which by assumption measures the agent's action without bias. Hence under the optimal contract the agent's compensation depends on market value of the firm. If we amend the optimal contract by limiting the agent's share of the payoff to positive domain, we arrive at a contract that combines fixed salary with a call option. This reasoning is based on the idea that it is optimal for

both parties to maximize the firm's market value. Further, in the long run, we have to assume that market value of equity and operational performance move in parallel. Therefore the fact that short-term fluctuations in equity values are often uncorrelated with fundamentals does not invalidate this model, since we treat the employee stock options as a long-term contract. We also assume that the agent has sufficient power to influence the firm's actions and hence operational performance is highly correlated with the agent's action. In summary, the principal knows ex post the agent's action and the firm's performance depends on the action.

In this model, which builds on Model 1 of Harris and Raviv (1979), the agent's utility is a concave function of his compensation and action (or effort). Compensation is determined by the contract function $S(z)$. Utility increases with compensation, but action causes disutility, as defined in Eq. (3.1).

$$U^A = f(S(z), a); \quad U_1^A > 0; \quad U_2^A < 0. \quad (3.1)$$

In Eq. (3.1) subscripts 1 and 2 denote partial derivatives of U with respect to first and second arguments. Market value of the firm is determined by the agent's action as well as an exogenous state variable θ . Compensation contract is signed and the agent chooses his action prior to knowing the realization of the state variable. We assume that both parties hold similar views about the distribution of θ . Random payoff to state θ , as defined in Eq. (3.2), increases with the agent's action. In our case the payoff is equal to change in the firm's market value.

$$x = X(a, \theta); \quad X_1 > 0. \quad (3.2)$$

Parties to this contract share the payoff; the agent's share is $S(z)$ and the principal's share is $x - S(z)$. The agent's problem is to maximize his expected utility, where uncertainty is generated by the state variable θ . In our binomial model it determines the distribution of equity returns. Because we will employ the binomial model, realizations of θ follow the binomial distribution. The agent's problem is formalized in Eq. (3.3). Arguments of the utility function are compensation (or contract function) and agent's action.

$$\max V^A = EU^A(S(z), a) \quad (3.3)$$

Note that taking the expectation over outcomes of θ is equivalent to calculating the certainty equivalent of utility. This yields an important result: because the utility function is concave with respect to compensation, its certainty equivalent becomes lower as the variance of state variable increases. If the agent gets to choose between two compensation schemes with equal means, but different variances, he will take the one with smaller variance because it yields higher certainty equivalent. This effect, illustrated in Figure 1 is what we call the *Jensen's effect*, referring to Jensen's inequality. [Fig. 1 here]

The implication to employee option pricing is that increased volatility has a two-way effect on option value. On one hand, option value increases with volatility. On the other, the certainty equivalent decreases, since the agent is

risk-averse. This is the intuition for the recently established result that it is not in the interest of employee to increase the volatility of employer stock without bounds. Proponents of this view include Carpenter (2000) as well as Lewellen (2003). In her dynamic model, Carpenter (2000) shows that the optimal share of wealth invested in risky asset converges to Merton constant for manager with CRRA utility. This implies that the optimal share of risky asset decreases as volatility increases. Lewellen (2003) looks at the connection of incentives and capital structure. She argues that volatility costs of debt are higher for managers with in-the-money options, and hence ESOs discourage adding leverage and hence increase risk aversion.

Harris and Raviv (HR, 1979) characterize in their Proposition 2 the Pareto-optimal contract in this setup. Since we assume that there is no uncertainty ex post about the agent's action, Pareto-optimal contract does not involve monitoring his action. Hence the contract function depends only on the realized payoff and the agent's action. HR show that any Pareto-optimal contract is of the form (3.4), where S_1 is (an arbitrary) Pareto-optimal contract, x is the realized payoff and $X^*(\theta)$ is the expected payoff in state θ .

$$S^*(X, \theta) = S_1(X^*(\theta), \theta) + x - X^*(\theta) \quad (3.4)$$

Hence the optimal contract combines a fixed salary, which may be a function of the state variable, with a state-dependent element tied to the actual payoff. If we limit the moving element to its positive domain, in other words replace $x - X^*(\theta)$ by $\max[0, x - X^*(\theta)]$ in Eq. (3.4), the contract function becomes a combination of fixed salary and call option on the payoff. While this contract is not Pareto-optimal, it can be viewed as a real-world proxy of the optimal contract or a second-best solution. In practice it is unsustainable that the employee would accept a contract that yields negative compensation with positive probability.

4 Employee stock option valuation with binomial model

4.1 Binomial model in brief

Derivatives pricing in discrete time, specifically the binomial model, builds on an arbitrage argument saying that the price of portfolio that replicates the option payoffs must be equal to the option price. Consider the classical set-up presented by Cox, Ross & Rubinstein (1979). The problem is to price a call option on stock that may take only two values one period from now. First step is to form a hedging portfolio that invests in the stock and a risk-free deposit. Portfolio weights are such that the value of the hedging portfolio is in both states equal to the value of option when it expires. Specifically, the weight of underlying stock is delta (Δ) and weight of risk-free deposit is B . Values of the call option and the hedging portfolio are given in the table below. In terms of notation, u and

d give the magnitudes of up and down movements and r is the risk-free rate. S denotes stock price and K the strike price. It is assumed that $d < 1 + r < u$. We use the notation $(x)^+ = \max(x, 0)$.

state	call option (C)	hedge portfolio ($\Delta S + B$)
u	$C_u = (uS - K)^+$	$\Delta uS + Be^{rt}$
d	$C_d = (dS - K)^+$	$\Delta dS + Be^{rt}$

Since we have two equations with two unknown variables, it is easy to solve for portfolio weights Δ (in equity) and B (in deposit). The exercise is completed by deriving the one-period option pricing formula with the idea that values of the hedging portfolio and the option must be equal. Basic binomial pricing relations are given by equations (4.1-4.3), where r is the risk-free rate for period t and n is the number of periods.

$$\Delta = \frac{C_u - C_d}{S(u - d)} \quad \text{and} \quad B = \frac{uC_d - dC_u}{e^{rt/n}(u - d)} \quad (4.1)$$

$$C_j = \frac{qC_{j+1,u} + (1 - q)C_{j+1,d}}{e^{rt/n}} \quad \text{for } j = 0, \dots, n - 1 \quad (4.2)$$

$$q = \frac{e^{rt/n} - d}{u - d} \quad (4.3)$$

$$u = \exp\left(\sigma\sqrt{t/n}\right); \quad d = \exp\left(-\sigma\sqrt{t/n}\right)$$

The pricing formula contains the risk-neutral probability q , even if we haven't assumed anything about probabilities. This is a key insight of the model: derivatives can be priced using the risk-neutral measure, even if investors use subjective probabilities p and $1 - p$. Cox, Ross & Rubinstein (1979, p. 236) state this explicitly: "Since the formula does not involve [subjective probability p] or any measure of attitudes toward risk, then it must be the same for any set of preferences, including risk neutrality."

The pricing kernel is an essential tool in valuing ESOs, which will materialize in the next section. In general, we argue that the fundamental advantages of the binomial model are its transparency and flexibility. Since the model breaks the lifetime of the option into a discrete number of periods and option value at is calculated recursively, intermediate value points become transparent. These values become useful when we investigate early exercise.

4.2 Valuation of employee stock options

The aim here is to show that the prices of ESOs and standardized options need not converge, when contract differences are accounted for. Specifically, three features reduce the ESO value (vis-à-vis standard options): vesting period, non-transferability and individual risk preferences. We will start by discussing vesting period and restricted transfer, and continue with an analysis of concave preferences.

The employee endowed with options does not face a complete market, which is a fundamental difference to the standard valuation framework. ESOs are not transferable during the vesting period, and in many instances it is impossible to take a short position in the underlying equity. These restrictions could arise because the option holder is unable to assume short position in his employer's stock, or the underlying stock is not traded. In general, shorting requires either stock borrowing or using derivatives (e.g. buying puts or selling calls). Use of both options is quite restricted for any employee who operates under insider trading rules. Such persons can usually trade only after report disclosure, and their trades will be subject to close scrutiny. Borrowing stock is quite difficult for private investors. When it comes to derivatives, frequent trading is a virtual impossibility for anyone operating under insider trading rules.

Incompleteness of the market challenges the usual arbitrage considerations that are fundamental in pricing any derivative. Value of the option is reduced, if it is impossible to form hedging portfolio when it requires taking short position in the underlying. To see this, think about a call option trading below fair value suggested by a standard (binomial or Black-Scholes) pricing model. In order to lock in the profit, the arbitrageur has to buy the "cheap" call option and sell the hedge portfolio of Fig. 2. But now the portfolio weights Δ (in equity) and B (in deposit) become negative, which is a breach of the short-selling constraint. Hence the arbitrage opportunity vanishes, and the option value must be lower than in complete markets. Detemple and Sundaresan (1999) show that the effect of no-short-sales constraint in the underlying asset can be incorporated in the option value by adjusting the risk-neutral measure. Their main conclusion is the following. In the presence of short-sales constraint, valuing a derivative asset on a non-dividend paying stock is similar to valuing a derivative on a dividend-paying asset without the constraint.

Constrained trading may also help to explain the evidence that ESO holders tend to exercise their options prematurely. For instance, Carpenter (1998) finds that the average exercise of a 10-year ESO takes place at 5.8 years. This has surprised academicians, because premature exercise is equal to giving up some time value of the option, and hence decreases the wealth of the option holder. However, if a great deal of the employee's wealth is invested in options, and the only way to reduce exposure is through exercise, especially the risk-averse agent is likely to give up some time value in order to smooth her consumption. This effect is analyzed in the next section.

4.3 The effect of concave preferences

The binomial model values derivatives as if investors were risk neutral. While this assumption is not unreasonable at the aggregate market level, it is not likely to hold for a single agent (i.e. an employee). If we drop risk-neutrality and assume that the option holder is risk-averse, her behavior will change. Consider a situation where the option has vested and it is in the money. Further, there is some time left to expiration. Since the underlying stock goes up or down every period, keeping the option exposes its holder to some probability that

the option will expire worthless. If the holder is risk-averse, she will strongly avoid adverse states where the option value is zero. The outcome is that she will prefer selling or exercising the option prior to expiration. More formally, the risk-averse employee uses a risk premium in valuing the contract, but the risk-neutral market does not. This effect is magnified when value of the option represents a significant part of the individual's wealth.

Assume that the employee has power utility function (4.4) with consumption (or wealth) W and risk aversion γ , for which we use the range $1.5 \leq \gamma \leq 2.5$. In this case, absolute and relative risk aversion measures equal γ/W and γ . Let us note that the limit of power utility with $\gamma \rightarrow 1$ is log utility.

$$U(W) = \frac{W^{1-\gamma} - 1}{1-\gamma} \quad (4.4)$$

$$A(W) = -\frac{U''(W)}{U'(W)} = \frac{\gamma}{W} \quad (4.5)$$

$$R(W) = -W \frac{U''(W)}{U'(W)} = \gamma \quad (4.6)$$

We apply risk-averse pricing by calculating the certainty equivalent (CE) price of option. CE could also be called the reservation price, because it is the price that makes the employee indifferent between holding the option or selling it and investing the proceeds at opportunity cost of capital¹. CE values are calculated below at time points of option grant and vesting.

4.3.1 Risk premium in ESO valuation

Let us define the CE value \bar{c} based on Eq. (4.7) as the subjective ESO value. Further, the difference between objective (i.e. risk-neutral) and subjective ESO values equals Pratt's risk premium, given by π in Eq. (4.7). Consider a risky payoff with expected value of $E(c) = \tilde{c}$ and its certainty equivalent \bar{c} . Other variables are the risk premium (in absolute terms) π and initial wealth w_0 .

$$U(w_0 + \bar{c} - \pi) = E[U(w_0 + \tilde{c})] \quad (4.7)$$

The Pratt risk premium equals the maximum discount at which the employee is ready to sell her option. Equation (4.8), derived by Pratt (1964), shows that risk premium increases in the variance of uncertain payoff. In this formula $A(\cdot)$ is absolute risk aversion defined by Eq. (4.5). Note that for power utility (4.4) $A(\cdot)$ is decreasing, a property which we call DARA.

$$\pi(x, \tilde{c}) = \frac{1}{2} \sigma^2 A(x + \tilde{c}) \quad (4.8)$$

The rest of this section is dedicated to demonstration of CE pricing. Our pricing framework is similar to Hall and Murphy (2000, App. A). To get started, assume that the employee has initial wealth w_0 , terminal value of the option

¹Assuming risk-neutrality, opportunity cost equals the risk-free rate.

is C_T and discount in option price is δ . Equations (4.9-4.11) define our CE framework. If the manager holds the option until expiration, her terminal wealth will be $W_T(0)$, sum of initial wealth invested at risk-free rate (because the market is risk-neutral) and terminal value of the option. If the manager takes the CE, terminal wealth $W_T(\delta)$ is the sum of initial wealth and certainty equivalent of the option price, invested at risk-free rate. Discount in option price can be calculated using Eq. (4.11) for any period. Left-hand side of (4.11) calculates the expected terminal wealth by integrating W_T over the binomial distribution of stock price S .

$$W_T(0) = w_0 e^{rT} + C_T = w_0 e^{rT} + (S_T - K)^+ \quad (4.9)$$

$$W_T(\delta) = [w_0 e^{ri} + (1 - \delta) C_i] e^{r(T-i)} \quad \text{for } i = 0, \dots, T-1 \quad (4.10)$$

$$\int_{S_0}^{S_T} U(W_T(\delta)) f(S_t) dS_t = \int_{S_0}^{S_T} U(W_T(0)) f(S_t) dS_t \quad (4.11)$$

We look at the behavior of an executive who gets a three-year option vesting after two years. The setup is modelled with 12-period binomial grid. The vesting period is accounted for by assuming that the manager may trade once every quarter. Given two-year vesting period, the manager may trade only in periods 8-12. The binomial model is calibrated with the pricing kernel given by equations (4.2-4.3).

		Stock volatility (% p.a.)		
		12	15	20
Risk	1.5	5.9	8.9	14.3
aversion	2.0	7.9	11.7	18.5
	2.5	9.8	14.5	22.5

Table 1. Discount (%) in ESO price at different levels of risk aversion and volatility.

Table 1 shows how the discount depends on risk aversion and equity volatility. Discounts are in the range of 6 – 23 %. The values are sustainable compared to actual trading prices of Finnish ESOs reported by Ikäheimo, Kuosa and Puttonen (2004). In line with intuition, as the person becomes more risk averse she will agree to higher discount. Naturally the chances of very favourable outcome increase as well, but this is unimportant with concave utility. Tian (2004) reports similar findings for the risk-averse option holder: discount to ordinary option values increases with risk aversion and volatility of stock returns. In general our results indicate smaller discounts than the “executive value lines” of Hall and Murphy (2000, 2002). Most likely the difference is due to the fact that we use risk-neutral probabilities, like Carpenter (1998), whereas Hall and Murphy (2002) employ a pricing kernel, where the expected return is equal to stock return. The downside of their model is that arbitrage possibilities cannot be ruled out, even if the employee is assumed risk-neutral. [Fig. 2 here]

We will now summarize the results of Table 1 in a proposition.

Proposition 1 (a) Assume that the option holder has an increasing and concave utility function $U(w)$. Then she applies a risk premium in valuing her wealth distribution given by the binomial grid. As a result her subjective value of the ESO is always lower than the market value calculated with risk-neutral option pricing model. (b) The risk premium increases, or her subjective value decreases as the level of risk aversion increases.

Proof. (a) Work out using Jensen's inequality. (b) work out using Eq. (4.8). ■

Let us now turn to the issue of how the certainty equivalent value develops during the option's lifetime. Most compensation schemes have a vesting period during which the manager cannot trade. Therefore it is relevant to look at the CE value when the option vests. At this point the variance of underlying return distribution is smaller compared to the start, and hence the Pratt risk premium is smaller. This implies that if we are interested in the actual trading prices or exercise profiles, we should calculate CE values when the options are sold or exercised. As shown in Figure 3, the discount in CE values declines rapidly during the option's lifetime. Three bars are drawn for each period corresponding to risk aversion coefficients of 2.5, 2.0 and 1.5 (going from left to right). For instance, the moderately risk averse manager with $\gamma = 2$ agrees to discount of 11.7% at the outset, but as the option vests after two years (in period 8) she is ready to sell at a meager 4.4 % discount. If the option is in the money, it is quite possible that this is more than the time value and hence the option will be exercised early. While our model incorporates early exercise, it is probably too unlikely compared to empirics. If we calibrate the model using 20 % volatility, expected exercise time for a 3-year in-the-money option with $(S, K) = (100, 80)$ varies between 2.6 and 2.9 years. This is assuming that the option vests after two years.

As shown by Figure 3, the relation of risk premium and time seems almost linear, in other words an increase in risk aversion increases the risk premium across the time axis. We have done a small regression study to complete the sensitivity analysis of Table 1. In this exercise, we regress the CE option values on three variables: simple and squared time to expiration as well as risk aversion parameter. The data consists of five sample paths (65 data points), which are generated by varying the level of risk aversion. This model results in the following least squares fit (with standard errors in parentheses):

$$d(t, \gamma) = \underset{(0.00713)}{-0.0958} + \underset{(0.00460)}{0.0797t} - \underset{(0.00148)}{0.00634t^2} + \underset{(0.00324)}{0.0486\gamma} + \varepsilon \quad (4.12)$$

$$R_{adj}^2 = 0.9768; \quad F_{3,61} = 663.8$$

Equation (4.12) use obvious notation: t is time to expiration (in years), γ is risk aversion coefficient and ε is the error term. We also tried to add squared risk aversion to the model (4.12), but it didn't work out, since the coefficients of both γ and γ^2 became statistically insignificant. In our opinion, the regression model (4.12) is useful in quantifying how the ESO risk premium diminishes as the expiration gets closer. This effect is plotted in the upper panel

of Fig. 4, messaging that in the case of 3-year option, in the early days the decay of risk premium is slow, but it accelerates towards expiration. To an extent this property is a result of assuming that in expiration, option price is given by its intrinsic value. Comparing the curves shows that an increase in risk aversion simply shifts the curve up, verifying the intuition that the time and risk aversion effects must be independent. Lower panel of Fig. 4 plots the residuals of (4.12), showing that the model fits pretty well for ‘intermediate’ values of risk aversion, but for very low and high risk aversion value the fit becomes poorer.

There is also empirical evidence that the risk premium in option price increases with time to expiration. In their empirical study of Finnish ESOs, Ikäheimo, Kuosa & Puttonen (2004) find that discount to BS value increases by 4.2 percent per year.

To our knowledge the least squares haven’t been applied yet to estimating the sensitivities of CE values, as we do. For example, Longstaff and Schwartz (2001) use OLS regression to determine the exercise timing for American options. Their heuristic is to exercise the option, if its intrinsic value is higher than expected cash flow on a given sample path. Longstaff and Schwartz calculate the expected cash flow by fitting a second-degree polynomial in underlying asset prices. [Fig. 3 here]

If one accepts that there must be a risk premium in ESO prices, the next thing is to connect the risk premium with early exercise. In the certainty equivalent framework the early exercise of the option is triggered by the Pratt risk premium being higher than the time value, and as a result it is rational in some cases. One cannot get this result in a binomial framework without CE modelling. For example, in the binomial model of Hull and White (2004) exercise occurs unconditionally when the stock price reaches some multiple M of strike price. However, Hull and White do not give an explicit rule for determination of M. In other words, they assume that employees tend to exercise options after a subjective limit, as they become in the money. Our calculations produce similar results. Early exercise takes place when the underlying stock has done very well, that is in the top nodes of the binomial grid.

4.3.2 Convexifying effect of call options

The effect of call options on managerial utility functions is a topical issue in finance (e.g. Carpenter 2000, Ross 2004). In formal terms, the discussion is about the impact of convex instruments on concave utility functions. Ross (2004) presents general conditions for a *contract* $f(x)$ to either convexify (U'' increases) or concavify (U'' decreases) the manager’s derived utility function $U \circ f$. According to his Theorem 1, contract $f(x)$ convexifies the manager if the condition (4.13), derived in the Appendix, holds. The derivation assumes that $f'(x) > 0$.

$$\frac{f''(x)}{f'(x)} \geq [A \circ f(x)] f'(x) - A(x) \quad (4.13)$$

As Ross shows convincingly, a convex compensation scheme does not unconditionally convexify the manager. In line with him, we assume positive time

value for the option, which ensures convexity of the pricing function and the contract $f(x)$. This is fair since the IFRS 2 values executive stock options when they are granted, and in general ESOs have long maturities.

Ross (2004) also develops a three-term decomposition of effect of contract $f(x)$ on absolute risk aversion $A(x) = -\frac{U''(x)}{U'(x)}$. To this end, denote the derived utility function by $V(x) = U \circ f(x)$. Equation (4.14), derived in the Appendix, gives the Ross decomposition, subject to numerical analysis below.

$$A_V(x) - A(x) = [A(f) - A(x)] + A(f)[f' - 1] + A_f(x) \quad (4.14)$$

where $A(f) = -\frac{U''(f(x))}{U'(f(x))}$, $A_V(x) = -\frac{V''(x)}{V'(x)}$ and $A_f(x) = -\frac{f''(x)}{f'(x)}$

Let us explain the terms on the right hand side of Eq. (4.14). The first one represents the *translation effect*. When the manager is under contract $f(x)$, utility function is evaluated at $f(x)$ instead of x . The second term, or *magnification effect*, maps the effect of option delta. The effect is negative assuming $0 < f'(x) < 1$. It gains power as the delta decreases, wherefore out-of-the-money (OTM) options induce the manager to accept more risk. In contrast, magnification effect for in-the-money options is smaller, since their delta is close to one. The third term, or *convexity effect*, adds the impact of the contract's convexity.

Figure 5 shows how Eq. (4.14) depends on the stock price. It shows two cases; in the first one (smooth lines) the manager's compensation is split 1:1 between fixed salary and options. In the second case (dashed lines), compensation is dominated by fixed salary with the ratio 2:1. There is nothing new to the result of Panel D that options convexify the manager. But the dynamics of risk aversion are somewhat surprising. Assuming DARA utility, one would guess that the call option's negative effect on $A(x)$ would increase with share price. But in fact the effect decreases with share price. A look at panels B and C shows that the convexity and magnification effects are working behind this feature. As the stock price increases, option delta approaches unity, and the option behaves like a stock. The third term (i.e. convexity effect) in Eq. (4.14) loses power as the option moneyness increases, given that option gamma, that is $f''(x)$, decreases in absolute value. Option values for Fig. 5 were calculated using standard Black-Scholes formula, since the focus is on incentive effects instead of option valuation. [Fig. 4 here]

To understand the strike price effect of **Fig. 5**, think about how option delta increases with stock price. The delta of in-the-money (ITM) option increases at a slower pace than the delta of out-of-the-money (OTM) option. In order to earn some money on her option, the manager who is granted OTM options has a stronger incentive to take risky projects and increase underlying volatility than the manager with ITM options. This might explain the fact, documented by Hall and Murphy (2000), that most option grants have at-the-money strike price. It is also in line with the findings of Lewellen (2003) that corporate executives holding in-the-money calls are not willing to increase leverage, since it would decrease the certainty equivalent of their wealth.

4.4 Empirical evidence on underpricing and early exercise

Recent empirical evidence of discount in ESO prices is presented in Ikäheimo et al. (2004), who employ the Black-Scholes model with historical volatility as a yardstick for fair value. They find that the average underpricing of Finnish ESOs is 15.5 % in a sample consisting of approximately 15,800 trades in ESOs of seven listed firms. However, as shown in their Table 4, discounts on a given issuer's options vary materially across different emissions. For instance, if we take the three options plans of Nokia, the 1995 and 1997 issues trade at less than 1 % average discount, but average discount on the 1999 plan is around 18 %. An interesting finding of [op.cit.] is that the discount is particularly high for ten days after listing. If we took the liberty to interpret these results, we would say that there is a population of highly risk-averse employees willing to dispose their options at the first instant. In countries where ESOs are not listed, these employees are forced to exercise their options, contributing data to the literature on early exercise. [Fig. 5 here]

Carpenter (1998) presents results on the exercise profiles on option plans of 40 firms listed in NYSE or AMEX. All the options have ten-year maturities. If we look at sample averages, vesting period is 1.96 years and exercise takes place at 5.83 years. Carpenter also reports some interesting correlations in her Table 1. It is surprising that the correlation of stock price (relative to strike) and time of exercise is only 0.14. This should be judged against the average stock price at exercise, which is 2.75 times the strike. A look at Carpenter's Figure 1 confirms that there is almost no association between time of exercise and stock price.

Finally, there is a survivorship issue that complicates interpreting these figures. All the published data concerns the sample of options that have finished in the money and not the full population. Hence it would be a great mistake to say that the average ESO holder in the US exercises her options at 2.75 times the strike price. A fair share of ESOs expires worthless, and because this amount is unknown, it is difficult to make conclusions on the behavior of the average employee endowed with options. Further, it is also possible that part of options in a single plan is exercised deep in the money and part of them expires worthless. Thus, we don't know the risk preferences of those who never exercise their options.

5 Discussion

IFRS 2 requires employee stock options to be expensed, decreasing reported profits as well as dividend payments. Further, in a case where period profit and stock price have a weak correlation an ESO plan does not automatically motivate managers to improve the operating performance of the reporting entity. According to agency theory, relatively long vesting period is preferable from the owner's point of view, as the risk-averse agent's commitment to the common goals is improved. If employee stock options form a substantial part of the manager's wealth, he may choose excessively conservative policies as a result of

risk aversion. If the maturity of employee stock options is relatively short, the management may invest all its time and effort in maximizing short-term profits, at the expense of the owner's long-term goals.

The effect of exercise price on the shape of resulting utility is often overlooked. When shareholders (the principal) give options to the manager (the agent), it should be understood that the manager's utility function will take a different shape. If the shareholders would like to see the manager taking more risk, out-of-the-money options provide correct incentives. On the other hand, if shareholders prefer the agent to stay risk-averse, in-the-money options should be used in compensation. Shareholders should also keep track of the dynamic incentive effects of options as time evolves. It is fully possible that a risk-inducing incentive becomes a risk-reducing incentive. This materializes when an option was granted out-of-the-money, but has become in-the-money with increased share price (see Fig. 6).

Adoption of IFRS 2 reduces the wealth of the owner in cases where substantial expense entries from ESO plans are disclosed and the distributable retained earnings diminish. Thus, the interests of the principal are more secured after the adoption of IFRS 2 while the eventual loss of wealth does not come as a surprise to the owners and the owners may assess the use of their investment compared with other possible investment opportunities. However, the power of the agent to the ESO valuation method may result in ambiguous profit and loss statements, although additional disclosure already demanded by the IFRS 2 may help to overcome these dilemmas. Nevertheless, the adoption of IFRS 2 will usually improve the comparability between entities with varying employee incentive schemes, but the increased disclosure will cause the size of financial statements to grow. Thus, increased effort from owners and all those working in the fields of finance, auditing or management is needed in order to understand the facts and figures presented.

Expensing the value of ESO plan means that the consideration of new share capital is ultimately the work performed by the employee or employee's commitment to the company. However, in some EU countries, for example in Finland, the company legislation prevents the actual issuance of shares in consideration of work. Also, the gap between tax accounting and financial reporting will probably grow. When accounting rules are changed to secure the owner's interests, the entity's point of view becomes less important and there is a gap between the reported costs for the owner and for the firm. One solution to the fair presentation problem is to use fair market value as the indicator of ESO value which in light of this study is lower than a risk-neutral valuation model predicts. For example, in a case where an option plan has become worthless the true and fair view is not likely to convey from the profit and loss statement. Hence, if it is found out that the options will probably not be exercised; the decline in the value of the option plan could be recognized as a deduction of employee costs as this diminishes the dilution effect.

6 Conclusions

We argue that if the employee is risk-averse, her subjective value of the options is lower than the objective value given by standard option pricing models. Second impact of risk-aversion is that the expected value of compensation does not uniquely determine the subjective value of compensation, because the risk premium depends on variance as well. Because the employee has limited opportunities to diversify and hedge, it is likely that when she faces any probability of zero outcomes, she will exercise at least some of her options prior to maturity. Ignoring vesting conditions and risk-averse behavior may result in substantial overestimation of the option plan value. Consequently, giving options to employees is likely to result in some deadweight loss, because the employee's value of the plan is below fair value for reasons given above. Therefore, in order for the value expensed in profit or loss statement to reflect any objective value, fair value stated as an expense in the financial statements of the company should usually not be equal, but lower than the value suggested by a risk-neutral option pricing model.

Shareholders should recognize that giving call options to managers convexifies them, i.e. makes all DARA managers less risk-averse. We argue that the choice of strike price is a key decision variable of shareholders. If shareholders want only a modest convexifying effect, the strike price should be in the money. However, if shareholders want managers to accept higher risk, options with out of the money strikes should be used. Corporate boards should also monitor the level of option-based incentives after grants. As share price develops and options become in or out of the money, they may impose different incentives than originally thought.

When it comes to choosing an option pricing model, we think that the binomial framework is flexible enough to account for two major complications in ESO pricing: risk-averse preferences and trading constraints in the underlying asset. Correct choice and calibration of valuation model is crucial, since ESO-related expenses may have a material impact on reported earnings. Having recognized the intricacies of valuing employee stock options, we find that wrong valuation choices, as well as market fluctuations, limit in some cases the attainment of the true and fair view from the owner's point of view. The grant date value of an ESO plan disclosed in IFRS financial statements is only as valid as the underlying assumptions.

A Proof of Equations (4.13) and (4.14)

We're interested in a manager, whose compensation is given by contract $f(x)$, for which we assume $f(x) \geq 0$ and $f'(x) > 0$. It may be either concave or convex, even though numerical experiments in Section 4 assume $f(x)$ to be convex. The manager's utility function is $U(x)$ being increasing and concave. Also, recall the definition of absolute risk aversion $A(x) = -\frac{U''(x)}{U'(x)}$.

Let us first prove Eq. (4.13). For a start, define derived utility as the

composite function $U \circ f(x) = U(f(x))$. The argument x is omitted for brevity when possible. Given the properties of U , there exists a smooth function G such that $U \circ f = G \circ U$. Differentiating this identity with respect to x twice yields Eqs. (A.1)-(A.2).

$$U'(f) f' = G'(U) U' \quad (\text{A.1})$$

$$U''(f) [f']^2 + U'(f) f'' = G''(U) [U']^2 + G'(U) U'' \quad (\text{A.2})$$

Under the current assumptions $G'(U)$ must be increasing; this follows from (A.1). However, the sign of $G''(U)$ is ambiguous and in our focus. The following lines work out an identity leading to Eq. (4.13).

$$\begin{aligned} & G''(U) [U']^2 \\ &= U''(f) [f']^2 + U'(f) f''(x) - \frac{U'(f)}{U'(x)} f' U''(x) \\ &= U'(f) f' \left[\frac{U''(f) f'}{U'(f)} + \frac{f''}{f'} - \frac{U''(x)}{U'(x)} \right] \\ &= U'(f) f' \left[-A(f) f' + \frac{f''}{f'} + A(x) \right]. \end{aligned} \quad (\text{A.3})$$

In the above display, the second line uses Eq. (A.1). Third and fourth lines re-arrange terms and use the definition of absolute risk aversion. Provided that U is increasing and $f'(x) > 0$, we can see from Eq. (A.3) that contract $f(x)$ convexifies the manager, i.e. $G''(U) \geq 0$, when $\frac{f''}{f'} \geq A(f) f' - A(x)$. This completes the proof of Eq. (4.13).

We will now turn to Eq. (4.14). Write $V(x)$ for the derived utility, that is $V(x) = U \circ f(x)$. Absolute risk aversion for the manager under contract $f(x)$ is in this notation $A_V(x) = -\frac{V''(x)}{V'(x)}$. This quantity is compared to $A(x) = -\frac{U''(x)}{U'(x)}$, being the risk aversion of a manager who receives x . (Note that if the contract is $f(x) = a + bC(x)$ where $C(x)$ is a call option on x , one can always choose constants a and b such that $f(x) = x$.) Differentiating $V(x)$ twice gives

$$A_V(x) = -\frac{V''(x)}{V'(x)} = -\frac{U''(f) [f']^2 + U'(f) f''}{U'(f) f'}.$$

Combining the previous equation with $A(x)$ yields the desired result after some calculations.

$$\begin{aligned} & A_V(x) - A(x) \\ &= -\frac{U''(f) [f']^2 + U'(f) f''}{U'(f) f'} + \frac{U''(x)}{U'(x)} \\ &= -\frac{U''(f) f'}{U'(f)} - \left(-\frac{U''(x)}{U'(x)} \right) - \frac{f''}{f'} \\ &= A(f) f' - A(x) + A_f(x) \\ &= A(f) - A(x) + A(f) [f' - 1] + A_f(x). \end{aligned}$$

Let us remark that f' and f'' represent option delta and gamma for the exemplary contract of Section 4. It follows that $0 < f' < 1$ and $A_f(x) < 0$.

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B Figures

Figure 1. The Jensen effect.

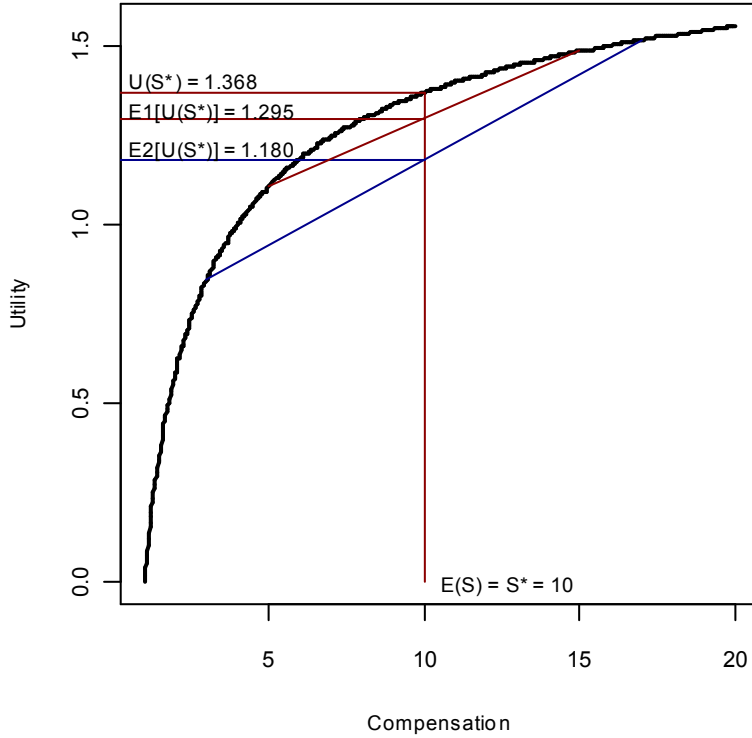


Figure B.1: The Jensen effect. Certainty equivalent of compensation decreases as variance increases. Compensation scheme 1 has range of $[5, 15]$, whereas compensation scheme 2 has range of $[3, 17]$. Both schemes offer the agent an expected compensation of 10 units. The figure is drawn using power utility function given by eq. (11).

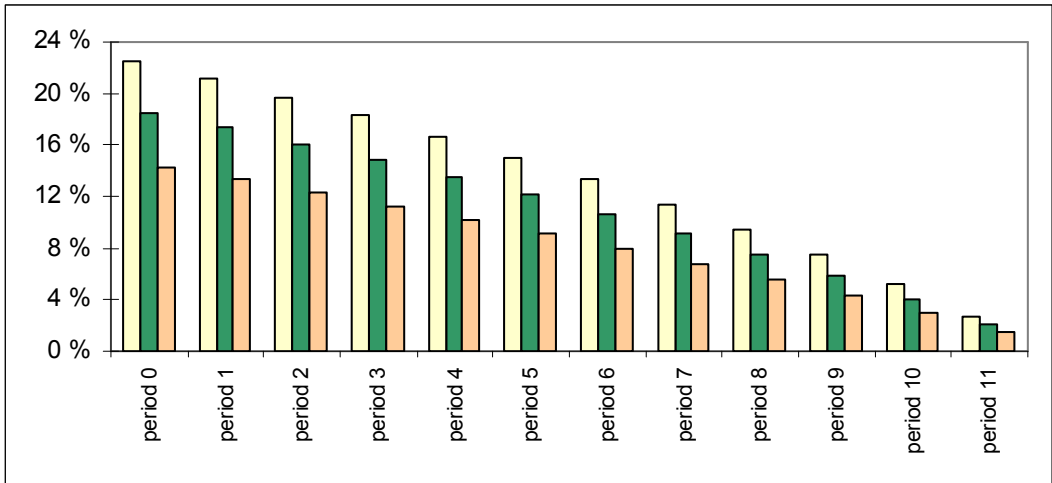


Figure B.2: Evolution of discount in the certainty equivalent option price calculated with 12-period binomial model. Three series are drawn using risk aversion coefficients of 2.5, 2.0 and 1.5. Volatility is held constant at 20 %. The figure shows how the discount to risk-neutral price decreases during the option's lifetime.

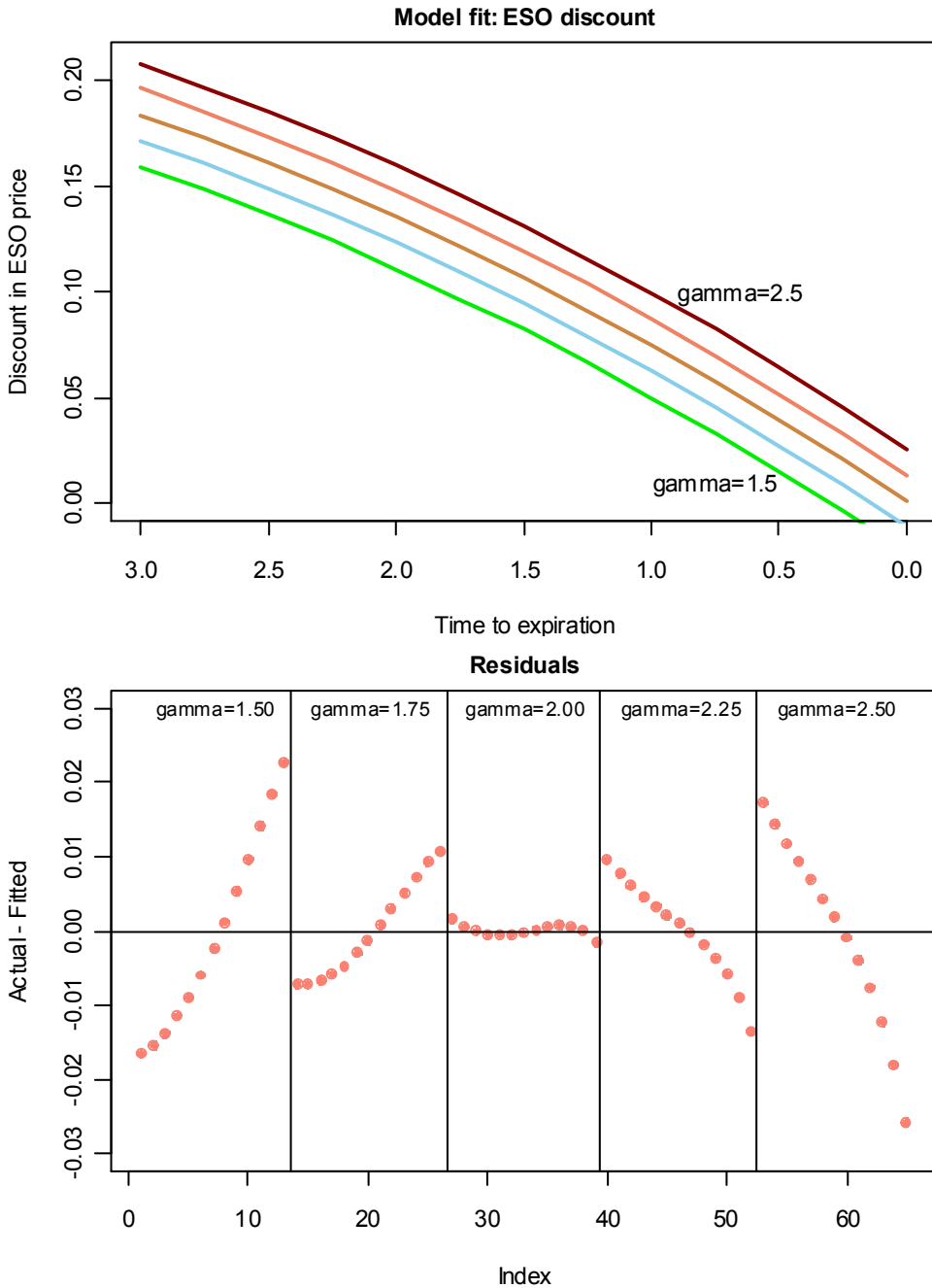


Figure B.3: Fit of the regression model (16). The upper panel plots the ESO risk premium as a function of time for five different values of risk aversion. The lowest curve uses risk aversion of 1.5, and the vertical distance between each two curves compares to 0.25 unit increase in risk aversion. The lower panel gives model residuals in the five scenarios used.

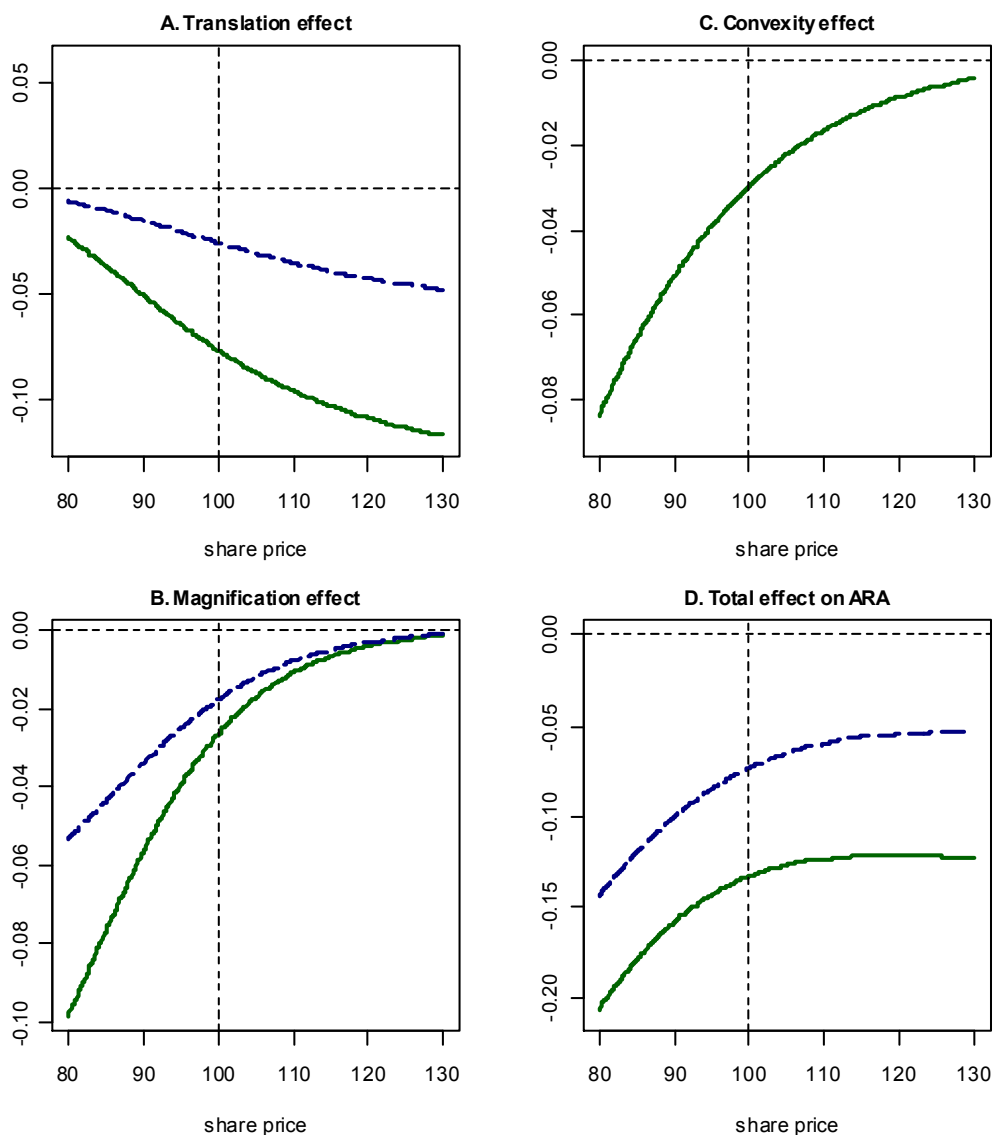


Figure B.4: Decomposition of the call option's convexifying effect, measured as decrease in absolute risk aversion. Continuous lines plot case 1, where a call option with B-S value of 10.45 units is added to fixed salary of 10 units. Dashed lines plot case 2, where the same option is added to fixed salary of 20 units. Note that the size of fixed salary has no effect on the convexity effect of panel C. Current share price is 100.

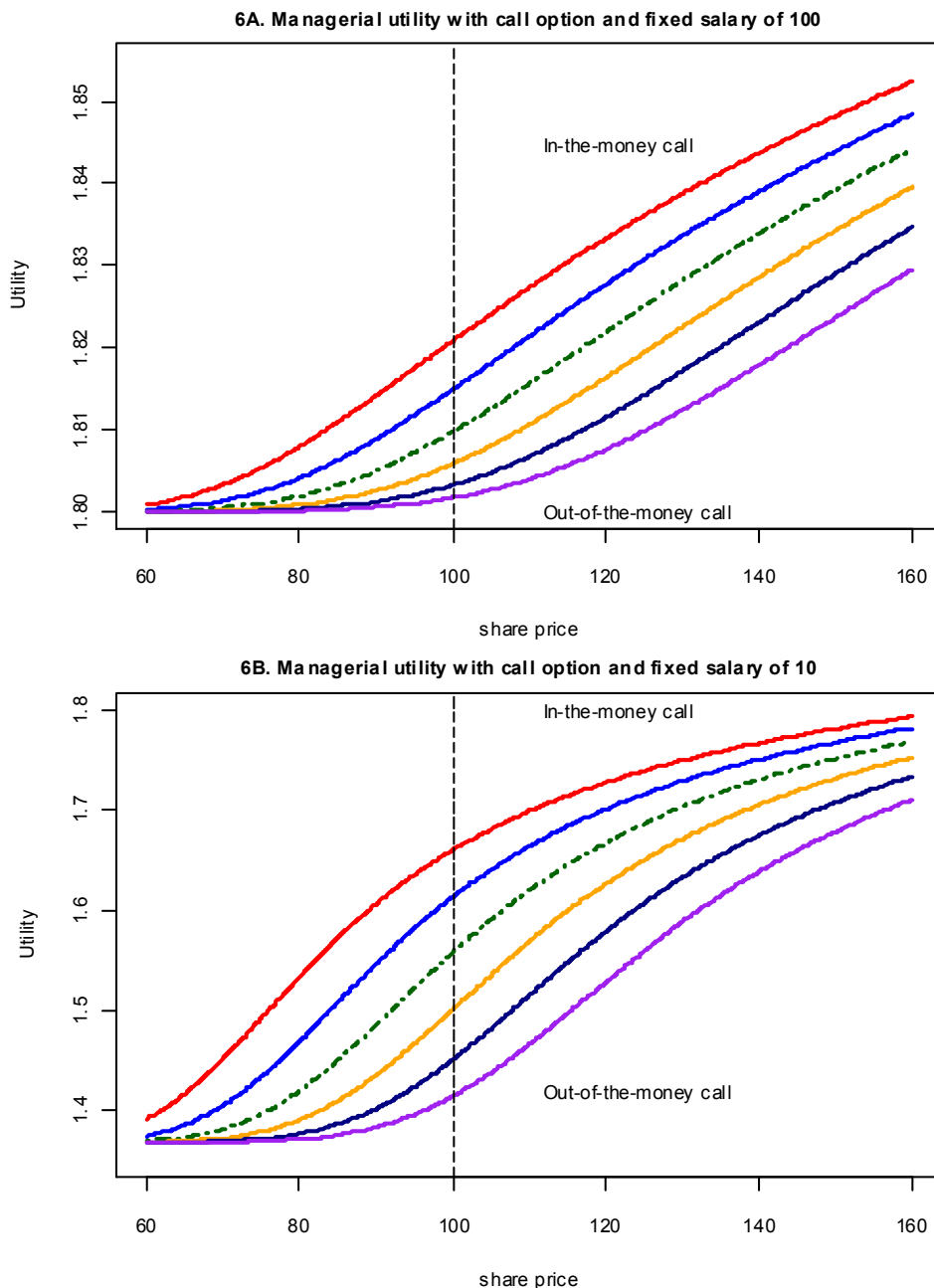


Figure B.5: Managerial utility functions $U \circ f(S)$, where $U(\cdot)$ is power utility given by eq. (4.4) and the contract $f(S) = a + c(S)$ is the sum of fixed salary a and call option value $c(S)$. The convexifying effect (decrease in risk aversion) induced by $f(S)$ is stronger for out-of-the money calls than in-the-money calls. Starting from top, strike prices increase from 80 to 130. Dashed line plots at-the-money option. Current stock price is 100.

Computational methods for incentive option valuation*

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Abstract

Employing stochastic programming, we provide a general framework for option pricing based on marginal bid/ask price valuation. It is applied to numerical analysis of options with European and American style exercise using a double binary tree. Incentive options are valued considering hedging restrictions and other market frictions, such as transaction and short position costs, and different borrowing and lending rates. The framework also includes correlated labor income. The possibility of partial sales is analyzed using ask price functions. Without friction costs and labor income, our model is the discrete-time equivalent of Ingersoll (2006). When labor income and/or market frictions are present, or a fraction of options is sold, the option values are materially different compared to Ingersoll (2006).

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1 Introduction

Valuation of incentive stock options (or executive stock options, ESOs) has received much attention in finance and microeconomics literature. This topic has both theoretical and practical importance given the major role of options in compensation packages. Practical importance follows from the fact that options generate a significant share of executive remuneration. According to a survey on CEO remuneration in the US, published by HR consulting firm Mercer, 265 out of 350 CEOs received stock options and they generated 52 percent of the value of long-run incentives ¹, which includes common and restricted stock as well as stock options. Note that restricted stock grants can be viewed as incentive stock options with zero strike price.

We stress that a single incentive item, such as an option, should not be valued in isolation of other items. Different forms of labor income like fixed salary and bonuses should be considered when valuing incentive options. Below we demonstrate that adding labor income in a consumption-investment problem has a large effect on incentive option values.

Based on valuation by indifference proposed by Pratt (1964), the valuation problem of ESOs is usually solved by calculating the certainty equivalent of option cash flows in a discrete-time framework. Such subjective value is the manager's ask price; i.e., the minimum price for the manager to trade. In their survey of option pricing, Broadie and Detemple (2004) discuss how to apply certainty-equivalent valuation in solving consumption-investment problems. Detemple and Sundaresan (1999) calculate incentive option values for the manager who cannot sell short the underlying asset. In their model, the certainty equivalent value is bounded above by the risk-neutral value, provided that the manager has concave utility. Detemple and Sundaresan develop a pricing kernel, where the short-sales constraint results in an implicit dividend yield reducing the ESO value. Hall and Murphy (2002) give a detailed review of incentive option pricing problems and literature, and present a basic ask price model that values the option using a lognormal distribution for the terminal stock price. However, this model is conditional on the assumption that the CAPM holds.

Investigating early exercise has been important in the valuation discussion. Carpenter (1998) presents a binomial model for American options with an exogenous probability for early exercise. Bettis et al. (2005) explain early exercise by fitting regression models in an exercise data from the US. Nevertheless, the discrete-time models cited above do not feature endogenous exercise decision, with the exception of Detemple and Sundaresan (1999).

There are also some studies solving the managerial portfolio problem and valuing incentive options using continuous time models. Typically these papers solve some extended version of the Merton (1971) portfolio problem. Generally speaking, the advantage of continuous-time models, when applicable, is that they lead to closed-form solutions, and in many cases these solutions are in-

¹The source is Mercer Human Resource Consulting 2005 CEO Compensation Survey. An interesting result of this survey is that other equity-based compensation, such as restricted shares, have increased their share relative to stock options.

tuitive, in allowing inference on how the correlation of income and investment risks affects portfolio choice. For example, Henderson (2005) solves the Merton portfolio problem with a random income stream in an exponential utility framework. The choice of utility function allows her to derive closed-form solutions for portfolio weights as well as a number of useful results regarding the manager's hedging demand. While exponential utility brings the advantage that closed-form solutions exist for portfolio weights and certainty equivalent, the disadvantage is a constant absolute risk aversion (CARA). CARA implies that if the manager's labor income is certain or the labor income risk is idiosyncratic², the resulting portfolio of risky assets is independent of labor income³. Continuous time models of Koo (1998) and Munk (2000) solve the portfolio problem using power utility without explicitly valuing options. Unfortunately, closed-form solutions are available only in the complete market case where income risk can be fully hedged in the financial market.

Power utility functions imply constant relative risk aversion. In this case, the level of income impacts portfolio choice, even if income and investment risks are uncorrelated; see Campbell and Viceira (2002). Ingersoll (2006) uses power utility and develops an extension of the Black-Scholes model assuming a mandatory holding of employer's stock applies to the manager. He derives a subjective pricing kernel, where the portfolio constraint decreases the risk-free rate, and as a result the subjective value of the option will be less than its objective value. Note that the Ingersoll (2006) solution is qualitatively similar to Detemple and Sundaresan (1999); in both papers the effect of portfolio constraints is to reduce the (subjective) risk-free rate, effectively decreasing the option value.

Our main topics, i.e market frictions and labor income, are not considered by either Detemple and Sundaresan (1999) or Ingersoll (2006). However, the relative impact of salary in a subjective option valuation can be tens of percents. Similarly, if the manager considers selling a part of his option endowment, then exercising the rest yields a stochastic cash flow stream, which has an impact in subjective valuation. This possibility is not treated by the two cited papers either; yet it may have a dramatic impact on valuation. The importance of market imperfections is investigated by numerical analysis and it is shown that transaction costs, bid-ask spreads and shorting costs cannot be ignored in subjective valuation.

Ask price valuation is implemented by using power or logarithmic utility. The numerical analysis starts from the discretized Ingersoll model and subsequently adds transaction costs, interest rate spreads between borrowing and lending, and shorting costs. The model also accounts for labor income that may be correlated with stock returns. Also, the possibility of selling some options initially and exercising the rest later is discussed below.

The rest of the article is organized as follows. Marginal ask price valuation is introduced in Section 2. Our main tool for valuation is a double binary tree that is able to accommodate different assumptions. Formal treatment of our

²In other words, there is no covariation in labor income and stock returns.

³For discussion of disadvantages of CARA utility, see pp. 166–167 of Campbell and Viceira [3].

valuation method is given in a separate Appendix, offered as Electronic Supplementary Material. Section 3 analyses the effects of friction costs, dividends and vesting period on European and American style options. Adding market frictions results in bounds for arbitrage free value. Moreover, early exercise may be optimal, even in absence of dividends.

Section 4 develops the ask price function for an option; i.e., the subjective ask price as a function of fraction sold. Numerical results indicate that the price can be highly sensitive to the sold fraction. In Section 5 we show that adding risky labor income in the consumption-investment problem has significant effect on subjective values. In the general case, where income and investment risks are correlated, the effect may be either positive or negative. Section 6 presents case studies of Fortum and Nokia with ESOs traded in the Helsinki Stock Exchange. Calibration of our pricing model to actual ESO prices suggests that the managers' subjective views of company prospects carry more weight in valuation than the stock market view.

2 Subjective valuation of an incentive option

In this section, we introduce marginal indifference pricing for option valuation and discuss data employed for numerical analysis throughout this paper. The aim is to determine the ask price; i.e., the price at which the manager is indifferent between selling and not selling an option. While the method originates from the certainty equivalent concept by Pratt (1964), the resulting option value is consistent with arbitrage theory. Numerical evaluation of the marginal ask price is based on a consumption-investment model with expected utility maximization.

2.1 A consumption-investment model

We employ three assets: a risk-free asset, the market (index) portfolio and the stock of the employer. The two risky assets follow a bivariate Geometric Brownian Motion. In addition, the model has risky labor income represented by an exogenous stochastic cash flow stream. In discrete time setting the time span of N years is subdivided into T periods of $\Delta = N/T$ years with stages $t = 0, 1, \dots, T$. The logarithm of the total risk free return in each period is a constant r . The return of the index and the stock price are stochastic and interdependent. With dividend yield q_m , expected cum dividend value ν_m and variance σ_m^2 , the logarithm of the index increases in each period by $\nu_m - q_m + \tilde{u}$, where \tilde{u} is a stochastic increment with $\text{var}(\tilde{u}) = \sigma_m^2$. Employing a stochastic increment \tilde{v} with $\text{var}(\tilde{v}) = v^2$, the logarithmic increment of the stock price is $\nu - q + \beta\tilde{u} + \tilde{v}$ with dividend yield q , drift ν and variance σ^2 . Assuming that \tilde{v} and \tilde{u} are independent, $\sigma^2 = \beta^2\sigma_m^2 + v^2$. The notation together with data employed in Sections 3–5 is summarized in Table 1.

Stochastic processes of the stock price, index and exogenous cash flow are approximated by a double binary event tree. It reveals realizations of prices

r	0.05	logarithm of the (total) risk free return
ν_m	0.04	logarithmic (total) increment of the index
q_m	0	dividend yield of the index
σ_m	0.4	volatility of the index
ν	$r + \beta(\nu_m - r)$	logarithmic (total) increment of the stock price
q	0.01	dividend yield of the stock
σ	0.3	volatility of the stock
v	0.2	idiosyncratic volatility of the stock
β	$[(\sigma^2 - v^2)/\sigma_m^2]^{0.5}$	beta of the stock with respect to index

Table 2.1: Single period parameters and annual data for the discrete-time model.

and exogenous cash flows. The nodes of the tree are denoted by $k = 0, 1, 2, \dots$, with a root node $k = 0$ at time $t = 0$. Node probabilities are $\pi_k > 0$. At each stage $t < T$, given a node k associated with a level of the index and a stock price, there are four successor nodes j at stage $t + 1$. For the index, there are two realizations σ_m and $-\sigma_m$ for \tilde{u} . Hence, there are two realizations for logarithmic increment $\nu_m - q_m + \tilde{u}$ of the index. Also for the stock, there are two realizations v and $-v$ for \tilde{v} . Hence, there are four realizations of increments $\nu - q + \beta\tilde{u} + \tilde{v}$ in nodes j . With an equal probability for each node j , one can readily check that our choice matches the expected values of $\nu_m - q_m$ and $\nu - q$, the variances σ_m^2 and v^2 , and the covariance relation $\sigma^2 = \beta^2\sigma_m^2 + v^2$ holds.

Endogenous variables of the model are defined for each node k as follows. For non-terminal nodes k , c_k denotes consumption in the period starting at node k . For terminal nodes, c_k is the total value of terminal positions. For all nodes k , asset positions taken at k are endogenous. Initial positions while entering time $t = 0$ are fixed. At each non-terminal node k , positions change due to purchases and sales. At terminal nodes no trading takes place.

Position dynamics equations for each asset are defined by node. We also consider subjective portfolio restrictions. Such restrictions may set bounds on portfolio positions, for instance. To conform to Ingersoll (2006), we require that the weight on stock in manager's investment portfolio is at least $\alpha \geq 0$.

For all k , let e_k denote a private exogenous endowment of the manager. Then for node k , the cash balance equation is given by in- and out-flows resulting from a number of sources: the level of consumption c_k is equal to the exogenous cash flow e_k incremented by the cash flow from changes in asset positions, dividends, and interest payments. Also transaction costs, interest rate margins for lending and borrowing, and charges for short positions may be taken into account.

The manager has preferences given by expected value of an additive utility function $\sum_{t=0}^T u_t(c_t)$ determined by consumption c_t over T periods and by the terminal portfolio value c_T . With a constant relative risk aversion $1 - \gamma > 0$, stage t utility function $u_t(c)$ is $\rho_t/\gamma c^\gamma$, for $\gamma \neq 0$, and $\rho_t \log c$, for $\gamma = 0$. Utility discounting factors are given by $\rho_t = \exp(-\rho t \Delta)$, where ρ is a constant. For node k at time t , denote the utility of consumption c_k by $u_k(c_k)$.

The consumption-investment problem is to find an investment strategy, levels of consumption and terminal wealth, to

$$\max \sum_k \pi_k u_k(c_k) \quad (2.1)$$

subject to constraints specifying cash balance equations, position dynamics equations and portfolio weight restrictions. *The consumption-investment problem is fully stated in (A.12) of the Appendix offered as Electronic Supplementary Material, where position dynamics are given by (A.7), cash balance by (A.10), and weight restrictions by (A.11).* In order to avoid excessive notation, repetition of mathematical formulation is omitted for the specialized model (2.1).

To deal with the optimal exercising of options, we append incentive call option as the fourth asset in the problem (2.1). Then, the single model (2.1) can be used throughout for numerical analysis. The initial position of options reveals the number of options held for exercising. Additional options in possession are sold initially at market price and the resulting revenue is incremented in the initial exogenous cash flow. For instance, in sections 3 and 5 below, all options are sold initially, while in Section 4 we parameterize the fraction of options sold. For the options held initially, in order to determine optimal exercising together with investment and consumption, we prohibit both short position in the option and an increase in long position; for implementation, see the discussion in the Appendix. The quantity sold is now interpreted as the number of options exercised, and the sales price is interpreted as the payoff of exercising one option.

Assuming that the problem (2.1) is feasible and no arbitrage opportunities exist, then an optimal solution exists and the optimal consumption stream (c_k) is unique. Furthermore, optimal dual multipliers $\lambda_k = \pi_k u'_k(c_k)$ for cash balance equations are strictly positive and unique.

2.2 Marginal ask price valuation

Consider valuation of an incentive call option on the stock with a maturity of $M \leq N$ years and exercise price X . The marginal ask price for such an option is the price at which the manager is indifferent between selling and not selling a small quantity of options in possession. In this section we introduce the methodology applied in Sections 3–6 for ask price valuation of incentive options. A general development of such methodology is presented in the Appendix, where we derive the valuation results and point out the relationship with arbitrage pricing theory.

Consider increments $\delta = (\delta_k)$ of the exogenous endowment in the cash balance equations, and let $\hat{U}(\delta)$ denote the resulting optimal expected utility. Then the gradient of $\hat{U}(\delta)$ with respect to δ exists at $\delta = 0$, and optimal multipliers λ_k of the cash balance equations yield marginal increments in the optimal expected utility for an increment in cash balance equation of node k . Hence, if an additional δ_k units of cash is provided at node k to relax the cash balance equation, then the optimal expected utility increases approximately by $\lambda_k \delta_k$.

For marginal ask price valuation, the manager is considering to sell an option in a small quantity ϵ . Given a price V for the option, a sales revenue ϵV is received at stage $t = 0$. Then, the marginal ask price V is the minimum unit price at which the manager is willing to sell a small share ϵ of the option. We consider various types of European and American options. Let $f = (f_k)$ be an option cash flow stream associated with a particular exercising strategy. Given all possible exercising strategies, there is a set F of attainable cash flow streams f . Consider a small share $\epsilon > 0$ of the option. If what is received is worth as much as what is given up (in terms of utility), the marginal ask price $V(f)$ of f satisfies the indifference equation

$$\lambda_0[\epsilon V(f)] = \sum_k \lambda_k[\epsilon f_k].$$

Hence, using optimal marginal utility u'_k of node k at time t , $\lambda_k = \pi_k u'_k(c_k)$ and

$$V(f) = \sum_k \kappa_k f_k, \quad (2.2)$$

where the state price κ_k are given by

$$\kappa_k = \lambda_k/\lambda_0 = \pi_k u'_k/u'_0 = \pi_k \rho_t (c_0/c_k)^{1-\gamma}. \quad (2.3)$$

Consequently, the marginal ask price of the option is

$$V = \max_{f \in F} V(f). \quad (2.4)$$

Optimality conditions for (2.1) imply that $\kappa_k > 0$ and $\kappa_0 = 1$ in (2.3). As shown in the Appendix, state prices κ_k constitute arbitrage free state prices. Hence, the marginal ask price valuation is consistent with arbitrage pricing theory. A unique subjective marginal option value V is obtained even if arbitrage free state prices are not unique.

For numerical evaluations in Section 3, we employ the following observations; see Lemma 3 in the Appendix. If the initial positions are zero and the exogenous endowment e_k is zero except at the root node $e_0 > 0$, then the values $V(f)$ in (2.2) and V in (2.4) are independent of the initial endowment $e_0 > 0$. If additionally there are no friction costs, then the values $V(f)$ and V are independent of the utility discounting factors ρ_t , and valuation of an option with a maturity of M years is independent of time horizon N , as far as $N \geq M$.

Arbitrage free bounds for option values are uniform applying to all utility functions considered above, and they are independent of the private endowment process. The smallest upper bound $V^+(f)$ and largest lower bound $V^-(f)$ for the value $V(f)$ is obtained by linear programming (see Appendix), and arbitrage free bounds for the option value are given by

$$\max_f V^-(f) \leq V \leq \max_f V^+(f). \quad (2.5)$$

For a European option maturing at time t , let \bar{f} be the expected value and σ_f the standard deviation of f_k . Define stochastic discounting factors (SDFs) z_k by $\pi_k z_k = \kappa_k$. At time t , let \bar{z} be the expected value and σ_z be the standard deviation of z_k . If ρ_{zf} denotes the correlation coefficient of the option cash flow and the SDF at time t , then by (2.2), we get the valuation equation

$$V(f) = E[z_k f_k] = \bar{z}\bar{f} + \rho_{zf}\sigma_z\sigma_f. \quad (2.6)$$

Given $z_k = \rho_t(c_0/c_k)^{1-\gamma}$, the SDF is proportional to the inverse of relative increase in consumption c_k/c_0 raised to power $1 - \gamma$. A high increase in consumption implies that the SDF and the state price are low. Also, if $c_k > c_0$, then the state price decrease as risk aversion increases. If there is a high increase in consumption with a high stock price, then we may expect ρ_{zf} to be negative.

2.3 Model data and implementation for computations

In Sections 3–5 we discuss a number of cases and compare some results with Ingersoll (2006). We use an initial price of the stock $S_0 = 100$, an exercise price $X = 100$ for the call, and a maturity of $M = 10$ years. The time horizon of the model is the maturity of the option. Hence, we set $N = M$. However, the impact of a longer horizon with $N > M$ is discussed. For the case studies of section 6 concerning Fortum and Nokia, data is provided in Table 6. The case studies account for proportional income and capital gains taxes to make the setup as realistic as possible.

Data for price processes is summarized in Table 1. Given the step size of Δ years in the model, the annual logarithmic risk free return 0.05 implies a return $r = 0.05\Delta$ for a single period. Similarly, the annual volatility 0.3 of the stock price implies a single period volatility $\sigma = 0.3\sqrt{\Delta}$. Unlike in continuous time analysis, we also need numerical values for ν_m , ν , β and σ_m . Conforming to Ingersoll (2006), we assume the CAPM relation $\nu = r + \beta(\nu_m - r)$, which provides ν given ν_m and β . Given ν_m , β is obtained from $\sigma^2 = \beta^2\sigma_m^2 + v^2$, where σ_m is determined such that the optimal level of investment in the stock is zero in the perfect market case. Numerically such value σ_m can be easily evaluated.

In subjective valuation, for comparison with results of Ingersoll (2006), we use $\gamma = -2$, for power utility, and $\gamma = 0$, for log utility. The weight limit on stock in manager's investment portfolio is α taking values 0, 0.1 or 0.5.

We utilize modern numerical analysis for multi-stage stochastic optimization; see Wets and Ziemba (1999). Computations reported in Sections 3–6 are carried out using AMPL (see www.ampl.com) with an interior point solver MOSEK; see Fourer, Gay and Kernighan (2003). AMPL is an algebraic modeling language used to implement optimization models such as (2.1) and (A.12). AMPL also reads data files specifying numerical values for model parameters and it calls for an optimization code (in our case MOSEK) to compute optimal primal and dual solutions. Precision of numerical optimization, given a particular optimization model, is specified by tolerance parameters, for which we used smallest possible

values (10^{-14}) leading to a highest possible precision. For instance, the optimal objective function value is computed with a relative error less than 10^{-10} .

3 Variations in European and American options

We begin the numerical analysis in Section 3.1 by a comparison of European option values. In a setup with no exogenous endowment or friction costs, European call values from our discrete time model agree with Ingersoll (2006). In Section 3.2 we demonstrate for European options the impact of friction costs: transaction costs, short position charges, and interest rate margins for borrowing and lending. In Section 3.3, American call values are discussed with and without friction costs, dividends on the stock, and a vesting period. A vesting period of τ years, $0 \leq \tau \leq M$, may apply to the incentive option. Problem dimensions and solution times are discussed below for the most demanding cases only (American options with friction costs; see Section 3.3). Valuation of basic European options with a large number of time steps can be done by solving one single period problem only; see Section 3.1.

3.1 Basic European options

For comparison with results of Ingersoll (2006) we first consider the case without friction costs. The marginal ask price of European call is calculated using 10 year maturity and exercise price of 100. It is assumed there is no labor income (that is $e_k = 0$, for all $k > 0$) and there are no dividends. Further, the manager wants to sell all his options⁴. In this case, the marginal ask price is the minimum price for the manager to trade. We consider two values for the portfolio weight limit $\alpha = 0.1$ and $\alpha = 0.5$, and two levels 1 ($\gamma = 0$) and 3 ($\gamma = -2$) for relative risk aversion. Note that Lemma 3 (i) in the Appendix

no longer holds; subjective option valuation is independent of the initial endowment $e_0 > 0$ and the utility discounting parameter ρ .

For (2.2), let $\kappa_k = \lambda_k/\lambda_0 = \pi_k z_k$ and $\bar{z} = \sum_{k \in K_T} \kappa_k = \sum_k \pi_k z_k = E_t [z_k]$ and define $\psi_k = \kappa_k/\bar{z}$. Then, by Lemma 3 (iii) in the Appendix, ψ_k is a multinomial distribution, which is obtained from optimal dual multipliers of equations (A.10) in a single period portfolio problem, and the option value $V(f)$ in (2.2) is $\bar{z} \sum_{k \in K_T} \psi_k f_k$. The discounting parameter \bar{z} is the subjective risk free discounting of Ingersoll (2006), and ψ_k is a subjective risk neutral probability. Perfect market prices are obtained from (2.5), where the arbitrage free price interval is a single value. Equivalently, in this case, the market prices are obtained with $\alpha = 0$, because the stock weight restriction is not binding.

The marginal ask prices for the European call option are shown in Table 2 for $\alpha = 0.1$ and $\alpha = 0.5$, and $\gamma = 0$ and $\gamma = -2$. The number of time steps T increases from 4 to 1000. The ask price is uniformly less than the market price. Hence, the manager is willing to sell all options at the market price. Two bottom

⁴If some options are exercised after $t = 0$, then the option cash flow interferes ask price valuation; see Section 3.

T	$\alpha = 0.1$		$\alpha = 0.5$		Market
	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	
4	49.97	43.73	42.35	23.15	52.88
6	49.23	43.48	41.86	23.15	52.55
8	49.35	43.35	41.58	23.32	52.36
10	49.33	43.40	41.63	23.64	52.34
100	49.49	43.73	41.77	24.60	52.56
1000	49.48	43.75	41.76	24.68	52.57
∞	49.48	43.75	41.76	24.69	52.57

Table 3.1: European call values without friction costs, dividends and salary. The values are ask prices for selling all options in possession at the initial stage. Minimum weight of the stock is given by w . There are no dividends nor salary. Market refers to perfect market prices and cases $T = \infty$ to continuous time values by Ingersoll (2006).

rows of Table 2 show that the call values converge at $T = 1000$ to those obtained with the continuous time model. Using a moderate number of time steps T causes the call values to deviate from continuous time values by a few percent, at most by six percent. This is a justification of the stochastic programming approach with moderate T below. Our results in subsequent sections show that the factors of practical importance are market frictions, labor income and proportion of sold options (total grant vs. part of it). These factors have major impact on subjective values but are disregarded in the papers of Detemple and Sundaresan (1999), Hall and Murphy (2002) and Ingersoll (2006).

3.2 European options with friction costs

We now modify the previous case introducing friction costs: proportional transaction cost of 0.1 percent for buying and selling, 1 percent interest rate margin between borrowing and lending, and short position cost of 2 percent p.a. For the European call option with 10 year maturity and exercise price of 100, marginal values are shown in Table 3 for $\alpha = 0$, $\alpha = 0.1$ and $\alpha = 0.5$, $\gamma = 0$ and $\gamma = -2$, and for the number of time steps T ranging from 4 to 8. The utility discounting parameter is $\rho = 0$, implying $\rho_t = 1$, for all t . There are no dividends, there is no salary and no vesting period. For computation of ask prices we use (2.2). Based on (2.5), also arbitrage free market prices are shown for some cases. By Lemma 3 (i)-(ii) in the Appendix, the ask price is independent of the initial endowment $e_0 > 0$.

A distinctive feature of Table 3 is that increasing risk aversion decreases option value, when the portfolio restrictions apply. Unlike in the preceding case, even with $\alpha = 0$, the stock weight restriction is binding and it has a small impact on the call value. However, for $\alpha = 0.5$ and $\gamma = -2$, the value decreases by about 15 percent due to friction costs. Even for $\alpha = 0$, a few percent decrease is observed.

T	$\alpha = 0$		$\alpha = 0.1$		$\alpha = 0.5$		V^-	V^+
	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$		
4	51.53	51.49	50.17	43.92	42.25	19.76	43.30	53.49
6	51.15	51.12	49.75	43.71	41.64	19.60	43.06	53.21
8	50.80	50.91	49.16	43.49	41.37	19.64	42.90	53.05
∞	52.57		49.48	43.75	41.76	24.69	52.57	

Table 3.2: European call value with friction costs in comparison with continuous time values without friction costs ($T=8$) by Ingersoll (2006). The values are ask prices for selling all options in possession at the initial stage. Minimum weight of the stock is given by α . Dividends and salary are absent from this model. The arbitrage free market price interval is $[V^-, V^+]$, and it reduces to a single value in the perfect market case.

Based on (2.5), arbitrage free market price intervals $[V^-, V^+]$ are computed omitting the weight restriction α on stock and assuming an equilibrium such that an agent is indifferent in investing and not investing a marginal amount in the stock. The market price interval $V^+ - V^-$ is relatively large, about 20 percent of the upper limit. For $\alpha = 0.5$, the call values are below the market price interval. Hence, as above, the manager is willing to sell all options at any market price in $[V^-, V^+]$. However, for $\alpha = 0$ and $\alpha = 0.1$, the ask price is within the market price interval. Hence, depending on the prevailing market price, the manager may or may not be willing to sell all options.

In addition to option values, solution to the portfolio problem involves the consumption and investment profiles of the manager. Figure 2 shows how the manager's consumption and investment portfolio evolve over time in the power utility case with risk aversion $\gamma = -2$, and portfolio restriction $\alpha = 0.5$. This model has eight periods and 4^8 nodes in the final period. Option values in this case are reported in the $T = 8$ row of Table 3.

In Figure 2, the top-left panel shows that under the circumstances the manager is able to increase consumption by 45 % during the 10-year horizon. Variance of consumption increases over time, because actual consumption depends on investment returns. Given the risk-averse nature of this case, the manager is conservative and invests (i.e. saves) about 40 units in the risk-free security. The rest is allocated to market portfolio and to the stock. Portfolio weights converge to zero, because all wealth is consumed at the end, there is no bequest. Continuing with Fig. 2, the bottom-right panel shows that investment to company stock is about as large as risk-free asset, implying that the portfolio constraint is binding and diversification is minimal at the outset. If we compare the profiles shown in Figure 2 to other cases of Table 2, changes have the following character. When risk aversion decreases, the impact on initial consumption is minor, but end-of-horizon consumption increases significantly. Unsurprisingly, the variance of consumption increases as well. These results can be traced to reduced allocation in the risk-free asset, and increased allocation to risky assets.

As indicated by the remark following Lemma 3 (see Appendix), in the fric-

tionless cases of Table 3 the option values remain unchanged if the planning horizon N is extended beyond maturity M of the option. Hence, in cases of Table 3 with friction costs, one might expect a minor impact by extended horizons. To demonstrate this, we made a quick test. The six cases of Table 3 were first computed with $N = M$ and $T = 3$ steps to obtain six call values. Thereafter, the planning horizon is doubled to $N = 20$ and valuation of the same options with maturity $M = 10$ were carried out using a $T = 6$ step model. As a result we obtain two sets of six call values. Comparing these values case by case indicate that the worst case difference is less than one per mille.

3.3 Early exercise, dividends and vesting

Next, we consider cases of Sections 3.1–3.2 modified to allow early exercise of the call option, dividends on stock and a vesting period of τ years. In case dividends are included, we have $q = 0.01\Delta$ and $q_m = 0$ allowing dividends yield on the stock only. In case a vesting period is imposed, we have $\tau = 4$ years. No salary is considered. For valuations we use (2.2) and (2.4) with standard backward recursion. Table 3 shows American call values with and without friction costs, each in three cases: $q = 0$ and $\tau = 0$, $q = 0.01\Delta$ and $\tau = 0$, and $q = 0.01\Delta$ and $\tau = 4$. Again, the values are ask prices for selling all options in possession.

As is well known, if there are no dividends nor frictions, then early exercise does not pay off. However, $\alpha > 0$ represents a friction and in this case one may benefit from early exercise even if there are no friction costs nor dividends. For instance, for $\alpha = 0.5$ and $\gamma = -2$, excluding friction costs and dividends, the American call value of 26.63 is well above the European call value of 23.32. If friction costs are included, then the gain from early exercise increases in comparison with the case without friction costs.

Also Detemple and Sundaresan (1999) point out that market frictions may cause early exercise to be optimal, even if there are no dividends. Their intuition is that early exercise increases the manager's utility because it helps to deal with the short-sales constraint. In fact there are two dimensions to this effect. On one hand, early exercise (which may be partial) reduces the need the hedge the incentive stock option. On the other hand, exercising the option increases the manager's liquid wealth, which helps to reduce the suboptimality of constrained portfolio.

If dividends are included, it is well known that one may benefit from early exercise even if the weight restriction on stock is omitted and there are no friction costs. For $q = 0.01\Delta$ and $\tau = 0$, call values are below the values obtained without dividends, because dividends decrease the stock price. The early exercise gain increases due to dividends, as expected. Moreover, if both dividends with $q = 0.01\Delta$ and a vesting period of $\tau = 4$ years are included, the sacrifice from the vesting period is relatively small in comparison with cases with $q = 0.01\Delta$ and $\tau = 0$. An exception is made by American option with $\alpha = 0.5$ and $\gamma = -2$, where vesting period has a considerable effect on call value.

In Table 4, for all cases where friction costs are omitted or $\alpha = 0.5$, the call

q/Δ	τ	$\alpha = 0.1$		$\alpha = 0.5$		V^-	V^+
		$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$		
European options, without friction costs							
0	0	49.35	43.35	41.58	23.32	52.36	
0.01	0	42.43	37.24	35.65	19.65	45.18	
0.01	4	42.43	37.24	35.65	19.65	45.18	
European options, with friction costs							
0	0	49.16	43.49	41.37	19.64	43.17	53.05
0.01	0	42.46	37.31	35.38	16.46	37.89	45.71
0.01	4	42.46	37.31	35.38	16.46	37.85	45.71
American options, without friction costs							
0	0	49.32	43.52	41.79	26.63	52.36	
0.01	0	42.72	38.16	36.69	23.98	45.18	
0.01	4	42.72	38.13	36.65	23.24	45.18	
American options, with friction costs							
0	0	49.16	43.59	41.52	23.93	43.17	53.05
0.01	0	42.70	38.12	36.41	21.79	37.89	45.71
0.01	4	42.70	38.09	36.38	20.48	37.85	45.71

Table 3.3: European and American call values with and without friction costs, dividends and a vesting period; q =dividend yield on stock, τ =vesting period (years). Reported values are ask prices for selling full option endowment at the start. The number of steps is $T = 8$. No salary is considered. The arbitrage free market price interval is $[V^-, V^+]$, which is a single value for a perfect market.

values are below the market price interval so that the manager is willing to sell all options at market price. For other cases, the call value is within the market price interval and the manager's willingness to sell depends on the prevailing market price. All runs of Table 4 were done with $N = M = 10$; i.e. with a planning horizon matching the maturity. The four cases of American options with friction costs, but without dividends and a vesting period, were tested for the impact in option values when the planning horizon is extended. As in Section 2.2, we first compute the four call values with $N = M = 10$ and $T = 3$. Thereafter, the four valuations are made with $N = 20$, $M = 10$ and $T = 6$. Again comparing the four pairs of option values indicates that the worst case difference is less than one per mille.

Table 5 shows problem dimensions and solution times for American options with friction costs, dividends $q/\Delta = 0.01$, vesting period $\tau = 0$ and three alternatives for the number of periods T . The number of nodes is $n = (4^{T+1} - 1)/2$, the number of columns is $13 \cdot n - 6 \cdot 4^T$ and the number of rows is $5 \cdot n - 4^T$.

				CPU time (sec)			
				$\alpha = 0.1$		$\alpha = 0.5$	
T	nodes	columns	rows	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$
4	341	2897	1449	0.1	0.1	0.1	0.1
6	5461	46417	23209	1.4	6.6	1.4	3.3
8	87831	742737	371369	33.3	136.4	31.7	51.0

Table 3.4: Problem dimensions and solution times (sec) for American calls with three alternatives for time steps T . Reported CPU times are obtained from MOSEK with Intel-P4 running under Linux/64-X86.

4 Option ask price functions

It has been assumed above that the manager sells all his options, if he trades. However, in practice the holdings may consist of several grants, vesting at different times. Therefore the options may be sold in parts. This will change the indifference value of remaining options, because cash flows from exercising are added in the model. Hence, Lemma 3 (i) in the Appendix no longer holds, and price and quantity become dependent in subjective valuation, pointing the need for a supply function of incentive options; i.e., the optimal quantity sold as a function of option price.

In contrast to risk-neutral pricing, valuation by indifference implies that price and quantity are no more independent. If only a small share of options are sold at time zero, there is obviously lower initial consumption compared to the case, where all options are sold at that point. In addition to initial consumption, the sold quantity affects consumption during later periods, and all state prices are subject to change. Specifically, variance of terminal consumption increases significantly, if nothing is sold initially.

Figure 3 shows inverse supply functions (price functions) in several cases, which have different risk aversion and portfolio restrictions. The calculations assume that the manager initially has a wealth of e_0 and a given number of options possession. This initial option position is determined such that its value is $e_0/2$ at perfect market price of European options. Further, he considers selling a fraction δ of the options, and the proceeds are invested optimally at time zero. After that no more option sales are allowed. The unsold fraction $1 - \delta$ may be exercised either at the end (for European options) or at an arbitrary time point (for American options). The optimal exercise is determined jointly with optimal consumption and investment. In line with intuition, the price functions of Figure 3 are upward sloping. The supply functions are generally more elastic for American options. One explanation for this is that American option holders are able to smooth their consumption by cashing in 'down the road', whereas European option holders have to wait until expiration.

Figure 4 shows explicitly, how consumption smoothing works. In the left panel, we plot the distribution of consumption in the case where only a differential fraction is sold at the start. In this case the options are European, risk

aversion is $\gamma = -2$ and the portfolio restriction is $\alpha = 0.5$. The right panel of Fig. 4 plots the case, where all options are sold at the start. It is clear that the consumption path is relative smooth in the latter case, however at the expense of significantly lower expected consumption at the end. Note also the different shapes of consumption distributions; keeping almost all options to expiration results in higher variance of consumption. It leads to higher variance of the SDF, which has a negative effect on the indifference value. The economic interpretation is unequivocal; the consumer dislikes uncertainty.

Finally, Figure 3 may help to clarify the early exercise puzzle, documented by Carpenter (1998), among others. Assume a setup where incentive options are not traded, but they may be exercised. A look at the price functions shows that the indifference value decreases if the manager considers exercising only a small fraction, and at some point this indifference value equals the intrinsic value. Then it is optimal to exercise this fraction. Again, the intuition can be found in the consumption smoothing effect.

5 Impact of risky labor income

In this section we examine the effect of labor income on valuation of incentive options. Income and asset dynamics are modeled as in Campbell and Viceira (2002) using three dimensional geometric Brownian motion. Labor income is allowed to correlate with stock and market returns. For income dynamics, we denote the drift by ν_l , volatility by σ_l , and correlations with market and stock returns by ρ_{ml} and ρ_{sl} . Note that Lemma 3 (i) in the Appendix no longer holds; subjective option valuation is influenced by income flows and initial cash holdings. The examples below assume that the initial annual salary is equal to initial cash holdings.

In an empirical study, Leone, Wu and Zimmermann (2006) find that the correlation of CEO cash compensation and stock returns is approx. 0.3 in the S&P Execucomp data for years 1993–2003. In fact, their Table 4 shows that cash compensation reacts asymmetrically to stock returns; the impact of stock returns is higher if they are negative.

Subsequently, five cases A–E are demonstrated. The income process is parameterized as follows: (A) labor income is deterministic with a drift $\nu_l = 0.03$; (B) with $\nu_l = 0.03$, the income volatility is increased to $\sigma_l = 0.10$ while $\rho_{ml} = 0$ and $\rho_{sl} = 0$ so that labor income volatility represents an idiosyncratic risk; (C) with $\nu_l = 0.03$ and $\sigma_l = 0.10$, correlation with stock return is increased to $\rho_{sl} = 0.40$ while the correlation with market return remains at $\rho_{ml} = 0$; (D) both correlations are positive with $\rho_{ml} = 0.20$ and $\rho_{sl} = 0.40$; and (E) the drift is reduced to $\nu_l = 0$, leaving other parameters to levels above: $\sigma_l = 0.10$, $\rho_{ml} = 0.20$ and $\rho_{sl} = 0.40$.

Valuation results for the five cases A – E are presented in Table 6. Values of European and American calls are shown for each case with friction costs, and with variations in the stock weight restriction α and in the risk aversion parameter γ . In case A with a deterministic salary, the option values are re-

Case	ν_l	σ_l	ρ_{ml}	ρ_{sl}	$\alpha = 0.1$		$\alpha = 0.5$	
					$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$
European options								
A	0.03	0.00	0.00	0.00	51.75	51.75	50.28	49.99
B	0.03	0.10	0.00	0.00	51.52	55.14	50.29	52.23
C	0.03	0.10	0.00	0.40	44.02	27.12	43.80	28.70
D	0.03	0.10	0.20	0.40	46.61	32.96	45.99	33.76
E	0.00	0.10	0.20	0.40	47.07	34.36	45.83	32.34
	no salary				49.16	43.49	41.37	19.64
American options								
A	0.03	0.00	0.00	0.00	51.75	51.79	50.32	50.07
B	0.03	0.10	0.00	0.00	51.52	55.15	50.36	52.53
C	0.03	0.10	0.00	0.40	44.06	28.82	43.87	30.04
D	0.03	0.10	0.20	0.40	46.62	33.53	46.02	34.17
E	0.00	0.10	0.20	0.40	47.07	34.74	45.87	33.08
	no salary				49.16	43.59	41.52	23.93

Table 5.1: European and American call values with salary and friction costs. The number of steps is $T = 8$. Reported values are ask prices for selling all options in possession at the initial stage. For comparison, call values excluding salary are also reported.

lately insensitive to changes in α and γ , and the values are above those in Table 4 obtained without salary. This is explained by a decreased absolute risk aversion due to increased consumption. Comparing cases A and B shows how idiosyncratic labor income risk affects option values. With moderate risk aversion ($\gamma = 0$), idiosyncratic risk has almost no effect on option value. But if we increase risk aversion to $\gamma = -2$, idiosyncratic risk actually increases the option value.

Correlated labor income risk is treated in cases C and D of Table 6. Compared to cases A and B, in cases C and D we add correlations with stock and market returns, respectively. In general, correlating labor income with stock and market returns reduces incentive option values. Assuming power utility, comparing cases C and D against the no salary case reveals a 24–38 percent reduction in incentive option value, with minimal stock weight of 0.1. A similar comparison using log utility shows that the effect of correlated risks on labor income is significantly smaller. Also, Munk (2000) finds that the value of labor income decreases when correlation with asset returns increases, because hedging becomes more difficult.

A less obvious effect is that adding a correlated salary has a positive effect on incentive option values, if the stock weight restriction is tightened to 0.5. The underlying intuition is clear; adding the salary in fact helps the manager to diversify his portfolio, since the correlations are far from perfect ($\rho_{ml} = 0.2, \rho_{sl} = 0.4$). Incentive options are more valuable as part of a better diversified portfolio. If labor income risk is idiosyncratic or absent, the positive effect on

option value is even more pronounced. Somewhat surprisingly, in case E the reduction of salary drift ν_l to zero has a relatively minor impact in comparison with case D.

To illustrate the consumption and investment profiles, Figure 5 shows case D with $\alpha = 0.5$ and $\gamma = -2$. For comparison, Figure 2 shows the profile with $\alpha = 0.5$ and $\gamma = -2$ without salary. As shown in Table 4, the option values with salary are well above those without salary. For European options the increase in value is 72 %. In Figures 2 and 5, growth rates of consumption are similar; however there is a major difference in levels. We also remark that saving at the start turns to borrowing, when labor income is included, which can be checked by comparing the risk-free asset allocations of Figures 2 and 5. Of course, we are not arguing that introducing labor income would unconditionally induce levered portfolios. Kahl, Liu and Longstaff (2003) solve the portfolio problem of a manager who has restricted stock (but no options or labor income) and also find that in some cases it is optimal to take levered positions in the market portfolio. Their explanation, agreeable to us, is that stock market risk is used to hedge various otherwise undiversifiable risks.

Using the stochastic discount factor we can relate consumption cum labor income and subjective option pricing. Large price differences in Table 6 are explained by the fact that labor income changes the distribution of state prices. Some analytics are provided by the valuation equation (2.6). For terminal nodes k , the SDF is $z_k = \rho_t(c_0/c_k)^{1-\gamma}$ so that the state price in (A.16) is $\kappa_k = \pi_k z_k$, where π_k is the probability of node k . If f_k is the cash flow from the European option, repetition of (2.6) yields for the option value $V(f) = \bar{z}\bar{f} + \rho_{zf}\sigma_z\sigma_f$, where \bar{z} and \bar{f} are expected values of z_k and f_k , σ_z^2 and σ_f^2 are variances, and ρ_{zf} is the correlation coefficient.

What is the impact on option value if salary is removed from case D? In the valuation equation (2.6) the term $\bar{z}\bar{f}$ increases as \bar{f} remains constant at 156.23 and the subjective discount factor \bar{z} increases from 0.66 to 0.89. In the covariance term the correlation ρ_{zf} remains approximately unchanged at -0.32 . Because the volatility of option cash flows, $\sigma_f = 272.33$, is the same in both cases, an increase in the volatility of the SDF σ_z from 0.79 to 1.39 remains the explanation of the price difference. To interpret this, note that consumption with salary in Case D is much higher than in case without salary. High consumption implies low marginal utility, a small value for the SDF z_k , and consequently, a small variance σ_z^2 . The correlation ρ_{zf} is negative, because for a high stock price at node k we expect a large cash flow f_k and a high consumption c_k , which leads to a small marginal utility and hence a small SDF z_k .

The runs of Table 6 are based on planning horizon that matches option maturity ($N = M = 10$). Similarly as in Section 2, we tried the impact of doubling the planning horizon. We took all 40 cases of Table 6, computed the call values first setting $N = M = 10$ and $T = 3$, to be compared with $N = 20$, $M = 10$ and $T = 6$. Comparison of the cases showed that doubling the horizon has only a minor effect. For the 20 cases with $\gamma = 0$, the worst case difference

is less than two per mille. For the 20 cases with $\gamma = -2$, the largest difference is 1.6 percent.

6 Case studies

This section shows how our approach to valuing incentive options can be used in practice. It is applied to incentive stock option programs in Fortum, a major Nordic power company, and Nokia, a leading producer of mobile phones and networks. Studying these cases is enabled by the fact that incentive options of Fortum and Nokia are actually traded in the Helsinki Stock Exchange. For description of the incentive options market see Ikäheimo, Kuosa and Puttonen (2006). For both companies there also exist active stock and ordinary option markets. In brief, the practice is that after a vesting period, incentive options are quoted in the exchange, and anyone can trade them. In our cases, the options are American in the sense, that after the vesting period they can be freely exercised subject to insider trading restrictions.

We demonstrate the usefulness of our model by calibrating it to empirical data given in Table 7. The option programs 2001AB of Fortum and 2002AB of Nokia are considered. These options expired in May 2007 and December 2007, respectively. Exercise prices of these programs were initially endogenous, however later they were fixed to values given in Table 6.

Due to privacy considerations, personal data, such as salary, wealth, etc. do not relate to any of the managers which we interviewed. For demonstration, we assume that the manager receives risky labor income and has some initial wealth. Risk preferences of the manager are given by log utility. The manager is allowed to trade in market portfolio and stock with the restriction that short positions in the stock are not allowed.

Compared to earlier examples, we add two modifications in the examples of Table 7. First, we set the time preference coefficient for terminal wealth to $\rho_T = 15$ to ensure that adequate wealth is retained at the end of horizon. Second, we add income and capital gains taxes to make the setup as realistic possible.

In case of Fortum, we look at subjective valuation dated back to October 2005, when the options were deep in the money. In case of Nokia, we think of valuation in May 2006, when the options were at the money. Hence, the maturities are 1.5 and 1.6 years, for Fortum and Nokia, respectively. The length of time horizon in the discrete time model is set to to maturity and divided in six three months periods. The intuition is that given the market regulation, managers usually find it prudent to trade in the stock and options only when interim reports are disclosed, i.e. once in a quarter.

We consider the manager selling a fraction δ of the options in possession. In table 6, δ is the optimal share given market price of the option. For Nokia, $\delta = 0.4$ is optimal and at this point the subjective option value 2.3 euro is the same as the market price shown for May 2006. In case of Fortum, $\delta = 0$ indicating that the manager would not be willing to sell. In fact, actual trade data of

October 2005 indicates a low trading volume for Fortum incentive options.

In order to gain practical insights we interviewed executives who have received the options of Table 7. We were especially interested in how managers evaluate incentive options. The interview revealed that a manager tends to estimate option value crudely as the intrinsic value; i.e., the difference between current stock price and exercise price. Naturally, the time value appears difficult to assess. Hence, we conclude that decision aid based on models such as ours, can be valuable if the time value relative to the intrinsic value is significant; see the Nokia case, for instance.

Given the market regulation, managers of both companies found trading in the stock and options feasible only after quarterly reports are disclosed. The representatives stressed that management sketches the firm's future by outlining scenarios. They are contingent on different realizations of various risks and they are discrete by nature. For instance, for an electricity firm different scenarios could refer to changes in regulation or decisions taken by competition authorities, since the firm was actively making acquisitions. When considering equity-based incentives, the manager's intuition is that stock price increases become more likely as the firm's prospects improve.

Based on the above, in terms of modeling, managers tend to apply subjective parameters for valuation. Specifically, the most influential parameters are risk aversion, stock price drift and volatility. To illustrate this with sensitivity analysis, consider stock drift increments of Nokia by one percent up and down; i.e., changes from 6 % to 7 % and 5 %. Then the relative changes in subjective option values are 3.5 % and -5.7 % for drift increase and decrease, respectively.

7 Conclusions

A manager's subjective value of an incentive option is the ask price at which the manager is indifferent between selling and not selling an option. While such valuation appears consistent with arbitrage pricing theory, it has the merit that the valuation principle is easy to explain to managers and a unique option value is obtained even in case of an incomplete and imperfect market. The underlying discrete time and discrete state stochastic processes are not restricted. Standard optimization methods are readily available for valuation, and consequently, the level of sophistication in option valuation is modest.

Our analysis indicates that friction costs and labor income can have a major impact in subjective option values. The effect of labor income on option values can be positive or negative, when labor income and stock market risks are correlated. Furthermore, we develop supply functions for the incentive options. If the manager considers selling only a fraction of options in possession and exercising the rest, then the ask price for a fraction can be significantly lower than the ask price for selling all options. Finally, we study ESOs in two case companies calibrating the model to actual market data. Interviews with managers revealed, for instance, that discrete-time analysis is supported due to insider trading restrictions. Specifically, the managers we interviewed could

		Fortum	Nokia
ESO	option program (maturity)	2001AB (5/07)	2002AB (12/07)
	exercise price	3.40 e	17.9 e
	time to maturity (years)	1.5	1.6
	market price of ESO (date)	11.7 e (10/05)	2.3 e (5/06)
Assets	risk free rate	3.5 %	3.5 %
	index: drift	6 %	6 %
	volatility	20 %	20 %
	stock: price (date)	14.9 e (10/05)	17.9 e (5/06)
	drift	6 %	6 %
	volatility	20 %	22 %
	dividend	1.0 %	1.0 %
beta	0.9	1.0	
Private	initial wealth	1.0 Me	1.0 Me
	salary: initial (annual)	0.2 Me	0.2 Me
	drift	3 %	3 %
	volatility	10 %	10 %
	initial number of ESOs (1000)	50	300
	relative risk aversion	1	1
	time preference coefficient	15	15
Model	time periods	6	6
	transaction cost	0.1 %	0.1 %
	interest rate spread	1 %	1 %
	short position cost	5 %	5 %
	no short position on stock		
	income tax	50 %	50 %
	capital gain tax	30 %	30 %
Results	fraction sold initially	0 %	40 %
	option value	11.7 e	2.3 e
	consumption (annual)	0.29 Me	0.28 Me
	terminal wealth	1.12 Me	1.21 Me

Table 6.1: Data for the case studies. Both Fortum and Nokia incentive options have been traded in the Helsinki Stock Exchange (nowadays part of the OMX Nordic Exchange).

trade only after disclosing interim reports, that is four times a year.

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A Marginal indifference valuation

We adopt the bid/ask price valuation from Kallio (2005) with some modifications and new results. We begin by formulating a suitable multi-stage portfolio model in Section 1.1, and present the consumption-investment problem in Sect. 1.2. In Sect. 1.3 we state the properties of the portfolio model, present the valuation results, and finally point out relationships with arbitrage pricing theory.

A.1 Portfolio dynamics

In the discrete time approach, the time span of N years is subdivided into T periods defined by stages $t = 0, 1, \dots, T$. The periods are of equal length. An index $t > 0$ also refers to a period between stages $t - 1$ and t .

An event tree specifies the probability measure and filtration describing how information is revealed. The price processes of securities, dividend processes,

as well as private exogenous endowment processes of the manager, e.g. salary, are adapted to the event tree. Let $k \geq 0$ denote a node of the event tree with $k = 0$ referring to the root. Let k_- denote the predecessor of node k , for $k > 0$, and let K_t be the set of nodes at time t . Hence K_T is the set of terminal nodes. Node k appears at stage $t_k \in \{0, 1, 2, \dots, T\}$. For the root, $t_0 = 0$, and for nodes $k \in K_t$, $t_k = t$. For $k > 0$, we assume $t_{k_-} = t_k - 1$ for the predecessor node k_- . Let J_k denote the set of successor nodes of k . Hence, for all $j \in J_k$, we have $j_- = k$, and for terminal nodes $k \in K_T$, J_k is empty. The probability of attaining node k is $\pi_k > 0$, for all k , and $p_j = \pi_j/\pi_k$ is the conditional probability of node $j \in J_k$ given k .

An option specifies a set of possible actions, and a choice of action yields a cash flow f_k , for all nodes k . We denote the stochastic cash flow stream by a vector $f = (f_k)$ and define an option by a set F of feasible choices f . We assume that the choice set F is nonempty, closed and bounded.

Consider finitely many assets i . These assets may refer to interest rate instruments, stock of companies, commodities market, financial derivatives, real estate, etc. Let \tilde{P}_t denote the stochastic vector of prices \tilde{P}_{ti} of all such assets i at stage t . A risk free asset $i = 0$ is included among assets. For this asset, $\tilde{P}_{t0} = 1$, and for simplicity, we assume the total return R over a single period is constant. The realizations of the stochastic price vector \tilde{P}_t are defined in our event tree. For each node k , P_k is the vector of prices at node k .

The vector $y_k^+ \geq 0$ denotes the asset quantities bought and the vector $y_k^- \geq 0$ the quantities sold at node k , for all k . The vector y_k^+ is interpreted as an increase in long positions or as a reduction in short positions and y_k^- is a reduction in long positions or as an increase in short positions.

Let $x_k = x_k^+ - x_k^-$ denote the vector of positions; i.e., quantities held in each instrument at node k , with $x_k^+, x_k^- \geq 0$ referring to long and short positions, respectively. Initial positions $x_{0-} = x_{0-}^+ - x_{0-}^- = \bar{x}$ are fixed. At terminal nodes all positions are closed so that $x_k^+ = x_k^- = 0$, for $k \in K_T$.

The quantities held at node k with initial conditions, for $k = 0$, and closing conditions, for terminal nodes k , satisfy

$$x_k^+ - x_k^- - x_{k_-}^+ + x_{k_-}^- - y_k^+ + y_k^- = 0, \quad (\text{A.7})$$

$$x_{0-}^+ - x_{0-}^- = \bar{x} \quad (\text{A.8})$$

and

$$x_k^+ = x_k^- = 0 \quad \forall k \in K_T. \quad (\text{A.9})$$

Price vectors P_k^+ of buying and P_k^- of selling include non-negative proportional transaction costs, such that $P_k^- \leq P_k \leq P_k^+$. Transaction costs of short selling are assumed the same as the transaction costs of reducing long positions, and transaction costs of reducing short positions are assumed the same as transaction costs of buying. If there are no transaction costs, then $P_k^- = P_k = P_k^+$. If an asset i cannot be bought at node k , we define $P_{ki}^+ = \infty$, and if it cannot be sold, we have $P_{ki}^- = -\infty$.

We also include dividends, charges for short positions, and periodic interest payments for lending and borrowing with an interest rate spread between borrowing and lending. For each node k , define vectors D_k , D_k^+ and D_k^- of proportional dividend and interest payments as follows. Let $D_k \geq 0$ denote the vector of frictionless proportional yield; i.e., interests at market rates or nominal dividends. Then, the frictionless yield $D_k x_{k-}$ at node k is determined by the position x_{k-} taken at the preceding node k_- . For long positions, D_k^+ is the frictionless vector D_k subtracted by friction costs, such as interest rate margin of lending. For short positions, D_k^- is the vector D_k added by friction costs, such as shorting costs and interest margins of borrowing. If long position is prohibited for asset i , we define $D_{ki}^+ = -\infty$. Similarly, short positions are excluded with $D_{ki}^- = \infty$. For all nodes k , we assume nonnegative friction costs so that $D_k^+ \leq D_k \leq D_k^-$. For the risk free asset $i = 0$, the lending rate is D_{k0}^+ and the borrowing rate D_{k0}^- with $D_{k0}^+ \leq R \leq D_{k0}^-$. If there are no friction costs, then $D_k^+ = D_k = D_k^-$.

In Sections 2–4, an asset i refers to an incentive call option with an initial position $\bar{x}_i \geq 0$. In order to determine optimal exercising together with investment and consumption, we define $D_{ki}^- = \infty$ and $P_{ki}^+ = \infty$ to prohibit short position in the option and an increase in long position, respectively. The payoff of exercising one option at node k is P_{ki}^- , and y_{ki}^- is the number of options exercised. There is no additional return so that $D_{ki}^+ = 0$.

For each node k , let c_k denote consumption and let e_k denote a private exogenous endowment of the manager. Given that taxes are excluded from the model, the net cash balance equations, for all k , are

$$c_k + P_k^+ y_k^+ - P_k^- y_k^- - D_k^+ x_{k-}^+ + D_k^- x_{k-}^- = e_k. \quad (\text{A.10})$$

We also consider subjective portfolio restrictions, for $k \notin K_T$, given by

$$E_k(x_k^+ - x_k^-) \leq 0, \quad (\text{A.11})$$

where E_k is a matrix. Such restrictions may set bounds on portfolio weights and prohibitions of short or long positions. Alternative formulations in place of (A.11) can be introduced in a straightforward manner to restrict additionally, for example, relative changes in each position over a single period.

A.2 The consumption-investment problem

The manager seeks to maximise her utility drawn from consumption in a dynamic setting. Her total utility equals $\sum_{t=0}^T u_t(c_t)$, where period t utility function is denoted $u_t(c_t)$, and $c = (c_t)$ denotes the consumption stream during periods $t = 0, 1, \dots, T$. For node k in the event tree, we denote $u_{t_k} = u_k$, hence the utility of consuming c_k is $u_k(c_k)$. The consumption-investment problem is to find an investment strategy $x_k^+, x_k^- \geq 0$, $y_k^+, y_k^- \geq 0$, and levels of consumption c_k , for all k , to

$$\max \sum_k \pi_k u_k(c_k) \quad s.t. \quad (A.7) - (A.11). \quad (A.12)$$

No arbitrage opportunities are allowed to exist in the event tree. Formally, we

assume that there is no homogenous solution $x_k^+, x_k^- \geq 0, y_k^+, y_k^- \geq 0$ and c_k of (A.7) - (A.10), satisfying (A.11), such that $c_k \geq 0$, for all k , and $c_k \neq 0$, for some k .

As shown by Kallio and Ziemba (2007), there are no arbitrage opportunities if and only if there exists prices $\kappa_k > 0, \mu_k$ and $\nu_k \geq 0$ for equations (A.7), (A.10) and (A.11), respectively, satisfying

$$-\nu_k E_k + \sum_{j \in J_k} (\kappa_j D_j^+ + \mu_j) \leq \mu_k \leq -\nu_k E_k + \sum_{j \in J_k} (\kappa_j D_j^- + \mu_j) \quad \forall k \notin K_T, \quad (A.13)$$

and

$$\kappa_k P_k^- \leq \mu_k \leq \kappa_k P_k^+ \quad \forall k. \quad (A.14)$$

Note that we always can scale prices such that $\kappa_0 = 1$ to obtain state prices. In the perfect market case with position constraints (A.11) omitted, $\mu_k = \kappa_k P_k$, by (A.14), and (A.13) yields the familiar result $\kappa_k P_k = \sum_{j \in J_k} \kappa_j (P_j + D_j)$.

Let C denote the set of feasible (attainable) consumption streams $c = (c_k)$ for (A.12) and let $U(c)$ be the expected utility given a consumption stream $c \in C$. Then (A.12) is restated as $\max_{c \in C} U(c)$. The valuation results below build on the following lemma.

Lemma 1 *Assume that an optimal solution exists for the problem (A.12) with a consumption stream $c = (c_k)$. Assume that stage t utility function u_t is increasing, strictly concave and differentiable. Then the optimal consumption stream c is unique. Furthermore, the optimal multiplier vector $\lambda = (\lambda_k)$ for (A.10) is strictly positive and unique, and optimal multipliers μ_k for (A.7) and $\nu_k \geq 0$ for (A.11) satisfy (A.13) - (A.14) with $\kappa_k = \lambda_k$.*

Proof: Based on standard optimization theory (see e.g. Mangasarian, 1971), the optimal consumption stream c is unique, because C is a convex set and $U(c)$ is strictly concave in C . Furthermore, optimality conditions imply existence of dual multiplier vectors $\lambda = (\lambda_k)$ for (A.10), μ_k for (A.7) and $\nu_k \geq 0$ for (A.11) satisfying (A.13) - (A.14). Finally, $\lambda_k = \pi u'_{t_k}(c_k)$, which is strictly positive by assumption and unique because c_k is unique. \square

Optimal multipliers λ_k in Lemma 1 yield marginal increments in the optimal expected utility given an increment in cash balance equation of node k ; i.e., if an additional δ_k units of cash is provided at node k to relax the cash equation (A.10), then the optimal expected utility increases approximately by $\lambda_k \delta_k$.

Let C denote the set of feasible (attainable) consumption streams $c = (c_k)$ for (A.12) and let $U(c)$ be the expected utility given a consumption stream

$c \in C$. Then (A.12) is restated as $\max_{c \in C} U(c)$. Because C is a convex set and $U(c)$ strictly concave in c , the optimal consumption stream c is unique. Consequently, the optimal multipliers $\lambda_k = \pi_k u'(c_k)$ are unique. The following result proves useful for marginal bid/ask price valuation⁵.

Lemma 2 *Assume that $U(c)$ is concave and differentiable and an optimal solution exists for (A.12) with an optimal consumption stream c^* in the interior of the domain of U . Increment the exogenous endowment vector in (A.12) by $\delta = (\delta_k)$. Assume that an optimal solution exists for $\max_{c \in C} U(c + \delta)$, for all δ in some open neighborhood of $\delta = 0$, and let $\hat{U}(\delta)$ denote the optimal expected utility. Then the gradient of $\hat{U}(\delta)$ with respect to δ exists at $\delta = 0$.*

Proof. By assumption, there is $\epsilon > 0$ such that $c^* + \delta$ is a feasible consumption stream, for all δ such that $\|\delta\| < \epsilon$. Hence, we obtain a lower limit $U(c^* + \delta) \leq \hat{U}(\delta)$. For $\delta = 0$, let $\lambda = \nabla U(c^*)$ denote the optimal multiplier vector. Then an upper limit is given by $\hat{U}(\delta) \leq \hat{U}(0) + \lambda\delta$. The assertion follows, because both limits are differentiable with respect to δ and they coincide at $\delta = 0$, and $\nabla_{\delta}\hat{U}(0) = \lambda$ with $\lambda = \nabla_c U(c^*)$. ■

A.3 Marginal bid. ask valuation

For marginal bid/ask valuation, the manager wants to buy or sell an option in a small quantity ϵ . Given a price V for the option, a buying cost or sales revenue ϵV is applied at stage $t = 0$. Then, the bid price V is the maximum price the manager is willing to pay for the option. Similarly, the ask price is the minimum price at which the manager is willing to sell the option.

Let $f = (f_k) \in F$ be an option cash flow stream associated with a particular exercising strategy. Consider a small share $\epsilon > 0$ of the option. Let ϵ approach zero and define the marginal ask price $V(f)$ as the limiting price at which the manager is indifferent between selling and not selling the share ϵ . Employing Lemma 2, such a limit $V(f)$ is obtained from the indifference equation $0 = \lambda_0[\epsilon V(f)] - \sum_k \lambda_k[\epsilon f_k]$ with optimal dual multipliers λ_k for (A.12). Hence, the marginal ask price of f is

$$V(f) = \sum_k \kappa_k f_k, \quad (\text{A.15})$$

where the state prices are given by

$$\kappa_k = \lambda_k / \lambda_0, \quad (\text{A.16})$$

and the marginal ask price of the option is

$$V = \max_{f \in F} V(f). \quad (\text{A.17})$$

Because F is nonempty and compact, and $V(f)$ is linear in f , the maximum in (A.17) is attained. Note that due to valuation at the margin, the bid price

⁵The authors are indebted to Teemu Pennanen for suggesting a simple mangle argument for the proof.

equals the ask price. Optimality conditions for (A.12) imply that $\kappa_k > 0$ and $\kappa_0 = 1$ in (A.16) satisfy (A.13) - (A.14) with some μ_k and $\nu_k \geq 0$. Hence, the marginal bid/ask price valuation is consistent with arbitrage pricing theory. A unique subjective marginal option value V is obtained even if arbitrage free state prices are not unique. We proceed by representing the valuation equation (A.15) in an asset pricing framework.

Denote $z_k = \kappa_k/\pi_k$. Then, using standard finance terminology, at each stage t , the vector (z_k) associated with nodes $k \in K_t$ at stage t forms stochastic discount factors (SDF) for time t . The standard properties of SDF hold: [i] the expected value of SDF is the reciprocal of the subjective risk-free rate, and [ii] the expected value of discounted returns is equal to one (cf. Campbell and Viceira (2002), pp. 38–39). The first property holds because the value of a riskless claim that pays one euro in all states at time t must be equal to one divided by the subjective risk-free return R_t . With $f_k = 1$, for $k \in K_t$, and $f_k = 0$, otherwise, (A.15) yields $V(f) = \sum_{k \in K_t} \kappa_k \cdot 1 = E_t[z_k] = 1/R_t$, where E_t refers to expectation with respect to node probabilities π_k , $k \in K_t$. Hence, the subjective risk free discounting factor $1/R_t$ is $\sum_{k \in K_t} \kappa_k$. To see that the second property holds, let $R_{0,k} = f_k/V(f)$ denote the return on claim f in state k , and rewrite (A.15) as

$$\sum_k \kappa_k f_k/V(f) = \sum_t \sum_{k \in K_t} \kappa_k R_{0,k} = \sum_t E_t[z_k R_{0,k}] = 1. \quad (\text{A.18})$$

The following proposition proves useful for some numerical evaluations; see Section 2.

Lemma 3 *Assume that an optimal solution for (A.12) exists. For exogenous endowments e_k , assume $e_0 > 0$ at the root node and $e_k = 0$ otherwise. For the initial positions, assume $\bar{x} = 0$. For all t , let stage t utility function $u_t(c_t)$ be $\rho_t/\gamma c_t^\gamma$, for $0 \neq \gamma < 1$, and, $\rho_t \log c_t$, for $\gamma = 0$, with a utility discounting factor $\rho_t > 0$, for all t . Then*

(i) *the state prices $\kappa_k = \lambda_k/\lambda_0$ in (A.16) and the value $V(f)$ in (A.15) are independent of the initial endowment $e_0 > 0$.*

If additionally there are no friction costs and single period logarithmic price increments as well as proportional dividend and interest yields are independent of time and state, then

(ii) *the state prices κ_k and the value $V(f)$ in (A.15) are independent of the utility discounting factors ρ_t ,*

(iii) *for optimal dual multipliers λ_k of equations (A.10), the set $\{\lambda_j/\lambda_k \mid j \in J_k\}$ is the same for all $k \notin K_T$, and*

(iv) *the optimal investment strategy is fix-mix; i.e., the vector of optimal portfolio weights is the same for all $k \notin K_T$*

Proof. For $e_0 = 1$, let $c_k, x_k = (x_k^+, x_k^-)$ and $y_k = (y_k^+, y_k^-)$, for all k , denote an optimal solution for (A.12), let $\lambda_k > 0$ denote the optimal dual multipliers for equations (A.10), and let U_0 be the optimal expected utility. Then for any $e_0 > 0$, with dual multipliers upgraded to $e_0^{\gamma-1} \lambda_k$ it follows that the optimal

solution is $e_0 c_k$, $e_0 x_k$ and $e_0 y_k$, and the state prices κ_k in (A.16) and value $V(f)$ in (A.15) are independent of e_0 , concluding (i). Furthermore, to aid the proof of (iii) below, the utility $u_k(e_0 c_k)$ of node k is $e_0^\gamma u_k(c_k)$, for $\gamma \neq 1$, and $\rho_t \log e_0 + u_k(c_k)$, for $\gamma = 0$. Consequently, optimal expected utility is $e_0^\gamma U_0$, for $\gamma \neq 0$, and $\sum_t \rho_t \log e_0 + U_0$, for $\gamma = 0$.

To show (ii), we restate problem (A.12) as follows. Let $x_k = x_k^+ - x_k^-$ and $y_k = y_k^+ - y_k^-$. For all k , (A.7) yields $y_k = x_k - x_{k-}$, (A.10) states $c_k + P_k x_k - P_k x_{k-} = e_k$, and problem (A.12) becomes

$$\begin{aligned} \max \left\{ \sum_k \pi_k u_k(c_k) \mid c_k + P_k x_k - P_k x_{k-} = e_k, \right. \\ \left. E_k x_k \leq 0, x_{0-} = 0, x_k = 0 \forall k \in K_T \right\} \end{aligned} \quad (\text{A.19})$$

If $\rho_t = 1$, for all t , let x_k and c_k , for all k , be the optimal for (A.19) with dual multipliers $\lambda_k > 0$ for the cash balance equations. For all t , it follows from (i) that $P_k x_k = w_t c_k$, for some constant w_t independent of $k \in K_t$, and $w_t > 0$, by no-arbitrage. For any $\rho_t > 0$, using backward recursion, we construct an optimal solution as follows. For all t and $k \in K_t$, scale c_k by $g_t = 1/\rho_t^{1/(\gamma-1)} > 0$. Then the gradient of the objective function as well as the dual multipliers remain unchanged. To satisfy $c_k - P_k x_{k-} = 0$ at stage T , we scale x_{k-} by $h_{T-1} = g_T > 0$. For $t = T-1, \dots, 1$ and $k \in K_t$, to determine the scaling factor h_{t-1} for x_{k-} , it follows from $P_k x_k = w_t c_k$ that $c_k + P_k x_k = (1 + w_t)c_k = P_k x_{k-}$ and $h_t P_k x_k = (h_t/g_t)w_t g_t c_k$. To meet the cash balance equation, we require $g_t c_k + h_t P_k x_k = h_{t-1} P_k x_{k-}$ or equivalently, $(1 + (h_t/g_t)w_t)g_t c_k = h_{t-1}(1 + w_t)c_k$. Solving for h_{t-1} yields $h_{t-1} = (g_t + h_t w_t)/(1 + w_t) > 0$. To satisfy the cash balance at the root node, we define the initial endowment by $g_0 c_0 + h_0 P_0 x_0 = (g_0 + h_0 w_0)c_0 > 0$. Consequently, (i) implies (ii).

To show (iii), consider optimization over a subtree with any node $k \in K_t$ as the root node. For initial endowment equal to 1 at k , let U_t denote the optimal expected utility over the subtree. In this notation, we reformulate problem (A.19) as

$$\max \left\{ u_0(c_0) + \sum_{j \in J_0} \pi_j \rho^* u_j(P_j x_0) \mid c_0 + P_0 x_0 = e_0, E_0 x_0 \geq 0 \right\}, \quad (\text{A.20})$$

where $\rho^* = (\gamma/\rho_1)U_1 > 0$, for $\gamma \neq 0$, and $\rho^* = 1/\rho_1$, for $\gamma = 0$. Let the optimal multipliers be λ_0 and $\lambda_j = \pi_j \rho^* (P_j x_0)^{(\gamma-1)}$. Then by (ii), state prices $\kappa_j = \lambda_j/\lambda_0$, for $j \in J_0$, are independent of ρ^* , and therefore, independent of T . We apply this observation for all t considering optimization over a subtree with a root node $k \in K_t$. After scaling, the price vector at k becomes P_0 . Then employing (i), we conclude (iii).

For (iv), let $x_0 = x^*$ be the optimal portfolio for (A.20) with $\rho^* \rho_1 = 1$. Then, for any positive ρ^* and ρ_1 , the optimal portfolio $x_0 = x^*/(\rho^* \rho_1)^{1/(\gamma-1)}$ is obtained via scaling, similarly as in case (ii) above. Consequently, the optimal portfolio weights at the root are independent of T , and by (i), independent of e_0 . We repeat these arguments for any subtree with root node $k \in K_t$, for $t < T$. After scaling, the price vector at k becomes P_0 , and we conclude (iv). ■

Remark 1 *Under assumptions of Lemma 3 (ii), consider the time horizon extended beyond T time steps by $n \geq 0$ steps. For a node k at time T , given an endowment equal to 1 for consumption and investment at k , let U_T denote the optimal expected utility over the subtree with root k . If e_k denotes the optimal endowment at k in the extended problem, then the optimal expected utility at k , for $\gamma \neq 0$, is $e_k^\gamma U_T = \rho^* / \gamma (e_k)^\gamma$, where $\rho^* = \gamma U_T > 0$. Hence, the extended problem can be solved using the T period problem and upgrading the utility discounting factor at the stage T by the factor ρ^* . Consequently, by Lemma 3 (ii), valuation of options with a maturity of at most T periods, is independent of $n \geq 0$. Similar arguments and conclusions apply to a logarithmic utility as well.*

Arbitrage free bounds for option values are uniform applying to all utility functions considered above, and they are independent of the private endowment process. The upper bound $V^+(f)$ and lower bound $V^-(f)$ for the value $V(f)$ is obtained by linear programming as the largest and smallest values of $\sum_k \kappa_k f_k$ such that $\kappa_k \geq 0$ and $\kappa_0 = 1$ satisfy (A.13)-(A.14) for some μ_k and $\nu_k \geq 0$.

Equivalently, taking the duals of these linear programs, strong duality theorem implies that $V^+(f)$ is the smallest value $f_0 - c_0$ such that $f_k - c_k \geq 0$, for $k > 0$, among all portfolio strategies with $e_k = 0$, for all k , and $\bar{x} = 0$. To interpret this, we observe that $-c_0$ is the investment expenditure at the root. Hence $V^+(f)$ is f_0 plus the smallest investment needed to super replicate f , i.e., to create cash flow $c_k \geq f_k$, for all $k > 0$. Similarly, $V^-(f)$ is the largest value $f_0 + c_0$ such that $f_k + c_k \geq 0$, for $k > 0$, for portfolio strategies with $e_k = 0$, for all k . Here c_0 is the cash flow created at the root, $-c_k$ is the repayment at node $k > 0$, and $f_k + c_k \geq 0$ requires that f super replicates repayments. Hence, if the price of contingent claim f is less than $V^-(f)$, then buying the claim and employing portfolio strategy determining $V^-(f)$ results in an arbitrage.

Bounds for the option value V in (A.17), equivalent to those by Harrison and Kreps (1979) for perfect markets, are given by

$$\max_f V^-(f) \leq V \leq \max_f V^+(f). \quad (\text{A.21})$$

An agent is not willing to pay a price above the upper limit in (A.21), because a smaller investment would super replicate the option cash flow. On the other hand, if the option price is below the lower limit in (A.21), an arbitrage opportunity is created for the agent. If F is a convex set, then the left side of (A.21) is a convex-concave saddle point problem, which can be solved using the method of Kallio and Ruszczyński [14]. The right side is an optimization problem with a pseudo concave objective function. This problem can be solved, for instance, using Minos; see Murtagh and Saunders (1978).

B Figures

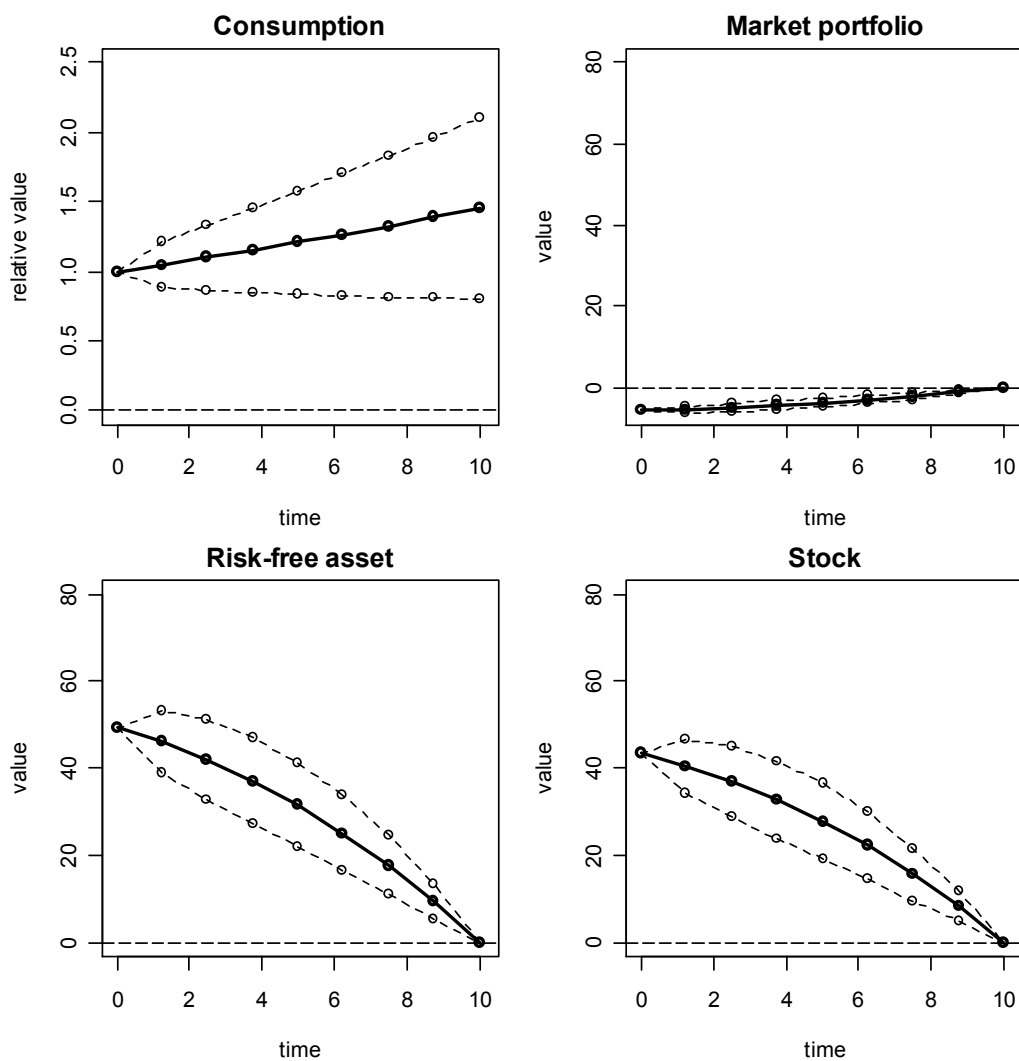


Figure B.1: Consumption and investment profiles of a manager with power utility ($\gamma = -2$) and portfolio restriction $\alpha = 0.5$. There is no labor income. In all four panels, solid lines point the mean and dashed lines point mean ± 1 st.dev.

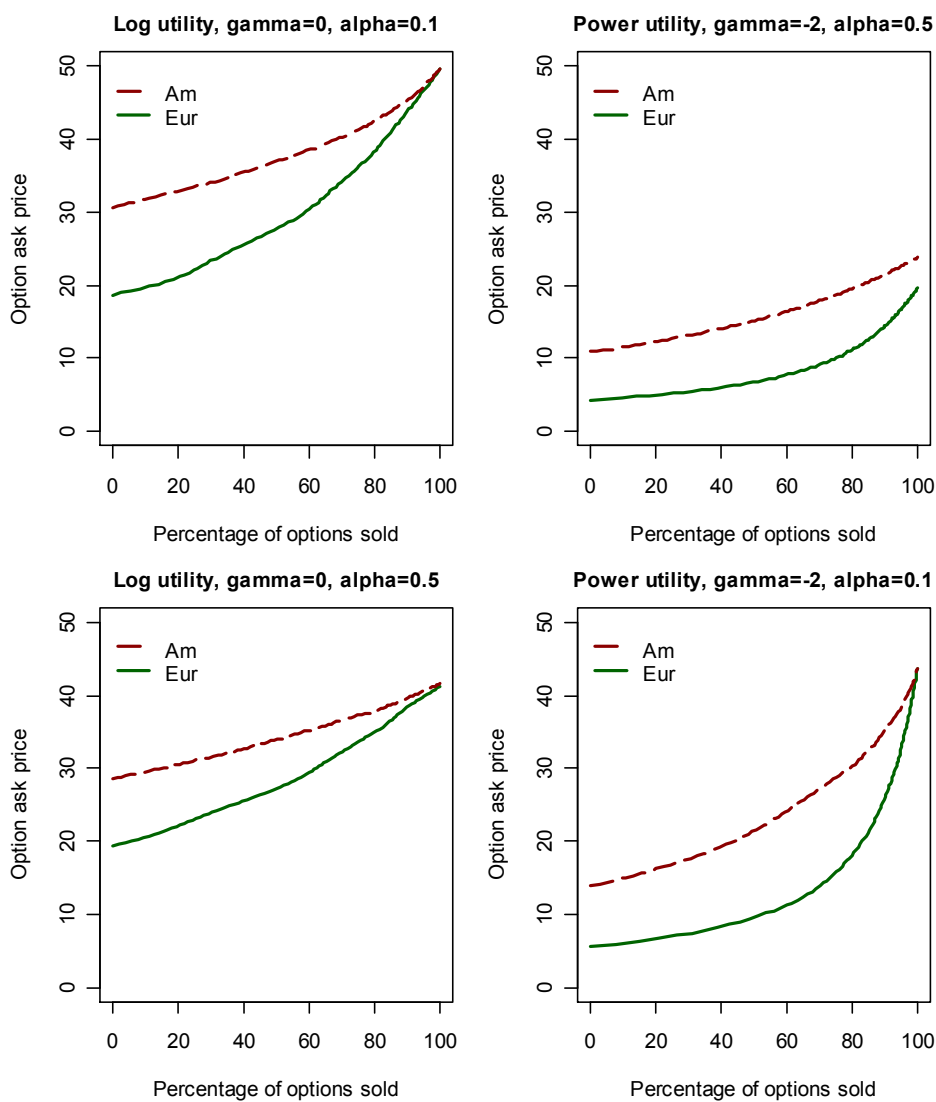


Figure B.2: Ask price functions, i.e. inverse supply functions for incentive stock options in four cases, where α is either 0.1 or 0.5 and relative risk aversion $1 - \gamma$ is either 1 or 3. Separate price functions are provided for American and European options.

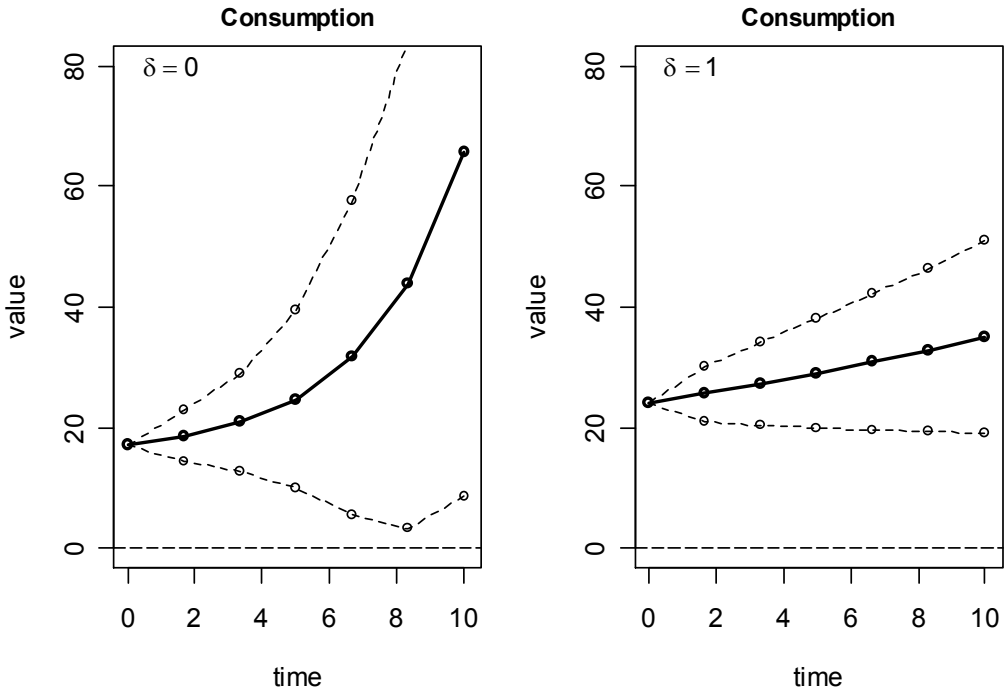


Figure B.3: Consumption profiles in two cases, where the manager sells either a small fraction of his options or all of them. In both panels, solid lines point the mean consumption, and dashed lines point mean ± 1 st.dev.

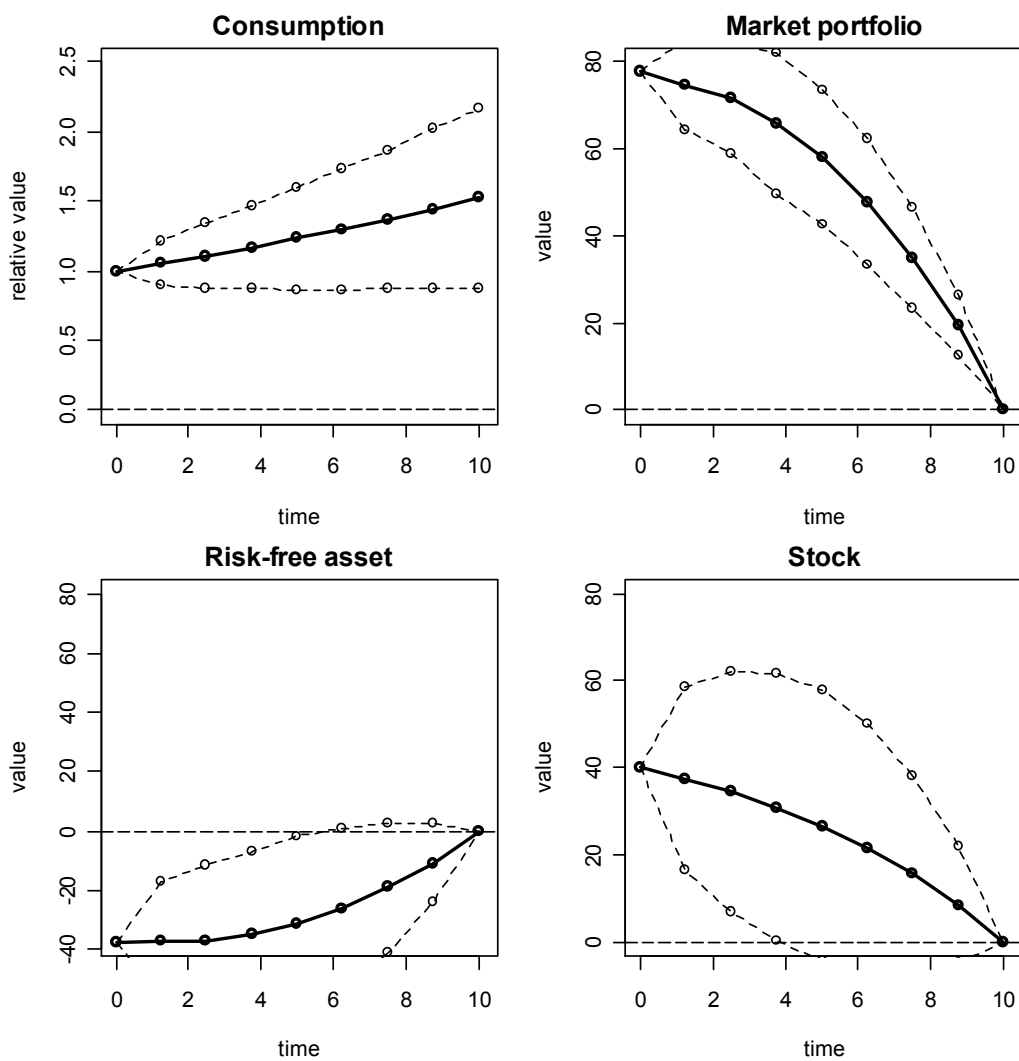


Figure B.4: Consumption and investment profiles of a manager with power utility and portfolio restriction. The manager receives labor income that is correlated with market and company stock returns. The correlations are 0.2 and 0.4, respectively. In all four panels, solid lines point the mean and dashed lines point mean \pm 1 st.dev.

Market pricing of executive stock options and implied risk preferences*

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Abstract

When managers get to trade in options received as compensation, their trading prices reveal several aspects of subjective option pricing and risk preferences. Two subjective pricing models are fitted to show that executive stock option prices incorporate a subjective discount. It depends positively on implied volatility and negatively on option moneyness. Further, risk preferences are estimated using the semiparametric model of Ait-Sahalia and Lo (2000). The results suggest that relative risk aversion is just above 1 for a certain stock price range. This level of risk aversion is low but reasonable, and it may be explained by the typical manager being wealthy and having low marginal utility. Related to risk aversion, it is found that marginal rate of substitution increases considerably in states with low stock prices.

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1 Introduction

When managers receive equity-based compensation, their decisions on investments and leverage are materially affected by the interaction of risk preferences and compensation structure (e.g. option moneyness and portfolio diversification). Holding executive stock options (ESOs) may increase or decrease managerial risk taking, as illustrated theoretically by Ross (2004) and empirically by Lewellen (2006), among others. Ross (2004) derives general conditions under which a compensation schedule concavifies or convexifies the manager's utility functions. Concentrating on volatility costs of debt, Lewellen (2006) finds that managers holding in-the-money options are typically worse off by an increase in leverage, based on certainty equivalent of wealth. Papers on ESO valuation, e.g. Hall and Murphy (2000, 2002) tend to use assumed risk aversion coefficients, because empirical estimates are few and exhibit significant variation [see Table 5 of Aït-Sahalia and Lo (2000)]. This paper contributes by estimating relative risk aversion (RRA) based on the executive stock option (ESO) data, employing the semiparametric RRA estimator of Aït-Sahalia and Lo (2000).

The paper contributes to existing evidence on subjective valuation and implied risk preferences. There is a voluminous literature on option-implied risk preferences, surveyed by Bliss and Panigirtzoglou (2002). However, most papers produce market-wide estimates of risk aversion, reporting average preferences. Unfortunately, these estimates offer little help in valuation of managerial incentives, and especially incentive options. These issues are in focus after the introduction of accounting standards IFRS 2 and FAS 123R, ordering listed firms to disclose the value of executive stock options. In contrast to common practice, we estimate the managerial risk aversion using actual trading prices of ESOs.

Relative pricing of ESOs is investigated by using two models allowing subjective pricing. The first alternative is the subjective option pricing model presented by Ingersoll (2006). It is called the BSI model, since it leads to a Black-Scholes type formula that accounts for idiosyncratic risk and portfolio constraint. These factors cause the resulting option values to be below Black-Scholes. Difference in subjective and market ('objective') prices is measured by the subjective risk premium, defined as the difference between market and subjective risk-free rates. NLS fit of the model shows that the subjective risk premium varies between 3.4% and 5.1% p.a.

As an alternative to the BSI model we try a generic diffusion model, where volatility depends on option delta and therefore on relevant option pricing parameters. Properties of such models are investigated by Bergman et al. (1996). Under this Generalized Black-Scholes (GBS) model option prices are given as composite function of Black-Scholes and volatility. Under the GBS model, subjectivity shows up in the form of volatility function. In fact, the GBS model gives the closest fit to the data. Qualitatively it provides similar results as the BSI model.

Risk preferences are estimated by applying the semiparametric method of Aït-Sahalia and Lo (1998, 2000) for estimation of the option-implied density

function in two steps. The first step estimates implied strike prices from the deltas as a smooth function, following Bliss and Panigirtzoglou (2004) and Kang and Kim (2006). The second step plugs the GBS volatility function in a lognormal density. Finally, The Aït-Sahalia and Lo estimator yields relative risk aversion and marginal rate of substitution, both depending on the stock price.

The data consists of 7610 trades in series 99 ESOs of Nokia Corporation, a global producer of mobile phones and networks. According to Nokia's annual report of 1999, these options were issued in March 1999 to about 5000 Nokia managers and other key employees around the world. These options had a vesting period of two years, after which they were listed on the Helsinki Stock Exchange until expiration at the end of 2004. After the vesting period, the employees were free to sell their ESOs, and in most cases the trades were made via Helsinki Stock Exchange. Given these facts our results should characterize the preferences of representative manager.

Our results on the risk aversion show a fairly stable RRA estimate of just above 1 for a wide price range of ESOs and a downward sloping marginal rate of substitution (MRS). High average moneyness implies that the managers enjoy high consumption levels consistent with low marginal utility. Our results on the RRA should have implications on agency theory based models as well as on structuring compensation packages for company managers, supporting the use of stock options.

Since ESO prices are shown to incorporate a subjective risk premium, questions about arbitrage possibilities need to be addressed. While arbitrage possibilities cannot be excluded, they are certainly limited given hedging error, other transaction costs and practical problems in hedging. These issues are investigated in Chapter 5. In the preceding material Chapter 2 reviews the option pricing models and Chapter 3 discusses the option-implied density function. Chapter 4 describes the data, deals with estimation issues, and provides the results. Conclusions are left for Chapter 6.

2 Framework for subjective option pricing

2.1 Subjective pricing kernel

Ingersoll (2006) solves the consumption-investment problem of a constrained manager and derives closed-form solutions for equity and option values. Two complications are added to a standard Merton problem. First, prior to retirement, the manager has to hold a positive proportion of wealth in the stock underlying the ESO. Second, idiosyncratic (stock-specific) risk cannot be hedged in the market, even if the manager is allowed to trade in the market portfolio. This brings the advantage that a no-arbitrage condition is satisfied, provided all assets are priced in the same subjective manner. The subjective pricing kernel is characterised by expected return below the market rate. It follows that subjective values of the stock and option will be less than their market values.

It takes two assumptions to put the above model in practical use. First,

all systematic risk is spanned by a stock market index. Second, the portfolio constraint θ is taken as a proxy for Nokia's weight in the manager's portfolio. Theoretically θ would be the weight in excess of market portfolio. Nonetheless, taking θ as proxy is justified by Nokia's low weight in a typical market index. For example, let us take DJ Stoxx 600 as a market proxy. At 30.12.1999 Nokia's weight was 2.76%, which increased to 4.87% by the of end 2001, and then decreased to 1.10 % by the end of 2004, when the series 99 ESOs expired. In short, low index weights support interpreting θ as the stock (i.e. Nokia) weight.

Managerial preferences are given by power utility $U(C) = C^\gamma/\gamma$, where C denotes consumption. Power utility implies relative risk aversion of $RRA = -\frac{U''(C)}{U'(C)} = 1 - \gamma$. The model has two risky assets, market portfolio M and stock S , on which the ESO is written, with expected returns μ_m and μ . Also, a risk-free bond B exists with return r . The manager is allowed to trade in the market index, whose return and the stock return are determined by continuous-time CAPM. Dividend yields on market index and the stock are denoted q_m and q . Both risky assets follow geometric Brownian motions, and they are correlated as shown by the system (2.1) below. Distributions of market and stock-specific returns are generated by standard Brownian motions W_m and W_s . Since they are independent, variance of the stock is given by $\sigma^2 = \beta^2\sigma_m^2 + \nu^2$. Also, the stock-specific risk W_s is assumed independent of stock-specific risks of *other* stocks.

$$\begin{aligned} \frac{dM}{M} &= (\mu_m - q_m) dt + \sigma_m dW_m & (2.1) \\ \frac{dS}{S} &= (\mu - q) dt + \beta\sigma_m dW_m + \nu dW_s \\ \frac{dB}{B} &= r dt \end{aligned}$$

Ingersoll (2006) solves the manager's problem in two stages to find out the effect of a positive holding constraint in the company stock. The first stage solves a standard Merton investment problem, where the resulting portfolio is unconstrained. The second stage involves solving the constrained problem. Managerial hedging demand is given by the difference between unconstrained and constrained market portfolio weights. Excess demand for the Nokia stock is zero in the unconstrained problem, based the mutual fund theorems of Merton (1971). When forced to hold Nokia equity in excess of market weight, the manager's will invest less than what is optimal to the market portfolio.

Solution of the Ingersoll (2006) consumption-investment problem yields a subjective pricing kernel, where the risk-free rate is adjusted to compensate for the portfolio constraint θ and idiosyncratic risk. The adjusted risk-free rate r^s is given by Eq. (2.2), where $1 - \gamma$ equals relative risk aversion. Note that r^s is decreasing in both portfolio constraint and idiosyncratic risk. The dividend rate q is also adjusted for the same reasons. The subjective dividend rate q^s is

given by Eq. (2.3).

$$r^s \equiv r - (1 - \gamma) \theta^2 \nu^2 \quad (2.2)$$

$$q^s \equiv q + (1 - \gamma) (1 - \theta) \theta \nu^2 \quad (2.3)$$

2.2 Ingersoll's model for subjective option pricing

This section combines the hedging portfolio approach of Björk (2004, esp. Th. 8.5) and subjective pricing of Ingersoll (2006), leading to a Black-Scholes-type pricing formula. The option price follows a B-S PDE with modified parameters. Given our empirical focus, the treatment is heuristic, with formal results given in the reference works. Here the leading role is played by the idiosyncratic (stock-specific) risk, hedging of which requires unconstrained trading in the stock. This is possible for option market makers, but not for the constrained managers.

It is shown here that an unconstrained trader can hedge the ESO risk by employing hedge portfolio $V(t)$. It invests the amounts h^0 and h^* in the stock and the bond, dynamics of which were given above by second and third lines of Eq. (2.1). Clearly, trading in the market index or other stocks is not sufficient for hedging. Value of the hedge portfolio at time $t \in [0, T]$ is given by

$$V(t) = h^0(t) B(t) + h^*(t) S(t). \quad (2.4)$$

Alternatively, one can use relative weights $u^0 = h^0(t) B(t) / V(t)$ and $u^* = h^*(t) S(t) / V(t)$ implying that $u^0 + u^* = 1$. Using relative weights the hedge portfolio dynamics are written as $\frac{dV(t)}{V(t)} = u^0 r + u^* \frac{dS(t)}{S(t)}$.

Now let us proceed by writing the Ito differential for option pricing function, conjecturing the hedge portfolio weights, and verifying the conjecture. If the hedge works, values of hedge portfolio and option must be equal, written as $V(t) = F(t, S(t))$, where F is a smooth pricing function¹.

Based on Ito calculus, it has the differential $dF = F_t dt + F_s dS + \frac{1}{2} F_{ss} (dS)^2$, where subscripts stand for partial derivatives. Considering the asset dynamics, and simplifying the notation as $\underline{\sigma} = (\beta \sigma_m, \nu)'$, $\underline{W} = (W_s, W_m)'$ and $\underline{1} = (1, 1)'$, the differential becomes

$$dF = \left(\frac{F_t + F_s S (\mu - q) + \frac{1}{2} (\underline{\sigma}' \underline{1})^2 S^2 F_{ss}}{F} \right) F dt + \left(\frac{F_s S}{F} \right) F \underline{\sigma}' d\underline{W}.$$

Given the diffusion term, we conjecture that the stock weight is $u^* = \frac{F_s S}{F}$, and rephrase the above equation as

$$dF = \left(\frac{F_t + \frac{1}{2} (\underline{\sigma}' \underline{1})^2 S^2 F_{ss}}{F} + u^* (\mu - q) \right) F dt + u^* F \underline{\sigma}' d\underline{W}.$$

¹We assume a priori that F exists, supported by the data of ESO trading prices.

Now it becomes clear that the bond weight is $u^0 = \frac{F_t + \frac{1}{2} F_{ss} S^2 (\underline{\sigma}' \underline{1})^2}{rF}$, which simplifies to $u^0 = \frac{F - F_s S}{F}$ once we recall that relative weights sum to unity, i.e. $u^0 + u^* = 1$. Actually, combining this statement and the weights (u^0, u^*) yields the pricing PDE (2.5), which is reminiscent of the Black-Scholes one². Specifically, the only difference is seen in the diffusion term, since here the risk is caused by exposure to Brownian motions W_s and W_m .

$$F_t + rSF_s + \frac{1}{2} (\underline{\sigma}' \underline{1})^2 S^2 F_{ss} = rF. \quad (2.5)$$

Before jumping to further conclusions, let us show that the hedge really works. Based on the previous conjecture, absolute weights of the hedge portfolio (2.4) become (time dependence is dropped for brevity)

$$\begin{aligned} h^0 &= \frac{u^0 V}{B} = \left(\frac{F - F_s S}{F} \right) \left(\frac{V}{B} \right) \\ h^* &= \frac{u^* V}{S} = \left(\frac{F_s S}{F} \right) \left(\frac{V}{S} \right) = \frac{F_s}{F} V. \end{aligned}$$

What proves the hedge is that Eq. (2.4) holds identically, if we plug in the above expressions for h^0 and h^* in it. Since F satisfies the pricing PDE, the conclusion is that $V(t) = F(t, S(t))$. Intuitively, the replacement $V = F$ shows that h^* is simply the stock delta, i.e. $h^* = F_s$.

In summary, hedging long option position requires selling the hedge portfolio, equivalent to shorting the stock. But this is not permitted for the manager acting under constraints. Hence, he becomes unwilling to hold the option, and agrees to sell it below the arbitrage price that equals the hedge portfolio value (2.4). In the Ingersoll's model, to be fitted in empirical data below, this effect is incorporated by cutting the risk-free rate, leading to lower option values.

Finally, we turn to option pricing. In order to compensate for hedging constraints and suboptimal portfolio, the constrained manager applies subjective risk-free and dividend rates in pricing. However, as shown in Ingersoll (2006, Eq. (9)), option pricing function still takes the form of the PDE (2.5). Therefore, it suffices to modify the inputs to Black-Scholes formula, which is a truly remarkable result. Equation (2.6) gives the Black-Scholes-Ingersoll (BSI) formula for European-style call options.

$$\begin{aligned} C^{BSI} &= S e^{-q^s T} N(d_1) - K e^{-r^s T} N(d_2) \\ d_1 &= \frac{\ln(S/K) + (r^s - q^s + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \sigma \sqrt{T} \end{aligned} \quad (2.6)$$

Application of the BSI formula results in option values that are generally below Black-Scholes, because the subjective risk-neutral rate is lower than the

²Let us emphasize that these are purely heuristic arguments. Their formal development is found in Chs. 7-8 of Björk (2004).

objective one. Hence, a valuation gap (or deadweight loss) prevails, and it measures how much higher the employer's cost of the option is compared to the employee's perceived value.

2.3 Generalized Black-Scholes model (GBS)

For comparison with BSI model we employ a generic diffusion model, where (implied) volatility is a smooth function of option delta. It is called the GBS model. Smoothness refers to using cubic splines in volatility modeling (see Section 3.3). Under the GBS model, risk-neutral stock dynamics are given by Eq. (2.7), where $\sigma_s(S/K, \dots)$ is the smooth volatility function depending on option delta, and consequently on moneyness (S/K) and other salient variables. B_m^* is Brownian motion generating risk-neutral market returns.

$$\frac{dS}{S} = rdt + \sigma_s(S/K, \dots) dB_m^* \quad (2.7)$$

The GBS model is equivalent to the one-dimensional diffusion case of Bergman, Grundy and Wiener (1996). It follows from their Theorem 8 that GBS option prices are calculated by Black-Scholes using either the predicted mean (as done below) or confidence bounds for volatility. In contrast to the BSI model, r in (2.7) is the market risk-free rate. Therefore, if ESO prices contain a subjective risk premium, implying $r^s < r$ in Eq. (2.2), GBS volatility has to be lower than BSI volatility. Figure B.2 shows that this holds in our data.

In fact, using the GBS provides an alternative method for checking if there is something anomalous in the ESO data. If ESOs were priced at parity with the market, $\sigma_s(S/K, \dots)$ in the diffusion model (2.7) would be equal to market volatility, and the subjective risk premium (SRP) would be zero. Table 1 shows that the fitted SRP is significantly positive. Moreover, BSI and GBS volatilities would agree, which is clearly not the case according to Fig. B.2. It shows that the GBS volatilities are consistently lower than their BSI counterparts.

3 Estimation of the option-implied PDF

3.1 State price density as valuation tool

We estimate the risk preferences of option-endowed managers. Measuring risk preferences requires the knowledge of managerial expectations, which are spanned by the probability density function implied by option prices. Further, using option prices enables us to characterize the development of preferences in time. As argued by Bliss and Panigirtzoglou (2004), option prices are useful in investigating market expectations, because options are risky assets with fixed expiration dates. This implies that prices of options with different maturities reflect the variation in expected returns over different time periods.

Our approach to calculating the risk-neutral density function (PDF) from option prices is based on estimating implied volatility as a smooth function

of option delta. The idea of estimating smooth volatility function was introduced by Malz (1997) and advanced by Bliss and Panigirtzoglou (2002, 2004), among others. In contrast to these papers, *we assume that the observed PDF for executive stock options (ESOs) reflects the subjective risk-free rate r^s* . This assumption is empirically motivated by the results of Ikäheimo et al. (2006), who find that actual trading prices for ESOs are considerably below Black-Scholes values. We stress that this approach requires empirical data on ESO prices.

The connection between the state price density and risk-neutral density is well-known. While we follow the exposition of Aït-Sahalia and Lo (2000), the ideas can be traced back to Harrison and Kreps (1979). To see how the stochastic discount factor is connected to the probability density function, start from Eq. (3.1), being the first-order condition for the representative agent. This relation defines the *state price density* (SPD), denoted by ζ . Let us remark that the empirical asset pricing literature is focused on testing implications of the restriction $E\left(\zeta_{t,T} \frac{X_T}{X_t}\right) = 1$, which goes back to at least Grossman and Shiller (1981).

$$U'(C_T) = e^{r(T-t)}U'(C_t)\zeta_{t,T} \quad (3.1)$$

$$\Rightarrow \zeta_{t,T} = \frac{e^{-r(T-t)}U'(C_T)}{U'(C_t)}$$

Second line of Eq. (3.1) defines the state price density ζ , also known as marginal rate of substitution (MRS) in microeconomics. Statistically it is equal to the Radon-Nikodym derivative of risk-neutral and cash market densities. This relation is highlighted by Eq. (3.3). Our primary motivation for introducing the SPD is to derive the relative risk aversion estimator of Aït-Sahalia and Lo (2000). The statistical form for SPD is given by Eq. (3.3). This representation³ is derived in Ch. 14.7 of Björk (2004). Equation (3.4) defines the market price of risk (ϕ).

$$f_t^*(S_T) = f_t(S_T)\zeta_{t,T} \quad (3.2)$$

$$\zeta_{t,T} = \exp\left[-\int_t^T \phi dB_m - \int_t^T \left(r + \frac{1}{2}\phi^2\right) ds\right] \quad (3.3)$$

$$\phi = \frac{\mu - r}{\sigma} \quad (3.4)$$

The SPD, as defined in (13), depends essentially on the market price of risk. In fact, the Black-Scholes-Ingersoll model should be seen as a modification of the market price of risk. Because the manager has to hold an asset that does not lie on the CAPM efficient frontier, his market price of risk is inferior compared to unconstrained investors.

³Björk (2004) uses the notion of stochastic discount factor instead of state price density.

3.2 PDF implied by option prices

This section shows how we extract the (risk-neutral) probability density function of the underlying asset from option prices. By assuming that security price dynamics follow geometric Brownian motion, Ait-Sahalia and Lo (1998, 2000) find that the PDF is given by Equation (15). Note that $\tau = T - t$ denotes time to expiration, other notation is as before. The first line of Eq. (3.5) is based on the results of Breeden and Litzenberger (1978), and the second line uses the lognormal density function.

$$\begin{aligned} f_{BS}^*(S_T) &= e^{r\tau} \left(\frac{\partial^2 C_{BS}}{\partial K^2} \right)_{K=S_T} \\ &= \frac{1}{S_T (2\pi\sigma^2\tau)^{1/2}} \exp \left[\frac{-\left(\ln(S_T/S_t) - (r - q - \frac{1}{2}\sigma^2)\tau\right)^2}{2\sigma^2\tau} \right] \end{aligned} \quad (3.5)$$

For later use we remark that the derivative of risk-neutral density function (3.5) is given by

$$\begin{aligned} (f_{BS}^*)' &= -\frac{1}{S_T^2 (2\pi\sigma^2\tau)^{1/2}} \left(\frac{1 + \ln(S_T/S_t) - (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma^2\tau} \right) \\ &\quad \times \exp \left[\frac{\left(\ln(S_T/S_t) - (r - q - \frac{1}{2}\sigma^2)\tau\right)^2}{2\sigma^2\tau} \right]. \end{aligned} \quad (3.6)$$

Equation (15) is the lognormal PDF implied by underlying stock dynamics. Under the Black-Scholes-Ingersoll model the risk-free rate (r) and dividend yield (q) are replaced by their B-S-I counterparts r^s and q^s . The B-S-I density function is a convenient tool for calculating both the theoretical and estimated densities. For example, Ait-Sahalia and Lo (1998, 2000) use the Black-Scholes PDF to calculate theoretical density using at-the-money volatility and ‘actual’ density by plugging in a nonparametric volatility function. In standard B-S economy the volatility function is a straight line, whereas in practical markets volatility is explained by a number of factors, for instance by time to expiration and moneyness.

3.3 Estimation of the smooth volatility function

We apply the volatility function method introduced by Malz (1997) and refined by Bliss and Panigirtzoglou (2002, 2004) to estimate the PDF by mapping option prices in delta-sigma space. The process starts by estimating deltas from option prices using (at-the-money) implied volatility. The next step is to estimate the smooth volatility function, which explains variation in volatility by variation in delta.

Malz (1997) has a rich data on three combinations of forex calls and puts, and he derives an arbitrage relation for their prices. As a result Malz gets a natural smoothing function, which is a second order polynomial of delta. We

cannot proceed like Malz and derive similar arbitrage relations, because our data concerns executive stock options with only call features. Therefore we smooth the data in delta-implied volatility space using nonparametric regression, using penalized splines to smooth the response variable (see chapter 3 of Ruppert, Wand and Carroll (2003)). In our case ‘nonparametric’ means that the interest is estimation of smooth volatility function. Functional form of the response, as well as parameter values are of lesser importance.

Our smooth volatility function is given by Equation (3.7). It is estimated using restricted maximum likelihood. The value of $\hat{\sigma}_s$ is the estimated volatility at a given level of option delta. The ‘cubic thin plate’ splines, or the third-order polynomials in Eq. (3.7), use radial basis functions⁴ of third order.

$$\begin{aligned}\hat{\sigma}_s &= h_s(x) + \varepsilon \\ h_s(x) &= E(\sigma_s|x) = \beta_0 + \beta_1 x + \sum_{k=1}^K |x - \kappa_k|^3\end{aligned}\tag{3.7}$$

In Eq. (3.7) ε is the error term with zero mean and constant variance, and κ_k are the knots located on the x-axis. In practice we estimate the smooth volatility function by fitting Eq. (3.7) in option deltas calculated using the standard B-S formula⁵. For software we use the SemiPar 1.0 package for R language [see Wand et al. (2005)].

The volatility function is fitted in delta-sigma space, but delta is not an input to the risk-neutral PDF formula introduced by Breeden and Litzenberger (1978) and developed by Ait-Sahalia and Lo (1998). Recall that the PDF is calculated using the second derivative of option price with respect to strike price. Hence we need to calculate strike prices as function of delta. This is done using Nokia’s daily at-the-money volatility σ_{atm} obtained from Thomson Datastream (datatype VI). Note that the ATM volatility is not used for option valuation. The implied strike price K is calculated numerically using Eq. (3.8).

$$\begin{aligned}\Delta &= g_s(S, K, \tau, r, \sigma_{atm}) \\ \Rightarrow K &= g_s^{-1}(\Delta; S, \tau, r, \sigma_{atm})\end{aligned}\tag{3.8}$$

Subscript s in Equation (3.8) indicates that we use similar smoothing as in the volatility function (3.7) to produce one-to-one correspondence of delta and strike prices. The next step is to calculate the density function (3.5), plugging in volatilities from the smooth volatility function (3.7) and strike prices from (3.8). We think that our smooth volatility function strikes a balance between flexibility and data requirements. Compared to the results of Ait-Sahalia and Lo (1998), specifically the ‘implied volatility surface’ of their Figure 3, our type of volatility function captures the moneyness effect.

The key issue in volatility function estimation is the extent of smoothing. There is no common norm as to how smooth the fitted curve should be. The aim

⁴To elaborate, the basis functions are $1, x, |x - \kappa_1|^3, \dots, |x - \kappa_K|^3$. This is called a radial basis, because the splines are radially symmetric about κ_k .

⁵We would like to stress that the Black-Scholes model is used here only for data transformation, as in Bliss and Panigirtzoglou (2004).

of smoothing is to decrease the amount of noise and clarify underlying trends. But the trade-off is that more and more information is lost as the amount of smoothing increases. To see the importance of choosing the smoothing parameters, note that volatility function estimation is not the only instance where we use smoothing. Below we will calculate the Aït-Sahalia–Lo risk aversion estimator, which requires us to compute derivatives of density functions. Unfortunately the derivatives of PDFs tend to be ‘wiggly’, and without smoothing the risk aversion estimates can be quite volatile.

Ruppert, Wand & Carroll (2003) recommend measuring the amount of smoothing with degrees of freedom of the fitted linear smoother. Nonetheless, the smoothing parameter used as smoothing measure in some studies, e.g. in Bliss and Panigirtzoglou (2002, 2004). Unfortunately, the smoothing parameter does not directly measure the extent of smoothing (Ruppert et al. (2003), p. 81). Further, different values of this parameter can result in quite similar fit, which makes choosing an ‘optimal’ value challenging. For example, Bliss and Panigirtzoglou (2004, p.416) their results are not sensitive to the choice of this parameter.

Instead, degrees of freedom (df_{fit}) has a clear interpretation; it is analogous to the number of parameters in a linear model. The generic spline model can be written as $\hat{y} = S_\lambda y$, where S_λ is called the hat matrix (or smoother matrix) and λ is the smoothing parameter. For linear regression, the hat matrix has degrees of freedom equal to number of parameters. For spline models, the parameters have different interpretation arising from Eq. (3.7). Hence, we need to calculate degrees of freedom using the trace (denoted Tr) of the hat matrix:

$$df_{fit} = \text{Tr}(S_\lambda). \quad (3.9)$$

If df_{fit} is chosen as the smoothing measure, the next question is: what is the optimal value for it? Theoretical answer is given by Ruppert, Wand & Carroll (2003), who recommend using the Generalized Cross-Validation GCV, which can be calculated as

$$GCV = \frac{RSS(\lambda)}{(1 - n^{-1}df_{fit})^2} \quad (3.10)$$

In Equation (3.10) $RSS(\lambda)$ is the residual sum of squares of the spline model and n is the number of observations. In theory, one should choose the model that minimizes GCV. However, in practice the amount of smoothing must be determined by the data at hand. In particular, estimated densities need to be smooth enough to have a smooth derivative; otherwise the risk aversion estimates may become unreasonably volatile.

3.4 Semiparametric estimator for relative risk aversion

This section presents the Aït-Sahalia and Lo (2000) estimator for relative risk aversion. It has the main benefit of producing the relative risk aversion (RRA) as a function of the observed option and cash market PDFs. Further, using the densities produces a mapping of RRA as a function of the state price and

not just a point estimate. In fact the shape of RRA function reveals important properties of implicit preferences. If the RRA function is flat, preferences are mapped by power utility, which is consistent with the Black-Scholes model. Ait-Sahalia and Lo (2000) take this argument further and show that all utility functions implied by the Black-Scholes model correspond to constant relative risk aversion⁶. Hence the shape of RRA function is indicative on of how well the applied pricing model fits to the empirical data.

Let us elaborate the notation. Probability density function is denoted by f , a hat ($\hat{\cdot}$) denotes an estimate and a star (\ast) denotes risk neutrality. In practice we estimate the risk-neutral PDF \hat{f}^\ast from Eq. (3.5) and the cash market PDF \hat{f} using a standard kernel regression method⁷. With this information we're able to compute the Ait-Sahalia RRA estimator, denoted $\rho(S_T)$. In the first step we differentiate Eq. (3.1) with respect to S_T and hold S_t constant, which gives $\zeta'(S_T) = e^{-r(T-t)} \frac{U''(S_T)}{U'(S_t)}$ and

$$\rho(S_T) = -S_T \frac{U''(S_T)}{U'(S_t)} = -S_T \frac{\zeta'(S_T)}{\zeta(S_T)}. \quad (3.11)$$

In the second step we take the statistical representation of SPD given by Eq. (3.2), i.e. $\zeta_{t,T} = f_t^\ast(S_T)/f_t(S_T)$, differentiate it with respect to S_T , and plug in the result in Eq. (3.11), which in turn gives the Ait-Sahalian and Lo (2000) estimator (3.12).

$$\hat{\rho}(S_T) = S_T \left(\frac{\hat{f}'}{\hat{f}} - \frac{\hat{f}^{\ast\prime}}{\hat{f}^\ast} \right) \quad (3.12)$$

Ziegler (2007) shows that risk aversion estimates are quite sensitive to errors in density estimation. In his example a small error in estimated standard deviation leads to a large perturbation in risk aversion estimates. In our empirical analysis a key requirement for estimated densities is to be smooth. Otherwise, 'bumps' in the estimated density turn into 'holes' in its derivative required by the risk aversion formula. In the current setup, smoothness of the density follows from smoothness of the volatility function.

4 Data description and empirical results

4.1 Data description

A unique feature of the Finnish market is that executive stock options are publicly traded on the HEX (nowadays the OMX Nordic Exchange). The main reason for the listing ESOs originates from taxation of share subscriptions. For employees, the subscription of ESO-based shares leads to a tax-based risk.

⁶This result applies to an economy with no intermediate consumption, hence the set-up is not exactly as in the Black-Scholes-Ingersoll model. See Ingersoll (2006).

⁷We estimate the Gaussian kernel using the 'density' function of R language with default bandwidth.

The taxable gain at exercise is the difference between the market value of the shares and the exercise price. The fair market value is usually deemed to be the average price of shares during the day (gross turnover of the day divided by the trade volume of the day). It may take several days or even weeks before the shares are registered and the executive can sell them. At the extreme, in January, executives (or anybody else holding these options at that time) do not receive shares until March-April, since these shares are ex-dividend and executives have to wait until March-April to receive the shares (ex-dividend). The tax benefit is determined at the exercise date but executives cannot sell the exercised shares immediately. However, the share price can decrease between the exercise date and the first possible trading date. This leads to a significant tax risk.

If, alternatively, the stock option is sold to a third party (on HEX), the taxable gain is simply the sales proceeds of the stock option less any price paid for the option. If ESOs are sold in the market, the employee receives cash within three days [settlement at T+3].

Other reasons for public listing are financing and liquidity. A number of companies have avoided the need to financial assistance by defining the stock option terms and conditions in a way that they are fully transferable to a third party (after vesting period). These options may be listed, with the advantage of better liquidity and continuous pricing through the participation of both executives and outside investors. From a company's point of view, outside investors can be assumed more probably to hold options to maturity ('Never exercise your option early!') in which case company avoids costs of issuing small amount of shares as a consequence of several early exercises. There were 53 ESO series issued by Finnish companies and listed on the HEX during years 2000-2002. Total ESO turnover exceeded 3.2 billion euros, of which Nokia options were responsible for 94.1%. A total of 45,600,977 options were traded in 34,443 transactions.

The data consists of Nokia executive stock options traded on the HEX during the period 2.4.2000 – 30.12.2002. The data spans 7611 trades in Nokia 1999 Stock option plan that expired on 31.12.2004. Average time to expiration is 3.10 years. A key property of this data is that most trades (80%) are done in-the-money. Median and average values of moneyness (S/K) are 1.54 and 1.49.

We investigate the pricing of Nokia 1999 (issued during respective year) ESO plan, even though the Nokia 1997 ESO plan traded at the same time and the trade data is available. However, during our research period the Nokia 1997 ESO plan was very deep in the money, which is shown by the strike price of 3.227 euros, whereas the Nokia 1999 ESO plan had a strike of 16.89 euros. (Variation of moneyness can be checked from Fig. 2B.) The strikes are adjusted for subsequent stock splits and euro conversion. Under these market conditions Nokia 1997 ESOs behaved very much like ordinary stock, and therefore we disregard the data. The interest rates are estimated using the Euribor interest rate and the Finnish zero-coupon bond yield curve. Using linear interpolation, we obtain all interest rates needed for discounting.

4.2 Empirical fit of option pricing models

This section discusses the empirical fit of the three models being Black-Scholes (BS), Generalized Black-Scholes (GBS), and Black-Scholes-Ingersoll (BSI). Detailed estimates and goodness of fit are reported in Table 1. In summary, the best fit is offered by the most flexible GBS model. Also, the BSI model fits the data substantially better than BS, based on comparison of residual standard error and bias. Given that the BSI subjective risk premium is highly significant and GBS volatilities are below market and BSI levels, we conclude that ESO prices incorporate a subjective discount. Regardless of the model, it should be associated with idiosyncratic risk of the Nokia stock that impacts the subjective valuation.

Estimation was done using nonlinear least squares (NLS), which yields consistent and asymptotically normal estimates by minimizing the criterion functions given below. In theory, this follows from the general properties of M-estimators discussed by van der Vaart (1998). Consistency and asymptotic normality follow from his Theorems 5.14 and 5.23. (See also Ex. 5.27 for discussion of NLS.) In this sense NLS is an efficient estimation method. It has been previously applied by e.g. Bates (1996). Asymptotic normality, or the convergence of estimated parameters to their true values, requires that partial derivatives of the NLS criterion functions are uncorrelated with residuals. This gives rise to moment conditions; their actual values are reported in Table 1.

Following Bates (1996), we pay attention to heteroscedasticity and autocorrelation (HAC), and use HAC consistent covariance matrices for NLS. They are discussed in detail by Zeileis (2004), who also describes the software that we use for computations. NLS residuals are likely to be heteroskedastic, because the data are not equispaced, i.e. the time between transactions varies. Nonetheless, in this data using HAC consistent covariances has no implications on parameter significance. Figure 1 plots the residual distributions. The main concern is that BS residuals are clearly dependent on stock price. Fortunately, such structure cannot be found for GBS and BSI residuals.

4.2.1 Black-Scholes

Fitting the basic BS model with constant volatility in our Nokia ESO data creates a benchmark for more flexible models. The estimation is done by NLS with volatility being the estimated parameter. The NLS estimator (4.1) minimizes the sum of squared pricing errors. Equation (4.2) gives a moment condition saying that fitted residuals and derivative of the criterion function w.r.t. estimated parameter should be zero. Here and below estimates are marked by a hat.

$$\min_{\sigma} \sum_{i=1}^n [C_i - C^{BS}(S_i, \dots; \sigma)]^2 \quad (4.1)$$

$$\sum_{i=1}^n \left(\frac{\partial \hat{C}^{BS}}{\partial \sigma} \right)_i (C_i - \hat{C}_i^{BS}) = 0 \quad (4.2)$$

Estimates and the goodness of fit statistics are reported in Table 1. Clearly, forcing volatility to be constant in the data does not produce a close fit, as shown by residual bias and standard error being higher than for the models that follow. Also, the residuals are correlated with option vega, implying that the moment condition (4.2) is not satisfied.

4.2.2 Generalized Black-Scholes (GBS)

The second candidate for ESO pricing is the GBS model that employs the volatility function (3.7). This specification is very flexible, since the fitted volatility function has roughly speaking 20 parameters [$df_{fit} = 20$ in Eq. (3.9)]. It is natural to predict volatility by delta, which is a composite measure of moneyness, risk-free rate and time to expiration. The criterion function for fitting the volatility model is given by Eq. (4.3). Note that the second term represents a smoothness penalty that is increasing in the smoothing parameter λ . Further, β denotes a parameter vector and K is a $K \times K$ matrix of spline metrics $[|\kappa_k - \kappa_{k'}|]$ for $1 \leq k < k' \leq K$. Further details are given in Ch. 3.7 of Ruppert et al. (2003). GBS option prices are computed using Eq. (4.4). It is a composite function of Black-Scholes and the volatility model. Option delta is denoted by x .

$$\min_{\beta} \sum_{i=1}^n [\sigma_i - h_s(x_i; \beta)]^2 + \lambda^3 \beta' K \beta \quad (4.3)$$

$$\widehat{C}_i^{GBS} = C^{BS} \circ h_s(x_i) \quad (4.4)$$

$$\sum_{i=1}^n \left(\partial \widehat{C}^{BS} / \partial x \right)_i \left(C_i - \widehat{C}_i^{GBS} \right) = 0 \quad (4.5)$$

The estimates given in Table 1B suggest that the GBS model affords an excellent fit. Residual bias and standard error are smaller than for the competitors, and also the moment condition (4.5) is quite well satisfied. In conclusion, this model produces the best fit.

4.2.3 Black-Scholes-Ingersoll (BSI)

The third candidate for ESO valuation is the BSI model (2.6). It has three unknown parameters: risk aversion (γ), required equity holding (θ) and idiosyncratic risk (ν). In fact, the case $\theta = 1$ is less extreme than it first appears. Think of an entrepreneur who has invested her net financial wealth in an enterprise. Using leverage it is certainly possible to have $\theta > 1$. In practice, we try values of 0.25, 0.5 and 1 for θ . Results reported in Table 1 shows that the subjective risk premium estimates are not awfully sensitive to θ .

The parameterization requires an explanation. Subjective risk-free rate r^s and subjective dividend yield q^s are defined by Eqs. (2.2)-(2.3). Recall that the *subjective risk premium* equals the difference between market and estimated risk-free rates, with estimated values reported in Table 1C. We also note that the SRP is related to relative risk aversion through and to idiosyncratic risk through

. This allows us to fit the BSI model using NLS without prior knowledge of the idiosyncratic risk and risk aversion parameters. The criterion function, using obvious notation, is given by (4.6) and moment conditions by (4.7)-(4.8).

$$\min_{\sigma} \sum_{i=1}^n [C_i - C^{BSI}(S_i, \dots; \sigma, r^s, q^s)]^2 \quad (4.6)$$

$$\sum_{i=1}^n \left(\partial \widehat{C}^{BSI} / \partial \sigma \right)_i (C_i - \widehat{C}_i^{BSI}) = 0 \quad (4.7)$$

$$\sum_{i=1}^n \left(\partial \widehat{C}^{BSI} / \partial r^s \right)_i (C_i - \widehat{C}_i^{BSI}) = 0 \quad (4.8)$$

The estimates of SRP vary between 3.4% and 5.1%, depending on the portfolio restriction θ . Given that the parameters are highly significant and the BSI model fits the data better than BS, one may conclude there is a subjective discount in ESO prices. These findings are consistent with Ikäheimo et al. (2006), who report major underpricing of Finnish ESOs relative to their expected Black-Scholes values. While the BSI model fits clearly better than BS, the fit is still inferior to the more flexible GBS model. Recall that the BSI model uses only two parameters, whereas the GBS model has appr. 20 parameters [$df_{fit} = 20$ in Eq. (3.9)].

4.2.4 Summary of empirical results

Depending on the fitted model, the results show that implied volatilities of ESOs are below market levels, or the risk-free rate is less than market rate. Either way, the conclusion is that ESOs are ‘cheaper’ than standard options in relative terms. This can be interpreted in a framework, where managers are risk-averse and the stock risk cannot be diversified. Assume that the subjective view of stock distribution is riskier than the market view. By ‘riskier’ we refer to the definition of Rotschild and Stiglitz (1970). This means that the subjective valuation involves a white noise risk factor (with zero mean) that the market does not perceive. Then it is possible that the subjective option value is lower than the market value, even if subjective stock distribution is riskier. Such an example is given in Jagannathan (1984). He shows that it is possible for two stocks to have the same risk-averse value, yet the option on the riskier stock is less valuable. ‘Two stocks’ in this example compare to subjective and market views of the stock. Moreover, Jagannathan’s result is consistent with absence of arbitrage.

4.2.5 Time series aspects of the SRP

It is natural to ask how the subjective risk premium (SRP) depends on option moneyness and volatility. The answers are provided by Figure 2. It presents rolling estimates of the BSI model (4.6) using the following sampling scheme. First the data is divided to moneyness quartiles, and 500 observations are sampled from each quartile. This yields a stratified sample of 2000 observations, which is ordered as time series. Then the BSI model parameters, being

SRP and volatility, are estimated using a rolling window of 600 observations. Average length of the window is 173 days (the median is 201). Because the data are not equispaced, the window could not have fixed length without wasting a lot of observations.

The upper row of Figure 3 plot rolling estimates for the SRP and moneyness. The upper left plot suggests a weak positive time trend for the SRP. The dependence of SRP and moneyness is negative with correlation of -0.58. The lower row of Fig. 3 compares BS and BSI volatilities and graphs the t copula (with correlation of 0.94 and 3.6 df) of SRP and BSI volatility (copulas are discussed in Ch. 5 of McNeil et al., 2005). Using the copula is motivated by the interest being in joint dependence rather than in marginal distributions. It is hardly surprising that BS and BSI volatilities exhibit correlation of 0.81. The intriguing finding is that the SRP and BSI volatilities are strongly dependent as shown by the t copula fit with correlation of 0.94 and 3.6 degrees of freedom.

4.3 Volatility skew, risk aversion estimates and model fit

Since the data concerns ESO prices and most trades are done in-the-money, the shape of volatility function may be different from the familiar volatility smile or skew. Figure 3A shows that the implied volatility decreases with option delta. A potential explanation is that the sensitivity of option price to volatility decreases as moneyness increases. The same applies to adjusting the risk-free rate in the BSI model: for high values of moneyness, the adjustment has a minor impact. Therefore, if prices of in-the-money options exhibit little time value, their implied volatilities are likely to be low, and volatility depends inversely on delta. Figure 3B shows how the moneyness of an ESO is associated with time value and implied volatility. On the upper right corner one can find first moneyness quartile observations, to be associated with high time value and high implied volatility. In contrast, from the lower left corner of Fig. 3B one can infer that high moneyness (Q4) observations associate with low time value and low implied volatility. Clearly, the clustering of observations shows that deep-in-the-money options are treated like common stock in subjective valuation.

We estimate the risk aversion of Nokia ESO holders using the Ait-Sahalia and Lo estimator (3.12). In this section we use the GBS based on smooth volatility function. Optimal amount of smoothing is determined by choosing the degrees of freedom (df_{fit}) of the smoother matrix S_λ defined by Eq. (3.9). Moreover, df_{fit} is chosen to minimize the generalized cross-validation (GCV) criterion (22), which yields $df_{fit} = 20$. This corresponds to GCV value of 6.92 and smoothing parameter $\lambda = 0.0761$.

The fitted volatility function is plotted in Figure 3A over the scatter plot of observations. The estimated curve seems to pick up nuances of our data, yet is it is smooth enough. Figure 3B shows that implied volatility increases with time value of the option, a property that arises from the BSI option pricing model.

The risk aversion estimator takes as inputs risk-neutral and cash market densities and their derivatives. We estimate the risk-neutral density by plugging in the implied volatility function in Eq. (3.5). Cash market density is

estimated from Nokia stock returns using ten year period that ends at the ESO expiration date. Mean daily return for the 10-year period equals +0.070% (19.1% p.a.). Daily volatility is 3.27%, yielding 51.70% p.a. These numbers reveal how volatile the Nokia stock has been, and suggest market makers need to be careful in hedging their positions. The annual cash market density is estimated using the stationary bootstrap⁸ of Politis and Romano (1994). This method does not assume any parametric distribution for the stock. First, 10,000 return sequences of 250 daily observations are sampled from the data. Each sequence produces a single observation for annual return, calculated as the sum of 250 daily log returns. The bootstrap yields a histogram for annual returns. Next, the cash market PDF is estimated using a Gaussian kernel and bandwidth $bw = 0.785 (\hat{q}_{0.75} - \hat{q}_{0.25}) n^{-1/5}$. This formula is recommended by Davidson and MacKinnon (2004).

Figure 4 plots the estimated risk-neutral and cash market densities. As expected, the cash market density has a flatter profile. The shape of risk-neutral density is close to lognormal. If the volatility function of Figure 3 was flat, the shape of risk-neutral PDF would be exactly lognormal. In this case relative risk aversion would be constant. Both estimated densities assume a time to expiration of one year, and they are calculated using average dividend yield. The upper panel of Figure 5 plots the marginal rate of substitution (MRS). Note that the MRS is higher for low stock prices, implying that in those states the manager is unwilling to transfer consumption. The lower panel of Figure 6 shows the relative risk aversion depending on terminal stock price. It is quite interesting that implied risk aversion seems to be slightly higher than one for most of the stock price range. We think this indicates two things. Firstly, constant relative risk aversion (CRRA) may be a reasonable assumption. This is comforting, since the Black-Scholes model is consistent with CRRA preferences, as pointed out by Ait-Sahalia and Lo (2000). Secondly, the data suggests that log utility approximates the average Nokia manager's preferences, and RRA is on average about 1.1.

When interpreting the results, we assume that a representative manager exists. This is justified by Ziegler (2007). He shows that if traders have homogenous beliefs, implied risk aversion at market level is the weighted harmonic mean of individual preferences, and the weights are determined by individual consumption shares. Since our data covers the managers of a single firm, the assumption of homogenous beliefs should be realistic. Further, Ziegler shows that relative risk aversion decreases as the proportion of wealthy consumers increases, which helps to explain why the risk aversion estimates are relatively low. Typically, managers are wealthy, so their risk aversion should be lower than the average investor's.

⁸We tested the stationarity of daily returns using Augmented Dickey-Fuller regressions. For software we used the R language and tseries package (R Development Core Team, 2008)

5 Analysis of hedging error

The above analysis deals with a manager looking to sell ESOs, with the result they have lower implied volatilities than standard (at-the-money) options. Now the focus moves from supply to the demand side of the market. For this purpose, consider an unconstrained trader, referred to as ‘market maker’, willing to buy ESOs from managers. Clearly, the market maker will not bid more than what it costs to hedge the option. Maintaining a delta-neutral hedge involves transaction costs and hedging error, which we estimate below by simulation.

The ESO market is characterized by managers offering and market makers bidding for options, giving rise to a bid-ask spread. Empirical studies take it as a measure of hedging costs. For example, Jameson and Wilhelm (1992) and Petrella (2006) show that the spread is largely explained by transaction costs and stochastic volatility risk. Our data does not include the spread, and even if it did, two reliability issues would arise. First, many trades occur within the spread, and second, the ESO market is arguably less competitive than standard option markets. Thereby, hedging error is simulated using the model of Hayashi and Mykland (2006). The results, reported in Table 2, should be taken as a lower bound of a market maker’s hedging costs. This is because the simulation does not account for e.g. cost of shorting the stock.

Three causes of hedging error are explored below. First, ESOs and standard options have different strike prices and maturity. Second, updates of the hedging portfolio are discrete. Third, implied volatilities of ESOs and standard options are subject to uncertainty. A simulation study shows that the market maker who buys ESOs and hedges the position in cash and derivatives markets cannot ignore hedging error. It causes substantial transaction costs and decreases the market maker’s bid price on Nokia ESOs.

5.1 Practical aspects of hedging

It is well known that the hedge portfolio weights are determined by deltas and gammas of the hedging assets. When long position in ESOs is hedged using the stock and a standard call, both hedging assets will be assigned negative weights. Short position in standard calls (traded in Eurex) is potentially problematic due to American-style exercise. When the call option is exercised, the writer has three days to deliver the stock. This obligation cannot be covered by converting ESOs to stock, because the process is slow as explained in our data description. Also, Nokia ESOs and standard options are not exchangeable in any way. Nokia stock can be shorted, with usual borrowing cost of around 2% p.a. For an ESO with maturity of 3 years, the shorting cost is material.

In practice, hedge ratios can be calculated parametrically from an option pricing model, but they can also be estimated from data using the local polynomial estimation (LPE) of Bossaerts and Hillion (1997). The idea of LPE is to estimate the hedge elasticity by local nonlinear least squares. The method has the advantage of producing very accurate local fit, but it is very data-intensive, since it uses stock and option returns, instead of prices.

5.2 Discrete and relative hedging error

What follows is based on the case study of Hayashi and Mykland (2005, in short HM05), where a trader applying the Black-Scholes model is exposed to gamma and stochastic volatility risks, formalized by Eqs. (5.1) and (5.2). Clearly, our data incorporates both risks (see Fig. 2). Also, they are shown to increase option bid-ask spreads by Jameson and Wilhelm (1992) and Petrella (2006).

For modeling purposes, hedging error is defined as the sum $L = L_1 + L_2$ of discrete error L_1 and relative error L_2 . When L is positive (negative), a delta hedging portfolio has higher (lower) terminal value than the derivative (see the Appendix). Let us first consider the discrete error, caused by discrete updates of the hedging portfolio. In a Black-Scholes environment it converges in distribution to the stochastic integral (5.1).

$$L_1 = \sqrt{\frac{T}{2}} \int_t^T C_{SS} \sigma^2 S_t^2 dW \quad (5.1)$$

In Eq. (5.1) T is time to maturity, subscripts of option price C denote partial derivatives and W is a standard Brownian motion. Squared volatility is σ^2 . Let us remark that L_1 has zero expected value, and it depends on the option gamma C_{SS} . In fact, Eq. (5.1) appears already in Bertsimas, Kogan and Lo (2000), who verify it for an arbitrary European-style contingent claim with smooth payoff.

Relative hedging error is caused by uncertainty in implied volatility. Here we refer to Proposition 6.1 of HM05. It says that relative hedging error converges in distribution to the stochastic integral L_2 as the number of trading times increases.

$$L_2 = - \int_t^T \frac{C_{S\Xi}}{C_\Xi} R_t dS_t \quad (5.2)$$

In Equation (25) Ξ is the cumulative variance, and R_t denotes the option pricing error. Alternatively, one could think of R_t as a component of the bid-ask spread. As we approach expiration (T), the error converges to zero. [This is a regularity condition for the stochastic integral, see HM05.]

What we measure empirically is a cross-section of errors associated with average option maturity of 3.10 years. Because the data does not extend to expiration of the options, an assumption is required to characterize error behavior approaching expiration. We assume that the error decays at a steady rate and multiply it with an exponential decay factor. The latter equals the probability of observing the error at time t , on the condition it is observed prior to expiration.

$$R_t = R_0 \frac{\Pr(x \leq T - t)}{\Pr(x \leq t)} = R_0 \left[\frac{1 - e^{-\psi(T-t)}}{1 - e^{-\psi t}} \right] \quad \text{for } 0 \leq t < T \quad (5.3)$$

Eq. (5.3) multiplies the pricing error by the decay factor. Note that R_t satisfies three properties; (1) it agrees with data, since we can set $R_{(t=0)} = R_0$, (2) R_t occurs with probability 1, and (3) R_t converges to zero as $t \rightarrow T$. Decay

rate of the pricing error is governed by the parameter ψ , which we set to $\psi = 1$ in simulations. The decay of relative pricing error is shown in Figure 6A.

It is convenient to measure the pricing error in terms of implied volatility, while holding other variables constant. As suggested above, we calculate empirical ESO pricing errors using the residuals of implied volatility function (IVF). The intuition is clear; pricing error distribution will converge to normal one with zero mean, if the IVF fit is unbiased and does not leave any structure in residuals.

In practice the IVF residuals are transformed to price residuals using first-order Taylor approximation, following Remark A.3 of HM05. Pricing error $R_{0,p}$ is calculated by multiplying the IVF residual ε_p by option price sensitivity to cumulative variance C_{Ξ} . Subscript p refers to the percentile of IVF residual distribution.

$$R_{0,p} = \Delta \Xi_p C_{\Xi} \left(S_t, \hat{\Xi}_t \right) \quad \text{where } C_{\Xi} \left(S_t, \hat{\Xi}_t \right) = \frac{S_t \varphi(d_1)}{2\sqrt{\hat{\Xi}_t}} \quad (5.4)$$

In Equation (5.4) $\Delta \Xi = \Xi - \hat{\Xi}$ is prediction error in cumulative variance, $\hat{\Xi}_t$ is the predicted variance for the remaining term and $\varphi(\cdot)$ is the standard normal density function. In order to calculate the relative hedging error, we need a formula for the ‘‘greeks’’. It is shown in the Appendix that the greek term in the integral (5.2) is given by the following expression.

$$\frac{C_{S\Xi}}{C_{\Xi}} = -\frac{\log(S_t/K) + r_t - \Xi_t/2}{S_t \Xi_t} \quad (5.5)$$

$$r_t = \int_t^T r_s ds \quad \text{and} \quad \Xi_t = \int_t^T \Xi_s ds$$

Note from Eq. (5.5) that the greek term is zero, i.e. relative hedging error is minimized (in absolute terms) when $S/K = \exp(-r_t + \Xi_t/2)$, because at this point option delta is always 0.5. Figure 6B plots the greek term at different levels of volatility, suggesting that the greek term is large at low levels of moneyness. The effect of volatility is a bit hard to read off the curves, but in general a decrease in volatility magnifies the greek term. Finally, we note that in Fig. 6B the curves coincide at the forward at-the-money point $S^* = Ke^{-r_t}$. At this point the greek term becomes a constant; $C_{S\Xi}/C_{\Xi} = 1/(2S^*)$.

5.3 Monte Carlo study of hedging error

Simulation is a natural tool for practical evaluation of discrete and relative hedging error. Given that the volatility function fit is (appr.) unbiased, we’re interested in hedging error induced by the tails of the residual distribution. This is somewhat similar to value-at-risk calculations that usually involve 5% or 1% quantiles. However, we need to consider both tails of the error distribution. As shown in the Appendix, negative hedging error is interpreted as a loss for a market maker who is long in ESO and short in hedge portfolio.

Estimation of hedging errors is carried out by simulating 5,000 paths for the stock, assuming it follows Geometric Brownian Motion. Eq. (5.1) and (5.2) are used to simulate L_1 and L_2 . Stock returns are normal with annual mean of 10% and volatility of 50%. Other parameters match the data; option maturity is set at sample average of 3.1 years and implied volatility at 0.517 p.a. The initial stock price is fixed at 1, and in-, at-, and out-of-the-money options are created using strikes of 0.9, 1.0 and 1.1, corresponding to initial deltas of 0.75, 0.71 and 0.68. Interest rate and dividend yield are set to their sample averages of 4.30% p.a. and 1.24% p.a. Each year consists of 250 trading days, and implied volatility is calculated daily using the estimated volatility function. Also the greek term $C_{S\Xi}/C_{\Xi}$ is updated daily using current parameter values. Option pricing error R_p starts at value based on volatility function residual, and then decays exponentially to zero at expiration [cf. Eqs. (5.3)-(5.4)].

Our simulation results are reported in Table 2. They are based on six quantile values of the IVF residual distribution. Since the hedging error $L_1 + L_2$ is considered a transaction cost to the market maker, all reported errors are scaled by corresponding option values.

Table 2 should be interpreted vis-à-vis the fact the discrete error has zero expectation. However, magnitude of standard errors shows that ignoring it could be costly. 95% confidence bands for means are calculated as mean $\pm 2*s.e.$ (see Jäckel, 2002, p. 20). Going forward, means of relative hedging error are in the range [-2.92%, 2.55%]. Moneyness has most severe impact on out-of-the-money options, a result that holds still if the scaling is removed. Largest hedging errors are expected in cases 1-3, where the volatility residual is negative. This happens as the hedger overestimates his delta and goes too short in the stock (or other hedging instrument, see the Appendix). Also, L1 and L2 are uncorrelated in all cases, as predicted by the theory of HM05. Figures 6C and 6D plot the means and 95% confidence bands for L1 and L2. The plotted case is at-the-money option, with data provided on the bottom row of Table 2.

6 Conclusions

Using actual price data on executive stock options produces a number of insights that have not been presented in the empirical literature. We fit Generalized Black-Scholes and Black-Scholes-Ingersoll models in the data and find that ESOs are priced below market levels. This is shown by GBS volatilities being consistently below market levels, or alternatively by BSI estimates of significantly positive subjective risk premium. The BSI model says that constrained managers use a subjective pricing kernel, i.e. they settle for lower return since they are unable to hedge the option position. The existence of subjective pricing kernel opens up limited arbitrage opportunities for unconstrained market makers, who buy the ESOs from managers and hedge their positions in the open market. Limits to arbitrage are set by transaction costs, limited supply of ESOs and competition among market makers.

Our data gives rise to a downward sloping implied volatility function. This

unusual shape occurs because prices of in-the-money options incorporate little time value. Semiparametric estimation of managerial preferences, or the marginal rate of substitution, suggests that the value of consumption is higher in states with low stock prices, consistent with concave utility functions. In fact, for a certain interval of stock prices, the relative risk aversion is estimated at just above one, indicating that the representative manager's preferences could be approximated by logarithmic utility. Further, our results suggest that log utility function, leading to constant relative risk aversion, provides a reasonable fit in the data.

In terms of optimal compensation, there is some theoretical support for using options if we accept that managerial preferences are approximated by log utility. Hemmer, Kim and Verrecchia (2000) use a principal-agent model to show that in the case of log utility, the optimal contract is linear in stock value, and if relative risk aversion is less than one, the optimal contract is convex in stock value. Also Aseff and Santos (2006) work with a principal-agent model and show that a the optimal contract involves a fixed salary and stock options, if one assumes log utility and some regularity conditions for the problem. Therefore, both the data and existing theory support the use of options in managerial compensation.

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A Hedging error definitions and option price sensitivities

A.1 Discrete and relative hedging errors

We're interested in European-style derivative security C that has delta θ_t . The price of C starts from C_0 and evolves as $dC_t = \theta_t dS_t$, where dS_t is the underlying stochastic process, with time index $0 \leq t \leq T$. Money market account is used as numeraire. Terminal value of the derivative security is $C_T = C_0 + \int_0^T \theta_u dS_u$. If the trader knew the underlying model⁹, his *true* hedging strategy in discrete time would be π and gains from trade $V^\pi = \sum_{t=0}^T \pi_t \Delta S_t$. But since he estimates implied volatility with error, the *actual* hedging strategy is $\tilde{\pi}$, producing gains from trade $V^{\tilde{\pi}} = \sum_{t=0}^T \tilde{\pi}_t \Delta S_t$. Referring to the true hedging strategy, let us define *discrete hedging error* as $L_1 = C_T - C_0 - V^\pi$. Referring to actual hedging strategy, define *relative hedging error* as $L_2 = V^\pi - V^{\tilde{\pi}} = \sum_{t=0}^T (\pi_t - \tilde{\pi}_t) \Delta S_t$.

In practice, the market maker buys executive stock option C at time t , and hedges his position using the imperfect strategy $\tilde{\pi}$. At terminal time T , when the option expires, his net gains from trade, equal to total hedging error are $C_T - C_t - V^{\tilde{\pi}} = (C_T - C_t - V^\pi) + (V^\pi - V^{\tilde{\pi}}) = L_1 + L_2$. When the hedging error is positive (resp. negative), the market maker makes a profit (resp. loss).

A.2 Call option sensitivities

Sensitivity to cumulative variance (C_Ξ): To get started, recall that the B-S call price sensitivity with respect to volatility is $C_\sigma = \frac{\partial C}{\partial \sigma} = S\varphi(d_1)\sqrt{T-t}$ (see Björk, 2004, p. 125). To proceed, write cumulative variance as $\Xi_t = \sigma^2(T-t) \Leftrightarrow \sigma = (+)\sqrt{\Xi_t/(T-t)}$. Formula C_Ξ for follows from the chain rule as

$$C_\Xi = \frac{\partial C}{\partial \sigma} \frac{\partial \sigma}{\partial \Xi_t} = \frac{S\varphi(d_1)}{2\sqrt{\Xi_t}}$$

where $\varphi(\cdot)$ denotes the standard normal density function.

The greek term ($C_{S\Xi}/C_\Xi$): Start by writing $\frac{C_{S\Xi}}{C_\Xi} = \frac{1}{C_\Xi} \frac{\partial}{\partial S} C_\Xi = \frac{\partial}{\partial S} \log C_\Xi$. Application of the above formula for C_Ξ , taking logs and differentiation with respect to S gives $\frac{\partial \log C_\Xi}{\partial S} = \frac{1}{S} - d_1 \frac{\partial d_1}{\partial S}$. Finally, using the definition of d_1 and some algebra gives the desired result being

$$\frac{C_{S\Xi}}{C_\Xi} = - \left[\frac{\log(S_t/K) + r_t - \Xi_t/2}{S_t \Xi_t} \right].$$

⁹In other words, he could predict implied B-S volatility exactly.

B Tables and figures

<i>Model</i>	<i>Portfolio restriction</i>	$\theta = 1$	$\theta = 0.5$	$\theta = 0.25$
A. Black-Scholes Eqs. (4.1)-(4.2)	volatility		0.4644	
	(s.e.)		(0.0036)	
	bias		0.0151	
	residual s.e.		0.4557	
	res. corr. w. vega		-0.2414	
	moment restriction (4.2)		0.0496	
B. Generalized Black-Scholes Eqs. (4.3)-(4.5)	volatility		given by Eq. (3.7)	
	(s.e.)		NA	
	bias		0.0037	
	residual s.e.		0.2911	
	res. corr. w. delta		-0.0017	
	moment restriction (4.5)		0.0029	
C. Black-Scholes-Ingersoll Eqs. (4.6)-(4.8)	subj. risk prem.	0.0484	0.0339	0.0511
	(s.e.)	(0.0020)	(0.0013)	(0.0019)
	volatility	0.5763	0.5428	0.5381
	(s.e.)	(0.0045)	(0.0034)	(0.0033)
	bias	0.0061	0.0076	0.0078
	residual s.e.	0.3179	0.3166	0.3167
	res. corr. w. vega	-0.2340	-0.2058	-0.1256
	moment restriction (4.7)	0.0045	-0.2363	-0.1215
	res. corr. w. rho	-0.0720	-0.0543	-0.1097
moment restriction (4.8)	-0.0098	-0.1573	-0.4912	

Table 1: NLS estimates of the option pricing parameters. Standard errors are calculated using heteroscedasticity and autocorrelation consistent (HAC) covariance matrices of White and Hansen type, as implemented in sandwich package of R language, see Zeileis (2004). The need for HAC covariances arises from BS and BSI residuals that return almost zero p-values in Ljung-Box autocorrelation test. Going forward, bias equals mean estimation error. Residual standard errors are the square roots of diagonal entries of covariance matrix. Moment restrictions are defined in Section 4.2. For the BSI model, parameter θ equals the weight of employer stock in the manager's portfolio, in excess of market weight. For other models θ has no impact.

Error quantile [ε_p]	Moneyness S/K	Option value	\bar{L}_1	$se(L_1)$ % of option value	\bar{L}_2	$se(L_2)$	Corr.
$p = 0.01$ [-0.0852]	0.91	0.3921	0.94	0.88	5.62	0.10	0.02
	1.00	0.3578	1.25	1.07	6.95	0.12	0.00
	1.11	0.3272	-1.24	1.31	7.84	0.14	-0.01
$p = 0.05$ [-0.0489]	0.91	0.3921	0.26	0.85	3.26	0.06	-0.02
	1.00	0.3578	1.60	1.04	4.03	0.07	0.00
	1.11	0.3272	1.10	1.29	4.66	0.08	-0.01
$p = 0.10$ [-0.0324]	0.91	0.3921	1.98	0.87	2.20	0.04	0.03
	1.00	0.3578	1.49	1.05	2.68	0.05	-0.03
	1.11	0.3272	-0.13	1.28	3.02	0.00	0.01
$p = 0.90$ [0.0321]	0.91	0.3921	-0.36	0.86	-2.41	0.04	0.00
	1.00	0.3578	-0.16	1.08	-2.87	0.05	0.00
	1.11	0.3272	-0.09	1.30	-3.26	0.06	0.00
$p = 0.95$ [0.0451]	0.91	0.3921	-0.51	0.84	-3.50	0.06	0.03
	1.00	0.3578	0.39	1.07	-4.12	0.06	0.00
	1.11	0.3272	0.43	1.28	-4.75	0.09	-0.04
$p = 0.99$ [0.0750]	0.91	0.3921	0.26	0.84	-5.86	0.10	-0.03
	1.00	0.3578	0.34	1.06	-7.10	0.12	0.02
	1.11	0.3272	1.01	1.24	-8.01	0.15	-0.03

Table 2: Simulation results of discrete (L_1) and relative (L_2) hedging error using 5,000 replications. (\bar{L}_1, \bar{L}_2) and $(se(L_1), se(L_2))$ denote the means and standard errors of L_1 and L_2 . The error quantiles (ε_p) correspond to residual distribution of the implied volatility function. All hedging errors are expressed as percentages of option values given in the third column. The rightmost column gives correlation of L_1 and L_2 .

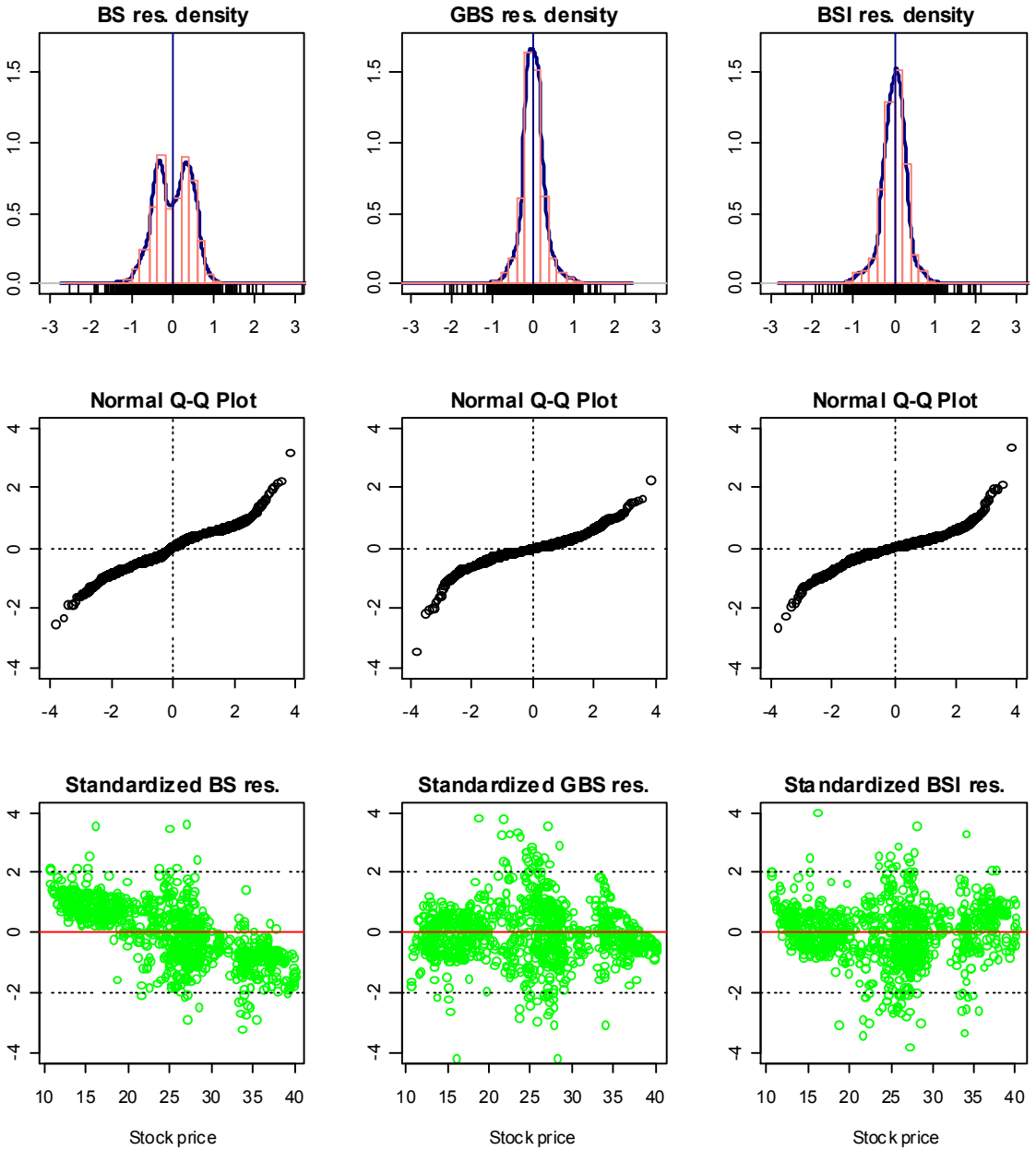


Figure B.1: Goodness of fit of the BS, GBS and BSI models. The first row plots estimated residual densities using identical scales. The second row provides Q-Q-plots against the normal distribution for the option price residuals. The third row gives scatter plots of standardized residuals, with 1000 points sampled from each series to keep the plots readable. Residual standard errors are reported in Table 1. The BSI fit assumes that $\theta = 0.5$.

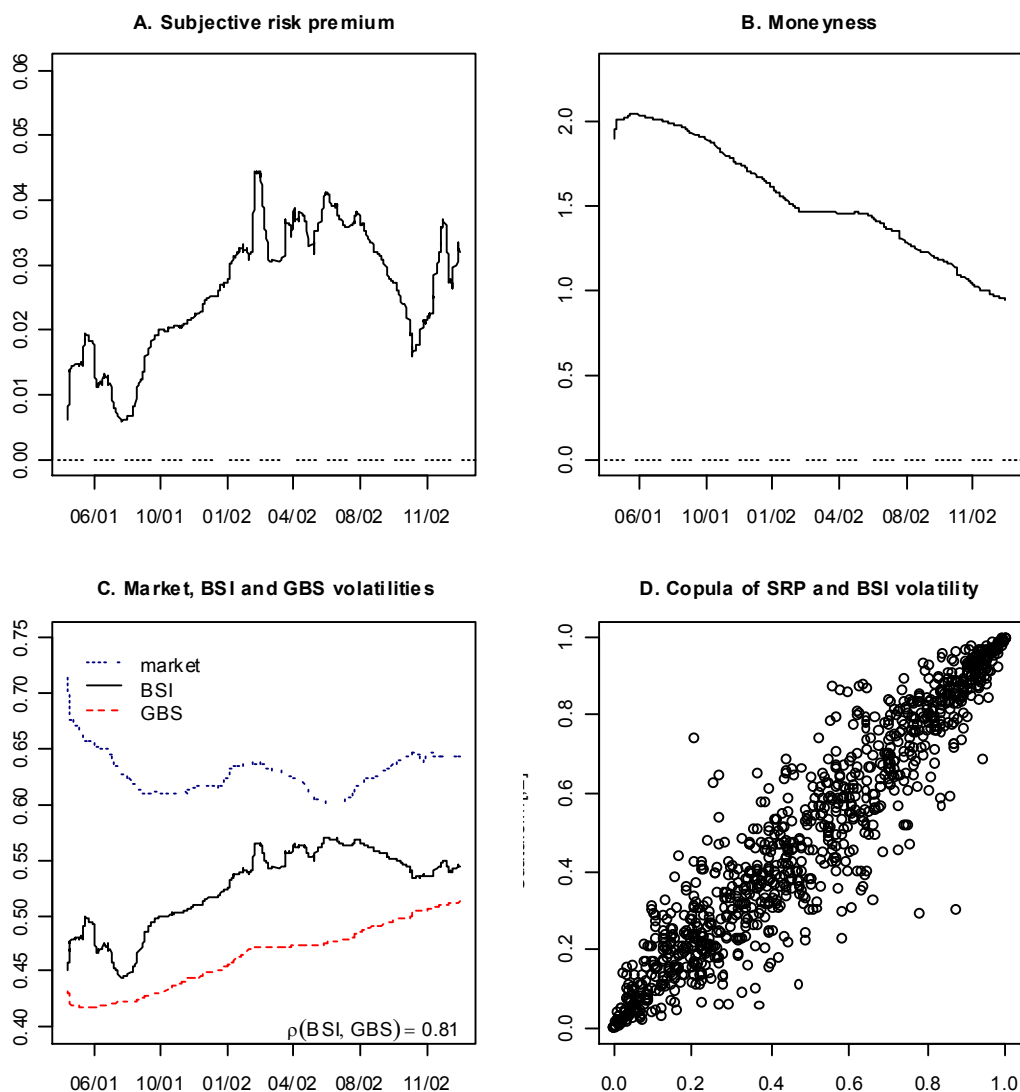


Figure B.2: Rolling estimates for SRP, moneyness and volatilities. A stratified sample of 2000 obs. is constructed by splitting the data to moneyness quartiles and sampling 500 obs. from each quartile. Consequently the BSI and GBS models are fitted using a rolling window of 600 observations. This window has average length of 173 days backwards from the x-axis date. Panel A gives the SRP, calculated as the market risk-free rate minus the subjective rate. Panel B plots average moneyness during each period. Panel C plots market volatility of ATM options, as well as estimated volatilities of BSI and GBS models. Finally, in panel D, the dependence of SRP and BSI volatility is illustrated by t copula with correlation of 0.94 and 3.6 df. For the BSI model, the portfolio restriction is set at $\theta = 0.5$.

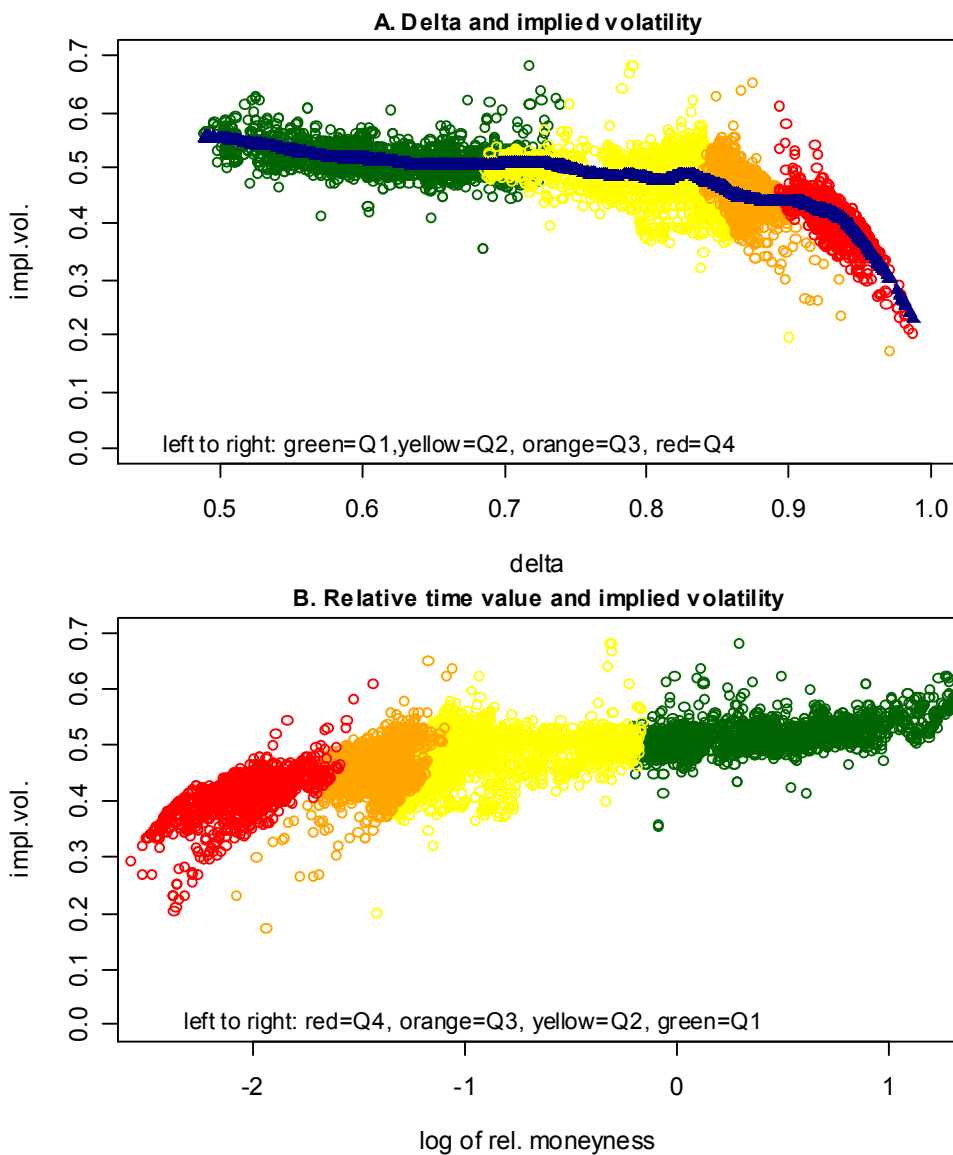


Figure B.3: Panel A is the scatter plot of delta and implied volatility. Going from left to right, the data is grouped by moneyness quartiles in increasing order. Clearly, the conclusion is that implied volatility decreases as delta increases. The black line in panel A draws the smooth volatility function. Panel B features the scatter plot of relative time value and implied volatility. Relative time value is calculated as log of time value divided by option price, that is $\log[(C - (S - K))/C]$. We use a logarithmic scale in order to make the increasing pattern more visible. The data are plotted in decreasing order of moneyness.

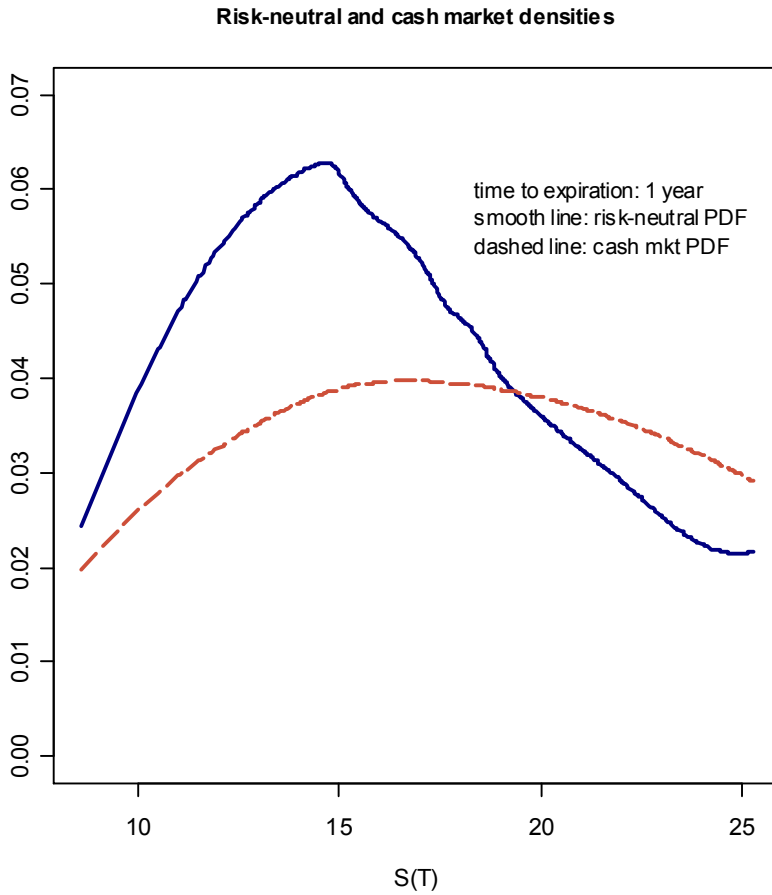


Figure B.4: Risk-neutral (smooth line) and cash market (dashed line) densities. The risk-neutral density is estimated from Nokia ESO trade data which contains 7610 trades in Nokia series 99 ESOs that vested in April 2001 and expired at the end of 2004. The cash market density is estimated using the bootstrap from daily returns of Nokia stock. Both densities have a time frame of one year.

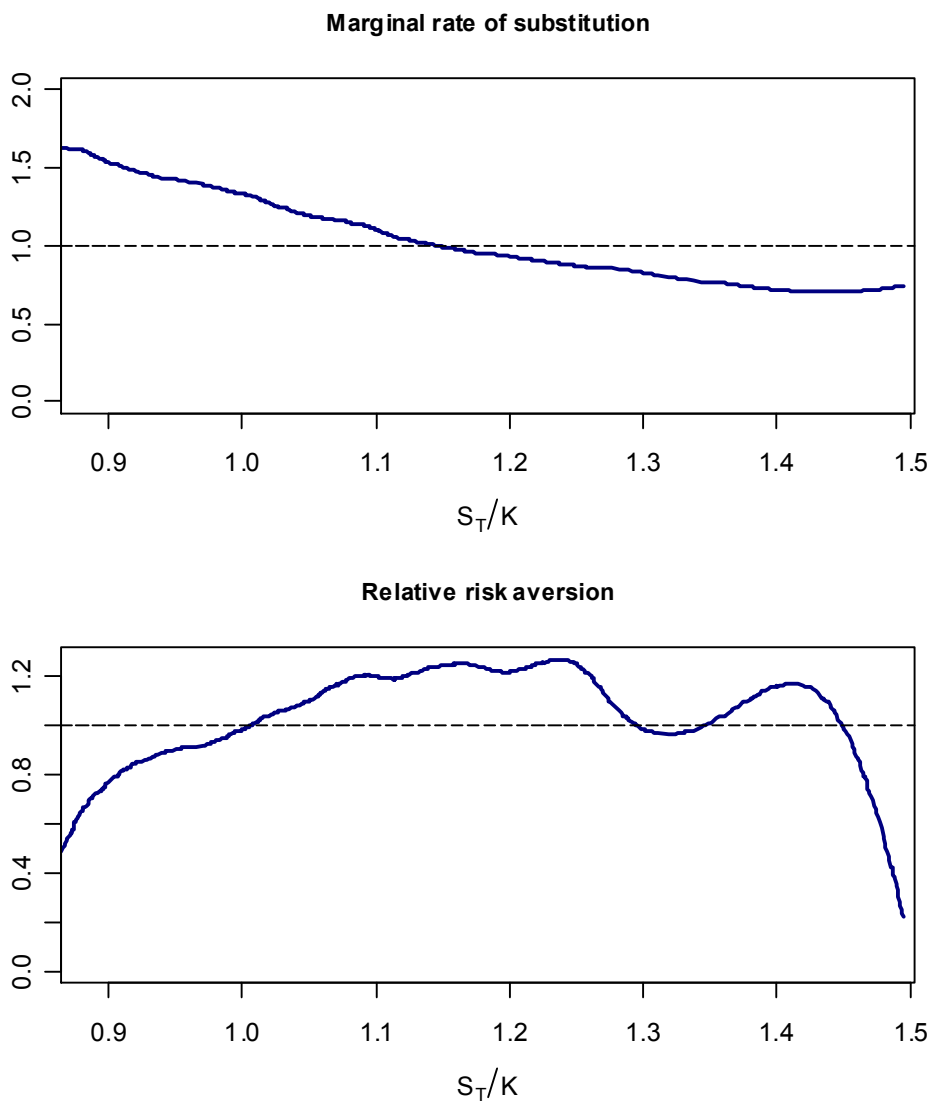


Figure B.5: The upper panel plots marginal rate of substitution and the lower panel plots relative risk aversion, plotted as a function of moneyness. The dashed line is drawn at $RRA=1$, equivalent to logarithmic utility function.

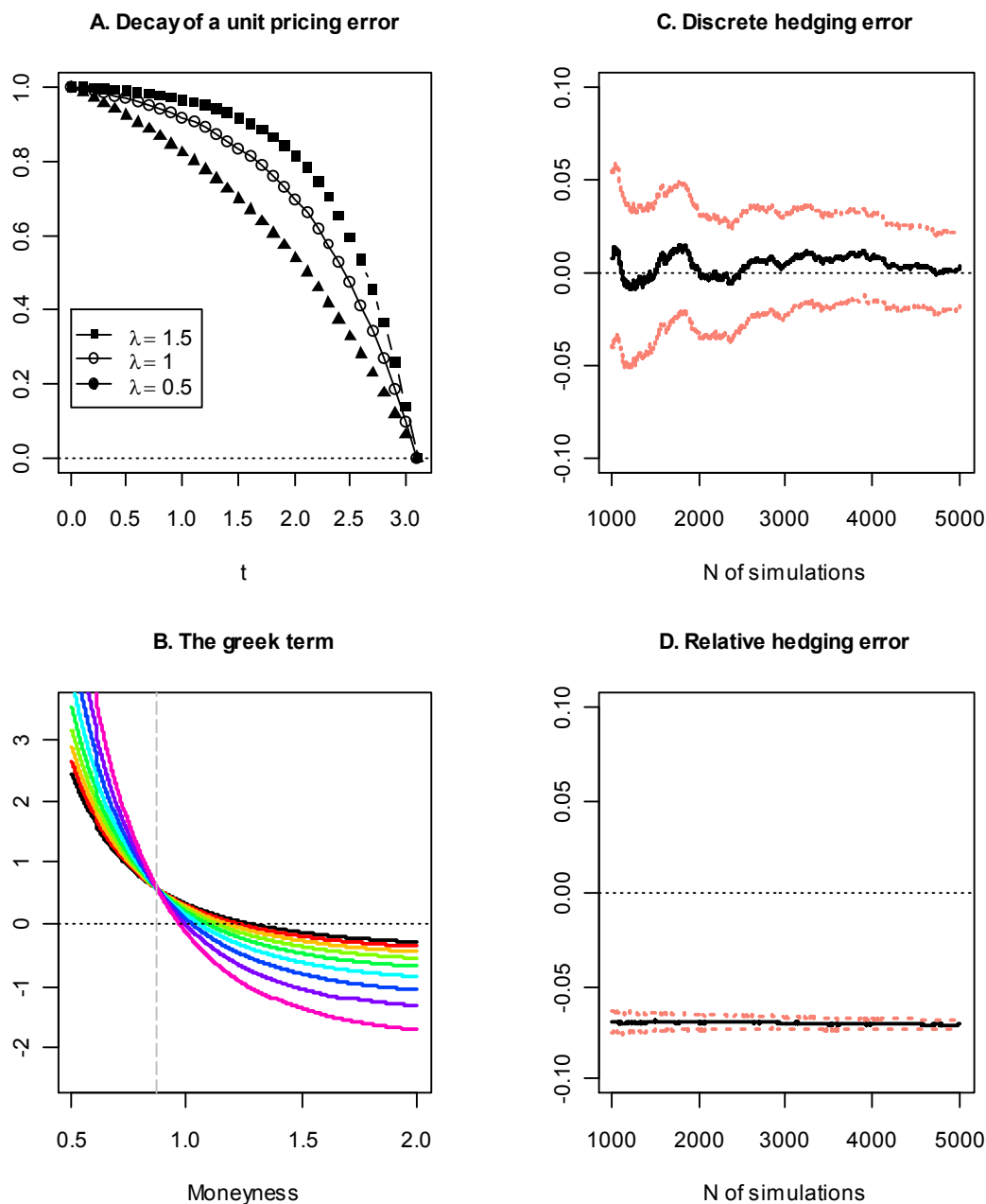


Figure B.6: Panel A shows the decay of a unit size pricing error. The curves are drawn using different values of λ . Panel B plots the greek term of Eq. (5.5) using different volatilities in the range $[0.26, 0.50]$. The curves coincide at forward at-the-money point $S^* = Ke^{-rt}$, marked by the dashed line. Panels C and D report the means of discrete and relative hedging error, depending on number of simulations. The plotted case is based on volatility function residual of $\varepsilon_{0.99} = 0.0750$, with details given on the bottom row of Table 2.

Empirical option pricing with copula-based correlation of stock returns and volatility

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Abstract

Correlation of stock returns and volatility is a key parameter in stochastic volatility models. An original method for computing this correlation is developed below. Under this method, a copula is fitted in pairwise data of option delta and volatility, and linear correlation is inferred from rank correlation that is measured by Kendall's tau. Fitness of the estimate can be evaluated using goodness of fit tests for copulas, based on bootstrap techniques. Also, an ANOVA procedure based on prediction error table is proposed for the purpose. The method is demonstrated using the Hull-White option pricing model and a smooth volatility function. Finally, in an empirical application it turns out that volatility and stock returns are negatively dependent, and both Gauss and t copulas provide acceptable dependence models.

Keywords: Copula, rank correlation, option pricing, stochastic volatility

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1 Introduction

Applications of stochastic volatility have become popular in the practice and theory of option pricing. The finance profession knows that a combination of flexible volatility function and Black-Scholes formula provides a close fit in most cases. However, such models are not always well-defined theoretically. Volatility is not a traded asset and therefore not priced by arbitrage, which renders the martingale measure non-unique. Also, it is difficult to find meaningful arbitrage price bounds (see e.g. Fouque et al. [12, Ch. 2.7]). Below we define arbitrage bounds using the results of Schweizer and Wissel [29]. For option pricing, we aim at a model that specifies the stochastic processes, yields a (non-unique) martingale measure, and accommodates non-Gaussian stock returns.

We calculate option prices using the Hull and White [19] model that features correlated stock returns and volatility. It belongs to the class of stochastic volatility models, as defined in the monograph of Fouque et al. [12]. The pricing principle is that risk-neutral expected return on traded assets equals risk-free rate, whatever the price of volatility risk. Specifically, we consider mean-reverting volatility with expected level given by smooth function of stock delta. Mean-reversion in volatility is also applied by Fouque et al. [13], however they have different asset dynamics, and the fitted volatility function is linear in log-moneyness. In contrast, we assume volatility is a smooth function of option delta. In this sense our model is similar to Bliss and Panigirtzoglou [3], who study risk preferences implied by option prices.

Although the idea of correlated stock price and volatility is intuitive, data requirements make it difficult to infer the correlation. Strictly speaking, one should estimate instantaneous correlation, or ρ in Eq. (2.1). In particular, simultaneous measurement of volatility and instantaneous returns is ambiguous and prone to errors. In contrast, because information vendors (e.g. Datastream) provide data on implied Black-Scholes delta and volatility, gathering data for the copula-based method is easy.

The solution proposed here is to infer linear correlation from rank correlation, which is a copula-based dependence measure. Rank correlation is measured by Kendall's tau. Main advantage of the method is that it allows to fit the copula in pairwise data of option delta and volatility. There is no need for instantaneous stock return data. This is enabled by the invariance property of copulas (McNeil et al. [23, Prop. 5.6]), saying that they are invariant with respect to strictly positive transformations of the margins. Option delta is an increasing function of the stock price in stochastic volatility context, implied by convexity of option prices shown by El Karoui et al. [10]. Therefore, the copula of delta and volatility is also the copula of stock price and volatility.

Fortunately, for Gauss and t copulas, (linear) correlation is linked to Kendall's tau. It follows that one can fit the copula in delta–volatility data and infer correlation for option pricing. For Archimedean copulas (Clayton and Gumbel), Kendall's tau is determined by the copula parameter, however there is no link to (linear) correlation. Reliability of the estimate can be tested by the copula's goodness of fit, with two tests applied below. First one is the Rosenblatt trans-

formation test, based on transforming marginal probabilities to normality, due to Breyman et al. [4] and Dobric and Schmid [8]. In fact, the same idea was earlier applied by Berkowitz [1] for testing the fit of univariate distributions. The second test is original and uses weighted ANOVA to check if the prediction errors form a white noise sequence. Mathematically, it is based on Donsker's theorem, but it can also be seen as a variant of stratified sampling. We also study large prediction errors by using a Kolmogorov-Smirnov statistic.

A brief outline of the paper follows. Chapter 2 provides tools for option pricing, with focus on stock return dynamics and the price of volatility risk. Chapter 3 formulates the copula-based method for inferring correlation and discusses goodness of fit. Chapter 4 provides a case study applying the computational tools in practical data. It consists of market prices of Nokia's executive stock options (ESO). The data was chosen based on the author's interests and earlier work. However, the methods advocated here apply generally to European-style options that may be quoted in terms of implied volatility. All computations of this paper have been implemented using the R language [26] available under public license.

2 Option pricing under stochastic volatility

2.1 When does an option complete an incomplete market?

It is known by now that stochastic volatility (SV) models are consistent with volatility smiles, or the empirical fact that implied volatilities co-move with moneyness (see e.g. §4.3 of Broadie and Detemple [5]). Looking at my ESO data, there emerge two properties aligned with stochastic volatility. First, implied volatility follows a certain pattern as function of delta, and second, it has a random character. Both properties are easily checked with Figure 1. Therefore, volatility is assumed to follow a diffusion process, and this diffusion may be correlated with the one driving the stock price. The SV model, where both stock return and variance follow geometric Brownian motion, can be traced back to Hull and White [19]. Uncertain variance causes market incompleteness in a standard setup with one risky and a risk-free asset.

However, it has been shown that an incomplete market may be completed by introducing a contingent claim on the risky asset (see Pham and Touzi [24], Romano and Touzi [27]). Specifically, subject to mild regularity conditions, Romano and Touzi show that a European option completes the market in a stochastic volatility model, if its arbitrage price is convex in the underlying asset price. Knowledge of market price is not required for the market to be completed; it suffices that the option pricing model is specified. Provided that the market is completed, one can analyse market equilibrium with a viable price system and a representative agent.

Moreover, with viable state prices, the agents optimally hold all securities supplied in the market. A key result of Pham and Touzi [24] says that the Hull-White model is consistent with log utility when stock return and volatility risks

are correlated. When the risks are independent, the model is consistent with power utility. In this context, consistency implies that the state price system is viable, and market clearing conditions are satisfied. The market clears when all securities are held and consumption equals the dividend on risky asset, subject to the assumption that the agent receives no other cash flows.

2.2 Stock dynamics with stochastic volatility

2.2.1 Objective dynamics

Under the objective (i.e. the statistical) probability measure \mathbf{P} , the dynamics of stock return and variance are defined by the following system. Assume that the stock is traded, but variance is not.

$$\begin{pmatrix} dS_t \\ dY_t \end{pmatrix} = \begin{pmatrix} \mu_t S_t \\ \eta_t Y_t \end{pmatrix} dt + \begin{bmatrix} \sqrt{Y_t} \sqrt{1 - \rho^2} S_t & \sqrt{Y_t} \rho S_t \\ 0 & \gamma_t Y_t \end{bmatrix} dW \quad (2.1)$$

System (2.1) employs standard notation. Stock price and variance at time t are denoted by S_t and Y_t , hence instantaneous volatility is $\sqrt{Y_t}$. State variable dynamics are driven by two-dimensional standard Wiener process $W := (W_S, W_Y)'$. (Note that W_S and W_Y are independent.) Stock price and variance drifts μ_t and η_t , as well as the diffusion parameter $\gamma_t > 0$ may be constants or deterministic functions, and the triplet $(\mu_t, \eta_t, \gamma_t)$ is adapted to filtration \mathcal{F}_t . My focus is in the case of mean-reverting volatility. The dynamics given by (2.1) imply that the (local) correlation of stock returns and variance equals ρ .

2.2.2 Risk-neutral dynamics

The risk-neutral dynamics corresponding to system (2.1) are determined following Romano and Touzi [27]. The pricing toolbox consists of a martingale measure $Q(\lambda)$ and market price of risk vector $\lambda := (\lambda_S, \lambda_Y)'$. Based on the Girsanov theorem, the P -Wiener process $W = (W_S, W_Y)'$ and $Q(\lambda)$ -Wiener process $\widetilde{W} := (\widetilde{W}_S, \widetilde{W}_Y)'$ are related by the Girsanov kernel $\varphi = -\lambda$ (see Theorem 11.3 of Björk [2]).

$$dW = \varphi dt + d\widetilde{W}$$

With this notation, the likelihood process $L := dQ(\lambda)/dP$ is calculated as $L_t = \exp \left[-\int_t^T \lambda' dW - \frac{1}{2} \int_t^T \|\lambda\|^2 dt \right]$. Following Def. 10.20 of Björk [2], the stochastic discount factor (SDF) is defined by Eq. (2.2). Risk-free rate is denoted r with time subscript suppressed, and r is assumed an adapted process.

$$M_t(\lambda) = e^{-r(T-t)} L_t = \exp \left[-\int_t^T \lambda' dW - \int_t^T \left(r + \frac{1}{2} \|\lambda\|^2 \right) dt \right] \quad (2.2)$$

Economic interpretation of the SDF is that it gives the time t risk-neutral value of one euro to be received at time T . As pointed out by Pham and Touzi [24] and Fouque et al. [12, Ch. 2.5], the market price of risk vector is unique,

but its components are not. When the market is completed by introducing the option, expected return under $Q(\lambda)$ equals the risk-free rate. In effect, this leaves one degree of freedom in Eq. (2.3), which has two unknown parameters λ_S and λ_Y .

$$\left(\lambda_S \sqrt{1 - \rho^2} + \lambda_Y \rho\right) \sqrt{Y_t} = \mu_t - r \quad (2.3)$$

Based on the above reasoning, we assume that a non-unique martingale measure exists. It can be called *minimal*, since the risk-neutral return on the stock equals the risk-free rate as long as Eq. (2.3) holds (cf. Hofmann et al. [18]). Recall that volatility is not priced by arbitrage, because it is not traded. An admissible price for the option is defined as the expected payoff under $Q(\lambda)$. In summary, risk-neutral dynamics of the risky asset and variance are given by the system (2.4).

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{Y_t} \left[\sqrt{1 - \rho^2} d\widetilde{W}_S + \rho d\widetilde{W}_Y \right] \\ \frac{dY_t}{Y_t} &= (\eta_t - \gamma_t \lambda_Y) dt + \gamma_t d\widetilde{W}_Y \end{aligned} \quad (2.4)$$

It is necessary to add two remarks. First, the current model is similar to Heston [17], with the difference that in Heston model variance follows a square-root process written as $dY_t = (\eta_t - \gamma_t \lambda_Y) Y_t dt + \gamma_t \sqrt{Y_t} d\widetilde{W}_Y$. In particular, definition of correlation is the same. Second, a semi-closed form solution is available in this setup. Romano and Touzi [27] derive a pricing formula assuming that the correlation ρ is constant, and the risk-neutral drift of variance $(\eta_t - \gamma_t \lambda_Y)$ is independent of stock price. While the first assumption is innocuous, the second one is not, since it does not allow mean reversion in volatility (the topic of Section 3.1).

2.2.3 Stock return distribution

Lemma 1 verifies that the stock return distribution is in the class of normal mean-variance mixtures, with the variance process playing the role of mixing variable. This is a flexible class of distributions that includes the hyperbolic and normal inverse gaussian (NIG) distributions (McNeil et al. [23, Ch. 3.2]). It follows that the current setup is consistent with skewed and kurtotic stock returns.

Lemma 1 *The risk-neutral distribution of stock returns implied by the system (2.4) is a normal mean-variance mixture. Denote stock return and variance as $X_T := \log(S_T/S_0)$ and $Y_T := T^{-1} \int_0^T Y_s ds$. In this notation, the mean and variance of returns are given by $E^Q(X_T) = [r - \frac{1}{2} E(Y_T)] T$ and $\text{var}(X_T) = E(Y_T) T + \frac{T^2}{4} \text{var}(Y_T)$. Further, conditional on realised variance, returns are normally distributed.*

Proof. According to Section 3.2.2 of McNeil et al. [23], random variable $X \in \mathcal{R}^d$ follows a normal mean-variance mixture distribution, if it can be written as

$$X \stackrel{d}{=} m(W) + \sqrt{W}AZ \quad (2.5)$$

where m is the mean function, W is a non-negative scalar r.v. known as mixing variable, A is a positive definite $d \times k$ matrix, and Z is a k -vector of standard normal variates. First note that the stock dynamics in the system (2.4) imply the following return process:

$$X_T = \left(r - \frac{1}{2} Y_T \right) T + \sqrt{Y_T T} \left(\sqrt{1 - \rho^2} \widetilde{W}_S + \rho \widetilde{W}_Y \right)$$

where \widetilde{W}_S and \widetilde{W}_Y are independent standard normal variates (time scaling is taken care by \sqrt{T}).

Now define cumulative variance as the mixing variable; $W := Y_T T$. As required, this is a non-negative r.v. For other parameters, choose $m = \left(r - \frac{1}{2} Y_T \right) T$, $A_{1 \times 2} = \left(\sqrt{1 - \rho^2}, \rho \right)$ and finally $Z_{2 \times 1} = \left(\widetilde{W}_S, \widetilde{W}_Y \right)'$. Using this parametrisation, the stock dynamics comply with the definition (2.5) of normal mean-variance mixture distribution. Following McNeil et al., the mean and variance of X_T are calculated as $E^Q(X_T) = \left(r - \frac{1}{2} E^Q(Y_T) \right) T$ and $\text{var}(X_T) = E^Q(\text{var}(X_T | W)) + \text{var}(E^Q(X_T | W)) = E(Y_T) T + \frac{T^2}{4} \text{var}(Y_T)$. Conditional on realised variance y_T , returns are distributed as $X_{T, (Y=y_T)} \sim N\left(\left(r - \frac{1}{2} y_T\right) T, y_T T\right)$. ■

2.3 Volatility model and implied price of risk

For volatility modelling I use a smooth function of option delta. This specification is due to Bliss and Panigirtzoglou [3]. Conveniently, delta is strictly increasing in the stock price, and it summarizes the option properties in one number. Equation (2.6) gives the volatility model, denoting option delta and implied volatility by x and y , and the knots (anchor points of the splines) by κ_j . Smoothness is induced by the *cubic thin plate splines* $|x - \kappa_j|^3$. The model is unbiased when $E(y|x) = f(x)$, and if this is true, the residuals are Gaussian. Estimation is done by restricted maximum likelihood¹; properties of the estimator are worked out in Ruppert et al. [28, Ch. 3-4]. The fit, along with the data, are plotted in Figure 1.

$$\begin{aligned} f(x) &= \beta_0 + \beta_1 x + \sum_{j=1}^J \beta_{1j} |x - \kappa_j|^3 + \varepsilon \\ \varepsilon &\sim N(0, \sigma_\varepsilon^2) \end{aligned} \quad (2.6)$$

Proposition 1 characterises the price of volatility risk (λ_Y) under mean-reverting volatility. In terms of data, expected volatility and delta are related by the above model (2.6). The baseline parametrization, or zero expected price of volatility risk, corresponds to option-implied volatility being an unbiased predictor of volatility. The case of non-zero expectation is treated in Remark 1, but simulations below use the baseline.

Proposition 1 *Assume that an unbiased volatility function $\sigma = f(\Delta) + \varepsilon$ has been fitted in empirical data, and risk-neutral dynamics of stock and volatility*

¹For software, I use the R language (R Development Core Team [26]) and *SemiPar* package.

are specified by the system (2.4). Further, a trader prices options using mean reversion in volatility, i.e. volatility drift (per unit of time) is given by

$$\frac{1}{2} \left(\eta_t - \gamma_t \lambda_Y - \frac{1}{4} \gamma_t^2 \right) = a(\sigma^* - \sigma) \quad (2.7)$$

where a is the speed of mean reversion. Moreover, expected and actual implied volatilities are denoted $\sigma^* := E^{\mathbf{Q}(\lambda)}[\sigma | \Delta]$ and σ . This setup allows a **general parameterization** $(\eta_t, \gamma_t, a) \in \mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+$ consistent with nonzero expected price of volatility risk;

$$\begin{aligned} E^{\mathbf{Q}(\lambda)}[\lambda_Y] &= \frac{1}{\gamma_t} \left(\eta_t - \frac{1}{4} \gamma_t^2 \right) \\ \text{var}(\lambda_Y) &= \frac{4a^2 \sigma_\varepsilon^2}{\gamma_t^2 Y_t} \end{aligned}$$

as well as a **baseline parameterization** $(\eta_t = \frac{1}{4} \gamma_t^2, \gamma_t = 2a)$ consistent with zero expected price of volatility risk:

$$\begin{aligned} E_t^{\mathbf{Q}(\lambda)}[\lambda_Y] &= 0 \\ \text{var}(\lambda_Y) &= \frac{\sigma_\varepsilon^2}{Y_t}. \end{aligned}$$

Proof. Expected value of λ_Y . Since the system (2.4) is defined for variance, but the observed variable is volatility, it is necessary to calculate a stochastic differential for volatility $V_t := \sqrt{Y_t}$ using Ito's Lemma (Björk [2, Prop. 4.11]). Starting from the second line of (2.4), tedious but straightforward calculations yield the SDE (2.8) for volatility.

$$\frac{dV_t}{V_t} = \frac{1}{2} \left[\left(\eta_t - \gamma_t \lambda_Y - \frac{1}{4} \gamma_t^2 \right) dt + \gamma_t d\widetilde{W}_Y \right] \quad (2.8)$$

If the implied volatility function (IVF) is correctly specified, actual volatility σ will converge to the predicted level σ^* at speed a . As a result, theoretical and empirical volatility dynamics are coupled by Eq. (2.9).

$$\frac{1}{2} \left(\eta_t - \gamma_t \lambda_Y - \frac{1}{4} \gamma_t^2 \right) dt = a(\sigma^* - \sigma) dt \quad (2.9)$$

Further, unbiasedness of the IVF implies zero expected value for the residual; $E^{\mathbf{Q}(\lambda)}[a(\sigma^* - \sigma)] = aE^{\mathbf{Q}(\lambda)}[\varepsilon] = 0$. Note that the expectation is taken under $\mathbf{Q}(\lambda)$, because the IVF is fitted in option prices. Taking expectations of both sides of Eq. (2.7) results in $E^{\mathbf{Q}(\lambda)}[\lambda_Y] = \frac{1}{\gamma_t} \left(\eta_t - \frac{1}{4} \gamma_t^2 \right)$, because η_t and γ_t are either constants or deterministic functions.

Variance of λ_Y . (i) Assume that the volatility function residuals have constant variance; it follows that $E^{\mathbf{Q}(\lambda)}[a(\sigma^* - \sigma)]^2 = a^2 E^{\mathbf{Q}(\lambda)}[\varepsilon^2] = a^2 \sigma_\varepsilon^2$.

(ii) Take expectations of the squared left-hand side of Eq. (2.7) and calculate the variance as follows.

$$\begin{aligned} E^{\mathbf{Q}(\lambda)} \left[\frac{1}{2} \sqrt{Y_t} \left(\eta_t - \gamma_t \lambda_Y - \frac{1}{4} \gamma_t^2 \right) \right]^2 &= \frac{1}{4} Y_t \gamma_t^2 E^{\mathbf{Q}(\lambda)} \left[\frac{1}{\gamma_t} \left(\eta_t - \frac{1}{4} \gamma_t^2 \right) - \lambda_Y \right]^2 \\ &= \frac{1}{4} Y_t \gamma_t^2 E^{\mathbf{Q}(\lambda)} \left[E^{\mathbf{Q}(\lambda)} (\lambda_Y) - \lambda_Y \right]^2 = \frac{1}{4} Y_t \gamma_t^2 \text{var}(\lambda_Y) \end{aligned}$$

The desired outcome follows from combining the results (i-ii); $\frac{1}{4} Y_t \gamma_t^2 \text{var}(\lambda_Y) = a^2 \sigma_\epsilon^2$ implies $\text{var}(\lambda_Y) = \frac{4a^2 \sigma_\epsilon^2}{\gamma_t^2 Y_t}$. ■

Remark 1 One may also assume that expected price of volatility risk is some constant c , i.e. that $E^{\mathbf{Q}(\lambda)}[\lambda_Y] = c$. This would be similar to Fouque et al. [13], where λ_Y is fixed (Assumption 2.1). It follows from the proof of Proposition 1 that in this case parameters γ_t and η_t are related as $\gamma_t = 2 \left(-c \pm \sqrt{c^2 - \eta_t} \right)$.

Remark 2 Note that the volatility dynamics with zero expected price of volatility risk, implied by Eqs. (2.8) and (2.9), became

$$V_t = V_{t-\Delta t} \exp \left[\left(a(\sigma^* - \sigma) - \frac{1}{4} \gamma_t^2 \right) \Delta t + \frac{1}{2} \gamma_t \widetilde{W}_Y \right]$$

where $\widetilde{W}_Y \sim N(0, \Delta t)$. Moreover, expected volatility is given by $E^{\mathbf{Q}(\lambda)}[V_t] = V_{t-\Delta t} e^{a(\sigma^* - \sigma)\Delta t}$, which implies mean reversion to level σ^* at speed a .

2.4 Schweizer-Wissel bound for implied volatility

In a recent paper, Schweizer and Wissel [29] provide conditions for arbitrage-free implied volatility, when stock returns and volatility are correlated. They assume a that a family of options, with different maturities but identical strikes, are traded along with a risk-free asset and the underlying stock. In this context absence of arbitrage requires that call prices increase with maturity, in other words all options must have positive forward implied maturities. In particular, Theorem 3.4 of Schweizer and Wissel applies to a European call option, saying that the drift conditions reduce to inequalities (2.10)–(2.11), where x denotes implied forward variance, defined as $x = \frac{\partial}{\partial T} [(T-t)\sigma_{t,T}^2]$, and $w > \epsilon > 0$ are small constants. Notation $\sigma_{t,T}^2$ means that the conditions apply to all options (with equal strike) that expire during the period $[t, T]$. c is a constant that bounds of the market price of risk vector i.e. $|\lambda_j| < c$ with $j = S, Y$. Therefore the first condition (2.10) is of practical interest, whereas the second condition (2.11) is purely technical. In contrast to the SW model I assume the risk-free

rate to be positive².

$$x - \frac{1}{4} \log^2 \left(\frac{S}{K e^{-rT}} \right) \gamma_t^2 \geq x\epsilon \quad (2.10)$$

$$|\gamma_t| \leq c \frac{1}{w + (1 + \sqrt{x})(1 + |\log S|)} \quad (2.11)$$

Remark that SW assume in their Section 3.3 (cf. Eq. (3.23)) that implied variance follows geometric Brownian motion, similar to both S and Y dynamics in the system (2.1). As hinted above, the first condition (2.10) has a practical interpretation. By taking the limit $\epsilon \rightarrow 0$ I get the lower bound (2.12) for implied volatility. It provides a necessary condition for positive implied volatility and also for absence of arbitrage.

$$\sigma_{t,T} \geq \sqrt{\frac{1}{4} \log^2 \left(\frac{S}{K e^{-rT}} \right) \gamma_t^2} \quad (2.12)$$

Behavior of the Schweizer-Wissel volatility bound (2.12) is illustrated in Figure 1. It has a palm-tree-like concave shape, and higher volatility of volatility values compare to upper leaves of the palm. Further, the slope of volatility stays positive even for high values of moneyness.

2.5 Monte Carlo method for option pricing

The simulation method for calculating option prices under mean-reversion in volatility is given below. Following Hull and White [19], antithetic variates are used for efficiency improvement. In practice, the standard normal distribution is sampled for a pair of outcomes (v, u) , and further analysis uses four different pairs of $(\pm v, \pm u)$. The option value is calculated by averaging over the four pairs.

The dynamics of stock price and volatility are determined as in system (2.4) and Proposition 1. Note that the simulated volatility path is implied by the V dynamics (2.8)-(2.9). Further, simulation parameters are chosen according to the baseline parameterization ($\eta_t = \frac{1}{4}\gamma_t^2, \gamma_t = 2a$). In fact, the variance drift η_t is not required for actual calculations, based on Remark 2. Note that the expected volatility σ^* in Eq. (2.14) may depend on option delta and therefore on stock price.

When the stock price and volatility processes are correlated, the distribution of S_T depends on the volatility path, and option prices become path-dependent. Hence, it is necessary to simulate both the stock price and volatility paths, which is done using the following three-step procedure. Time is indexed as $i = 0, 1, \dots, n$. One year consists of 250 trading days, i.e. $n = 250$ for one year maturity (for three years $n = 750$). Remark that the drift parameter η_t does not appear in simulation formulas (2.13)-(2.14); this follows from Eq. (2.9).

²Equation (2.10) is the equivalent of Eq. (3.28) of Schweizer and Wissel [29], when risk-free rate is not zero. This can be verified by replacing K with its present value $\tilde{K} = K e^{-rT}$ in Eq. (2.29) of SW and in the Black-Scholes formula.

1. Generate n standard normal variates $\tilde{z}_{1,i}$ and $\tilde{z}_{2,i}$. Then simulate the paths of stock price and variance using Eqs. (2.13)-(2.14). They follow from the risk-neutral stock and variance dynamics given by the system (2.4).

$$S_i = S_{i-1} \exp \left[(r - Y_{i-1}/2) \Delta t + \left(\sqrt{1 - \rho^2} \tilde{z}_{1,i} + \rho \tilde{z}_{2,i} \right) \sqrt{Y_{i-1} \Delta t} \right] \quad (2.13)$$

$$V_i = V_{i-1} \exp \left[\left(a(\sigma^* - \sigma) - \frac{1}{4} \gamma_t^2 \right) \Delta t + \frac{1}{2} \gamma_t \tilde{z}_{2,i} \sqrt{\Delta t} \right] \quad (2.14)$$

2. Proceed by calculating the raw option price $C = e^{-R} (S_T - K)^+$ for each case. S_T and R denote final stock price and the cumulative interest rate. The result is a vector of raw simulated prices with length n .
3. Employ antithetic variates, i.e. repeat steps 1–2 by replacing \tilde{z}_1 and \tilde{z}_2 sequentially with $-\tilde{z}_1$ and $-\tilde{z}_2$. Further, calculate the antithetic option price as $\tilde{C} = (C_{z_1, z_2} + C_{-z_1, z_2} + C_{z_1, -z_2} + C_{-z_1, -z_2})/4$ and standard error as $\text{se}(\tilde{C}) = \sqrt{\text{var}(\tilde{C})/n}$. Length of the vector \tilde{C} will be $n + 1$.

3 Copula-based correlation modelling

This chapter takes as objective to infer the correlation of stock returns and variance processes using the copula of implied volatility and delta. The modeling approach is mathematically legitimized by the invariance property that justifies using delta instead of stock returns. Moreover, invariance implies that one can use Black-Scholes delta and volatility as proxies of true parameters from the stochastic volatility model. Practically speaking, it is beneficial to replace stock returns with delta, because simultaneous quotes of delta and volatility are available from market data vendors. The copula-based method is applicable for any model that involves the correlation of volatility and stock price processes, like stochastic volatility models of Hull and White [19] and Heston [17]. The following subsections discuss first general properties of option prices, and then move on to copulas and goodness of fit.

3.1 Joint distribution of implied volatility and delta

In my empirical setting volatility is estimated as a smooth function of option delta. Moreover, delta and stock price have the same copula by the invariance property defined in Section 4.2. Having the same copula turns out to be useful in estimating the correlation of stock price and volatility. El Karoui et al. [10] prove that delta is a strictly increasing function of stock price. Their Theorem 5.2 posits that the European option value is convex in the underlying stock price. In fact, convexity extends to American options, if the risk-free rate is positive and the payoff function is bounded from below (see Corollary 9.5). Because the

option price is convex ($\frac{\partial^2 C}{\partial S^2} > 0$), the delta ($\frac{\partial C}{\partial S}$) must be positive and strictly increasing in S .

3.2 Key properties of copulas

The notion of copula was launched in 1959 by Abe Sklar, who reminisces the "birth of copula" in Sklar [30]. To define a copula, assume that continuous distribution functions for delta (Δ) and volatility function residuals (ε) are given by $F_1(\Delta)$ and $F_2(\varepsilon)$, respectively. Then a unique copula $C : [0, 1]^2 \rightarrow [0, 1]$ is defined as the joint distribution function of probabilities and denoted by $C(F_1(\Delta), F_2(\varepsilon))$. Further, marginal probabilities $u_1 = F_1(\Delta)$ and $u_2 = F_2(\varepsilon)$ are uniformly distributed; u_1 and u_2 follow $U(0, 1)$. This property, known as Rosenblatt's transformation, will be used in Section 4.3 for goodness of fit testing. Candidates for the dependence model consist of four copulas, being Gaussian, t, Clayton and Gumbel, whose basic properties are summarized in Table 1.

In the copula context, I will use Kendall's tau for dependence measure, motivated by the fact that rank correlations (Kendall's tau and Spearman's rho) are invariant under strictly increasing transformations that need not be linear. As shown in McNeil et al. [23, Prop. 5.37], for Gauss and t copulas, linear correlation (ρ) and Kendall's tau (ρ_τ) are related by Eq. (3.1). This formula can also be applied for calibration.

$$\rho_\tau = \frac{2}{\pi} \arcsin \rho \quad (3.1)$$

Unfortunately, no such formula exists for Archimedean copula family, represented here by Clayton and Gumbel copulas. However, according to Genest and Rivest [14], Kendall's tau is related to the copula parameter θ by Eqs. (3.2)-(3.3), and they also show these formulas can be applied for fast calibration of Archimedean copulas.

$$\text{Clayton: } \rho_\tau = \theta/(\theta + 2) \quad (3.2)$$

$$\text{Gumbel: } \rho_\tau = 1 - 1/\theta \quad (3.3)$$

The *invariance property* is crucial, since it implies that the copula of delta and volatility is also the copula of stock price and volatility. In formal terms, a copula is invariant with respect to strictly increasing transformations of the arguments. What this means is highlighted below. Assume a strictly increasing function $y = G(x)$ and denote its (cumulative) distribution by $\tilde{F}(y) = \Pr(G(x) \leq y) = \Pr(x \leq G^{-1}(y))$. Applying the identity $x = G^{-1} \circ G(x)$ for x , where $G^{-1} \circ G$ denotes composition of G and G^{-1} , yields Eq. (3.4).

$$\tilde{F}(G) = \Pr(G^{-1} \circ G(x) \leq G^{-1}(y)) = F \circ G^{-1}(y) \quad (3.4)$$

Now Sklar's Theorem (McNeil et al. [23, Th. 5.3]) implies that a copula is invariant to changes in marginal distributions, formally written as $C(u_1, u_2) =$

$\Pr(F_1(x_1) \leq u_1, F_2(x_2) \leq u_2)$, which equals $\Pr(\tilde{F}_1(G(x_1)) \leq u_1, \tilde{F}_2(G(x_2)) \leq u_2)$.

Fortunately, Eq. (3.4) allows to simplify the previous probability to $\tilde{F}_i \circ G(x_i) = F_i \circ G^{-1} \circ G(x_i) = F_i(x_i)$ for $i = 1, 2$. As a result, the invariance property is verified.

3.3 Goodness of fit

In order to find the copula that fits best in the current data, we use the Rosenblatt transformation test (RTT), as well as an ANOVA test for prediction errors original to this paper. As a third issue, a Kolmogorov-Smirnov statistic is applied to evaluate the frequency of large prediction errors. RTT was introduced in the copula context by Breymann et al. [4] and modified by Dobric and Schmid [8], whose bootstrap version is implemented below. Their insight was that when empirical distribution functions are used to estimate copula parameters, distribution of the test statistic must be computed by resampling.

3.3.1 Rosenblatt transformation test

The null hypothesis of RTT is that the copula C_θ , using parameters θ , agrees with actual joint probabilities, wherefore the marginal probabilities are $U(0, 1)$ distributed. Now consider C_θ and marginal probabilities $u_1 = F_1(x_1)$ and $u_2 = C_\theta(F_2(x_2) | F_1(x_1))$. The Rosenblatt transformation implies that $\Phi^{-1}(u_1)$ and $\Phi^{-1}(u_2)$ became independent standard normal variates. It follows that the sum of squares $S = [\Phi^{-1}(u_1)]^2 + [\Phi^{-1}(u_2)]^2$ is chi-square distributed with two degrees of freedom under the RTT null hypothesis (3.5).

$$H_0 : S(u_1, u_2) \sim \chi_2^2 \text{ where } S(u_1, u_2) = [\Phi^{-1}(u_1)]^2 + [\Phi^{-1}(u_2)]^2 \quad (3.5)$$

Given a sample of n observations, the null is tried using the Anderson-Darling (AD) statistic (3.6). The notation $S_{(i)}$ calls for sorting S in ascending order; viz. $S_{(1)} \leq \dots \leq S_{(n)}$. Also, $F_0(\cdot)$ is the chi-square distribution function with two degrees of freedom.

$$AD = -n - \frac{1}{n} \sum_{i=1}^n \{(2n-1) \log F_0(S_{(i)}) + \log [1 - F_0(S_{(n-i+1)})]\} \quad (3.6)$$

Following Dobric and Schmid [8], a parametric bootstrap is used to find out the distribution of the AD statistic (3.6) under the null hypothesis. The first step is to estimate distribution of the test statistic by bootstrapping 2,000 runs of 1,000 observations from the estimated copula C_θ , and then refit the same copula by estimating Kendall's tau, which yields C_θ^* . Next, calculate the AD statistic for C_θ^* , which yields the bootstrap distribution F_{AD^*} for AD_i^* where i runs from 1 to 2000. In the second step, a similar bootstrap is run by resampling 2,000 runs of 1,000 observations from the empirical copula (i.e. actual data), and calculating the AD statistic using the estimated copula C_θ . If the average AD for C_θ is denoted simply by \bar{x} , p-value for the test is given by $1 - F_{AD^*}(\bar{x})$.

3.3.2 ANOVA test for prediction errors

The objective of this test is to find out if the fitted copula converges in distribution³ (i.e. converges weakly) to the empirical one, denoted by " \rightsquigarrow ". Naturally, the empirical copula will be taken as representation of the true one. This is made legitimate by Theorem 3 of Fermanian et al. [11]. It proves that the empirical and true copula converge in distribution, if the latter has continuous margins and partial derivatives.

To proceed, let us define formally *empirical process*. For some distribution function F , the empirical process is given by $\mathbb{F}_n = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}}$. Of particular interest is the case where an estimated copula $C_\theta(F_1, F_2)$ implies marginal distributions (F_1, F_2) , and \mathbb{F}_n refers to the empirical process of either F_1 or F_2 . In order to establish the null hypothesis, we recall *Donsker's theorem* from van der Vaart [31, Th. 19.3]. It says that if X_1, X_2, \dots are random variables from distribution function F , then the empirical process converges weakly to a Gaussian process, in particular a Brownian bridge \mathbb{G}_F . Employing standard notation, Donsker's theorem is stated as

$$\sqrt{n}(\mathbb{F}_n - F) \rightsquigarrow \mathbb{G}_F. \quad (3.7)$$

The Brownian bridge \mathbb{G}_F is a Gaussian process with zero mean and covariance function $E\mathbb{G}_F(s)\mathbb{G}_F(t) = F(\min(s, t)) - F(s)F(t)$. It takes value zero at the extreme points ∞ and $-\infty$. Moreover, it can be written as function of a standard Brownian motion $\mathbb{B}(t)$ as $\mathbb{G}_F(t) = \mathbb{B}(t) - tZ$, where Z is a standard normal variate. Formally, our null hypothesis is that the parametric copula $C_{\theta, n}$ converges to the true (i.e. empirical) copula C .

$$H_0 : C_{\theta, n} \rightsquigarrow C \quad (3.8)$$

If the above null is true, also the marginal distributions converge. This follows from Lemma 21.2 of van der Vaart [31], saying that (weak) convergence of the distribution function process implies (weak) convergence of the quantile transformation. Given that the margins of a copula are quantile transformations, we can apply the lemma and write the null in terms of the margins as

$$H_0 : F_{\theta, n} \rightsquigarrow F.$$

For testing purposes, note that Eq. (3.7) represents prediction errors of a marginal distribution of a copula. If the fit is good, prediction errors follow a zero-mean Gaussian distribution. In practice, this is tested by following three-step procedure. First, marginal probabilities are simulated from the empirical copula; $(u_1, u_2) = C^{-1}(\pi)$. Accordingly (u_1, u_2) becomes an $n \times 2$ matrix. In the second step, the first one is repeated for the estimated copula; $(\hat{u}_1, \hat{u}_2) =$

³We say that random variable X converges in distribution to X_n if $\lim_{n \rightarrow \infty} E[f(X_n)] = E[f(X)]$, where f is a bounded and continuous function (Jacod and Protter [21, Th. 18.1]). This is satisfied by copula margins, since they are continuous by assumption and also bounded as maps from $[0, 1]$ to $[0, 1]^2$.

$C_\theta^{-1}(\pi)$. In the third step, actual and predicted probabilities are classified in two-way frequency tables. Finally, the null is tested by fitting the analysis of variance (ANOVA) model (3.9) in $I \times J$ table of prediction errors. They are denoted by f_{ij} , with α_i and β_j corresponding to row and column effects. Given the parameter restrictions, the number of estimated parameters is $I + J - 1$.

$$\begin{aligned} f_{ij} &= \mu + \alpha_i + \beta_j + \varepsilon_{ij} & (3.9) \\ (i, j) &= (1, 1) \dots (I, J); \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2) \\ \sum_{i=1}^I \alpha_i &= \sum_{j=1}^J \beta_j = 0 \end{aligned}$$

Eq. (3.9) is fitted using weighted least squares, with the weights given by actual cell probabilities (see Table 5). Under the null hypothesis f_{ij} are white noise, so both row and column effects should be insignificant. This is tested using a standard F statistic. Demonstration of the test is given in Section 5.4. In fact, the ANOVA procedure corresponds to stratified sampling, using marginal probabilities as stratification variables as in Glasserman [15, Ch. 4.3]. Moreover, his analysis shows that the prediction error variance can be estimated using Eq. (3.10), where p_{ij} and σ_{ij}^2 denote the the probability and variance of cell (i, j) .

$$\text{var}(f_{ij}) = \sum_{i,j} p_{ij} \sigma_{ij}^2 \quad (3.10)$$

The ANOVA approach yields several benefits. It is compatible with prediction errors (technically cell counts) that are close or equal to zero. In particular, they would be problematic for the chi-square test for goodness of fit (cf. van der Vaart [31, Ch. 17.5]). Also, comparison of actual and predicted frequencies reveals the areas in the unit square where the fit is compromised. Such information cannot be produced by tests like the RTT that only provide a p-value.

3.3.3 Kolmogorov-Smirnov statistic for copula margins

For risk management purposes, it is important to know the quality of copula fit in the tails. This can be checked using the Dvoretzky-Kiefer-Wolfowitz [9] inequality that gives tail probabilities for a Kolmogorov-Smirnov statistic, i.e. the distance $\sup_x |F_{\theta,n}(x) - F(x)|$. Specifically, we use Massart's [22] refinement of the DKW inequality given by Eq. (3.11), where $\psi > 0$ is a parameter.

$$\Pr \left(\sqrt{n} \sup_{x \in [0,1]} |F_{\theta,n}(x) - F(x)| > \psi \right) \leq 2 \exp(-2\psi^2) \quad (3.11)$$

The contribution of Massart [22] was to prove that the constant 2 holds and it cannot be improved. In practice, if the copula fits well, observed tail probabilities of copula margins should not exceed those from Eq. (3.11). Denoting the exceedance probability by π^* yields $\psi/\sqrt{n} = \sqrt{(-\frac{1}{2n} \log \frac{\pi^*}{2})}$. This formula implies that in a sample of 1000 observations, there is 5% (resp. 1%) probability of seeing prediction errors exceeding 0.0429 (resp. 0.0515). If large errors are observed more frequently, the copula does not fit well in the tails. We use this rule in section 4.4 and Fig. 7 to check the copula fit.

4 Case study using executive stock option data

4.1 Short description of the data

The data includes 7611 trades of Nokia series 99 executive stock options (ESOs), recorded in the Helsinki Stock Exchange (nowadays part of the OMX Nordic Exchange) between April 2, 2000 and December 30, 2002. The option plan was approved by the firm's Annual General Meeting of 1999, the options became exercisable in three tranches beginning April 1, 2001, and the plan expired at the end of calendar year 2004. Details of the ESO plan can be checked from Nokia's 1999 annual report⁴. Given the success of Nokia in mobile handset business and buoyant stock market, most trades were done in the money, with average and median moneyness (S/K) values of 1.54 and 1.49. The risk-free rate was estimated by interpolation, based on Euribor interest rates and the Finnish zero-coupon yield curve. Implied (B-S) volatilities and the estimated volatility function are plotted in Figure 1; it makes clear that implied volatility decreases with moneyness.

Because these options are traded, selling them is generally preferable to exercising for two reasons. First, the market price incorporates some time value, which is lost by exercising. Time value is important, because average maturity (when traded) is 3.1 years. Second, Finnish tax code encourages selling. If the option is sold, taxable income equals the (net) sales proceeds. If the option is exercised, taxable income equals current stock price less the exercise price, but receiving the shares takes from a few weeks to few months, during which they cannot be sold. This is known as *tax-based risk* (Ikäheimo et al. [20]).

4.2 A casual review of ESO pricing literature

ESO valuation has been a popular subject in finance since late 1990s (see Hall and Murphy [16] and references therein); what follows is a review of a few subjectively chosen topics. The standard approach to valuing ESOs is to apply the certainty equivalent (CE) method introduced by Pratt [25]; the application to ESOs is worked out in Broadie and Detemple [5]. What the CE method yields is a managerial ask price, which may not be consistent with market equilibrium (assuming it exists). This occurs when the ESO and the underlying stock are traded in a market, where some traders are constrained (managers), while others are unconstrained (market makers). To simplify the picture, assume that managers generate the supply of ESOs, and market makers make up the demand by providing bids. What justifies this view is that managerial ask prices are in general lower than risk-neutral option values, for reasons given below.

In theory, subjective ESO valuation starts from the manager's consumption-investment problem, and the outcome depends on initial wealth, labor income, and correlation of income and investment risks. Often the manager cannot short the underlying asset, and his risk-neutral discount rate is below the market's

⁴The details are given in *Convertible bonds and stock options*, Nokia's Financial Statements 1999, pp. 31–32. This document can be downloaded from the company's web site.

risk-free rate. As a result, his subjective value is lower than the risk-neutral value. This is verified by Detemple and Sundaresan [7] under the assumption of no short positions in the underlying stock. Hall and Murphy [16] list topical issues in the ESO literature; it is concentrated on restrictions on diversification and hedging, as well as explaining early exercise. However, the possibility of *traded* ESOs seems to be forgotten, with the exception of Ikäheimo et al. [20].

Contract design of ESOs differs from standard options in terms of exercise policy, strike price and maturity. The exercise policy is a mix of European and American styles. When ESOs are granted, there is an initial vesting period, during which the options cannot be exercised or sold to a third party. After vesting, ESOs are traded in the Helsinki Stock Exchange. Ikäheimo et al. [20] explain the workings of this market and provide evidence of underpricing relative to standard options. ESOs have typically long maturities up to five years. In some cases, this implies that ESOs are deep in the money, whereas in standard options liquidity is found close to the money. Differences in moneyness and maturity imply that ESOs cannot be statically hedged using standard options, using the liquid contracts.

4.3 Option values and implied volatilities

Table 2 prices call options representing the Nokia ESO characteristics, using the mean-reverting Hull-White model. The intuition is that the ESO is traded in the market, and expected volatility is inferred using the model (2.6) shown in Fig. 1. In line with the data, maturity of the option is extended to three years. The model is parameterised using Proposition 1. Note that the prices are arbitrage-free, because the price of risk vector is well-defined and a martingale measure exists.

The dynamics of volatility imply that option values do not converge to Black-Scholes even when the correlation is zero. Mean reversion in volatility decreases option values, and also the sign of correlation comes into play. Relative to no correlation, positive correlation has negative effect on out-of-the-money options, which turns positive moving onto at-the-money and in-the-money options. Negative correlation causes the opposite effect; out-of-the-money options gain and in-the-money options lose value. These effects are statistically significant in most cases given the standard errors. [*Table 2 here*]

Figure 2 plots relative prices produced by the HW model with mean reversion. A price below 1 means the option price is below Black-Scholes. The exercise is repeated for three levels of the volatility of volatility (γ) and a wide range of moneyness. The main impression is that relative option prices decrease as γ increases. Further, this effect is decreases with moneyness. Correlation is given by x-axis labels, with the correlation effect being similar as in Table 2. Note that the distribution of λ_Y has the same mean and variance in all cases, because $\gamma = 2a$ (see Proposition 1). Yet the relative prices show significant variation. Qualitatively similar results are shown in Fig. 3 of Hull and White [19], however they only plot the zero correlation case.

The impact of correlation on option values suggests a closer look at how

the stock distribution is affected. In fact, positive correlation induces positive skewness and kurtosis, as can be seen from Table 3 and Figure 3. It plots the densities of present value of the stock, being more or less concentrated around one. Table 3 reports four first moments of the stock distribution, the median and risk-neutral drift for different levels of correlation. The risk-neutral drift, which represents the actual risk-free return, is calculated by adding back one-half of variance to the mean. Note that while the risk-free return is (nearly) constant, higher moments of the stock return distribution are quite sensitive to correlation. [*Table 3 here*]

Note that the skewness of the stock distribution increases with variance. This follows from the properties of geometric Brownian motion (Glasserman [15, Ch. 3.2]). Given the stock dynamics (2.4), simulated sample paths converge with probability one to

$$\frac{\log(S_t/S_0)}{t} \rightarrow r - \frac{1}{2}Y_t$$

because the standard Brownian motions converge to zero ($t^{-1}W_t \rightarrow 0$ as $t \rightarrow \infty$). When the Y process produces large variances ($r < Y_t/2$), the stock sample paths characteristically converge to zero, i.e. $\frac{\log(S_t/S_0)}{t} \rightarrow -\infty$ as $t \rightarrow \infty$, regardless of the fact that expected return equals the risk-rate, i.e. $E^Q[\log S_t/S_0] = rt$. The positive mean is generated by rare but large positive realisations of S_t . As a result, intermediate outcomes of S_t became less frequent and the stock distribution becomes more skewed. Based on Table 2, this 'skewness penalty' falls especially on out-of-the money contracts.

Figure 4 provides the implied (B-S) volatility distributions for an at-the-money option, as well as a quantile-quantile plot. It verifies that positive correlation increases the likelihood of high implied volatilities. In summary, the correlation effect on option values is ambiguous. On one hand, positive correlation adds value by coupling positive outcomes of stock price and volatility. On the other hand, it reduces value by inducing skewness, and decreases the probability that the contract is in-the-money at expiration. The latter effect is severe only for options that are initially out-of-the money.

4.4 Copula estimates and goodness of fit

Estimated copulas, parameters and goodness of fit are discussed below. However, two remarks are due before that. First, all four copulas are meta-models in the sense that marginal probabilities are given by empirical distribution functions. For example, the Gauss copula, or meta-Gaussian model, does not assume that stock returns and volatility were normally distributed.

The second remark is that the copula estimation was done using negative delta. The reason is that Archimedean copulas assume tail dependence in either (0,0) or (1,1) corners of the unit square. As illustrated in Figure 5, this appears in the data if negative delta is used. Therefore, linear and rank correlation estimates for positive delta and volatility are obtained from those of Table 4 simply by changing the sign. These estimates also apply to the dependence of

(positive) stock return⁵ and volatility.

Different aspects of the fitted copulas are summarized in Table 4. Starting from the top, Gauss and t copulas yield almost identical estimates that imply negative correlation of 0.8 for stock returns and volatility. In fact, there seems to be no pronounced dependence of extremes, as confirmed by both the t copula fit with 9.4 df and Fig. 5. In terms of likelihood values, the Clayton and Gumbel copulas score lower, suggesting that the elliptic models (Gauss and t) provide better fit. Reported parameters are maximum likelihood (ML) estimates⁶. Details for the ML estimators are given in McNeil et al. [23, Ch 5.5].

Given that copula-based methods are data-intensive, one should not ignore sampling variation. It is examined in Figure 6 that plots bootstrap estimates for correlation and degrees of freedom. The correlations of Gauss and t copulas are found very stable; bootstrapping 2,000 runs of 1,000 observations yields standard deviation of 0.01 in both cases. For t model, standard deviation of df was 3.15. Sampling variation is subdued also for Archimedean copulas; standard deviation of θ was 0.10 for Clayton and 0.06 for Gumbel.

Based on the results of Genest and Rivest [14] and McNeil et al. [23, Prop. 5.37], all four copulas can be consistently fitted using rank correlation, with the advantage of reduced computational effort. Therefore it is interesting to compare sample values of rank correlations to those implied by ML estimates. Table 4 shows that actual and implied values of Kendall's tau are somewhat different. Clearly, ML is the preferred method to estimate a single set of parameters, for it returns higher log-likelihood values than using rank correlation. However, when a large number of estimates are needed, employing rank correlations becomes a competitive alternative.

Goodness of fit is measured by the Rosenblatt transformation test (RTT) yielding the AD statistic, as well as using the convergence test for prediction errors. Given the similarity of correlation estimates, it is hardly surprising that the tests do not produce strong evidence against any of the copulas. The AD statistics imply that the null is maintained in all cases with p-values around 0.50. These p-values are based on bootstrap distributions of the AD statistic.

Table 4 also reports the average prediction errors and their standard deviations, with the latter corresponding to a sample of 1000 observations. These numbers suggest that all four copulas are unbiased. However, the Gauss copula yields most efficient estimates (measured by st. dev. of errors). Note that one cannot "cheat" this method by deliberately adding low frequency cells, because the weighted variance estimator (3.10) is used. Compliant with this view, the F statistics find no dependence in prediction error tables. The weighted ANOVA procedure is best understood by looking at exemplary data for the Gauss copula given in Table 5. In summary, the results suggest that best fit to the data is achieved by Gauss and t copulas. Practical value of these models is enhanced by

⁵The invariance property is still valid. Note that the mapping $-S \rightarrow -\frac{\partial C}{\partial S}$ is strictly increasing in $-S$. It follows that *negative* delta and *negative* stock price have the same copula. Moreover, the correlations are $\rho(-S, \hat{\sigma}) = -\rho(S, \hat{\sigma})$ and $\rho_{\tau}(-S, \hat{\sigma}) = -\rho_{\tau}(S, \hat{\sigma})$.

⁶The estimates were obtained using the R language (by R Development Core Team [26]) and in particular the packages *QRMLib* and *copula*.

them providing correlation estimates for option pricing. [Tables 4 and 5 here]

Finally, a critical point follows. The weak spot of Gaussian and t copulas is that their Kolmogorov-Smirnov statistics take large values more often than they should, as illustrated in Fig. 7. It gives the histograms of K-S statistics for copula margins. The DKW inequality (3.11) implies that 95% point of the histogram should be 0.0429 with sample size 1,000. However, in practice the 95th percentile is just below 0.06. Therefore, while the average fit is good, in some cases the prediction errors can be large.

5 Conclusions

This paper proposes an original method to estimate correlation for option pricing, assuming that the underlying stock and volatility follow diffusion processes linked by correlated Brownian motions. The asset dynamics are similar to the Hull and White [19] model. Applicable arbitrage conditions, in particular a change of measure via definition of price of risk vector, are well known and made rigorous by e.g. Romano and Touzi [27]. The model is parameterised using Proposition 1, assuming that the volatility function is unbiased, and the change of measure is applicable.

A key parameter of the model is the correlation of stock returns and volatility. Unfortunately, it cannot be directly estimated from data. This problem is solved here by estimating the correlation as a copula parameter, using the copula of option delta and volatility. This is permitted by the invariance property common to all copulas. While copula modelling is based on rank correlation, linear correlation and Kendall's tau are closely related for Gaussian and t copulas. Either this relation or maximum likelihood estimation are applicable for computing the correlation. Reliability of the estimate can be judged using goodness of fit tests for copulas.

The above method is illustrated in a case study using rare data of traded executive stock options. Option values and sensitivities to stock price and are calculated using simulation. It turns out that volatility and stock returns are negatively correlated, and both Gaussian and t copulas provide well-fitting dependence models. Also, it is shown that sample variation in the estimates is not a concern. While both Gauss and t models are unbiased, large prediction errors occur more often than they should.

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A Tables and figures

Copula	Distribution function ¹⁾	Parameters ²⁾	Tail dependence
Gaussian	$C_\rho^{Ga} = \Phi_\rho(\widehat{F}_1^{-1}(u_1), \widehat{F}_2^{-1}(u_2))$	$ \rho \leq 1$	$\lambda_l = \lambda_u = 0$
t	$C_{\rho,\nu}^t = t_{\rho,\nu}(\widehat{F}_1^{-1}(u_1), \widehat{F}_2^{-1}(u_2))$	$ \rho \leq 1$ $\nu > 2$	$\lambda_l = \lambda_u =$ $2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$
Clayton	$C_\theta^{Cl} = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$	$\theta > 0$	$\lambda_l = 2^{-1/\theta}$ $\lambda_u = 0$
Gumbel	$C_\theta^{Gu} = \exp\left(-\psi^{1/\theta}\right)$ $\psi = (\ln u_1^{-1})^\theta + (\ln u_2^{-1})^\theta$	$\theta \geq 1$	$\lambda_l = 0$ $\lambda_u = 2 - 2^{1/\theta}$

Table 1: *Notes:* 1) Bivariate normal and t distribution functions are denoted Φ_ρ and $t_{\rho,\nu}$, with correlation ρ and degrees of freedom ν . Also, \widehat{F}_1^{-1} and \widehat{F}_2^{-1} denote empirical quantile functions of option delta and volatility. 2) For t distribution it is required that $\nu > 2$; otherwise the covariance matrix is not defined (McNeil et al. [23], Ex. 3.7).

Hull-White option prices						
Correlation		S/K=0.8	S/K=0.9	S/K=1.0	S/K=1.1	S/K=1.2
0	mean	0.1220	0.1697	0.2181	0.2748	0.3192
	(se)	(0.0027)	(0.0024)	(0.0023)	(0.0029)	(0.0024)
0.5	mean	0.1084	0.1664	0.2244	0.2767	0.3261
	(se)	(0.0015)	(0.0016)	(0.0015)	(0.0015)	(0.0014)
-0.5	mean	0.1296	0.1747	0.2141	0.2660	0.3107
	(se)	(0.0038)	(0.0070)	(0.0031)	(0.0041)	(0.0035)
Black-Scholes		0.1425	0.1904	0.2384	0.2849	0.3288

Table 2: Means and standard errors of simulated option values (using 5000 replications) from the Hull-White model with mean reversion. The priced claim is a European call with three years maturity. Expected volatility is assumed to be 25%, speed of mean reversion $a = 0.5$ and volatility of volatility $\gamma = 1$. The Black-Scholes prices are provided for comparison.

	$\rho = 0.5$	$\rho = 0.3$	$\rho = 0$	$\rho = -0.3$	$\rho = -0.5$
Mean return	0.0370	0.0430	0.0441	0.0440	0.0467
r-n drift	0.0468	0.0504	0.0493	0.0488	0.0510
median	-0.0047	0.0046	0.0133	0.0345	0.0509
st.dev.	0.1398	0.1214	0.1023	0.0979	0.0928
skewness	4.6224	4.0249	2.7149	1.2939	0.0174
kurtosis	43.0557	30.7747	17.0261	12.6745	7.0372

Table 3: Distributional characteristics of the stock return (i.e. $\log(S_T/S_0)$) corresponding to different correlations (ρ). R-n drift is the actual risk-neutral return, calculated by adding back one-half of variance to the mean. All figures are based on 5000 replications. Other parameters are $\gamma = 2a = 1$, $r = 0.05$, and $T = 3$. Skewness and kurtosis would take values of zero and three in the case of normal distribution.

	Copula	Gauss	t	Clayton	Gumbel
distribution function		C_ρ^{Ga}	$C_{\rho,\nu}^t$	C_θ^{Cl}	C_θ^{Gu}
parameters: ρ		0.7918	0.7992	.	.
ν or θ		.	9.448	2.001	2.196
Kendall's tau: actual		0.6110	0.6110	0.6110	0.6110
implied		0.4184	0.4184	0.5002	0.5447
lower tail: λ_l		0	0.4527	0.7073	0
upper tail: λ_u		0	0.4527	0	0.6289
log likelihood		3731.5	3822.5	3308.3	3267.6
RTT: AD statistic		335.9	334.7	335.1	335.5
p-value		0.4963	0.5061	0.4836	0.4778
critical values: p=0.05		384.9	385.1	385.6	384.5
p=0.01		403.5	407.6	405.3	404.9
ANOVA: mean(f_{ij})		0.00	0.00	0.00	0.00
se(f_{ij})		2.29	2.41	2.39	3.07
sd(f_{ij}) (1000 obs.)		13.71	14.48	14.33	18.42
$F_{5,5}$:row; column		0.63; 0.38	0.49; 0.35	1.18; 1.43	0.46; 0.24
p-values: row; column		0.68; 0.86	0.78; 0.88	0.36; 0.26	0.80; 0.94

Table 4: The table reports ML estimates of copula parameters, followed by goodness of fit tests. For Kendall's tau, 'actual' is the empirical value and 'implied' is calculated from the copula parameters using Eqs. (3.1)-(3.3). Tail indices are based on Table 1. Output of the copula tests of Section 4.3 is given under the headings RTT and ANOVA. Numbers under ANOVA describe the prediction errors. F statistics (with 5 and 5 df) and p-values correspond to row and column effects of Eq. (3.9). ANOVA data for Gauss copula is reported in Table 5.

Actual frequencies (1,000 obs.)						
u_1/u_2	0.05	0.25	0.50	0.75	0.95	1
0.05	29.06	20.79	0.00	0.00	0.00	0.00
0.25	11.74	106.67	77.19	3.39	0.53	0.27
0.50	6.19	56.46	115.02	57.66	11.86	2.26
0.75	2.61	15.83	54.19	93.03	72.90	12.29
0.95	0.26	0.41	3.21	91.15	90.16	14.89
1	0.00	0.00	0.12	4.63	24.87	20.34
Predicted frequencies (1,000 obs.)						
u_1/u_2	0.05	0.25	0.50	0.75	0.95	1
0.05	24.38	21.88	3.39	0.31	0.01	0.00
0.25	21.94	99.25	60.28	17.11	1.75	0.01
0.50	3.49	60.26	100.97	67.91	16.91	0.32
0.75	0.32	17.00	68.05	101.29	59.71	3.48
0.95	0.01	1.78	17.06	60.04	98.89	22.14
1	0.00	0.01	0.33	3.40	21.99	24.34
Prediction errors (1,000 obs.)						
u_1/u_2	0.05	0.25	0.50	0.75	0.95	1
0.05	-4.67	1.09	3.38	0.31	0.01	0.00
0.25	10.20	-7.41	-16.91	13.71	1.22	-0.25
0.50	-2.70	3.80	14.05	10.25	5.05	-1.93
0.75	-2.29	1.17	13.85	8.25	-13.18	-8.80
0.95	-0.25	1.37	13.84	-31.11	8.72	7.25
1	0.00	0.01	0.21	-1.23	-2.88	4.01

Table 5: Actual and predicted frequencies based on resampling the empirical and fitted Gauss copulas. In this experiment 2,000 bootstraps of 1,000 paired observations are taken, followed by classifying the outcomes in 6x6 table. Reported numbers are averages per 1,000 observations. Prediction errors are calculated as Predicted minus Actual. The categories give upper limits of marginal probabilities, for example (0.05, 0.05) refers to $(0, 0) \leq (u_1, u_2) < (0.05, 0.05)$. Weights for ANOVA are given by Actual frequencies divided by 1000.

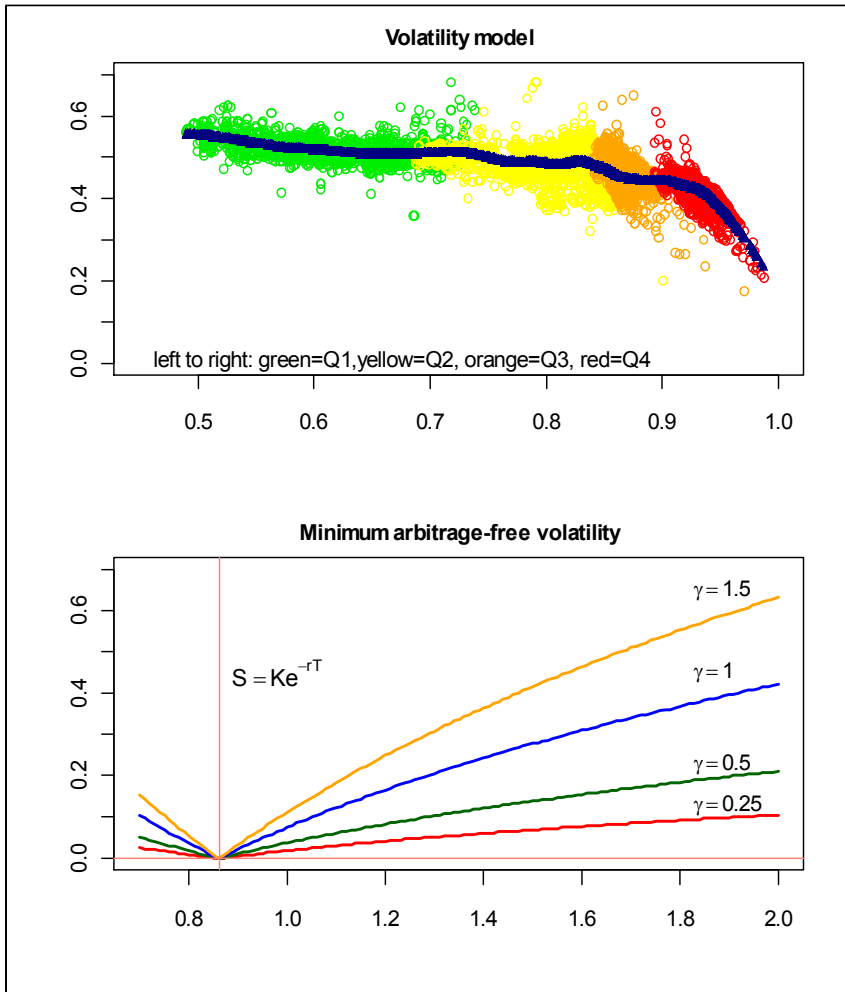


Figure A.1: *The upper plot* presents fit of the volatility model (2.6) with option delta on x-axis. Actual data is presented by the scatterplot, using different colors for moneyness quartiles. *In the lower plot*, Schweizer-Wissel volatility bounds (2.12) are drawn as function of moneyness. The curves are produced by increasing parameter γ (volatility of volatility) from 0.25 to 1.5. Other parameters: $T = 3, r = 0.05$.

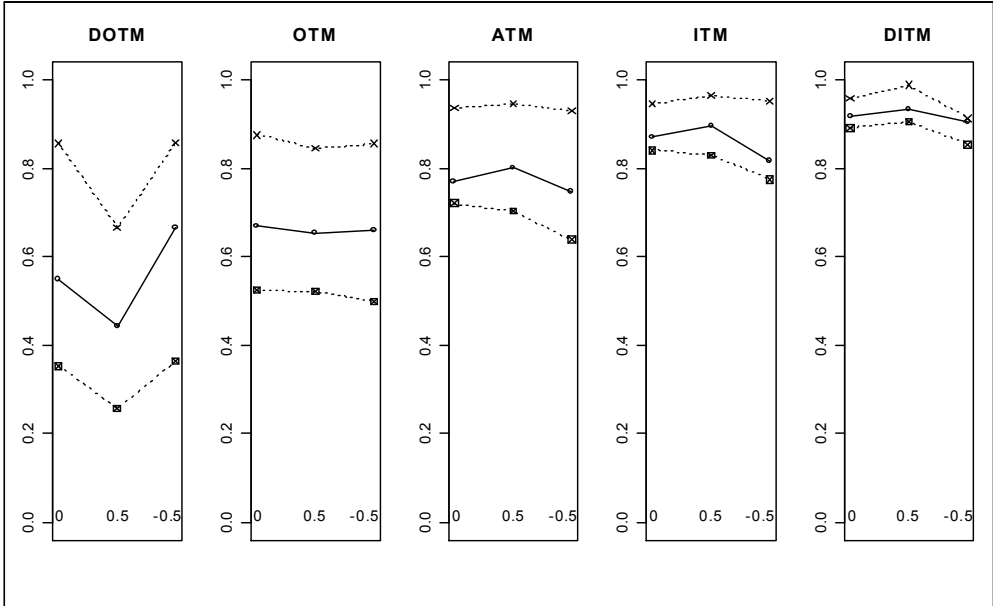


Figure A.2: Relative option prices, i.e. mean-reverting HW values scaled by Black-Scholes. The plots are titled by moneyness, ranging from DOTM ($S/K = 0.8$) to DITM ($S/K = 1.2$). The plotting symbols correspond to three parameter sets. The dotted lines with crosses use $(\gamma = 1, a = 0.5)$, solid lines with points use $(\gamma = 2, a = 1)$, and dashed lines with boxes use $(\gamma = 3, a = 1.5)$. Note that $\gamma = 2a$ in all cases. Finally, x-axis labels (0, 0.5, -0.5) give the correlation. Other parameters are as in Table 2B.

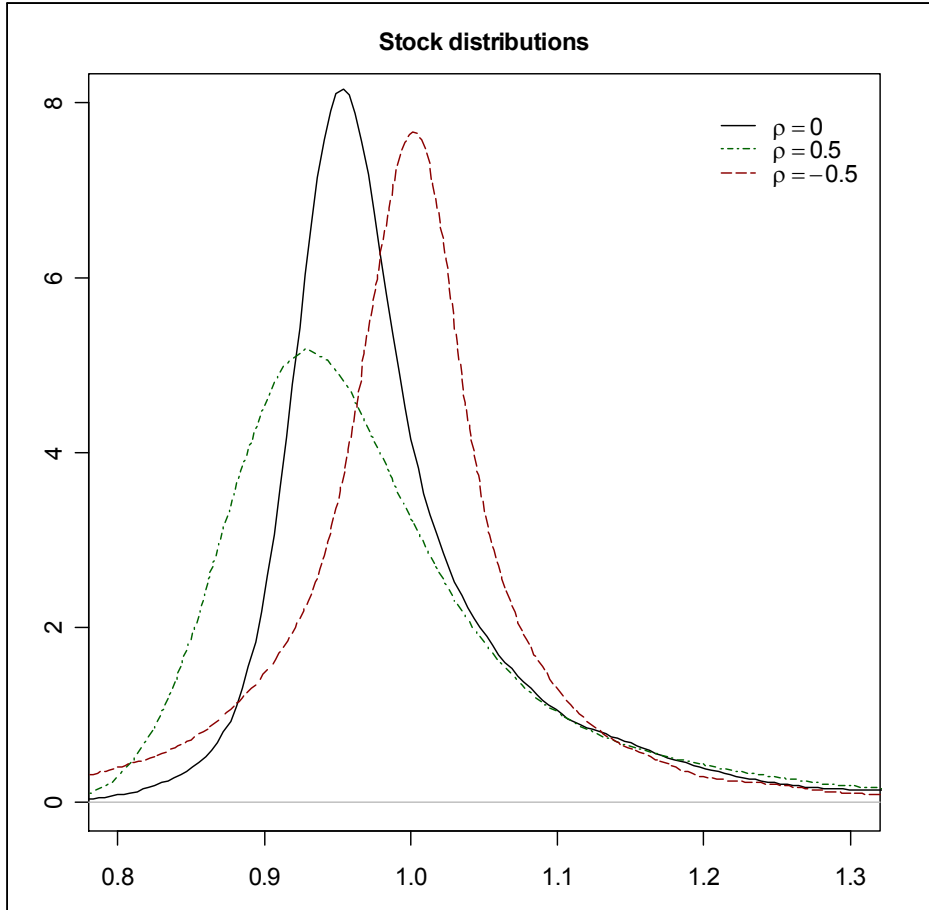


Figure A.3: Estimated stock price densities based on 5,000 simulations. The densities are plotted for present value of the stock price, and therefore concentrated around value of one. Bandwidths are calculated as $bw = 1.059 \text{sd}(x)N^{-1/5}$, where $\text{sd}(x)$ is the standard deviation of outcomes and $N = 5000$. Other parameters are the same as in Table 2. Note the heavy right tail and skewness of the case $\rho = 0.5$.

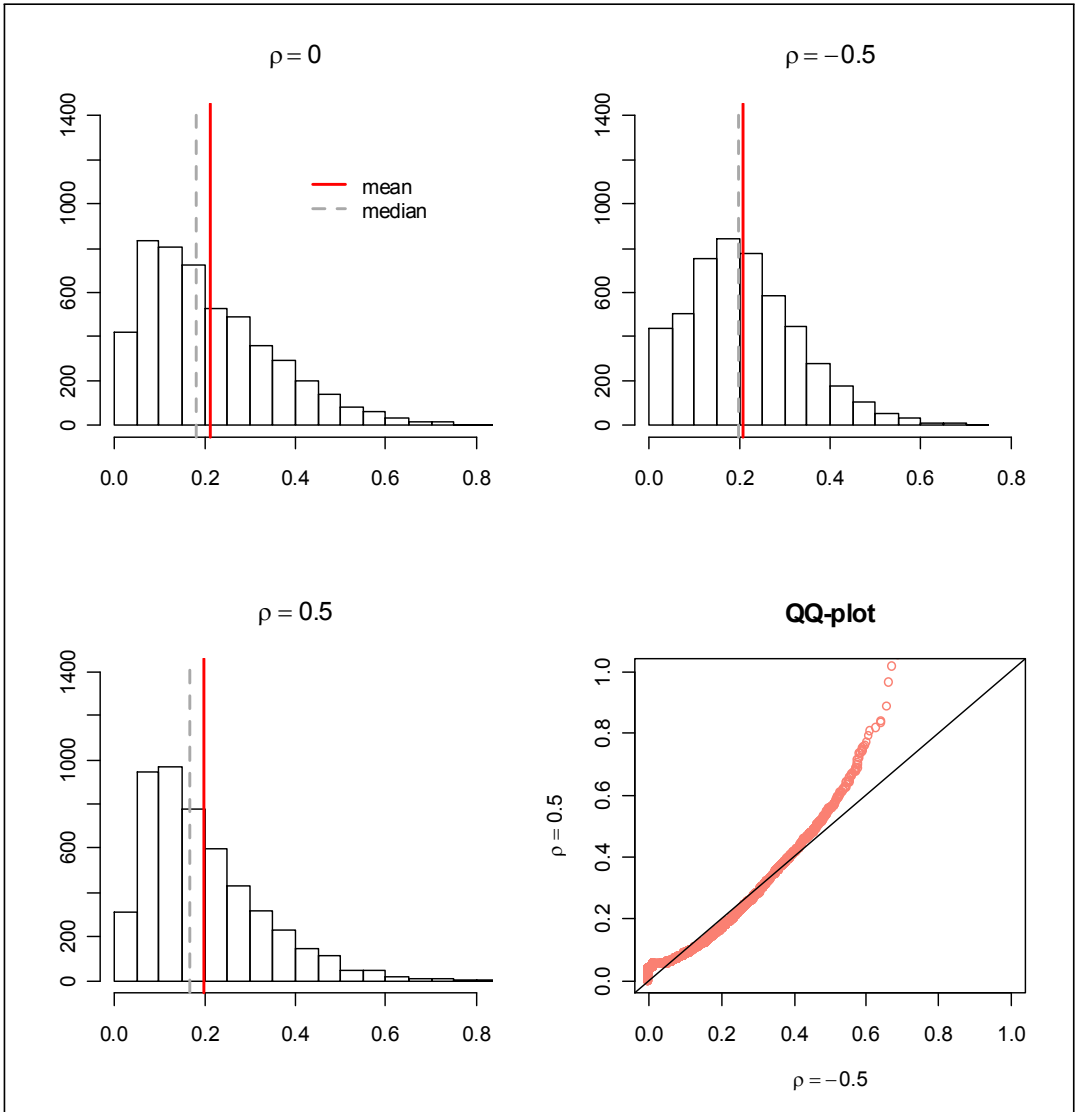


Figure A.4: Histograms of implied volatility (IV) of an at-the-money option for different correlations. Solid and dashed lines plot the mean and median IVs. Moreover, on bottom right there is a quantile-quantile plot of IVs, comparing the cases $\rho = 0.5$ and $\rho = -0.5$. Other parameters: $a = 0.5$, $\gamma = 1$, $r = 0.05$, $\sigma^* = 0.25$, $T = 3$.

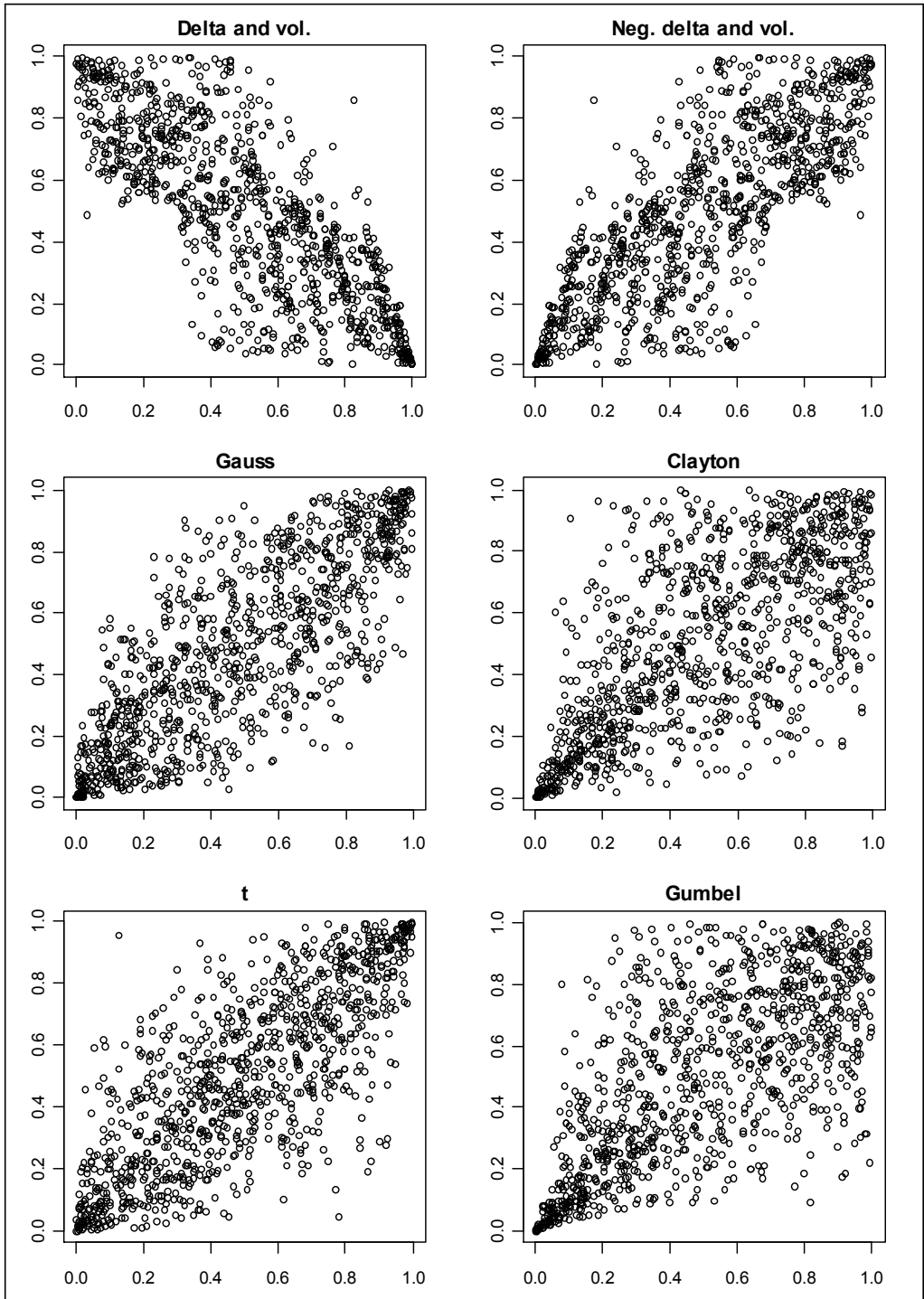


Figure A.5: The first row presents empirical copulas of the delta and (implied) volatility, using positive and negative delta. The second and third rows plot samples from the fitted copulas with parameters reported in Table 4. All six plots are based on sampling 1,000 pairs from the respective copulas.



Figure A.6: Sampling variation is investigated by taking 2,000 bootstraps of 1,000 observations from the data, followed by fitting Gauss and t copulas in the samples. *The left figure* is scatterplot of t copula fit with degrees of freedom and correlation on on x- and y-axes. *The right figure* is quantile-quantile plot of Gauss and t copula correlations, illustrating that the distributions are closely aligned. Full sample estimates for df and correlation are marked by lines.

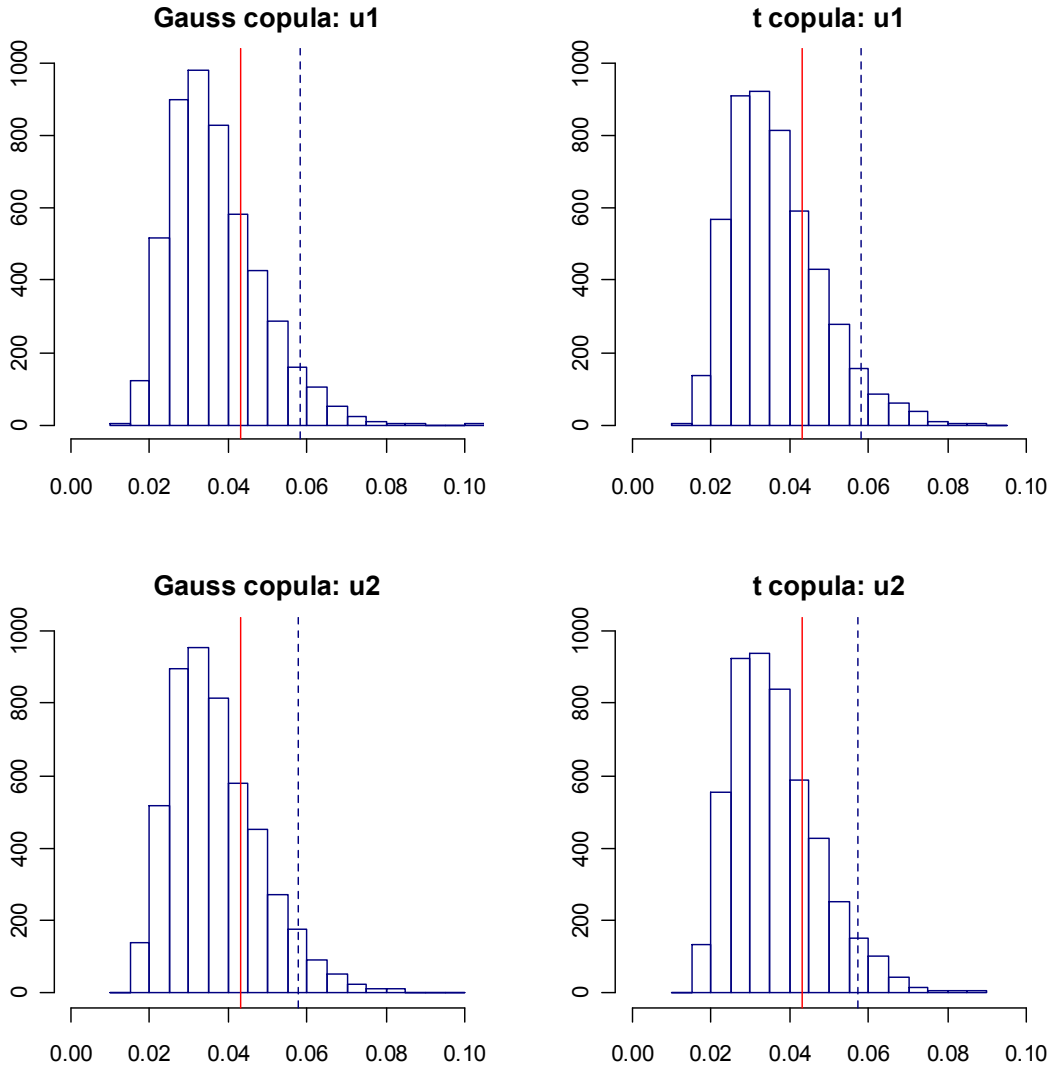


Figure A.7: Bootstrap distributions of Kolmogorov-Smirnov statistic, drawn for Gaussian and t copula margins. X- and y-variables are the K-S statistic and frequency in a bootstrap based on 5,000 runs of 1,000 observations. The solid bar is the theoretical 95th percentile implied by Eq. (3.11), which equals 0.0429, and the dashed bar is the actual 95th percentile.

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