A NOTE ON WIENER–HOPF MATRIX FACTORIZATION

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SUMMARY

In this paper the most general class of 2×2 matrices is determined which permit a Wiener–Hopf factorization by the procedure of Rawlins and Williams (**1**). According to this procedure, the factorization problem is reduced to a matrix Hilbert problem on a half-line, where the matrix involved in the Hilbert problem is required to have zero diagonal elements.

1. Introduction

IN THE paper of Rawlins and Williams (**1**) a Wiener–Hopf factorization of the matrix

$$\mathbf{A}(\alpha)=\begin{pmatrix} F(K) & G(K)F(K) \\ H(K) & -G(K)H(K)\end{pmatrix} \tag{1}$$

was carried out. In (1), F, G and H are analytic functions (except possibly at $K=0$) of the variable $K=(k^2-\alpha^2)^{\frac{1}{2}}$, where α is a complex variable and k is a constant with positive real and imaginary parts. The branch of the square root chosen is such that $K=k$ at $\alpha=0$, with the branch cuts C and C' lying along the half-lines $\alpha=-k-\delta$ and $\alpha=k+\delta$, $\delta\geqslant 0$, respectively. It was shown in (**1**) that, provided F, G and H do not have any zeros in the cut α-plane, and $G(K)=-G(-K)$, then the matrix (1) can be factorized in the form

$$\mathbf{A}(\alpha)=\mathbf{U}(\alpha)\mathbf{L}^{-1}(\alpha),$$

where $\mathbf{U}(\alpha)$ and $\mathbf{L}(\alpha)$ are non-singular matrices whose elements are analytic for $\operatorname{Im}(\alpha)>-\operatorname{Im}(k)$ and $\operatorname{Im}(\alpha)<\operatorname{Im}(k)$ respectively.

The crux of the technique of factorization depends on being able to assume that $\mathbf{U}(\alpha)$ is analytic everywhere except along the branch cut C through $\alpha=-k$, whilst $\mathbf{L}(\alpha)$ is analytic everywhere except along the branch cut C' through $\alpha=k$, and then to show that

$$\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha)=\begin{pmatrix} 0 & g(\alpha) \\ h(\alpha) & 0\end{pmatrix}, \tag{2}$$

where $g(\alpha)$, $h(\alpha)$ are specific functions, and the suffices $\pm$ denote values evaluated on the upper side and lower side of the branch cut $C:\alpha=-k-\delta$, for $\delta\geqslant 0$.

Professor J. Boersma has posed the question as to whether (1) is the most general matrix, with the given branch cuts, for which the matrix product $\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha)$ takes the form (2). He conjectured that it would not be. In this note we confirm his conjecture, and give the most general form of the class of 2×2 matrices which produce zeros in the diagonal for the Hilbert problem. Explicitly, we show that the most general form is

$$\begin{aligned}\mathbf{A}(\alpha)&=\begin{pmatrix}a_{11}(\alpha) & a_{11}(\alpha)\{F_1(\alpha)+(k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\}\\ a_{21}(\alpha) & a_{21}(\alpha)\{F_1(\alpha)-(k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\}\end{pmatrix}\\ &=\begin{pmatrix}a_{11}(\alpha) & a_{11}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\\ a_{21}(\alpha) & -a_{21}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}1 & F_1(\alpha)\\ 0 & F_2(\alpha)\end{pmatrix},\end{aligned}\qquad (3)$$

with $a_{11}(\alpha)a_{21}(\alpha)F_2(\alpha)\neq 0$ in the cut plane, where $a_{11}(\alpha)$, $a_{21}(\alpha)$ are analytic functions in the cut plane (with branch cuts C and C'), and $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire α-plane except possibly along the branch cut C'. If, further, $\mathbf{A}(\alpha)=\mathbf{A}(-\alpha)$ then $F_1(\alpha)=E_1(\alpha)$, $F_2(\alpha)=E_2(\alpha)$, where $E_1(\alpha)$ and $E_2(\alpha)$ are analytic in the entire α-plane. Since post-multiplication of $\mathbf{A}(\alpha)$ by an entire or $\mathbf{L}$-matrix will not affect $\mathbf{A}_+(\alpha)\mathbf{A}^{-1}(\alpha)$, the basic general form can be taken to be

$$\begin{pmatrix}a_{11}(\alpha) & a_{11}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\\ a_{21}(\alpha) & -a_{21}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\end{pmatrix}.\qquad (3')$$

This matrix may be post-multiplied by an arbitrary $\mathbf{L}$-matrix and/or pre-multiplied by an arbitrary $\mathbf{U}$-matrix, yielding a matrix that can also be factorized.

2. Derivation of the general form (3)

Consider the matrix

$$\mathbf{A}(\alpha)=\begin{pmatrix}a_{11}(\alpha) & a_{12}(\alpha)\\ a_{21}(\alpha) & a_{22}(\alpha)\end{pmatrix},$$

where we suppose that $a_{11}(\alpha)$, $a_{12}(\alpha)$, $a_{21}(\alpha)$, $a_{22}(\alpha)$ are analytic functions in the cut α-plane, and $\det\mathbf{A}(\alpha)\neq 0$ in the cut α-plane. Then

$$\mathbf{A}_+(\alpha)\mathbf{A}_-^{-1}(\alpha)=\frac{1}{\det\mathbf{A}_-(\alpha)}\begin{pmatrix}a_{11}^+a_{22}^- - a_{12}^+a_{21}^- & a_{12}^+a_{11}^- - a_{11}^+a_{12}^-\\ a_{21}^+a_{22}^- - a_{22}^+a_{21}^- & a_{22}^+a_{11}^- - a_{21}^+a_{12}^-\end{pmatrix},\qquad (4)$$

where $\det\mathbf{A}_-(\alpha)=(a_{11}^-a_{22}^- - a_{12}^-a_{21}^-)\neq 0$. In order that (4) should have the same form as (2), that is, with zeros on the main diagonal, we require that

$$a_{11}^+a_{22}^-=a_{12}^+a_{21}^-,\qquad a_{22}^+a_{11}^-=a_{21}^+a_{12}^-,$$

or, ignoring the trivial situation where $a_{11}^{\pm}(\alpha)\equiv 0$ and/or $a_{21}^{\pm}(\alpha)\equiv 0$,

$$\left(\frac{a_{12}}{a_{11}}\right)^{+}-\left(\frac{a_{22}}{a_{21}}\right)^{-}=0, \tag{5}$$

$$\left(\frac{a_{22}}{a_{21}}\right)^{+}-\left(\frac{a_{12}}{a_{11}}\right)^{-}=0, \tag{6}$$

where $a_{21}(\alpha)\neq 0$ and $a_{11}(\alpha)\neq 0$ on C.

Adding and subtracting (5) and (6) gives

$$\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{+}-\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{-}=0, \qquad \alpha\in C, \tag{7}$$

$$\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)^{+}+\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)^{-}=0, \qquad \alpha\in C. \tag{8}$$

Using the fact that $[(k^2-\alpha^2)^{\frac{1}{2}}]^{\pm}=\pm i\,|k-\alpha|^{\frac{1}{2}}\,(k-\alpha)^{\frac{1}{2}}$ we can rewrite (8) in the form

$$\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{+}-\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{-}=0, \qquad \alpha\in C. \tag{9}$$

Now, provided $a_{11}(\alpha)$ and $a_{21}(\alpha)$ are non-zero in the cut plane and satisfy the conditions

$$\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}=O[(k^2-\alpha^2)^{\mu}] \qquad \text{as } \alpha\to\pm k, \qquad 0\leqslant\mu<1,$$

$$\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}=O[(k^2-\alpha^2)^{\nu-\frac{1}{2}}] \qquad \text{as } \alpha\to\pm k, \qquad 0\leqslant\nu<1,$$

the most general solution of (7) and (9) which has no pole singularity at $\alpha=\pm k$ and no other singularities in the cut plane except a branch cut along C' is given by (**2**)

$$\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}=2F_1(\alpha), \tag{10}$$

$$\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}=2F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}, \tag{11}$$

respectively, where $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire plane except possibly along the branch cut C'. Adding and subtracting (10) and (11) gives

$$a_{12}(\alpha)=a_{11}(\alpha)\{F_1(\alpha)+F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\},$$
$$a_{22}(\alpha)=a_{21}(\alpha)\{F_1(\alpha)-F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\}.$$

If $\mathbf{A}(\alpha)=\mathbf{A}(-\alpha)$ then $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire complex plane, as the following analysis will show.

If $\mathbf{A}(\alpha)=\mathbf{A}(-\alpha)$ then $a_{ij}(\alpha)=a_{ij}(-\alpha)$, for $i,j=1,2$, and in an exactly analogous way one obtains equations similar to (7) and (9) on carrying out evaluations on the branch cut C':

$$\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{+}-\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{-}=0, \qquad \alpha\in C', \tag{7'}$$

$$\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{+}-\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{-}=0, \qquad \alpha\in C', \tag{9'}$$

where now $\pm$ denote values on the lower and upper sides of C', respectively. Adding (7) to (7′) and (9) to (9′) gives

$$\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{+}-\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{-}=0, \qquad \alpha\in C\cup C', \tag{7''}$$

$$\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{+}-\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{-}=0, \qquad \alpha\in C\cup C'. \tag{9''}$$

Thus the most general solution of (7″) and (9″) which has no pole singularity at $\alpha=\pm k$ and no other singularities in the cut α-plane is given by

$$a_{12}(\alpha)=a_{11}(\alpha)\{E_1(\alpha)+E_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\},$$
$$a_{22}(\alpha)=a_{21}(\alpha)\{E_1(\alpha)-E_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\},$$

where $E_1(\alpha)$ and $E_2(\alpha)$ are analytic in the entire α-plane. If in particular we let $a_{11}(\alpha)=F(K)$, $a_{21}(\alpha)=H(K)$, $E_1(\alpha)=0$ and $E_2(\alpha)=KG(K)$ (the condition that $G(K)=-G(-K)$ ensures that $KG(K)$ is an entire function), we obtain the special form considered in (**1**).

Following the procedure outlined in Rawlins and Williams (**1**), a particular factorization of the matrix (3), which will be useful in applications, is given by $\mathbf{A}(\alpha)=\mathbf{U}^{(0)}(\alpha)[\mathbf{L}^{(0)}(\alpha)]^{-1}$, where

$$\mathbf{U}^{(0)}(\alpha)=\begin{pmatrix}[W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}} & (k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}}\\ [W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}} & -(k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}}\end{pmatrix}.$$

Here $W_1(\alpha)$ and $W_2(\alpha)$ are solutions of the standard Hilbert problems on the half-line C:

$$[\ln W_1(\alpha)]^{+}-[\ln W_2(\alpha)]^{-}=\ln[g(\alpha)h(\alpha)],$$
$$[(k+\alpha)^{\frac{1}{2}}\ln W_2(\alpha)]^{+}-[(k+\alpha)^{\frac{1}{2}}\ln W_2(\alpha)]^{-}=i\,|k+\alpha|^{\frac{1}{2}}\ln[g(\alpha)/h(\alpha)],$$

where

$$g(\alpha)=(a^{+}_{12}(\alpha)a^{-}_{11}(\alpha)-a^{+}_{11}(\alpha)a^{-}_{12}(\alpha))/\det\mathbf{A}_{-}(\alpha)=a^{+}_{11}(\alpha)/a^{-}_{21}(\alpha),$$
$$h(\alpha)=(a^{+}_{21}(\alpha)a^{-}_{22}(\alpha)-a^{+}_{22}(\alpha)a^{-}_{21}(\alpha))/\det\mathbf{A}_{-}(\alpha)=a^{+}_{21}(\alpha)/a^{-}_{11}(\alpha).$$

The set of solutions for $W_1(\alpha)$, $W_2(\alpha)$ is further restricted by the requirements that the factor matrix $\mathbf{L}^{(0)}(\alpha)$ is non-singular at $\alpha=-k$ and its elements are analytic in the region $\mathrm{Im}(\alpha)<\mathrm{Im}(k)$. It is interesting to note that the functions $F_1(\alpha)$, $F_2(\alpha)$ have dropped out completely. This means that for all matrices of the form (3) the factorization problem reduces to the same Hilbert problem! This follows from the justification given for (3′).

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