Blow-up in reaction-diffusion equations with exponential and power-type nonlinearities

Aappo Pulkkinen

Aalto University

DOCTORAL DISSERTATIONS

Aalto University publication series
DOCTORAL DISSERTATIONS 45/2011

Blow-up in reaction-diffusion equations with exponential and power-type nonlinearities

Aappo Pulkkinen

Doctoral dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the School of Science for public examination and debate in Auditorium E at the Aalto University School of Science (Espoo, Finland) on the 15th of June 2011 at 12 noon.

Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervisor
Prof. Gustaf Gripenberg

Instructor
Prof. Stig-Olof Londen

Preliminary examiners
Prof. Peter Poláčik, University of Minnesota, USA
Dr. Julia Matos, Université d'Evry-Val-d'Essonne, France

Opponent
Prof. Pavol Quittner, Comenius University, Slovakia

Aalto University publication series
DOCTORAL DISSERTATIONS 45/2011

ISBN 978-952-60-4139-1 (pdf)
ISBN 978-952-60-4138-4 (printed)
ISSN-L 1799-4934
ISSN 1799-4942 (pdf)
ISSN 1799-4934 (printed)

Aalto Print
Helsinki 2011

Finland

The dissertation can be read at http://lib.tkk.fi/Diss/

Aalto University

Abstract

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

Author
Aappo Pulkkinen

Name of the doctoral dissertation
Blow-up in reaction-diffusion equations with exponential and power-type nonlinearities

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 45/2011

Field of research Mathematical analysis

Manuscript submitted 1 March 2011 **Manuscript revised** 10 May 2011

Date of the defence 15 June 2011 **Language** English

☐ **Monograph** ☒ **Article dissertation (summary + original articles)**

Abstract

In this dissertation we study blow-up phenomena in semilinear parabolic equations with both exponential and power-type nonlinearities. We study the behavior of the solutions as the blow-up moment in time and the blow-up point in space are approached.

Our focus is on the supercritical case; however, we also give some results on the subcritical case. We prove results concerning the blow-up rate of solutions, and we obtain the blow-up profile for limit L^1-solutions both with respect to the similarity variables and at the blow-up moment. We use techniques that are applicable both for the exponential and power nonlinearities. We also consider immediate regularization for minimal L^1-solutions and improve on some earlier results.

We are also interested in the behavior of selfsimilar solutions and we prove the existence of regular selfsimilar solutions that intersect the singular one arbitrary number of times.

Keywords semilinear parabolic equation, supercritical case, exponential nonlinearity, power-type nonlinearity, blow-up, selfsimilar solutions, blow-up rate, blow-up profile, regularity, semigroup estimates

ISBN (printed) 978-952-60-4138-4 **ISBN (pdf)** 978-952-60-4139-1

ISSN-L 1799-4934 **ISSN (printed)** 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Espoo **Location of printing** Helsinki **Year** 2011

Pages 144 **The dissertation can be read at** http://lib.tkk.fi/Diss/

A! Aalto-yliopisto

Tiivistelmä

Aalto-yliopisto, PL 11000, 00076 Aalto www.aalto.fi

Tekijä
Aappo Pulkkinen

Väitöskirjan nimi
Blow-up reaktio-diffuusio yhtälöissä eksponentiaalisella sekä potenssi-tyyppisellä epälineaarisuudella

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

Sarja Aalto University publication series DOCTORAL DISSERTATIONS 45/2011

Tutkimusala Matemaattinen analyysi

Käsikirjoituksen pvm 01.03.2011 **Korjatun käsikirjoituksen pvm** 10.05.2011

Väitöspäivä 15.06.2011 **Kieli** Englanti

☐ **Monografia** ☒ **Yhdistelmäväitöskirja (yhteenveto-osa + erillisartikkelit)**

Tiivistelmä

Tässä väitöskirjassa tutkimme blow-up ilmiötä semilineaarisissa parabolisissa yhtälöissä, joiden epälineaarisuus on joko eksponentiaalinen tai potenssi-tyyppinen. Tutkimme ratkaisujen käyttäytymistä blow-up hetkeä ja pistettä lähestyttäessä.

Keskitymme ylikriittiseen tapaukseen, mutta osa tuloksista on voimassa myös alikriittisessä tapauksessa. Tuloksemme koskee ratkaisujen blow-up nopeutta ja profiilia. Löydämme muun muassa heikkojen L^1-ratkaisujen blow-up profiilin similaarisuusmuuttujien suhteen sekä blow-up profiilin blow-up hetkellä. Tuloksien todistamisessa käytämme tekniikoita, jotka toimivat myös potenssi-tyyppisillä epälineaarisuuksilla. Todistamme myös ratkaisujen saavuttavan säännöllisyyden heti blow-up hetken jälkeen parantaen aikaisempia tuloksia.

Olemme kiinnostuneita myös itsesimilaarisista ratkaisuista ja todistamme tuloksen koskien säännöllisten itsesimilaaristen ratkaisujen olemassaoloa.

Avainsanat semilineaarinen parabolinen yhtälö, ylikriittinen tapaus, eksponentiaalinen epälineaarisuus, potenssi-tyyppinen epälineaarisuus, blow-up, itsesimilaariset ratkaisut, blow-up nopeus, blow-up profiili, säännöllisyys, puoliryhmä estimaatit

ISBN (painettu) 978-952-60-4138-4 **ISBN (pdf)** 978-952-60-4139-1

ISSN-L 1799-4934 **ISSN (painettu)** 1799-4934 **ISSN (pdf)** 1799-4942

Julkaisupaikka Espoo **Painopaikka** Helsinki **Vuosi** 2011

Sivumäärä 144 **Luettavissa verkossa osoitteessa** http://lib.tkk.fi/Diss/

Preface

I would like to express my gratitude to the advisor of my thesis, Stig-Olof Londen, for inspiring me to start the postgraduate studies as well as for carefully proofreading the manuscripts and for the many valuable comments.

I owe a great debt to Marek Fila who has always been very keen in sharing his wide knowledge about mathematics; thus, giving me the opportunity to learn a great deal about it. He has also been very kind for having me to visit Bratislava several times during the completion of this dissertation. I would also like to thank all the people from the community of blow-up research that have shared their time by discussing mathematics with me.

I am of course grateful for the Department of Mathematics and Systems Analysis at the Aalto University School of Science and the co-workers there for the help during the time of my postgraduate studies. I am also thankful to Gustaf Gripenberg for taking care of the duties as my supervisor during the last years. For the financial support, I am indebted to Magnus Ehrnrooth Foundation and the Finnish Cultural Foundation.

The importance of the encouragement of my friends and the love and support of my family and Saara is something that goes beyond my ability to express here.

Espoo, May 16, 2011,

Aappo Pulkkinen

Contents

Preface **7**

Contents **9**

List of Publications **11**

Author's Contribution **13**

1 Introduction **15**
1.1 Critical exponents and dimensions 19

2 Weak solutions **23**

3 Selfsimilar solutions **25**

4 Blow-up profiles of solutions **29**
4.1 Blow-up rate 29
4.1.1 Subcritical case 30
4.1.2 Supercritical case 31
4.2 Blow-up set 33
4.3 Selfsimilar profile 34
4.3.1 Constant selfsimilar profile 35
4.3.2 Nonconstant selfsimilar profile 38
4.4 Final time blow-up profile 39

5 Regularity of weak solutions **43**

Bibliography **47**

Errata **51**

Publications **53**

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I M. Fila, A. Pulkkinen, Backward selfsimilar solutions of supercritical parabolic equations, Applied Mathematics Letters **22** (2009), 897-901.

II M. Fila, A. Pulkkinen, Nonconstant selfsimilar blow-up profile for the exponential reaction-diffusion equation, Tohoku Mathematical Journal **60** (2008), 303-328.

III A. Pulkkinen, Blow-up profiles of solutions for the exponential reaction-diffusion equation, arXiv:1102.4158v1 [math.AP] (2011), 1-29, (accepted for publication in Mathematical Methods in the Applied Sciences).

IV A. Pulkkinen, Some comments concerning the blow-up of solutions of the exponential reaction-diffusion equation, arXiv:1102.4275v2 [math.AP] (2011), 1-18.

Author's Contribution

Publication I: "Backward selfsimilar solutions of supercritical parabolic equations"

The ideas for the proofs as well as a significant part of the writing is due to the author.

Publication II: "Nonconstant selfsimilar blow-up profile for the exponential reaction-diffusion equation"

The author is responsible for a substantial part of the writing and analysis, excluding the proof of Lemma 4.5.

Publication III: "Blow-up profiles of solutions for the exponential reaction-diffusion equation"

The author is responsible for the entire article.

Publication IV: "Some comments concerning the blow-up of solutions of the exponential reaction-diffusion equation"

The author is responsible for the entire article.

1. Introduction

In this dissertation I study the nonlinear reaction diffusion equation

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

To avoid unnecessary difficulties, let us assume for now that the initial function u_0 belongs to $C^1(\Omega)$ and that Ω is either an open domain in $\mathbb{R}^N$ with smooth boundary or the whole space $\mathbb{R}^N$ with $N \geq 1$. In the case of bounded Ω we will assume Dirichlet boundary condition throughout the current treatise.

Equation (1.1) is a semilinear example of more general reaction-diffusion equations, where, instead of the Laplace operator, any second-order elliptic operator can be used and the function f may also depend of x, t and the derivatives of u. There exists an extensive theory for the existence and uniqueness of solutions of these general types of equations; therefore, the behavior of the solutions of (1.1) is also well understood, at least locally in time. The problem is known to be well posed in many Lebesgue spaces $L^q(\Omega)$, by assuming, say, that the nonlinearity is a Lipschitz continuous function, and so a unique classical solution of (1.1) is known to exist for $t \in (0, T(u_0))$ with some maximal existence time $T(u_0) > 0$; see [49], [57].

On the other hand, the above equation is a simple nonlinear version of the fundamental linear parabolic equation $u_t = \Delta u$. The behavior of the solutions of this basic equation is well understood - they exist for every $t > 0$ and converge to zero as time tends to infinity, under rather weak assumptions on the initial data. However, whether the maximal existence time $T(u_0)$ of the nonlinear counterpart (1.1) is finite or infinite and what happens to the solution at $t = T(u_0)$ or after that time moment (in case $T(u_0) < \infty$) can not always be deduced from the classical theory of parabolic equations. The possibility of an occurrence of a singularity in some sense at $t = T(u_0)$ is one of the most important characteristics of nonlinear equations.

The formation of singularities that arise from the nonlinear character of the equation can be understood in an elementary way by considering the ordinary differential equation

$$u_t = f(u), \tag{1.2}$$

where the function f is positive and continuous and satisfies

$$\int_1^\infty \frac{1}{f(u)}\,\mathrm{d}u < \infty. \tag{1.3}$$

This ordinary differential equation has a unique solution $u(t)$ that tends to infinity at a certain rate, determined by the nonlinearity, as t tends to some finite T, depending on the initial data. Even though this example does not have any spatial structure and it therefore does not seem to be very interesting as such, the behavior of its solutions is characteristic to many solutions of the corresponding partial differential equation (1.1). As will be seen below, there exists a large class of solutions of (1.1) whose blow-up rate is the same as that of the solutions of (1.2).

Whether a solution of (1.1) is global, i.e., exists for every $t > 0$, or blows up at some finite time $t = T < \infty$, meaning that

$$\limsup_{t\to T} \|u(\cdot,t)\|_\infty = \infty,$$

depends on which part of the equation eventually dominates; the non-linear reaction $f(u)$ or the diffusion given by the Laplacian. The strength of the diffusion is affected by the geometry of Ω - the smaller the domain is, the stronger is the diffusion - so some qualitative properties of the solution can be formulated in terms of the size of the initial data, the geometry of Ω and the form of the nonlinearity f. The property of global existence was studied in [21] and [43], which are considered to be the pioneering works on blow-up, published in the 60's by Fujita and Kaplan.

In the paper [21], Fujita considers equation (1.1) with the nonlinearity $f(u) = u^p$ and $\Omega = \mathbb{R}^N$ in a seminal way. The main result is that when $p \in (1, p_F)$, then all positive solutions blow-up in finite time, whereas if $p > p_F$, then all solutions exist globally, presuming the initial data are small enough. Here p_F is an exponent (to be defined in the next section) depending on the dimension of the space. See [32, 44] for results on the critical case $p = p_F$. As a motivation for the study of such an equation, Fujita mentions the simplification of more involved nonlinear parabolic equations, such as the Navier-Stokes equation, still preserving some of the most important characteristics, such as the strong interplay between the spatial dimension and the degree of the nonlinearity. Actually, in general, the value of the critical exponent p_F is affected not only by the dimension, but also by the geometry of Ω. This kind of interplay between the geometry of Ω and the strength of the nonlinearity is very distinctive for nonlinear parabolic equations as a whole.

The nonlinearity

$$f(u) = e^u \tag{1.4}$$

arising, e.g., from thermal ignition processes, as proposed by Gelfand [26], was also studied by Fujita in [22], where sufficient conditions on blow-up either in finite or infinite time were obtained.

Kaplan studied, in [43], equation (1.1) with rather general nonlinearities in bounded domain Ω, assuming only (1.3) and that f is convex. He proved that then the solutions of (1.1) blow up in finite time, provided that the initial data is positive and large enough. This result was achieved by the famous method of multiplying equation (1.1) by the first eigenfunction of the Laplacian. See also [48], by Levine, for related results.

In this thesis the focus is on the nonlinearity (1.4), although many results are also valid for

$$f(u) = u|u|^{p-1}. \tag{1.5}$$

This is the most frequently studied nonlinearity in the context of blow-up and it is of course nothing more but the nonlinearity studied by Fujita, only generalized to non-positive solutions. Even though equation (1.1) with (1.4) and (1.1) with (1.5) share many properties, there are also some significant differences. There exist many results that are well known in the power case, but remain to be proved for the exponential.

One important difference, from the point of view of this treatise, is the following. Considering equation (1.1) in a ball, if the nonlinearity is the exponential $f(u) = \lambda e^u$, with $N \in [3, 9]$ and suitable $\lambda > 0$, then there exists a family of steady states, which all are saddle points of the system. This enables one to study solutions connecting two different steady states, and, in particular, such connections that blow-up in finite time, see [13] and references therein. However, equation (1.1) with the power nonlinearity has exactly one, or none, positive steady state, depending on the value of p. Therefore, such connections do not exist. The existence of this type of connecting orbits can be seen as one motivating factor for proving results for (1.4) in Publication II.

Another difference, that makes the analysis in some cases more difficult in the case of the exponential nonlinearity, is the form of the intrinsic rescaling associated to the problem. In the exponential case, it does not, for example, preserve positivity. Therefore, the methods used for the nonlinearity (1.5) do not always work when the reaction term is of type (1.4). In Publication III and Publication IV one cornerstone of the analysis has been to develop techniques that are applicable to both nonlinearities (1.4) and (1.5).

In Publication I we consider the properties of certain selfsimilar solutions both for the nonlinearity (1.4) and (1.5). Contrary to the above mentioned differences, the theory concerning the selfsimilar solutions, including our analysis in Publication I, is almost entirely analogous for both of the nonlinearities.

Our interest in blow-up phenomena is primarily mathematical. It should, however, be noted that already in the 1940's physicists were intrigued by this phenomenon which occurs, for example, in studies on combustion theory, in the solid fuel ignition model, and in thermonuclear combustion in plasma; see [5], [25] and references therein. In addition, many problems in mathematical biology have been formulated in terms of nonlinear parabolic partial differential equations - thus, the analysis of these models falls within the scope of blow-up studies.

In recent years, the study of blow-up has been extended to large classes of nonlinear parabolic equations and systems - by far bypassing the simple type of equations considered by Fujita and Kaplan. The studied problems include, e.g., equations with gradient terms such as the viscous Hamilton-Jacobi equation, quasilinear equations such as the porous medium equation, the p-parabolic equation, and different models from mathematical biology such as the Keller-Segel model describing chemotaxis, see, e.g., [25, 42, 57]

In addition, hyperbolic equations, such as the nonlinear wave equation, and equations of higher order, such as the fourth order Cahn-Hilliard equation or the thin film equation, have been studied from the point of view of considering the properties of blow-up solutions; see [12, 41]. In some of these cases, blow-up signifies singularity formation in some of the derivatives of the solution.

Considering the large class of equations for which blow-up in finite time may occur, a natural question is to try to understand the mechanism of blow-up and the behavior of blow-up solutions in more detail. However, it took nearly two decades, after the works of Kaplan and Fujita, for mathematicians to turn their interest into how blow-up takes place. In the mid 80's a surge of papers on this and related topics appeared, in particular on the blow-up rate and the profile of solutions, see Friedman, McLeod [20], Giga, Kohn [27, 28] and Weissler [63, 64]. Following these pioneering papers, numerous problems regarding blow-up of solutions have been posed and answered. Typical key questions asked have been whether blow-up occurs, where it occurs, how it occurs, and what happens after the blow-up time.

Even though the classical theory of parabolic regularity give satisfactory results when $t < T(u_0)$, it is unable to give any information on solutions that escape from the space where the problem is well posed. Therefore, it has been a challenge to develop new methods, which would give insight into what happens to the solutions after the blow-up time. It may happen, that a solution blows up completely, meaning that it can not exist after the blow-up time in any reasonable sense. However, as demonstrated in the papers [17], [24], [45], [56], solutions exist, which blow-up in finite time, and become regular, that is, bounded and smooth, immediately after the blow-up time. At the beginning of the 21st century, rather general solutions of (1.1) with nonlinearities (1.4) and (1.5) were considered in [15], and it was proved that certain minimal solutions become regular immediately after blow-up. More recently, the uniqueness of the continuation of blow-up solutions, solutions with multiple blow-up moments and regularization of nonminimal solutions have been under active research, just to mention a few topics.

1.1 Critical exponents and dimensions

Let us first consider equation (1.1) with the nonlinearity (1.5). As already the results of Fujita imply, the properties of the equation depend strongly on the strength of the nonlinearity together with the geometry of Ω. Let us define some of the critical values of the parameters p and N that have an influence on the characteristics of the equation.

Whether a solution blows up or not, depends on the initial data, and the competition between the diffusion and the nonlinearity. The strength of the nonlinearity is determined by p, and the strength of the diffusion depends on the geometry of Ω. In the case of $\Omega = \mathbb{R}^N$, this becomes visible through the critical exponent defined as

$$p_F = \frac{N+2}{N}$$

and named after Fujita. This exponent divides the p-range in two regions; $p > p_F$ with global solutions and $p < p_F$ without global solutions, [21].

The Sobolev critical exponent, defined through

$$p_S = \begin{cases} \frac{N+2}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N \leq 2, \end{cases}$$

draws a line between existence and nonexistence of the useful imbedding $H^1(\mathbb{R}^N) \subset L^{p+1}(\mathbb{R}^N)$. For $p < p_S$, one can often use functional analytic tools to attack some of the problems related to blow-up, whereas for $p > p_S$, one has to use more refined tools, such as intersection comparison. This leads to placing additional assumptions, such as radial symmetry, on the solutions. In what follows, the term subcritical refers to the case $p < p_S$, and supercritical to $p > p_S$.

The so-called Joseph-Lundgren exponent is given by

$$p_{JL} = \begin{cases} \infty & \text{for } 3 \leq N \leq 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{for } N \geq 11. \end{cases}$$

It was found in [40], where the existence of stationary solutions of (1.1), (1.5) was considered. This exponent is also related to the existence of so-called selfsimilar solutions.

There is one more exponent that is crucial to our analysis in this treatise. This is the Lepin exponent

$$p_L = \begin{cases} \infty & \text{for } 1 \leq N \leq 10, \\ 1 + \frac{6}{N-10} & \text{for } N > 10. \end{cases}$$

This exponent, discovered by Lepin in [47], is crucial for the existence of selfsimilar solutions.

When considering the exponential nonlinearity (1.4), the only parameter is the dimension N, so the situation is at least seemingly more simple. To find the critical dimensions for the exponential, the nonlinearity (1.4) is to be compared with the nonlinearity $f_p(u) = (1 + u/p)^p$ for which the critical exponents are also p_S and p_{JL}, see [40]. The exponential nonlinearity corresponds then to the limit $\lim_{p\to\infty} f_p(u)$. Therefore, it can formally be argued that the exponent p_S corresponds to the critical dimension $N = 2$ and the Joseph-Lundgren exponent corresponds to $N = 10$. See [10, 40, 60] for results that demonstrate some differences between the subcritical case $N \leq 2$, the supercritical case $N > 3$, and the case $N > 10$ for the exponential nonlinearity.

The structure of the dynamical system described by equation (1.1) is very different in the sub- and supercritical cases. The existence of steady states of (1.1), for example, depends on which of these cases we consider. Also, the behavior of the dynamical system obtained by rescaling, which determines the asymptotic behavior of many blow-up solutions as the blow-up moment is approached, is very different in the sub- and supercritical cases.

In the analysis below, the reader will be made more familiar with these critical exponents and dimensions and on how they affect the properties of the equation.

2. Weak solutions

By standard parabolic theory, the problem (1.1) is well posed in many Lebesgue spaces, provided f is assumed to be locally Lipschitz continuous. However, the classical solutions that are described by this theory are in many cases required to be too regular. The initial data may be too singular to fall in the space for which the problem is known to be well posed. In this case, the classical theory is unable to provide a solution.

The situation is the same if a solution blows up and escapes from the space where the problem is well posed. To that end, one can define a weaker type of solutions. Solutions of this type may have singular initial data and may exist also after the blow-up time.

For bounded Ω, we may define a weak solution of (1.1) on $[0, \mathcal{T}]$ to be a function $u \in C([0, \mathcal{T}];\ L^1(\Omega))$ such that $f(u) \in L^1(Q_\mathcal{T})$, where $Q_\mathcal{T} = \Omega \times (0, \mathcal{T})$ and such that the equality

$$\int_\Omega [u\psi]_{t_1}^{t_2}\, dx - \int_{t_1}^{t_2} \int_\Omega u\psi_t \, dx\, dt = \int_{t_1}^{t_2} \int_\Omega (u\Delta\psi + f(u)\psi)\, dx\, dt$$

holds for any $0 \le t_1 < t_2 \le \mathcal{T}$ and $\psi \in C^2(\bar{Q}_\mathcal{T})$ verifying $\psi = 0$ in $\partial\Omega \times [0, \mathcal{T}]$.

A priori it is not clear whether this or some other definition of a weak solution would make it possible for a solution to blow-up and still continue to exist after the blow-up time. However, Ni, Sacks and Tavantzis considered a weak solution of (1.1), (1.5) with bounded and convex Ω that is obtained as a limit of an increasing sequence of classical solutions belonging to the domain of attraction of the solution $u \equiv 0$, see [56]. By this approach they in fact constructed global unbounded weak solutions for $p \ge p_S$. But it was left open whether this solution becomes unbounded in finite or in infinite time. This question was answered, in the case of radially symmetric solutions in a ball, by Galaktionov and Vázquez in [24], by using intersection comparison method with suitable backward selfsimilar solutions. Later this work was completed by Mizoguchi in [54]. The results yield that these global unbounded weak solutions blow up in finite time whenever $p > p_S$.

For the exponential nonlinearity similar results were obtained by Lacey and Tzanetis in [45], where global unbounded weak solutions were discovered. Then, Fila and Poláčik proved in [17] that radially symmetric global unbounded weak solutions blow-up in finite time for $N \in [3, 9]$.

In the subcritical case with the nonlinearity (1.5), blow-up is always complete, which means that if a solution blows up in finite time, then no continuation of u exist beyond the blow-up time, see [1].

The above definition of a weak solution is relatively general, and as the construction of weak solutions in [56] demonstrates, there is a need for a more restrictive definition of a weak solution. To that end, let us define a limit L^1-solution of (1.1) to be a weak solution u on $[0, \mathcal{T})$ that can be approximated by a sequence $\{u_n\}_{n=1}^{\infty}$ of functions, for which u_n verifies (1.1) on $[0, \mathcal{T})$ in the classical sense with initial data $u_{0,n} \in C(\overline{\Omega})$, and is such that

$$u_{0,n} \to u_0 \quad \text{in } C(\overline{\Omega})$$

and

$$\begin{aligned} &u_n \to u \quad \text{in } L^1(\Omega) \text{ for every } t \in [0, \mathcal{T}), \\ &f(u_n) \to f(u) \quad \text{in } L^1(\Omega \times (0,t)) \text{ for every } t \in [0, \mathcal{T}). \end{aligned}$$

Moreover, a limit L^1-solution is said to be a minimal L^1-solution if the approximating initial data $\{u_{0,n}\}_n$ verifies

$$0 \leq u_{0,1}(x) \leq u_{0,2}(x) \leq u_{0,3}(x) \leq \dots$$

for every $x \in \Omega$.

Any limit L^1-solution is classical, and therefore unique, as long as the solution stays bounded. However, uniqueness after the blow-up time is a priori not given. Thereby, it is an interesting question whether a classical solution that blows up in finite time can be continued as a weak solution beyond the blow-up time in more than one way. For a long time this question was left open until, in [16], it was proved that the continuation does not have to be unique. But, even weak solutions do not have to be unique after the blow-up time, the minimal limit L^1-solution is unique.

3. Selfsimilar solutions

Considering equation (1.1) with nonlinearities (1.4) or (1.5), one notices that the equation exhibits some rescaling properties. Namely, if u is a solution, then

$$u_a(x,t) = \begin{cases} 2\log(a) + u(x_0 + a(x-x_0), t_0 + a^2(t-t_0)) & \text{for } f(u) = e^u, \\ a^{2/(p-1)} u(x_0 + a(x-x_0), t_0 + a^2(t-t_0)) & \text{for } f(u) = u|u|^{p-1} \end{cases} \tag{3.1}$$

is also a solution for any $a > 0$ at least locally around (x_0, t_0), when $(x_0, t_0) \in \Omega \times (0, T)$. This rescaling property is a quality that depends on the particular form of the equation, and it implies that there exists a special class of solutions of (1.1) with (1.4) or (1.5), called selfsimilar solutions.

A solution is said to be selfsimilar around (x_0, t_0) if it satisfies (1.1) with (1.4) or (1.5) for $(x,t) \in \mathbb{R}^N \times (-\infty, t_0)$ or $(x,t) \in \mathbb{R}^N \times (t_0, \infty)$ and is invariant under the above rescaling, that is, $u_a(x,t) = u(x,t)$ for every $a > 0$ and $(x,t) \in \mathbb{R}^N \times (-\infty, t_0)$ or $(x,t) \in \mathbb{R}^N \times (t_0, \infty)$. After some computations, one may conclude that in this case u has to be either a backward selfsimilar solution, defined for $x \in \mathbb{R}^N$ and $t < T = t_0$, satisfying

$$u_-(x-x_0,t) = \begin{cases} -\log(T-t) + \varphi_-\left(\frac{x}{\sqrt{T-t}}\right) & \text{for } f(u) = e^u, \\ (T-t)^{-1/(p-1)} \varphi_-\left(\frac{x}{\sqrt{T-t}}\right) & \text{for } f(u) = u|u|^{p-1}, \end{cases} \tag{3.2}$$

or a forward selfsimilar solution, defined for $x \in \mathbb{R}^N$ and $t > T = t_0$, verifying

$$u_+(x-x_0,t) = \begin{cases} -\log(t-T) + \varphi_+\left(\frac{x}{\sqrt{t-T}}\right) & \text{for } f(u) = e^u, \\ (t-T)^{-1/(p-1)} \varphi_+\left(\frac{x}{\sqrt{t-T}}\right) & \text{for } f(u) = u|u|^{p-1}. \end{cases} \tag{3.3}$$

Here

$$\begin{cases} \Delta\varphi_\mp \mp \frac{y}{2}\nabla\varphi_\mp + G_\mp(\varphi_\mp) = 0, \\ \varphi_\mp(0) = \alpha,\ \nabla\varphi_\mp(0) = 0 \end{cases} \tag{3.4}$$

and

$$G_{\mp}(\varphi) = \begin{cases} e^{\varphi} \mp 1 & \text{for } f(u) = e^u, \\ \mp\frac{1}{p-1}\varphi + \varphi|\varphi|^{p-1} & \text{for } f(u) = u|u|^{p-1}. \end{cases} \tag{3.5}$$

To restrict our attention to such selfsimilar solutions that belong to some reasonable function space, we assume that either $\varphi_{\mp}$ has to be a constant, or that the asymptotic conditions

$$C_\alpha = \begin{cases} \lim_{|y|\to\infty} \left(\varphi_{\mp}(y) + 2\log|y|\right) & \text{for } f(u) = e^u, \\ \lim_{|y|\to\infty} |y|^{2/(p-1)}\varphi_{\mp}(y) & \text{for } f(u) = u|u|^{p-1} \end{cases} \tag{3.6}$$

hold for some constant $C_\alpha \in \mathbb{R}$.

In what follows, we use the term backward selfsimilar solution both for the solutions of the elliptic equation (3.4)-(3.5) with the minus sign satisfying the correct asymptotics (3.6), and for the solutions of the parabolic equation defined through (3.2). Similarly, we slightly abuse the use of the term forward selfsimilar solution.

When $N > 2$ and $p > \frac{N}{N-2}$, there is an important class of selfsimilar solutions. These are the singular solutions

$$\varphi^*(y) = \begin{cases} -2\log|y| + \log(2(N-2)) & \text{for } f(u) = e^u, \\ \left(\frac{2}{p-1}(N-2-\frac{2}{N-2})\right)^{1/(p-1)} |y|^{-2/(p-1)} & \text{for } f(u) = u|u|^{p-1}. \end{cases} \tag{3.7}$$

These singular solutions are invariant under the rescalings in (3.2) and (3.3) and so they also satisfy the parabolic equation (1.1), with corresponding nonlinearities, for $|x| = |y| > 0$. Actually these singular solutions are stationary, global weak solutions of the equation, for $N > 2$ and $p > \frac{N}{N-2}$.

The existence of backward selfsimilar solutions is dependent on the sub- or supercriticality of the equation. In the subcritical case the only backward selfsimilar solutions are constants; for the exponential case $\varphi = 0$ and in the power case $\varphi \in \{0, \pm\kappa\}$, where $\kappa = (\frac{1}{p-1})^{1/(p-1)}$, see [10] and [27].

The supercritical case, however, is different since there exist many other backward selfsimilar solutions in addition to the constant ones. If $f(u) = e^u$ and $N \in (2, 10)$ or if $f(u) = u|u|^{p-1}$ and $p \in (p_S, p_{JL})$, then there exists at least a countable set $I \in (0, \infty)$ and a family $\{\varphi_-(\cdot;\alpha)\}_{\alpha\in I}$ of backward selfsimilar solutions satisfying (3.6) and $\varphi_-(0;\alpha) = \alpha$; see [8], [11], [45], [46], [47], [59].

For $p \in (p_{JL}, p_L)$, Lepin proved in [47] that there exists at least a finite number of backward selfsimilar solutions verifying (3.6). These results on the existence of selfsimilar solutions is again a good reminder on how the structure of the dynamical system related to the problem changes as p and N are varied.

The behavior of forward selfsimilar solutions is somewhat different. They exist, and satisfy (3.6), provided that $\alpha \in \mathbb{R}$, $N \in (2, 10)$ and $f(u) = e^u$, or $\alpha > 0$, $p \in (p_S, p_L)$ and $f(u) = u|u|^{p-1}$, see [31], [45], [60].

The selfsimilar solutions form an important class of solutions of (1.1) that have singularities. As we will see below, many other solutions of (1.1) behave almost like these selfsimilar solution, when the behavior is examined close to where blow-up takes place. In fact, many solutions are asymptotically selfsimilar as the blow-up moment $t = T$ and the blow-up point $x = x_0$ is approached. Knowledge about the properties of the selfsimilar solutions is therefore essential to the study of general blow-up solutions.

Our initial motivation for the study of backward selfsimilar solutions in Publication I was to determine if there exist selfsimilar solutions that blow-up completely. Since it is known that only those selfsimilar solutions that intersect the singular solution an odd number of times can blow-up completely, we determined the behavior of the selfsimilar solutions with respect to α as α is increased.

For the power nonlinearity there were already some results available. Lepin proved in [46] that if $p \in (p_S, p_{JL})$, then for every even integer, or large odd integer k, there exists a backward selfsimilar solution verifying (3.6), which intersects the singular solution k times. However, for the exponential nonlinearity, there were no results concerning the intersection number of selfsimilar and singular solutions.

In Publication I we consider backward selfsimilar solutions of equation (1.1) both for the nonlinearity $f(u) = e^u$ and for the nonlinearity $f(u) = u|u|^{p-1}$ and obtain some asymptotic properties of these solutions that complement the results in [11], [45] and [46]. The conclusion is that for every $k \geq 2$, and for $p \in (p_S, p_{JL})$, when the power nonlinearity is in question, or for $N \in (2, 10)$, when considering the exponential, there exists a backward selfsimilar solution that has k intersections with the singular solution φ^* and satisfies (3.6). Our result also holds for $p \in (p_{JL}, p_L)$ in the form that if there exists a backward selfsimilar solution with k_0 intersections with the singular solution, then there also exists a solution with k intersections for every $2 \leq k \leq k_0 - 3$.

The proof is based on comparison arguments and the idea is the following. First, by the results in [45], there exists a backward selfsimilar solution with two intersections with the singular solution. Then, as we increase α, first two intersections appear from infinity, after which the second one of these tend to infinity again. Considering the solution at the specific α for which the second intersection has just vanished, it can be shown that this solution has three intersection and it satisfies the correct asymptotics. When α is again increased, one intersection appears from infinity and at some point two additional intersections appear. The solution just before the appearance of the two intersections is our solution with four intersections and the correct asymptotics. By continuing this reasoning we obtain the conclusion.

Since our result is positive, in the sense that there exist backward selfsimilar solutions with an odd number of intersections with the singular solutions, it does not give an answer to the interesting and open problem concerning complete blow-up of backward selfsimilar solutions. At this point we can merely state that complete blow-up is possible and some further analysis is needed. An open problem is also whether the set $I \subset (0, \infty)$ of initial values α, for which the backward selfsimilar solutions $\varphi_-(\cdot;\alpha)$ satisfy (3.6) and $\varphi_-(0;\alpha) = \alpha$, is discrete or if it contains an interval (α_1, α_2) for some $0 < \alpha_1 < \alpha_2 < \infty$. This question in turn is related to the stability properties of the backward selfsimilar solutions.

4. Blow-up profiles of solutions

In this section we want to give a survey of the results in Publication II and Publication III.

We consider weak solutions of (1.1) on $[0, \mathcal{T}]$, with the nonlinearity (1.4), that blow-up in finite time $t = T < \mathcal{T}$. Our aim is to describe the behavior of such solutions as the blow-up moment is approached, and at the blow-up moment, as precisely as possible.

These solutions are known to exist in the supercritical range $N \in [3, 9]$, as explained in Section 2. Therefore, our focus is on the supercritical case, but naturally we also discuss some results valid in the subcritical range.

To obtain any information about how blow-up takes place, we must first know the blow-up rate. This question is considered in the next section.

4.1 Blow-up rate

Blow-up of solutions is categorized in two classes with respect to its rate. Considering the ordinary differential equation (1.2), we obtain solutions

$$u(t) = \begin{cases} -\log(T-t) & \text{for } f(u) = e^u, \\ (T-t)^{-1/(p-1)} & \text{for } f(u) = u^p \end{cases}$$

with some $T = T(u_0)$. Blow-up is said to be of type I if a solution of (1.1) blows up with the same rate as the solution of the corresponding ordinary differential equation, namely, if there exist constants $C_1, C_2 \in \mathbb{R}$ such that for every $(x,t) \in \Omega \times (0,T)$ one has

$$C_1 \leq \log(T-t) + \|u(\cdot,t)\|_\infty \leq C_2 \qquad \text{for } f(u) = e^u, \tag{4.1}$$

or

$$(T-t)^{1/(p-1)} \|u(\cdot,t)\|_\infty \leq C_1 \qquad \text{for } f(u) = u|u|^{p-1}. \tag{4.2}$$

Such blow-up is also referred to as selfsimilar blow-up since the blow-up rate is the same as the blow-up rate of selfsimilar solutions.

In Publication II, we first want to determine the blow-up rate of solutions of (1.1), (1.4) that blow-up and continue to exist as weak solutions after blow-up.

The only earlier results that are applicable for the exponential nonlinearity in the appropriate parameter range $N \geq 3$ are from the mid 80's. At that time researchers grew more interested in the question of how blow-up occurs as a result of the widely cited paper [20] by Friedman and McLeod. In this paper one of the first results regarding the blow-up rate of solutions was established. Friedman and McLeod studied equation (1.1) with rather general nonlinearities, including $f(u) = e^u$ and $f(u) = u^p$, and they obtained results concerning, among others, the location of the blow-up set, the blow-up rate and a priori bounds for solutions.

Their results imply type I blow-up for solutions of (1.1) with (1.4) or (1.5) when Ω is a convex domain. Their method utilizes the maximum principle and a clever auxiliary function, thus forcing an additional requirement of monotonicity in time on the solution. This additional assumption, however, implies that blow-up is complete, see [1]. Consequently, this result is of no use in our setting.

We refer also to [64] for an earlier result.

To explain our approach in determining the blow-up rate, let us review some earlier advances.

4.1.1 Subcritical case

A few years after the paper [20], Giga and Kohn considered the problem from a different perspective and noticed that in fact blow-up is of type I, provided that u is nonnegative, $f(u) = u|u|^{p-1}$ and p is subcritical. Their technique is more involved than that of Friedman and McLeod, but it captures better the essential features of these types of equations since it utilizes some specific information that is only valid in the subcritical range.

By using an intrinsic rescaling of the equation and energy methods, they avoid the use of the rather restrictive assumption of monotonicity appearing in [20]. However, because of the functional analytic energy methods, and because of the use of the nonexistence of positive stationary solutions of (1.1), (1.5), they need to assume either that the solution is nonnegative and $p \in (1, p_S)$ or, allowing sign changing solutions, $N \geq 2$ and $p \in (1, \frac{3N+8}{3N-4})$ or $N = 1$.

These assumptions are not purely technical since some formal arguments can be used to construct sign changing solutions with type II blow-up, when $p = p_S$, [18]. However, if u is radially symmetric and positive, then blow-up is of type I also for $p = p_S$, see [50].

Fifteen years later the technical assumption $p < \frac{3N+8}{3N-4}$ was made redundant, and type I blow-up was obtained for every solution of (1.1) if p is subcritical, see [30]. This was made possible by refining the argument in [28] and introducing localized energies and applying a bootstrap argument for improved L^p estimates.

A completely different method for obtaining the blow-up rate was used by Galaktionov and Posashkov in [23] to prove type I blow-up for non-negative radially symmetric solutions of (1.1), (1.5) and subcritical exponents p. The one-dimensional character of the equation allowed them to use the intersection comparison technique in their proof. The same method was used in [38] for the exponential nonlinearity with $N = 1$, where type I blow-up is proved for any positive solution. Filippas, Herrero and Velazquez used similar ideas in [18] to prove, for the exponential equation with $N = 2$, that blow-up is of type I, provided that the solution is positive, radially symmetric and radially decreasing.

In Publication IV we consider the subcritical case $N = 2$ and $\Omega = B(R)$ or $\Omega = \mathbb{R}^2$ and prove that radially symmetric solutions of (1.1), (1.4) blow-up with type I rate, provided that the maximum of the solution is attained at the origin. The proof is based on combining techniques from [17] and [50], where the supercritical case was considered. Let us discuss the results from these papers and summarize some results on the supercritical case in the next section.

4.1.2 Supercritical case

The case of $p > p_S$ remained open until the early 21st century. The problem with the supercritical case is the lack of the Sobolev embeddings, which, in the subcritical case, allow one to use interior regularity arguments to obtain boundedness of solutions once certain L^p estimates are verified. Instead of these powerful techniques, one has to use a refined version of the maximum principle, namely, intersection comparison. This is, however, a tool for one-dimensional problems, which forces one to consider only radially symmetric solutions.

In the paper [50], Matano and Merle were able to prove type I blow-up by assuming radial symmetry and restricting the values of p to be supercritical but smaller than the Joseph-Lundgren exponent. Since they assume radial symmetry, they are able to use one-dimensional regularity arguments and energy estimates to obtain a priori upper bounds for solutions. These estimates allow them to prove that if u blows up with type II rate, then the sequence $\{u_{a_n}\}_n$ of rescaled solutions, as defined in (3.1), converges to a steady state of equation (1.1), (1.5) along some sequence $\{a_n\}_n$ tending to infinity. By using intersection comparison and the fact that every steady state of (1.1), (1.5) intersects with the singular solution φ^* in (3.7) infinitely many times if $p \in (p_S, p_{JL})$, they obtain a contradiction. By this method, they were able to prove type I blow-up for solutions both in a ball and in $\mathbb{R}^N$.

Our method in Publication II is based on these same ideas. We prove that the blow-up rate is of type I also in the exponential case, when radial solutions in a ball are in question, the maximum of the solution is attained at the origin, and the dimension of the space is between three and nine. We use the same rescaling as in [50] to obtain a sequence $\{u_{a_n}\}_n$, but because the rescaling in the exponential case does not automatically provide a lower bound, we need to use a result from [20] to achieve the convergence of that sequence. Another advantage of the power case over the exponential is the possibility to use energy methods. Since the usual rescaling in the exponential case does not preserve positivity, it is far more difficult to use the energy of the solution and thus obtain a priori bounds for the solutions. This forces us to assume that the solution attains the maximum at the origin.

To prove that the sequence $\{u_{a_n}\}_n$ converges to a steady state, we use a technique from [9], which utilizes the intersection comparison method and gives us a more concise proof. This approach allows us to obtain the rate even for more general nonlinearities that behave like the exponential far away from the origin, but do not allow such rescaling as in (3.1). Moreover, for such nonlinearities and $N \in [3, 9]$, we obtain also the boundedness of radially symmetric global classical solutions, which improves on the results in [17]. The same was proved for the power nonlinearity in [9].

In the two-dimensional case, discussed in Publication IV, we use these same ideas to prove type I blow-up. As in Publication II, we know that type II blow-up would imply the convergence of the sequence $\{u_{a_n}\}_n$ to some steady state of the equation. Following the treatise in [50], where the power case is considered for $p = p_S$, and using the intersection diminishing property, one concludes that this convergence implies the existence of two steady states that do not intersect. However, every two steady states are known to intersect, see [58], where explicit formulas for radial steady states of (1.1), (1.4) are found. Therefore, type II blow-up does not occur.

This method is reliant on the assumption that u is radially symmetric because intersection comparison is being used. Furthermore, we have to assume that u attains its maximum at the origin for the same reasons as in Publication II. Proving the blow-up rate for radially non-symmetric solutions of (1.1), (1.4) with $N = 2$ remains an open problem.

Moreover, it is not known if type II blow-up can take place for radially nonsymmetric solutions in the parameter $N \in [3, 9]$, for the nonlinearity (1.4), or $p \in (p_S, p_{JL})$, for the nonlinearity (1.5). The restriction $p < p_{JL}$ is, however, strict since there are known to exist type II blow-up solutions for $p > p_{JL}$, see [39], [53], [54]. Type II blow-up generally speaking corresponds to solutions that behave asymptotically as the singular selfsimilar solution. The existence of type II blow-up is then obtained by proving that the singular selfsimilar solution attracts some solutions. This method will only work for $p > p_{JL}$, thus suggesting that type II blow-up would only be present for $p > p_{JL}$.

Different rates for type II blow-up are obtained in [55].

4.2 Blow-up set

In this treatise we will not overly emphasize where blow-up occurs. Nevertheless, let us give some basic properties of the blow-up set.

A point $x_0 \in \Omega$ is defined to be a blow-up point if there exists a sequence $\{(x_i, t_i)\}_i \subset \Omega \times (0, T)$ such that $x_i \to x_0$ and $t_i \to T$ and $u(x_i, t_i)$ tends to infinity as i approaches infinity. It is known that the blow-up set, consisting of the blow-up points, of any solution of (1.1) with (1.4) or (1.5) is a compact set of Ω, provided Ω is a convex and bounded domain in $\mathbb{R}^N$, see [20]. Moreover, if $\Omega = B_R(0)$ and u is radially symmetric, with u_0 radially decreasing, then $x = 0$ is the only blow-up point. The same holds for radially symmetric solutions of (1.1) with (1.4) or (1.5) if the blow-up is not complete, see Publication IV and [50].

In general, however, the blow-up set can be a finite number of points, a region in Ω or the whole set $\overline{\Omega}$. If $\Omega = \mathbb{R}^N$, the solution u can also remain bounded in compact subsets of $\mathbb{R}^N$ and blow-up only at space infinity, i.e., $u(x_i, t_i) \to \infty$ for $|x_i| \to \infty$ and $t_i \to T$, see [57]

In the following sections, we will most of the time assume that the solution blows up at $(x,t) = (0,T)$. Some of the results also work under the assumption that blow-up takes place at an arbitrary $a \in \Omega$.

4.3 Selfsimilar profile

In what follows, we will discuss two kinds of blow-up profiles. On one hand, we will discuss final time blow-up profiles, by which we mean the pointwise limit profile $u(x,T) = \lim_{t\to T} u(x,t)$. On the other hand, we will consider selfsimilar profiles, in which case the profile of a solution is determined by the convergence to a selfsimilar solution.

Assuming that a solution u of (1.1) blows up with type I rate at $(x,t) = (0,T)$, it is convenient to use this information together with the intrinsic rescaling arising from the selfsimilarity of the equation to define

$$w(y,s) = \begin{cases} \log(T-t) + u(\sqrt{T-t}y, t) & \text{for } f(u) = e^u, \\ (T-t)^{-1/(p-1)} u(\sqrt{T-t}y, t) & \text{for } f(u) = u|u|^{p-1}, \end{cases} \tag{4.3}$$

where the new similarity variables are defined as $s = -\log(T-t)$ and $y = \frac{x}{\sqrt{T-t}}$. This rescaling implies that w verifies the equation

$$w_s = \Delta w - \frac{y}{2}\nabla w + G_-(w) \tag{4.4}$$

in some domain $\Omega_s \subset \mathbb{R}^N$ depending on s and with some initial data. Here G_- is as in (3.5). The properties of u at the blow-up moment near the blow-up point are then reflected in the asymptotic behavior of w as $s \to \infty$.

In other words, to study the asymptotics of u as the blow-up moment is approached, one needs to consider whether w converges as $s \to \infty$. For $F(w) = \int_0^w G_-(v)\mathrm{d}v$, the energy

$$E(w) = \int_{\Omega_s} \left(\frac{|\nabla w|^2}{2} + F(w) \right) \rho \, \mathrm{d}y,$$

where Ω_s is the domain of w and $\rho(y) = e^{-|y|^2/4}$, is a Lyapunov functional for the problem. Since type I blow-up, and some additional assumption for the exponential nonlinearity, imply that the energy is bounded from below, it is a relatively easy matter to show that w approaches the set of equilibria of (4.4) as s tends to infinity, see [27].

4.3.1 Constant selfsimilar profile

In the subcritical case, the only backward selfsimilar solutions are the constants 0 and $\pm\kappa$, and it can be argued that w converges to one of these steady states as s tends to infinity. In [29], this result is refined by demonstrating that if $f(u) = u|u|^{p-1}$, then w cannot converge to zero. Consequently, we have the conclusion that if u is a solution of (1.1), with (1.5), and $p < p_S$, or with (1.4), and $N \leq 2$, that blows up at $(x,t) = (0,T)$ with type I rate, then

$$\lim_{s\to\infty} w(y,s) = \begin{cases} 0 & \text{for } f(u) = e^u, \\ \pm\kappa & \text{for } f(u) = u|u|^{p-1} \end{cases} \tag{4.5}$$

uniformly for y in compact sets. If (4.5) holds, we say that u has a constant selfsimilar blow-up profile. We refer to [2], [4], [27], [28] for related results.

Now two questions arise. Although the selfsimilar profile gives a certain description of the behavior of u near the blow-up point, it does not give any direct information about the solution at the blow-up moment. Therefore, we can ask whether we can say something about the final time blow-up profile $u(x,T)$ by assuming a constant selfsimilar blow-up profile. Secondly, since in the supercritical case there exist many selfsimilar solutions different from the constants, do there exist some solutions that have nonconstant selfsimilar blow-up profiles?

Our aim in Publication II is to answer both of these questions, when $f(u) = e^u$.

As concerns the first question, note that there are many results already available, but mainly for the nonlinearity $f(u) = u|u|^{p-1}$. These results are obtained by following the same basic steps, which we also obey in our approach in Publication II.

The first step is to find the leading term of the convergence (4.5). The outcome is that the rate of the convergence is either algebraic or exponential. In the literature this has been achieved essentially in two different ways. The method used by Herrero and Velázquez, in a series of papers, see [34] and [62], is based on some perturbation techniques and the maximum principle.

Our method, however, is based on the approach of Filippas, Kohn and Bebernes, Bricher, see [3], [19]. Their idea is to utilize some ideas from center manifold theory for infinite-dimensional dynamical systems, even though the problem does not strictly speaking fit in that framework. Since the linear part of the operator on the right hand side of (4.4) has zero as an eigenvalue, the convergence of w corresponds either to convergence along the center stable manifold or the stable manifold (see [33]). In the former case the convergence rate of w is algebraic and in the latter case w converges with exponential rate.

The second step is to extend the domain, in which the convergence (4.5) takes place, from bounded to some larger sets depending on s. This is done by using the variation of constants formula and certain semigroup estimates with respect to shifted L^p-norms. The calculations are rather straightforward, but technically complicated. The principal idea is to carefully analyze the interplay between the rate of the convergence and the convection caused by the term $-\frac{y}{2}\nabla w$ in (4.4). The result is that the convergence holds in expanding domains with radius of the order $\sqrt{s}$, instead of only in bounded sets. This part of the proof goes through along the lines of [61].

In the last step we use these refined convergence results to obtain the behavior of $u(x,T)$ close to the blow-up point $x = 0$. This is essentially a consequence of the stability of certain solutions. The problem in the exponential case, when compared to the case of power nonlinearity, is that the scaling (4.3) does not produce a bounded function even though the blow-up rate is assumed to be of type I. Another problem arises from the fact that the linear part of the operator in (4.4) has only negative eigenvalues in the power case, whereas zero is an eigenvalue in the exponential case. This makes it more complicated to prove stability and we are forced to use an upper bound for the solution. This upper bound (obtained in [15]) is, however, valid only if the solution is radially nonincreasing.

Nonetheless, we are able to prove that if u is a radially nonincreasing solution of (1.1) with the nonlinearity (1.4) in a ball, and u blows up with type I rate and has a constant selfsimilar blow-up profile, that is, w tends to zero as s tends to infinity, then

$$\lim_{|x|\to 0}\left(u(x,T) + 2\log|x| - \log|\log|x|| - \log(8)\right) = 0. \tag{4.6}$$

The final time blow-up profile for the power nonlinearity was first considered in [34], [35] and [36], when $N = 1$. It was proved that if u is a solution of (1.1) in $\mathbb{R}$, with the nonlinearity (1.5), and u blows up with type I rate and has a constant selfsimilar blow-up profile, then either

$$\lim_{x\to 0}\left(\frac{|x|^2}{\log|x|}\right)^{1/(p-1)} u(x,T) = [8p/(p-1)^2]^{1/(p-1)}, \tag{4.7}$$

or there exists a constant $C > 0$ and an even integer $m \geq 4$, such that

$$\lim_{x \to 0} |x|^{m/(p-1)} u(x,T) = C. \tag{4.8}$$

Velázquez then generalized these results to higher space dimensions in [62, 61] while the Dirichlet problem in a ball was covered by Matos in [52].

In the papers [34] and [38], Herrero and Velázquez treated also the case of exponential nonlinearity in one dimension with similar techniques and obtained comparable results. Their technique in the one-dimensional case is, however, different from the higher dimensional since they use suitable subsolutions and comparison methods not applicable in higher dimensions.

See also [6] for a result on the existence of solutions with the profile (4.6).

It is interesting to note that we do not obtain profiles of the type (4.8) in Publication II. This is because the profiles (4.8) correspond to solutions that have m maxima which all converge to zero at the blow-up moment, [36]. Our assumption that u is radially nonincreasing thus rules out such solutions. It is in fact known that all the profiles (4.8) with even $m \geq 4$ do occur, see [7] and [36], and that the profile (4.7) is the generic one and stable under certain perturbations, [37].

In Publication III we are able to improve our result and show that it is unnecessary to assume that the solution is radially nonincreasing, provided we assume that blow-up is of type I and that the solution blows up at the origin. With these assumptions we obtain that if w converges to zero as s tends to infinity, then either (4.6) holds or

$$\lim_{|x| \to 0} \left(u(x,T) + m \log|x| - C\right) = 0 \tag{4.9}$$

for some $m \geq 4$ and constant C. This improvement is obtained by using a recursive argument together with a derivative estimate originating from [20], thereby overcoming some technical difficulties faced in Publication II.

In addition, as mentioned in the previous section, it is proved in Publication II that a solution blows up with type I rate at the origin, if $N \in [3,9]$ and the maximum of the solution is attained at the origin.

4.3.2 Nonconstant selfsimilar profile

In the supercritical case, there are many other possible selfsimilar profiles besides the constant ones. To prove that some solutions actually have a nonconstant profile, one needs to consider solutions that continue to exist as weak solutions beyond the blow-up time.

By the results presented in the previous section, if a solution has a constant selfsimilar blow-up profile, then the precise form of $u(x,T)$ near the blow-up point is known and given by (4.6) or (4.9) in the case of the exponential nonlinearity and (4.7) or (4.8) when the power-type nonlinearity is in question. These types of profiles are, however, too large near the blow-up point in order for a weak continuation of the solution after the blow-up time to exist, see [52] and [60]. Therefore, by proving that w converges to the equilibria of (4.4), one can deduce that if a solution blows up with type I rate and continues to exist as a weak solution after the blow-up time, then the selfsimilar blow-up profile has to be a nonconstant one. This method was used by Matos in [52], where the above mentioned result was proved for $f(u) = u^p$.

We demonstrate in Publication II that this same approach works also for the exponential nonlinearity. As explained in the previous section, the solutions that have a constant selfsimilar blow-up profile and blow-up with type I rate, verify either (4.6) or (4.9) at the blow-up moment. These final time profiles imply complete blow-up by a result of Vázquez from [60]. Therefore, a nonconstant selfsimilar blow-up profile is the only possibility for weak solutions that blow-up in finite time. To be more precise, if u is a radially symmetric weak solution of (1.1), (1.4) on $(0, \mathcal{T})$ that blows up at $t = T < \mathcal{T}$ with type I rate, then

$$w(y,s) \to \varphi_-(y) \quad \text{as } s \to \infty \tag{4.10}$$

uniformly on compact sets, where φ_- is a solution of (3.4)-(3.6).

It is a different matter to explain to which of the infinitely many backward selfsimilar solutions φ_- the solution does converge. In Publication II we were able to give an example of a solution for which this profile can be determined. A singular connection u is a global weak solution of equation (1.1) that blows up in finite time and connects two stationary solutions v_- and v_+, i.e., $\lim_{t\to-\infty} u(x,t) = v_-(x)$ and $\lim_{t\to\infty} u(x,t) = v_+(x)$. These singular connections of (1.1) with nonlinearity (1.4), were considered in [14]. Taking one particular such a connection we proved that it has a nonconstant selfsimilar blow-up profile, which intersects with the singular solution φ^* exactly two times. This is the first example of a solution for which such information can be retrieved.

We can do this only for one type of connections, i.e., for such that connect two steady states with a prescribed number of intersections with the singular steady state. There are, however, many other connections between different steady states. We cannot show which one of the selfsimilar blow-up profiles these connections have in general. To find the selfsimilar profile of a general connection, one should be able to consider the number of those intersections of the connecting solution and the singular solution that vanish at the blow-up moment.

4.4 Final time blow-up profile

Since every weak solution that exists on $[0, \mathcal{T})$ and blows up at $t = T < \mathcal{T}$ has a nonconstant selfsimilar blow-up profile, one may ask if the final time blow-up profile of such a solution can be found.

If one chooses to approach the problem as in the case of a constant selfsimilar profile, the first thing to consider is the convergence rate of (4.10). To attack this problem, one linearizes the operator on the right hand side of (4.4) around φ_- and hopes to find some properties of the eigenvalues of the obtained operator

$$\Lambda = \Delta - \frac{y}{2}\nabla + e^{\varphi_-}. \tag{4.11}$$

Especially, the interest here lies in whether zero is an eigenvalue or not. This question turns out to be quite delicate, and, as far as the author is aware, no answer has been found, neither for the exponential nor for the power nonlinearity.

Consequently, the above approach does not seem to produce any results. Fortunately, there are other methods, whose outcome has been satisfactory. The analysis in Publication III was done in part simultaneously with [51] where the question was answered for $f(u) = u|u|^{p-1}$ and a classification of blow-up of radial solutions was given in the following sense. Blow-up is of type I with constant selfsimilar profile if and only if the limit

$$\lim_{|x|\to 0} |x|^{2/(p-1)} u(x, T)$$

is infinite. Blow-up is of type I with nonconstant selfsimilar profile if and only if the above limit is finite and not equal to plus or minus one or zero. Blow-up is of type II if and only if the above limit equals plus or minus one. There is no blow-up if the limit equals zero.

Our result in Publication III is only partial when compared to that in [51], but it has some advantages. It states that if a solution of (1.1) with (1.4) blows up with type I rate and the selfsimilar blow-up profile is nonconstant, then

$$\lim_{|x|\to 0}\left(u(x,T)+2\log|x|\right)=C_{\alpha}. \tag{4.12}$$

This can be reformulated together with the results from the previous section to give that if a solution of (1.1), with (1.4), blows up at $(x,t)=(0,T)$ with type I rate, then it has a nonconstant selfsimilar profile if and only if (4.12) holds. It has a constant selfsimilar profile if and only if (4.6) or (4.9) holds.

One advantage of our approach to that in [51] is that our proof is rather general using only semigroup estimates and variation of constants formula, and it works also with the nonlinearity (1.5), whereas the proof in [51] cannot be directly applied to the exponential nonlinearity. The reason for this is that the analysis in [51] is vitally based on the a priori estimates in [50]. Those estimates, as already mentioned in the previous sections, are based on energy methods, which do not seem to apply to the exponential case.

Our method, on the other hand, is based on a simple observation. Since the convergence (4.10) is assumed to hold for y in compact sets, and because φ_- verifies (3.6), we only need to show that the convergence (4.10) holds approximately for $y=e^{s/2}\xi$, when $|\xi|$ is small. This gives the behavior (4.12) as s tends to infinity by the definition of w and the asymptotics (3.6).

To carry through this analysis, we consider the semigroup generated by the operator Λ in (4.11) with respect to weighted L^p-norms, whose weight has been shifted away from the origin by a factor of $e^{s/2}$. Since $e^{\varphi_-(y)}$ tends to zero as y tends to infinity, we are able to prove that the semigroup generated by Λ behaves more or less like the semigroup generated by the Hermite operator $\Delta-\frac{y}{2}\nabla$ with respect to those shifted norms. Then a careful stability type analysis with respect to the shifted norms give us the desired result.

Another advantage of our approach is that, assuming that blow-up is of type I and that the solution has a nonconstant selfsimilar blow-up profile, we do not need to assume the radial symmetry of the solution. We only work with semigroup estimates and variation of constants formula, thus being independent of any arguments related to the dimension or symmetry. In [51], on the other hand, parabolic regularity results in one dimension are being used, which make the assumption on radial symmetry crucial.

5. Regularity of weak solutions

As demonstrated in the previous sections, there exist weak solutions that are global but blow-up in finite time. Moreover, by assuming this property, we can recover the selfsimilar and final time blow-up profiles of the solutions.

However, the regularity of the weak solutions after the blow-up time is far from being transparent since it cannot be deduced from the standard parabolic theory. This is because the appearing singularity may be such that the solution escapes from the space where the problem is well posed. For example, the problem (1.1), (1.5) with $p > 1$ is well-posed in $L^q(\Omega)$ if $q > \frac{n(p-1)}{2}$, see [57]. This means that the singularity of u must be weaker than that of $|x|^{-2/(p-1)}$ in order for u to belong to L^q, ruling out the backward selfsimilar solutions for example.

The problem of regularity of weak solutions can be approached by constructing examples of some weak solutions that behave, in a certain sense, decently. Galaktionov and Vázquez constructed, in [24], special weak solutions that are obtained by gluing together backward, see (3.2), and forward, see (3.3), selfsimilar solutions having the same profile at the blow-up moment. They proved that starting from the initial data

$$u_0(x) = C|x|^{-2/(p-1)}$$

corresponding to the case (1.5), the so-called proper solution of (1.1) in $\Omega = \mathbb{R}^N$ is the forward selfsimilar solution given by (3.3), when the asymptotics (3.6) is satisfied with $C_\alpha = C$. Therefore, by taking a backward selfsimilar solution such that there exists a forward selfsimilar solution that shares the same asymptotic behavior (3.6), we can match them at the blow-up moment and obtain a weak solution that blows up at $(x,t) = (0,T)$. The blow-up rate of such a solution is of course selfsimilar and the solution becomes regular immediately after blow-up, also with the selfsimilar rate. Naturally, the construction only works in $\mathbb{R}^N$ since the selfsimilar solutions do not satisfy reasonable boundary conditions.

This kind of solutions were treated in [45] in the case of the exponential nonlinearity, while solutions with singular initial data were considered also in [60]. These are only some examples of blow-up solutions, but as we have already seen, the selfsimilar behavior is very characteristic of many other solutions as well.

In Publication IV we use our results from Publication III to prove that if u is a radially symmetric minimal limit L^1-solution of (1.1), (1.4) on $(0, \mathcal{T})$ that blows up at $t = T < \mathcal{T}$ and has a nonconstant selfsimilar blow-up profile, then it becomes regular immediately after the blow-up time $t = T$. The same technique can be used in the case of the power-type nonlinearity as well. Apart from one special case, we also obtain that the regularization is asymptotically selfsimilar. Hence, the behavior of such solutions around the blow-up moment $t = T$ is the same as the behavior of the peaking selfsimilar solutions treated in [45]. We prove the results by using a comparison principle from [60], based on the existence of forward selfsimilar solutions, and refining the techniques in [15].

In [60] Vázquez gives necessary conditions for complete blow-up. He proves that there exists a constant μ_e such that if

$$u_0(x) > -2\log|x| + \mu_e + \epsilon$$

for $|x| \leq \gamma$, for some $\gamma, \epsilon > 0$, then there is no weak solution of (1.1), (1.4). On the other hand, if the above inequality holds with reversed inequality sign $<$, and $\epsilon = 0$, then the minimal limit L^1-solution can be compared with the maximal forward selfsimilar solution. Thus, it is regular at least locally for $t > 0$, and the rate of regularization is selfsimilar. For analogous results in the case of (1.5), see [52].

Since our results in Publication III imply that every solution with a nonconstant selfsimilar blow-up profile satisfies (4.12) for some constant C_α, we may apply the above result of Vázquez. If such a solution can be continued beyond the blow-up time, then one has that either $C_\alpha < \mu_e$ or $C_\alpha = \mu_e$. In the former case, $u(x, T) < -2\log|x| + \mu_e$ near the origin and so the solution is regular for $t > T$, at least locally. In this case, we can also prove that the regularization is asymptotically selfsimilar, i.e.,

$$\log(t - T) + u(\sqrt{t - T}y, t) \to \varphi_+(y)$$

uniformly on compact sets as t tends to T from above. In the latter case we cannot always use the comparison of Vázquez, but we have to use ideas from [15] suitably modified. In this case, the selfsimilarity of the regularization is out of reach. However, we obtain that the solution becomes regular immediately without assuming that u is radially nonincreasing, which is assumed in [15].

The regularity of weak solutions that blow-up and continue to exist after the blow-up time was considered by Fila, Matano and Poláčik in [15]. They proved that the minimal limit L^1-solution becomes regular immediately after the blow-up time if the solution is radially symmetric and either $f(u) = e^u$, u is radially nonincreasing and $N \in [3,9]$, or $f(u) = u^p$ and $p \in (p_S, p_{JL})$. Their method was to first prove a priori bounds for the solution and then apply these to obtain regularity. This allowed them to use comparison with forward selfsimilar solutions and energy methods together with intersection comparison to prove the result. In the case of the power-type nonlinearity, the a priori bounds were already proved in [51]. For the exponential, however, they needed to assume that the solution is radially nonincreasing. This was assumed in order to apply a refined version of Kaplans eigenfunction method first to get a weaker type of a priori bounds. Assuming the minimality of the continuation is crucial in the proof since they use the comparison method between the approximating sequence and the forward selfsimilar solutions.

Since we do not assume that the solution is radially nonincreasing, we cannot proceed as in [15]. Instead, we consider the solution for t strictly larger than the blow-up moment $t = T$. Then we use an intersection comparison method with a solution having the blow-up profile (4.6). This gives us the desired upper bound. We can also prove that if the solution does not become regular immediately after blow-up, then it has to be locally nonincreasing close to the blow-up point. By these results, we are able to proceed along the lines of [15].

In the recent paper [51], immediate regularization is proved for (1.1), (1.5) without the minimality assumption. The proof is based on the same a priori estimates of [50] that were used in [15], and on a clever rescaling method. This approach is not applicable for the exponential case because of the energy methods used in [50].

The technique that we utilized in Publication IV cannot be used to attack the case of nonminimal continuations that were treated in [51]. This is because of the use of a comparison method with forward selfsimilar solutions which requires that the approximating sequence related to the continuation stays below the solution it converges to. The regularization of nonminimal continuations in the exponential case therefore remains an open problem. Recall that nonuniqueness of continuations of blow-up solutions was proved in [16].

Here we can also observe a connection to the question of complete blow-up of backward selfsimilar solutions, discussed in Section 3. If one could prove that for every backward selfsimilar solution φ_-, the constant C_α in (3.6) is smaller than μ_e, then there is no complete blow-up for backward selfsimilar solutions. In that case, regularization of the minimal limit L^1-solutions, with nonconstant selfsimilar blow-up profile, would be a direct consequence of our results in Publication III and the results of Vázquez. Moreover, the method would be completely independent of radial symmetry, if type I blow-up is assumed to take place at $x = 0$.

There is one more important observation. We prove in Publication IV that if u is radially symmetric and continues to exist as a weak solution beyond the blow-up time, then $x = 0$ is the only blow-up point. Therefore, if a solution is a radially symmetric, minimal limit L^1-solution that blows up at $t = T < \mathcal{T}$ with type I rate, then it blows up at $x = 0$, and, by the results in Publication II, it has a nonconstant selfsimilar blow-up profile. Thereby, the results of Publication IV can be applied and it becomes regular immediately after blow-up. Type I blow-up, on the other hand, is given if $N \in [3, 9]$ and the solution attains the maximum at the origin. Therefore, our results improve on those of [15].

Bibliography

[1] P. Baras, L. Cohen, *Complete blow-up after T_{max} for the solution of a semilinear heat equation*, J. Funct. Anal. **71** (1987), 142-174.

[2] J. Bebernes, A. Bressan, D. Eberly, *A description of blowup for the solid fuel ignition model*, Indiana Univ. Math. J. **36** (1987), 295-305.

[3] J. Bebernes, S. Bricher, *Final time blow-up profiles for semilinear parabolic equations via center manifold theory*, SIAM J. Math. Anal. **23** (1992), 852-869.

[4] J. Bebernes, D. Eberly, *A description of self-similar blow-up for dimensions* $n \geq 3$, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), 1-21.

[5] J. Bebernes, D. Eberly, *Mathematical Problems from Combustion Theory*, Springer-Verlag, New York, 1989.

[6] A. Bressan, *Stable blow-up patterns*, J. Differential Equations **98** (1992), 57-75.

[7] J. Bricmont, A. Kupiainen, *Universality in blow-up for nonlinear heat equations*, Nonlinearity **7** (1994), 539-575.

[8] C. J. Budd, Y.-W. Qi, *The existence of bounded solutions of a semilinear elliptic equation*, J. Differential Equations **82** (1989), 207-218.

[9] X. Chen, M. Fila, J.-S. Guo, *Boundedness of global solutions of a supercritical parabolic equation*, Nonlinear Anal. **68** (2008), 621-628.

[10] D. Eberly, *Nonexistence for the Kassoy problem in dimensions* 1 *and* 2, J. Math. Anal. Appl. **129** (1988), 401-408.

[11] D. Eberly and W. C. Troy, *Existence of logarithmic-type solutions to the Kassoy-Kapila problem in dimensions* 3 *through* 9, J. Differential Equations **70** (1987), 309-324.

[12] J. D. Evans, V. A. Galaktionov, J. F. Williams, *Blow-up and global asymptotics of the limit unstable Cahn-Hilliard equation*, SIAM J. Math. Anal. **38** (2006), 64-102.

[13] M. Fila, *Blow-up of solutions of supercritical parabolic equations*, Handbook of Differential Equations: Evolutionary Equations, Elsevier, Amsterdam, 2005.

[14] M. Fila, H. Matano. P. Poláčik, *Existence of L^1-connections between equilibria of a semilinear parabolic equation*, J. Dynam. Differential Equations **14**, (2002), 463-491.

[15] M. Fila, H. Matano, P. Poláčik, *Immediate regularization after blow-up*, SIAM J. Math. Anal. **37** (2005), 752-776.

[16] M. Fila, N. Mizoguchi, *Multiple continuation beyond blow-up*, Differential Integral Equations **20** (2007), 671-680.

[17] M. Fila, P. Poláčik, *Global solutions of a semilinear parabolic equation*, Adv. Differential Equations **4** (1999), 163-196.

[18] S. Filippas, M. A. Herrero, J. J. J. Velázquez, *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **456** (2000), 2957-2982.

[19] S. Filippas, R. Kohn, *Refined asymptotics for the blow-up of $u_t - \Delta u = u^p$*, Comm. Pure Appl. Math. **45** (1992), 821-869.

[20] A. Friedman, B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425-447.

[21] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. A **13** (1966), 109-124.

[22] H. Fujita, *On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$*, Bull. Amer. Math. Soc. **75** (1969), 132-135.

[23] V. A. Galaktionov, S. A. Posashkov, *Application of new comparison theorems in the investigation of unbounded solutions of nonlinear parabolic equations*, Diff. Urav. **22** (1986), 1165-1173. (English translation: Diff. Equations **22** (1986), 809-815).

[24] V. A. Galaktionov, J. L. Vázquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. **50** (1997), 1-67.

[25] V. A. Galaktionov, J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete Contin. Dyn. Syst. **8** (2002), 399-433.

[26] I. M. Gelfand. *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. Ser. 2 **29** (1963), 295-381.

[27] Y. Giga, R. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297-319.

[28] Y. Giga, R. Kohn, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1-40.

[29] Y. Giga, R. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989), 845-884.

[30] Y. Giga, S. Matsui, S. Sasayama, *Blow-up rate for semilinear heat equation with subcritical nonlinearity*, Indiana Univ. Math. J. **53** (2004), 483-514.

[31] A. Haraux, F. B. Weissler, *Nonuniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982), 167-189.

[32] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503-505.

[33] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840**, New York 1981.

[34] M. A. Herrero, J. J. L. Velázquez, *Blow-up behaviour of one-dimensional semilinear parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 131-189.

[35] M. A. Herrero, J. J. L. Velázquez, *Blow-up profiles in one-dimensional, semilinear parabolic problems*, Comm. Partial Differential Equations **17** (1992), 205-219.

[36] M. A. Herrero, J. J. L. Velázquez, *Flat blow-up in one-dimensional semilinear heat equation*, Differential and Integral Equations **5** (1992), 973-997.

[37] M. A. Herrero, J. J. L. Velázquez, *Generic behaviour of one-dimensional blow up patterns*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 **19** (1992), 381-450.

[38] M. A. Herrero, J. J. L. Velázquez, *Plane structures in thermal runaway*, Israel J. Math. **81** (1993), 321-341.

[39] M. A. Herrero, J. J. L. Velázquez, *A blow up result for semilinear heat equations in the supercritical case*, preprint

[40] D. D. Joseph, T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972), 241-269.

[41] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235-268.

[42] W. Jäger, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819-824.

[43] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. **16** (1963) 305-330.

[44] K. Kobayashi, T. Sirao, H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan **29** (1977), 407-424.

[45] A. A. Lacey, D. E. Tzanetis, *Global, unbounded solutions to a parabolic equation*, J. Differential Equations **101** (1993), 80-102.

[46] L. A. Lepin, *Countable spectrum of the eigenfunctions of the nonlinear heat equation with distributed parameters*, Diff. Urav. **24** (1988), 1226-1234. (English translation: Diff. Equations **24** (1988), 799-805).

[47] L. A. Lepin, *Self-similar solutions of a semilinear heat equation*, Mat. Model. **2** (1990) 63-74 (in Russian).

[48] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au+F(u)$*, Arch. Rational Mech. Anal. **51** (1973), 371-386.

[49] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhauser, Basel, 1995.

[50] H. Matano, F. Merle, *On nonexistence of type II blowup for a supercritical nonlinear heat equation*, Comm. Pure Appl. Math. **57** (2004), 1494-1541.

[51] H. Matano, F. Merle, *Classification of type I and type II behaviors for a supercritical nonlinear heat equation*, J. Funct. Anal. **256** (2009), 992-1064.

[52] J. Matos, *Self-similar blow up patterns in supercritical heat equations*, Commun. Appl. Anal. **5** (2001), 455-483.

[53] N. Mizoguchi, *Type II blowup for a semilinear heat equation*, Adv. Differential Equations **9** (2004), 1279-1316.

[54] N. Mizoguchi, *Boundedness of global solutions for a supercritical semilinear heat equation and its application*, Indiana Univ. Math. J. **54** (2005), 1047-1059.

[55] N. Mizoguchi, *Rate of type II blowup for a semilinear heat equation*, Math. Ann. **339** (2007), 839-877.

[56] W.-M. Ni, P. E. Sacks, J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations **54** (1984), 97-120.

[57] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*, Birkhuser Advanced Texts, Birkhuser, Basel, 2007.

[58] J. I. Tello, *Stability of steady states of the Cauhy problem for the exponential reaction-diffusion equation*, J. Math. Anal. Appl. **324** (2006), 381-396.

[59] W. C. Troy, *The existence of bounded solutions of a semilinear heat equation*, SIAM J. Math. Anal. **18** (1987), 332-336.

[60] J. L. Vázquez, *Domain of existence and blowup for the exponential reaction-diffusion equation*, Indiana Univ. Math. J. **48** (1999), 677-709.

[61] J. J. L. Velázquez, *Higher dimensional blow up for semilinear parabolic equations*, Comm. Partial Differential Equations **17** (1992), 1567-1596.

[62] J. J. L. Velázquez, *Classification of singularities for blowing up solutions in higher dimensions*, Trans. Amer. Math. Soc. **338** (1993), 441-464.

[63] F. B. Weissler, *Single point blow-up for a semilinear initial value problem*, J. Differential Equations **55** (1984), 204-224.

[64] F. B. Weissler, *An L^∞ blow-up estimate for a nonlinear heat equation*, Comm. Pure Appl. Math. **38** (1985), 291-295.

Errata

Publication II

In this article on page 307 the inequality (2.5) is incorrect. From the inequality preceding (2.5), we obtain that

$$\int_{u(r,t)}^{u(0,t)} (u(0,t_i) - z)^{-1/2} \mathrm{d}z \leq 2e^{u(0,t_i)/2} r,$$

for every large i, and so

$$-2\sqrt{u(0,t_i) - u(0,t)} + 2\sqrt{u(0,t_i) - u(r,t)} \leq 2e^{u(0,t_i)/2} r.$$

This implies

$$u(0,t_i) - u(r,t) \leq 4(e^{u(0,t_i)} r^2 + u(0,t_i) - u(0,t)) \leq 4(e^{u(0,t_i)} r^2 + e^{u(0,t_i)}(t_i - t)),$$

where we used the estimate

$$u(0,t_i) - u(0,t) \leq \int_t^{t_i} u_t(0,\tau) \mathrm{d}\tau \leq e^{u(0,t_i)}(t_i - t).$$

Therefore, $w_i(\rho,\tau)$ is bounded for $(\rho,\tau) \in [0,C_1] \times [-C_2, 0]$ for every $C_1, C_2 > 0$. The rest of the proof of Theorem 1.1 proceeds as in Publication II.

Also, on page 322 the definition of the energy should be

$$E[w](s) = \int_{|y| \leq R_1 e^{s/2}} \left(\frac{1}{2} |\nabla w|^2 - e^w + w \right) e^{-|y|^2/4} \mathrm{d}y.$$

Publication IV

In Theorem 3 it should be noted that the constant $c^{\#}$ is the constant from Proposition 2.1. Therefore, it can be considered as given, and the case $C_\alpha = c^{\#}$ can not be excluded just by increasing $c^{\#}$.

Aalto-DD 45/2011

ISBN: 978-952-60-4139-1 (pdf)
ISBN: 978-952-60-4138-4
ISSN-L: 1799-4934
ISSN: 1799-4942 (pdf)
ISSN: 1799-4934

Aalto University
School of Science
Department of Mathematics and Systems Analysis
www.aalto.fi

BUSINESS +
ECONOMY

ART +
DESIGN +
ARCHITECTURE

SCIENCE +
TECHNOLOGY

CROSSOVER

DOCTORAL
DISSERTATIONS