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CRYPTOGRAPHIC PROTOCOL DESIGN

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ABSTRACT: In this work, we investigate the security of interactive computations. The main emphasis is on the mathematical methodology that is needed to formalise and analyse various security properties. Differently from many classical treatments of secure multi-party computations, we always quantify security in exact terms. Although working with concrete time bounds and success probabilities is technically more demanding, it also has several advantages. As all security guarantees are quantitative, we can always compare different protocol designs. Moreover, these security guarantees also have a clear economical interpretation and it is possible to compare cryptographic and non-cryptographic solutions. The latter is extremely important in practice, since cryptographic techniques are just one possibility to achieve practical security. Also, working with exact bounds makes reasoning errors more apparent, as security proofs are less abstract and it is easier to locate false claims.

The choice of topics covered in this thesis was guided by two principles. Firstly, we wanted to give a coherent overview of the secure multi-party computation that is based on exact quantification of security guarantees. Secondly, we focused on topics that emerged from the author's own research. In that sense, the thesis generalises many methodological discoveries made by the author.

As surprising as it may seem, security definitions and proofs mostly utilise principles of hypothesis testing and analysis of stochastic algorithms. Thus, we start our treatment with hypothesis testing and its generalisations. In particular, we show how to quantify various security properties, using security games as tools. Next, we review basic proof techniques and explain how to structure complex proofs so they become easily verifiable. In a nutshell, we describe how to represent a proof as a game tree, where each edge corresponds to an elementary proof step. As a result, one can first verify the overall structure of a proof by looking at the syntactic changes in the game tree and only then verify all individual proof steps corresponding to the edges.

The remaining part of the thesis is dedicated to various aspects of protocol design. Firstly, we discuss how to formalise various security goals, such as input-privacy, output-consistency and complete security, and how to choose a security goal that is appropriate for a specific setting. Secondly, we also explore alternatives to exact security. More precisely, we analyse connections between exact and asymptotic security models and rigorously formalise a notion of subjective security. Thirdly, we study in which conditions protocols preserve their security guarantees and how to safely combine several protocols. Although composability results are common knowledge, we look at them from a slightly different angle. Namely, it is irrational to design universally composable protocols at any cost; instead, we should design computationally efficient protocols with minimal usage restrictions. Thus, we propose a three-stage design procedure that leads to modular security proofs and minimises usage restrictions.

KEYWORDS: asymptotic security, data authentication, exact security, homomorphic encryption, secure multi-party computation, sequential composability, subjective security, time-stamping, universal composability.

TIIVISTELMÄ: Tässä työssä tutkitaan vuorovaikutteisen laskennan turvallisuutta. Erityisesti painotetaan matemaattisia menetelmiä, joita tarvitaan erilaisten turvallisuusominaisuuksien määrittelyyn ja analysointiin. Perinteisistä käsittelytavoista poiketen usean osapuolen laskennan turvallisuutta mitataan tässä työssä tarkoilla suureilla. Vaikka tarkkojen rajojen käyttö arvioitaessa laskennallista vaativuutta ja onnistumisen todennäköisyyttä on teknisesti vaativampaa, sillä on myös useita etuja. Se tekee mahdolliseksi eri protokollien välisen vertailun. Lisäksi, tällaisilla turvallisuuden mitoilla on selkeä kustannustaloudellinen tulkinta, mikä tekee mahdolliseksi vertailla salaustekniikkaa käyttävien ja muiden tietoturvallisuusratkaisujen kustannuksia. Tällä on merkitystä käytännön kannalta, koska salaustekniikkaan perustuvat menetelmät ovat usein vain yksi vaihtoehto käytännön turvallisuusjärjestelmiä toteutettaessa. Lisäksi tarkkojen rajojen käyttö tekee turvallisuustodistuksista selkeämpiä ja siten helpottaa todistusten päättelyvirheiden havaitsemista.

Tämän työn aiheiden valinta perustuu kahteen periaatteeseen. Ensimmäisen mukaan tavoitteena on luoda johdonmukainen katsaus usean osapuolen laskennan turvallisuuteen, joka perustuu turvallisuustakuiden tarkkaan määrittämiseen. Toisen periaatteen mukaan keskitytään tarkastelemaan aiheita, jotka ovat olleet tekijän tutkimusten kohteena. Tässä väitöskirjassa esitetään yleistetyssä muodossa monia tekijän tekemiä menetelmiä koskevia löydöksiä.

Niin yllättävältä kuin se tuntuukin, turvallisuuden määritelmissä ja todistuksissa käytetään tilastollisen päättelyn ja stokastisten algoritmien menetelmiä. Siksi tämän työn aluksi tarkastellaan hypoteesien testausta ja sen yleistyksiä. Erityisesti osoitetaan kuinka erilaisille turvallisuusominaisuuksille voidaan antaa numeerinen arvo turvallisuuspelejä käyttäen. Seuraavaksi tarkastellaan todistuksen perustekniikoita ja esitetään kuinka todistus tulee rakentaa, jotta se on helposti todennettavissa. Kiteytettynä tämä tarkoittaa, että kuvataan turvallisuuspeli puuna, jonka jokainen kaari vastaa yksinkertaista askelta todistuksessa. Näin esitetty todistus voidaan todentaa tarkastelemalla ensin sen syntaktista kokonaisrakennetta ja sen jälkeen todentamalla jokaista puun kaarta vastaava todistusaskel.

Väitöskirjan loppuosassa tarkastellaan salausteknisten protokollien suunnittelun eri piirteitä. Ensiksi käsitellään erilaisten turvallisuustavoitteiden, kuten syötteen yksityisyys, tuotoksen oikeellisuus ja täydellinen turvallisuus, täsmällistä määrittelyä ja sitä, kuinka turvallisuustavoite tulee asettaa vastaamaan konkreettista tilannetta. Toiseksi tutkitaan tarkan turvallisuuden vaihtoehtoja. Tarkemmin sanottuna analysoidaan tarkan ja asymptoottisen turvallisuusmallin välisiä yhteyksiä ja annetaan täsmällinen määritelmä subjektiiviselle turvallisuudelle. Kolmanneksi tarkastellaan ehtoja, joilla protokolla säilyttää turvallisuusominaisuutensa, ja kuinka useita protokollia voidaan yhdistää turvallisesti. Salausteknisten protokollien teoriassa yhdistettävyyttä koskevat tulokset tunnetaan yleisesti, mutta tässä työssä niitä tarkastellaan uudesta näkökulmasta. Nimittäin, ei ole mielekästä rakentaa universaalisti yhdisteltävissä olevia protokollia mihin hintaan hyvänsä, vaan tuloksena olevien protokollien tulee olla tehokkaita ja käytön rajoitusten niin pieniä kuin mahdollista. Tässä työssä esitetään kolmivaiheinen menettely, jolla saavutetaan modulaariset turvallisuustodistukset ja minimaaliset käytön rajoitukset.

AVAINSANAT: asymptoottinen ja tarkka turvallisuus, datan autentikointi, homomorfinen salaus, turvallinen usean osapuolen välinen laskenta, peräkkäinen yhdistettävyys, subjektiivinen turvallisuus, aikaleimaus, universaali yhdistettävyys.

CONTENTS

PREFACE

This thesis is a result of studies and research at the Laboratory for Theoretical Computer Science of Helsinki University of Technology. I remember vividly my first days in Finland when I was a master student working under the supervision of Helger Lipmaa. Much has changed during my stay in Finland, but the TCS lab has always been a great place to work.

Although these last four years have been quite intense and sometimes even overly stressful, I have learnt a lot during my studies. For that I am indebted to my supervisors Helger Lipmaa and Kaisa Nyberg. They gave out enough puzzles and problems to keep me intrigued. At the same time, they were always helpful and taught me many secrets of research, scientific writing and publishing. They also helped me to get funding from the Academy of Finland and managed the corresponding CryDami research project.

Besides my supervisors, I had pleasure to work together with many other beautiful minds. I am especially grateful to my co-authors Ahto Buldas, Bart Goethals, Emilia Käsper, Taneli Mielikäinen and Sylvain Pasini, who produced many excellent ideas and were patient enough to work with me. I would also like to thank Pekka Orponen and Jaakko Hollmén for their excellent book suggestions, which have made my literature exploration phases much easier.

Also, it would have been much more difficult to write the thesis without comments from demanding readers. For that I am grateful to Billy Brumley, Ahto Buldas, Emilia Käsper, Phillip Rogaway, Berry Schoenmakers and Jan Willemson, who took the time to read the drafts of this thesis, raised many interesting questions, and pointed out many errors and inconsistencies.

Last but not least, I would like to thank my family and friends from Tartu, Otaniemi and Espoo, who kept me in balance during tough times.

Tartu, March 2008

Sven Laur

CONTRIBUTIONS OF THE AUTHOR

This thesis consists of a mini-monograph followed by four articles. The articles were carefully selected out of seven articles published by the author to illustrate the theoretical methodology developed in the first part of the thesis. In short, this thesis gives a systematic overview of secure multi-party computations in the framework of exact security. To the author's knowledge, no similar work has been published in cryptographic literature, although the need for such treatment is apparent. We emphasise that the first part of the thesis is not just a summary but contains several technical novelties and methodological advances:

- We develop a consistent methodology for quantifying the security of interactive computations. The corresponding security definitions are much more descriptive than classical asymptotic security definitions.

- We show that not all security proofs can be formalised as game chains and explain how game trees and new deduction rules solve this problem. Additionally, we describe how to verify the overall structure of a proof by looking at the syntactic changes in the game tree.

- We are the first to consistently use time-success and risk profiles to characterise quantitative properties of cryptographic reductions.

- We explicitly discuss intermediate levels of security between security in the semi-honest model and complete security in the malicious model. Such models provide cost-efficient solutions for many practical tasks.

- We establish a rigorous formalisation of subjective security. As a result, one can formulate and prove security claims for individual public keys and hash functions and thus escape limitations of classical formalisations.

- Finally, we show how to combine protocols with different composability guarantees. The corresponding three-stage design methodology can significantly simplify the design of complex protocols.

The included articles [P1, P2, P3, P4] provide solutions to specific but practically important computational tasks and thus nicely illustrate different theoretical aspects covered in the first part of the thesis.

[P1]: Bart Goethals, Sven Laur, Helger Lipmaa, and Taneli Mielikäinen. On Private Scalar Product Computation for Privacy-Preserving Data Mining. In Choonsik Park and Seongtaek Chee, editors, *Information Security and Cryptology - ICISC 2004, 7th International Conference, Seoul, Korea, December 2-3, 2004, Revised Selected Papers*, volume 3506 of *Lecture Notes in Computer Science*, pages 104–120. Springer, 2004.

[P2]: Sven Laur and Helger Lipmaa. A New Protocol for Conditional Disclosure of Secrets and Its Applications. In Jonathan Katz and Moti Yung, editors, *Applied Cryptography and Network Security, 5th International Conference, ACNS 2007, Zhuhai, 5-8 June, 2007*, volume 4521 of *Lecture Notes in Computer Science*, pages 207–225. Springer, 2007.

[P3]: Ahto Buldas and Sven Laur. Knowledge-Binding Commitments with Applications in Time-Stamping. In Tatsuaki Okamoto and Xiaoyun Wang, editors, *The International Conference on Theory and Practice of Public-Key Cryptography, PKC 2007, Beijing, 16-20 April, 2007*, volume 4450 of *Lecture Notes in Computer Science*, pages 150–165. Springer, 2007.

[P4]: Sven Laur and Kaisa Nyberg. Efficient Mutual Data Authentication Using Manually Authenticated Strings. In David Pointceval, Yi Mu, and Kefei Chen, editors, *The 5th International Conference on Cryptology and Network Security, CANS 2006, Suzhou, Dec. 8-10, 2006*, volume 4301 of *Lecture Notes in Computer Science*, pages 90–107. Springer, 2006.

The author has made significant contributions to all of the abovementioned articles. In short, the author derived most proofs and attacks in the article [P1]. The basic methodology used in the article [P2] was also discovered by the author, although the current exposition is a result of extensive joint work with H. Lipmaa. Similarly, the author established the notion of knowledge-binding commitments and corresponding reductions in the article [P3]. Finally, the original problem statement and the initial solution in the article [P4] is due to K. Nyberg and N. Asokan. However, the author derived all security proofs and made several technical refinements that were essential for achieving provable security.

1 INTRODUCTION

Good design is always a trade-off between conflicting requirements regardless of whether we talk about physical objects, such as buildings and devices, or mental constructions, such as mathematical objects and computer programs. Some of those design constraints are easy to describe and formalise, whereas others are either subjective or inexplicable. Therefore, the design process itself is often a mixture of creative and engineering steps and thus not completely expressible using formal methodology. Nevertheless, we can use mathematical models as powerful tools to formalise design requirements, to limit the search space and to validate the resulting prototypes against objective constraints.

In this thesis, we describe how to formalise and analyse the security of distributed computations. Since a computational process is always inseparable from physical phenomena that make it observable, simplified mathematical models, such as finite automata and Turing machines have their limitations. Namely, some artefacts that are observable in theoretical models might not be present or meaningful in practice. Therefore, we must make a constant conscious effort to assure that theoretical models adequately represent the reality.

In particular, note that security as a concept cannot be viewed as a thing in itself, rather it has a meaning only in a larger economical, political or military context, where entities and organisations compete against each other. Consequently, these entities have to withstand internal and external attacks in order to achieve their goals. As reliable information transfer and processing are usually among the key factors to success, we have to protect computations against wire-tapping and device corruption. Essentially, there are three main types of protection methods: physical, organisational and cryptographic countermeasures. Physical countermeasures limit the abilities of a potential attacker to compromise critical computational processes. For example, servers that run mission critical software are often located in physically guarded rooms. Organisational countermeasures change the common process logic in order to demotivate potential attackers, limit potential damage, or provide fail-safe recovery procedures. For example, making regular back-ups from mission critical data can both limit the potential damage and simplify the recovery procedure.

Cryptographic countermeasures are somewhat different, since they modify the computational process itself and not the surrounding context. Although this approach can be immensely flexible, there are certain inherent limitations. Most importantly, cryptographic methods are applicable only in distributed settings. Indeed, if an adversary gets full control over the computational process, then he or she can ignore all cryptographic countermeasures built into the process. Moreover, successful recovery from such attacks is impossible, as the adversary can corrupt the entire internal state. In particular, note that back-up strategies make sense only in a distributed setting, otherwise the adversary can corrupt the back-ups, too. To minimise such risks, mission-critical processes are often divided into well-isolated sub-processes or accompanied with unerasable logging procedures. But the latter does not automatically imply that we need many computational devices. In fact, we can still use a single device for all computations, provided that a potential adversary cannot corrupt the entire device and all subtasks look like different computational entities.

We emphasise that one cannot achieve security without combining various countermeasures against the attacks. In fact, one must find a proper balance between various protection methods, otherwise a single weakness can make the entire defence meaningless. In short, cryptographic countermeasures achieve their purpose when attacks against non-cryptographic measures are more attractive to the adversary. In many cases, the cost of cryptographic countermeasures is marginal compared to the other measures and thus people tend to use them as cheap substitutes for real solutions. Although cryptographic constructions must have an appropriate safety margin, it is impractical and wasteful to make them overly strong compared to the other protection measures that users are willing to accept. For the same reason, the total cost of countermeasures must match the potential losses and sometimes the best strategy is to ignore threats.

To summarise, finding an appropriate balance between different protection methods is mostly an economical decision that should be based on approximate cost estimates. Unfortunately, precise estimation of potential losses and gains currently resembles more to magic than science. In fact, we still cannot answer even the simplest questions like what would it cost to break a modern 128-bit block cipher, or when the use of 1024-bit RSA becomes a considerable liability issue in e-commerce. Although such a sad state of affairs is really discouraging, we should still try to capture practical requirements in theoretical models, or at least try to develop solutions that are meaningful in practice.

1.1 CRYPTOGRAPHY AS AN ENGINEERING DISCIPLINE

Cryptography as a general term unites a wide spectrum of different subfields ranging from empirical design of various base constructions to branches of pure mathematics that focus on complexity theoretical aspects and fundamental limitations in cryptography. Hence, there are many valid viewpoints and possible research goals. In this thesis, we view cryptography mostly as an engineering discipline that should provide formal methodology for making economically sound and scientifically justified decisions. More precisely, we study how to design protocols with desired properties using cryptographic primitives and how the achievable security level depends on the properties of basic primitives. We are mostly interested in the *exact* quantification of security and thus state all security guarantees in terms of concrete running times and success probabilities. Although such a quantification does not directly reflect the corresponding monetary losses and gains, we can easily overcome this by estimating the cost of computations and the perceived utility of a successful attack. The corresponding cost estimation phase is beyond our scope, since it is a purely economical problem. Nevertheless, the existence of such transformations is important, as it gives a clear interpretation to all security estimates and allows us to compare cryptographic and non-cryptographic countermeasures.

In many cases, it is beneficial to combine cryptographic and non-cryptographic countermeasures. Consequently, our formalism must be flexible enough to model a wide range of security goals and adversarial powers. Moreover, none of these theoretical models is a priori better than the others, since the corresponding costs and gains depend on practical details. A solution that is tuned for a particular setting may be completely unsuitable for other settings. As a result, there are literally hundreds, not to say thousands, of meaningful security models. Since we cannot treat each model separately, we must limit ourselves to the most important aspects that are shared by most of the models.

In particular, it makes sense to use the same formal model of interactive computations. Although all standard formalisations are based on Turing machines, there are some technical details that make Random Access machines more appealing. We remark here that the difference is only in exact resource consumption and both models are qualitatively equivalent, see Chapter 2. Also, note that security properties are commonly described in terms of computational indistinguishability or advantage. Both notions have their origins in statistics. Chapter 3 covers essential properties of hypothesis testing that are necessary to quantify and analyse various security properties. Moreover, the knowledge of these basic concepts is sufficient to carry out most common security proofs as long as we abstract away all complexity theoretical details.

We can always specify complex constructions in terms of abstract primitives that are known to satisfy certain functionality and security guarantees. In other words, we actually specify a template that can be instantiated by replacing abstract primitives with practical constructions. Therefore, the corresponding security proofs are free from implementation details and rely mostly on the elementary probability theory. Of course, the complexity theoretical problems do not disappear entirely. They re-emerge as soon as we try to establish whether a concrete realisation of an abstract primitive indeed satisfies the necessary security requirements or not. However, the latter is the main concern of primitive design and thus beyond our scope.

Such an approach leads to a hierarchy of cryptographic abstractions. At the lowest level of this hierarchy, we find many low level primitives that formalise our current beliefs and knowledge about computational complexity. For example, we can talk about abstract Diffie-Hellman groups and abstract sets of RSA composites that formalise certain complexity theoretical beliefs about the hardness of discrete logarithm and integer factorisation. Again, we must use the common methodology of primitive design to establish whether such abstract constructions can be implemented in practice. Primitives that realise some well-specified functionality, such as encryption or authentication, are often specified in terms of lower level primitives. For example, ElGamal and Rabin encryption schemes are built on top of Diffie-Hellman groups and RSA composites. The next abstraction level consists of lower level protocols that are designed and optimised for well-specified tasks, such as authentic message delivery or entity identification. These basic protocols can be combined in order to achieve more complicated tasks like online banking and digital rights management. The hierarchy of abstractions continues—we can combine these protocols to achieve even more complex tasks. Since there are so many different levels, it is essential to have a flexible formalism that makes it easy to change the levels of abstraction, otherwise we are either trapped in the ocean of unnecessary technical details or the viewpoint is too coarse for performance optimisations.

Still, many proofs are inherently complex, even if we choose a correct abstraction level, and occasionally we must increase the granularity to discover performance bottlenecks. In these cases, we must handle proofs that are tens, if not hundreds, of pages long. For obvious reasons, such proofs quickly become unmanageable, unless they are extremely well structured. Ideally, these proofs should be automatically verifiable. Such a goal seems extremely hard to achieve, as cryptographic proofs are often based on complex reductions and thus require creative manipulation of probabilities. In Chapter 4, we explore this issue further and show that all proofs share the same meta-structure. Any proof can be formalised as a game tree, where each edge corresponds to an elementary proof step. Although each elementary proof step itself may rely on complex reasoning, it is easy to describe the related syntactical change in the game. More importantly, proof steps that lead to the same kind of syntactic changes also reuse the same proof template. Therefore, we can formalise such steps as reduction schemata. Notably, the total number of reduction schemata is rather limited, since each primitive introduces few elementary reduction schemata and other more complex reductions can be expressed as mixtures of those.

As reduction schemata abstract away complex technical details, the resulting proof skeleton is compact and easily verifiable. In fact, a person who applies reduction schemata does not have to know the exact technical details as long as he or she applies reduction schemata correctly. In short, derivation of security proofs is mostly an engineering task, whereas formalisation and systematisation of reduction schemata requires a deeper understanding of cryptography. We believe that such a representation of basic concepts brings us closer to computer-aided design of complex cryptographic primitives and protocols. The need for such design assistants becomes self-evident when we compare cryptography to other engineering disciplines, where hundreds or even thousands of general and special purpose design programs are routinely used by engineers. Although there are many experimental protocol analysers, most of them put too much effort on automatic derivation of proofs. The author believes that the latter is a secondary goal and the most critical part is a human readable representation of proofs and the ability to apply various reduction schemata without errors. Of course, the development of such a computer environment is a major undertaking and it may take several man-years to achieve significant progress.

1.2 INTRODUCTION TO CRYPTOGRAPHIC PROTOCOL DESIGN

A conceptual difference between cryptographic primitives and protocols lies in the security guarantees. Primitives are designed to preserve security in few well-defined scenarios, whereas protocols must resist different attacks aimed at different targets. Often, there are literally thousands and thousands of potentially relevant security goals and we cannot analyse them separately. Instead, we should compare different protocols π_α and π_β for the same functionality. Beaver was the first to state the corresponding resilience principle [Bea91a]:

If any adversary attacking π_α cannot gain more information or wield more influence than when it attacks π_β, then π_α is at least as secure and reliable—i.e. as resilient—as π_β.

Evidently, a protocol π_α attains a maximal security level if π_α is as resilient as any other protocol π_β that correctly implements the same functionality. As a direct comparison with any other imaginable protocol π_β is infeasible in practice, we must use a methodological shortcut. Namely, we can easily define an ideal implementation π° that is as resilient as any other protocol π_β. Since resilience is transitive by definition, it is sufficient to compare only protocols π_α and π° to establish optimality. The corresponding real versus ideal world paradigm is a central methodological approach in cryptographic protocol design.

However, the exact nature of these security guarantees depends on many technical details. Most importantly, we must define comparability between attacks against real and ideal world implementations. In particular, note that participation in a protocol is economically justified if potential gains outweigh associated risks. Thus, we must assure that all relevant economical aspects are also captured by the ideal implementation, otherwise real and ideal implementations are not truly comparable. In Chapter 5, we study this problem thoroughly in the simplest stand-alone setting, where a protocol is executed in isolation and the adversary cannot use external information. Although the resulting security definitions are equivalent to the standard formalisations [Can00a, Gol04], our treatment also reveals underlying causes why the definitions must be formalised in such a way.

Again, note that an appropriate solution for a specific task is often a balanced mixture of cryptographic and non-cryptographic measures. Therefore, different applications utilise different cryptographic properties. In Chapter 5, we describe the most common security objectives. We start from the semi-honest model, where all participants follow the protocol, and finish with the malicious model, where the set of malicious participants can arbitrarily deviate form the protocol. Since we consider input-privacy and output-consistency in addition to complete security, we obtain four security models that capture the most essential security requirements. Although practical settings are often more complex, they are usually just intelligent variations of these basic models.

The biggest advantage of the stand-alone setting is conceptual simplicity, which makes it easy to formalise and analyse various security objectives. However, the corresponding security models explicitly assume that adversaries are isolated from external influences. The assumption rarely holds in practice, where protocols are commonly used as subroutines to achieve more complex goals. Hence, it is essential to know which computational contexts preserve the security of the protocol. First, if we know the corresponding usage restrictions, we can combine various protocols without unexpected consequences. Secondly, usage restrictions are actually security guarantees that describe when it is secure to execute a protocol. Since the whole computational context is often unknown to the participants, it is important to state usage restrictions in local terms. We emphasise that there is a natural trade-off between usage restrictions and the efficiency of a protocol. In other words, we cannot rank protocols only according to the usage restrictions, since protocols with the most liberal usage restrictions might have sub-optimal performance and unjustified deployment costs.

Although it is relatively straightforward to understand what a computational context is, the corresponding formalisation is cluttered with many tedious technical details. Hence, we try to present the complex computational model as modularly as possible in Chapter 7. We also study the true limitations of stand-alone security and establish the corresponding usage restrictions. Briefly, stand-alone security is sufficient for all computational contexts, as long as we are willing to assume that all external participants[1] are corrupted during the protocol. Al-

[1]More formally, external participants are parties that do not participate in the protocol.

though the result is a minor enhancement of classical sequential composability theorems [Ore87, Can00a], it bears a huge conceptual significance. Namely, it implies that stand-alone security is sufficient for all end-to-end applications that treat all external participants as enemies.

However, the corresponding usage restrictions are also quite limiting, since participants in a stand-alone secure protocol cannot perform any side-computations during the execution of the protocol. As a result, we obtain a centralised execution model that treats the whole network as a single virtual processor. Since network-wide synchronisation is often impossible or causes significant performance penalties, we need protocols with more liberal usage restrictions. Surprisingly, we can significantly relax usage restrictions for stand-alone secure protocols as soon as the correspondence between real and ideal world adversaries satisfies certain structural restrictions already suggested in early works [Bea91b, MR91b]. In fact, such protocols are called universally composable, as they preserve security in any reasonable context. Moreover, all-or-nothing results by Lindell [Lin04] indicate that a general class of protocols with simple and natural usage restrictions consists of protocols that are either secure only in the stand-alone model, or universally composable.

The structural restrictions mentioned above are difficult to fulfil. In fact, it is impossible to achieve universal composability when honest participants are in minority, i.e., malicious participants have more control over the protocol than the coalition of honest parties. Fortunately, there is a loophole that allows us to bypass the all-or-nothing nature of usage restrictions. Namely, we can often locate critical regions in a stand-alone protocol that must be executed in isolation, whereas the other regions are indeed universally composable. Consequently, our task is to minimise the duration and count of these critical regions without losing the overall efficiency. Chapter 7 covers also the corresponding three-stage design methodology and studies protocols with a trusted setup.

As a final detail, note that cryptographic protocol design borrows many concepts from hypothesis testing and thus it also inherits many limitations of classical statistics. In particular, classical statistics describes only average-case behaviour and cannot be used for analysing individual experiments. Analogously, classical cryptography describes only collective security properties and fails when we want to analyse the security of specific instances. In other terms, design and usage of cryptographic protection measures are two separate things. As an illustrative example, consider a nation-wide public-key infrastructure. A well-designed signature scheme assures that the number of weak signing keys and potential monetary risks are negligible for the organisation that issues keys for all individuals. However, the latter does not imply that a particular signing key is secure to use. In fact, we cannot even formalise the corresponding design goal by using classical statistics. The latter is a serious drawback as a key holder is only interested in the security level provided by his or her signing key. In Chapter 6, we show how to formalise such security goals by using the notion of subjective probability. While we acknowledge that such concepts and security goals are somewhat non-standard in cryptographic literature, the resulting formalism is mathematically sound and well-aligned with practice.

1.3 BENEFITS OF FINITE SET POLICY

There are many ways to obtain mathematically sound security definitions; exact quantification of security properties is only one of them. In particular, various asymptotic security models are widely used in theoretical cryptography, since these models are well aligned with complexity theory and also hide many tiresome technical details. Nevertheless, there are several compelling reasons why we should stick to exact quantification of security. First, connections between theoretical results and practical consequences are significantly weaker and we must address the limitations already raised by Yao [Yao82]:

> *In this theory, the complexity of a computational problem is measured by the asymptotic behaviour of algorithm as the input length becomes large. To apply the theoretical result to input of a particular length, we tacitly assume that this length is large enough that the asymptotic results can be used.*

Secondly, asymptotic behaviour can be surprisingly brittle and counterintuitive. Misunderstanding of limiting processes has created countless errors and paradoxes throughout the history of mathematics. There is no reason to believe that cryptography or computer science are special in that respect and thus we should always follow the finite set policy [Jay03]:

> *Apply the ordinary process of arithmetic and analysis only to expressions with a finite number of terms. Then, after the calculation is done, observe how the resulting finite expressions behave as the number of terms increases indefinitely.*

Less formally, we first formulate models with precise bounds and only then observe limiting behaviour. Indeed, violation of this simple principle has lead to subtle errors or inconsistencies in cryptography. For example, aesthetic reasons urged theoreticians to consider also algorithms that run in expected polynomial time, see [Gol07] for further discussion. As a result, we have now at least four different notions of polynomial security, which are incompatible. To make things worse, several important results, such as composability theorems, do not hold for all of these models, see the first half of Chapter 6. At the same time, there exists basically one finite security model and we can safely apply a theorem as soon as all quantitative requirements of the theorem are satisfied.

Another notorious example is the definition of universal composability. Depending on subtle details in definitions, some important equivalence results either hold or not, see the discussion in [Can00b]. In fact, there are at least four different asymptotic definitions of universal composability [DKMR05], whereas there is a single finite security model for universal composability. All these results clearly demonstrate that skipping the finite modelling step is dangerous.

Finally, we remark that many asymptotic models use a complex limiting process over several variables to characterise many aspects in a single stroke. For example, a zero-knowledge proof can be characterised by the complexity of the statement to be proved and three other quantitative properties: correctness, soundness and zero-knowledge property. However, the corresponding asymptotic model is unidimensional and thus many details that are important in practice are erased by an inherent dimension reduction.

1.4 CONTRIBUTIONS OF THE AUTHOR

This thesis has a slightly non-standard structure. Most theses are either monographs or contain a brief summary and a listing of published articles, whereas this thesis consists of a monograph followed by a few selected articles. Such a choice was made mostly because the author felt compelled to first systematically present essential theoretical concepts and then use the articles as illustrations for the covered topics. The author believes that such a systematic approach is justified, since an engineering view on cryptography is currently underrepresented in mainstream literature. Although exact quantification of security properties is quite common in cryptographic primitive design, the corresponding formalisations are often very application specific. Moreover, all standard treatments of secure multi-party computations [MR91b, Can00a, Can00b, Gol04] are based on asymptotic settings and do not place results into a larger economical context. Hence, the need for systematical treatment of exact security is real and hopefully this thesis is a step to the right direction. Also, most of the topics covered in this thesis are closely connected with the problems that the author has faced in his research. However, we put the main emphasis on the methodological issues, since protocol design itself becomes a mundane engineering task as soon as methodological issues are resolved and design criteria are fixed.

As this thesis covers many essential results, it is somewhat difficult to separate the author's original contribution from prior art. Therefore, we list the most important novelties and their connections to the author's published articles. Conceptually, the most important novelty is an engineering viewpoint to cryptographic protocol design and the coherent use of exact security guarantees throughout this thesis. Differently from standard treatments, we use hypothesis testing as a main tool and design a primitive or a protocol only from abstract primitives that completely formalise all relevant security properties. Such an approach allows us to present proofs in a more systematic manner and use (semi-)formal methods to derive and verify security proofs. The corresponding formalism established in Chapter 4 is a novel addition to game-playing proofs popularised by Shoup and Rogaway, see for example [Sho04, BR06]. First, we demonstrate that some security proofs cannot be converted to game chains and show how to formalise these proofs by using game trees. Although such proofs are quite rare in primitive design, they are often required in the security analysis of protocols. In fact, the author experienced these limitations by analysing authentication protocols in [LN06] and the proof methodology presented in Section 4.4 is just a direct generalisation of the techniques used in [LN06]. Secondly, we go one step further than the standard game-playing proofs [Sho04, BR06] and show how to abstract away all technical details and express the skeleton of the proof via syntactic changes formalised as reduction schemata. The approach differs from classical security calculi, as the proof skeleton is informative enough to reconstruct all technical details, including the bounds on advantages and running times. The author has found the corresponding formal approach invaluable for deriving and verifying technically complex reductions such as [LP08].

Thinking in terms of reduction schemata forces us to generalise individual proof steps into parametric proof templates. The corresponding meta-level analysis often reveals analogies between proof steps that are formulated for completely different primitives, but use technically identical arguments. For example, the author's study of specific collision extraction techniques [KLL06, BL07] has shown that the corresponding security analysis deals with a certain combinatorial problem. Hence, the results obtained in [KLL06, BL07] are not specific to a certain primitive; but can be used in the contexts of hash functions, commitment schemes, proofs of knowledge and digital signatures. We acknowledge here that these results contain mostly technical improvements, but the generality and optimality of the obtained bounds has a definite engineering value.

Time-success and risk profiles introduced in Chapters 4 and 5 is another methodological advance that improves our understanding about cryptographic reductions. In many security proofs, we must choose between alternative reduction methods, which lead to incomparable security guarantees. Consequently, we need a methodology for comparing radically different reduction strategies. Although the problem caught the author's attention already in the article [BL07], the issue was fully resolved during the preparation of the manuscript [KLL06] that lead to the current formalisation presented in Section 4.2.

Also, note that standard formalisations of secure multi-party computations just postulate the security criterion without explaining whether it also makes sense from an economical viewpoint. Now, considering the problem in terms of time-success and risk profiles provides an elementary explanation why these security definitions are meaningful in the economical context as well.

As the third methodological advance, we explicitly discuss intermediate security levels between security in the semi-honest model and complete security in the malicious model. Such models provide cost-efficient solutions for many practical settings. In particular, we have studied many privacy-preserving data mining tasks [GLLM04, LLM05, LLM06] that become practically intractable in the malicious setting, if we want to achieve complete security. Still, we can guarantee the privacy of inputs by using the techniques developed in the article [LL07]. Moreover, we have shown that one can detect cheating with a significantly lower overhead than is needed for complete security [LL06].

Another, more debatable but still conceptually novel enhancement by the author is a rigorous formalisation of subjective security. In Chapter 6, we define the corresponding notion and explain why classical security definitions are inadequate for certain settings. Moreover, we also explain why random oracle or generic group models provide only subjective security guarantees and why all standard arguments, such as [CGH04b], against these models are unconvincing.

A final and probably the most important methodological advance is a coherent and practice-oriented treatment of modular protocol design. Differently from the standard approach, we treat universal composability as a tool and not as the final aim. In particular, we present an efficient three-stage design methodology that significantly simplifies the construction of complex protocols without losing efficiency. The corresponding design process often leads to protocols that share the same trusted setup phase. As all classical composability theorems are inadequate in such settings, we must use more fine-grained methods. The corresponding methodology presented in Section 7.7 is a direct generalisation of proof methods used in the article [LL07], where we proved composability theorems for very specific two-round protocols with a shared key generation phase. Similar theoretical concerns emerge also in the articles [LN06, LP08] where protocols are proved to be secure in the common reference string model.

On the choice of selected articles. As explained above, the first part of this thesis covers and generalises the most important methodological advances made by the author. Thus, we include only four articles [GLLM04, LN06, BL07, LL07] out of eight publications into this thesis. This does not mean that the remaining four publications [LLM05, BL06, LLM06, LP08] are technically less sound or otherwise inferior. Neither was the choice based on the author's impact on the articles. The author has made a significant contribution to all eight articles. The choice was made mainly on the basis that the selected articles illustrate certain theoretical aspects and methods more vividly.

Published articles cover three different sub-fields of protocol design: privacy-preserving data mining, time-stamping and message authentication. To be precise, we consider only such tasks of privacy-preserving data mining that are special instances of secure multi-party computation. In a nutshell, we assume that the data is split between several participants and they want to learn global patterns in the data without revealing more information than absolutely necessary. Many data mining algorithms are based mostly on linear vector and matrix operations. Therefore, private evaluation of scalar products is one the most important tasks in privacy-preserving data mining and many authors have used ad hoc methods to solve the problem. The first included article [GLLM04] shows that these ad hoc protocols are not only cryptographically insecure but can completely reveal the private input if used in practical applications.

We remark here that ad hoc methods are still widely used in privacy-preserving data mining, since cryptographic solutions are practically infeasible and people are actually willing to reveal more information during the execution than absolutely necessary. Consequently, the cryptographic models must be flexible enough to model various trade-offs between privacy and efficiency, in order to be useful. See Section 5.4 for the corresponding remarks. At the same time, the article [GLLM04] also emphasises that small leakages can accumulate and cause total disclosure and thus one must be extremely careful with such specifications. As a positive application of cryptographic techniques, we show that scalar products can be privately computed by using homomorphic encryption. Moreover, the follow-up articles [LLM05, LLM06, LL07] use homomorphic encryption to implement more complex privacy-preserving tasks.

The second included article [LL07] presents a general methodology for evaluating complex predicates by using homomorphic encryption. More precisely, we study a hybrid security model, where the owner of the private key is maliciously corrupted but all opponents are guaranteed to be semi-honest. We show that complex protocols can be methodically designed by using three elementary operations based on homomorphic encryption. In terms of Chapter 7, we establish that these protocols are universally composable even if they share the same trusted setup phase. We also show how to enhance the security of semi-honest protocols so that they become secure in the hybrid model described above. Although the article presents methodology for certain two-round protocols, the methodology can be generalised for more complex protocol classes.

The third included article [BL07] explores various properties of time-stamping schemes. Briefly, a time-stamping scheme is a multi-party protocol that implements an unalterable digital register of events. As such, time-stamping must be based on binding commitments, such as hash trees. However, state of the art results [BS04] indicate that not all binding commitments are suitable for time-stamping. More formally, there exists a binding commitment scheme that is not provably secure for time-stamping if one considers only black-box reductions. In this article, we prove that bounded list and set commitment schemes are sufficient for time-stamping. Moreover, our security definitions are more precise and well aligned with the ideal versus real world paradigm. Secondly, the article also provides a nice example of how white-box reduction techniques can be built on top of constructive combinatorial arguments.

The last included article [LN06] contains practical protocols for manual data authentication and manual key agreement. The article was greatly motivated by the recent advances in wireless communication. Nowadays, many handheld devices, such as cell phones, music players, keyboards send and receive data over wireless links and are thus vulnerable to active attacks. Moreover, practical limitations prevent the use of pre-shared keys or public-key infrastructure. Hence, we can assure authenticity only with the help of human operators who transfer short messages from one device to another. The article presents protocols for message authentication and key agreement that achieve a maximal security level in such settings. Moreover, these protocols are now used in the Bluetooth and Wireless-USB standards, see the discussion in [LN06].

2 COMMON NOTATION AND BASIC CONCEPTS

History has repeatedly shown that notation is extremely important in mathematics. Hence, we try to make the notation as simple and mnemonic as possible. Sometimes we intentionally omit details that can be grasped from the context. On the other hand, many authors try to capture all details and leave no room for misinterpretation. Although such presentation is pleasingly rigorous, it is often tedious, not to say boring and incomprehensible. We, readers, tend to have a limited amount of memory and thus exposing all details often makes the notations so complex that it overshadows the concepts behind them.

Modern cryptography is built on various subfields of mathematics and computer science, such as computational algebra, complexity theory, probability theory and statistics. As this thesis considers only the design of cryptographic protocols, we can narrow the set of necessary mathematical concepts and build the theory on basic properties of algorithms, reductions, probability and hypothesis testing. Surprisingly, we can actually abstract away all complexity theoretical details and provide a clean conceptual design of protocols for many tasks without any knowledge of complexity theory. We emphasise here that we still need complexity theoretical conjectures to reason about security. However, these complexity theoretical conjectures come into play later, when we must construct cryptographic primitives used in the design.

Next, we briefly list basic notations and then discuss the central concepts, such as models of computation and probability, more thoroughly.

2.1 BASIC MATHEMATICAL CONCEPTS

Functions and domains. Cryptography deals mostly with discrete transformations; real-valued functions are used only in the security analysis. As usual the set of all finite bit strings is denoted by $\{0,1\}^*$. The set of finite bit string sequences is denoted by $\{0,1\}^{**}$ and is identified with $\{0,1\}^*$. Therefore, any uniquely determined discrete transformation can be viewed as a *deterministic function* $f : \{0,1\}^* \to \{0,1\}^*$ regardless of how many arguments the transformation has. Some discrete transformations are randomised. A *randomised function* is defined as $f : \{0,1\}^* \times \Omega \to \{0,1\}^*$, where Ω is a corresponding sample space and the output $f(x) := f(x;\omega)$ depends on the *random choice* $\omega \in \Omega$. A randomised function without explicit arguments $X : \Omega \to \{0,1\}^*$ is known as a *discrete random variable*. We always assume that the random choices for different functions are independent unless explicitly specified otherwise.

Distributions and probability. Randomised functions are inherently non-deterministic, as the random choice ω is not known ahead. Probability theory provides qualitative methods for assessing the corresponding uncertainty. Usually, the probability is formalised according to the axiomatisation given by Kolmogorov [Bil95]. The latter requires a formal definition of measurable subsets $\mathcal{F}(\Omega)$ and the corresponding probability measure $\Pr[\cdot] : \mathcal{F}(\Omega) \to [0,1]$. Fortunately, we can omit these measure theoretical details and give a more elementary definition for discrete random variables and functions. Any *discrete* *distribution* can be viewed as a random variable $X : \Omega \to \{0,1\}^*$ together with a sequence of non-negative numbers $(p_x)_{x\in\{0,1\}^*}$ such that $p_x = 0$ for all $x \notin X(\Omega)$ and

$$\sum_{x\in X(\Omega)} p_x = 1 \ . \tag{2.1}$$

Usually, one can only observe the output $X(\omega)$ and not the underlying random choice $\omega \in \Omega$. Therefore, we can limit our attention to externally observable *events* $X(\omega) \in \mathcal{A}$ where $\mathcal{A} \subseteq \{0,1\}^*$ and define the probability measure as

$$\Pr\left[\omega \leftarrow \Omega : X(\omega) \in \mathcal{A}\right] := \sum_{x\in\mathcal{A}} p_x \ . \tag{2.2}$$

The equations (2.1) and (2.2) assure that the corresponding probability measure is indeed well-defined and satisfies Kolmogorov's axioms.

To simplify notation, we use the following conventions. A shorthand $x \leftarrow X$ denotes that x is chosen according to the distribution X. Since a randomised function $f : \{0,1\}^* \times \Omega \to \{0,1\}^*$ defines a distribution $F_z(\omega) = f(z;\omega)$ for every input z, we can use analogous shorthand $y \leftarrow f(z)$ to denote that y is chosen according to the distribution F_z. Finally, let $\mathcal{A}$ be a finite subset of $\{0,1\}^*$ consisting of n elements. Then a shorthand $x \leftarrow \mathcal{A}$ denotes a uniform distribution over $\mathcal{A}$, i.e., $\Pr\left[x \leftarrow \mathcal{A} : x = a\right] = \frac{1}{n}$ for all $a \in \mathcal{A}$.

Asymptotics and time complexity. Cryptographic proofs often contain many lower or upper bounds. In many cases, we are only interested in the asymptotic behaviour of these bounds. Then so called Landau notation can significantly simplify the derivation and the form of corresponding bounds.

Let f and g be non-negative functions that are defined over the positive real numbers. Then $f(x) = \mathrm{O}(g(x))$ denotes that there exists constants $c, x_0 > 0$ such that $f(x) \leq c \cdot g(x)$ for all $x > x_0$. Similarly, $f(x) = \Omega(g(x))$ denotes that there exists constants $c, x_0 > 0$ such that $c \cdot g(x) \leq f(x)$ for all $x > x_0$. Now $f(x) = \Theta(g(x))$ stands for $f(x) = \mathrm{O}(g(x))$ and $f(x) = \Omega(g(x))$. Finally, symbols $\mathrm{o}(\cdot)$ and $\omega(\cdot)$ denote differences in the growth rates: $f(x) = \mathrm{o}(g(x))$ if $f(x)/g(x) \to 0$ and $f(x) = \omega(g(x))$ if $f(x)/g(x) \to \infty$, as $x \to \infty$.

Asymptotic behaviour can be studied in two different contexts. First, we can investigate how the exact security bounds of the basic primitives influence the overall security of a particular construction. For example, we can characterise how a discovery of a new more efficient factoring algorithm influences the overall security if a construction is based on 1024-bit primes. Secondly, we can consider construction families that are indexed by a *security parameter* $\Bbbk$ and observe asymptotic behaviour of security guarantees as a function of $\Bbbk$. Such an approach helps us to choose optimal constructions for specific security levels. In particular, we can consider various complexity classes for different time bounds. A time bound $f(\Bbbk)$ is *polynomial* if $f(\Bbbk) = \mathrm{O}(\Bbbk^c)$ for $c > 0$ and *polylogarithmic* if $f(\Bbbk) = \mathrm{O}(\log(\Bbbk)^c)$ for $c > 0$. The corresponding shorthands are $f \in \mathbf{poly}(\Bbbk)$ and $f \in \mathbf{polylog}(\Bbbk)$. Similarly, a function $\varepsilon(\Bbbk)$ is *asymptotically negligible* denoted by $\varepsilon \in \mathbf{negl}(\Bbbk)$ if it decreases faster than a reciprocal of any polynomial, i.e., $\varepsilon(\Bbbk) \in \mathrm{o}(\Bbbk^{-c})$ for all $c > 0$. We deliberately emphasise the asymptotic nature of this notion and reserve the term *negligible* for values that are insignificant in practice, say less than 2^{-80}. Similarly, *overwhelming probability* is a constant sufficiently close to 1 and not an asymptotic notion.

2.2 DIFFERENT INTERPRETATIONS OF PROBABILITY

Most of the cryptographic security definitions are explicitly based on the concepts of hypothesis testing. Therefore, cryptography also inherits many problems and shortcomings from statistics. In particular, we must interpret probabilities. Interpretation of probabilities is by no means an easy and unambiguous task as there are at least five different interpretations. Basic differences of interpretations concern objectivity or subjectivity of probabilities.

Imagine that you throw an unbiased coin. Then objective probability describes the potential of seeing heads and subjective probability your rational opinion about the odds of seeing heads. In particular, the objective probability disappears when the coin has landed even if you do not see it. The subjective probability can exist also after the landing provided that you do not see the outcome. In other words, objective probability describes a potential that realises in the future, whereas subjective probability can handle completed events with unknown outcomes. In that sense, objective probability describes always collective properties—characteristics of possible future events—and subjective probability individual properties—characteristics of a fixed event. This conflict between individual and collective properties is an unresolved theme in statistics.

Frequentists believe that probability is an *inherently collective* property of distributions that manifests itself as a limiting relative frequency of independent repeated trials. This line of thought was initiated by Cournot, Venn and Ellis in the middle of the 19th century. These ideas were extended further by von Mises in 1919 who explicitly stated that probability is a property of infinite sequences (*collectives*) that satisfy certain conditions. In particular, all "admissibly selected" sub-sequences must have the same limiting frequency as the sequence itself, i.e., no gambler can devise a winning strategy with fixed bets. Such a strong form of frequentism was refuted in 1937 by the contemporary mathematicians, as the theory had several "inconsistencies" and Kolmogorov had provided a better alternative in 1933, see [Fie92, vL96, Bin00, Sti86, LV97] for further comments and historical remarks. Such development boosted the popularity of mathematical statistics as a more liberal branch of frequentism and it became a prevalent school of thought. Starting from seminal works of Pearsons, Neyman and Fisher the main emphasis shifted to the design inference procedures (algorithms) that behave well on average over the distributions.

To illustrate the tension between objective and subjective probability in the context of cryptography, consider the security of a cryptosystem. Recall that a cryptosystem is a collection of sub-instances indexed by a randomly chosen secret key. In particular, we can fix an optimal attack algorithm and ask how well the cryptosystem as a collection of sub-instances survives the attack on average. Such a frequentistic viewpoint is appropriate in the design and evaluation of cryptosystems. Assume that an attack succeeds with negligible probability on average. Then by the law of large numbers, we expect that only a tiny, not to say non-existent, fraction of users to suffer from the attack. However, if we take the position of a user then the tables are turned. A user could not care less about the overall security as long as his or her particular sub-instance is immune against the attacks. Such a problem is *ill-posed* in the frequentistic setting: the secret key has already been chosen, we just do not know if the *particular* key is easier or harder to break than the average key.

To answer the questions about individual instances, it is *inevitable* to accept the notion of subjective probability. This line of thought was started by Bayes and was later rediscovered by Laplace. According to them, the probability is a quantitative measure of uncertainty that can be rationally updated by observing empirical data. During the early twenties and thirties of the 20th century the notion subjective probability was intensively studied by economists and mathematicians. Keynes and Ramsey were the first to propose formal theories that treated probability as a measure of uncertainty. The process was later completed by de Finetti who showed in 1937 that any rationally behaving entity must assign prices to bets that confirm Kolmogorov's axioms and satisfy Bayes' theorem. These results were later reproved by Savage in 1954 and Cox in 1961 under different but natural assumptions about rational behaviour. We refer to [Fie05] for an excellent treatment of historical developments and to [Jay03, Jef04, BT92] for a thorough treatment of subjective probability.

Essentially, there are two schools of Bayesianism. In statistics the main goal is objectivity and thus the objective Bayesianism tries to minimise the impact of the subjectivity and make the end results universally acceptable. The subjective Bayesianism makes no efforts to achieve objectivity and is more common in economics and game theory, where the buyers or players may indeed have biased but still rational preferences. Since all individual decisions about a particular choice of cryptographic primitives *can* and *should* be treated in the framework of economical game theory, we follow the school of subjective Bayesianism.

Finally, we emphasise that both interpretations of probabilities and their corresponding extensions in cryptography are equally important. The objective treatment is a natural choice when we assess the general design of cryptographic solutions, whereas one cannot avoid subjectivity if we want to make theoretically well-founded user level decisions, see the thorough discussion in Chapter 6.

2.3 BASIC PROPERTIES OF RANDOM VARIABLES

Independence and product space. Two random variables $X : \Omega_1 \to \{0,1\}^*$ and $Y : \Omega_2 \to \{0,1\}^*$ are independent if the corresponding random choices are independent. Formally, we must form a joint sample space $\Omega = \Omega_1 \times \Omega_2$ with random variables $X(\omega_1, \omega_2) := X(\omega_1)$ and $Y(\omega_1, \omega_2) := Y(\omega_2)$ such that

$$\Pr\left[X \in \mathcal{A} \wedge Y \in \mathcal{B}\right] := \Pr\left[\omega_1 \leftarrow \Omega_1 : X \in \mathcal{A}\right] \cdot \Pr\left[\omega_2 \leftarrow \Omega_2 : Y \in \mathcal{B}\right] \quad . \tag{2.3}$$

Alternatively, we can postulate that two random variables X and Y defined over the same sample space Ω are independent if for all events $\mathcal{A}$ and $\mathcal{B}$

$$\Pr\left[\omega \leftarrow \Omega : X \in \mathcal{A} \wedge Y \in \mathcal{B}\right] = \Pr\left[\omega \leftarrow \Omega : X \in \mathcal{A}\right] \cdot \Pr\left[\omega \leftarrow \Omega : Y \in \mathcal{B}\right] \quad . \tag{2.4}$$

Conditional probability. Let random variables $X = X(\omega)$ and $Y = Y(\omega)$ be defined over the same sample space Ω. Then the conditional probability of the event that $X \in \mathcal{A}$ given that $Y \in \mathcal{B}$ is defined as

$$\Pr\left[X \in \mathcal{A} | Y \in \mathcal{B}\right] := \frac{\Pr\left[\omega \leftarrow \Omega : X(\omega) \in \mathcal{A} \wedge Y(\omega) \in \mathcal{B}\right]}{\Pr\left[\omega \leftarrow \Omega : Y(\omega) \in \mathcal{B}\right]} \tag{2.5}$$

provided that the denominator is not zero. In the Bayesian framework, a shorthand $\Pr[X \in \mathcal{A} | Y \in \mathcal{B}]$ quantifies a subjective belief that $X \in \mathcal{A}$ when $Y \in \mathcal{B}$ is known to hold and *a priori* it is not clear that the equation (2.5) must hold. Nevertheless, it is possible to prove that the famous Bayes' theorem

$$\Pr[X \in \mathcal{A} \wedge Y \in \mathcal{B}] = \Pr[X \in \mathcal{A} | Y \in \mathcal{B}] \cdot \Pr[Y \in \mathcal{B}] \tag{2.6}$$

and all other Kolmogorov axioms still hold for any *rational belief system*, see the handbooks [Jay03, Jef04] for further details. In other words, all manipulation rules are formally the same for objective and subjective probabilities.

Statistical distance. Let X and Y be random variables. Then statistical distance (also known as statistical difference) between X and Y is defined as

$$\mathsf{sd}(X, Y) := \sup_{\mathcal{A} \subseteq \{0,1\}^*} |\Pr[X \in \mathcal{A}] - \Pr[Y \in \mathcal{A}]| \enspace , \tag{2.7}$$

where the probabilities are computed over the corresponding distributions. It is straightforward to prove that statistical difference has an alternative form[1]

$$\mathsf{sd}(X, Y) = \frac{1}{2} \cdot \sum_{x \in \{0,1\}^*} |\Pr[X = x] - \Pr[Y = x]| \tag{2.8}$$

and thus can be viewed as ℓ_1-norm. Hence, it also satisfies the triangle inequality

$$\mathsf{sd}(X, Z) \leq \mathsf{sd}(X, Y) + \mathsf{sd}(Y, Z) \enspace . \tag{2.9}$$

Intricate connections between statistical distance and hypothesis testing and their extensions to computationally bounded cases are discussed in Chapter 3.

Random tape model. Many cryptographic models use coin-flipping as the only source of randomness, i.e., the probability space is $\Omega_* = \{0,1\}^*$ where each consecutive binary digit $\omega_i \in \{0,1\}$ is generated by flipping a fair coin. Note that it is fairly straightforward to show that any discrete distribution can be constructed in such a way. That is, for any sequence of non-negative numbers $(p_x)_{x \in \{0,1\}^*}$ that satisfy the equation (2.1) there exists $X : \Omega_* \to \{0,1\}^*$ such that $\Pr[X = x] = p_x$. Moreover, the random variable X can be approximated with arbitrary precision using enough coin-flips. More formally, we can define a sequence of random variables $X_n : \Omega_* \to \{0,1\}^*$ such that $X_n(\omega)$ depends only on the first n coin-flips $(\omega_1, \ldots, \omega_n)$ and $\mathsf{sd}(X_n, X) \to 0$, as $n \to \infty$.

Surprisingly, a similar result holds in general. The Skorohod Representation Theorem assures that any probability distribution on an arbitrary set $\mathcal{X}$ can be represented as a random variable $X : \Omega_* \to \mathcal{X}$. See the handbook [Bil95] for further details. Such a result is philosophically very pleasing, as it implies that the random tape model can still adequately represent reality.

[1]The sum on the right is absolutely convergent, since all terms are non-negative and the sum can be upper bounded by 2.

2.4 DIFFERENT FORMAL MODELS OF COMPUTATION

Computability is another central concept in cryptography, which is essentially a science of secure communication and computation. The notion of computability emerged in the first half of 20th century during the systematisation and re-evaluation of the foundations of mathematics. Initially, the emphasis was on the limitations of formal proof systems but later the attention turned to fully automatic derivation of numeric results. The remarkable breakthrough came in 1936 when Alonzo Church and Alan Turing independently showed that it is impossible to algorithmically decide whether a certain claim holds in arithmetic or not [Chu36, Tur37]. Both authors postulated that a rather simple model can be used to describe any fully mechanistic computational processes.

These postulates were quickly accepted and the corresponding thesis explicitly stated by Stephen Kleene [Kle43]. The Church-Turing thesis states that all "imaginable" algorithmic computations can be specified by Turing machines. Of course, the Church-Turing thesis can never be proved, since the notion of "imaginable" algorithmic computations is inherently vague.

We emphasise that the Turing machine is only a mathematical abstraction that allows us to formally model some aspects of modern computational devices. In the light of all known and foreseeable technologies, it is impossible to actually construct a Turing machine. The problem lies in the fact that Turing machine is an infinite object, whereas the observable universe is finite.[2] Briefly, a Turing machine is only a simple model of a modern computing device and not the other way around. Moreover, in many situations Random Access Machines reflect reality more precisely than Turing machines.

Turing Machines. A Turing machine can be viewed as a *finite automaton* with access to an infinite stack of memory cells. The stack is usually modelled by a one-sided infinite tape and a moving head that can read or write symbols underneath it. The movement of the head is controlled by an automaton with a finite number of states and a finite set of transition rules. A transition may depend on the symbol read by the head and it can specify a replacement symbol. However, the controller must be deterministic, i.e., there can be only a single transition rule for each state and a symbol underneath the head. A more formal treatment of Turing machines is given in textbooks [Pap93, HMU00].

Nevertheless, some peculiarities of Turing machines are worth stressing. By common convention, the *running time* of a Turing machine is just the number of transitions made by the controller before halting. In other words, the speed of an elementary read-write operation is independent of the size of finite controller. Therefore, for any predicate $f : \{0,1\}^n \to \{0,1\}$ there exist a Turing machine that outputs the correct answer for any input $x \in \{0,1\}^n$ in n steps. Basically, we can hardwire the binary search for the 2^n element truth table. The corresponding 2^{n+1} state finite controller finds the right entry in the truth table

[2]Astrophysicist have estimated that the mass of the Universe ranges from 10^{53}–10^{60} kg. Even if the Universe is infinite, then our current knowledge indicates that mankind can observe only a finite portion of the Universe during a finite time interval. Therefore, all modern computing devices are actually finite automatons whether we like it or not.

Another justification comes from the fundamental von Neumann-Landauer limit [Lan61]: any logically irreversible bit operation leads to unavoidable energy dissipation. Hence, only a finite number of logically irreversible operations can be carried out.

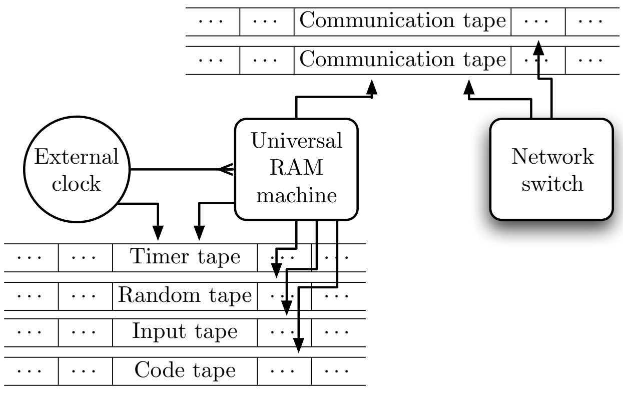


Figure 2.1: A schematic model of a computational node in a network

with n steps and outputs the corresponding result. More generally, any function $f : \{0,1\}^n \to \{0,1\}^m$ can be computed in $n + m$ steps.

To resolve the issue, we fix a universal Turing machine and require that any algorithm must be specified as a read-only bit-string (*program code*) on the special input tape. Such a model coincides with the *non-uniform complexity model* where the Turing machine is fixed but one can feed a special advice string that depends only on the input size n. Alternatively, we can directly penalise the Turing machine based on the size of the hardwired program. Interestingly enough, the latter leads to *circuit complexity*, as the computational logic can be completely encapsulated in the controller design. Although circuit complexity provides a better insight into how well an algorithm can be parallelised, the corresponding formalism is also much more difficult to understand.

Random Access Machines. Although Turing machines have no limits on the amount of accessible memory, it takes time to move the head to a random location. Often such linear memory access cost is not justified, since modern computing devices provide a large chunk of volatile memory that can be randomly accessed in a constant time. Similarly, read and write operations of modern hard disks can be considered location independent. Consequently, if a program has a small enough memory footprint, then a *Random Access Machine* (RAM machine) models computations more precisely. Recall that a RAM machine consists of a finite controller (*processor*) that can manipulate an infinite array of registers by sequentially processing a finite hardwired program. Informally, a RAM machine is just a program in some assembler dialect, i.e., it can do basic arithmetic operations with registers and conditional jumps. More formal definitions of RAM machines together with comparison to other computational models can be found in textbooks [Smi94, WW01, HMU00]. Similarly to Turing machines, we fix a single universal RAM machine to avoid an artificial speed up that is caused by hardwired binary search techniques.

Standardised interface. Although the computing process itself is handled differently by universal Turing and RAM machines, it is still possible to standardise how these *computing nodes* interact with the surrounding environment. Briefly, only three types of external events can influence computations: random choices, messages from other computing devices and timer interrupts. In the following, we gradually specify all these aspects of interaction, see Fig. 2.1.

Randomised computations. A computing node has two one-sided read-only input tapes: one for program code and one for randomness $\omega \in \Omega_*$. It also has a one-sided read-write tape for input and output. Given a code of $\mathcal{A}$, an input x and a random tape ω, the interpreter $\mathcal{U}$ starts to execute code. The computation ends successfully if the input is replaced by the output $\mathcal{A}(x;\omega)$ and $\mathcal{U}$ halts in the canonical end configuration. Otherwise, the output of $\mathcal{A}$ is denoted by $\perp$. An algorithm $\mathcal{A}$ is *deterministic* if no cells from the random tape ω are read. A *randomised algorithm* $\mathcal{A}$ can access a prefix of ω proportional to running time. Hence, we can talk about output distribution of $\mathcal{A}$ and the distribution of running time $t(\omega)$ together with all statistical estimators.

Interactive aspects of computing. Besides classical input tapes a computing node has a pair of read-write tapes: one for communication and the other for timer services. The communication tape is connected to a *network switch*. A network switch is a dedicated machine that delivers messages between different nodes. More formally, a network switch has access to all communication tapes and whenever a new message is output by a node, it determines the recipient, copies the messages to the communication tape of the recipient and sends a notification message to the sender after that. Obviously, a certain communication protocol between nodes and network switch must be fixed, or otherwise some messages may get lost or infinitely delayed. Let us neglect the exact details and just postulate properties of a correct implementation: (a) no messages are dropped, inserted or modified; (b) each node has a physical network address; (c) a node always waits for a notification message that indicates a successful transfer; (d) messages are transferred in the same order as they are created.

Note that the actual time needed to copy messages is linear in the size of the message, whereas the network delay depends on the exact configuration and behaviour of the network. Finally, reliable point-to-point networking is only a stepping stone: more complex network topologies such as non-reliable and asynchronous communication can be built on top of the basic network layer.

Complexity theoretical setting. Turing and RAM machines model the timing aspects of computations differently, as they use different models of memory. More precisely, a random memory access takes $\Theta(n)$ elementary operations for Turing machines and $\Theta(1)$ operations for RAM machines, where n is the size of the memory bank. Neither of these estimates is truly accurate, as we can arrange memory cells into a three dimensional lattice. Thus, the maximal physical distance between two memory cells is $\Theta(\sqrt[3]{n})$ and the corresponding access time is $o(n)$. Consequently, Turing machines overestimate and RAM machines underestimate the actual time needed to compute the outcome.

These time differences are irrelevant in the classical complexity theoretical setting that neglects polynomial factors. Although such an approach hides many tedious details, the resulting estimates are often too crude in practice. For example, algorithms with a quadratic asymptotic complexity are considered intractable in the context of large numerical computations, whereas algorithms with worst-case exponential complexity are successfully used in compilers. In reality, the tractability of an algorithm depends on the number of required elementary steps. Moreover, the effective time bound depends on the context. It can range from 10–10^{18} elementary operations depending whether it is a mission critical real-time software or an earthquake prediction algorithm.

Secondly, setting up a correct asymptotic feasibility definition is not a straightforward task at all, since there are so many choices. Often these choices are irrelevant and lead to equivalent descriptions, whereas sometimes they make a big difference, see the discussion in manuscripts [Can00b, Gol07]. To avoid such surprises, we persistently operate with exact time bounds and consider asymptotic behaviour whenever complexity theoretic results are needed.

Exact time complexity. Unfortunately, the definition of a Turing machine is not robust if we consider exact time complexity. Namely, the running times of different algorithms depend on whether a Turing machine has zero, one or more than one internal working tape, see the discussion and further references in [LV97, p. 432–444]. Although the difference in running times can be at most quadratic, it is still very noticeable in practice. The same effect emerges also when the computations are divided between several Turing machines. Then the whole network has more working tapes at its disposal than a single Turing machine. Consequently, simulating an interactive computation on a single Turing machine takes significantly more time steps than the total running time of all the nodes in the network. Hence, the running time of an attack depends on whether an attacker controls a single or a cluster of Turing machines. Of course, the overhead is only polynomial in the total running time it but makes security analysis more complex. Hence, we state security results only for RAM machines, as they are more robust in terms of exact time complexity.

Let $\mathcal{A}$ be an *algorithm*, that is, a finite program code for the universal RAM machine. Then the running time of $\mathcal{A}$ is counted as follows. First, the program code of $\mathcal{A}$ is red command by command to the main memory. Next, the program is interpreted according to a fixed instruction set such as [CR72, p. 75]. Each basic operation should require time at least proportional to the bit-length of the corresponding operands. Otherwise, we get paradoxical results such as a factoring algorithm that works in linear time in the input size [Sha79]. Now the exact time complexity of an algorithm can be defined either in a strict or liberal sense. An algorithm $\mathcal{A}$ is a *t-time algorithm in the strict sense* if $\mathcal{A}$ always halts after t elementary steps regardless of inputs and random coins. In reality, computations are often halted based on an external clock, i.e., the output of $\mathcal{A}$ is set to $\perp$ when the time limit t is exceeded. The latter gives a rise to the liberal definition of t-time computations that is used throughout the thesis.

Obviously, the definitions can be generalised to Turing machines but then one must explicitly fix the number of Turing machines controlled by an attacker and the number of working tapes inside a Turing machine.

Time-bounded subroutines. Most of the cryptographic proofs are based on black-box reductions, where an algorithm $\mathcal{B}$ calls an algorithm $\mathcal{A}$ as a subroutine to complete its task. Now the liberal definition of t-time computations causes a slight trouble. Imagine that a t-time routine $\mathcal{A}$ succeeds with probability $\frac{1}{2}$ and otherwise stays in the infinite loop. Then $\mathcal{B}$ that calls $\mathcal{A}$ must stop the subroutine $\mathcal{A}$ after t steps, or otherwise $\mathcal{B}$ also stays in the infinite loop. Obviously, such an external clock can be included into $\mathcal{B}$ with the cost of polynomial slowdown. However, such slowdown is highly undesirable in our setting. Thus, we include timers explicitly into the computational model. Note that algorithms can use many timers, since subroutines can also call subroutines with timers.

For clarity, we use an infinite timer model, where an algorithm can set potentially unbounded number of timers. If a timer is fired then the algorithm is stopped and an interrupt handler is called. More formally, an algorithm $\mathcal{B}$ can always write an address $\mathfrak{L}$ of an interrupt handler and a time bound t on the dedicated timer tape. Given $\mathfrak{L}$ and t, the timer decreases t until it reaches zero. Then the execution is continued from the $\mathfrak{L}^{\text{th}}$ command of $\mathcal{B}$.

Even more formally, an *external clock* is a dedicated machine with a single timer t, an address register $\mathfrak{L}$ and a potentially infinite stack containing time difference and address pairs $(\Delta t_i, \mathfrak{L}_i)$. The external clock is coupled with the interpreter and each elementary step decreases the value of t until it reaches zero. If the timer value reaches zero, then the interpreter seeks the $\mathfrak{L}^{\text{th}}$ command from the code and continues execution from it. Next, the timer and the address registers are updated by the topmost pair $(\Delta t_i, \mathfrak{L}_i)$ in the stack. If the stack is empty then the external clock waits for a an incoming query $(t_i, \mathfrak{L}_i)$ to update the timer and address registers. If a query $(t_i, \mathfrak{L}_i)$ arrives and the timer is already activated, then the clock computes $\Delta t = t - t_i$. If $\Delta t \leq 0$ then the query is discarded, otherwise the values $(\Delta t, \mathfrak{L})$ are pushed on top of the stack and registers $(t, \mathfrak{L})$ are replaced by $(t_i, \mathfrak{L}_i)$. As a final detail, note that computation of Δt must take into account the time needed to do all computations connected to stack updates to get timer interrupts at precise time moments.

The discussion above also shows that the infinite timer model is just a convenience and adversaries cannot use the timers for speeding up their computations. Indeed, the use of timers cannot cause a significant speedup, as the external clock can be realised by a single Turing or RAM machine.

3 HYPOTHESIS TESTING

One of the main objectives in cryptography is confidentiality. Throughout history various encryption schemes have been used to scramble communication so that eavesdroppers cannot reconstruct the original messages. However, sometimes even partial message recovery can be as disastrous as full disclosure. Generally, secrecy is preserved only if an adversary cannot reject or accept non-trivial hypotheses about secret inputs given access to public outputs. Thus, we must formalise secrecy goals in the framework of hypothesis testing.

More formally, let s be a secret input and $f(s)$ the corresponding public output x where f is a publicly known randomised function. Then we can state at least one trivial hypothesis $s \in \{0,1\}^*$ and many non-trivial hypotheses such as $s \in \{1,4,7\}$. We say that $\mathcal{H}$ is a *simple hypothesis* if it uniquely determines the output distribution, i.e., probabilities $\Pr[x|\mathcal{H}]$ are well-defined for any $x \in \{0,1\}^*$. For obvious reasons, any hypothesis can always be represented as a union of simple hypotheses. Therefore, we start with a question of how to choose between two simple hypotheses $\mathcal{H}_0$ or $\mathcal{H}_1$ and then gradually extend the approach to consider more and more complex settings.

3.1 SIMPLE HYPOTHESIS TESTING

Statistical tests. Assume that an adversary must choose one of two simple hypotheses $\mathcal{H}_0$ or $\mathcal{H}_1$. A *statistical test* $\mathcal{A} : \{0,1\}^* \to \{0,1\}$ is a randomised predicate that specifies whether to accept or reject the *null hypothesis* $\mathcal{H}_0$. The rejection probability of $\mathcal{H}_0$ is often denoted by $\phi(x) = \Pr[\mathcal{A}(x) = 1]$. A statistical test $\mathcal{A}$ can make two types of errors: *false negatives* and *false positives.*[1] The ratio of false negatives is commonly denoted by $\alpha(\mathcal{A})$ and the ratio of false positives is denoted by $\beta(\mathcal{A})$. In other words

$$\alpha(\mathcal{A}) = \Pr[\mathcal{A}(x) = 1|\mathcal{H}_0] \quad , \tag{3.1}$$

$$\beta(\mathcal{A}) = \Pr[\mathcal{A}(x) = 0|\mathcal{H}_1] \quad . \tag{3.2}$$

A common task in statistics is to find a strategy $\mathcal{A}$ that minimises the ratio of false positives $\beta(\mathcal{A})$ given a bound on false negatives $\alpha(\mathcal{A}) \le \alpha_0$. A statistical test $\mathcal{A}_*$ is *optimal w.r.t. the bound α_0 on false negatives* if

$$\beta(\mathcal{A}_*) = \inf\{\beta(\mathcal{A}) : \alpha(\mathcal{A}) \le \alpha_0\} \quad . \tag{3.3}$$

The most obvious way to choose between hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ is based on the likelihood of outputs. This test is known as the *likelihood ratio test*:

$$\phi_{\mathrm{lrt}}(x) = \begin{cases} 1 & \text{if } \Pr[x|\mathcal{H}_0] < \eta \cdot \Pr[x|\mathcal{H}_1] \quad , \\ \rho & \text{if } \Pr[x|\mathcal{H}_0] = \eta \cdot \Pr[x|\mathcal{H}_1] \quad , \\ 0 & \text{if } \Pr[x|\mathcal{H}_0] > \eta \cdot \Pr[x|\mathcal{H}_1] \quad . \end{cases} \tag{3.4}$$

Observe that $\alpha(\phi_{\mathrm{lrt}})$ decreases monotonically when $\eta \in [0,\infty)$ is decreased and $\rho \in [0,1]$ only determines how the ties are handled, see Fig. 3.1. Hence, for any

[1]Due to historical reasons, statisticians use terms Type I and Type II Errors instead.

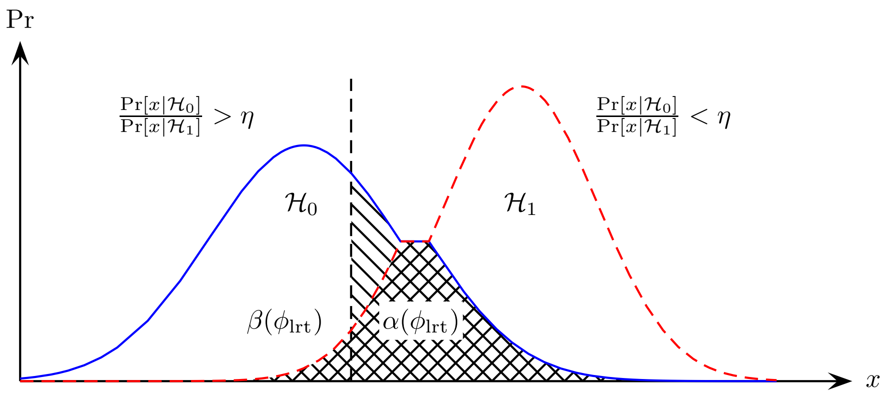


Figure 3.1: Likelihood ratio test. Decreasing η moves the decision border to the right and thus decreases $\alpha(\phi_{\mathrm{lrt}})$ and increases $\beta(\phi_{\mathrm{lrt}})$. The aggregate error $\gamma(\phi_{\mathrm{lrt}})$ is minimised when it coincides with the double crossed area.

α_0 there exists a likelihood ratio test that achieves $\alpha(\phi_{\mathrm{lrt}}) = \alpha_0$. Moreover, the famous Neyman-Pearson Theorem states that the likelihood ratio test is optimal and every optimal test is functionally indistinguishable from the corresponding likelihood ratio test. See statistical handbooks such as [CB02] for more details.

Depending on the task at hand, the adversary can either reduce the ratio of false positives $\alpha(\mathcal{A})$ by increasing the ratio of false negatives $\beta(\mathcal{A})$ or vice versa. Although different tests can provide different trade-offs, we can still establish lower bounds for the *aggregate error* $\gamma(\mathcal{A}) = \alpha(\mathcal{A}) + \beta(\mathcal{A})$. Indeed, define a statistical distance between $\mathcal{H}_0$ and $\mathcal{H}_1$ w.r.t. the output x as

$$\mathsf{sd}_x(\mathcal{H}_0, \mathcal{H}_1) = \sup_{\mathcal{A}} |\Pr[\mathcal{A}(x) = 1|\mathcal{H}_0] - \Pr[\mathcal{A}(x) = 1|\mathcal{H}_1]| \enspace . \tag{3.5}$$

Then $\gamma(\mathcal{A})$ of a statistical test $\mathcal{A}$ is lower bounded as follows:

$$\gamma(\mathcal{A}) = 1 + \Pr[\mathcal{A}(x) = 1|\mathcal{H}_0] - \Pr[\mathcal{A}(x) = 1|\mathcal{H}_1] \geq 1 - \mathsf{sd}_x(\mathcal{H}_0, \mathcal{H}_1) \enspace . \tag{3.6}$$

The inequality (3.6) also provides nice geometrical proof that the formula (2.8) indeed computes the statistical distance. By the construction

$$\gamma(\phi_{\mathrm{lrt}}) \geq \sum_{x \in \{0,1\}^*} \min\{\Pr[x|\mathcal{H}_0], \Pr[x|\mathcal{H}_1]\} \tag{3.7}$$

where the equality holds for parameters $\eta = 1$ and $\rho = \frac{1}{2}$. Again, by geometrical considerations the right hand side of (3.7) can be expressed as the formula (2.8). Since the likelihood test is optimal, the formulae (2.7) and (2.8) must be equal.

Computational tests. Although Fig. 3.1 correctly illustrates properties of optimal statistical tests, it is also slightly misleading. Namely, not all distribution pairs have a simple decision boundary nor are the probabilities $\Pr[x|\mathcal{H}_i]$ efficiently computable. In short, likelihood ratio tests are often intractable and thus it makes sense to consider only computationally feasible tests. Random variables X and Y are (t, ε)-*indistinguishable* if for any t-time algorithm $\mathcal{A}$

$$\mathsf{Adv}^{\mathrm{ind}}_{X,Y}(\mathcal{A}) = |\Pr[x \leftarrow X : \mathcal{A}(x) = 1] - \Pr[y \leftarrow Y : \mathcal{A}(y) = 1]| \leq \varepsilon \enspace . \tag{3.8}$$

We emphasise that the computational effort needed to create a sample x is irrelevant, as the running time of $\mathcal{A}$ depends only on its input and not on the way the input was created. The *computational distance* $\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1)$ between two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ w.r.t. the variable x is the minimal value[2] of ε such that the corresponding distributions are (t, ε)-indistinguishable. Observe that there is a correspondence between statistical and computational distance:

$$\mathsf{sd}_x(\mathcal{H}_0, \mathcal{H}_1) = \lim_{t\to\infty} \mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1) \ . \tag{3.9}$$

Secondly, we can easily derive a new triangle inequality

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{H}_0,\mathcal{H}_2}(\mathcal{A}) \le \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{H}_0,\mathcal{H}_1}(\mathcal{A}) + \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{H}_1,\mathcal{H}_2}(\mathcal{A}) \ , \tag{3.10}$$

from the classical triangle inequality $|a - c| \le |a - b| + |b - c|$ by substituting $a = \Pr[\mathcal{A}(x)\!=\!1|\mathcal{H}_0]$, $b = \Pr[\mathcal{A}(x)\!=\!1|\mathcal{H}_1]$ and $c = \Pr[\mathcal{A}(x)\!=\!1|\mathcal{H}_2]$ into it. As the inequality holds for any algorithm $\mathcal{A}$ and any time bound t, we have

$$\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_2) \le \mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1) + \mathsf{cd}^t_x(\mathcal{H}_1, \mathcal{H}_2) \ . \tag{3.11}$$

Thirdly, note that if $\mathcal{H}_0$ and $\mathcal{H}_1$ determine different distributions, there exists a value x_0 such that $\Pr[x_0|\mathcal{H}_0] \ne \Pr[x_0|\mathcal{H}_1]$. Consequently, an algorithm $\mathcal{A}_{x_0}$ that outputs 1 only if $x = x_0$ and 0 otherwise achieves

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{H}_0,\mathcal{H}_1}(\mathcal{A}_{x_0}) = |\Pr[x_0|\mathcal{H}_0] - \Pr[x_0|\mathcal{H}_1]| > 0 \ . \tag{3.12}$$

Now, if the time bound t is large enough to compare an input x and the hardwired constant x_0, then the operator $\mathsf{cd}^t_x(\cdot,\cdot)$ is positively definite

$$\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1) > 0 \quad \Longleftrightarrow \quad \mathcal{H}_0 \not\equiv \mathcal{H}_1 \ . \tag{3.13}$$

As $\mathsf{cd}^t_x(\cdot,\cdot)$ is clearly symmetric, we have indeed established that $\mathsf{cd}^t_x(\cdot,\cdot)$ is a distance measure between output distributions for sufficiently large t.

3.2 NEGLIGIBLE EVENTS AND SEMANTIC SECURITY

Computational distance between $\mathcal{H}_0$ and $\mathcal{H}_1$ characterises the selectivity of feasible computational tests. If $\mathcal{A}$ is a t-time test then analogously to the inequality (3.6) we can prove a lower bound $\gamma(\mathcal{A}) \ge 1 - \mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1)$. However, the bound on aggregate error is very non-intuitive. Hence, we pose the question differently and ask whether the best feasible computational test is worth implementing at all. Here, we view cryptology as a tool to make economically sound and scientifically justified decisions. Usually, decisions are made by ignoring events with insignificant probabilities. For example, a probability of an airplane crash is roughly 2^{-23} but most of us are still willing to use air transportation.

More formally, let *negligible* denote a threshold for probabilities such that a change of probability by a negligible amount does not affect economical decisions. Clearly, the threshold depends on the exact setting. Nevertheless, the famous Borel's Law gives some rough estimates: probability 2^{-20} is negligible on the personal scale, probability 2^{-50} is negligible on the terrestrial scale and

[2]Since there are only finite number of t-time algorithms, the minimum always exists.

probability 2^{-166} is negligible on the cosmic scale [Bor62]. Note that many authors use the term negligible also for characterising the asymptotic growth rate of variables. In such settings, we always use the term *asymptotically negligible* in order to avoid confusion, see Section 2.1 for the precise definition.

Let us now return to simple hypothesis testing and consider the classical scenario where the null hypothesis $\mathcal{H}_0$ holds with a probability $\Pr[\mathcal{H}_0] \geq \Pr[\mathcal{H}_1]$. Then a t-time test $\mathcal{A}$ provides a correct answer with probability

$$\begin{aligned}\Pr[\mathsf{success}] &= \Pr[\mathcal{H}_0] \cdot \Pr[\mathcal{A}(x) = 0|\mathcal{H}_0] + \Pr[\mathcal{H}_1] \cdot \Pr[\mathcal{A}(x) = 1|\mathcal{H}_1] \\ &\leq \Pr[\mathcal{H}_0] + \Pr[\mathcal{H}_1] \cdot \big(\Pr[\mathcal{A}(x) = 1|\mathcal{H}_1] - \Pr[\mathcal{A}(x) = 1|\mathcal{H}_0]\big) \\ &\leq \Pr[\mathcal{H}_0] + \Pr[\mathcal{H}_1] \cdot \mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1) \ .\end{aligned} \tag{3.14}$$

Now $\Pr[\mathsf{success}] = \Pr[\mathcal{H}_0]$ for a trivial test $\mathcal{A}_* \equiv 0$ and the optimal t-time test $\mathcal{A}$ can exceed the trivial success probability only by $\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1)$. Consequently, the effort needed to implement $\mathcal{A}$ is not justified if $\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1)$ is negligible.

Simple hypothesis testing is often inadequate for many real world scenarios. Usually there are many plausible secret inputs s instead of two alternatives and a partial disclosure of secrets is also harmful. Thus, given a public input $f(s)$ it should be infeasible to reliably predict the output $g(s)$ for any non-constant function g. Such inability to infer non-trivial information about secret inputs is called *semantic security*. The corresponding security notion was first proposed by Goldwasser and Micali in the context of probabilistic encryption [GM82] and later extensively studied by others [Yao82, MRS88, Lub96, BDJR97].

The concept of semantic security can be modelled in the framework of hypothesis testing. Let $g(\cdot)$ be a deterministic function. Then any hypothesis about $g(s)$ naturally splits all plausible values of s into two sets $\mathcal{S}_0$ and $\mathcal{S}_1$ so that the null hypothesis $\mathcal{H}_0$ holds for all $s \in \mathcal{S}_0$ and the complementary alternative hypothesis $\mathcal{H}_1$ for all $s \in \mathcal{S}_1$. Strictly speaking, the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ alone do not determine the output distribution and we must assume the existence[3] of some unknown probability distributions over $\mathcal{S}_0$ and $\mathcal{S}_1$. Then the simple hypothesis testing scenario still adequately describes the process and the inequality (3.14) holds whenever $\Pr[\mathcal{H}_0] \geq \Pr[\mathcal{H}_1]$. Note that we can only upper bound the value of $\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1)$, as the distributions on $\mathcal{S}_0$ and $\mathcal{S}_1$ are unknown. Let $\mathcal{H}_{[s=s_i]}$ denote an elementary hypothesis that $s = s_i$ for $i \in \{0, 1\}$. Then

$$\mathsf{cd}^t_x(\mathcal{H}_0, \mathcal{H}_1) \leq \sup_{s_i \in \mathcal{S}_i} \mathsf{cd}^t_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_1]}) \ , \tag{3.15}$$

since we can upper bound $|\Pr[\mathcal{A}(x)=1|\mathcal{H}_1] - \Pr[\mathcal{A}(x)=1|\mathcal{H}_0]|$ by

$$\sum_{s_0 \in \mathcal{S}_0} \sum_{s_1 \in \mathcal{S}_1} \Pr[s_0] \cdot \Pr[s_1] \cdot |\Pr[\mathcal{A}(x) = 1|s_1] - \Pr[\mathcal{A}(x) = 1|s_0]| \tag{3.16}$$

and upper bound the absolute value by $\sup_{s_i \in \mathcal{S}_i} \mathsf{cd}^t_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_1]})$. By combining the inequalities (3.14) and (3.15), we obtain the final upper bound

$$\Pr[\mathsf{success}] \leq \Pr[\mathcal{H}_0] + \Pr[\mathcal{H}_1] \cdot \sup_{s_i \in \mathcal{S}_i} \mathsf{cd}^t_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_1]}) \ . \tag{3.17}$$

[3]In our context, such a semi-Bayesian assumption is completely natural although orthodox statisticians would use the notion of universally most powerful tests instead.

The inequality (3.17) holds also for randomised functions $g(s) = g(s;\omega)$ when the random choices of $f(s)$ and $g(s)$ are independent. For proof, we must just consider elementary hypotheses $\mathcal{H}_{[s=s_i,\omega=\omega_i]}$ instead of $\mathcal{H}_{[s=s_i]}$, repeat the derivation and use the independence assumption to verify

$$\mathsf{cd}^t_x(\mathcal{H}_{[s=s_0,\omega=\omega_0]}, \mathcal{H}_{[s=s_1,\omega=\omega_1]}) = \mathsf{cd}^t_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_1]}) \ . \tag{3.18}$$

Finally, consider the scenario where an adversary does not test hypotheses but predicts the output of $g : \{0,1\}^* \to \{0,\ldots,n\}$ instead. Let $\mathcal{H}_i$ denote the hypothesis $g(s) = i$ and $\mathcal{S}_i = \{s : g(s) = i\}$. For simplicity, assume $\Pr[\mathcal{H}_0] \geq \Pr[\mathcal{H}_i]$ for $i \in \{1,\ldots,n\}$. Then we can derive

$$\Pr[\mathsf{success}] \leq \Pr[\mathcal{H}_0] + \sum_{k=1}^{n} \Pr[\mathcal{H}_i] \cdot \bigl(\Pr[\mathcal{A}(x)=k|\mathcal{H}_k] - \Pr[\mathcal{A}(x)=k|\mathcal{H}_0]\bigr) \ ,$$

where the terms in the parenthesis are again averages over unknown but existing distributions. Obviously, we can test $\mathcal{A}(x) = k$ in time $\mathrm{O}(\log n)$ and thus

$$|\Pr[\mathcal{A}(x)=k|\mathcal{H}_k] - \Pr[\mathcal{A}(x)=k|\mathcal{H}_0]| \leq \sup_{s_i \in \mathcal{S}_i} \mathsf{cd}^\tau_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_k]}) \ , \tag{3.19}$$

where $\tau = t + \mathrm{O}(\log n)$. Therefore, we can conclude

$$\Pr[\mathsf{success}] \leq \Pr[\mathcal{H}_0] + \Pr[\neg\mathcal{H}_0] \cdot \sup_{s_0,s_1} \mathsf{cd}^\tau_x(\mathcal{H}_{[s=s_0]}, \mathcal{H}_{[s=s_1]}) \ , \tag{3.20}$$

where the supremum is taken over all possible secret inputs. To summarise, we have established that for any function $g(s)$ with a reasonable output length, the inability to distinguish simple hypotheses implies the inability to predict the output significantly better than choosing the most probable output. Hence, we have established the classical result that indistinguishability implies semantic security. Still, note that the proof technique is strictly non-constructive and the function $g(\cdot)$ itself can be intractable. The latter is somewhat different from the strictly constructive approach pursued in the article [BDJR97]. We re-examine these issues in Sections 6.6 and 6.7 in the context of subjective security.

3.3 INTERACTIVE INFERENCE AND SECURITY GAMES

Thus far we have considered non-interactive inference where an adversary has no control over public outputs. This model is adequate only if the adversary is an external party in the computational process, for example listens to network traffic or breaks into a computational node after the computations have been completed. Alternatively, the adversary can actively participate in the computations and thus influence the sampling procedure itself. Such interactive scenarios are known as games or experiments. Although games have been implicitly around since 1982 when Goldwasser, Micali and Yao first used hybrid arguments [GM82, Yao82], a more formal treatment of games is rather recent. It gradually emerged together with the notion of exact security and reached its maturity around 2004, when several game-playing proof methodologies [BR04, Sho04, Hal05] were explicitly outlined. In some sense, this was an inevitable change forced by the ever-growing complexity of cryptographic proofs and the nit-picking argumentation style demanded by exact security.

Although the basic ideas behind game-playing proofs are rather simple, it is always nice to start with illustrative examples. Various security definitions of public-key cryptosystems are a good starting point, since they have a simple structure and are still complex enough to illustrate all the basic ideas.

Security of cryptosystems as an example. A public-key cryptosystem is specified by three efficient randomised algorithms $\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec}$. A key generation algorithm Gen outputs a key pair that consists of a secret key sk and a public key pk. Two other algorithms $\mathsf{Enc}_{\mathsf{pk}} : \mathcal{M} \rightarrow \mathcal{C}$ and $\mathsf{Dec}_{\mathsf{sk}} : \mathcal{C} \rightarrow \mathcal{M} \cup \{\perp\}$ are used to encrypt and decrypt messages. Sets $\mathcal{M}$ and $\mathcal{C}$ are known as message and ciphertext spaces, respectively. A cryptosystem is functional if, for all key pairs $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ and messages $m \in \mathcal{M}$, the equality

$$\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m)) = m \qquad (3.21)$$

holds with overwhelming probability. Message and ciphertext space can depend on the public key pk. Moreover, the decryption algorithm $\mathsf{Dec}_{\mathsf{sk}}(\cdot)$ can return an error value $\perp$ if non-ciphertext is used as an input.

Now, there are many security definitions for cryptosystems but all of them are formalised using specific attack scenarios. Goldwasser and Micali were the first to formalise the weakest classical security notion as *indistinguishability under chosen plaintext attack* (IND-CPA security), see [GM84]. In a chosen plaintext attack, an adversary $\mathcal{A}$ has partial control over the encrypted messages. The corresponding attack scenario is formally captured by security games $\mathcal{G}_0$ and $\mathcal{G}_1$ that have oracle access to a stateful *adversarial* algorithm $\mathcal{A}$.

```
$\mathcal{G}_0^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    $\textbf{return } \mathcal{A}(c)$

$\mathcal{G}_1^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    $\textbf{return } \mathcal{A}(c)$
```

Since random variables $\mathcal{G}_0^{\mathcal{A}}$ and $\mathcal{G}_1^{\mathcal{A}}$ describe the output of $\mathcal{A}$ under two different hypotheses, we can generalise the indistinguishability definition. We say that a cryptosystem is (t, ε)-IND-CPA secure if for any stateful t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A}) = |\Pr[\mathcal{G}_0^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_0^{\mathcal{A}} = 1]| \leq \varepsilon \ . \qquad (3.22)$$

Sometimes an adversary has limited access to the decryption procedure and thus can conduct chosen ciphertext attacks. Therefore, we can also consider *indistinguishability under chosen ciphertext attacks* (IND-CCA security). There are two possible formalisations: IND-CCA1 security proposed by Naor and Yung [NY90] and IND-CCA2 security proposed by Rackoff and Simon [RS91]. The corresponding indistinguishability games are very similar to the IND-CPA games except $\mathcal{A}$ has black-box access to oracles $\mathcal{O}_1$ and $\mathcal{O}_2$.

```
$\mathcal{G}_0^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1}(\mathsf{pk})$
    $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    $\textbf{return } \mathcal{A}^{\mathcal{O}_2}(c)$

$\mathcal{G}_1^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1}(\mathsf{pk})$
    $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    $\textbf{return } \mathcal{A}^{\mathcal{O}_2}(c)$
```

In the IND-CCA2 games both oracles $\mathcal{O}_1$ and $\mathcal{O}_2$ decrypt inputs, i.e, $\mathcal{O}_i(\hat{c}) = \mathsf{Dec}_{\mathsf{sk}}(\hat{c})$. To avoid trivial attacks, the game is halted with $\perp$ if $\mathcal{A}$ submits the challenge ciphertext c to $\mathcal{O}_2$. In IND-CCA1 game, the oracle $\mathcal{O}_1$ decrypts inputs and $\mathcal{O}_2$ does nothing. A cryptosystem is (t, ε)-IND-CCA1 or (t, ε)-IND-CCA2 secure if for any t-time adversary $\mathcal{A}$ the inequality (3.22) holds.

Basic properties of games. Formally, a game is a two-party protocol between an honest challenger $\mathcal{G}$ and a malicious adversary $\mathcal{A}$ that satisfies some special rules. Most importantly, the challenger and the adversary are executed in turns, so that they are not active at the same time. The game is started by $\mathcal{G}$ that sends some message to $\mathcal{A}$ and then stops. Next $\mathcal{A}$ wakes up reads the message, composes a reply and stops. Then $\mathcal{G}$ wakes up, reads the message, composes a reply and stops, and so on. The ping-pong with messages continues until either $\mathcal{A}$ halts with $\perp$ or $\mathcal{G}$ halts with $\mathsf{out} \in \{0, 1, \perp\}$. The output of the game $\mathcal{G}^{\mathcal{A}}$ is out, unless $\mathcal{A}$ outputs $\perp$ then $\mathcal{G}^{\mathcal{A}} = \perp$. Note that such a ping-pong execution ensures that $\mathcal{A}$ cannot measure the computational effort made by the challenger and thus eliminates the possibility of timing-based side-channel attacks.[4]

Now it is straightforward to generalise the notion of computational distance to games. The computational distance $\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1)$ between games $\mathcal{G}_0$ and $\mathcal{G}_1$ is the minimal value ε such that for any t-time algorithm $\mathcal{A}$

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A}) = |\Pr[\mathcal{G}_0^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_1^{\mathcal{A}} = 1]| \le \varepsilon \ . \tag{3.23}$$

The asterisk in the symbol $\mathsf{cd}^t_\star(\cdot,\cdot)$ emphasises the fact that the resulting interaction pattern may vary depending on the actions of the adversary. Clearly, $\mathsf{cd}^t_\star(\cdot,\cdot)$ is a semi-distance, as it is symmetric and the triangle inequality

$$\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_2) \le \mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1) + \mathsf{cd}^t_\star(\mathcal{G}_1, \mathcal{G}_2) \ . \tag{3.24}$$

follows from the basic triangle inequality $|a - c| \le |a - b| + |c - b|$ similar to the inequality (3.11). Again, we can define statistical distance as a limit

$$\mathsf{sd}_\star(\mathcal{G}_0, \mathcal{G}_1) = \lim_{t\to\infty} \mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1) \ . \tag{3.25}$$

We say that games $\mathcal{G}_0$ and $\mathcal{G}_1$ are equivalent if $\mathsf{sd}_\star(\mathcal{G}_0, \mathcal{G}_1) = 0$, since any adversary $\mathcal{A}$ has an equal probability to win both games $\mathcal{G}_0$ and $\mathcal{G}_1$. Again, we can set restrictions for the games so that for large enough t, the double implication $\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1) > 0 \Leftrightarrow \mathcal{G}_0 \not\equiv \mathcal{G}_1$ always holds and $\mathsf{cd}^t_\star(\cdot,\cdot)$ becomes a distance measure. Hence, the name computational distance is still justified.

Different description styles. Although a game is just a two-party protocol between an honest challenger $\mathcal{G}$ and a malicious adversary $\mathcal{A}$, different cryptographic formalisms use different ways to describe games, see Fig. 3.2.

One viable option is to fully specify actions of the challenger together with a rigid messaging schedule for the protocol as we specified for IND-CPA games. Such a description style is the oldest and most widespread in cryptographic literature. More formally, a *challenger-centric* description specifies a game as an oracle algorithm $\mathcal{G}$ that internally calls out a stateful subroutine $\mathcal{A}$ until $\mathcal{G}$ reaches some output or is halted due to abnormal behaviour of $\mathcal{A}$. Sometimes,

[4]Here, we just eliminate accidental exposure of timing artefacts, since we can re-introduce timing information in a controlled way by modifying the description of the game.

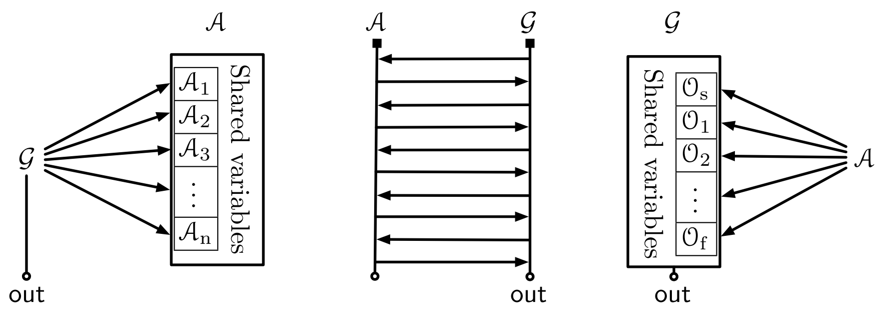


Figure 3.2: Duality between challenger- and adversary-centric game descriptions. Both formalisations describe the interactive process shown in the middle.

the adversary $\mathcal{A}$ is even represented as a collection of different stateless algorithms $\mathcal{A}_1, \ldots, \mathcal{A}_n$ that communicate by using shared variables.

The main virtue of the challenger-centric description is simplicity. For games with a deterministic scheduling, it is the most natural approach. However, for non-deterministic scheduling, the approach quickly becomes cumbersome and tedious. A reader can easily verify this by formalising the description of IND-CCA2 games without explicit use of decryption oracles.

Another compelling alternative is an *adversary-centric* description of games. Compared to the challenger-centric description, the tables are turned—the algorithm $\mathcal{A}$ internally calls out a subroutine $\mathcal{G}$, until $\mathcal{G}$ provides the desired output. Formally, the challenger code is split into smaller threads $\mathcal{O}_s, \mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_f$ that are scheduled according to the adversary's requests. The game is started by an oracle $\mathcal{O}_s$ that initialises all necessary variables and provides input to the adversary $\mathcal{A}$. Next, $\mathcal{A}$ can interact with oracles $\mathcal{O}_1, \ldots, \mathcal{O}_n$ by submitting queries and receiving replies. To end the game, $\mathcal{A}$ calls out the finalisation procedure $\mathcal{O}_f$ that computes the output of the game $\mathsf{out} \in \{0, 1, \perp\}$. Note that oracles $\mathcal{O}_s, \mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_f$ as threads of the same program can communicate through shared variables and thus force constraints on the adversarial behaviour.

As an example, consider the adversary-centric description of the IND-CPA games. The corresponding initialisation oracle $\mathcal{O}_s$ runs the key generation algorithm Gen and stores the corresponding key pair $(\mathsf{sk}, \mathsf{pk})$. The oracle $\mathcal{O}_s$ always replies pk to all queries that $\mathcal{A}$ makes. The second oracle $\mathcal{O}_1$ can be queried only once. Given a message pair (m_0, m_1), the oracle $\mathcal{O}_1$ replies $\mathsf{Enc}_{\mathsf{pk}}(m_0)$ or $\mathsf{Enc}_{\mathsf{pk}}(m_1)$ depending on whether we are in the game $\mathcal{G}_0$ or in the game $\mathcal{G}_1$. The finalisation oracle $\mathcal{O}_f$ always outputs the first message sent by $\mathcal{A}$.

We emphasise that both description styles have their advantages and drawbacks and it is only a matter of taste which syntactic sugar we choose. In short, the challenger-centric approach forces constraints explicitly through message scheduling, whereas the adversary-centric approach forces constraints implicitly through cooperative behaviour of oracles. Generally, the adversary-centric approach is better in the context of complex interactions, since it is well suited for non-deterministic scheduling. However, a suitable compromise between the two approaches can often be even more descriptive and concise.

Indistinguishability and semantic security. A game corresponds to an interactive hypothesis testing scenario if the challenger $\mathcal{G}$ always outputs the last reply from the adversary $\mathcal{A}$. Note that the argumentation given in Section 3.2

also holds for interactive hypothesis testing. More precisely, let $\{\mathcal{G}_s\}_{s\in\mathcal{S}}$ be a set of games such that any pair of them is computationally indistinguishable. In that case, the corresponding proofs in Section 3.2 still hold and consequently bounded adversaries cannot restore even partial information about s. Hence, confidentiality of secret inputs can be stated in terms of indistinguishability.

Naturally, the exact formulation is not as simple as that. Protocols usually leak some information about secret inputs and thus the pairwise indistinguishability of games is not always achievable. Moreover, interactive hypothesis testing forms only a small subclass of games, i.e., games can specify more complex properties, such as consistency and verifiability of protocol outputs. We discuss these issues further in Chapters 5 and 7.

4 CRYPTOGRAPHIC PROOF TECHNIQUES

Games are powerful tools for analysing interactive computations, as they are flexible enough to capture any attack pattern needed to define a specific design goal, such as authenticity, confidentiality or fairness. Moreover, the strength of a cryptographic construction is often quantified as an advantage

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \tag{4.1}$$

against a specific game $\mathcal{G}$. Therefore, we must often upper bound advantages in order to prove the security of a primitive or a protocol. To establish these bounds, we can use either direct or conditional proofs. Direct proofs use Peano axiomatisation of arithmetic and nothing more, whereas conditional proofs rely on additional assumptions, such as the hardness of factoring or $\mathbf{P} \neq \mathbf{NP}$.

Note that direct proofs are somewhat trivial in the settings where the running times of all participants are bounded. Since there is only a finite number of different adversaries and a finite number of relevant random bits, maximising the advantage $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A})$ is a discrete optimisation task over a finite set. Hence, for most cryptographic problems there exists either an exhaustive proof or an explicit counterexample. However, the verification time for such proofs is gargantuan, compared to the time bounds established in them. Hence, direct proofs are useful only if they are compact enough to be efficiently verifiable.

Unfortunately, no compact direct proofs have been discovered so far in cryptography. One of the few non-trivial results is a construction of a permutation that can be inverted approximately 2 times slower than computed [Hil92] and any significant advance in this field is believed to be hard [Mas96].

As a way out, we take the existence of certain cryptographic primitives for granted and build everything else on top of these basic primitives. As a result, we can use indirect conditional proofs to establish security. More precisely, it is sufficient to prove that a construction can be insecure only if some basic primitive fails to meet the specification. Moreover, the approach separates abstract design goals form practical implementation details. Still, we cannot completely ignore complexity theoretical details, or otherwise we end up with basic primitives that cannot be implemented at all. Consequently, the choice of basic primitives should always reflect our *beliefs* in various complexity theoretical conjectures. Also, note that the choice of basic primitives determines the abstraction level. For example, we can use basic primitives to model high-level properties, such as one-way functions or IND-CPA secure cryptosystems, or more specific computational assumptions, such as intractability of factoring RSA composites.

In this chapter, we show how to decompose conditional proofs into small elementary steps that are easy to apply and verify. The verification procedure itself can be completely formalised. In fact, it is possible to construct a hypothetical proof system CryProof that automatically verifies a proof and computes the corresponding security guarantees, see the discussion in Section 4.5.

4.1 REDUCTIONS AS REWRITING RULES

Although the use of abstract security properties simplifies proofs and makes them more universal, it also brings along some drawbacks. In particular, such a choice restricts the structure of possible proofs. Let us consider the IND-CPA cryptosystem as an illustrative example. Recall that a public-key cryptosystem is specified by efficient algorithms $\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec}$ and a functional requirement that $\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m)) \equiv m$ for every key pair $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$. There are no restrictions as to how a cryptosystem must operate or what its internal structure is. Hence, a new construction built on top of the (t, ε)-IND-CPA cryptosystem cannot use any other operations than $\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec}$ to manipulate keys, messages and ciphertexts. Of course, such restrictions do not extend to potential adversaries. As the construction assumes nothing about the implementation details, the provable security guarantees must hold for all implementations of (t, ε)-IND-CPA secure cryptosystems regardless of how bizarre they might seem. Thus, the only way to prove security is to show that the insecurity of the construction implies an explicit contradiction with the IND-CPA security premise.

More generally, the abstraction of security properties always leads to indirect proofs with identical structure where one must prove $\mathsf{cd}_{\star}^{t_1}(\mathcal{G}_0, \mathcal{G}_1) \leq \varepsilon_1$ from premises of type $\mathsf{cd}_{\star}^{t_0}(\mathcal{Q}_0, \mathcal{Q}_1) \leq \varepsilon_0$. The corresponding indirect proof shows that an assumption $\mathsf{cd}_{\star}^{t_1}(\mathcal{G}_0, \mathcal{G}_1) > \varepsilon_1$ implies a contradiction $\mathsf{cd}_{\star}^{t_0}(\mathcal{Q}_0, \mathcal{Q}_1) > \varepsilon_0$. The heart of the proof is a code transformation rule (*reduction*), which converts any t_1-time adversary $\mathcal{A}$ to a t_0-time adversary $\mathcal{B}$ such that

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A}) > \varepsilon_1 \qquad \Longrightarrow \qquad \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathcal{B}) > \varepsilon_0 \ . \tag{4.2}$$

Most of the reductions are *black-box reductions*, where $\mathcal{B}$ internally runs $\mathcal{A}$ in the black-box manner to play the games $\mathcal{Q}_0$ and $\mathcal{Q}_1$. That is, $\mathcal{B}$ analyses only the messages that $\mathcal{A}$ writes on the communication and input-output tapes. However, $\mathcal{B}$ controls all external factors influencing $\mathcal{A}$: inputs, random coins, received messages and used timers. In particular, $\mathcal{B}$ can always restart $\mathcal{A}$ with different inputs, replies and random coins or just rewind $\mathcal{A}$ back to any other suitable state. In *white-box reductions*, the dependence between $\mathcal{A}$ and $\mathcal{B}$ can be arbitrary, for example $\mathcal{B}$ can actively alter the internal variables of $\mathcal{A}$.

If game pairs $(\mathcal{G}_0, \mathcal{G}_1)$ and $(\mathcal{Q}_0, \mathcal{Q}_1)$ are similar enough, it is often advantageous to construct a reduction where $\mathcal{B}$ simulates the game $\mathcal{G}_i$ to $\mathcal{A}$. Fig. 4.1 depicts an illustrative example where games $\mathcal{G}_0$ and $\mathcal{G}_1$ differ only by a single line $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$ versus $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$. More formally, in the game $\mathcal{G}_i$ the challenger generates $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$, interacts with $\mathcal{A}$ and somehow obtains a valid message pair $m_0, m_1 \in \mathcal{M}$. Next, the challenger computes an encryption $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_i)$ and then continues with the game $\mathcal{G}_i$. Now consider an adversary $\mathcal{B}$ against IND-CPA games that first obtains pk from the challenger and then simulates the game $\mathcal{G}_i$ to $\mathcal{A}$ until the line $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_i)$ is reached. Next $\mathcal{B}$ outputs (m_0, m_1) and uses the reply as c and finally outputs $\mathcal{G}_i^{\mathcal{A}}$.

Note that the simulation of $\mathcal{G}_i$ is possible only if sk is not used in the game. Similarly, the line $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_i)$ must be reachable only once, or otherwise $\mathcal{B}$ violates the rules of IND-CPA games $\mathcal{Q}_i$. If these two preconditions are satisfied, $\mathcal{B}$ simulates the game $\mathcal{G}_i$ perfectly and thus $\Pr[\mathcal{G}_i^{\mathcal{A}} = 1] = \Pr[\mathcal{Q}_i^{\mathcal{B}} = 1]$. Consequently, we have proven that $\mathsf{cd}_{\star}^{t}(\mathcal{G}_0, \mathcal{G}_1) \leq \varepsilon$ if the cryptosystem is $(t + \tau + \mathrm{O}(1), \varepsilon)$-IND-CPA secure, where τ is the running time of $\mathcal{G}_i$. More precisely,

```
$\mathcal{B}$
  ...
  Ask for pk.
  ...
  $m_0 \leftarrow$ ...
  $m_1 \leftarrow$ ...
  Output $(m_0, m_1)$ and ask for $c$.
  ...
  return ...
```

```
$\mathcal{G}_0^{\mathcal{A}}$
  ...
  $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
  ...
  $m_0 \leftarrow$ ...
  $m_1 \leftarrow$ ...
  $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
  ...
  return ...
```

```
$\mathcal{G}_1^{\mathcal{A}}$
  ...
  $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
  ...
  $m_0 \leftarrow$ ...
  $m_1 \leftarrow$ ...
  $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
  ...
  return ...
```

Figure 4.1: A schematic description of an IND-CPA reduction. By construction, the adversary $\mathcal{B} = \mathcal{B}^{\mathcal{A}}$ on the left simulates perfectly a game $\mathcal{G}_i^{\mathcal{A}}$ on the right.

the claim only holds for RAM machines, since the simulation of $\mathcal{G}_i^{\mathcal{A}}$ can take more time on a Turing machine due to tape rewinding.

Note that the reasoning above holds for any game pair $(\mathcal{G}_0, \mathcal{G}_1)$ that satisfies the preconditions. Moreover, such semi-transparent reasoning is common in cryptography. In fact, any valid reduction proof implicitly specifies a code transformation $\mathcal{T}$ that preserves computational closeness under certain preconditions $\mathcal{P}$. The closeness guarantee $\mathcal{S}$ is often in the form: if the condition $\mathcal{P}$ holds for $\mathcal{G}_0$ and $\mathcal{G}_1 = \mathcal{T}(\mathcal{G}_0)$ then $\mathsf{cd}_\star^t(\mathcal{G}_0, \mathcal{G}_1) \leq \varepsilon$. For such reductions, we use shorthands $\mathcal{G}_0 \xRightarrow[\varepsilon]{\mathcal{T}} \mathcal{G}_1$ to emphasise the bound on the computational distance.

Definition 1. *A reduction schema is a triple $(\mathcal{P}, \mathcal{T}, \mathcal{S})$, where $\mathcal{P}$ is a precondition, $\mathcal{T}$ is a code transformation and $\mathcal{S}$ is the corresponding security guarantee.*

These reduction schemata significantly simplify the derivation of security proofs. It is much easier to test whether a certain precondition holds than to construct the same reduction over and over again. However, we should note that reduction schemata are not usually rigorously formalised and applied, rather they are used informally to construct complex reductions as proofs.

Generally, each basic primitive introduces at least one reduction schema, but there can be more than one schema for each of them. Since reductions schemata can have different preconditions and security guarantees, the major task in cryptography is to identify triples $(\mathcal{P}, \mathcal{T}, \mathcal{S})$ that are optimal, i.e., $\mathcal{P}$ cannot be weakened or $\mathcal{T}$ generalised without changing $\mathcal{S}$. Note that compiling a security proof, given a proper set of optimal reduction schemata, is just a plain engineering task. In other words, such an approach can be viewed as a further modularisation of cryptographic proofs into trivial and non-trivial portions.

4.2 REDUCTIONS AND TIME-SUCCESS PROFILES

Any attack against a cryptographic construction or a protocol is a trade-off between desired goals and available resources. It is possible to break any construction by applying a sufficient amount of computational resources, unless it is information-theoretically secure. At the same time, additional computational resources increase the cost of the attack and thus a rationally behaving attacker must find a proper balance between estimated losses and potential gains.

The time-success profile is the most obvious way to describe trade-offs between the losses and gains. For many security objectives, we can construct a game $\mathcal{G}$ such that the advantage $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) = \Pr[\mathcal{G}^{\mathcal{A}} = 1]$ quantifies the success probability of an adversary $\mathcal{A}$. Now a *time-success profile* is a function

$$\varepsilon(t) = \max_{\mathcal{A} \in \mathfrak{A}(t)} \mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) \ , \tag{4.3}$$

where $\mathfrak{A}(t)$ consists of all t-time adversaries. Note that the time-success profile is well-defined for every concrete construction or primitive. In fact, a time-success profile is even computable by exhaustive inspection of all possibilities. However, such a process is clearly intractable for all practical constructions and thus we can only approximate time-success profiles. One possibility is to view a cryptographic reduction as a rule that combines time-success profiles of basic primitives into a time-success profile of a new construction. More precisely, we get an upper bound on the time-success profile in terms of basic primitives.

Unfortunately, even the approximate shapes of time-success profiles are unknown for all cryptographic primitives used in practice. Only some indicative results have been established for uninstantiable generic models. For example, the time-success profile of the discrete logarithm and Diffie-Hellman problem is known to be $\Theta(t^2)$ in generic groups [Nec94, Sho97, SJ00]. Analogous results are known for the RSA root finding problem in generic rings [DK02, LR06].

Time-success ratio is a more conservative alternative for approximating time-success profiles. The term itself was first proposed in the context of asymptotic security [HL92, Lub96], but it has a natural interpretation also for a fixed primitive in the exact security setting. Namely, one can estimate the minimal *time-success ratio* α for a primitive and thus fix a linear upper bound:

$$\alpha = \min_t \frac{t}{\varepsilon(t)} \qquad \Longleftrightarrow \qquad \forall t: \ \varepsilon(t) \leq \frac{t}{\alpha} \ . \tag{4.4}$$

If the time-success profile has a globally convex-cup shape, then α is the minimum time needed to completely break the primitive, see Fig. 4.2. For many low level primitives, the adversary can compute the output $\mathcal{G}^{\mathcal{A}}$ by himself, e.g., verify that he or she has factored a large number or has found a discrete logarithm. For such primitives, the time-success profile becomes approximately convex-cup if the adversary can restart[1] the security game at any point. Then an estimate on the minimal breaking time is a theoretically sound estimate of α.

A time-success profile captures the essential information about the cryptographic construction. In particular, one can assess how secure the construction is in real world scenarios. For a moment, consider the security against rational adversaries that always choose the most beneficial attack. To analyse such behaviour, we must rescale the time and success axes so that they represent expected losses and gains, see Fig. 4.2. Obviously, the cost of an attack depends on the computational complexity. Let $x(t)$ be the correspondence between running time and computing costs. Similarly, there is a connection between success probability and the potential utility $y(\varepsilon)$ in dollars. Note that besides direct objective costs, an adversary may have indirect subjective costs, such as a threat

[1]If the problem is randomly self-reducible, the adversary can restart the game internally. Therefore, the profile is approximately convex-cup for the discrete logarithm problem.

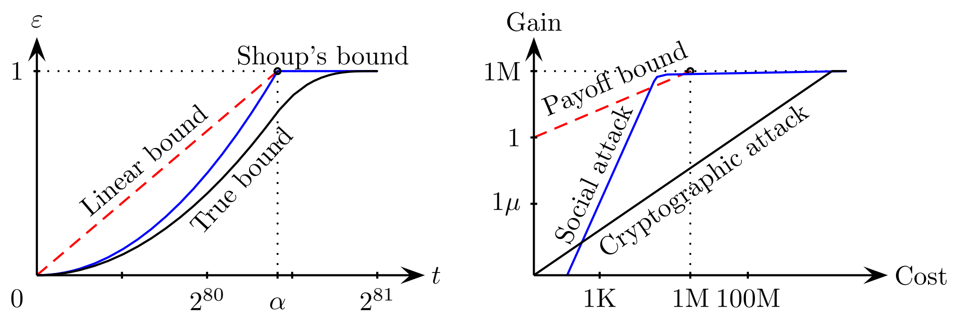


Figure 4.2: A hypothetical time-success profile for a discrete logarithm problem and the corresponding cost-gain profile in the case of 1 million dollar award. The left graph is given in the linear and the right in the logarithmic scale.

to ruin his or her reputation by attacking. Analogously, the correspondence between success probability and potential gains may be nonlinear.

Evidently, a construction is secure against rational adversaries if the costs are always greater than the potential gains. Since an adversary may also conduct physical and social engineering attacks, the whole system may still be insecure, even if the cryptographic construction is secure, see Fig. 4.2. Ideally, the cryptographic strength should be comparable to other weaknesses, as the use of overly complex cryptographic primitives leads to unjustified maintenance costs.

Not all entities behave rationally. Some of them may want to cause maximal achievable damage even at any cost. To analyse such scenarios, we must turn the tables and consider the situation from the perspective of legitimate users. In particular, we must use different rescaling such that the $x(t)$ axis still represents the potential costs of an attack, whereas $y(\varepsilon)$ counts potential losses of legitimate users, including the resources needed to run the system. As a result, we can estimate the losses in terms of available resources and thus compare different protection mechanisms with different profiles. In particular, it may turn out that the most cost-efficient solution is to have no protection at all.

Since time-success profiles are so important, it is crucial to understand how cryptographic reductions change these profiles. In the simplest case, one can prove that a (t_1, ε_1)-attack against a construction can be converted to a (t_0, ε_0)-attack against the basic primitive. The corresponding *tightness factor*

$$\beta(\varepsilon_1, t_1) = \frac{\alpha_1}{\alpha_0} = \frac{t_1\varepsilon_0}{t_0\varepsilon_1} \tag{4.5}$$

characterises the inevitable security loss, i.e., how much does the bound on the total breaking time decrease. Again, the tightness factors have been used in the context of asymptotic security [HL92, Lub96], but the corresponding classification [Lub96, p. 21–34] of reductions is too coarse for our purposes.

For more precise classification, consider the correspondence between time-success profiles. Fix a linear target bound $\varepsilon_1 = \Theta(t_1)$. In such case, a reduction is *linear* if it leads to a linear bound $\varepsilon_0 = \Theta(t_0)$. Most cryptographic reductions are linear but complex reductions can lead to convex-cup bounds $\varepsilon_0 = \Theta(t_0^k)$. The constant $k \geq 1$ determines the order of the reduction, for example, a reduction is *quadratic* if $\varepsilon_0 = \Theta(t_0^2)$. Since we normally start with linear bounds $\varepsilon_0 = \Theta(t_0)$ for basic primitives, we get convex-cap security guarantees $\varepsilon_1 = \Theta\left(\sqrt[k]{t}\right)$ for the derived construction or primitive. Such a shape is

```
$\mathcal{G}_0^{\mathcal{A}}$
    $(\mathsf{sk},\mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    $b \leftarrow \mathcal{A}(c_1)$
    if $b \in \{0,1\}$ then return $b$
    $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    return $\mathcal{A}(c_2)$
```

```
$\mathcal{G}_2^{\mathcal{A}}$
    $(\mathsf{sk},\mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    $b \leftarrow \mathcal{A}(c_1)$
    if $b \in \{0,1\}$ then return $b$
    $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    return $\mathcal{A}(c_2)$
```

Figure 4.3: Two-stage attack against a public-key cryptosystem.

quite unsuitable from game theoretical viewpoint, as it increases the potential payoff in a low costs range and also increases the bound on α_1.

Sometimes, it is possible to provide several reductions for the same problem. Often, these reductions have incomparable structure and security bounds. Then the comparison of tightness factors $\beta(\varepsilon_1, t_1)$ provides some insight. Assume that basic primitives have linear time-success bounds. Then a reduction with the largest tightness ratio allows us to choose a primitive with the smallest α_0 and thus we gain efficiency without changing the security level.

Finally, the tightness ratio characterises the robustness of the bounds. As all current estimates on the security of cryptographic primitives are heuristic, they are likely to change in the future. The ratio β characterises how big the corresponding changes in derived security guarantees are.

4.3 SURPRISING PROPERTIES OF CONDITIONAL PROBABILITIES

Before we investigate standard proof techniques, it is instructive to consider common dangers first. In particular, note that conditional probabilities often behave counter-intuitively and the corresponding confusion can cause subtle argumentation flaws in security proofs. Let us start with a simple example and analyse games $\mathcal{G}_0$ and $\mathcal{G}_2$ in Fig. 4.3 under the assumption that a cryptosystem is (t,ε)-IND-CPA secure. Note that the adversary $\mathcal{A}$ must distinguish between ciphertexts of messages m_0 and m_1 but differently from IND-CPA games, the adversary $\mathcal{A}$ can choose not to answer. Namely, if $b \in \{0,1\}$ then the challenger accepts b as a final guess, otherwise a new ciphertext c_2 is generated and the adversary $\mathcal{A}$ gets a second chance. In other words, if $b \in \{0,1\}$, then $\mathcal{A}$ plays the IND-CPA game with the challenge c_1 and otherwise $\mathcal{A}$ plays the IND-CPA game with challenge c_2. Hence, it would be natural to assume that the following inequalities should hold for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}_1(\mathcal{A}) = |\Pr[\mathcal{G}_0^{\mathcal{A}} = 1 | b \in \{0,1\}] - \Pr[\mathcal{G}_2^{\mathcal{A}} = 1 | b \in \{0,1\}]| \leq \varepsilon \ , \tag{4.6}$$

$$\mathsf{Adv}_2(\mathcal{A}) = |\Pr[\mathcal{G}_0^{\mathcal{A}} = 1 | b \notin \{0,1\}] - \Pr[\mathcal{G}_2^{\mathcal{A}} = 1 | b \notin \{0,1\}]| \leq \varepsilon \ . \tag{4.7}$$

However, neither of these inequalities (4.6) and (4.7) holds, as efficient adversaries $\mathcal{A}_1$ and $\mathcal{A}_2$ depicted in Fig. 4.4 achieve $\mathsf{Adv}_1(\mathcal{A}_1) = \mathsf{Adv}_2(\mathcal{A}_2) = 1$. For the proof, note that if messages differ $m_0 \neq m_1$, then a collision $\mathsf{Enc}_{\mathsf{pk}}(m_j) = \mathsf{Enc}_{\mathsf{pk}}(m_i)$ implies $m_i = m_j$. Therefore, $\mathcal{A}_1$ outputs a correct answer if $b \in \{0,1\}$

```
$\mathcal{A}_1$
  given pk
  reply $m_0 \neq m_1$

  given $c_1$
  $\hat{c}_0 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
  $\hat{c}_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
  if $c_1 = \hat{c}_0$ then reply $0$
  if $c_1 = \hat{c}_1$ then reply $1$
  if $c_1 \notin \{\hat{c}_0, \hat{c}_1\}$ then reply $2$

  given $c_2$
  return $0$
```

```
$\mathcal{A}_2$
  given pk
  reply $m_0 \neq m_1$

  given $c_1$
  $\hat{c}_0 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
  $\hat{c}_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
  if $c_1 \in \{\hat{c}_0, \hat{c}_1\}$ then reply $2$
  if $c_1 \notin \{\hat{c}_0, \hat{c}_1\}$ then reply $0$

  given $c_2$
  if $c_1 = \hat{c}_0$ then return $0$
  if $c_1 = \hat{c}_1$ then return $1$
```

Figure 4.4: Adversaries $\mathcal{A}_1$ and $\mathcal{A}_2$ which achieve $\mathsf{Adv}_1(\mathcal{A}_1) = \mathsf{Adv}_2(\mathcal{A}_2) = 1$.

and $\mathcal{A}_2$ outputs a correct answer if $b \notin \{0,1\}$. Hence, we have proven that

$$\Pr\left[\mathcal{G}_0^{\mathcal{A}_1} = 1 | b \in \{0,1\}\right] = 0 \ , \qquad \Pr\left[\mathcal{G}_2^{\mathcal{A}_1} = 1 | b \in \{0,1\}\right] = 1 \ , \tag{4.8}$$
$$\Pr\left[\mathcal{G}_0^{\mathcal{A}_2} = 1 | b \notin \{0,1\}\right] = 0 \ , \qquad \Pr\left[\mathcal{G}_2^{\mathcal{A}_2} = 1 | b \notin \{0,1\}\right] = 1 \ . \tag{4.9}$$

Note that the drastic discrepancy between superficial estimates (4.6) and (4.7), and true bounds (4.8) and (4.9) is caused by the unintentional mismatch of prior probabilities in the reasoning. Namely, the inequality (4.6) follows from IND-CPA security only if $\Pr[b \in \{0,1\}] = 1$, but the actual probability $\Pr[b \in \{0,1\}]$ can be arbitrarily small. Flaws in the derivation of (4.7) are analogous.

A similar flaw, but in a much more subtle form appears in the classical proofs of the PRP/PRF Switching Lemma (Theorem 5.1 in [IR88]). The error itself was discovered alarmingly late, in 2004, by Kohno, see the discussion in [BR04]. The theorem estimates a statistical distance between two games $\mathcal{G}_0$ and $\mathcal{G}_1$. In the game $\mathcal{G}_0$ the challenger chooses a function f uniformly from the set of all functions $f : \mathcal{X} \to \mathcal{X}$, whereas f is a uniformly chosen permutation on $\mathcal{X}$ in the game $\mathcal{G}_1$. The task of an adversary $\mathcal{A}$ is do distinguish these games by adaptively querying at most q values of $f(x_i)$. Note that any deterministic strategy $\mathcal{A}$ can be formalised as a tree, where the nodes represent queries $f(x_i)$ and the edges correspond to the responses as illustrated in Fig. 4.5. In the game $\mathcal{G}_0$ all paths are possible, whereas no path that leads to a collision $f(x_i) = f(x_j)$ is possible in $\mathcal{G}_1$. In both games, the probability to reach a plausible node depends only on the number of queries. Now if $\mathcal{A}$ always makes *exactly* q distinct queries, then

$$\Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right] = \Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1 | \neg\mathsf{Collision}\right] \tag{4.10}$$

as nodes are sampled with uniform probability. However, if $\mathcal{A}$ decides to stop earlier in some paths, then conditional probabilities $\Pr[node|\neg\mathsf{Collision}]$ start to differ between the games. For example, the conditional probabilities from left to right on the decision border in Fig. 4.5 are $\left(\frac{0}{7}, \frac{1}{7}, \frac{1}{7}, \frac{0}{7}, \frac{1}{7}, \frac{1}{7}, \frac{3}{7}\right)$ in the game $\mathcal{G}_0$ and $\left(\frac{0}{6}, \frac{1}{6}, \frac{1}{6}, \frac{0}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6}\right)$ in the game $\mathcal{G}_1$. Therefore, the equation. (4.10) is not guaranteed to hold when $\mathcal{A}$ halts prematurely in some paths. Consequently, in the strict sense, the proofs given in [IR88, BKR94] are correct as they consider adversaries that make *exactly* q distinct queries. If we consider adversaries that

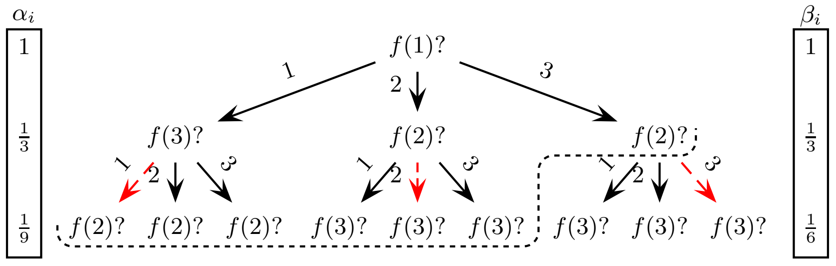


Figure 4.5: A querying strategy for $\mathcal{X} = \{1, 2, 3\}$ with premature stopping. The dashed line denotes the decision border and α_i, β_i denote the probabilities of plausible queries $f(x_1), \ldots, f(x_i)$ in the games $\mathcal{G}_0$ and $\mathcal{G}_1$, respectively.

can make *up to* q queries, then these proofs are indeed incorrect [BR04]. Nevertheless, the final claim is still correct, as adversaries that make up to q queries cannot outperform adversaries that make exactly q queries.

To summarise, reasoning about conditional probabilities is counterintuitive and non-robust against microscopic changes in the security claims.

4.4 FROM GAME CHAINS TO PROOF TREES

Let us return to the first example posed in the previous section. Observe that the description of $\mathcal{G}_0$ and $\mathcal{G}_2$ differ in two places, namely, ciphertexts c_1 and c_2 are computed differently. As the secret key sk is not used in the games, we can apply the IND-CPA reduction schema twice to $\mathcal{G}_0$ and obtain a game chain

```
$\mathcal{G}_0^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    $b \leftarrow \mathcal{A}(c_1)$
    if $b \in \{0,1\}$ then ...
    $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    return $\mathcal{A}(c_2)$
```

```
$\mathcal{G}_1^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    $b \leftarrow \mathcal{A}(c_1)$
    if $b \in \{0,1\}$ then ...
    $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0)$
    return $\mathcal{A}(c_2)$
```

```
$\mathcal{G}_2^{\mathcal{A}}$
    $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
    $(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk})$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    $b \leftarrow \mathcal{A}(c_1)$
    if $b \in \{0,1\}$ then ...
    $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1)$
    return $\mathcal{A}(c_2)$
```

where the grey boxes emphasise changes. To represent reductions, we can write

$$\mathcal{G}_0 \xRightarrow[\varepsilon]{\text{IND-CPA}} \mathcal{G}_1 \xRightarrow[\varepsilon]{\text{IND-CPA}} \mathcal{G}_2 \quad . \tag{4.11}$$

Now if the cryptosystem is (t, ε)-IND-CPA secure, there exists a time bound $t_1 = t - \mathrm{O}(1)$ such that $\mathsf{cd}_\star^{t_1}(\mathcal{G}_0, \mathcal{G}_1) \leq \varepsilon$ and $\mathsf{cd}_\star^{t_1}(\mathcal{G}_1, \mathcal{G}_2) \leq \varepsilon$. Hence, the triangle inequality yields

$$\mathsf{cd}_\star^{t_1}(\mathcal{G}_0, \mathcal{G}_2) \leq \mathsf{cd}_\star^{t_1}(\mathcal{G}_0, \mathcal{G}_1) + \mathsf{cd}_\star^{t_1}(\mathcal{G}_1, \mathcal{G}_2) \leq 2\varepsilon \quad . \tag{4.12}$$

More generally, we can use game chains as a systematic way to estimate computational distances. Moreover, note that for any t-time algorithm $\mathcal{A}$

$$\mathsf{Adv}_{\mathcal{G}_0}(\mathcal{A}) = \Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] \leq \mathsf{cd}_\star^{t}(\mathcal{G}_0, \mathcal{G}_n) + \mathsf{Adv}_{\mathcal{G}_n}(\mathcal{A}) \tag{4.13}$$

```
$\mathcal{G}_0^{\mathcal{A}}$
    $(\mathsf{sk}_0, \mathsf{pk}_0) \leftarrow \mathsf{Gen}$
    $(\mathsf{sk}_1, \mathsf{pk}_1) \leftarrow \mathsf{Gen}$
    $x \leftarrow \{0,1\}$
    $q \leftarrow \mathcal{A}(\mathsf{pk}_0, \mathsf{pk}_1)$
    **if** $q \notin \{0,1\}$ **then return** $\perp$
    $c_0 \leftarrow \mathsf{Enc}_{\mathsf{pk}_0}(x)$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}_1}(x)$
    **return** $\mathcal{A}(\mathsf{sk}_q, c_0, c_1)$
```

```
$\mathcal{G}_1^{\mathcal{A}}$
    $(\mathsf{sk}_0, \mathsf{pk}_0) \leftarrow \mathsf{Gen}$
    $(\mathsf{sk}_1, \mathsf{pk}_1) \leftarrow \mathsf{Gen}$
    $x \leftarrow \{0,1\}$
    $q \leftarrow \mathcal{A}(\mathsf{pk}_0, \mathsf{pk}_1)$
    **if** $q \notin \{0,1\}$ **then return** $\perp$
    $c_0 \leftarrow \mathsf{Enc}_{\mathsf{pk}_0}(x)$
    $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}_1}(1-x)$
    **return** $\mathcal{A}(\mathsf{sk}_q, c_0, c_1)$
```

Figure 4.6: A game pair derived from the Bellare-Micali protocol.

and thus we can use game chains to also upper bound advantages against individual games. Often, such proofs are compressed further by combining the elementary reductions into a single aggregate construction. The latter is known as a constructive hybrid argument, see Section 6.7. However, this is just a trade-off between compactness and clarity: complex reductions are much harder to comprehend and verify than the corresponding elementary reductions.

It is important to note that not all security proofs can be represented as game chains. See Fig. 4.6 contains for a specific counterexample. We remark that the game pair is not arbitrary—slightly more sophisticated game pairs appear in the security proof of the Bellare-Micali oblivious transfer protocol [BM89].

As the adversary $\mathcal{A}$ can decrypt only a single value c_i, he or she cannot learn whether c_1 and c_2 are ciphertexts of the same message or not. However, we cannot use the IND-CPA transformation, as we do not know in advance whether $\mathcal{A}$ uses a secret key sk_0 or sk_1. Hence, we must define four auxiliary games $\mathcal{G}_{ij}$ where $\mathcal{G}_{ij}$ denotes a game $\mathcal{G}_i$ that is halted with $\perp$ if $q \neq j$. As

$$\Pr[\mathcal{G}_0^{\mathcal{A}} = 1] = \Pr[\mathcal{G}_0^{\mathcal{A}} = 1 \wedge q = 0] + \Pr[\mathcal{G}_0^{\mathcal{A}} = 1 \wedge q = 1] \enspace , \tag{4.14}$$
$$\Pr[\mathcal{G}_1^{\mathcal{A}} = 1] = \Pr[\mathcal{G}_1^{\mathcal{A}} = 1 \wedge q = 0] + \Pr[\mathcal{G}_1^{\mathcal{A}} = 1 \wedge q = 1] \enspace , \tag{4.15}$$

the straightforward application of the triangle inequality yields

$$\mathsf{cd}_\star^t(\mathcal{G}_0, \mathcal{G}_1) \le \mathsf{cd}_\star^t(\mathcal{G}_{00}, \mathcal{G}_{10}) + \mathsf{cd}_\star^t(\mathcal{G}_{01}, \mathcal{G}_{11}) \enspace . \tag{4.16}$$

The analysis simplifies as games $\mathcal{G}_{00}, \mathcal{G}_{10}$ do not use sk_1 and $\mathcal{G}_{01}, \mathcal{G}_{11}$ do not use sk_0. Consequently, we can now use IND-CPA transformation for both games:

$$\mathcal{G}_{00} \xRightarrow[\varepsilon]{\text{IND-CPA}} \mathcal{G}_{10} \qquad \text{and} \qquad \mathcal{G}_{01} \xRightarrow[\varepsilon]{\text{IND-CPA}} \mathcal{G}_{11} \enspace . \tag{4.17}$$

Again, if the cryptosystem is (t, ε)-IND-CPA secure, there exists $t_1 = t - \mathrm{O}(1)$ such that $\mathsf{cd}_\star^{t_1}(\mathcal{G}_{00}, \mathcal{G}_{10}) \le \varepsilon$, $\mathsf{cd}_\star^{t_1}(\mathcal{G}_{01}, \mathcal{G}_{11}) \le \varepsilon$ and thus $\mathsf{cd}_\star^{t_1}(\mathcal{G}_0, \mathcal{G}_1) \le 2\varepsilon$.

This horizon-splitting technique can be generalised to handle any finite set of exhaustive but mutually exclusive hypotheses $\mathcal{H}_1, \ldots, \mathcal{H}_n$ about the interaction pattern in the game $\mathcal{G}$. Namely, we can define narrowings of the game $\mathcal{G}$:

$$\mathcal{G}^{\mathcal{A}}\big|_{\mathcal{H}_i} = \begin{cases} \mathcal{G}^{\mathcal{A}}, & \text{if } \mathcal{H}_i \text{ holds} \enspace , \\ \perp, & \text{otherwise} \enspace . \end{cases} \tag{4.18}$$

$$\begin{tikzcd}
& \mathcal{G}_0 \arrow[dl, "\mathcal{H}_1"'] \arrow[d, "\mathcal{H}_2"] \arrow[dr, "\mathcal{H}_3"] & \\
\mathcal{G}_1 \arrow[d, Rightarrow, "\varepsilon_1"] & \mathcal{G}_2 \arrow[d, Rightarrow, "\varepsilon_2"] & \mathcal{G}_3 \arrow[d, phantom, "\equiv"] \\
\mathcal{G}_4 \arrow[d, phantom, "\equiv"] & \mathcal{G}_5 \arrow[d, phantom, "\equiv"] & \perp \\
\perp & \perp &
\end{tikzcd}$$

$$\boxed{\mathsf{Adv}_{\mathcal{G}_0}(\mathcal{A}) \le \varepsilon_1 + \varepsilon_2}$$

$$\begin{tikzcd}
& (\mathcal{G}_0, \mathcal{G}_1) \arrow[dl, "\mathcal{H}_1"'] \arrow[dr, "\mathcal{H}_2"] & \\
(\mathcal{G}_2, \mathcal{G}_3) \arrow[d, Rightarrow, "\varepsilon_{23}"] & & (\mathcal{G}_4, \mathcal{G}_5) \arrow[d, Rightarrow, "\varepsilon_{45}"] \\
\dfrac{(\mathcal{G}_6, \mathcal{G}_3)}{\mathcal{G}_6 \equiv \mathcal{G}_3} & & \dfrac{(\mathcal{G}_4, \mathcal{G}_7)}{\mathcal{G}_4 \equiv \mathcal{G}_7}
\end{tikzcd}$$

$$\boxed{\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1) \le \varepsilon_{23} + \varepsilon_{45}}$$

Figure 4.7: Game trees together with the corresponding bounds. Simple arrows represent horizon-splitting and double arrows represent reduction schemata.

where the verification of $\mathcal{H}_i$ is done by the challenger. Now, the ordinary triangle inequality induces two tight splitting bounds

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) \le \mathsf{Adv}_{\mathcal{G}|_{\mathcal{H}_1}}(\mathcal{A}) + \cdots + \mathsf{Adv}_{\mathcal{G}|_{\mathcal{H}_n}}(\mathcal{A}) \ , \tag{4.19}$$

$$\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1) \le \mathsf{cd}^t_\star(\mathcal{G}_0|_{\mathcal{H}_1}, \mathcal{G}_1|_{\mathcal{H}_1}) + \cdots + \mathsf{cd}^t_\star(\mathcal{G}_0|_{\mathcal{H}_n}, \mathcal{G}_1|_{\mathcal{H}_n}) \ . \tag{4.20}$$

Horizon-splitting technique can be used whenever no elementary transformation is applicable due to the unknown behaviour of $\mathcal{A}$. In a sense, splitting bounds are more detailed analogs of the formal derivation rule

$$\frac{\Gamma, F \vdash A \qquad \Gamma, G \vdash A}{\Gamma, F \vee G \vdash A} \tag{4.21}$$

in logic that allows us to strengthen the assumptions by splitting the proof into several branches. In our case, we get a game tree instead of a proof tree.

To summarise, we can bound the advantage $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A})$ by constructing a game tree such that all leaf level games $\mathcal{G}_i$ have trivial bounds $\mathsf{Adv}_{\mathcal{G}_i}(\mathcal{A}) \le \varepsilon_i$, since we can use triangle inequalities (4.13), (3.23) and splitting bounds (4.19)–(4.20) to estimate $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A})$. Moreover, we can find a tight upper bound on the running times by aggregating all time constraints corresponding to the edges of the game tree. Similarly, we can bound a computational distance $\mathsf{cd}^t_\star(\mathcal{G}_0, \mathcal{G}_1)$ but the corresponding proof tree consists of game pairs. Namely, the pair $(\mathcal{G}_0, \mathcal{G}_1)$ is a root node and we can derive new nodes by applying reductions schemata to individual components of a pair, or alternatively create several siblings by using the horizon-splitting technique. Again, if all leaf level nodes $(\mathcal{G}_i, \mathcal{G}_j)$ are bounded $\mathsf{cd}^t_\star(\mathcal{G}_i, \mathcal{G}_j) \le \varepsilon_{ij}$, then we can use inequalities (3.23) and (4.20) to compute the final bound $\mathsf{cd}^t_\star(\mathcal{G}_i, \mathcal{G}_j)$. See Fig. 4.7 for illustrative examples. Note that such a proof technique is sound and complete. More formally, if we cannot represent a valid handcrafted proof as a game tree, then we can view the proof itself as a new reduction schema and thus obtain a two-element game chain.

This approach is particularly useful when we have to derive technically complex proofs, since it is straightforward to apply and it is immune to argumentation flaws. At the same time, the corresponding proofs can be extremely boring to read, since they are quite long and mechanical. Moreover, it is often easier to construct reductions for specific games than to formalise respective reduction schemata $(\mathcal{P}, \mathcal{T}, \mathcal{S})$. Thus, it is common to compress proof trees by providing more complex handcrafted reductions after the proof is derived.

4.5 FORMAL VERIFICATION OF CRYPTOGRAPHIC PROOFS

The methodology outlined above has slowly evolved together with the concept of exact security. As the main emphasis has always been on practical results, the meta-analysis of proof methodology itself is rather recent. Only in the year 2004, Bellare, Rogaway and Shoup started to talk about game-playing proofs as a general methodology [BR04, Sho04]. Although some interesting proof methods, such as [LMMS98, Mau02], have been published before, none of them has claimed to be universal, as the approach [BR04] proposed by Bellare and Rogaway. Next, Halevi took a step further and suggested that cryptographers should build an automatic tool for generating, analysing and verifying the game-playing proofs [Hal05]. The first proposal CryptoVerif [BP06] for such a formal proof system is now available. Notably, the authors have taken effort to transform the ordinary program analysis lingo into an adversary-centric description language, which is almost understandable without any prior preparation.

Despite the existence of CryptoVerif and the corresponding extensive documentation [Bla06b, Bla07] and numerous other approaches that work in the asymptotic setting (e.g. [CV01, Bla01, Low97, MRST06]), the author still feels compelled to explain the high-level structure of such a formal proof system. Firstly, a majority of these systems are quite unnatural from a cryptographer's viewpoint, as they are formalised by logicians, who have put proof theoretical aspects in the first place. Secondly, all of them are rather limited as they utilise only reduction schemata and not the horizon-splitting technique. In fact, the linearity of proofs is one of the main reasons why current automatic provers are so successful. To be fair, we note that horizon splitting is not a well known technique. In fact, only a weak form of it has been semi-officially published by Dent [Den06]. Finally, thinking in terms of proof systems makes certain properties and limitations of conditional proofs more apparent.

In the following, we describe a proof system in which the emphasis is placed on cryptographic aspects. In particular, note that the derivation of proofs is not the main problem in cryptography, the actual problem lies in the verification of subtle details. Hence, the automatic derivation of proofs is not a key feature at all, although the system should handle mundane tasks in the proofs, such as verification of the preconditions for various reduction schemata.

High-level description. Our hypothetical computer environment CryProof should be a proof system in a strict sense, i.e., it is just an interactive environment for proving cryptographic statements about advantages and computational distances of different games. Secondly, CryProof should directly manipulate games that are given in challenger- or adversary-centric description style similar to [BR04, App. B]. Since the imperative programming paradigm is dominant in cryptography, all alternative description styles based on various forms of process calculi have an extremely high entrance cost.[2] Finally, CryProof should do the automatic bookkeeping and output the resulting time-success profiles given a characterisation of basic primitives used in the construction.

Let us now briefly describe how one should use CryProof. As explained in the previous section, any security proof is a game tree and the CryProof envi-

[2]Repeated switching between different programming paradigms is annoying both from a technical and cognitive viewpoint. Engineers and cryptographers just do not do subtitles!

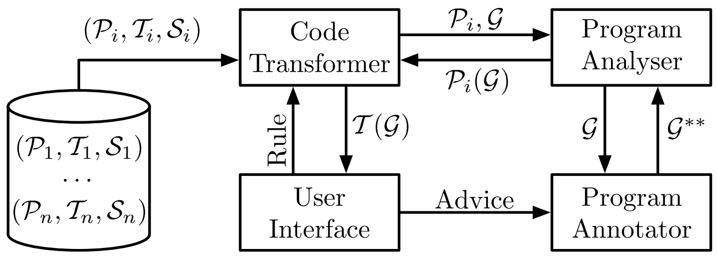

$(\mathcal{P}_i, \mathcal{T}_i, \mathcal{S}_i)$
Code Transformer
$\mathcal{P}_i, \mathcal{G}$
$\mathcal{P}_i(\mathcal{G})$
Program Analyser
$(\mathcal{P}_1, \mathcal{T}_1, \mathcal{S}_1)$
$\dots$
$(\mathcal{P}_n, \mathcal{T}_n, \mathcal{S}_n)$
Rule
$\mathcal{T}(\mathcal{G})$
$\mathcal{G}$
$\mathcal{G}^{**}$
User Interface
Advice
Program Annotator

Figure 4.8: High-level description of the proof system CRYPROOF.

ronment should assure that a game tree is properly formed. Hence, a single step in CRYPROOF is either an application of a reduction schemata or a horizon-splitting step. Testing the validity of horizon-splitting step is more straightforward, as we must check that hypotheses $\mathcal{H}_1, \dots, \mathcal{H}_m$ are mutually disjoint and exhaustive. The application of a reduction schema is more complex, see Fig. 4.8. Essentially, we must check that a precondition $\mathcal{P}_i$ holds for the game $\mathcal{G}$. Since a precondition describes a non-trivial property of a program $\mathcal{G}^{\mathcal{A}}$, the predicate $\mathcal{P}_i(\mathcal{G})$ might be undecidable. Consequently, a user must be able to prove $\mathcal{P}_i(\mathcal{G})$ by annotating the program $\mathcal{G}$ when automatic methods fail.

Any formal annotation method that makes the verification of $\mathcal{P}_i(\mathcal{G})$ easy is suitable for the task. Hoare logic is one of the best choices, since it resembles the way we actually reason about properties of a program. In a nutshell, Hoare logic is a systematic way to label the code statements with pre- and postconditions. For example, a labelling $\{y \leq 5\}\, y \leftarrow y + 3\, \{y \leq 8\}$ means that if $y \leq 5$ before the statement $y \leftarrow y+3$ then $y \leq 8$ afterwards. Notably, only a small language dependent set of derivation rules determines the labelling for the whole algorithm. Thus, one can prove the correctness of the labelling by giving the corresponding derivation tree. For more detailed discussion see [NN92].

Note that the challenger can always test whether $\mathcal{P}_i(\mathcal{G})$ holds for any particular run. Let $\mathcal{G}^*$ be the description of a modified game, where the challenger sets a flag $\mathsf{bad} \leftarrow 1$ if the precondition $\mathcal{P}_i(\mathcal{G})$ does not hold. Then a valid Hoare triple $\{\emptyset\}\, (\mathcal{G}^*)^{\mathcal{A}}\, \{\mathsf{bad} \neq 1\}$ proves that the condition $\mathcal{P}_i$ holds for all runs of $\mathcal{G}^{\mathcal{A}}$. Consequently, a valid annotation $\mathcal{G}^{**}$ that proves $\{\emptyset\}\, (\mathcal{G}^*)^{\mathcal{A}}\, \{\mathsf{bad} \neq 1\}$ is also a proof for $\mathcal{P}_i(\mathcal{G})$. Moreover, all provable properties of an algorithm are provable also in Hoare logic provided that one allows a sufficiently rich description language for the pre- and postconditions [NN92]. Therefore, the annotation $\mathcal{G}^{**}$ exists if and only if the condition $\mathcal{P}_i(\mathcal{G})$ is indeed provable.

All other operations, such as applying the transformations, constructing the proof trees and computing various bounds, are straightforward. Clearly, the database of reduction schemata $\{(\mathcal{P}_i, \mathcal{T}_i, \mathcal{S}_i)\}$ is the central part of CRYPROOF, since it captures the domain knowledge. Also, it is likely to be incomplete at the beginning. Therefore, we must update it when handcrafted security proofs reveal missing reduction schemata, i.e., some proofs cannot be formalised in the CRYPROOF environment. The task of the system maintainer is to carefully verify handcrafted proofs to extract and generalise the corresponding reduction schema. The latter is the only non-engineering task in the long run.

Axioms and deduction rules. Normally, a domain of knowledge is formalised by axioms. Indeed, all reduction schemata $(\mathcal{P}_i, \mathcal{T}_i, \mathcal{S}_i)$ are axiomatic theorems. However, it is far easier to interpret them as deduction rules and postulate that one must apply reduction schemata until the leafs of game trees contain only pairs $\mathcal{G}_i \equiv \mathcal{G}_j$ or single games $\mathcal{G}_i \equiv \perp$, see again Fig. 4.7.

Note that our proof system must have deduction rules for code simplifications that do not change the behaviour of $\mathcal{G}^{\mathcal{A}}$, such as algebraic manipulations and dead code elimination. For advantage computations, we need a special WRAP rule that formalises the intuition that advantage cannot decrease if we push some computations from the honest challenger into the malicious adversary. Similarly, the computational distance cannot increase if some messages sent out by the challenger are dropped and never reach the adversary $\mathcal{A}$.

The second group of transformations is used for manipulating cryptographic primitives like $\mathsf{Enc}_{\mathsf{pk}}(\cdot)$ or $\mathsf{Hash}(\cdot)$. As such symbols represent unknown functions, we must finally eliminate them from a game $\mathcal{G}$, otherwise it is formally impossible to establish bounds on $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A})$. Therefore, for every legitimate usage of a primitive, there must be an elimination rule. Consider a cryptosystem $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ as an example. If a game $\mathcal{G}$ uses only the Gen symbol, then the WRAP rule is sufficient to remove Gen. If a game $\mathcal{G}$ does not contain $\mathsf{Dec}_{\mathsf{sk}}(\cdot)$ symbol, then IND-CPA and WRAP rule together are sufficient to eliminate Gen and $\mathsf{Enc}_{\mathsf{pk}}(\cdot)$ symbols. For the $\mathsf{Dec}_{\mathsf{sk}}(\cdot)$ symbol, there are three alternatives. The WRAP rule is sufficient if there are no $\mathsf{Enc}_{\mathsf{pk}}(\cdot)$ symbols, since key generation together with decryption can be wrapped into $\mathcal{A}$. Secondly, in code lines $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1), \ldots, m_2 \leftarrow \mathsf{Dec}_{\mathsf{sk}}(c)$, we can simplify $m_2 \leftarrow m_1$. In the remaining cases, we need IND-CCA2 transformation, as $\mathcal{A}$ can alter ciphertexts.

The third class of transformations is used in the probability analysis after all cryptographic primitives are removed from the games. Such transformations are extensively used in primitive design, where the elimination of all cryptographic primitives can create difficult combinatorial problems. Notably, most of these reductions are based on the BAD reduction schema, see [BR04]. In a nutshell, we can substitute any code block seg_0 with a block $\mathsf{bad} \leftarrow 1; \mathsf{seg}_1$ provided that seg_0 is reached with low probability. Let $\mathcal{G}_0^{\mathcal{A}}(\omega)$ and $\mathcal{G}_1^{\mathcal{A}}(\omega)$ denote random variables that correspond to the transformation $\mathcal{G}_0 \xRightarrow[\varepsilon]{\text{BAD}} \mathcal{G}_1$, where $\omega = (\omega_{\mathcal{A}}, \omega_{\mathcal{G}})$ consists of all random coins used by the adversary and the challenger. Then values $\mathcal{G}_0^{\mathcal{A}}(\omega)$ and $\mathcal{G}_1^{\mathcal{A}}(\omega)$ can differ only if $\mathsf{bad} = 1$ and we obtain

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A}) \leq \Pr\left[\omega \leftarrow \Omega_* : \mathcal{G}_0^{\mathcal{A}}(\omega) \neq \mathcal{G}_1^{\mathcal{A}}(\omega)\right] \leq \Pr\left[\omega \leftarrow \Omega_* : \mathsf{bad} = 1\right] \quad .$$

To bound $\Pr[\mathsf{bad} = 1]$, we have to specify a set $\Omega_\Delta \supseteq \{\omega \in \Omega_* : \mathsf{bad} = 1\}$ and an estimate $\Pr[\Omega_\Delta] \leq \varepsilon_\Delta$. Now if we omit the probability calculations, then a valid annotation of the Hoare triple $\{\omega \notin \Omega_\Delta\}\, \mathcal{G}_1^{\mathcal{A}}(\omega)\, \{\mathsf{bad} = 0\}$ is sufficient. Hence, we can formally verify combinatorial transformations, which seem to be the major hurdle in symmetric primitive design [Mau02, Hal05].

Relativised models. A conditional security proof is meaningful only if there exists a plausible instantiation of abstract primitives. Since direct proofs are practically out of reach, we must indirectly verify that various abstract demands for the primitive are not contradictory. More precisely, we need a computational model, where all instantiations of all basic primitives do exist and are provably secure. As we use *black-box specification* for all primitives, it is straightforward to build such a model by employing external oracles.

For concreteness, we construct a model for IND-CPA secure cryptosystem. Assume that an external oracle $\mathcal{O}$ provides replies to Gen, Enc and Dec queries. To serve the i^{th} query Gen, the oracle draws $r_i \leftarrow \{0,1\}^n$, stores (i, r_i) and outputs a key pair $((i, r_i), i)$. To serve $\mathsf{Enc}(i, m)$, the oracle stores m into the first open slot of an array $(m_{ij})_{j=1}^{\infty}$ and returns the corresponding index number j_0. Let $\mathsf{Dec}((i, r), c) = \perp$ if no recorded secret key coincides with (i, r) and $\mathsf{Dec}((i, r), c) = m_{ic}$ otherwise. Note that the cryptosystem is functional and $(t \cdot 2^{-n}, t)$-IND-CPA secure, since any t-time adversary can try out only t potential secret keys and the ciphertext is independent from a message.

Limitations of conditional proofs. It is not trivial to guess whether a conditional proof provides optimal security bounds or whether a security claim is provable at all. To solve such conundrums, we can provide counterexamples for establishing that we cannot have a conditional proof unless we add new assumptions. Commonly, these proofs describe a relativised world, where all basic primitives exist but a new construction is insecure. As a result, there are no valid security proofs provided that the proof system itself is consistent. However, these *oracle separation* results do not eliminate the possibility of direct proofs, as the additional usage of Peano axioms might reveal additional relations between abstract security properties that make these proofs possible.

For stronger separation results, we need to construct a counterexample from the basic primitives that are assumed to exist. Namely, we must construct a secondary set of weakened primitives that explicitly invalidates the hypothetical security claim. Such a *black-box separation* proves that there are no valid security proofs as long as the initial set of basic primitives exists and the proof system itself remains consistent. The separation itself can be as artificial as needed. For example, we can prove that IND-CPA security is insufficient for IND-CCA2 security by defining a new encryption rule $\overline{\mathsf{Enc}}_{\mathsf{pk}}(m) = b \| \mathsf{Enc}_{\mathsf{pk}}(m)$ where $b \leftarrow \{0,1\}$. To decrypt a message, one just has to omit the first bit. Clearly, the new cryptosystem is (t, ε)-IND-CPA secure whenever the original cryptosystem is $(t + \mathrm{O}(1), \varepsilon)$-IND-CPA secure, but not IND-CCA2 secure, as we can flip the first bit of the encryption to fool the decryption oracle.

As there are many sub-optimal ways to prove security, separation techniques are often employed to show that obtained security guarantees are tight or at least near-optimal. At the same time, such separation results do not guarantee that bounds remain optimal if we take a more fine-grained view on the construction and break some basic primitives into smaller ones. Neither do these separation results prove that some constructions are inherently insecure, sometimes they just indicate underspecification of security premises.

5 SECURITY OF INTERACTIVE COMPUTATIONS

For obvious reasons, we can rigorously analyse only the security of well-specified distributed computations. Such well-specified distributed algorithms are commonly referenced as *protocols*. Security analysis of protocols is one of the main tasks in cryptography. Essentially, we can talk about security in two different contexts. We can analyse the security of a protocol in a *stand-alone setting*, where participants do not execute any other protocols. Alternatively, we can consider the security of a protocol in a wider computational context, where several protocols are executed to achieve more complex computational goals. In this chapter, we consider only stand-alone security.

Superficially, the stand-alone setting is quite restrictive, since protocol outputs cannot be processed further. In Chapter 7, we show that this restriction is purely formal and stand-alone security is sufficient to achieve sequential composability. In layman's terms, a protocol remains secure if no other protocols are executed in parallel. For many settings, such a requirement is naturally forced by the design of the system and thus the provided security level is sufficient. We discuss these issues more thoroughly in the following chapters, as even the stand-alone security model is complex enough to confuse non-experts.

The number of potential security models is humongous even in the stand-alone setting. There are literally hundreds of different security models that characterise various practical settings. Although only few of them are well studied, the others are equally meaningful. Therefore, we concentrate on the real versus ideal world paradigm that forms the core of all security models.

The real versus ideal world concept itself is easy to formalise—a protocol instance is secure if it is sufficiently similar to an ideal implementation. However, it is somewhat nontrivial to describe what the similarity between real and ideal world exactly implies or what happens if we change some minor details in the formalism. Thus, we start with basic consistency principles that must hold for any reasonable formalisation. Next, we consider what security levels are achievable in the ideal setting. Although the latter may seem a rather absurd question to ask as we cannot do any better, it is actually one of the most important ones. If the risks caused by joining an ideal world protocol are not rewarded by the potential gains, then the whole task is hopeless from the start and there is no reason to design a cryptographic solution at all. Secondly, the ideal world behaviour also reveals what kind of properties must be preserved by the correspondence between real and ideal world adversaries.

As a result, we can justify all details of the ideal versus real world approach from a game theoretical viewpoint. The latter provides stronger validity guarantees, since all security definitions that contradict or ignore economical principles are meaningless in practice. Secondly, only a deep understanding of desired security goals provides a justified way to choose a proper security model. Essentially, there are four basic levels of security ranging form input-privacy to complete security and all of them are covered in this chapter. More advanced topics, like asymptotic security and setup assumptions, are left out and are treated separately in remaining chapters.

5.1 FORMAL REQUIREMENTS TO SECURITY DEFINITIONS

Before we actually start formalising various security goals, it is wise to contemplate what software and hardware designers mean by the terms 'attack' and 'security' and what kind of results are expected from cryptographers. These questions are often considered trivial and thus ignored. The ignorance is often rewarded by aesthetically pleasing formalisations and beautiful theories. But at the same time, these results can be cumbersome, not to say completely unrelated to the demands of everyday life. Beautiful theories that are touching the limits of mathematical reasoning are not something to be ashamed of. However, we choose a viewpoint that is as close to practice as possible.

Obviously, all security definitions must be well defined and non-contradictory with itself. Briefly, a cryptographic construction either meets the security definition or not and alternative descriptions of a same construction are either all secure or all insecure. This property is known as internal consistency.

However, the definition has to be appropriate also for practical settings, or otherwise we have hopelessly departed from reality. To assure that a security definition has a clear interpretation in practice, we postulate four restrictions to capture external consistency. First, the effect of an attack must be observable and unambiguously detectable, as emphasised by the restrictions (C1)–(C2). Second, a security notion must be defined as a resistance level against potential attacks, as emphasised by the remaining restrictions (C3)–(C4).

(C1) The effect of a successful attack must be observable. If an attack induces changes that manifest only in the internal retrospection of a participant and not in any observable outcome, then the attack is not successful.

(C2) For each protocol run and each attack against the protocol, it must be possible to determine whether the attack was successful or not. More formally, the overall success of an adversarial strategy $\mathcal{A}$ is quantified in terms of an appropriate game $\mathcal{G}$ such that $\mathcal{G}^{\mathcal{A}} \in \{0, 1, \perp\}$ and the advantage $\Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right]$ is the corresponding quantitative measure.

(C3) A construction is secure if no plausible adversarial algorithm is successful enough. For example, a construction is (t, ε)-secure if the advantage $\Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon$, for all plausible t-time adversaries $\mathcal{A}$.

(C4) Adding restrictions to attack algorithms can only increase the security level. For example, if a construction is (t_1, ε)-secure and not (t_2, ε)-secure for $t_1 > t_2$, the corresponding success criterion is inconsistent.

Let us emphasise some details before going any further. First, it is really impossible to detect how a person or an algorithm internally perceives the situation when there is no measurable outcome. Moreover, we as designers are often interested only in certain behavioural aspects. For example, the output of a poker automate does not have to be truly pseudorandom. A player may even register certain regularities, but the latter is completely irrelevant as long as it does not change his or her gaming behaviour or winning odds.

Secondly, the external consistency requirements are crucial only in the context of practical security, where one has to make economical decisions based on cryptographic results. Consequently, cryptographic definitions may violate some of the postulates (C1)–(C4), as long as they are used as tools to achieve externally consistent security goals. For example, the original definition of universal composability [Can01] violates assumptions (C2)–(C3), and at first glance some asymptotic definitions of statistical security [Gol04, Can00a] contradict postulates (C3)–(C4). For clarity, we note that universal composability has also an externally consistent formulation [Lin03b, Can00b], and that the problems with statistical security are caused by ambiguities in the interpretation, see Section 7.6 and Section 6.2. Nevertheless, one can view the external consistency postulates as a litmus test for detecting possible errors in the formalisations.

Thirdly, a bound of the total running time is often a too coarse measure for defining a class of plausible adversarial strategies. For most protocols, an adversary must provide replies to the other participants in a certain time frame and it is justified to quantify online and offline complexity of the attack. Now, if we add some additional constraints, then the conditions (C3)–(C4) must be modified accordingly to capture all monotonicity requirements.

5.2 SECURITY OF IDEALISED COMPUTATIONS

Suitability of a cryptographic construction can be viewed as any other design problem: we have to state some clear objectives and then assess the construction in respect to them. For example, we may state that an e-voting scheme is secure if one cannot submit incorrect votes, valid votes are correctly counted, and it is impossible to disclose the choice made by an individual voter. However, such a specification is incomplete, as it does not rule out other possible abuses. For example, a voter may still be able to sell her vote, if she can disclose her vote in an indisputable way to a potential buyer. We can add this restriction to the black-list of malicious behavioural patterns but we can never be sure that the list is complete. Also, it may be difficult to verify that such a black-list specification is internally consistent, i.e., does not contain non-resolvable conflicts between security goals. Therefore, a white-list based specification that explicitly states what behavioural patterns are plausible is a far better alternative.

Let us start with some simplifying conventions. First, let symbols $\mathcal{P}_1, \ldots, \mathcal{P}_n$ always denote the participants of a protocol. Second, a protocol must implement some, possibly randomised, functionality $f_1, \ldots, f_n$ in order to be useful, i.e., each party should obtain $y_i = f_i(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the inputs of all participants. The computational process itself can be either non-adaptive or adaptive. For non-adaptive protocols, the functionality $f_1, \ldots, f_n$ is known ahead, whereas f_i is determined dynamically during an adaptive protocol. In this chapter, we consider only non-adaptive computations. Adaptive protocols are normally composed from smaller non-adaptive basic blocks and thus it makes sense to consider them together with composability in Chapter 7.

Additionally, we have to resolve the question whether the inputs $x_1, \ldots, x_n$ are automatically deleted after the completion of a protocol or not. Both choices are equally expressive but preserving the inputs is technically cleaner and looks more natural. Therefore, we require that each non-corrupted node $\mathcal{P}_i$ outputs $z_i = (x_i, y_i)$ at the end of the protocol. Such requirement makes it straightforward to consider the security in the scenarios, where the output z_i is further

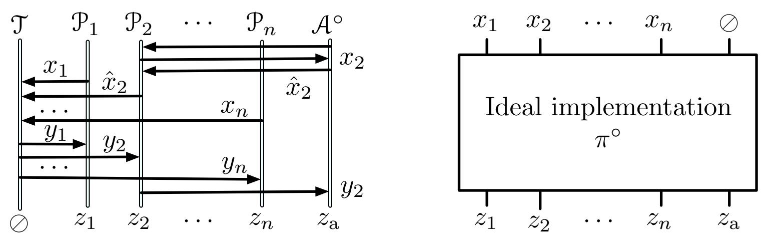


Figure 5.1: Ideal implementation of a protocol π. An example attack on the left and a schematic description as an abstract computational block on the right.

processed by $\mathcal{P}_i$ locally, or used as an input to another protocol.

Idealised computational process. Note that a protocol must reveal some information about the inputs as long as it implements a desired functionality $f_1, \ldots, f_n$. Namely, if parties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ run a protocol π, then each of them must learn $f_i(x_1, \ldots, x_n)$. Moreover, a malicious party $\mathcal{P}_i$ can always change the outcome of $\mathcal{P}_j$ by choosing a different input $\hat{x}_i$ instead of x_i. Finally, a party $\mathcal{P}_i$ may decide to halt in the middle of the protocol or not to participate at all. Obviously, no cryptographic countermeasure can avoid these actions.

Cryptographers use a concept of ideal implementation to model such unavoidable attacks. The ideal implementation π° of a protocol is defined using a trusted third party $\mathcal{T}$ that does all computations on behalf of the others, see Fig. 5.1. That is, all participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ first submit their inputs $x_1, \ldots, x_n$ securely to the trusted third party $\mathcal{T}$. Next, the trusted third party $\mathcal{T}$ computes the desired outcomes $y_i \leftarrow f_i(x_1, \ldots, x_n)$ and sends them securely back to the participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$. In particular, if a party $\mathcal{P}_i$ does not submit his or her input x_i, the protocol is halted and $\mathcal{T}$ sends $\perp$ to all participants. As a result, either all participants will get their desired outputs or neither of them will.

Alternatively, we can fix a schedule how $\mathcal{T}$ must deliver the replies and allow participants to stop $\mathcal{T}$. More formally, a participant $\mathcal{P}_i$ must send zero-one verdict to $\mathcal{T}$ after receiving y_i. If a participant $\mathcal{P}_i$ sends 1, $\mathcal{T}$ continues as usual, otherwise $\mathcal{T}$ halts and sends $\perp$ instead of all remaining replies y_j.

The second idealised model does not guarantee *fairness*, since a malicious party $\mathcal{P}_i$ can force honest parties to compute $f_i(x_1, \ldots, x_n)$ and halt computations before the others obtain their outputs. Clearly, it is easier to design protocols without fairness. Moreover, fundamental limits already outlined by Cleve [Cle86] imply that fairness is unachievable for two-party protocols and protocols with honest minority. Although certain solutions like gradual release of secrets [LMR83, BG89, GL90] and use of external referees [ASW98, CC00] can alleviate the problem, none of them is universally applicable.

Tolerated adversarial behaviour. In many settings, participants can form coalitions to achieve their malicious goals. We model such behaviour by considering an ideal world adversary $\mathcal{A}^\circ$ which can corrupt some of the nodes $\mathcal{P}_1, \ldots, \mathcal{P}_n$, see again Fig. 5.1. The set of possible coalitions can be specified by listing all possible subsets of parties that can be corrupted. The corresponding set of sets $\mathcal{A}$ is called an *adversary structure*. Obviously, if it is possible to corrupt a subset

$A \in \mathcal{A}$, it is also possible to corrupt a subset $B \subseteq A$. Hence, we can always assume that the adversarial structure $\mathcal{A}$ is monotone.

Exact details about the nature of a plausible corruption determine the maximal achievable damage. Adversarial behaviour can be static, adaptive or mobile. A *static adversary* corrupts all participants already before the beginning of the protocol. An *adaptive adversary* corrupts parties adaptively, based on the acquired information; a *mobile adversary* can additionally retreat from corrupted nodes. A mobile adversary respects the adversarial structure $\mathcal{A}$ if at any moment in time, the adversary controls a subset of participants that belongs to $\mathcal{A}$. We emphasise here that adaptive and mobile adversaries can corrupt participants also after protocol values y_i are sent out by $\mathcal{T}$.

In a sense, the mobile adversary models the reality most precisely in large infrastructures, whereas static and adaptive adversaries are more suited for small well-protected networks. Two-party protocols always have static adversaries, as the adversarial structure is simplistic $\mathcal{A} = \{\{\mathcal{P}_1\}, \{\mathcal{P}_2\}\}$.

Non-standard ideal world models. Sometimes all parties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are guaranteed to be honest in the real world and the ideal adversary $\mathcal{A}^\circ$ represents the effects of malicious outsiders. For example, key agreement protocols [BR93a, PV06b, LN06] try to resolve problems caused by malicious message transmission. In such cases, it is more natural to include $\mathcal{A}^\circ$ as a direct participant in the ideal computation instead of introducing a dummy corrupted party $\mathcal{P}_0$.

There are many similar minor details that can be varied to express design goals more succinctly. For example, the idealised computational process for adaptive tasks may have more than two rounds. In all these non-standard cases, the ideal world model can still be decomposed into a model of idealised computational behaviour and a model of tolerable adversarial behaviour.

Highest achievable security level. An ideal world implementation itself does not guarantee privacy of inputs nor correctness of outputs. The model just shows, which security goals are achievable for all protocols that correctly implement the desired functionality. For example, if two parties both want to obtain $y = x_1 + x_2$, then the output y reveals the opponent's input and thus privacy of inputs cannot be preserved. Therefore, it is important to understand what kind of security guarantees are achievable at all. Although we cannot answer these questions without making additional assumptions about the adversarial behaviour, it is still an important aspect to consider. If the ideal implementation fails to provide a desired security level, there is no reason to design the corresponding cryptographic protocols. Secondly, a proper understanding of ideal world attacks provides an important insight into what kind of correspondence between ideal and real world implementations is needed.

Recall that any externally consistent security goal can be formalised as a game, see Section 5.1. Although protocol inputs are normally fixed, it is more instructive to first consider a game that measures average-case security.

```
$\mathcal{G}^{\mathcal{A}^\circ}$
    $\boldsymbol{x} \leftarrow \mathfrak{D}$
    $\boldsymbol{z}_{\mathrm{obs}} \leftarrow \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}}(\boldsymbol{x})$
    return $\mathcal{B}(\boldsymbol{z}_{\mathrm{obs}})$
```

That is, the challenger first draws inputs $\boldsymbol{x} = (x_1, \ldots, x_n)$ from an input distribution $\mathfrak{D}$ and then runs a sub-game $\mathcal{G}_{\text{id-atk}}(\boldsymbol{x})$ to simulate the ideal world

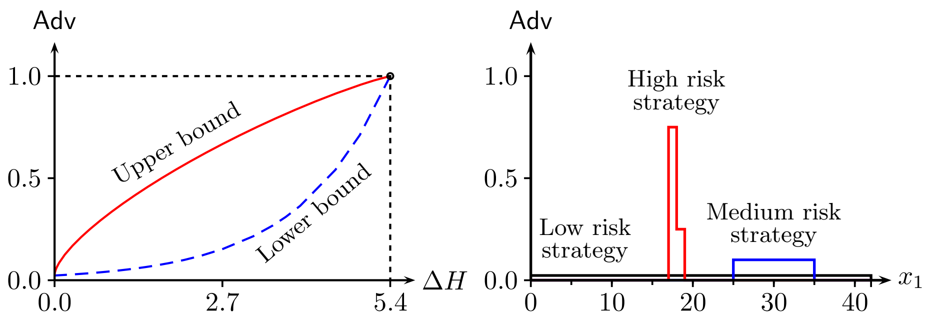


Figure 5.2: Analysis of ideal addition protocol for an attack target $x_1 + \hat{x}_2 = 42$. Dependence between the prior knowledge and the advantage on the left and dependence between the advantage and the true value of x_1 on the right.

attack, and finally evaluates a predicate $\mathcal{B}(\cdot)$ on the set of observable outcomes $\boldsymbol{z}_{\text{obs}} = (z_1, \ldots, z_n, z_{\text{a}})$ to determine whether an attack was successful or not. Now the maximal achievable advantage depends on the input distribution $\mathfrak{D}$ and on the maximal running time of $\mathcal{A}^\circ$. Let us first consider the uniform distribution $\mathfrak{D}_{\text{u}}$ where all plausible inputs are equiprobable. Then

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ) = \frac{1}{|\mathfrak{D}|} \cdot \sum_{\boldsymbol{x} \in \mathfrak{D}} \mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x}) \enspace , \tag{5.1}$$

where

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x}) = \Pr\left[\boldsymbol{z}_{\text{obs}} \leftarrow \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}}(\boldsymbol{x}) : \mathcal{B}(\boldsymbol{z}_{\text{obs}}) = 1\right] \tag{5.2}$$

quantifies the success probability for a fixed input $\boldsymbol{x}$.

As a concrete example, consider a scenario, where a cheating party $\mathcal{P}_2$ wins if the output of ideal addition protocol is 42. For clarity, assume also that the trusted third party $\mathcal{T}$ provides outputs to participants only if submitted inputs $x_1, \hat{x}_2$ are in the range $\{0, 1, 2, \ldots, 42\}$. As the output of $\mathcal{P}_1$ is $z_1 = (x_1, y_1)$, the security goal $\mathcal{B}(z_1, z_2, z_{\text{a}}) = 1$ iff $y_1 = x_1 + \hat{x}_2 = 42$ is indeed legitimate. Now the right sub-figure of Fig. 5.2 depicts graphically the dependence of $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|x_1)$ and shows clearly that different strategies can lead to different trade-offs between maximal and minimal success. A sharp and peaky *risk profile* $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x})$ indicates that the strategy is risky, since the success probability is high for a few inputs. As the input $\boldsymbol{x}$ will rarely fall to the high success region, the adversary executes an all-or-noting attack strategy. Conservative uniformly flat risk profiles minimise the potential losses with the cost of decreasing the maximal possible winning probability. That is, it is possible to win the game for each input $\boldsymbol{x}$, but the corresponding success probability is equally low.

As the time-success profile $\varepsilon(t)$ w.r.t. $\mathfrak{D}_{\text{u}}$ limits the area under a risk profile, it also determines the basic properties of various trade-offs. In particular, no t-time strategy $\mathcal{A}^\circ$ can guarantee $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x}) > \varepsilon(t)$ for all $\boldsymbol{x}$ and $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x}) \approx \varepsilon(t)$ for conservative strategies. Moreover, $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ|\boldsymbol{x}) \geq c \cdot \varepsilon(t)$ only for $\frac{1}{c}$ fraction of inputs and thus achieving high c values means taking a huge risk.

Situation can change drastically if an adversary has additional prior information about the inputs. Formally, such prior information is modelled as a *subjective* input distribution $\mathfrak{D}$ that characterises the adversaries *beliefs* on the likelihood of various inputs. For example, the knowledge $7 \leq x_1 \leq 17$ is represented by the uniform distribution over $\{7,\ldots,17\}$. As the distribution $\mathfrak{D}$ is fixed, the adversary can find an optimal strategy $\mathcal{A}^\circ$ that maximises $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}^\circ)$ or evaluate the subjective risk of missing a ε_0-success region

$$\Pr\left[\boldsymbol{x} \leftarrow \mathfrak{D} : \mathsf{Adv}(\mathcal{A}^\circ|\boldsymbol{x}) \leq \varepsilon_0\right] \quad . \tag{5.3}$$

Again, the time-success profile $\varepsilon(t)$ w.r.t. $\mathfrak{D}$ describes basic choices for trade-offs between risk and success. We emphasise here that using prior information leads to sharper risk profiles and consequently a potential attacker may seriously underperform if the used information is fraudulent. In other words, the potential attacker must really believe in the validity of information.

For many tasks, we can even quantify the direct effect of prior information to success probability. Namely, we can measure the information content in bits as a difference between maximal and actual Shannon entropy of variables unknown to the adversary, and compute the corresponding success bounds, see Fig. 5.2. Similarly, we can quantify the inevitable privacy loss of a protocol as

$$\Delta H = H(\mathcal{X}_{\mathrm{prior}}) - H(\mathcal{X}_{\mathrm{post}}) \quad , \tag{5.4}$$

where $\mathcal{X}_{\mathrm{prior}}$ is the prior and $\mathcal{X}_{\mathrm{post}}$ is the posterior distribution of unknown variables and $H(\cdot)$ denotes Shannon entropy. Note that such bounds characterise only average properties and may underestimate the privacy loss for concrete inputs. Nevertheless, if an ideal implementation causes a huge privacy loss measured in hundreds or thousands of bits, then it might be acceptable to have a leakage of few extra bits in the real protocol to gain higher efficiency.

So far we have considered the situation from the adversarial viewpoint. The choices of honest parties are even more difficult, as the expected attack strategy depends on the risk tolerance and the prior knowledge of a potential adversary. Therefore, one must make even more subjective assumptions about adversarial behaviour. In principle, such choices can be summarised as a single randomised adversarial strategy $\mathcal{A}^\circ$. Consequently, we can use the corresponding security profiles to make justified economical decisions about potential losses. In practice, one often makes the decision based on a hunch or has no choice at all. Still, it is important to know that the basic security and the game theoretical properties of the ideal model are captured by time-success and risk profiles.

5.3 THE REAL VERSUS IDEAL WORLD PARADIGM

Evidently, a protocol provides the highest achievable security level if it satisfies all externally consistent security goals that are achievable in the ideal world. Such a reductionist view is quite old and can be traced back to the early works of Beaver, Micali and Rogaway [Bea91b, MR91b]. Our approach is based on the formalisation given by Canetti and Goldreich [Can00a, Gol04]. However, there are minor differences between our approach and the one pursued by Canetti and Goldreich. We discuss these matters explicitly at the end of this section.

Formal definition of proximity. As a starting point, recall that any externally consistent security goal can be formalised as a game between an adversary and a challenger. More precisely, such a game can be split into online and offline phases. The online phase of a game models the attack against the protocol, where

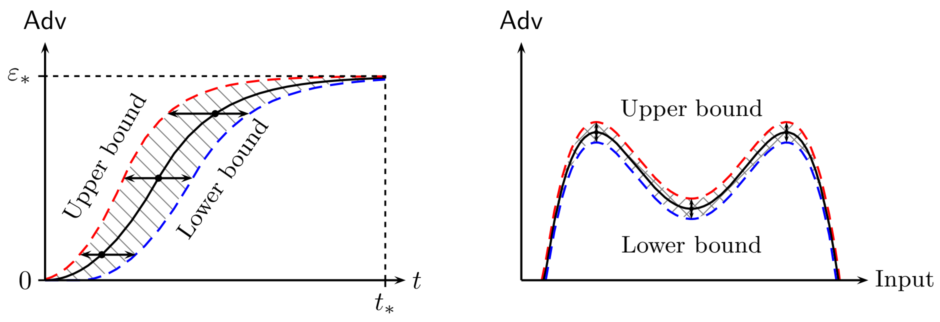


Figure 5.3: Correspondence between ideal and real world models. A real world time-success profile can be approximated with the corresponding ideal world profile. Similarly, the risk profile of $\mathcal{A}^\circ$ induces bounds for the risk profile of $\mathcal{A}$.

all participants reach a final observable outcome z_i. In the subsequent offline phase, the challenger decides, based on the observable outcomes $\boldsymbol{z}_{\mathrm{obs}}$ whether the attack was successful or not. Hence, the offline phase can be formalised as a predicate $\mathcal{B}(\cdot)$ that is independent of the implementation details and thus we can compare ideal and real world implementations.

More formally, let $\mathcal{G}_{\text{re-atk}}$ and $\mathcal{G}_{\text{id-atk}}$ denote sub-games that model the execution of a real and an ideal protocol. Let $\mathcal{A}$ and $\mathcal{A}^\circ$ be the corresponding real and ideal world adversaries. Then, for any input distribution $\mathfrak{D}$, we can compare the advantages of the corresponding security games:

```
$\mathcal{G}^{\mathcal{A}}_{\mathrm{real}}$
    $\boldsymbol{x} \leftarrow \mathfrak{D}$
    $\boldsymbol{z}_{\mathrm{obs}} \leftarrow \mathcal{G}^{\mathcal{A}}_{\text{re-atk}}(\boldsymbol{x})$
    return $\mathcal{B}(\boldsymbol{z}_{\mathrm{obs}})$

$\mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}}$
    $\boldsymbol{x} \leftarrow \mathfrak{D}$
    $\boldsymbol{z}_{\mathrm{obs}} \leftarrow \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}}(\boldsymbol{x})$
    return $\mathcal{B}(\boldsymbol{z}_{\mathrm{obs}})$
```

The sub-game $\mathcal{G}_{\text{id-atk}}(\boldsymbol{x})$ is played as follows. The challenger runs the participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$, $\mathcal{T}$ with inputs $\boldsymbol{x} = (x_1, \ldots, x_n)$ and interacts with the adversary $\mathcal{A}^\circ$ according to the specification of the ideal world. In particular, the challenger immediately halts the game if $\mathcal{A}^\circ$ violates the description of tolerable adversarial behaviour. Also, the challenger gives control over a participant $\mathcal{P}_i$ to the adversary $\mathcal{A}^\circ$ as soon as $\mathcal{A}^\circ$ corrupts $\mathcal{P}_i$. Finally, all participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and $\mathcal{A}^\circ$ halt with some output values $z_1, \ldots, z_n$ and z_{a} and the corresponding vector $\boldsymbol{z}_{\mathrm{obs}} = (z_1, \ldots, z_n, z_{\mathrm{a}})$ is returned as an output of $\mathcal{G}_{\text{id-atk}}(\boldsymbol{x})$.

The specification of the sub-game $\mathcal{G}_{\text{re-atk}}(\boldsymbol{x})$ is similar. The challenger runs the participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ with inputs $\boldsymbol{x} = (x_1, \ldots, x_n)$ and interacts with the adversary $\mathcal{A}$ according to the description of the protocol. Again, the challenger halts the game immediately if $\mathcal{A}$ violates the description of tolerable adversarial behaviour. Finally, all participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and the adversary $\mathcal{A}$ halt and the vector of observable outputs $\boldsymbol{z}_{\mathrm{obs}}$ is returned as an output of $\mathcal{G}_{\text{re-atk}}(\boldsymbol{x})$.

Now consider a situation where, for a fixed security goal $\mathcal{B}(\cdot)$ and for any real world adversary $\mathcal{A}$, there exists a comparable ideal world adversary $\mathcal{A}^\circ$ such that the corresponding advantages are negligibly close

$$|\Pr[\mathcal{G}^{\mathcal{A}}_{\mathrm{real}} = 1] - \Pr[\mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}} = 1]| \leq \varepsilon \tag{5.5}$$

for any input distribution $\mathfrak{D}$. Then the corresponding time-success profiles in the real and ideal world are also close as illustrated in Fig. 5.3. More precisely, for any fixed input distribution and a time bound, we can execute the corresponding optimal ideal world attack with a constant overhead in the real world. The latter gives a lower bound for the time-success profile. The the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ that satisfies the inequality (5.5) explicitly determines the corresponding upper bound. Moreover, as the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ preserves closeness for all input distributions, then for any real world attack $\mathcal{A}$, there exists a generic attack $\mathcal{A}^\circ$ with a similar risk profile and comparable running time that is applicable against any protocol that implements the functionality. In other words, if an honest participant believes that a protocol π is insecure due to the prior information acquired by other parties, then no other protocol that implements the same functionality can be secure.

Note that the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ must be independent of the input distribution $\mathfrak{D}$, otherwise the ideal world attacker has more prior information than the real world attacker. Namely, if the attack $\mathcal{A}^\circ$ in the ideal world uses the whole description of $\mathfrak{D}$, then the dependence leaks information about inputs and we lose privacy guarantees. Similarly, one should assume that $\mathcal{A}^\circ$ corresponding to $\mathcal{A}$ corrupts the same set of participants, otherwise the attack strategies $\mathcal{A}$ and $\mathcal{A}^\circ$ are incomparable. At the same time, the dependence on $\mathcal{B}(\cdot)$ does not introduce any side-effects, since the predicate $\mathcal{B}(\cdot)$ is assumed to be known to all attackers. In fact, an attacker is assumed to behave optimally w.r.t. $\mathcal{B}(\cdot)$.

More generally, let the set of relevant security objectives $\mathcal{B}(\cdot)$ be denoted as $\mathfrak{B}$. Then a real and an ideal world model are $(t_{\mathrm{re}}, t_{\mathrm{id}}, \varepsilon)$*-close w.r.t. the predicate set* $\mathfrak{B}$ if for any t_{re}-time $\mathcal{A}$ and $\mathcal{B}(\cdot) \in \mathfrak{B}$, there exists a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any input distribution $\mathfrak{D}$

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} = 1]| \leq \varepsilon \ . \tag{5.6}$$

Alternatively, we can view $(t_{\mathrm{re}}, t_{\mathrm{id}}, \varepsilon)$-closeness w.r.t. $\mathfrak{B}$ as a reduction schema[1]

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \xRightarrow[\varepsilon]{\text{MPC-SEC}} \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} \tag{5.7}$$

that can be applied to simplify analysis of security games, where the predicate $\mathcal{B}(\cdot) \in \mathfrak{B}$ is computed interactively after the completion of the protocol π; t_{re} limits the online complexity of an attack $\mathcal{A}$ and the inequality (5.6) limits the discrepancy between $\mathcal{G}_{\mathrm{real}}^{\mathcal{A}}$ and $\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ}$. See Sections 7.2-7.5 for further details.

Finally, note that the exact dependencies $t_{\mathrm{id}} = t_{\mathrm{id}}(t_{\mathrm{re}})$ and $\varepsilon = \varepsilon(t_{\mathrm{re}})$ are important, as they determine the correspondence between the ideal and the real world time-success profiles $\varepsilon_0(t)$ and $\varepsilon_1(t)$. Indeed, let t_π be the total running time of the protocol π and $\varepsilon(t_{\mathrm{re}})$ the right hand side of (5.6). Then any ideal world attack can be carried out in the real world with a constant t_π-time overhead. Similarly, the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ assures that any t_{re}-time real world attack has t_{id}-time ideal world counterpart. Consequently, we get

$$\varepsilon_0(t_{\mathrm{re}} - t_\pi) \leq \varepsilon_1(t_{\mathrm{re}}) \leq \varepsilon_0(t_{\mathrm{id}}(t_{\mathrm{re}})) + \varepsilon(t_{\mathrm{re}}) \ . \tag{5.8}$$

In short, the time-success profiles are comparable $\varepsilon_0(t) \approx \varepsilon_1(t)$ only if $\varepsilon(t_{\mathrm{re}})$ is small and time bounds t_{re} and t_{id} are comparable. Note that the inequality (5.8)

[1]The name of the reduction schema MPC-SEC emphasises the fact that the reduction schemata can be applied as the protocol π is secure in the multi-party sense.

is more precise if we assume that the computational effort done by the trusted third party $\mathcal{T}$ is free for the ideal world adversary $\mathcal{A}^\circ$.

The choice of relevant predicates. The most conservative alternative is to include all predicates $\mathcal{B} : \{0,1\}^{n+1} \to \{0,1\}$ into the set $\mathfrak{B}$. Such a choice implicitly states that the real world and the ideal world implementations lead to comparable outputs even if the computational process following the protocol is unbounded. We can talk about *statistical* $(t_{\text{re}}, t_{\text{id}}, \varepsilon)$*-closeness* of the real world and the ideal world if $\mathfrak{B}$ consists of all possible predicates. Alternatively, we can assume that the adversary is computationally unbounded. Then bounds on the online complexities t_{re} and t_{id} become irrelevant and we can talk about ε*-closeness* of the real and the ideal world. Both notions of statistical closeness lead to externally consistent security models. However, the nature of these security guarantees is completely different, see Section 6.2 for further discussion.

Statistical closeness is unachievable for most cryptographic tasks, since many cryptographic primitives, like (asymmetric) encryption, hash functions and digital signatures, are secure only due to our limited computational abilities. Also, one often needs security guarantees only for limited time intervals. For example, in many countries secret governmental information is usually declassified after 25–75 years. Therefore, it makes sense to consider the set of time-bounded predicates $\mathfrak{B}$ or add even more severe restrictions. In particular, we can talk about *computational* $(t_{\text{re}}, t_{\text{pr}}, t_{\text{id}}, \varepsilon)$*-closeness* between the real and the ideal world when the set $\mathfrak{B}$ consists of all t_{pr}-time predicates. The latter gives security guarantees for all scenarios, where the total time needed to post-process the protocol outputs and to evaluate the success criterion is less than t_{pr}.

Comparison with the standard approach. As said before, our approach is slightly different from the mainstream formalisation [Can00a, Gol04]. In the classical formalisation, the correspondence $\mathcal{A} \mapsto \mathcal{A}^\circ$ between real and ideal adversaries must be universal for all predicates $\mathcal{B}(\cdot) \in \mathfrak{B}$ and input distributions $\mathfrak{D}$. In other words, given an adversary $\mathcal{A}$, we can find a corresponding universal adversary $\mathcal{A}^\circ$ that works equally well for all possible security objectives $\mathcal{B}(\cdot)$ and we get a seemingly stronger security guarantee. However, such an appearance is deceptive, since for each security goal $\mathcal{B}(\cdot)$, time bound t_{id}, level of prior information specified as $\mathfrak{D}$ and risk tolerance, there exists an optimal generic attack $\mathcal{A}^\circ_*$ that already depends on $\mathcal{B}(\cdot)$. Hence, even if the map $\mathcal{A} \mapsto \mathcal{A}^\circ$ is independent of $\mathcal{B}(\cdot)$, we must still estimate the success in terms of $\mathcal{A}^\circ_*$:

$$\mathsf{Adv}_{\mathcal{G}_{\text{real}}}(\mathcal{A}) \le \mathsf{Adv}_{\mathcal{G}_{\text{ideal}}}(\mathcal{A}^\circ) + \varepsilon \le \mathsf{Adv}_{\mathcal{G}_{\text{ideal}}}(\mathcal{A}^\circ_*) + \varepsilon \ . \tag{5.9}$$

Consequently, the independence from $\mathcal{B}(\cdot)$ does not have any effect on time-success or risk profiles. Also, one cannot rationally eliminate the possibility that an attacker actually knows his or her attack target $\mathcal{B}(\cdot)$ and can thus use both schemes to convert his or her attack to the universal attack $\mathcal{A}^\circ$.

In Section 7.3 we also show that both formalisations are quantitatively equivalent for static adversaries, i.e., the choice of formalism is just a matter of taste. From a practical viewpoint, there is no difference anyway, as standard security proofs provide a construction of $\mathcal{A}^\circ$ that depends only on $\mathcal{A}$. Recall that reduction schemata are usually stated in terms of computational distance and consequently we get natural bounds for $\mathsf{cd}^{t_{\text{pr}}}_{\boldsymbol{z}_{\text{obs}}}(\mathcal{G}^{\mathcal{A}}_{\text{re-atk}}, \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}})$.

Another small but still important difference lies in the way we treat corruption. Note that corruption is not objectively observable, since it describes the internal state of a participant, which may or may not influence the observable outcomes. In particular, a predicate $\mathcal{B}(\cdot)$ cannot treat z_i differently depending on whether $\mathcal{P}_i$ is corrupt or not, since such a dependence would lead to externally inconsistent security goals. In practice, one can detect corruption with high probability by forcing each participant $\mathcal{P}_i$ to prove in zero-knowledge that he or she knows an input x_i and randomness ω_i that would lead to the observed protocol transcript. However, the latter is already a new protocol.

Various levels of security. The correspondence between the real and the ideal world implementations is amazingly flexible. The description of the ideal world model, the tolerated adversarial behaviour and the set of relevant predicates $\mathfrak{B}$ are the three fundamental factors that determine the corresponding security guarantees. Intelligent variation of these basic factors can produce myriads of relevant security models. However, four of them are more essential than the others, since they formalise somewhat canonical security objectives. The following four sections are dedicated to these fundamental models.

5.4 SECURITY IN SEMI-HONEST MODEL

Semi-honest model provides the weakest but still meaningful security guarantees. Briefly, semi-honest participants follow the protocol description, but quietly store all intermediate values and later try to deduce some extra information about the remaining inputs or outputs. Hence, semi-honest behaviour is always possible, unless we take extreme organisational measures to assure that internal computational results cannot be stored. Moreover, note that the following formal definition captures only the basic privacy requirements, as malicious behaviour is prohibited and the outputs cannot be tampered with.

Definition 2. *A protocol π is $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-private w.r.t. the idealised functionality $f_1, \ldots, f_n$ and the adversarial structure $\mathcal{A}$ in the semi-honest model if the corresponding real and ideal world models are $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-close. That is, for any t_{re}-time adversary $\mathcal{A}$ and for any t_{pr}-time predicate $\mathcal{B}(\cdot)$, there exists a t_{id}-time ideal world adversary $\mathcal{A}^{\circ}$ such that for any input distribution $\mathfrak{D}$*

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^{\circ}} = 1]| \leq \varepsilon \ . \tag{5.10}$$

Some important details in the definition are worth stressing. First, the time difference $t_{\mathrm{id}} - t_{\mathrm{re}}$ quantifies the maximal computational gain that the adversary may achieve. Second, the bound t_{pr} determines the maximal time complexity of a post-processing phase that preserves security. Third, ideal world model is flexible enough to accommodate any general purpose security objective.

Non-cryptographers often complain that the formalism is overly restrictive in practice, since participants may tolerate bigger information leakage than is needed to compute the outputs. However, it is surprisingly simple to model such protocols by providing a proper description of an ideal world model. More precisely, assume that participants need to compute a functionality $\hat{f}_1, \ldots, \hat{f}_n$, however, they are willing to leak $f_1, \ldots, f_n$ instead. This setting is meaningful only if the desired outputs can be computed from the leaked ones:

$$\hat{f}_i(x_1, \ldots, x_n) = g_i\big(f_i(x_1, \ldots, x_n)\big) \ . \tag{5.11}$$

Consequently, we can consider an ideal world model, where the participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ first use a trusted third party $\mathcal{T}$ to get replies $f_i(x_1, \ldots, x_n)$ and after that locally compute $\hat{y}_i = g_i(y_i)$. That is, a non-corrupted party $\mathcal{P}_i$ deletes y_i and outputs $z_i = (x_i, \hat{y}_i)$. The computational closeness between the real and the ideal world models implies that the semi-honest node $\mathcal{P}_i$ cannot learn anything beyond y_i but at the same time the protocol π must correctly reproduce only the values $\hat{y}_i$. Therefore, we can easily analyse protocols where the outputs are strictly less informative than potentially leaked information. In the limiting case, one can reveal enough information, so that each participant can compute all protocol messages, i.e., the methodology is universally applicable.

As a concrete example, consider private evaluation of randomised approximation algorithms [FIM+06]. In such settings, the distribution of approximations $\hat{f}_i(x_1, \ldots, x_n)$ can be simulated knowing only $f_i(x_1, \ldots, x_n)$ and we can formalise the maximal leakage as $f_1, \ldots, f_n$. Similarly, we can formalise security for the protocols that sometimes fail to produce the correct output. In fact, we can state the correctness requirement as a separate property.

Definition 3. *A failure probability δ for a protocol π is the maximal probability that the obtained outputs $\hat{y}_i$ differ from the intended outputs $f_i(x_1, \ldots, x_n)$ in the semi-honest model. A protocol π is called correct if there are no failures.*

One can often achieve significant gains in efficiency by allowing negligible failure probability. For example, many protocols for private predicate evaluation based on homomorphic encryption and oblivious polynomial evaluation have a small tunable failure probability. See the articles [Fis01, FNP04, BK04, KS05] for the most prominent examples and [LLM05, LL07] for the author's results.

We emphasise that any efficiently computable functionality can be privately evaluated. Yao was the first theoretician to state these completeness theorems for the semi-honest model [Yao82, Yao86]. We refer to the manuscript [LP04] for the corresponding proofs and historical remarks. For the multi-party protocols, these completeness theorems were first proven by Goldreich, Micali and Wigderson for the computational setting [GMW87]. The information-theoretical setting where the adversary is unbounded was covered only a year later [BOGW88, CCD88]. See the articles [MR91b, Bea91b, Can00a, Gol04] for further historical references.

5.5 INPUT-PRIVACY IN MALICIOUS MODEL

Before going any further, note that the privacy in the semi-honest model captures two different security goals: input-privacy and output-privacy. Moreover, these goals are quite loosely coupled. As an example, consider a randomised protocol, where participants generate outputs locally without using inputs. Then the protocol preserves the privacy of these inputs even if participants broadcast all outputs. Similarly, the privacy of outputs is preserved when participants broadcast all inputs. For deterministic functionalities, output-privacy implies input-privacy, since additional knowledge about inputs leaks information about outputs. Also, it is evident that output-privacy and correctness guarantees are tightly coupled when the adversary can deviate from the protocol description, since the ability to influence outputs violates privacy constraints.

Consequently, input-privacy is one of the weakest security objectives that still makes sense in the *malicious model*, where a coalition of corrupted parties can arbitrarily deviate from the protocol description. Moreover, input-privacy itself is an economically justified design goal for some settings.

Consider a video on demand service as an illustrative example. Assume that a single subscription server delivers content decryption keys to all potential clients. The functionality and the threats are asymmetric for this setting. As there are thousands, if not millions, potential subscribers, it is impossible to guarantee semi-honest behaviour for all clients. However, we can still use organisational methods to assure that the server is semi-honest. Also, the service provider itself is economically motivated to ensure that an honest client gets the desired output. On the other hand, the service provider must limit potential information leakages against a coalition of malicious clients.

Input-privacy is formally defined in terms of interactive hypothesis testing, where a malicious adversary tries to verify hypotheses about protocol inputs. Although it is sufficient to bound the total running time, we separate online and offline attack complexities to achieve more precise classification. More formally, let the set $\mathfrak{B}_{z_\mathrm{a}}(t_\mathrm{pr})$ consists of all t_pr-time predicates $\mathcal{B}(z_1, \ldots, z_n, z_\mathrm{a}) = \mathcal{B}(z_\mathrm{a})$ that can be computed offline by the adversary in the following definition.

Definition 4. *A protocol is $(t_\mathrm{re}, t_\mathrm{id}, t_\mathrm{pr}, \varepsilon)$-input-private w.r.t. the idealised functionality $f_1, \ldots, f_n$ and the adversarial structure $\mathcal{A}$ in the malicious model if the corresponding real and ideal world models are $(t_\mathrm{re}, t_\mathrm{id}, \varepsilon)$-close w.r.t. $\mathfrak{B}_{z_\mathrm{a}}(t_\mathrm{pr})$. That is, for any t_re-time adversary $\mathcal{A}$ and for any t_pr-time predicate $\mathcal{B}(\cdot)$ in the form $\mathcal{B}(z_1, \ldots, z_n, z_\mathrm{a}) = \mathcal{B}(z_\mathrm{a})$, there exists a t_id-time ideal world adversary $\mathcal{A}^\circ$ such that for any input distribution $\mathfrak{D}$*

$$|\Pr[\mathcal{G}_\mathrm{real}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_\mathrm{ideal}^{\mathcal{A}^\circ} = 1]| \leq \varepsilon \ . \tag{5.12}$$

Input-privacy is strictly weaker than the complete security in the malicious model and thus input-private protocols are often more efficient. In particular, no non-trivial protocol can be secure in the malicious model and have only two rounds, see the discussions in [GK96b, KO04]. Nevertheless, many two-round protocols do achieve input-privacy. For example, standard security definitions of oblivious transfer [NP99a, AIR01] require input-privacy with respect to unbounded receivers and computationally bounded senders. The same security requirements are often posed for other two-round client-server protocols, where only one participant obtains the output. One of the articles included into the thesis [LL07] investigates a certain subclass of two round client-server protocols and provides an automatic compilation technique, which converts any private protocol in the semi-honest model to an input-private protocol in the malicious model. The transformation preserves round complexity and has a relatively small overhead for many practical protocols.

5.6 OUTPUT-CONSISTENCY IN MALICIOUS MODEL

For many applications, input-privacy is not enough as honest parties need also explicit correctness guarantees. In particular, malicious behaviour should be detectable and one should be able to issue verifiable complaints about cheating.

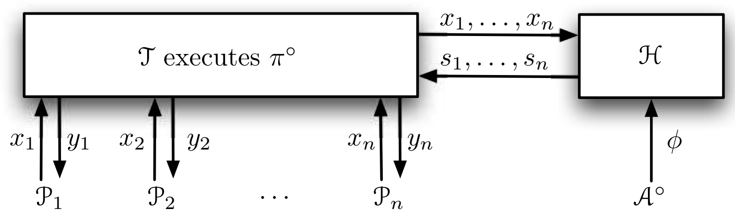


Figure 5.4: Ideal world model of output-consistent computations.

Now such correctness guarantees come in two flavours: honest parties may either detect malicious behaviour in general, or only the malicious changes in the outcomes. The second design goal can be viewed as output-consistency.

Observe that output-consistency is strictly weaker than complete security. Namely, if a malicious participant can alter his or her behaviour so that the outputs are changed only for few inputs $\boldsymbol{x} \in \mathcal{X}$, then a public complaint about such a change reveals that $\boldsymbol{x} \in \mathcal{X}$. The latter is not true for complete security, since the corresponding complaint only implies that the outputs could have been altered for some inputs. Therefore, output-consistency usually causes a significantly smaller overhead than complete security. Moreover, output-consistency is an economically justified design goal for scenarios, where trustworthy reputation is worth more than the potential gain achieved by cheating.

Informally, a protocol is output-consistent if a malicious adversary cannot tamper the outputs without causing honest parties to halt. More formally, an adversary has partial control over the ideal protocol π°. A non-corruptible trusted party $\mathcal{T}$ still computes the outcomes of a protocol, however, a *halting-machine* $\mathcal{H}$ controls the delivery of outputs, see Fig. 5.4. That is, the halting-machine $\mathcal{H}$ gets inputs $x_1, \ldots, x_n$ from $\mathcal{T}$ and sends back zero-one verdicts $s_1, \ldots, s_n$. Next, the trusted third party $\mathcal{T}$ sends the output y_i to the participant $\mathcal{P}_i$ only if $s_i = 1$. The communication between the halting-machine $\mathcal{H}$ and the adversary $\mathcal{A}^\circ$ is unidirectional—only the adversary $\mathcal{A}^\circ$ can send messages to $\mathcal{H}$. Formally, it is sufficient to assume that $\mathcal{H}$ is a universal computing device such that the adversary $\mathcal{A}^\circ$ can start $\mathcal{H}$ by sending the initial program code ϕ and possibly later fine-tune $\mathcal{H}$ by sending additional instructions.

Definition 5. *A protocol is $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-output-consistent w.r.t. the idealised functionality $f_1, \ldots, f_n$ and the adversarial structure $\mathcal{A}$ in the malicious model if the corresponding real and ideal world models are $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-close. That is, for any t_{re}-time adversary $\mathcal{A}$ and for any t_{pr}-time predicate $\mathcal{B}(\cdot)$, there exists a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any input distribution $\mathfrak{D}$*

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} = 1]| \le \varepsilon \ . \tag{5.13}$$

First, observe that an output-consistent protocol must also be private in the semi-honest model and input-private in the malicious model. Second, the exact interaction pattern between $\mathcal{A}^\circ$ and $\mathcal{H}$ determines the maximal expressiveness of leaked predicates when an honest participant issues a rightful complaint. Therefore, it makes sense to characterise the underlying structure of these predicates. Sometimes one can show that $\mathcal{H}$ always evaluates predicates that have a simple structure, see our manuscript [LL06] for further discussion. Finally, the computational effort done by $\mathcal{H}$ should be free for $\mathcal{A}^\circ$ to make the correspondence between the ideal and the real world time-success profiles more tight.

The concept of output-consistency can be traced back to the classical works on secure multi-party computation [FY92, CFGN96, CO99], where authors consider detectability of malicious behaviour. However, none of them clearly formulates the ideal world model. The concept has also implicitly surfaced as a natural design property for adaptive oblivious transfer [NP99b], since it is almost trivial to achieve. Recently, Aumann and Lindell generalised these early notions and proposed three alternative security models [AL07]. However, their formulations are strictly weaker than output-consistency, as none of them guarantees even input-privacy nor the ability to detect altered outputs.

Output-consistency also alleviates the problem of false accusations. When a complaint is issued about malicious behaviour in an input-private protocol, one has to prove his or her innocence. Since tampering with outputs is detectable in an output-consistent protocol, the burden of proof lies on persons who raise complaints. In particular, malicious coalition cannot frame honest participants if one can use digital signatures to protect the authenticity of protocol messages. This property is specially useful in the client-server scenarios, where clients have different access rights and the server may rightfully fail for some queries, e.g. private inference control problem [WS04]. However, there is a slight difference between two- and multi-party protocols. In two-party protocols, an output failure exposes the malicious participant, whereas the latter is not a priori true in the multi-party setting. Hence, we can talk about a stronger form of consistency that also reveals some maliciously acting participants.

5.7 COMPLETE SECURITY IN MALICIOUS MODEL

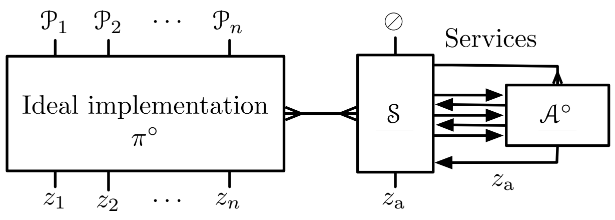
The highest attainable security level in the malicious model is of course the computational closeness between the real world and ideal world without a halting-machine $\mathcal{H}$. In this case, malicious participants can cause only non-selective protocol failures and public complaints reveal nothing about inputs.

Definition 6. *A protocol is $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-secure w.r.t. the idealised functionality $f_1, \ldots, f_n$ and the adversarial structure $\mathcal{A}$ in the malicious model if the corresponding real and ideal world models are $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$-close. That is, for any t_{re}-time adversary $\mathcal{A}$ and for any t_{pr}-time predicate $\mathcal{B}(\cdot)$, there exists a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any input distribution $\mathfrak{D}$*

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} = 1]| \leq \varepsilon \ . \tag{5.14}$$

A major drawback of secure protocols is the overhead needed to achieve security, since all participants have to prove that they obey the protocol without disclosing their inputs and random coins. The latter can be done with zero-knowledge proofs of correctness, using the famous GMW-compiler [GMW87], or with verifiable secret sharing [BOGW88, CCD88] in the information theoretical setting. See also the state of the art results [Pin03, KO04, LP07] for Yao's two-party computation and its extensions to multi-party case [BMR90].

It is well known that secure protocols can be practically intractable when the input size is large and functionalities have a large circuit complexity. Also, the



$\mathcal{P}_1$ $\mathcal{P}_2$ $\cdots$ $\mathcal{P}_n$
Ideal implementation π°
z_1 z_2 $\cdots$ z_n
$\emptyset$
Services
$\mathcal{S}$
$\mathcal{A}^\circ$
z_a
z_a

Figure 5.5: Canonical constructive correspondence between adversaries.

strength of organisational and physical methods is often enough to protect the computational processes and thus lower security levels are sufficient.

In particular, the difference between output-consistency and complete security can be marginal. Recall that issuing a complaint in output-consistent protocols reveals one bit of extra information. Consequently, such a leak may be a reasonable sacrifice for additional efficiency, if the protocol output itself reveals hundreds or thousands of bits. Moreover, adversaries must take a tremendous subjective risks to learn beyond 20 bits of new information. Namely, let $\mathfrak{D}$ be a subjective probability distribution that describes the adversaries prior knowledge about the inputs. Then the maximal entropy loss due to selective halting is $H(s_1, \ldots, s_n)$, where $s_1, \ldots, s_n$ are halting decisions made by $\mathcal{H}$ and the entropy is computed with respect to $\mathfrak{D}$. Hence, the adversary must take a risk of being exposed with a subjective probability 2^{-k} to learn k bits.

Another interesting aspect that is worth commenting is the issue of correctness guarantees raised by Micali and Rogaway [MR91a]. Namely, the computational closeness between the real and the ideal world models does not guarantee that honest participants obtain the correct values, rather they are t_{pr}-indistinguishable from the values obtained in the ideal world. We can naturally strengthen the correctness constraints, if we considers only the subclass of nice adversaries that append a flag $\mathsf{corrupted}$ to outputs of all corrupted participants. Namely, we must consider an additional set of predicates $\mathfrak{B}_{\mathsf{cor}}$ that consists of all predicates that ignore the inputs z_i with the flag $\mathsf{corrupted}$, i.e., require that the outputs of honest parties are statistically indistinguishable from the ideal world. Formally, this strengthening is externally inconsistent, since the corruption is not an externally observable phenomenon. Nevertheless, any hostile adversary can be converted to the corresponding nice adversary with minimal overhead so that the end result of a security game does not change. Hence, in principle such a strengthening can be incorporated into the model.

5.8 CANONICAL CONSTRUCTIVE CORRESPONDENCE

Note that all security definitions given above require a correspondence between the real and the ideal world adversaries. A canonical way to achieve such a correspondence is based on code wrappers that internally run a real world adversary and translate their actions into legitimate ideal world attacks, see Fig. 5.5. Briefly, a code wrapper Sim is guaranteed to succeed if it manages to simulate the protocol execution for an adversary $\mathcal{A}$. Thus, it is more common to use a term *simulator* instead. We emphasise that a simulator Sim has complete control over the real adversary $\mathcal{A}$. In particular, the simulator Sim provides random coins and timer services to $\mathcal{A}$. The simulator Sim can also restart or rewind $\mathcal{A}$ to acquire extra knowledge or to bypass security checks in the protocol. All other participants see the algorithm pair $(\mathsf{Sim}, \mathcal{A})$ as an ideal world adversary $\mathcal{A}^{\circ}$, i.e., Sim can corrupt ideal world participants and interact with the halt machine $\mathcal{H}$.

Most security proofs are based on black-box simulation, where the simulator Sim does not inspect nor manipulate the internal state and random coins of the adversary $\mathcal{A}$. Necessary information is gathered from the protocol messages exchanged between Sim and $\mathcal{A}$. The latter is not compulsory in white-box reductions, where Sim may act arbitrarily. Nevertheless, white-box reductions must be somehow constructive, or otherwise we are no closer to the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^{\circ}$. Generally, white-box reductions use some sort of adversary dependent information to interact with the adversary $\mathcal{A}$.

Simulatability as a security criterion appeared only after the discovery of zero-knowledge proofs [GMR85] and early definitions of secure multi-party computations [Yao82, Yao86] were given in terms of maximal indistinguishability. For clarity, consider two-party computations with a corrupted node $\mathcal{P}_1$ in the semi-honest model. Then an adversary $\mathcal{A}$ should be unable to distinguish protocol runs corresponding to the inputs x_2 and $\hat{x}_2$ whenever $f_1(x_1, x_2) = f_1(x_1, \hat{x}_2)$. Now, if the adversary $\mathcal{A}$ is good in predicting a function $g(x_1, x_2)$, then g must be constant over the class of inputs

$$\mathcal{M}_{x_1,y} = \{(x_1, x_2) : f_1(x_1, x_2) = y\} \tag{5.15}$$

or we get a contradiction with the basic properties of semantic security, see Section 3.2. Consequently, $\mathcal{A}$ cannot predict anything that is not expressible in terms of x_1 and y. However, the limitation does not apply for the functions g that can be computed from x_1 and y. For example, assume that F is a permutation that is hard to invert and consider a protocol, where the participant $\mathcal{P}_2$ sends a message $e = F^{-1}(x_2)$ to $\mathcal{P}_1$ who outputs $y_1 = F(e) = x_2$. Then the protocol runs corresponding to the same output y_1 are indistinguishable by the construction. However, the participant $\mathcal{P}_1$ learns $F^{-1}(x_2)$, which is difficult to obtain from the output x_2 alone. Therefore, the approach based on the maximal indistinguishability is unsuitable, since it does not provide adequate security guarantees for all target functions $g(x_1, x_2)$.

Simulating the protocol execution in the ideal world can be quite challenging. Fortunately, the semi-honest model has two important properties that make the task easier. First, the adversary $\mathcal{A}$ cannot change the inputs of corrupted parties. Second, the adversary $\mathcal{A}$ must follow the protocol and it is sufficient if the simulator Sim can simulate all messages sent to corrupted parties. Let $\mathcal{C}$ be the set of corrupted parties and $\overline{\mathcal{C}}$ be its complement. Then simulating a static adversary $\mathcal{A}$ is particularly straightforward. The simulator Sim first corrupts nodes that belong to $\mathcal{C}$, submits their inputs $(x_i)_{i \in \mathcal{C}}$ to the trusted third party $\mathcal{T}$ and uses the received values $(f_i(\boldsymbol{x}))_{i \in \mathcal{C}}$ to simulate all messages sent to $\mathcal{C}$ in the real execution. Adaptive and mobile corruption makes the task more difficult, since the simulator Sim must decide when to submit inputs to $\mathcal{T}$ and assure that the adversary's view remains consistent after a new corruption instruction.

Providing a good simulation in the malicious model is more demanding, as a simulator Sim cannot automatically submit inputs $(x_i)_{i \in \mathcal{C}}$. Namely, a malicious adversary $\mathcal{A}$ can always replace the initial input x_i of a corrupted node $\mathcal{P}_i$ by

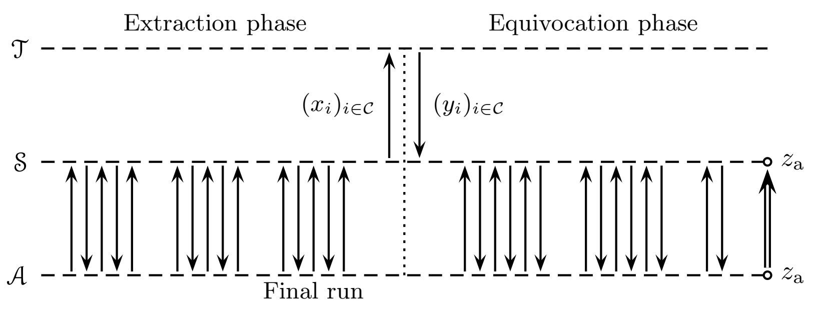


Figure 5.6: Canonical structure of a simulation procedure.

a new value $\hat{x}_i$ and thus Sim must reconstruct the actual inputs $(\hat{x}_i)_{i\in\mathcal{C}}$. Consequently, the simulation must follow a specific structure depicted in Fig. 5.6. Briefly, the simulator must first extract all inputs of corrupted parties and then, given corresponding outputs, continue the simulation appropriately.

Extractability. If we want to achieve a black-box simulation, the simulator must extract inputs from the messages sent by corrupted participants. In particular, one should be able to extract the inputs of a semi-honest coalition. Such an extraction property seems to contradict input-privacy. However, such impression is misleading, as the powers of a simulator and adversary are always asymmetric. In the multi-party setting, the asymmetry is achieved by the clever choice of the adversarial structure $\mathcal{A}$. Namely, the adversarial structure $\mathcal{A}$ is often chosen so that a complement $\overline{\mathcal{C}}$ of a plausible coalition $\mathcal{C} \in \mathcal{A}$ can never be corrupted. Thus, the simulator can form a malicious coalition $\overline{\mathcal{C}}$ that is large enough to extract the inputs $(x_i)_{i\in\mathcal{C}}$ from the received protocol messages, whereas the coalition $\overline{\mathcal{C}}$ is impossible in the real protocol.

The same trick does not work for protocols with honest minority, where an adversary can corrupt as many or even more parties than a simulator. Corresponding security proofs use another important asymmetry. Namely, the simulator Sim can always rewind $\mathcal{A}$ to previous states and then change messages sent to $\mathcal{A}$. Thus, the simulator Sim can restore $\hat{x}_i$ by observing many correlated protocol transcripts, whereas the latter is impossible for an adversary $\mathcal{A}$.

Equivocability. Observe that a protocol is input-private only if a protocol binds all participants to their future outputs. For the sake of contradiction, assume that an adversary $\mathcal{A}$ can legitimately change the inputs after the extraction of inputs $(\hat{x}_i)_{i\in\mathcal{C}}$. Namely, $\mathcal{A}$ can efficiently find yet another set of inputs $(\tilde{x}_i)_{i\in\mathcal{C}}$ and random coins $(\omega_i)_{i\in\mathcal{C}}$ such that messages exchanged so far are consistent with the protocol. Then the adversary $\mathcal{A}$ can clearly learn $(\tilde{y}_i)_{i\in\mathcal{C}}$ instead of $(\hat{y}_i)_{i\in\mathcal{C}}$ obtained by the simulator. Consequently, the new set of outputs $(\tilde{y}_i)_{i\in\mathcal{C}}$ must be efficiently computable from the old ones $(\hat{y}_i)_{i\in\mathcal{C}}$. For most protocols, this implies that $(\tilde{y}_i)_{i\in\mathcal{C}} \equiv (\hat{y}_i)_{i\in\mathcal{C}}$ for all legitimate input changes $(\tilde{x}_i)_{i\in\mathcal{C}}$.

For many protocols, such an implicit or explicit commitment to the outputs creates another problem, since the simulator Sim itself does not know $(\hat{y}_i)_{i\in\mathcal{C}}$ in the extraction phase. As a result, Sim must be able to legitimately retrofit the outputs $(\hat{y}_i)_{i\in\mathcal{C}}$. More precisely, the simulator Sim must produce a continuation for the protocol run that leads to $(\hat{y}_i)_{i\in\mathcal{C}}$ if $\mathcal{A}$ behaves honestly after the extraction. Hence, we have a similar asymmetry as before. For malicious parties, the protocol must be binding, and for the simulator, the protocol must be equivocable. Again, a clever choice of the adversarial structure $\mathcal{A}$ can create the desired asymmetry of powers for multi-party protocols. For two-party protocols, we must use rewinding to bypass consistency checks that force $(\tilde{y}_i)_{i\in\mathcal{C}} \equiv (\hat{y}_i)_{i\in\mathcal{C}}$. Moreover, it is often impossible to rewind the entire extraction phase to retrofit the desired outputs, as the latter may change the extractable inputs $(\hat{x}_i)_{i\in\mathcal{C}}$.

Trusted setup. Many protocols use a special setup procedure, where a trusted dealer either broadcasts a single message or provides a trustworthy public key infrastructure. Note that the trusted dealer exists only in the real world model and thus a simulator can corrupt the trusted dealer in the ideal world to achieve extractability and equivocability. Hence, we can construct non-rewinding simulators also for two-party protocols where rewinding is otherwise unavoidable. Since the rewinding techniques do not scale well for parallel compositions, the trusted setup procedures are the only known way to achieve composability in the two-party setting, see Section 7.7 for more detailed discussion.

To summarise, the canonical constructive correspondence is achievable only by explicit embedding of trapdoor mechanisms that violate both input-privacy and output-consistency. The latter causes a tension between the security and simulation, which is always resolved by introducing the asymmetry of powers between the simulator and the adversary. In less formal terms, no protocol can be “unconditionally” secure in all settings.

6 ALTERNATIVE SECURITY MODELS

Exact quantification of security properties is only one of many paradigms used in cryptography. Therefore, it is important to know the connections between exact security and its alternatives. We start from asymptotic security, since it is probably the most popular and widely studied alternative approach. Note that the initial simplicity of asymptotic security models is highly deceptive. In fact, even the notion of polynomial security varies in cryptographic literature. Therefore, we first formalise the limiting process that is needed to define asymptotic security and then explore common formalisations. As a result, we obtain the description of eight asymptotic security models and a corresponding hierarchy that characterises the qualitative relations between these models.

Subjective security models are another common alternative to exact security. The main difference between objective and subjective security models lies in the choice of possible adversaries. Objective security models can only restrict available computing resources, whereas subjective security models can place additional constraints on the set of potential adversaries. For example, all security definitions that consider only uniform adversaries are actually subjective, since a potential attacker is not allowed to choose algorithms that are optimal for concrete security parameters. We remark here that subjective security models are quite popular in cryptographic literature. For example, the random oracle, the ideal cipher and the generic group model are all subjective security models, since the corresponding security guarantees are meaningful in practice only if we are willing to make certain subjective assumptions about the set of possible adversarial algorithms. Finally, subjective security notions are also appropriate when we consider protocols with trusted setup, see Section 6.4.

Subjective and objective security models share many technical similarities but there are also important conceptual differences. Most importantly, it is impossible to formalise certain natural security properties without using the notion of subjective security. In Section 6.4, we discuss these limitations more thoroughly and explain why objective approach is destined to fail in some settings. In Section 6.5, we rigorously formalise the notion of subjective security that allows us to escape these limitations. We also show that the corresponding security proofs are technically very close to standard proofs. Namely, a standard security proofs is also valid in the subjective setting if it is strictly constructive. Nevertheless, it might be challenging to find strictly constructive proofs, especially since the proof methodology presented in Chapter 4 is actually non-constructive. Fortunately, most of these non-constructive proof steps can be eliminated. We describe the corresponding semi-automatic techniques in Section 6.7. As a result, we can still describe proofs by using game trees and reduction schemata, and thus hide tedious technical details.

6.1 SCALABILITY AND POLYNOMIAL SECURITY

It is easy to see that all conditional security proofs just reduce the amount of unjustified trust by isolating few essential simplistic assumptions that guarantee the security of the entire construction. However, all instantiations of these basic primitives are currently heuristic and the corresponding security guarantees are essentially unknown. Although there are some numerical security estimates for these basic primitives, they are based on empirical experiments or on heuristic claims, and thus likely to change over time. Moreover, the amount of available computing power is also likely to increase in time. Therefore, it is not wise to design protocols so that the underlying primitives cannot be replaced.

Fortunately, cryptographic constructions and protocols are seldomly tied to specific building blocks, such as the 128-bit MD5 hash function, and can thus be "regularly" updated to take advantage of more powerful computing devices and ensure the proper safety margin against plausible attacks. As conditional security proofs abstract away all unnecessary implementation details, we can view a cryptographic construction as a parametrised template, i.e., we can increase the security level by using stronger but slower basic primitives.

Let $\Bbbk$ be a *security parameter* that indexes both a sequence of basic primitives and the construction in question. For clarity, assume that security can be characterised by the advantage

$$\varepsilon(\Bbbk) = \max_{\mathcal{A}\in\mathfrak{A}(\Bbbk)} \mathsf{Adv}_{\mathcal{G}_\Bbbk}(\mathcal{A}) \enspace , \tag{6.1}$$

where $\mathcal{G}_\Bbbk$ captures the attack against the construction $\pi_\Bbbk$ and $\mathfrak{A}(\Bbbk)$ is the set of all $t(\Bbbk)$-time adversaries. W.l.o.g. we can assume that $t(\Bbbk)$ is an increasing function and $\varepsilon(\Bbbk)$ is a decreasing function. Let us also define a *relative security margin*

$$\delta(\Bbbk) = \frac{t(\Bbbk)}{\tau(\Bbbk)} \enspace , \tag{6.2}$$

where $\tau(\Bbbk)$ denotes the overall running time of $\pi_\Bbbk$. Note that essential algorithmic and cryptographic properties of the *construction family* $(\pi_\Bbbk)$ are determined by the functions $\tau(\cdot), t(\cdot), \varepsilon(\cdot), \delta(\cdot)$. For example, one can determine appropriate security level $\Bbbk$. To achieve (t_0, ε_0)-security, one has to choose minimal $\Bbbk_0$ such that $\varepsilon(\Bbbk_0) \le \varepsilon_0$ and $t(\Bbbk_0) \le t_0$ provided that running time $\tau(\Bbbk_0)$ is still feasible in practice. To make similar economical decisions, one must consider rescalings that characterise true costs and gains. Since the underlying principle of analysis remains the same, we consider only the algorithmic aspects.

The choice of the security parameter is quite arbitrary, usually it describes important implementation details, such as the bit length of the RSA modulus. From a theoretical viewpoint it is advantageous to identify $\Bbbk$ with the maximal running time $\tau(\Bbbk)$ that is needed to complete the protocol $\pi_\Bbbk$. Then we must consider only the properties of $\varepsilon(\cdot)$ and $\delta(\cdot)$. For obvious reasons, a decreasing safety margin $\delta(\cdot)$ implies that the construction becomes more vulnerable in the future. More generally, a construction is sustainable in the long run only if the function $\delta(\Bbbk)$ is increasing and $\varepsilon(\Bbbk)$ is decreasing fast enough. Hence, it is tempting to analyse only the limiting behaviour of $\delta(\cdot)$ and $\varepsilon(\cdot)$.

When one considers asymptotic behaviour, polynomial-time algorithms are special. Mostly, because the class of polynomially bounded functions $\mathbf{poly}(\Bbbk)$ is the first nontrivial class of functions that contains the identity function and is closed under the multiplication and superposition. The latter is sufficient to prove that the notion of polynomial-time algorithms is machine independent and the corresponding set of algorithms is closed under recursive composition. Mainly because of these reasons, Cobham and Edmonds postulated already in

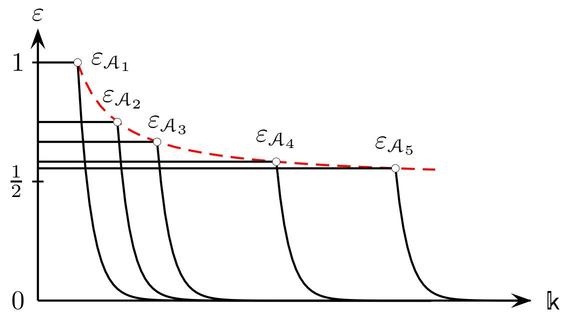


Figure 6.1: A construction may be completely insecure, although for each polynomial time bound $t(\Bbbk)$ the advantage $\varepsilon_{\mathcal{A}_i}(\Bbbk) \in \mathbf{negl}(\Bbbk)$ for any algorithm $\mathcal{A}_i$.

the early 1960s that the set of feasible and polynomial algorithms must coincide [Cob64, Edm65]. This description of tractability was quickly adopted by other theoreticians and it led to the birth of the complexity theory.

Cryptographers reached an analogous consensus and postulated that a construction family $(\pi_\Bbbk)$ is *asymptotically secure* if for all polynomial safety bounds $\delta(\Bbbk) \in \mathbf{poly}(\Bbbk)$ the corresponding advantage decreases faster than a reciprocal of any polynomial, i.e., $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. More precisely, we can talk about uniform and non-uniform polynomial security model. The essential difference between the models is in the definition of the set $\mathfrak{A}(\Bbbk)$. In the *non-uniform model*, the set $\mathfrak{A}(\Bbbk)$ consists of all $t(\Bbbk)$-time algorithms and thus represents objective security. The *uniform model* is a subjective security model, since the model assumes that an adversary uses a single self-tunable algorithm $\mathcal{A}$ for all attacks. Formally, the set $\mathfrak{A}(\Bbbk)$ consists of a single $t(\Bbbk)$-time algorithm $\mathcal{A}_\Bbbk$ that is obtained from $\mathcal{A}$ by feeding $\Bbbk$ as the first argument. As the universal algorithm $\mathcal{A}$ is unknown, the advantage $\varepsilon_{\mathcal{A}}(\Bbbk)$ must be asymptotically negligible for all algorithms $\mathcal{A}$ and polynomial time bounds $t(\Bbbk) \in \mathbf{poly}(\Bbbk)$.

The uniform security model is completely natural under a super-strong artificial intelligence assumption. If we *believe* that any intelligent behaviour can be outperformed by an artificial intelligence—an algorithm $\mathcal{A}$, then we may indeed set $\mathfrak{A}(\Bbbk) = \{\mathcal{A}_\Bbbk\}$ and study the limitations of intelligent reasoning. For more conservative readers, it is evident that uniform security model provides questionable security guarantees. Namely, security in the uniform model does not exclude the possibility that a potential adversary may achieve constant advantage by cleverly choosing algorithms for each instance, as depicted in Fig. 6.1. From this perspective, the naming convention is awfully misleading, as the uniform model assures only pointwise convergence of security guarantees, whereas the non-uniform model assures fast uniform convergence over all algorithms.

On a more practical note, observe that advances in computing technology show rather interesting trends. Although available computing power has grown in an exponential pace, the approximate ratio between ordinary and maximal available computing power has always been more or less constant with occasional fluctuations. Fig. 6.2 illustrates these trends by comparing the computational capabilities of widely accessible computing devices and the fastest supercomputers throughout the entire history of computing. In particular, note that the available adversarial power t has always been in the range

$$10 \cdot \tau \le t \le 100,000 \cdot \tau \quad , \tag{6.3}$$

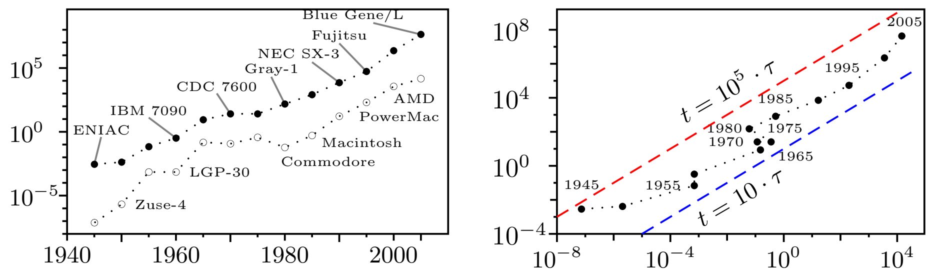


Figure 6.2: Comparison of conventional and supercomputers on the left and the corresponding dependence between honest and adversarial power on the right. Computing power measured in MIPS and the data is taken from [Mor98].

where τ is the power of an ordinary computing device. Of course, the capabilities of the fastest supercomputer do not adequately characterise the total computing power available in the world. On the other hand, this measure clearly indicates how much monetary resources organisations are willing to spend on extremely important computational tasks and thus the estimate is adequate for practical purposes. Consequently, one can argue that the polynomial security model is overly conservative in the context of short-term security. In particular, non-existence proofs in the asymptotic security models are not convincing, unless they hold for all superlinear time bounds $t(\Bbbk) \in \omega(\Bbbk)$. The latter was the main reason why we established such strong non-existence proofs in [LN06].

Note that the ratio between ordinary and adversarial power is bounded even for longer time periods provided that the growth rate is not super-exponential. Of course, we cannot make irrefutable conclusions, since the empirical evidence is rather limited and not collected for cryptographic purposes. Still, one should be extremely careful in interpreting results that are obtained in the asymptotic setting. Both positive and negative results obtained in the asymptotic security model are usually inconclusive, unless they have explicitly stated and practical time bounds. For example, the first proposal for the public key cryptosystem [Mer78]—so called Merkle's puzzle system—has only a quadratic relative safety margin and thus it is insecure in polynomial security model. However, in practice an ordinary computer can do 10^{11} operations in minutes, whereas 10^{22} operations takes roughly 1 year on Blue Gene/L. In other words, a quadratic security margin is already sufficient for short time security and the gap is growing with every year, since the computing power becomes cheaper and cheaper.

On the other hand, many constructions that are asymptotically secure under reasonable assumptions, such as Ajtai-Dwork cryptosystem [AD97], are insecure for all practical parameter sizes [NS98]. Going even further, note that a generic encoding of an **NP**-problem as a CNF-SAT formula leads to quadratic blowup and thus a quadratic safety margin might be preserved even if CNF-SAT problems can be solved in linear time. As a result, even a constructive proof for the statement **P** = **NP** does not rule out the existence of practical cryptographic primitives. Similarly, the existence of asymptotically secure cryptography does not rule out the existence of efficient adversarial algorithms for all practical time bounds. To summarise, life is not so dull as Impagliazzo's famous five world model [Imp95] predicts: cryptography might be a booming business even in

$$\begin{tikzcd}
& \text{strong-n-stat} \arrow[dl, Rightarrow, "/" marking] \arrow[d, Rightarrow, "/" marking] \arrow[dr, Rightarrow, "/" marking] & \\
\text{stat-u-post-proc} \arrow[d, Rightarrow, "/" marking] & \text{weak-n-stat} & \text{stat-n-post-proc} \arrow[ll, Rightarrow] \arrow[d, Rightarrow, "/" marking] \\
\text{strong-u-poly} \arrow[d, Rightarrow] & & \text{strong-n-poly} \arrow[ll, Rightarrow] \arrow[d, Rightarrow] \\
\text{weak-u-poly} & & \text{weak-n-poly} \arrow[ll, Rightarrow]
\end{tikzcd}$$

Figure 6.3: Relations between asymptotic security models. Hatched arrows denote implications with known separations, the others are believed to be strict. Symbols -u- and -n- denote uniform and non-uniform complexity.

Algorithmica, where the statement $\mathbf{P} = \mathbf{NP}$ holds; whereas obtaining exact security bounds might be troublesome in Cryptomania, where asymptotically secure one-way trapdoor permutations are guaranteed to exist.

6.2 ASYMPTOTIC SECURITY FOR PROTOCOLS

Classifying protocols according to asymptotic properties is not as simple as it seems at first glance. As before, assume the maximal total running time of all honest participants $\tau(\Bbbk) = \Bbbk$. We emphasise here that the bound τ must hold for any plausible attack pattern, i.e., τ quantifies the maximal workload that attacker can cause. Now regardless of the ideal world description, the security of a protocol π is characterised by a quadruple $(t_{\text{re}}, t_{\text{id}}, t_{\text{pr}}, \varepsilon)$. More precisely, the standard way to define the advantage[1] is the following

$$\varepsilon(\Bbbk) = \mathsf{cd}_{\star}^{t_{\text{pr}}(\Bbbk)}(\mathcal{G}^{\Bbbk}_{\text{re-atk}}, \mathcal{G}^{\Bbbk}_{\text{id-atk}}) \enspace , \tag{6.4}$$

where games $\mathcal{G}^{\Bbbk}_{\text{re-atk}}$ and $\mathcal{G}^{\Bbbk}_{\text{id-atk}}$ capture the effects of active attacks. The game $\mathcal{G}^{\Bbbk}_{\text{re-atk}}$ models the attack of $t_{\text{re}}(\Bbbk)$-time real world adversary $\mathcal{A}$ against the protocol $\pi_{\Bbbk}$. Similarly, $\mathcal{G}^{\Bbbk}_{\text{id-atk}}$ models the attack of $t_{\text{id}}(\Bbbk)$-time adversary $\mathcal{A}^{\circ}$ in the ideal world. Observe that time bounds $t_{\text{re}}(\cdot)$, $t_{\text{id}}(\cdot)$ and $t_{\text{pr}}(\cdot)$ can be arbitrarily fixed and thus there are several ways to define the limiting process $\Bbbk \to \infty$.

Strong and weak polynomial security model. Recall that actual security goals are always specified by individual security games and the ideal world model is just a mind experiment that simplifies the reasoning for such games. Hence, the polynomial security model for protocols should be consistent with the definition given in the previous section. Indeed, the standard definition postulates that a protocol family $(\pi_{\Bbbk})$ is *secure in a (strong) polynomial model* if for any $t_{\text{re}}(\Bbbk), t_{\text{pr}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists $t_{\text{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ such that $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. Hence, for any polynomial-time security game the advantage in the real and the ideal world differ by a negligible amount. As the ideal world adversary is also polynomial, the corresponding advantage $\varepsilon_{\mathcal{A}^{\circ}}(\Bbbk)$ cannot be avoided for any cor-

[1]In the previous chapter, we used a slightly more liberal notion, where the construction of $\mathcal{A}^{\circ}$ could depend on the target relation $\mathcal{B}$. But Section 7.3 shows that the latter is equivalent to the classical definition if we discard polynomial factors.

rect implementation of the desired functionality. Thus, a properly scaled advantage that omits the trivial success probability is indeed negligible.

However, soon after the wide acceptance of polynomial security model, cryptographers failed to prove the security of many interesting protocols. Remarkably, the first zero-knowledge proof [GMR85] is not secure in the strict polynomial model. The corresponding journal article [GMR89] had to explicitly relax security requirements and thus implicitly defined a weaker security model.

The weakened model bounds only the expected running time $\hat{t}_{\mathrm{id}}$ of the ideal adversary $\mathcal{A}^{\circ}$, where the average is taken over its coin tosses. The corresponding security definitions for zero-knowledge [GMR89, GMW91] and proofs of knowledge [BG92] require that for any $t_{\mathrm{re}}(\Bbbk), t_{\mathrm{pr}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists a bound on average running time $\hat{t}_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ such that $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. Although this change is seemingly marginal, it leads to several unexpected consequences, see [BL04, KL05]. The main drawback lies in the fact that a sequential composition theorem for zero-knowledge proofs [GO94] holds only in the strong polynomial model. Moreover, Barak and Lindell showed that there can be no constant round black-box zero-knowledge proofs for non-trivial problems in the strong polynomial model [BL04] and thus achieving constant round complexity and composability simultaneously is difficult. As a small consolation, they also showed that under reasonable cryptographic assumptions, there exists a constant round zero-knowledge protocol for any $\mathbf{NP}$ language with a white-box reduction but the corresponding security guarantees were quite loose.

Such a sad state of affairs has led to several technical refinements [KL05, Gol07] that avoid the unpleasant dilemma. Briefly, the refinements show that if the distribution of running time satisfies certain specific constraints, then the composition theorems hold for the model, where only the expected running times $\hat{t}_{\mathrm{re}}(\Bbbk)$ and $\hat{t}_{\mathrm{id}}(\Bbbk)$ are known to be polynomial. More importantly, almost all zero-knowledge proofs, including classical constant round proofs [GMW91, GK96a, FS89] for all $\mathbf{NP}$ languages, satisfy these constraints. Therefore, one can indeed avoid the choice between round efficiency and composability.

However, if one looks deeper into the issue, he or she soon discovers that the expected running time is just a camouflage for hiding unpleasant facts. Namely, certain properties of proofs of knowledge make the strict polynomial time black-box zero-knowledge impossible. In a nutshell, if a cheating prover succeeds with a probability $\varepsilon_{\mathrm{pok}}$, then the corresponding knowledge-extraction algorithm must work in time $\Theta(\varepsilon_{\mathrm{pok}}^{-1})$ to succeed with overwhelming probability. Such a lower bound on the running time has a natural explanation: a black-box knowledge-extractor can succeed only if it gets a valid proof of knowledge and the latter requires on average $\Omega(\varepsilon_{\mathrm{pok}}^{-1})$ repetitions. A corresponding zero-knowledge proof that uses a proof of knowledge as a sub-protocol runs in expected polynomial time only because with probability $1 - \varepsilon_{\mathrm{pok}}$ a cheating verifier fails in the proof of knowledge, and thus one has to run the slow knowledge-extraction algorithm with probability $\varepsilon_{\mathrm{pok}}$. Consequently, the unknown probability $\varepsilon_{\mathrm{pok}}$ cancels out and we are left with the expected running time $\hat{t}_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$.

However, such a trick does not solve the underlying intrinsic difficulties with knowledge extraction, since "useful computations" are done only in long runs. Moreover, the corresponding security guarantees are machine dependent, provide only average-case security, and have thus questionable interpretation in practice [BL04, p. 785]. Therefore, we choose a more obvious and sincere approach to relax polynomial security model. We say that a protocol family $(\pi_{\Bbbk})$ is *weakly secure in the polynomial model* if for any asymptotically non-negligible bound $\varepsilon(\Bbbk) \in \Omega(\Bbbk^{-c})$ and for any $t_{\mathrm{re}}(\Bbbk), t_{\mathrm{pr}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists an ideal world adversary with a strict running time $t_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$.

The corresponding security model is not novel at all. In fact, the original definition of proofs of knowledge [FFS87] was indeed given in the weak polynomial security model. Later the same concept resurfaced under the name of ε-knowledge [DNS04]. It is straightforward to show that the classical definition for proof of knowledge [BG92] also satisfies the original definition [FFS87] and thus all troublesome zero-knowledge proofs are secure in the weak model. More importantly, the security guarantees have clear interpretations and all composition theorems that hold in strong polynomial model also hold in the weak model. Hence, the author believes that the explicit definition of weak polynomial security model is better than various technical tweaks [KL05, Gol07].

Evidently, the strong polynomial security model is contained in the weak polynomial security model. Moreover, Barak and Lindell have proven the non-existence of constant round black-box zero-knowledge proofs for all languages $\mathcal{L} \notin \mathbf{BPP}$ [BL04]. Thus, the aforementioned inclusion is strict for black-box reductions under a widely believed assumption $\mathbf{NP} \neq \mathbf{BPP}$. For white-box reductions, it is not known whether the inclusion is strict or not. At the same time, it is straightforward to prove that if a protocol is secure in a weak non-uniform polynomial security model, then for any $t_{\mathrm{re}}(\Bbbk), t_{\mathrm{pr}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists an ideal world adversary with super-polynomial but sub-exponential running time $t_{\mathrm{id}}(\Bbbk) \in \Bbbk^{\omega(1)} \cap 2^{\mathrm{o}(\Bbbk)}$ such that the advantage is negligible $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$.

The proof is based on a simple diagonalisation argument. Fix bounds on the running times $t_{\mathrm{re}}(\Bbbk), t_{\mathrm{pr}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ and consider the sequence of decreasing bounds $\varepsilon_i(\Bbbk) = \Bbbk^{-i}$. Then, by the definition, we also get a corresponding sequence of asymptotic polynomial time bounds $t_{\mathrm{id},i}(\Bbbk) \leq \Bbbk^{c_i}$ for large enough $\Bbbk$. Now, if we fix a sequence of switching points $(\Bbbk_i)_{i=1}^{\infty}$ and use the i^{th} reduction in the range $\Bbbk \in [\Bbbk_i, \Bbbk_{i+1})$, then by the construction the resulting bound on the advantage $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. To complete the proof, we must choose switching points so that the running time is sub-exponential. If we take $\Bbbk_i = 2^{c_{i+1}}$, then $t_{\mathrm{id}} \in \mathrm{O}(\Bbbk^{\log \Bbbk})$ but in principle, any super-polynomial bound is achievable.

Analogously, we can define the strong and the weak uniform polynomial security model. However, these notions make sense only if the protocol inputs are generated by a uniform polynomial algorithm. Otherwise, the adversary can use the inputs of corrupted participants as an external advice and we are still in the standard non-uniform setting. This additional requirement for inputs makes it difficult to interpret the resulting security guarantees. Hence, uniform security models are seldomly used. Nevertheless, it is the only plausible choice when the basic primitives are secure solely in the uniform model.

Three alternatives for statistical security. The first and the most obvious choice is to consider adversaries that are computationally bounded on the online phase of the attack, but afterwards have infinite time to analyse the obtained results. More formally, let us consider redefined advantage

$$\varepsilon(\Bbbk) = \mathsf{sd}_{\star}(\mathcal{G}^{\Bbbk}_{\text{re-atk}}, \mathcal{G}^{\Bbbk}_{\text{id-atk}}) \ . \tag{6.5}$$

Then a protocol family $(\pi_{\Bbbk})$ is *secure against polynomial online and unbounded post-processing attacks* if for any $t_{\mathrm{re}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists $t_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$

such that $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. Similarly to the discussion above, we can relax the security conditions, i.e., a protocol family $(\pi_\Bbbk)$ is *weakly secure against polynomial online and unbounded post-processing attacks* if for any non-negligible error bound $\varepsilon(\Bbbk) \in \Omega(\Bbbk^{-c})$ and for any $t_{\mathrm{re}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$ there exists a polynomial bound $t_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk)$. Since the adversary can store all received messages, two-party protocols that implement non-trivial functionality cannot be secure against unbounded post-processing attacks, see for example [CK89]. Nevertheless, these models are meaningful for certain two-party protocols that implement limited functionality. Various settings for statistical and perfect zero-knowledge (such as [BC86, Cha86, For87]) are most utilised examples but there are others, such as data authentication, see for example [BR93a].

Many multi-party protocols can also handle online attacks with unbounded time complexity. However, there are two ways of relaxing the online complexity. The most simplistic way is to postulate that the adversary has an indeed infinite computing power and thus the computational complexity is irrelevant. Under such assumptions it is natural to say that a protocol family $(\pi_\Bbbk)$ is *secure in the weak statistical security model* if for any time bound $t_{\mathrm{re}}(\Bbbk)$ there exists a corresponding time bound $t_{\mathrm{id}}(\Bbbk)$ that achieves $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. Although the definition limits the amount of extra information the adversary can gain, it does not limit the extra computational power that he or she can achieve by participating in the protocol. As a result, a protocol that is secure in the weak statistical model can still be insecure in the polynomial model. An excellent example of such a protocol is given in the seminal article [Can00a, p. 159].

Alternatively, we can require that the running times of the real and the ideal world adversaries are comparable for all functions $t_{\mathrm{re}}(\Bbbk)$. More formally, a protocol family $(\pi_\Bbbk)$ is *secure in the strong statistical security model* if for any function $t_{\mathrm{re}}(\Bbbk)$ there exists a bound $t_{\mathrm{id}}(\Bbbk) \in \mathbf{poly}(\Bbbk) \cdot t_{\mathrm{re}}(\Bbbk)$ such that $\varepsilon(\Bbbk) \in \mathbf{negl}(\Bbbk)$. This definition assures that the adversary cannot use the protocol to significantly speed up his or her computations. Thus, the strong statistical security model is a natural strengthening of all other asymptotic security models. The latter makes it the most dominant in the literature [Gol04, Can00a]. Also, all generic multi-party computation techniques [BOGW88, CCD88, GRR98] satisfy this definition. We emphasise that the strong statistical security model limits the discrepancy between the time-success profiles of the real and the ideal world. Intuitively, strong statistical security assures that a collapse of a computational basic primitive cannot cause abrupt jump in the time-success profile.

The weak statistical security model loses this correspondence between the time-success profiles of the real and the ideal world. As a result, the weak statistical security model has several weird and unexpected properties. Nevertheless, the definition is externally consistent. If one really assumes that the adversary has infinite computational resources, the gain in computational power is irrelevant and the honest parties must accept consequences of all possible attacks in the ideal world. In some practical applications, honest parties indeed care about specific properties that hold in the ideal world even for unbounded adversaries. For example, the concept of weak statistical security is often used in the context of oblivious transfer [AIR01, Lip05] and other two-party privacy-preserving operations, see for example [FNP04, FIPR05, KS05, LL07].

6.3 GENTLE INTRODUCTION TO SUBJECTIVE SECURITY

At first glance, the concept of subjective security seems a bit odd, not to say inappropriate, for a rational scientific discipline, such as cryptography. As scientists, we should seek impartial and well-justified descriptions of security threats and avoid any subjective and questionable assumptions on the adversarial behaviour. However, a closer inspection reveals that such models are quite common in mainstream literature. Even the celebrated uniform security model is strongly subjective, as there are no *objective* reasons to assume that an attacker will always use the same generic solution for all instances of a protocol.

The random oracle model is another widely used but often misunderstood security model that provides only subjective security guarantees. Bellare and Rogaway proposed the random oracle model [BR93b] as an idealised computational model to mimic hash functions. More formally, assume that a function h is chosen uniformly from the set $\mathcal{H}_{\text{all}}$ where $\mathcal{H}_{\text{all}} = \{h : \mathcal{M} \to \mathcal{T}\}$ consists of all functions from the set $\mathcal{M}$ to the set $\mathcal{T}$. Now, let $\mathcal{O}$ be an oracle that provides black-box access to the function h, i.e., given $m \in \mathcal{M}$ the oracle replies $h(m)$. As a result, we have defined a computational model, where the oracle $\mathcal{O}$ can be used as an idealised hash function from the set $\mathcal{M}$ to the set $\mathcal{T}$.

Assume that a game $\mathcal{G}$ describes the desired properties of a construction or a protocol. Then the definition of the random oracle model yields

$$\mathsf{Adv}^{\text{rom}}_{\mathcal{G}}(\mathcal{A}) = \frac{1}{|\mathcal{H}_{\text{all}}|} \cdot \sum_{h \in \mathcal{H}_{\text{all}}} \Pr\left[\mathcal{G}^{\mathcal{A},\mathcal{O}} = 1 | \mathcal{O}(\cdot) = h(\cdot)\right] \enspace . \tag{6.6}$$

In practice, one must substitute $\mathcal{H}_{\text{all}}$ with another function family $\mathcal{H} \subseteq \mathcal{H}_{\text{all}}$ that has a more succinct description and can be efficiently evaluated. As a result, the advantage in the standard model is computed over a much smaller sample

$$\mathsf{Adv}^{\text{std}}_{\mathcal{G}}(\mathcal{A}) = \frac{1}{|\mathcal{H}|} \cdot \sum_{h \in \mathcal{H}} \Pr\left[\mathcal{G}^{\mathcal{A},\mathcal{O}} = 1 | \mathcal{O}(\cdot) = h(\cdot)\right] \enspace . \tag{6.7}$$

A small sample size $|\mathcal{H}| \ll |\mathcal{H}_{\text{all}}|$ by itself is not a problem. If a family $\mathcal{H}$ is randomly sampled from $\mathcal{H}_{\text{all}}$ then by the central limit theorem

$$\mathsf{Adv}^{\text{rom}}_{\mathcal{G}}(\mathcal{A}) \approx \mathsf{Adv}^{\text{std}}_{\mathcal{G}}(\mathcal{A}) \tag{6.8}$$

holds with overwhelming probability over the choice of $\mathcal{H}$. However, the function family $\mathcal{H}$ must have a very short description and the latter introduces a possible bias into the estimate (6.8). Hence, a security proof in the random oracle model is convincing only under a *subjective* equivalence assumption that the approximation (6.8) still holds for the set of all *relevant* adversaries $\mathfrak{A}$.

Here, the concept of *relevant adversary* $\mathcal{A} \in \mathfrak{A}$ needs further clarification. Although humans have admirable intellectual abilities, these powers are still limited. Therefore, humans are likely to choose a suboptimal adversarial strategies and one should not worry about optimal attacks that never materialise. The set $\mathfrak{A}$ is meant to denote all algorithms that a mankind can devise. Of course, it is impossible to describe the set $\mathfrak{A}$ ahead and thus we can *objectively* verify the claim (6.8) only after the attacks have taken place.

Nevertheless, one can try to describe a set of algorithms such that the subjective equivalence assumption is satisfied for this set. Assume that $\mathcal{H}$ is a (t, ε)-pseudorandom function family, i.e., the security games

$$\begin{array}{ll} \mathcal{Q}_0^{\mathcal{A}} & \qquad\qquad \mathcal{Q}_1^{\mathcal{A}} \\ \left[\begin{array}{l} h \leftarrow \mathcal{H} \\ \textbf{return } \mathcal{A}^{\mathcal{O}} \end{array}\right. & \qquad\qquad \left[\begin{array}{l} h \leftarrow \mathcal{H}_{\mathrm{all}} \\ \textbf{return } \mathcal{A}^{\mathcal{O}} \end{array}\right. \end{array}$$

are (t, ε)-indistinguishable. Now, consider a class of generic t-time attack algorithms $\mathfrak{A}_{\mathrm{bb}}$ that always evaluate h in a *black-box* manner. By definition

$$\mathsf{Adv}_{\mathcal{G}}^{\mathrm{std}}(\mathcal{A}) \leq \mathsf{Adv}_{\mathcal{G}}^{\mathrm{rom}}(\mathcal{A}) + \varepsilon, \tag{6.9}$$

for any $\mathcal{A} \in \mathfrak{A}_{\mathrm{bb}}$, otherwise we can use the adversary $\mathcal{G}^{\mathcal{A},\cdot}$ to distinguish between the games $\mathcal{Q}_0$ and $\mathcal{Q}_1$. Thus, any attack that significantly violates the bound (6.9) must use the description of h in an essential way. As such an attack is specialised for h, it is likely not to succeed for other functions $h^* \in \mathcal{H}$. At the same time, a black-box adversary $\mathcal{A} \in \mathfrak{A}_{\mathrm{bb}}$ can still exploit the description of the family $\mathcal{H}$ and thus the corresponding attacks can be quite specific. Hence, security in the random oracle model is a reasonable starting point whenever the hash function must exhibit pseudorandom properties in the construction.

The fact that the random oracle model is a subjective security model makes it quite different from objective security models. In particular, the validity of the subjective equivalence assumption (6.8) is not automatic and must be individually re-evaluated for each specific construction. Even a single decision is not universal, since different persons may reach different conclusions. As a result, a specific subjective equivalence assumption that is associated with a specific construction π can be falsified only by showing a concrete algorithm $\mathcal{A}_0$ that for this concrete construction π achieves

$$\mathsf{Adv}_{\mathcal{G}_\pi}^{\mathrm{rom}}(\mathcal{A}_0) \ll \mathsf{Adv}_{\mathcal{G}_\pi}^{\mathrm{std}}(\mathcal{A}_0) \enspace . \tag{6.10}$$

Since one must make an individual subjective decision for each construction separately, the subjective equivalence assumption cannot be falsified by showing the existence of specially crafted protocols π_* that indeed satisfy (6.10). If we acknowledge this essential property of the model, the wave of critique [CGH04b, GK03, BBP04, MRH04, CGH04a] started by Canetti, Goldreich and Halevi falls apart. Essentially, all these results show that the subjective equivalence assumption (6.8) is not universal. But the latter is an expected result, otherwise the assumption (6.8) would be a valid mathematical claim.

The random oracle model is not the only widely used computational model that forces cryptographers to make subjective assumptions in order to interpret the results. All so-called black-box models starting from the ideal cipher model already proposed by Shannon [Sha49] and ending with the generic group and ring models [Nec94, Sho97, LR06] force cryptographers to make subjective equivalence assumptions similar to the approximation (6.8). Again, several extensions [Den02, Bla06a] of basic techniques [CGH04b, MRH04] indeed show that the corresponding equivalence assumption is not universal.

Finally, let us show that such a subjective equivalence assumption can be disputed in a more persuading manner. Assume that instead of a random oracle one uses standard iterative hash function family like SHA-1 and WHIRLPOOL. The

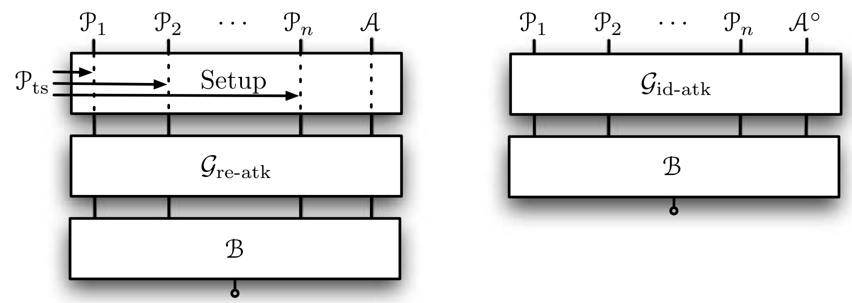


Figure 6.4: Real and ideal world models for a protocol with a trusted setup phase.

vanilla hash function f^*_{iv} without an appropriate padding is computed iteratively according to the Merkle-Damgård construction

$$h(x) = f^*_{\mathsf{iv}}(x_1, \dots, x_k) = f(f^*_{\mathsf{iv}}(x_1, \dots, x_{k-1}), x_k) \ , \tag{6.11}$$

where $f : \{0,1\}^m \times \{0,1\}^n \to \{0,1\}^m$ is a round function, iv is a m-bit initial value and x is parsed as a vector $(x_1, \dots, x_k)$ of n-bit blocks. Consequently, we have a hash function family that is indexed by the initial value iv:

$$\mathcal{H}_f = \{f^*_{\mathsf{iv}} : \{0,1\}^{n^*} \to \{0,1\}^n\}_{\mathsf{iv} \in \{0,1\}^m} \ . \tag{6.12}$$

Now given a value $f^*_{\mathsf{iv}}(x_1)$ and the black-box access to the round function f, it is straightforward to compute $f^*_{\mathsf{iv}}(x)$ and thus it becomes trivial to distinguish between the function families $\mathcal{H}_f$ and $\mathcal{H}_{\text{all}}$.[2] Hence, the inequality (6.9) fails for a general class of adversaries that access functions f^*_{iv} and f in a black-box manner. In more explicit terms, there might exist a universal black-box algorithm $\mathcal{A}$ that breaks the construction for all possible iterative hash functions. Hence, one should use a one-way function g to get the final hash value $h(x) = g(f^*_{\mathsf{iv}}(x_1, \dots, x_n))$ in order to reduce the amount of subjective trust.

6.4 SETUP ASSUMPTIONS AND SUBJECTIVE SECURITY

In a certain sense, it is impossible to avoid subjective security. Even if we try to prove the security in the objective non-uniform computational model, we soon reach fundamental limitations of hypothesis testing. Many cryptographic protocols are specified as a part of a communication standard, such as the DSS standard [Nat00] for digital signatures, or the TLS v1.1 standard [Net06] for secure web browsing. Such standards almost always explicitly fix some cryptographic primitives, for example, state that messages must be hashed with SHA-1. Hence, contrary to the belief of most theoreticians, almost all cryptographic applications are implemented in the *common reference string model,* where all protocol participants can access authentically distributed setup parameters.

Such a setting is formally modelled by a *trusted setup procedure*, where a non-corruptible party $\mathcal{P}_{ts}$ sends some initial parameters to the participants of the protocol. Usually, it is sufficient if the communication between the dealer $\mathcal{P}_{ts}$ and other participants is authentic but some settings also require confidentiality.

[2]The argumentation can be generalised for the actual hash functions with a padding.

For example, the Digital Signature Standard fixes some recommended parameters including SHA-1 as a preferred hash function. Another class of similar examples consists of various theoretical models for public-key infrastructure.

From a technical viewpoint, analysis of a protocol with a trusted setup procedure is not different from ordinary protocol, except that the real world model has an extra phase that is missing in the ideal world, see Fig. 6.4. A bigger, not to say an essential, problem lies in the fact that the corresponding objective security model does not faithfully reflect the reality:

- For many standards, the setup procedure is run only once. Consequently, one really cares how successful a potential adversary is for this *particular* outcome of the setup procedure, whereas the objective security finds only an *average-case* success bound over all possible runs of the setup.
- Public setup parameters psp are often fixed for years. Thus, an adversary can and should *in principle* devise a specific attack that utilises the properties of psp. No objective security model can capture such a dependence, as the algorithms are always chosen *before* the parameters are generated.

We emphasise that the problem lies in the fundamental mismatch between individual and collective properties of the setup procedure and thus cannot be resolved by clever modifications of security games.

It is instructive to consider a couple of concrete examples before giving more abstract reasoning. Recall that a function family $\mathcal{H}$ is (t, ε)-collision resistant if

$$\mathsf{Adv}^{\mathrm{cr}}_{\mathcal{H}}(\mathcal{A}) = \Pr\left[h \leftarrow \mathcal{H}, (x_0, x_1) \leftarrow \mathcal{A}(h) : h(x_0) = h(x_1) \wedge x_0 \neq x_1\right] \leq \varepsilon$$

for any t-time adversary $\mathcal{A}$. Observe that collision resistance is a collective property of the function family $\mathcal{H}$. For a fixed non-injective function h, the notion of collision resistance does not make sense, since there exists a trivial adversary $\mathcal{A}_{x_0,x_1}$ that outputs a fixed collision $h(x_0) = h(x_1)$. Such discrepancy between individual and collective properties leads to a hashing paradox:

- It is impossible to *objectively use* collision resistant hash functions, even if one has an explicit description of a collision resistant function family $\mathcal{H}$.
- Any protocol that is compromised by a hash collision $h(x_0) = h(x_1)$ becomes insecure in the *objective sense* as soon as h is chosen from $\mathcal{H}$, as we can then use a fixed algorithm $\mathcal{A}_{x_0,x_1}$ to break the protocol.

The problem is not specific to any practical hash function, such as SHA-1 or WHIRLPOOL, nor to the way they have been designed. The hashing paradox holds for any imaginable compressing function. In particular, it is impossible to talk about the *objective security* of a digital signature scheme that uses a fixed hash function to compress messages before signing them.

As a second example, we present a similar paradox for asymmetric encryption. Observe that IND-CPA security is again a collective property of a cryptosystem. For a fixed public key pk there exists a trivial adversary $\mathcal{A}_{\mathsf{sk}}$ that uses a hardwired secret key sk to successfully decrypt all messages encrypted by $\mathsf{Enc}_{\mathsf{pk}}(\cdot)$. Consequently, we can state an encryption paradox:

- It is impossible to *use* an *objectively* confidential asymmetric enciphering method even if one has explicit description of an IND-CPA cryptosystem.

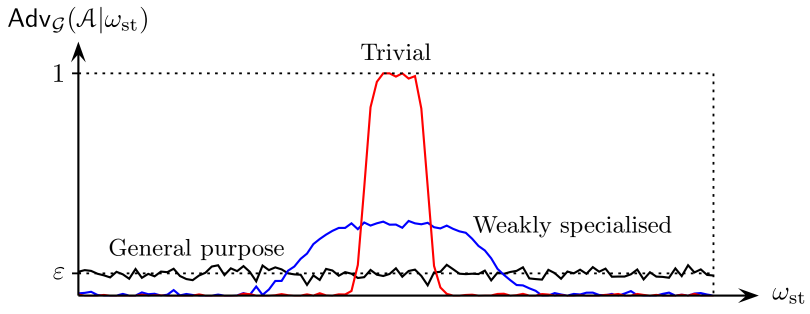


Figure 6.5: The advantage $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}})$ as a function of randomness ω_{ts} used by the dealer $\mathcal{P}_{\mathrm{ts}}$ for various different types of attacking algorithms.

- Any protocol that uses asymmetric encryption becomes insecure in the *objective sense* as soon as the public key pk is officially published.

A similar paradox emerges for digital signatures. Moreover, this problem cannot be ignored as a weird theoretical observation that has no practical consequences, since the public-key infrastructures are common in everyday life. We have enjoyed secure web-browsing from 1994, when Netscape developed the first version of SSL protocol, and many countries are deploying country-wide official public-key infrastructures. All such infrastructures collapse if one obtains a master secret key that is used to guarantee the authenticity of all public keys. Hence, a potential adversary needs to crack a single *fixed* master key.

For clarity, assume that the master secret key is a full factorisation of an RSA modulus $N = p \cdot q$. Now finding a factorisation of a specific integer N might be much easier than the factoring problem in general. For example, the state of the art factoring efforts have produced a full factorisation of a 1039-bit Mersenne number, whereas the current record of factoring RSA moduli is 640-bits [AFK$^+$07, RSA07]. However, if an adversary can specify an algorithm for the concrete modulus N, then we cannot exclude a trivial algorithm $\mathcal{A}_{p,q}$ that just prints out the corresponding factors p and q. Thus, no *objective* security model can capture the intractability of factoring for a particular modulus N.

The essence of the trusted setup problem is revealed, if we consider an advantage of a t-time algorithm $\mathcal{A}$ as a function of random coins used by $\mathcal{P}_{\mathrm{ts}}$

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1 | \mathcal{P}_{\mathrm{ts}} \text{ uses randomness } \omega_{\mathrm{ts}}\right] \enspace . \tag{6.13}$$

As the overall advantage can be computed as an average

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) = \frac{1}{|\Omega_{\mathrm{ts}}|} \cdot \sum_{\omega_{\mathrm{ts}} \in \Omega_{\mathrm{ts}}} \mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}}) \enspace , \tag{6.14}$$

the objective security bounds limit only the area under the profile, as illustrated in Fig. 6.5. Now the advantage for a fixed setup run is just a single point in the profile of $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}})$. Consequently, objective setting for a fixed setup run is meaningful only if for all t-time adversaries $\mathcal{A}$ and randomness $\omega_{\mathrm{ts}} \in \Omega_{\mathrm{ts}}$

$$\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}}) \le \varepsilon_0 \ll 1 \enspace . \tag{6.15}$$

But then the setup procedure is redundant, as we can fix all random coins $\omega_{\mathrm{ts}} = 0 \ldots 0$ and let all parties compute the setup messages by themselves. Analogously, we can remove the setup procedure even if the inequality (6.15) holds only for a single choice of random coins $\omega^*_{\mathrm{ts}} \in \Omega_{\mathrm{ts}}$.

Hence, for any *meaningful* setup procedure, there must exist a trivial attack for any protocol run. We can eliminate them, but the corresponding choice between weakly specialised and trivial attacks is inherently *subjective*. Fortunately, there is a *heuristic* bridge between objective and subjective security.

Fairness Postulate. *If the setup procedure has been properly carried out, it is rational to assume that the value* $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}})$ *exceeds an objective bound* $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}) \leq \varepsilon$ *only by several magnitudes for all relevant algorithms* $\mathcal{A} \in \mathfrak{A}$.

Of course, it is impossible to prove the postulate, we can only justify it. Since the area under the profile is bounded by ε, the advantage $\mathsf{Adv}_{\mathcal{G}}(\mathcal{A}|\omega_{\mathrm{ts}})$ can exceed $c \cdot \varepsilon$ only for a fraction $\frac{1}{c}$ of all $\omega_{\mathrm{ts}} \in \Omega_{\mathrm{ts}}$. Hence, algorithms with ultra-high advantages must be extremely specialised and should be left out. Also, if an attacker manages to find an ultra-specialised algorithm $\mathcal{A}$ for a significant fraction of $\omega_{\mathrm{ts}} \in \Omega_{\mathrm{ts}}$, then the attacker itself as a self-tuning attacking strategy exceeds the objective security bound. Evidently, we cannot exclude such super-intelligent adversaries but in that case there is nothing we could do anyway.

Finally, we emphasise that objective and subjective security are not conflicting concepts but rather complementary formalisations. Objective security provides a rigorous way to design protocols that are resistant to general purpose attacks, whereas subjective security is unavoidable if we want to analyse security guarantees after some parameters are fixed. A similar duality between the design and the actual usage also appears in the statistics. It is possible to design statistical estimation algorithms that behave well on average, but a meaningful interpretation of the resulting outputs must use subjective probabilities.

6.5 RIGOROUS FORMALISATION OF SUBJECTIVE SECURITY

The fairness postulate has the same drawbacks as the subjective equivalence assumption used in the random oracle model. Both of them rely on inherently subjective decisions, which might turn out to be inappropriate in retrospection. Therefore, one should apply them only in very simplistic settings and use the resulting subjective security premises as stepping stones in the analysis of more complex constructions. Such an approach reduces the amount of subjectivity by locating the essential subjective security premises one must believe.

A proper formalisation of subjective security premises and proofs is needed for other more technical reasons as well. First, it is not evident at all that excessive application of the fairness postulate or its analogues does not lead to inconsistencies. Moreover, there are no objectively secure cryptographic primitives known so far and thus the fairness postulate is *de facto* inapplicable. Additionally, it is not apparent at all that compact direct proofs do exist in the computational setting, even if primitives like one-way functions exist. In other words, we do not know whether one can ever provide a complete security proof for a specific cryptographic primitive that is verifiable in reasonable time. As a result, the subjective security might be truly unavoidable. Finally, a proper formalisation of subjective security is interesting in its own right, as it clarifies what we actually mean by subjective security. In particular, what is the main difference between security proofs in objective and subjective settings.

It turns out that most proofs can be freely translated from the objective setting to the subjective setting and vice versa, and that the main difference is only in the exact bounds on the running times. The latter is somewhat expected, as the uniform polynomial security model is also subjective and so far most of the published security proofs hold for both polynomial security models.

Formalisation of subjective security. At first glance, it seems impossible to formalise the intuition that some algorithms are not accessible to adversary, since we do not know how the algorithms are generated. Indeed, we showed above that such a formalisation is impossible in the classical frequentistic setting. However, if we accept the subjectivity of probabilities, the solution becomes evident. Namely, we must assign *subjective occurrence properties* to all possible adversarial algorithms to exclude ultra-specialised attacks.

More formally, one must specify a distribution of algorithms $\mathfrak{A}$ for a particular problem. A choice of the corresponding distribution is entirely subjective, as it reflects personal beliefs on which algorithms are more likely to be used in potential attacks. In practice, one does not have to specify the entire distribution, it is sufficient to fix several *unquestionable beliefs* about adversarial distribution and then use principles of coherent reasoning. A fully determined distribution $\mathfrak{A}$ is needed to formally define a *relative advantage against a game* $\mathcal{G}$

$$\mathsf{Adv}_{\mathcal{G}}(\mathfrak{A}) = \Pr\left[\mathcal{A} \leftarrow \mathfrak{A} : \mathcal{G}^{\mathcal{A}} = 1\right] \tag{6.16}$$

and a relative *distinguishing advantage for a game pair* $\mathcal{G}_0$ *and* $\mathcal{G}_1$

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathfrak{A}) = \left|\Pr\left[\mathcal{A} \leftarrow \mathfrak{A} : \mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{A} \leftarrow \mathfrak{A} : \mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \ . \tag{6.17}$$

We emphasise that the distribution $\mathfrak{A}$ does not have to be not uniform. Hence, the success probabilities of more probable algorithms $\mathcal{A}$ have a much bigger impact to $\mathsf{Adv}_{\mathcal{G}}(\mathfrak{A})$ and $\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathfrak{A})$ than other less probable algorithms.[3]

Observe that it is reasonable to specify different distributions for different time bounds t, as some attacks might require a certain amount of time to be useful at all. Hence, we use a shorthand $\mathfrak{A}(\cdot)$ to denote a family of distributions, where $\mathfrak{A}(t)$ denotes a distribution of t-time adversaries. Now a *relative computational distance with respect to a time bound* t is defined as

$$\mathsf{cd}^{\mathfrak{A}(t)}_{\star}(\mathcal{G}_0,\mathcal{G}_1) = \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathfrak{A}(t)) \ . \tag{6.18}$$

As an example, recall that in the uniform polynomial model, a distribution family $\mathfrak{A}(\cdot)$ consists of a single adversary $\mathcal{A}$ that gets a security parameter $\mathbb{k}$ as an extra argument. Also, note that even a non-uniform adversary can be modelled by a distribution family, where $\mathfrak{A}(t)$ consists of a single adversary $\mathcal{A}_t$ that is optimal over the set of all t-time adversaries.

Now, as the distribution family $\mathfrak{A}(\cdot)$ puts restrictions on the set of plausible adversaries, the security with respect to fixed parameters becomes meaningful.

[3]Of course, there are no objective measures for determining the right occurrence probabilities for all algorithms, since we cannot objectively predict what "useful" algorithms will be discovered and used in the future. Hence, these notions are inherently subjective.

In particular, the following instantiations of IND-CPA games

$$
\begin{array}{ll}
\mathcal{Q}_0^{\mathcal{A}} & \mathcal{Q}_1^{\mathcal{A}} \\
\left[\begin{array}{l}(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_0) \\ \textbf{return } \mathcal{A}(c)\end{array}\right. & \left[\begin{array}{l}(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_1) \\ \textbf{return } \mathcal{A}(c)\end{array}\right.
\end{array}
$$

where pk is a fixed valid public key, might lead to nontrivial bounds on the relative computational distance $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{Q}_0, \mathcal{Q}_1)$. In other words, the subjective notion of IND-CPA security is meaningful also for a fixed public key pk.

Principles of coherent reasoning. Since a rational entity is free to choose subjective probabilities, it is impossible to formally prove the (in)validity of security assumptions. We can only try to eliminate inconsistencies between various beliefs in order to achieve coherence. Therefore, we are going to mimic the classical approach that is used to formalise subjective probabilities, see handbooks [Jay03, Jef04]. In a nutshell, we assume that all subjective probability distributions are assigned by a *rational entity* who is willing to correct his or her assignments if he or she finds out that they contradict common sense. We specify this vague definition by fixing a set of axioms (R1)–(R3) that directly follow from the common-sense understanding of attack strategies.

For brevity, we only consider probability assignments for game pairs, since a relative advantage $\mathsf{Adv}_{\mathcal{G}}(\mathfrak{A})$ can be restated as a relative computational distance $\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G},\perp}(\mathfrak{A})$, where $\perp$ denotes a game that always ends with $\perp$. Assume that a rational entity has specified different adversarial distributions $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ for different problems. In other words, the rational entity acknowledges that an attacker can sample an algorithm from these *candidate distributions* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$. Then it is irrational to assume that a potential attacker cannot use the candidate distribution $\mathfrak{A}$ for other problems. Similarly, it is irrational to assume that an attacker cannot use a well-known code transformation $\mathcal{T}$ to convert a sampled adversarial algorithm $\mathcal{A}$ to a different algorithm $\mathcal{B} \leftarrow \mathcal{T}(\mathcal{A})$. For example, there is no reason to refute the possibility that an attacker may encode a factoring task as a Boolean formula if he or she has access to a very efficient solver for Boolean formulae. In other terms, well-known transformations introduce new candidate distributions $\mathcal{T}(\mathfrak{A}), \mathcal{T}(\mathfrak{B}), \mathcal{T}(\mathfrak{C}), \ldots$. However, such transformations are not for free, since the application of the transformation itself takes time. Shortly, if $\mathcal{B} \leftarrow \mathcal{T}(\mathcal{A})$, then the time needed to compute $\mathcal{T}(\mathcal{A})$ is added to the running time of $\mathcal{B}$. As a result, we have discovered three basic coherence axioms that reveal inconsistencies between subjective probability assignments:

(R1) Let $\mathcal{T}$ be a code transformation *known* to a rational entity and $\mathfrak{A}$ be a candidate distribution for distinguishing $\mathcal{G}_0$ and $\mathcal{G}_1$. Then a distribution $\mathfrak{B} = \mathcal{T}(\mathfrak{A})$ induced by $\mathcal{T}$ and $\mathfrak{A}$ must also be a candidate distribution.

(R2) Let $\mathfrak{B} = \mathcal{T}(\mathfrak{A})$ as specified above. Then the time needed to compute $\mathcal{T}(\mathcal{A})$ from a sample $\mathcal{A} \leftarrow \mathfrak{A}$ must be added to the running time of $\mathcal{T}(\mathcal{A})$.

(R3) Let $\mathfrak{A}(t)$ and $\mathfrak{B}(t)$ be two candidate distributions for distinguishing $\mathcal{G}_0$ and $\mathcal{G}_1$. Now if the entity *knows* that $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_1) < \mathsf{cd}_\star^{\mathfrak{B}(t)}(\mathcal{G}_0, \mathcal{G}_1)$ then the entity must prefer the distribution $\mathfrak{B}(t)$ to the distribution $\mathfrak{A}(t)$.

The verb ‘know’ and the term ‘knowledge’ are used here to denote explicit awareness, i.e., an entity who reasons about security must literally know the description of a transformation $\mathcal{T}$ or possess a valid proof that the inequality $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{G}_0,\mathcal{G}_1) < \mathsf{cd}_\star^{\mathfrak{B}(t)}(\mathcal{G}_0,\mathcal{G}_1)$ holds. In particular, a person who knows nothing about cryptography can arbitrarily assign adversarial distributions to various problems, since any assignment is coherent with his or her knowledge about these problems. When the person gains more knowledge, then certain assignments become irrational and thus he or she must sometimes accept that certain constructions are secure. Nevertheless, note that a set of facts rarely determines unique candidate distributions for all problems and thus entities can reach different conclusions even if they share the same set of facts. In other words, a rational entity can and should discover inconsistencies according to his or her knowledge, but a certain amount of subjectivity is unavoidable in his or her assignments.

Note that all security proofs must be strongly constructive in the subjective setting, since a rational entity must explicitly know a distribution $\mathfrak{B}$ that is a better alternative than a distribution $\mathfrak{A}$. A mere existence of a better alternative does not cause inconsistencies as long as the rational entity is not informed. Such a position is unavoidable, or otherwise we cannot escape the objective setting. If we require consistency for all possible transformations, then we also allow trivial transformations that ignore the input code and output an optimal adversary for a particular problem and we back in the objective setting.

Finally, observe that the axiomatisation cannot lead to contradictory restrictions for adversarial distributions as long as the objective security model itself is non-contradictory. Namely, a rational entity can always assume that a distribution $\mathfrak{A}(t)$ for a game pair $\mathcal{G}_0$ and $\mathcal{G}_1$ consist of a single t-time adversary $\mathcal{A}$ that maximises the advantage $\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A})$. Consequently, there can be no better alternatives for the distribution $\mathfrak{A}(t)$ and none of the axioms (R1)–(R3) is violated. Note that this conservative assignment of adversarial distributions corresponds to the objective setting and thus must be well defined.

6.6 REDUCTIONS AND SUBJECTIVE SECURITY PREMISES

The axioms of coherent reasoning (R1)–(R3) alone are not sufficient to prove the subjective security of a protocol or a construction, as they just reveal the inconsistencies. A decision how such inconsistent assignments should be changed is also a subjective decision and can depend on personal preferences of rational entities. To solve the ambiguity, one must fix a small set of *fundamental beliefs* that is never changed if it is possible to achieve coherence otherwise. These beliefs will play the role of subjective security assumptions. Formally, a *basic security premise* $[\![\mathcal{Q}_0 \cong \mathcal{Q}_1|(t,\varepsilon)]\!]$ is an unquestionable belief that the relative computational distance satisfies $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{Q}_0,\mathcal{Q}_1) \leq \varepsilon$ for any achievable distribution family $\mathfrak{A}(\cdot)$. As a result, we can formalise the subjective (t,ε)-collision resistance for a function h and (t,ε)-IND-CPA security for a public key pk by adding the corresponding security premises into the set of fundamental beliefs.

We emphasise here that fundamental beliefs are empirically falsifiable. For example, the MD5 hash function was believed to be collision resistant in 1992 but the publication of successful attacks has forced us to change our beliefs.

Technically, subjective security proofs are very similar to classical proofs. Assume that a security premise $[\![\mathcal{Q}_0 \cong \mathcal{Q}_1|(t_0,\varepsilon_0)]\!]$ is fundamental. Then an efficient constructive reduction $\mathcal{T}$ that maps an algorithm $\mathcal{A}$ to $\mathcal{B}$ so that

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A}) > \varepsilon \qquad \Longrightarrow \qquad \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathcal{B}) > \rho(\varepsilon) \tag{6.19}$$

for a known function $\rho : [0,1] \to [0,1]$ introduces an upper bound on the computational distance $\mathsf{cd}^{\mathfrak{A}}_{\star}(\mathcal{G}_0,\mathcal{G}_1)$. Namely, if the security guarantee ρ is convex-cup, the corresponding distribution $\mathfrak{B} = \mathcal{T}(\mathfrak{A})$ satisfies

$$\mathsf{cd}^{\mathfrak{B}}_{\star}(\mathcal{Q}_1,\mathcal{Q}_1) = \sum_{\mathcal{A}\in\mathfrak{A}} \Pr\left[\mathcal{A}\right] \cdot \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathcal{T}(\mathcal{A})) \geq \rho\bigl(\mathsf{cd}^{\mathfrak{A}}_{\star}(\mathcal{G}_0,\mathcal{G}_1)\bigr) \enspace , \tag{6.20}$$

as the definition of ρ and Jensen's inequality assure

$$\sum_{\mathcal{A}\in\mathfrak{A}} \Pr\left[\mathcal{A}\right] \cdot \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathcal{T}(\mathcal{A})) \geq \sum_{\mathcal{A}\in\mathfrak{A}} \Pr\left[\mathcal{A}\right] \cdot \rho\left(\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A})\right) \enspace , \tag{6.21}$$

$$\sum_{\mathcal{A}\in\mathfrak{A}} \Pr\left[\mathcal{A}\right] \cdot \rho\left(\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A})\right) \geq \rho\left(\sum_{\mathcal{A}\in\mathfrak{A}} \Pr\left[\mathcal{A}\right] \cdot \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathcal{A})\right) \enspace . \tag{6.22}$$

Therefore, we obtain an average-case security guarantee

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_0,\mathcal{G}_1}(\mathfrak{A}) > \varepsilon \qquad \Longrightarrow \qquad \mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathfrak{B}) > \rho(\varepsilon) \enspace . \tag{6.23}$$

If the security guarantee ρ is not convex-cup, the pointwise guarantee (6.19) does not automatically imply the average-case guarantee (6.23) and we have to bound $\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{Q}_0,\mathcal{Q}_1}(\mathfrak{B})$ by using other methods. Given a fundamental security premise $[\![\mathcal{Q}_0 \cong \mathcal{Q}_1|(t_0,\varepsilon_0)]\!]$ and an average case security guarantee (6.23) that respects the time bound t_0, we can conclude

$$\mathsf{cd}^{\mathfrak{A}}_{\star}(\mathcal{G}_0,\mathcal{G}_1) \leq \rho^{-1}(\varepsilon_0) \enspace , \tag{6.24}$$

otherwise we violate either the axiom (R3) or the security premise. Hence, subjective security proofs are technically identical to traditional objective security proofs, provided that the code transformation $\mathcal{T}$ has an explicit description. Moreover, as all common pointwise reductions have convex-cup security guarantees, and even the quantitative success bounds of traditional and subjective security estimates coincide. The difference appears only in running times, since we must also consider the complexity of the code transformation $\mathcal{T}$.

We remark here that the same issues have also been addressed by Rogaway who tried to solve the hashing paradox described in Section 6.4. However, the corresponding article [Rog06] does not provide a proper mathematical foundation of subjective security (human ignorance). In fact, Rogaway believes that rigorous formalisation of subjective security is impossible [Rog06]:

> *What is meant is that there is no efficient algorithm known to man that outputs a collision in H. But such a statement would seem to be unformalizable—outside the realm of mathematics.*

We have clearly shown that such a formalisation is possible as soon as one is willing to accept subjective interpretation of probabilities. Moreover, our treatment of subjective security also provides a formal justification to the methodology used in [Rog06]. More precisely, Rogaway required that all reductions should be strictly constructive and accompanied with appropriate pointwise security guarantees (6.19). As a result, both formalisations give qualitatively equivalent security guarantees for most constructions and protocols.

However, there is still a major conceptual difference that makes our formalisation different from the traditional objective setting and the formalisation given by Rogaway. Namely, the axioms (R1)–(R3) are applicable only if a *concrete* security premise is *known* to be violated. Therefore, we cannot prove

$$\mathsf{cd}_{\star}^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_2) \leq \varepsilon_1 + \varepsilon_2 \tag{6.25}$$

from premises $[\![\mathcal{G}_0 \cong \mathcal{G}_1 | (t, \varepsilon_1)]\!]$ and $[\![\mathcal{G}_1 \cong \mathcal{G}_2 | (t, \varepsilon_2)]\!]$, although the violation of the inequality (6.25) implies that one of the inequalities

$$\mathsf{cd}_{\star}^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_1) > \varepsilon_1 \qquad \text{or} \qquad \mathsf{cd}_{\star}^{\mathfrak{A}(t)}(\mathcal{G}_1, \mathcal{G}_2) > \varepsilon_2 \tag{6.26}$$

holds. The problem lies in the fact that we *do not know* which disjunct is true and thus the distribution $\mathfrak{A}(t)$ that violates the inequality (6.25) does not contradict the axioms (R1)–(R3) nor security premises. Hence, a non-constructive use of triangle inequality is not allowed. One might think that such a strong constructivity requirement is artificial and redundant. However, a consistent formalisation of subjective security without this assumption is impossible, otherwise we end up in the objective setting. For example, we can always consider singleton distributions $\mathfrak{A}_i(t)$ that consist of a single t-time algorithm $\mathcal{A}_i$. Now if $\mathfrak{A}_1(t), \ldots, \mathfrak{A}_s(t)$ cover all t-time algorithms, the formula

$$\mathsf{cd}_{\star}^{\mathfrak{A}_1(t)}(\mathcal{G}_0, \mathcal{G}_1) > \varepsilon \vee \mathsf{cd}_{\star}^{\mathfrak{A}_2(t)}(\mathcal{G}_0, \mathcal{G}_1) > \varepsilon \vee \cdots \vee \mathsf{cd}_{\star}^{\mathfrak{A}_s(t)}(\mathcal{G}_0, \mathcal{G}_1) > \varepsilon \tag{6.27}$$

holds whenever $\varepsilon < \mathsf{cd}_{\star}^{t}(\mathcal{G}_0, \mathcal{G}_1)$. Hence, we cannot conclude the contradiction from the formula (6.27), or otherwise the non-contradictory assignment must satisfy

$$\mathsf{cd}_{\star}^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_1) = \mathsf{cd}_{\star}^{t}(\mathcal{G}_0, \mathcal{G}_1) \ . \tag{6.28}$$

Since formulae (6.26) and (6.27) are analogous, we must treat them similarly.

In many occasions, we can circumvent the problem by finding more explicit reductions, but when a construction is based on different basic primitives, it is often impossible to eliminate non-constructivity. Therefore, we allow explicit strengthenings of fundamental beliefs. More formally, let

$$[\![\mathcal{Q}_0 \cong \mathcal{Q}_1 | (t_1, \varepsilon_1)]\!] \wedge [\![\mathcal{Q}_2 \cong \mathcal{Q}_3 | (t_2, \varepsilon_2)]\!] \tag{6.29}$$

denote a *strengthened premise* that distributions $\mathfrak{A}$ and $\mathfrak{B}$ are inconsistent if

$$\mathsf{cd}_{\star}^{\mathfrak{A}(t_1)}(\mathcal{Q}_0, \mathcal{Q}_1) > \varepsilon_1 \qquad \text{or} \qquad \mathsf{cd}_{\star}^{\mathfrak{B}(t_2)}(\mathcal{Q}_2, \mathcal{Q}_3) > \varepsilon_2 \tag{6.30}$$

even if it is not known which disjunct of the statement holds. Such explicit way of strengthening keeps the amount of non-constructiveness under tight control and we do not lose the connection with theoretical foundations.

Moreover, it is possible to express strengthened beliefs as a belief about a special game if we use a signed advantage

$$\overline{\mathsf{cd}}_{\star}^{\mathfrak{A}(t_1)}(\mathcal{G}_0, \mathcal{G}_1) = \Pr\left[\mathcal{A} \leftarrow \mathfrak{A} : \mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{A} \leftarrow \mathfrak{A} : \mathcal{G}_1^{\mathcal{A}} = 1\right] \tag{6.31}$$

instead of the computational distance $\mathsf{cd}_{\star}^{\mathfrak{A}(t_1)}(\mathcal{G}_0, \mathcal{G}_1)$. We omit the corresponding technical details, since it is just a theoretically pleasing reassurance that the axioms (R1)–(R3) are expressive enough and does not give additional insight.

Subjective security for protocols. Recall that the classical ideal real world comparison was just a convenient way to define security with respect to all relevant security goals $\mathfrak{B}$. More precisely, the classical definition allowed to replace the real world adversary $\mathcal{A}$ with an ideal world adversary $\mathcal{A}^{\circ}$ in order to get a contradiction. The subjective security setting also adds an efficiency restriction. The real and ideal world are *constructively* $(t_{\mathrm{re}}, t_{\mathrm{id}}, \varepsilon)$*-close w.r.t. a distribution of adversaries* $\mathfrak{A}$ *and a distribution of security goals* $\mathfrak{B}$ if there exists a code transformation $\mathcal{T}$ that satisfies the following constraints. First, $\mathcal{T}$ transforms a distribution of real world adversaries $\mathfrak{A}$ into a distribution of ideal world adversaries $\mathfrak{A}_{\circ}$. Secondly, the time needed to compute $\mathcal{A}^{\circ} = \mathcal{T}(\mathcal{A})$ is included in the running time of $\mathcal{A}^{\circ}$. Finally, for the time bounds t_{re} and t_{id} on the running times of $\mathcal{A}$ and $\mathcal{A}^{\circ}$ and for any feasible input distribution $\mathfrak{D}$:

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^{\circ}} = 1]| \leq \varepsilon \enspace , \tag{6.32}$$

where the probability is also taken over the distributions $\mathfrak{A}$, $\mathfrak{A}_{\circ}$ and $\mathfrak{B}$.

Observe that we must limit the set of security goals $\mathcal{B}$ that are used to compute the outputs of $\mathcal{G}_{\mathrm{real}}$ and $\mathcal{G}_{\mathrm{ideal}}$ or otherwise the requirements may be too strict. For example, consider a protocol where an honest party $\mathcal{P}_1$ sends an encryption $\mathsf{Enc}_{\mathsf{pk}}(x_1)$ to corrupted $\mathcal{P}_2$. If pk is a fixed public parameter, then there exists an efficient predicate $\mathcal{B}$ that uses hardwired sk to decrypt the output of $\mathcal{A}$. Hence a trivial adversary $\mathcal{A}$ that outputs $\mathsf{Enc}_{\mathsf{pk}}(x_1)$ clearly violates input-privacy. To put it in another way, the distribution $\mathfrak{B}$ must be restricted as it captures also the further offline behaviour of an adversary.

Now assume that one has provided a universal simulator construction Sim and shown that a t_{re}-time distribution $\mathfrak{A}$ and t_{pr}-time distribution $\mathfrak{B}$ that violate inequality (6.32) for some input distribution $\mathfrak{D}$ can be used to construct an adversary distribution that violates a subjective security premise. Then for all coherent distributions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{D}$ the reduction schema

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \xrightarrow[\varepsilon]{\text{MPC-SEC}} \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^{\circ}} \tag{6.33}$$

still holds and the real and the ideal worlds are *subjectively* $(t_{\mathrm{re}}, t_{\mathrm{id}}, t_{\mathrm{pr}}, \varepsilon)$*-close w.r.t. the security premises.* To summarise, the proximity is still defined by the reduction schema (6.33) but the latter may fail for incoherent distributions.

Consider the uniform polynomial security model as an illustrative example. Evidently, not all input distributions are allowed or otherwise an adversary can use the inputs of corrupted parties as an external advice and we are back in the non-uniform model. Hence, all inputs should be generated by uniform polynomial algorithm. Also, note that the security goal $\mathcal{B}(\cdot)$ must be a uniform algorithm or otherwise reductions may fail, too.

6.7 STRICTLY CONSTRUCTIVE PROOF TECHNIQUES

Classical security proofs can be invalid in the context of subjective security, since they may contain non-constructive steps. In particular, one cannot use game trees in the classical manner, as an application of a triangle inequality or a horizon splitting is a non-constructive proof step. Fortunately, most non-constructive proof steps can be substituted by constructive counterparts.

Let us start with the constructive hybrid argument. The technique itself is quite old and can be traced back to the early works of Yao and Goldreich [Yao82, Gol98]. Namely, consider a classical non-constructive game chain

$$\begin{tikzcd}
\mathcal{G}_0\Longrightarrow\mathcal{G}_1 \arrow[d, "\mathcal{T}_1"] & \mathcal{G}_1\Longrightarrow\mathcal{G}_2 \arrow[d, "\mathcal{T}_2"] & \cdots & \mathcal{G}_{n-1}\Longrightarrow\mathcal{G}_n \arrow[d, "\mathcal{T}_n"] \\
\mathcal{Q}_{00}, \mathcal{Q}_{01} & \mathcal{Q}_{10}, \mathcal{Q}_{11} & \cdots & \mathcal{Q}_{n-1,0}, \mathcal{Q}_{n-1,1}
\end{tikzcd}$$

where each game has an efficient reduction $\mathcal{T}_i$ to another more elementary game pair $\mathcal{Q}_{i,0}$ and $\mathcal{Q}_{i,1}$ and the reduction provides perfect simulation

$$\mathcal{Q}_{i,0}^{\mathcal{T}_i(\mathcal{A})} \equiv \mathcal{G}_{i-1}^{\mathcal{A}} \qquad \text{and} \qquad \mathcal{Q}_{i,1}^{\mathcal{T}_i(\mathcal{A})} \equiv \mathcal{G}_{i}^{\mathcal{A}} \quad . \tag{6.34}$$

Then we can express $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_n)$ as a telescopic sum

$$\left|\sum_{\mathcal{A}\in\mathfrak{A}}\sum_{i=1}^{n} \Pr[\mathcal{A}]\cdot\Pr[\mathcal{Q}_{i,0}^{\mathcal{T}_i(\mathcal{A})}=1] - \sum_{\mathcal{A}\in\mathfrak{A}}\sum_{i=1}^{n} \Pr[\mathcal{A}]\cdot\Pr[\mathcal{Q}_{i,1}^{\mathcal{T}_i(\mathcal{A})}=1]\right| \quad . \tag{6.35}$$

In the simplest case, all elementary game pairs coincide $(\mathcal{Q}_{i,0}, \mathcal{Q}_{i,0}) \equiv (\mathcal{Q}_0, \mathcal{Q}_1)$. A randomised transformation that given a code sample $\mathcal{A}$ from $\mathfrak{A}$ applies the transformation $\mathcal{T}_i$ with probability $\frac{1}{n}$ produces a distribution $\mathfrak{B}$ such that

$$\mathsf{cd}_\star^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_n) \le n \cdot \mathsf{cd}_\star^{\mathfrak{B}}(\mathcal{Q}_0, \mathcal{Q}_1) \quad , \tag{6.36}$$

since the term $n \cdot \mathsf{cd}_\star^{\mathfrak{B}}(\mathcal{Q}_0, \mathcal{Q}_1)$ can be also expressed as the sum (6.35). Generally, we have n_j different transformations $\mathcal{T}_i$ that lead to the same game pairs $(\mathcal{Q}_{2j-2}, \mathcal{Q}_{2j-1})$. Now for each $j \in \{1, \dots, k\}$, we can generate an adversarial distribution $\mathfrak{B}_j$ by choosing uniformly at random a transformation $\mathcal{T}_i$ that leads to the game pair $(\mathcal{Q}_{2j-2}, \mathcal{Q}_{2j-1})$. The resulting adversarial distributions $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ satisfy the inequality

$$\mathsf{cd}_\star^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_n) \le n_1 \cdot \mathsf{cd}_\star^{\mathfrak{B}_1}(\mathcal{Q}_0, \mathcal{Q}_1) + \cdots + n_k \cdot \mathsf{cd}_\star^{\mathfrak{B}_k}(\mathcal{Q}_{2k-2}, \mathcal{Q}_{2k-1}) \quad , \tag{6.37}$$

since the right hand side can again be lower bounded by the sum (6.35).

To complete the picture, we also consider a non-constructive horizon splitting step for $\mathsf{cd}_\star^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_1)$ under the analogous simulatability assumption

$$\mathcal{Q}_{i,0}^{\mathcal{T}_i(\mathcal{A})} \equiv (\mathcal{G}_0|_{\mathcal{H}_i})^{\mathcal{A}} \qquad \text{and} \qquad \mathcal{Q}_{i,1}^{\mathcal{T}_i(\mathcal{A})} \equiv (\mathcal{G}_1|_{\mathcal{H}_i})^{\mathcal{A}} \quad . \tag{6.38}$$

Again, we can express $\mathsf{cd}_\star^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_n)$ as a telescopic sum (6.35) and use similar regrouping techniques to obtain

$$\mathsf{cd}_\star^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_n) \le n_1 \cdot \mathsf{cd}_\star^{\mathfrak{B}_1}(\mathcal{Q}_0, \mathcal{Q}_1) + \cdots + n_k \cdot \mathsf{cd}_\star^{\mathfrak{B}_k}(\mathcal{Q}_{2k-2}, \mathcal{Q}_{2k-1}) \quad , \tag{6.39}$$

where the distributions $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ are generated by analogous uniform choice over the transformations corresponding to the same game pair.

To summarise, under the perfect simulation assumptions (6.34) and (6.38), we can use both game chains and horizon splitting to bound the computational distance $\mathsf{cd}_{\star}^{\mathfrak{A}(t)}(\mathcal{G}_0, \mathcal{G}_n)$ by using only a strengthened security premise

$$[\![\mathcal{Q}_0 \cong \mathcal{Q}_1|(t_1, \varepsilon_1)]\!] \wedge \ldots \wedge [\![\mathcal{Q}_{2k-2} \cong \mathcal{Q}_{2k-1}|(t_k, \varepsilon_k)]\!] \enspace .$$

Naturally, we must take into account the running times of $\mathcal{T}_i$ and additional $\mathrm{O}(\log n)$ time needed for random sampling, but these are minor details.

Note that the perfect simulatability assumptions is satisfied for elementary reductions that define computational indistinguishability for a primitive, for example various encryption schemes and pseudorandom generators have the corresponding reductions. Other primitives are normally defined in terms of advantage in a single game $\mathcal{Q}_i$ like collision resistance and one-wayness. Consequently, the corresponding game hops rely on constructive reductions

$$\mathsf{Adv}^{\mathrm{ind}}_{\mathcal{G}_{i-1},\mathcal{G}_i}(\mathfrak{A}) > \varepsilon_i \qquad \Longrightarrow \qquad \mathsf{Adv}_{\mathcal{Q}_i}(\mathcal{T}_i(\mathfrak{A})) > \rho_i(\varepsilon_i) \enspace . \tag{6.40}$$

Again, consider the simplest case where all games coincide $\mathcal{Q}_i \equiv \mathcal{Q}$. Then the corresponding triangle inequality

$$\mathsf{cd}_{\star}^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_n) \leq \mathsf{cd}_{\star}^{\mathfrak{A}}(\mathcal{G}_0, \mathcal{G}_1) + \cdots + \mathsf{cd}_{\star}^{\mathfrak{A}}(\mathcal{G}_{n-1}, \mathcal{G}_n) \tag{6.41}$$

for a game chain still holds, although we do not know the computational distances $\varepsilon_i = \mathsf{cd}_{\star}^{\mathfrak{A}}(\mathcal{G}_{i-1}, \mathcal{G}_i)$. Nevertheless, we can get an adversarial distribution $\mathfrak{B}$ by applying a transformation $\mathcal{T}_i$ with uniform probability $\frac{1}{n}$. The corresponding advantage can be lower bounded by solving the optimisation task

$$\mathsf{Adv}_{\mathcal{Q}}(\mathfrak{B}) \geq \min_{\varepsilon_1 + \cdots + \varepsilon_n \geq \varepsilon} \frac{\rho_1(\varepsilon_1) + \cdots + \rho_n(\varepsilon_n)}{n} \enspace , \tag{6.42}$$

where $\varepsilon = \mathsf{cd}_{\star}^{\mathcal{A}}(\mathcal{G}_0, \mathcal{G}_n)$. This approach can be directly generalised to handle several premises or horizon splittings. Linear bounds $\rho_i(\cdot)$ lead us back to classical inequalities, otherwise we get slightly more loose bounds.

To summarise, most non-constructive black-box proofs can be made constructive by following the corresponding game tree. Moreover, the type of required security assumptions do not change, although the resulting time bounds and advantages may differ. The latter also applies for the white-box reductions that are efficiently constructible, i.e., the borderline between translatable and non-translatable proofs is not determined by the reduction type.

7 MODULAR DESIGN OF COMPLEX PROTOCOLS

Appropriate abstraction level and modular design are the key factors to success in cryptographic protocol design. In particular, it is not enough to have a definition that adequately describes the desired security goals. We also need intermediate security definitions that support modular design and analysis. More precisely, the security of a compound protocol should automatically follow from the security of its sub-protocols, or otherwise the analysis of complex protocols quickly becomes intractable. To achieve such modularity, we must consider more complex attack models than used for defining stand-alone security.

Since there are various decomposition methods, we also need protocols with different usage restrictions. In the simplest case, small sub-protocols are executed one by one to accomplish the final goal, and therefore we need a security definition that is closed under *sequential composition*. Alternatively, we can consider compound protocols that schedule sub-protocols dynamically and possibly execute them in parallel to minimise the total running time. In such cases, we need *universally composable* protocols that preserve security in such ultra-liberal settings. Of course, there are many other alternatives between these two extremes and each of them induces a new related security definition.

We emphasise that the choice of an appropriate protocol is a trade-off between various conflicting requirements and usage restrictions are an important but not the only relevant aspect. In a nutshell, various composability theorems [Ore87, Can00a, DM00, Can01] provide just a formal methodology that is needed to establish simple and natural-looking usage restrictions. Moreover, the precise knowledge of the expected usage patterns is important, as it poses structural restrictions to the corresponding protocol, see Sections 7.5–7.6.

Alternatively, we can use composability theorems for making security proofs more modular. In particular, we can use universal composability as a tool to design complex protocols in a systematic way. The corresponding methodology has three stages. First, we construct universally composable protocols for all necessary sub-tasks. More precisely, we can employ trusted setup to achieve universal composability with minimal overhead. In the second stage, we combine these sub-protocols into a round optimal solution and utilise universal composability in the corresponding security proofs. As the final design step, we replace the trusted setup phase with a corresponding sub-protocol that is secure in the stand-alone model. As a result, we obtain a round efficient protocol that is secure in the stand-alone model and has a short and modular security proof.

The formalism needed to describe and analyse various attack models is rather complicated. Hence, we start from the main concepts in Section 7.1 and only then formalise the corresponding computational model in Section 7.2. Afterwards, we gradually describe various subtle details in Sections 7.3 and 7.4 until we can establish important composability results. Last two sections are dedicated to the non-standard issues arising from our design methodology.

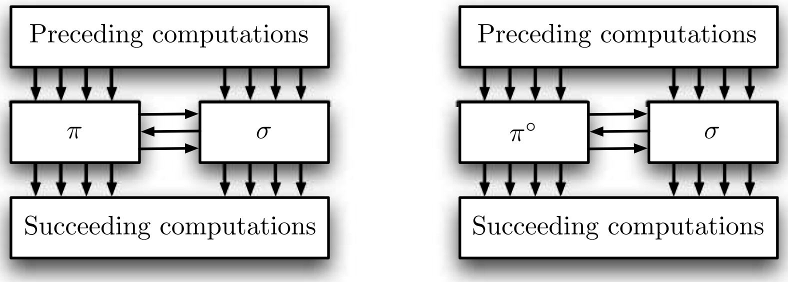


Figure 7.1: Computational context $\varrho\langle\cdot\rangle$ is a template, which describes all computations preceding, co-occurring and following the protocols π and π°.

7.1 DUALITY BETWEEN PROTOCOL DESIGN AND ANALYSIS

A protocol that implements a complex functionality is not only difficult to design but also difficult to analyse. To alleviate the underlying complexity, such protocols are commonly split into a collection of well-defined sub-protocols. The corresponding security proof is modular if it gradually replaces all sub-protocols with the ideal implementations until we obtain the desired ideal functionality. As a result, we have to consider the security of a protocol π in a *computational context* $\varrho\langle\cdot\rangle$ that uses the protocol π in order to compute something else.

In the following, we always consider contexts that use protocols in a black-box manner, i.e., a context first provides inputs to the protocol and later uses the resulting outputs in subsequent computations. In fact, a context $\varrho\langle\cdot\rangle$ is like a template that can be instantiated by specifying the missing protocol. The resulting compound protocols $\varrho\langle\pi\rangle$ share the same general structure. Participants first precompute some values, then execute the protocol π together with side-computations σ and finally do post-processing, see Fig. 7.1 for schematic description. In particular, note that the adversarial behaviour can cause an information flow between the computational processes π and σ, although they are formally separated from each other by the specification. Such coupling effects can make the security analysis extremely difficult.

The security of a protocol π in a context $\varrho\langle\cdot\rangle$ is defined by comparing the corresponding real and ideal world implementations. Let $\varrho\langle\pi\rangle$ denote the real world implementation and $\varrho\langle\pi^\circ\rangle$ the corresponding ideal world implementation. Then, for any input distribution $\mathfrak{D}$ and for any security objective $\mathcal{B}(\cdot)$, we can define the corresponding security games

$$
\begin{array}{ll}
\mathcal{G}^{\mathcal{A}}_{\text{real}} & \mathcal{G}^{\mathcal{A}^\circ}_{\text{ideal}} \\
\left[\begin{array}{l} \boldsymbol{x} \leftarrow \mathfrak{D} \\ \boldsymbol{z}_{\text{obs}} \leftarrow \mathcal{G}^{\mathcal{A}}_{\text{re-atk}}(\boldsymbol{x}) \\ \textbf{return } \mathcal{B}(\boldsymbol{z}_{\text{obs}}) \end{array}\right. &
\left[\begin{array}{l} \boldsymbol{x} \leftarrow \mathfrak{D} \\ \boldsymbol{z}_{\text{obs}} \leftarrow \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}}(\boldsymbol{x}) \\ \textbf{return } \mathcal{B}(\boldsymbol{z}_{\text{obs}}) \end{array}\right.
\end{array}
$$

where $\boldsymbol{z}_{\text{obs}}$ is the vector of observable outcomes and the sub-games $\mathcal{G}_{\text{re-atk}}$ and $\mathcal{G}_{\text{id-atk}}$ model online attacks against the protocols $\varrho\langle\pi\rangle$ and $\varrho\langle\pi^\circ\rangle$. We omit the exact details here and discuss them separately afterwards.

More precisely, note that a context $\varrho\langle\cdot\rangle$ defines only the usage of a protocol π and not the exact scheduling of protocol messages. As a result, we must formally define security in terms of game pairs, since each context might create several game pairs $(\mathcal{G}_{\text{real}}, \mathcal{G}_{\text{ideal}})$. More formally, fix a set of relevant contexts $\varrho\langle\cdot\rangle$, a set of relevant security objectives $\mathfrak{B}$ and possible message schedulings. Let $\mathfrak{G}$ be the set of corresponding game pairs. Then the ideal and real world implementations are $(t_{\text{re}}, t_{\text{id}}, \varepsilon)$-*close* w.r.t. all game pairs $\mathfrak{G}$ if for any pair $(\mathcal{G}_{\text{real}}, \mathcal{G}_{\text{ideal}}) \in \mathfrak{G}$ and for any t_{re}-time real world adversary $\mathcal{A}$ there exists a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any input distribution $\mathfrak{D}$

$$|\Pr[\mathcal{G}_{\text{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\text{ideal}}^{\mathcal{A}^\circ} = 1]| \le \varepsilon \ . \tag{7.1}$$

Alternatively, we can view $(t_{\text{re}}, t_{\text{id}}, \varepsilon)$-closeness w.r.t. $\mathfrak{G}$ as a reduction schema

$$\mathcal{G}_{\text{real}}^{\mathcal{A}} \xRightarrow[\varepsilon]{\mathfrak{G}\text{-SEC}} \mathcal{G}_{\text{ideal}}^{\mathcal{A}^\circ} \tag{7.2}$$

that can be applied to any game pair $(\mathcal{G}_{\text{real}}, \mathcal{G}_{\text{ideal}}) \in \mathfrak{G}$. Moreover, one often defines the set of relevant contexts implicitly by specifying usage restrictions for the protocol π. For example, we can consider all game pairs, where the participants execute the protocol π sequentially or in parallel with some other protocols, i.e., we can talk about sequential and parallel composability.

In principle, we can arbitrarily choose the set of game pairs. However, only a few of these choices lead to modular security proofs. We remark that a standard design methodology creates a chain of hybrid protocols

$$\pi^\circ \equiv \varrho_0 \xRightarrow{\pi_1} \varrho_1 \xRightarrow{\pi_2} \dots \xRightarrow{\pi_{s-1}} \varrho_{s-1} \xRightarrow{\pi_s} \varrho_s \equiv \pi \tag{7.3}$$

that gradually change the ideal implementation into a practical protocol without calls to the trusted third party. More precisely, each step in the chain introduces a new sub-protocol π_i that implements some task needed to complete the desired functionality. Note that the sub-protocol π_i itself may utilise additional ideal sub-protocols that are implemented with the help of the trusted third party. These calls must be eliminated by the subsequent steps in the chain, so that the final protocol π does not depend on the trusted third party.

The corresponding modular security proof uses appropriate reduction schemata to traverse the chain in the opposite direction

$$\mathcal{G}_{\text{real}}^{\mathcal{A}} \equiv \mathcal{G}_s^{\mathcal{A}_s} \xRightarrow[\varepsilon_s]{\mathfrak{G}_s\text{-SEC}} \mathcal{G}_{s-1}^{\mathcal{A}_{s-1}} \xRightarrow[\varepsilon_{s-1}]{\mathfrak{G}_{s-1}\text{-SEC}} \dots \xRightarrow[\varepsilon_2]{\mathfrak{G}_2\text{-SEC}} \mathcal{G}_1^{\mathcal{A}_1} \xRightarrow[\varepsilon_1]{\mathfrak{G}_1\text{-SEC}} \mathcal{G}_0^{\mathcal{A}_0} \equiv \mathcal{G}_{\text{ideal}}^{\mathcal{A}^\circ} \ , \tag{7.4}$$

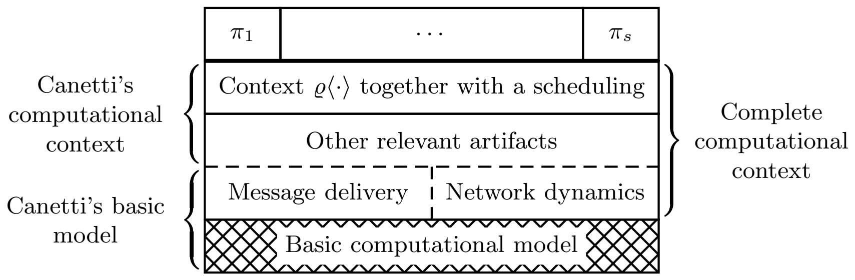
where the game $\mathcal{G}_i$ captures the tolerated adversarial behaviour against the hybrid protocol $\varrho_i\langle\pi_i\rangle$. As a result, the game chain converts a real world adversary $\mathcal{A}$ step by step to an ideal world adversary $\mathcal{A}^\circ$ such that

$$|\Pr[\mathcal{G}_{\text{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\text{ideal}}^{\mathcal{A}^\circ} = 1]| \le \varepsilon_1 + \dots + \varepsilon_s \tag{7.5}$$

and thus there exists a composite reduction schema

$$\mathcal{G}_{\text{real}}^{\mathcal{A}} \xRightarrow[\varepsilon_s + \dots + \varepsilon_1]{\star\star\star} \mathcal{G}_{\text{ideal}}^{\mathcal{A}^\circ} \tag{7.6}$$

provided that $(\mathcal{G}_i, \mathcal{G}_{i-1}) \in \mathfrak{G}_i$ for all $i \in \{1, \dots, s\}$ and all reduction steps can be carried out. As a result, security guarantees of all protocols $\pi_1, \dots, \pi_s$ implicitly determine a set of valid game pairs $\mathfrak{G}$, where the compound protocol π remains $(t_{\text{re}}, t_{\text{id}}, \varepsilon)$-secure. For example, we often want to establish that the compound protocol π is secure in the stand-alone model.



π_1
$\cdots$
π_s
Canetti's computational context
Context $\varrho\langle\cdot\rangle$ together with a scheduling
Other relevant artifacts
Complete computational context
Canetti's basic model
Message delivery
Network dynamics
Basic computational model

Figure 7.2: Layered description of a compound protocol $\varrho\langle\pi_1, \ldots, \pi_s\rangle$.

Note that there are many trade-offs between efficiency and generality. Evidently, we obtain the most efficient solution when we tailor a protocol π specifically for a pre-specified context $\varrho\langle\cdot\rangle$. However, such design cannot be reused and the corresponding security proof might be difficult to find. Alternatively, we may design a protocol π that is secure in almost any context. However, such a protocol might be impossible to devise or it might be highly inefficient. Thus, one has to find a proper balance between these extreme cases.

As a way out from this anarchy of security models, we require that each protocol is specified with usage restrictions that determine the supported contexts. Namely, one should fix a set of *composition rules* that determine the basic structural properties of plausible contexts; for example, whether the protocol shares some variables with the context or what kind of pre-, post- and side-computations are supported. Secondly, one should fix a tolerated adversarial behaviour and the maximal running time for computational contexts. Of course, any other unambiguous specification methodology is also appropriate.

7.2 LAYERED DESCRIPTION OF COMPUTATIONAL CONTEXTS

Although the general concept of computational context is quite intuitive, it is somewhat difficult to give a formalisation that is not plagued by the abundance of technical details. Various types of interactive computations share the same general description but differ in many minor details, for example compare the manuscripts [MR91b, Can00b, PSW00, BPW04]. To make technical details more tractable, we specify the computational model step by step. First, we define a basic model of computations that is absolutely minimal. Next, we formalise all other more complex artefacts, like asynchronous and non-authentic communication, in terms of this basic model. More precisely, we model these artefacts by adding extra nodes into the network and by adjusting the description of protocols and contexts. As a result, we can gradually formalise complex settings by grouping similar technical details together into separate layers, as illustrated in Fig. 7.2. Moreover, our task here is to specify only the basic computational model, since an appropriate formalisation of higher level artefacts is the responsibility of the person who analyses a concrete protocol or a context. Secondly, we can state and prove many theorems independently from higher level artefacts, as we can always analyse the basic level description of the protocol.

Basic computational model. The basic model consists of a fixed number of participants $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and a fixed number of trusted third parties $\mathcal{T}_1, \ldots, \mathcal{T}_s$ that have no inputs and no outputs. A trusted third party $\mathcal{T}_i$ is active only in the context, where participants are supposed to execute an ideal implementation π_i°. All participants of a context are modelled as interactive Random Access machines that run simultaneously at the same clock speed. Communication between participants is synchronous and secure. Moreover, a sender cannot proceed further until the message has been written on the recipients communication tape. Hence, the communication tape behaves as an unlimited buffer and the recipient does not have to process messages at the arrival time. The low-level message delivery procedure is a deterministic and a fairly low cost operation, which is described more thoroughly in Section 2.4.

Adversaries and security games. We treat adversaries as external entities and not as participants of a computational context. There are several important reasons for such a choice. First, our main aim here is to study attacks that utilise only acquired data and do not employ timings or other side information. Second, we formalised a computational context as a real-time system, where participants obey many real-time constraints. Hence, putting an adversary into the computational context would create many implicit limitations on the adversarial behaviour, which are normally neglected in cryptography. Third, timing-based attacks can be formalised also in our simplified model.

In our model, the adversary can influence computations by sending oracle calls to the challenger. Namely, the adversary can either corrupt participants, send some messages, or give his or her final output z_a directly to the challenger. A semi-honest corruption call provides read-only access to the internal state of the participant. A malicious corruption query gives a complete control over the participant $\mathcal{P}_i$ and stops the local computations, i.e., the adversary must do all computations instead of $\mathcal{P}_i$. Additionally, we require that the adversary can send messages only on behalf of maliciously corrupted participants.

The challenger's primary task in the game is to correctly simulate the execution of a computational context and to serve the queries of an adversary. For the faithful execution, the challenger can allocate a separate thread for each participant and then simulate the execution in micro-rounds, where each thread completes only a single basic operation. Additionally, the challenger tests all queries and halts with $\perp$ if the adversarial behaviour is not tolerable. As before, the security game itself is a ping-pong protocol started by the adversary. To avoid leakage of temporal information, the adversary is activated only when the challenger provides a reply or a corrupted participant receives a message.

Computational complexity. As a final detail, we must specify several important time bounds. Let t_π denote the time complexity of a protocol π, i.e., the number of the elementary steps needed to complete π. Similarly, let t_cnt denote the time complexity of a computational context $\varrho\langle\cdot\rangle$, i.e., the number of elementary steps made by all non-adversarial parties in the game $\mathcal{G}_\mathrm{ideal}$. The latter can be further decomposed into the online and the offline time complexity:

$$t_\mathrm{cnt} = t_\varrho + t_\mathrm{pr} \quad , \tag{7.7}$$

where t_ϱ counts all elementary steps made by $\mathcal{P}_1, \ldots, \mathcal{P}_m, \mathcal{T}_1, \ldots, \mathcal{T}_s$ for evaluating $\varrho\langle\pi^\circ\rangle$ and t_pr is the time complexity of the corresponding security objective

$\mathcal{B}(\cdot)$. As before, let t_{re} and t_{id} denote the running times of real and ideal world adversaries. Also, note that t_ϱ may depend on the adversarial behaviour.

Message delivery. Different algorithmic models of message delivery can be easily specified sending messages through a dedicated *courier* node $\mathcal{P}_{\mathrm{nw}}$. The courier node $\mathcal{P}_{\mathrm{nw}}$ is an ordinary participant in the basic model, where communication is synchronous and secure. However, as all other participants send their messages via $\mathcal{P}_{\mathrm{nw}}$, the effects of insecure, non-reliable, non-instantaneous and non-authentic communication can be modelled by an appropriate choice of the message delivery algorithm executed by $\mathcal{P}_{\mathrm{nw}}$. For example, non-authentic communication can be formalised by using maliciously corrupted $\mathcal{P}_{\mathrm{nw}}$. Similarly, a semi-honest corruption of $\mathcal{P}_{\mathrm{nw}}$ represents the ability to eavesdrop all communication channels without the possibility to alter messages.

Most importantly, the courier node $\mathcal{P}_{\mathrm{nw}}$ is an ordinary protocol participant and thus the accuracy and correctness of all these message delivery models is not our responsibility. Of course, the delivery model has to be specified, but such non-essential technical details do not clutter the basic formalism.

Dynamic networks. Dynamic evolution of computational contexts is another technical detail, which is emphasised a lot in Canetti's manuscript [Can00b]. Again, such behaviour can be modelled by a dedicated *administrator* node $\mathcal{P}_{\mathrm{na}}$ that creates and deletes network nodes. More formally, only few nodes are initially active in the context and the remaining nodes are waiting for start-up messages from $\mathcal{P}_{\mathrm{na}}$. Now any active node $\mathcal{P}_i$ can send a special message with a program code to $\mathcal{P}_{\mathrm{na}}$ to activate a participant. The administrator $\mathcal{P}_{\mathrm{na}}$ loads the code into first free node $\mathcal{P}_j$ and sends the *node label* j back to $\mathcal{P}_i$. Analogously, we can give nodes the ability to halt other nodes. Note that the bound on the total running time also limits the maximal number of activated nodes and thus $\mathcal{P}_{\mathrm{na}}$ never runs out of free nodes if the context has enough participants.

Message scheduling. Recall that a computational context $\varrho\langle\pi\rangle$ specifies only a black-box usage of π. Such a specification is sufficient for the ideal implementation π°, since the protocol execution is determined by a black-box usage pattern. However, normal protocols usually have a more complex structure and thus the message scheduling is not uniquely fixed. There are two principal ways to solve this ambiguity. First, we might fix the explicit scheduling for $\varrho\langle\pi\rangle$ and consider the corresponding game pairs. However, such a choice is often unsuitable for practical applications, where messages are scheduled dynamically in order to minimise network delays. Similarly, a rigid scheduling is unsuitable for adaptive computations, where the execution itself depends dynamically on the non-deterministic choices made by the participants. As a way out, we allow flexible schedulings as long as they are completely deterministic. More formally, if we fix random coins of all participants and the adversary, then the message scheduling should be uniquely fixed for the protocol $\varrho\langle\pi\rangle$. We remind here that all artefacts are defined in the basic model and thus these random coins uniquely fix the networking behaviour and message delays.

Other relevant artefacts. Many other factors can influence the outcome of interactive computations starting from different clock speeds and ending with power consumption and sound made by the computational device. Nevertheless, all these artefacts can be implemented as dedicated nodes with a prescribed corruption level, or as extra messages in protocols. Nevertheless, the layered description of computational contexts is not a silver bullet. It just provides a more modular way to specify features and at the end of the day the overall complexity is comparable to Canetti's formalism [Can00b]. In fact, Canetti's model can be viewed as a canonical formalisation of interactive computations.

7.3 TWO FLAVOURS OF STAND-ALONE SECURITY

Although the stand-alone security model is a quite restrictive computational context, it is still a good starting point for illustrating many intricate concepts. In fact, simplicity is the main virtue of the stand-alone model, since the corresponding security proofs are quite straightforward and without subtle details. Hence, complex security notions have often alternative description in the stand-alone model to escape tedious technical details. In particular, it is important to characterise computational contexts that preserve stand-alone security.

First, note that participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ may possess more information than is actually used in the protocol. That is, let $\boldsymbol{x} = (x_1, \ldots, x_n)$ denote partial inputs used by the protocol and let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ denote the corresponding complete inputs. We emphasise here that such a distinction is purely syntactic and not a redefinition of the stand-alone security model. Namely, we can always investigate protocols that first extract sub-inputs $\boldsymbol{x}$ from the true inputs $\boldsymbol{\phi}$ and then use $\boldsymbol{x}$ in later computations. Similarly, we use $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n, \psi_{\mathrm{a}})$ to denote the final output states of all participants and the adversary.

Second, note that our and the classical formalisation of stand-alone security are qualitatively equivalent for static adversaries. Recall that a correspondence between the real and the ideal world adversaries $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ may depend on the security goal $\mathcal{B}(\cdot)$ in our formalism, whereas the classical formalism requires a mapping $\mathcal{A} \mapsto \mathcal{A}^\circ$ that is independent of $\mathcal{B}(\cdot)$, see Section 5.3. Now assume that $\mathcal{P}_i$ is a non-corruptible node with an input $\phi_i = (\beta_i, x_i)$ and let $\psi_i = (\beta_i, x_i, y_i)$ be the corresponding output. Then we can consider a universal security goal $\mathcal{B}^i_{\mathrm{u}}(\cdot)$ that interprets β_i as a formal description of $\mathcal{B}(\cdot)$ and outputs $\mathcal{B}(\psi_1, \ldots, \psi_n, \psi_{\mathrm{a}})$. Due to the properties of RAM machines, there exists a constant $c > 0$ such that any t_{pr}-time predicate $\mathcal{B}(\cdot)$ can be interpreted in time $c \cdot t_{\mathrm{pr}}$. Hence, if for any input distribution $\mathfrak{D}$ and $c \cdot t_{\mathrm{pr}}$-time predicate $\mathcal{B}^i_{\mathrm{u}}(\cdot)$

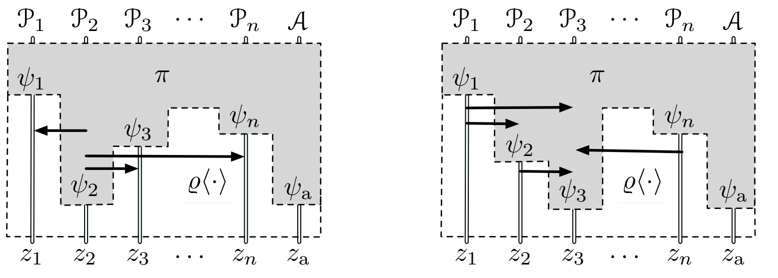
$$|\Pr[\mathcal{G}^{\mathcal{A}}_{\mathrm{real}} = 1] - \Pr[\mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}} = 1]| \leq \varepsilon \enspace , \tag{7.8}$$

then the mapping $(\mathcal{B}^i_{\mathrm{u}}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ produces ideal world adversaries such that

$$\mathsf{cd}^{t_{\mathrm{pr}}}_{\star}(\mathcal{G}^{\mathcal{A}}_{\text{re-atk}}, \mathcal{G}^{\mathcal{A}^\circ}_{\text{id-atk}}) \leq \varepsilon \tag{7.9}$$

whenever $\mathcal{P}_i$ remains uncorrupted. Consequently, there also exists a universal mapping $\mathcal{A} \mapsto \mathcal{A}^\circ$ for all adversaries that corrupt a fixed set of nodes, as always assumed in the classical formalisations [Can00a, Gol04]. We emphasise that the equivalence does not hold for input-privacy in the malicious model.

Third, note that it is irrelevant whether a predicate $\mathcal{B}(\cdot)$ is computed in centralised or distributed manner. Hence, stand-alone security is sufficient for all computational contexts $\varrho\langle\cdot\rangle$, where all participants first execute a protocol and then interactively post-process the outputs $\boldsymbol{\psi}$. However, the models are equivalent only if we can embed the post-processing phase into $\mathcal{B}(\cdot)$. For obvious



$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}_3$ $\cdots$ $\mathcal{P}_n$ $\mathcal{A}$
π
ψ_1 ψ_2 ψ_3 ψ_n ψ_{a}
$\varrho\langle\cdot\rangle$
z_1 z_2 z_3 $\cdots$ z_n z_{a}
$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}_3$ $\cdots$ $\mathcal{P}_n$ $\mathcal{A}$
π
ψ_1 ψ_2 ψ_3 ψ_n ψ_{a}
$\varrho\langle\cdot\rangle$
z_1 z_2 z_3 $\cdots$ z_n z_{a}

Figure 7.3: Synchronisation errors that break the equivalence with the stand-alone setting. Escaping messages are on the left and invading messages on the right. The execution time goes from top to bottom.

reasons, such an embedding does not exist for input-privacy but there are other more surprising exceptions. Namely, the adversarial behaviour might cause synchronisation errors, where protocol messages are mixed together with the post-processing phase, see Fig. 7.3. The grey area in the figure represents the time when the participants execute the protocol, the white area represents the post-processing phase and the arrows represent misinterpreted messages.

In short, messages can either escape or invade the stand-alone setting. A message *escapes* from the stand-alone setting if it is sent to a participant that has already completed the protocol. A message *invades* the stand-alone setting if it is sent by a node that has already completed the protocol. To avoid ambiguity, we say that all maliciously corrupted nodes finish the protocol together with the last (semi-)honest node. Escaping messages have no influence on outputs $\boldsymbol{\psi}$ in the stand-alone model, since the low level message delivery just writes them on the communication tapes of halted participants. Therefore, we cannot construct the corresponding predicate, since the post-processing phase may actually use these messages. Invading messages cause a similar incompatibility with the stand-alone model. Moreover, it is straightforward to construct examples, where such synchronisation errors cause the complete failure of a protocol that is secure in the stand-alone model. We leave this as an exercise to the reader.

We emphasise here that it is relatively easy to eliminate problems caused by escaping messages. For example, we can use special tags to denote the protocol messages and drop all these messages in the post-processing stage. Invading messages, on the other hand, cause more severe problems, since they bring extra information to the stand-alone setting. In particular, a post-processing context $\varrho\langle\cdot\rangle$ may force a participant $\mathcal{P}_i$ to publish the state ψ_i after completion of the protocol and such synchronisation errors are extremely dangerous.

As an illustrative example consider a lottery protocol π_{sl} depicted in Fig. 7.4, where $\mathcal{P}_2$ tries to guess a random number s_1 generated by $\mathcal{P}_1$ and the number s_1 is sent to the arbiter $\mathcal{P}_3$ to prevent $\mathcal{P}_1$ from cheating. In the ideal implementation, the trusted third party does all computations by herself and sends the results back to the participants. Now assume that only $\mathcal{P}_2$ can be corrupted and that all communication channels are secure. Then the protocol is 2^{-k+1}-secure in the malicious model. The corresponding simulator Sim just ignores the messages sent by $\mathcal{T}$ and $\mathcal{P}_2$ and outputs whatever $\mathcal{A}$ does. However, the protocol π_{sl} is clearly insecure in the post-processing context, where $\mathcal{P}_2$ queries s_1 from $\mathcal{P}_3$.

$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
Send $s_1 \leftarrow \{0,1\}^k$ to $\mathcal{P}_3$	Send $s_2 \leftarrow \{0,1\}^k$ to $\mathcal{P}_1$	Receive s_1 from $\mathcal{P}_1$
return $s_1 \overset{?}{=} s_2$	**return** s_2	**return** s_1

Figure 7.4: Protocol π_{sl} that models a simple lottery with an arbiter $\mathcal{P}_3$.

Namely, $\mathcal{P}_2$ can first pretend that π_{sl} is completed to get the *invading* message s_1 from $\mathcal{P}_3$ and then forward s_1 to $\mathcal{P}_1$ and thus always win the lottery. At the same time, the ideal implementation remains secure in this context. Moreover, the underlying problem cannot be solved with message tagging.

Such protocol failures are extremely dangerous and should be avoided by design. A protocol π has a *robust message scheduling* if no tolerable attack can force (semi-)honest nodes to make synchronisation errors. In particular, honest participants can always detect when all other honest participants have finished the protocol. A robust scheduling is always achievable in the presence of an authentic broadcast channel, as each participant can broadcast a specific message when he or she completes the protocol. Of course, there are other alternatives, such as fixed round protocols or limits on the duration of π.

Strong stand-alone security model. The example above clearly shows that a protocol π must have a robust scheduling, or otherwise it might lose security in post-processing contexts. However, the latter is not always sufficient when a context has a pre-processing phase. In particular, note that the adversary might gain some knowledge ϕ_{a} in the pre-computation phase that helps to attack the protocol afterwards. In the following, we describe a *strong stand-alone security model* that together with robust messaging assures security in all contexts $\varrho\langle\cdot\rangle$, where the protocol π is executed without any side-computations.

The ideal attack phase $\mathcal{G}_{\text{id-atk}}$ coincides with the standard formalisation described in Section 5.3 but there is a minor change in the real attack phase $\mathcal{G}_{\text{re-atk}}$. Namely, the challenger must always notify the adversary when some participant halts. Secondly, we consider the extended inputs $\boldsymbol{\phi}_{\mathrm{e}} = (\phi_1, \ldots, \phi_n, \phi_{\mathrm{a}})$ instead of normal inputs $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ but otherwise there are no other differences with Section 5.3. In particular, a real and an ideal world model are $(t_{\mathrm{re}}, t_{\mathrm{id}}, \varepsilon)$-close w.r.t. a set of relevant predicates $\mathfrak{B}$ in the strong stand-alone model if for any security objective $\mathcal{B}(\cdot) \in \mathfrak{B}$ and for any t_{re}-time real world adversary $\mathcal{A}$ there exists a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any extended input distribution $\boldsymbol{\phi}_{\mathrm{e}} \leftarrow \mathfrak{D}_{\mathrm{e}}$:

$$|\Pr[\mathcal{G}^{\mathcal{A}}_{\mathrm{real}} = 1] - \Pr[\mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}} = 1]| \le \varepsilon \ . \tag{7.10}$$

As a result, we get a slightly stronger reduction schema

$$\mathcal{G}^{\mathcal{A}}_{\mathrm{real}} \xRightarrow[\varepsilon]{\text{MPC-SEC}^+} \mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}} \ . \tag{7.11}$$

Note that we can always pack an extended distribution $\mathfrak{D}_{\mathrm{e}}$ into a standard input distribution $\mathfrak{D}$ by treating the input of $\mathcal{P}_i$ as a pair $(\phi'_i, \phi'_{\mathrm{a}})$ and the other way around. As a result, it is straightforward to show that the additional knowledge ϕ_{a} obtained in the pre-computation stage does not increase the adversary's ability to attack a protocol in the case of static corruption. Consequently, the extra input

ϕ_a is essential only if an adaptive or mobile adversary uses it to find "optimal" candidates for further corruption, see [Can00a, p.179].

Secondly, we can make a mapping $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ algorithmic for any fixed security goal $\mathcal{B}(\cdot)$. Let $\mathcal{A}_u$ be a universal adversary that interprets ϕ_a as a formal description of an adversarial strategy $\mathcal{A}$. Then the corresponding ideal adversary $\mathcal{A}_u^\circ$ can be viewed as a compiler that devises an appropriate attack strategy for the code ϕ_a. However, the algorithmic correspondence is not for free, since the adversaries $\mathcal{A}_u^\circ(\phi_a)$ and $\mathcal{A}^\circ$ have different running times $t_{id}(c \cdot t_{re})$ and $t_{id}(t_{re})$ where the constant $c > 0$ is the interpreting overhead.

For static adversaries, we can combine two results by considering the universal security goal $\mathcal{B}_u^i(\cdot)$ and the universal adversary $\mathcal{A}_u$. The corresponding compiler $\mathcal{A}_u^i(\cdot)$ assures also the computational indistinguishability of outputs when node $\mathcal{P}_i$ is not corrupted. Hence, our and classical formulation of stand-alone security coincide even if the set of corrupted parties is randomly fixed by the adversary. For the proof, consider a super-compiler $\mathcal{A}_u^*(\cdot)$ that first fixes random coins ω for the adversary $\mathcal{A}$ and obtains the list of corrupted participants $\mathcal{C}$. Next, $\mathcal{A}_u^*(\cdot)$ chooses i such that $\mathcal{P}_i \notin \mathcal{C}$ and uses the compiler $\mathcal{A}_u^i(\cdot)$ to transform the deterministic algorithm $\mathcal{A}(\omega)$ and executes the end result.

Finally, we want to emphasise that the existence of efficient compiler constructions is not formally sufficient for the constructive proofs needed for subjective security. Although such claim seems to be contradictory at first sight, it has a straightforward explanation. The classical security proof might prove non-constructively that the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ exists. The latter proves also the existence of the universal compiler $\mathcal{A}_u^\circ(\cdot)$ but does not provide an explicit code for it. Consequently, the subjective security proof is still incomplete as we cannot *explicitly* write down the code of a universal compiler $\mathcal{A}_u^\circ(\cdot)$. The latter is not a mere theoretical discussion, as some cryptographic proofs actually use inefficient counting arguments, e.g. [BL07].

Also, the existence of such compilers $\mathcal{A}_u^\circ(\cdot)$ does not contradict classical impossibility results for program analysis, since the compiler handles only programs with a fixed size. As a result, the compiler $\mathcal{A}_u^\circ(\cdot)$ may use an ultra-compact advice string—Philosopher's Stone—to speed up the online code analysis phase. Buldas and Laur [BL07] showed that such a Philosopher's Stone must exist for a very specific problem, but the corresponding counting argument itself is quite universal. In a nutshell, white-box proofs exist even for cryptographic primitives that satisfy only abstract security properties.

7.4 CANONICAL DECOMPOSITION TECHNIQUES

High-level security proofs often split interactive computations into isolated subphases to simplify the security analysis. For example, stand-alone security guarantees are sufficient for all interactive post-processing phases that can be embedded into the predicate $\mathcal{B}(\cdot)$. Although this claim seems trivial, the corresponding formal proof must separate post-processing phase from the protocol execution and formalise it as a sub-formula in the predicate $\mathcal{B}(\cdot)$. However, the robust scheduling only assures that the computational process is logically separable, i.e., messages are always sent and received in the same computational phase and we can view internal states $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_m, \psi_a)$ at the end of the phase as

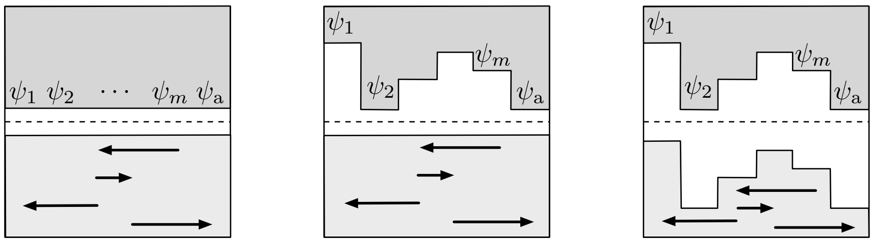


Figure 7.5: Three basic ways to decompose interactive computations into two sub-phases without altering the final output vector $\boldsymbol{z}_{\mathrm{obs}}$.

inputs $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m, \phi_a)$ in the next phase. However, different participants can finish the same phase in different time moments, as depicted in Fig. 7.5. Therefore, the knowledge of the internal states $\boldsymbol{\psi}$ alone might be insufficient to reconstruct the computations in the following phase. Only a slight change in timings may influence the ordering of messages and thus have an effect on the final outputs $\boldsymbol{z}_{\mathrm{obs}}$ even if the protocol and the context are specified in terms of received messages and their order, and not on their arrival times.

There are three principal solutions for this problem. First, we can guarantee that all participants always finish sub-phases simultaneously and thus the timing issues become irrelevant. In the corresponding synchronised communication model, messages are sent in only during short rounds that occur in fixed time intervals, and the exact order of messages in a round is discarded. Such a simple model for message delivery is used in many classical works, see for example [GMW87, BOGW88, CCD88, Bea91b, MR91b, DM00, Can00a].

As a second alternative, we can include timing effects directly to our computational model. More formally, we assume that the challenger is equipped with a clock[1] that is set to zero at the beginning of computations. In this model, observable outputs are pairs (z_i, τ_i), where z_i is a classical output and τ_i is the exact halting time. As a result, we can talk about time-preserving correspondences between real and ideal world models, since a security goal $\mathcal{B}(\cdot)$ now depends on exact halting times $\tau_1, \ldots, \tau_m$. Naturally, such a correspondence is possible only if the adversary has some control over timings. In the corresponding formal model, the challenger always adds timing information to his or her replies and the adversary can specify the exact time when the challenger must carry out the corresponding oracle queries. However, the corresponding *real-time model of computations* is quite technical and takes us away from our main goal—the study of time-invariant protocols and contexts. Hence, we do not discuss this model further, although the following proofs can be directly generalised to handle exact timing in computations.

A third alternative is to eliminate all timing artefacts from the model so that knowledge of internal states $\boldsymbol{\psi}$ alone is sufficient to reconstruct the proceeding computations. For example, we can postulate that the order of messages is independent of the inputs and the adversarial behaviour. Then the knowledge of $\boldsymbol{\psi}$ uniquely determines the output, since the order of messages is fixed but unknown. However, adversarial forces can often influence message delivery and

[1]The clock models a global physical time in the actual computations.

thus such models with fixed message schedulings are overly optimistic. Evidently, we can route messages through special courier nodes $\mathcal{P}_{\mathrm{nw}}^{1},\ldots,\mathcal{P}_{\mathrm{nw}}^{v}$ in order to model network delays. However, slight changes in timings can change the order of messages received by courier nodes $\mathcal{P}_{\mathrm{nw}}^{u}$ and the correct decomposition is still impossible without the exact timing information $\boldsymbol{\tau}$.

In more formal terms, a computational process with several parallel threads is often time-sensitive even if the individual threads are time-invariant. Therefore, we cannot escape timing artefacts, unless we consider pseudo-parallel executions, where only a single participant is active at the same time moment. Let $\mathcal{P}_{\mathrm{sc}}$ be a special malicious *scheduler node* that can activate other participants in any order as long as no two nodes are simultaneously active. That is, all other participants are initially waiting for a special activation messages from $\mathcal{P}_{\mathrm{sc}}$. When a semi-honest node receives an activation message, the node does some computations and gives control back to $\mathcal{P}_{\mathrm{sc}}$. If the activated node is malicious, the control goes back to the adversary. As a result, the adversary has full control over the scheduling and the knowledge of internal states $\boldsymbol{\psi}$ is sufficient for correct decompositions as desired. The corresponding *asynchronous model of computations* has many variations, since there are many reasonable alternatives for scheduling and network behaviour. Still, some formalisations are more fundamental than the others. In particular, all classical computational models [BOCG93, Can00b] require that an activated node stops when it starts to read the next incoming message. Such a scheduling has optimal granularity, since the adversary can dynamically choose any valid message ordering, but the scheduling is coarse enough to hide exact timing.

7.5 CHARACTERISATION OF SEQUENTIAL COMPOSABILITY

In the following section, we study the limitations of strong stand-alone security. In particular, we formally prove that strong stand-alone security is sufficient for all computational contexts, where honest participants stop all other computations during the protocol. Although such an *isolated execution* policy is quite restrictive, it is still adequate for many practical settings. In a sense, this result formally validates the strong stand-alone security model, as it shows that the model provides practically meaningful security guarantees.

For clarity, we divide the corresponding security analysis into two phases. First, we formally prove that any post-processing phase can be embedded into the security goal $\mathcal{B}(\cdot)$ if a protocol has a robust message scheduling. Second, we show that the effects of pre-computation phase can always be modelled as an extended input distribution $\mathfrak{D}_{\mathrm{e}}$. As a result, the security of an isolated protocol execution follows from the strong stand-alone security. In particular, a compound protocol that executes sub-protocols one by one remains secure in the strong stand-alone model as long as sub-protocols are also secure in the strong stand-alone model, i.e., strong stand-alone security is *sequentially composable*. We emphasise here that these claims hold only for the computational models that support *errorless decompositions*, i.e., the knowledge of internal state $\boldsymbol{\psi}$ is sufficient to determine the final outputs $\boldsymbol{z}_{\mathrm{obs}}$.

Lemma 1. *If a protocol has a robust message scheduling, then any post-processing phase can be embedded into the security goal. The corresponding overhead* *is linear in the total complexity of the post-processing phase.*

Proof sketch. For the proof, we split a pair $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ of corresponding security games into two sub-phases. In the first phase, (semi-) honest participants execute the original context until the real protocol π or the ideal protocol π° is completed and output their internal states ψ_i. After all participants have finished, the corresponding output vector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_m, \psi_{\mathrm{a}})$ is used as an input vector for the second phase, where (semi-)honest participants restore their internal states and continue with the execution. The vector of final outputs $\boldsymbol{z}_{\mathrm{obs}}$ is used as an input to the original security goal $\mathcal{B}(\cdot)$. Let us denote the corresponding two-phase security games by $\mathcal{G}^*_{\mathrm{real}}$, $\mathcal{G}^*_{\mathrm{ideal}}$ and the corresponding sub-adversaries by $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}^\circ_1$, $\mathcal{A}^\circ_2$. Next, we define constructive mappings $\mathcal{A} \mapsto (\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{A}^\circ_1, \mathcal{A}^\circ_2) \mapsto \mathcal{A}^\circ$ that achieve perfect correspondence

$$\mathcal{G}^{\mathcal{A}}_{\mathrm{real}} \equiv (\mathcal{G}^*_{\mathrm{real}})^{\mathcal{A}_1, \mathcal{A}_2} \qquad \text{and} \qquad (\mathcal{G}^*_{\mathrm{ideal}})^{\mathcal{A}^\circ_1, \mathcal{A}^\circ_2} \equiv \mathcal{G}^{\mathcal{A}^\circ}_{\mathrm{ideal}} \ . \tag{7.12}$$

More precisely, if we additionally assume that the challenger notifies $\mathcal{A}_1$ in the game $\mathcal{G}^*_{\mathrm{real}}$ when a (semi-)honest participant halts, then the correspondence $\mathcal{A} \mapsto (\mathcal{A}_1, \mathcal{A}_2)$ is evident. Namely, the sub-adversary $\mathcal{A}_1$ can run $\mathcal{A}$ until all (semi-)honest nodes are halted[2] and output the corresponding internal state as ψ_{a}. In the second phase, the sub-adversary $\mathcal{A}_2$ restores the internal state of $\mathcal{A}$ and continues the execution of $\mathcal{A}$. For the second mapping $(\mathcal{A}^\circ_1, \mathcal{A}^\circ_2) \mapsto \mathcal{A}^\circ$, note that the adversary $\mathcal{A}^\circ$ can always detect the change of phases in the ideal world and thus the adversary $\mathcal{A}^\circ$ can sequentially run $\mathcal{A}^\circ_1$ and $\mathcal{A}^\circ_2$.

Now it is straightforward to prove that all decompositions are well defined and the equivalences (7.12) indeed hold if π has a robust scheduling and the computational model facilitates errorless decomposition. Also, the difference in running times is linear. Thus, we can analyse game pairs $(\mathcal{G}^*_{\mathrm{real}}, \mathcal{G}^*_{\mathrm{ideal}})$ and mappings $(\mathcal{A}_1, \mathcal{A}_2) \mapsto (\mathcal{A}^\circ_1, \mathcal{A}^\circ_2)$ instead of $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ and $\mathcal{A} \mapsto \mathcal{A}^\circ$. As the real and ideal world are identical after the first phase, we can always choose $\mathcal{A}^\circ_2 = \mathcal{A}_2$. Consequently, the second phase together with $\mathcal{B}(\cdot)$ can be viewed as a new security objective $\mathcal{B}'(\cdot)$ and the claim follows.

□

Lemma 2. *Any pre-processing phase with robust scheduling can be replaced with an extended input distribution $\mathfrak{D}_{\mathrm{e}}$. The corresponding overhead in running times is linear for the context and for the adversary.*

Proof sketch. Again, we can split a pair $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ of corresponding security games into two sub-phases so that the first phase captures the pre-processing phase and the second phase captures the succeeding computations. Let us denote the corresponding two-phase game pair by $(\mathcal{G}^*_{\mathrm{real}}, \mathcal{G}^*_{\mathrm{ideal}})$ and the vector of internal states between the sub-phases by $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m, \phi_{\mathrm{a}})$. Similarly, to the proof of Lemma 1, we can analyse game pairs $(\mathcal{G}^*_{\mathrm{real}}, \mathcal{G}^*_{\mathrm{ideal}})$ and mappings $(\mathcal{A}_1, \mathcal{A}_2) \mapsto (\mathcal{A}^\circ_1, \mathcal{A}^\circ_2)$ instead of $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ and $\mathcal{A} \mapsto \mathcal{A}^\circ$. We omit the corresponding argumentation, since it is completely analogous to the previous proof. Finally, note that the first phase can be replaced with sampling $\boldsymbol{\phi}_{\mathrm{e}} \leftarrow \mathfrak{D}_{\mathrm{e}}$ in the games $\mathcal{G}^*_{\mathrm{real}}$ and $\mathcal{G}^*_{\mathrm{ideal}}$, if we consider only mappings with $\mathcal{A}^\circ_1 = \mathcal{A}_1$. Again such a choice is natural, since the ideal and real world coincide in the first phase.

□

[2]The escaping messages produced by the first stage of the adversary should be re-sent in the second stage to assure equivalence.

These decomposition lemmas alone are sufficient to prove several sequential composability results, provided that the computational context $\varrho\langle\cdot\rangle$ and the protocol π share the same set of participants. However, this assumption is rarely valid in practical applications, where only a few participants $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_n}$ of the global context $\varrho\langle\cdot\rangle$ take part in the protocol π. In fact, participants $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_n}$ usually have little, if any, knowledge about the global context $\varrho\langle\cdot\rangle$. Also, it is unlikely that participants $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_n}$ can force some global restrictions on the other participants. However, local restrictions concerning only the participants of the protocol π are enforceable. In particular, participants can always run the protocol π in *isolation* by forcing the following local restrictions:

(S1) The context $\varrho\langle\cdot\rangle$ uses the protocol π in black-box manner.

(S2) The protocol π and pre-processing phase have robust message scheduling when we consider only the participants of the protocol π.

(S3) The participants of the protocol π send out only the messages of π and delete all non-protocol messages during the execution of the protocol π.

Note that there are several ways how the participants of the protocol π can relate to other participants. If the protocol π provides an end-to-end solution, then it is also reasonable to assume that all non-participants are corrupted, as these external entities are out of our control. For example, all those millions and millions of computational devices that are connected to Internet, but do not participate in our protocol, are potentially hostile non-participants that we cannot control.

In the two-party setting, it is natural to assume that non-participants are corrupted, since each party trusts only him- or herself. The same is true for multiparty protocols, where one controls a fixed set of nodes and thus the adversary can potentially abuse all signals that are sent out of this group. On the other hand, the group of semi-controllable nodes that follow the restrictions (S1)–(S3) might change in time. For example, some new members that are believed to be honest can join the group. Therefore, it makes sense to slightly relax the assumption and postulate the following:

(A1) All nodes that do not participate in the protocol π are compromised during the execution of π, afterwards the adversary might retreat from them.

(A2) All trusted third parties that are guaranteed to finish before or start after the execution of π can be considered as participants of the protocol π.

Now it is straightforward to establish a correspondence between the isolated protocol execution and the strong stand-alone security model, i.e., the strong stand-alone security is sufficient for all security levels except for input-privacy. For clarity, we state only the qualitative results. The exact quantitative relations between the time bounds can be extracted from the proofs if necessary.

Theorem 1. *A context $\varrho\langle\cdot\rangle$ and a protocol π that satisfy restrictions (S1)–(S3) and (A1)–(A2) preserve strong stand-alone security. The corresponding overhead is linear for the running times and the bound on the advantage does not change.*

Proof sketch. Let us first consider the simplified case, where the participants of the context $\varrho\langle\cdot\rangle$ and the protocol π coincide. Since the assumptions of Lemmata 1 and 2 are satisfied, we can split the respective game pair $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ into three sub-phases, where the first phase corresponds to the pre-processing phase and the third one to the post-processing phase. Moreover, there are constructive mappings $\mathcal{A} \mapsto (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $(\mathcal{A}_1^\circ, \mathcal{A}_2^\circ, \mathcal{A}_3^\circ) \mapsto \mathcal{A}^\circ$ such that

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \equiv (\mathcal{G}_{\mathrm{real}}^{*})^{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3} \qquad \text{and} \qquad (\mathcal{G}_{\mathrm{ideal}}^{*})^{\mathcal{A}_1^\circ, \mathcal{A}_2^\circ, \mathcal{A}_3^\circ} \equiv \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} \ . \tag{7.13}$$

To conclude the proof, we must construct a mapping $\mathcal{A}_2 \mapsto \mathcal{A}_2^\circ$ such that

$$\left|\Pr\left[(\mathcal{G}_{\mathrm{real}}^{*})^{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3} = 1\right] - \Pr\left[(\mathcal{G}_{\mathrm{ideal}}^{*})^{\mathcal{A}_1, \mathcal{A}_2^\circ, \mathcal{A}_3} = 1\right]\right| \leq \varepsilon \ . \tag{7.14}$$

Lemma 1 assures that the third phase can be embedded into the security goal with a linear overhead. Lemma 2 assures that the first phase can be replaced with an appropriate extended distribution $\mathfrak{D}_{\mathrm{e}}$. As the second phase corresponds to the strong stand-alone security model, the existence of a mapping $\mathcal{A}_2 \mapsto \mathcal{A}_2^\circ$ is implied by the assumptions of the theorem. The claim holds even if the exact roles in the protocol are determined dynamically by the context, since we can always assume that the internal states $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ also contain the formal specification (code) of the computational context.

For the general case, consider a new extended protocol $\hat{\pi}$, where the non-participants of the protocol π just output their initial states and halt. Then the original security game $\mathcal{G}_{\mathrm{real}}$ can be viewed as an execution of $\hat{\pi}$, where the adversary corrupts the non-participants during the execution to do some malicious side-computations. Note that the protocol $\hat{\pi}$ and the pre-processing phase formally have a robust scheduling, since all trusted third parties are guaranteed to be inactive and other non-participants are assumed to be corrupted during the execution of the protocol $\hat{\pi}$. Thus, we have reached the simplified case.

To conclude, note that the modified protocol $\hat{\pi}$ preserves strong stand-alone security possibly with linear overhead. First, the presence of inactive trusted third parties is irrelevant, since their inputs and outputs are empty. Second, there is one-to-one correspondence between the security games for the protocols π and $\hat{\pi}$. Namely, the inputs of other non-participants can be embedded into the adversarial input ϕ_{a} and the outputs can be embedded into the output ψ_{a}. Similarly, any security goal $\mathcal{B}(\cdot)$ for the protocol $\hat{\pi}$ can be translated to a predicate $\mathcal{B}'(\cdot)$ that first extracts all embedded outputs from ψ_{a} and then evaluates $\mathcal{B}(\cdot)$. Hence, it is straightforward to construct an appropriate mapping $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ for the protocol $\hat{\pi}$ given only a mapping $(\mathcal{B}', \mathcal{A}) \mapsto \mathcal{A}^\circ$ for the protocol π. The corresponding overhead is linear, as needed.

□

Corollary 1. *Security in the strong stand-alone security model is sequentially composable, provided that all sub-protocols have robust scheduling.*

Proof. Let a protocol π be a sequential application of protocols $\pi_1, \ldots, \pi_s$ that share the same set of participants. Then Theorem 1 assures that we can substitute all protocols π_i step by step with the corresponding ideal implementations π_i°. Although each substitution step introduces a new trusted third party, the assumption (A2) still holds and we can continue. The corresponding game chain completes the proof and provides necessary bounds on the advantage. □

Further discussion. The correspondence between the stand-alone security and the isolated protocol execution implies that the strong stand-alone security is a sufficient and necessary security objective for end-to-end solutions. Also, note that protocol designers control only a tiny part of the whole world and the distinction between internal and external entities is unavoidable. The assumption (A1) captures the obvious judgement that external entities do not have to follow any restrictions. In that light, Theorem 1 is just a description that explains how and what kind of protocols one should execute in unknown hostile environments. We emphasise that the global computational context itself can reveal much more information than the ideal implementation of the protocol. Nevertheless, we cannot relax security requirements further to optimise the protocol performance, since we do not know how much extra information is revealed by the computational context. Consequently, the equivalence between ideal and real world implementations is the best we can achieve.

Historically, cryptographers have always taken the stand-alone security model for granted and have seldom explored its limitations in practical applications. The latter is the main reason why Theorem 1 or its analogues have never been proved in the mainstream cryptographic literature, although the result has been implicitly known for decades. In that sense, various composability results that simplify the design and analysis of protocols are more important. Thus, classical treatments of secure computations, such as [Can00a, Gol04], emphasise only sequential composability results and the corresponding design methodology.

Also, note that the equivalence result holds only if we assume that all external parties are corrupted. This assumption is often too conservative and can significantly weaken the corresponding security guarantees. The latter is another reason why the equivalence result has remained unnoticed, as other more stringent security notions provide more precise security guarantees.

Secondly, note that the usage restrictions (S1)–(S3) and (A1)–(A2) are optimal for the strong stand-alone security. In particular, stand-alone security is generally insufficient to guarantee parallel composability, see [GK96b, Lin04]. Hence, the assumption (A1) is essential and cannot be relaxed, since the restrictions (S1)–(S3) do not eliminate the possibility of parallel execution. For example, consider a context where $\mathcal{P}_1, \mathcal{P}_2$ execute a protocol π_1 and $\mathcal{P}_3, \mathcal{P}_4$ execute a protocol π_2 at the same time. Then the restrictions (S1)–(S3) and (A2) are formally satisfied, although the real implementation cannot be replaced with the ideal protocol. A concrete counterexample is given in [GK96b] as Theorem 5.1. The same counterexample can be modified to show that all trusted third parties must be inactive during the execution of the protocol.

Although the usage restrictions are optimal, we can still do some cosmetic relaxations. In particular, we can bypass the restriction (A1) by artificially increasing the number of participants. Let $\hat{\pi}$ be an extended protocol where all non-participants are inactive during the execution of π. Then it is straightforward to prove that the protocol $\hat{\pi}$ preserves strong stand-alone security if and only if the correspondence $(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{A}^\circ$ is independent of the security goal $\mathcal{B}$. Consequently, if the protocol π is secure according to classical definitions, there is no need to corrupt honest participants that remain inactive during the execution of π. However, this relaxation holds only for those honest participants that are guaranteed to be inactive, since the pre-processing phase and the protocol $\hat{\pi}$ must have a robust message scheduling.

Centralised task scheduling. Note that strong stand-alone security is sufficient for modular protocol design, since sub-tasks can always be executed one by one.

As an advantage, we do not have to strengthen the security requirements for sub-protocols but there are also many important drawbacks.

Firstly, we must implement a global synchronisation service that reserves time slots for individual sub-tasks. As a result, all nodes must either participate in a protocol or wait until the protocol is completed. Thus, the whole network acts as a single virtual processor and load balancing is impossible. Moreover, the overall performance is determined by the slowest node in the network.

Secondly, note that modern computing devices do millions of operations in few milliseconds. Consequently, network delays can cause significant performance penalties. Moreover, the relative impact of network delays only increases in the future, as these delays are likely to remain constant, whereas the computational throughput is likely to grow in the future.

Thirdly, secure function evaluation is not the only practical design goal. In many cases, we need to implement secure services that have several rounds or are non-terminating. For example, access control and secure storage systems must function continuously. Centralised task scheduling makes it impossible to run two or more continuous services in parallel. Instead, we must divide the corresponding ideal functionalities into micro-rounds and use some kind of fair interleaving mechanism to assure the availability of all services. However, such a solution is artificial and inherently decreases the responsiveness.

7.6 CHARACTERISATION OF UNIVERSAL COMPOSABILITY

In many cases, it is impossible to guarantee that sub-protocols are not executed simultaneously or the corresponding performance penalty is prohibitively large. Hence, it is advantageous to consider more liberal task schedulings. The latter again requires more strict security definitions, since the stand-alone security does not assure the security of parallel executions [GK96b, Lin04]. Evidently, various restrictions on computational contexts lead to different security requirements and feasibility results, for example, compare the results of [Pas04] and [Lin04]. However, the fundamental limitations obtained in [Lin03a, Lin04] indicate that additional usage restrictions do not make it easier to design protocols, unless these restrictions are really specific. Hence, it makes sense to consider protocols that preserve the security in all time-bounded computational contexts.

Formally, we must first fix all lower level artefacts, such as message delivery and timing models. In particular, we can study synchronised, asynchronous or real-time model of computations. As usual let π be a protocol and π° denote the corresponding ideal implementation. Let $\mathfrak{G}_{\mathrm{u}}$ be the set of all game pairs $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}})$ that are induced by all computational contexts $\varrho\langle\cdot\rangle$ and by all security objectives $\mathcal{B}(\cdot)$. Finally, fix an adversarial structure $\mathcal{A}_{\mathrm{loc}}$ that specifies the tolerable adversarial behaviour in local terms. The latter is necessary as contexts may assign different roles in the protocol π to different participants. Now the protocol π is $(t_{\mathrm{re}}, t_{\mathrm{pr}}, t_{\mathrm{cnt}}, \varepsilon)$-*universally composable* w.r.t. the adversarial structure $\mathcal{A}_{\mathrm{loc}}$ if for any t_{cnt}-time game pair $(\mathcal{G}_{\mathrm{ideal}}, \mathcal{G}_{\mathrm{real}}) \in \mathfrak{G}_{\mathrm{u}}$ and for any t_{re}-time real world adversary $\mathcal{A}$ that respects $\mathcal{A}_{\mathrm{loc}}$, there exist a t_{id}-time ideal world adversary $\mathcal{A}^\circ$ such that for any extended input distribution $\mathfrak{D}_{\mathrm{e}}$, we have

$$|\Pr[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1] - \Pr[\mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} = 1]| \leq \varepsilon \ . \tag{7.15}$$

The corresponding reduction schema is denoted as

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \xrightarrow[\varepsilon]{\text{UC-SEC}} \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} \quad . \tag{7.16}$$

We remark that the original definition of universal composability [Can01] was given in different technical terms. The computational context $\varrho\langle\cdot\rangle$ is substituted with an autonomous malicious entity $\mathcal{Z}$ that has black-box access to the protocol π and interacts with the adversary. However, the difference is purely syntactical, as the environment $\mathcal{Z}$ can always simulate the computational context $\varrho\langle\cdot\rangle$ and vice versa, see also [Lin03b]. We prefer the notion of computational context, since it is conceptually simpler and closer to reality.

Also, note that our definition is based on a mapping $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}}, \mathcal{A}) \mapsto \mathcal{A}^\circ$, whereas the original definition [Can01] requires universality $\mathcal{A} \mapsto \mathcal{A}^\circ$ for all t_{cnt}-time game pairs $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}}) \in \mathfrak{G}_{\mathrm{u}}$. As a result, the original definition also violates the external consistency restrictions (C1)–(C2), whereas our definition is externally consistent. Still, these definitions are qualitatively equivalent. More formally, consider a universal context $\varrho_{\mathrm{u}}\langle\cdot\rangle$, where all participants interpret their inputs as a code. Then the corresponding ideal world adversary $\mathcal{A}_{\mathrm{u}}^\circ$ is universal for all game pairs $(\mathcal{G}_{\mathrm{real}}, \mathcal{G}_{\mathrm{ideal}}) \in \mathfrak{G}_{\mathrm{u}}$ and the quantitative changes in the time bounds are linear. However, this equivalence result was first rejected in the asymptotic security model, since the initial definition of contexts with polynomial running times[3] was too strict, see [Can00b, p. 49–50].

Finally, note that universal composability is closed under concurrent composition. More formally, if a compound protocol $\varrho_0\langle\pi_1^\circ\rangle$ is a universally composable implementation of a functionality π_0° and a protocol π_1 is a universally composable implementation of a functionality π_1°, then the protocol $\varrho_0\langle\pi_1\rangle$ is also a universally composable implementation of the functionality π_0° regardless of the message scheduling. Indeed, fix a computational context $\varrho\langle\cdot\rangle$ and consider the corresponding chain of compound protocols

$$\varrho\langle\pi^\circ\rangle \xrightarrow{\pi_0^\circ} \varrho\langle\varrho_0\langle\pi_1^\circ\rangle\rangle \equiv \varrho_1\langle\pi_1^\circ\rangle \xrightarrow{\pi_1} \varrho_1\langle\pi_1\rangle \quad , \tag{7.17}$$

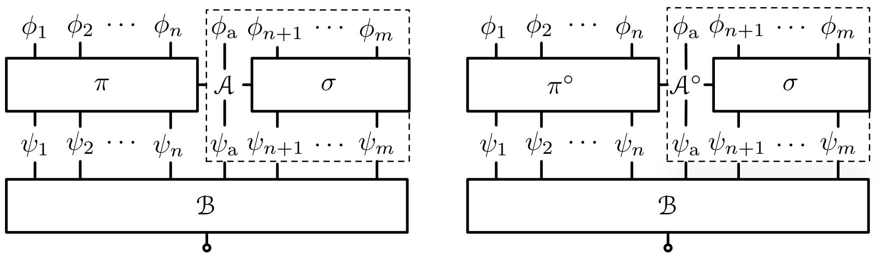
where $\varrho_1\langle\cdot\rangle = \varrho\langle\varrho_0\langle\cdot\rangle\rangle$ is a shorthand for another computational context. Note that we can still complete the corresponding reversed game chain

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \equiv \mathcal{G}_3^{\mathcal{A}_3} \xrightarrow{\text{UC-SEC}} \mathcal{G}_2^{\mathcal{A}_2} \equiv \mathcal{G}_1^{\mathcal{A}_1} \xrightarrow{\text{UC-SEC}} \mathcal{G}_0^{\mathcal{A}_0} \equiv \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^\circ} \quad , \tag{7.18}$$

where the security games $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ correspond to the protocols and contexts from left to right in the chain (7.17) and $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{G}_1 = \mathcal{G}_2$. To be precise, the game chain exists only if the adversarial structures for both protocols π_0 and π_1 are compatible. Evidently, the result holds also for longer chains of hybrid protocols, provided that all necessary quantitative bounds are compatible.

Criterion for universal composability. Although Canetti was the first to define the universal composability for the asynchronous model of computations, the concept itself was known already in the early 1990s. Micali and Rogaway defined and studied the corresponding security objective as *reducibility* in the synchronised model of computations, see articles [MR91a, MR91b] for further discussion. Also, the description of universally composable protocols has been

[3]The latter is another example that illustrates how dangerous it is to define asymptotic security models directly without first considering the exact formalisation.



ϕ_1 ϕ_2 $\cdots$ ϕ_n ϕ_a ϕ_{n+1} $\cdots$ ϕ_m
π
$\mathcal{A}$
σ
ψ_1 ψ_2 $\cdots$ ψ_n ψ_a ψ_{n+1} $\cdots$ ψ_m
$\mathcal{B}$
ϕ_1 ϕ_2 $\cdots$ ϕ_n ϕ_a ϕ_{n+1} $\cdots$ ϕ_m
π°
$\mathcal{A}^\circ$
σ
ψ_1 ψ_2 $\cdots$ ψ_n ψ_a ψ_{n+1} $\cdots$ ψ_m
$\mathcal{B}$

Figure 7.6: Correspondence between parallel execution and stand-alone model. The dashed box represents the compound adversary in the stand-alone model.

implicitly known for decades, starting from the early works [MR91b, Bea91b]. All subsequent works, such as [DM00, PW00, Can01, Lin03b], have just refined technical details. Nevertheless, we provide the characterisation together with a proof sketch, since the latter underlines some intricate connections with sequential composability and emphasises some unexpected details.

Again, we restrict ourselves to the computational models that support errorless decomposition. Consequently, we can ignore the pre- and post-processing phase and concentrate on the interactions between the protocol π and the side-computations σ, see Fig. 7.6. Observe that a real world adversary $\mathcal{A}$ can always break the conceptual separation between π and σ by selectively transferring data from one process to another. Such behaviour is impossible in the ideal world, where the idealised protocol π° itself is shielded from the adversary $\mathcal{A}^\circ$. Note that the proof of Theorem 1 uses just some obvious although very crude bounds on the information flow between these processes. Let $\mathcal{I}_\sigma$ denote the set of nodes that participate in the side computations. Then the information leakage from the process σ is bounded by the corresponding inputs $(\phi_i)_{i\in\mathcal{I}_\sigma}$. Similarly, we can overestimate the reverse flow of information from π to σ by giving the control over the output states $(\psi_i)_{i\in\mathcal{I}_\sigma}$ to the adversary $\mathcal{A}^\circ$.

Of course, the resulting security estimate is very crude, as real world adversaries normally have only partial control over the states $(\phi_i)_{i\in\mathcal{I}_\sigma}$ and $(\psi_i)_{i\in\mathcal{I}_\sigma}$. More precisely, the stand-alone adversary $\mathcal{A}_\mathrm{s} = (\mathcal{A}, \mathcal{P}_i)_{i\in\mathcal{I}_\sigma}$ can be viewed as a tuple, where $\mathcal{A}$ has no direct control over non-corrupted participants. Hence, we must place structural restrictions to the correspondence $(\mathcal{B}, \mathcal{A}_\mathrm{s}) \mapsto \mathcal{A}_\mathrm{s}^\circ$ in the stand-alone model, or otherwise the adversarial structures differ in the real and the ideal world. One possible solution was put forward by Beaver [Bea91b]. Namely, consider a non-rewinding *interface* $\mathcal{I}$ between the ideal implementation π° and the real world adversary $\mathcal{A}$ that translates the corresponding communication. Briefly, the interface $\mathcal{I}$ must simulate all protocol messages for the adversary $\mathcal{A}$ given only access to the ideal implementation π°. As a result, we can view the interface as a computational context $\mathcal{I}\langle\cdot\rangle$ that uses the real world adversary $\mathcal{A}_\mathrm{s}$ in a black-box manner to define the corresponding ideal world adversary[4] $\mathcal{A}_\mathrm{s}^\circ = \mathcal{I}\langle\mathcal{A}_\mathrm{s}\rangle = \mathcal{I}\langle\mathcal{A}, \mathcal{P}_i\rangle_{i\in\mathcal{I}_\sigma}$. More formally, we say that a protocol π is $(t_\mathrm{re}, t_\mathrm{id}, \varepsilon)$-close w.r.t. the predicate set $\mathfrak{B}$ and interface $\mathcal{I}\langle\cdot\rangle$ in the strong stand-alone model, if for any t_re-time adversary $\mathcal{A}_\mathrm{s}$ and for any predicate $\mathcal{B}(\cdot) \in \mathfrak{B}$, the

[4]Canetti uses a term shell simulator to denote the compound adversary $\mathcal{I}\langle\mathcal{A}_\mathrm{s}\rangle$ in [Can00b].

ideal t_{id}-time adversary $\mathcal{A}_{\mathrm{s}}^{\circ} = \mathcal{I}\langle\mathcal{A}_{\mathrm{s}}\rangle$ achieves

$$\left|\Pr\left[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}_{\mathrm{s}}} = 1\right] - \Pr\left[\mathcal{G}_{\mathrm{real}}^{\mathcal{I}\langle\mathcal{A}_{\mathrm{s}}\rangle} = 1\right]\right| \leq \varepsilon \tag{7.19}$$

for any extended input distribution $\mathfrak{D}_{\mathrm{e}}$. As the interface $\mathcal{I}\langle\cdot\rangle$ preserves the adversarial structure, it is straightforward to relax requirements in Theorem 1. However, the proof holds only if the communication of the protocol π is separable from the remaining messages. Here, we assume that all messages of π are routed via a dedicated courier node $\mathcal{P}_{\mathrm{nw}}^{\pi}$ but there are many alternatives. For example, the standard formalisation given by Canetti requires that each message contains a special identifier that uniquely determines its purpose [Can00b, p. 21–32].

Theorem 2. *A context $\varrho\langle\cdot\rangle$ and a protocol π that satisfy restrictions (S1)–(S2) preserve strong stand-alone security w.r.t. the interface $\mathcal{I}\langle\cdot\rangle$. The corresponding overhead is linear for running times and the advantage does not change.*

Proof sketch. For clarity, we first consider a context $\varrho\langle\cdot\rangle$, where participants of π do no side-computations. As the restrictions (S1)–(S2) still hold, we can follow the proof of Theorem 1 and split the execution into three phases. In particular, constructive mappings $\mathcal{A} \mapsto (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $(\mathcal{A}_1^{\circ}, \mathcal{A}_2^{\circ}, \mathcal{A}_3^{\circ}) \mapsto \mathcal{A}^{\circ}$ such that

$$\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} \equiv (\mathcal{G}_{\mathrm{real}}^{*})^{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3} \qquad \text{and} \qquad (\mathcal{G}_{\mathrm{ideal}}^{*})^{\mathcal{A}_1^{\circ}, \mathcal{A}_2^{\circ}, \mathcal{A}_3^{\circ}} \equiv \mathcal{G}_{\mathrm{ideal}}^{\mathcal{A}^{\circ}} \tag{7.20}$$

are still the same as in the proof of Theorem 1. Thus, the mapping $\mathcal{A}_2 \mapsto \mathcal{I}\langle\mathcal{A}_2\rangle$ defines the correspondence used in the proof of Theorem 1 and we obtain

$$\left|\Pr\left[(\mathcal{G}_{\mathrm{real}}^{*})^{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3} = 1\right] - \Pr\left[(\mathcal{G}_{\mathrm{ideal}}^{*})^{\mathcal{A}_1, \mathcal{I}\langle\mathcal{A}_2\rangle, \mathcal{A}_3} = 1\right]\right| \leq \varepsilon\ . \tag{7.21}$$

Recall that $\mathcal{A}_2$ is actually a compound adversary that captures the behaviour of $\mathcal{A}$ and non-participants $(\mathcal{P}_i)_{i \in \mathcal{I}_\sigma}$ during the execution of the protocol π. Since $\mathcal{I}\langle\mathcal{A}_2\rangle$ does not change the interaction patterns between $\mathcal{A}$ and non-participants $(\mathcal{P}_i)_{i \in \mathcal{I}_\sigma}$, the resulting correspondence $\mathcal{A} \mapsto \mathcal{A}^{\circ}$ also preserves the adversarial structure and the claim follows from the proof of Theorem 1.

The general case is slightly more complex, as nodes can concurrently participate in π and σ. Since all protocol messages are routed via the node $\mathcal{P}_{\mathrm{nw}}^{\pi}$, the correspondence $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \mapsto (\mathcal{A}_1, \mathcal{I}\langle\mathcal{A}_2\rangle, \mathcal{A}_3)$ described above is formally valid and the bound (7.21) still holds. For the formal proof, we split each node into two virtual sub-nodes $(\mathcal{P}_i, \mathcal{P}_i^{*})$ with a shared[5] internal state, where $\mathcal{P}_i$ does all the computations except for π and $\mathcal{P}_i^{*}$ is reserved for the protocol π. That is, all messages transferred by the dedicated courier node $\mathcal{P}_{\mathrm{nw}}^{\pi}$ are sent out and received by sub-nodes $\mathcal{P}_i^{*}$. All other messages are sent and received by normal nodes $\mathcal{P}_i$. Also, the adversary $\mathcal{A}$ can corrupt the node pair $(\mathcal{P}_i, \mathcal{P}_i^{*})$ only simultaneously, as labels $\mathcal{P}_i$ are $\mathcal{P}_i^{*}$ refer to the same node.

As a result, we have reached the simplified case, where participants of π do not engage in any side-computations σ. The claim follows, as the corresponding adversary $\mathcal{I}\langle\mathcal{A}, \mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}_1^{*}, \ldots, \mathcal{P}_n^{*}\rangle$ behaves exactly as $\mathcal{I}\langle\mathcal{A}_2\rangle$.

□

[5]It is just a mind experiment that gives two labels for the same node to simplify the reasoning in the proof.

Further relaxations. First, note that there is no difference between the normal non-corruptible participant $\mathcal{P}_i$ and a trusted third party $\mathcal{T}_j$. Hence, the proof also holds if there are many trusted third parties in the context.

Secondly, we emphasise that the restriction (S2) is essential or otherwise the decomposition can be ill-defined and the reasoning may fail. For example, the lottery protocol π_{sl} in Fig. 7.4 has a non-rewinding interface $\mathcal{I}\langle\cdot\rangle$ in the strong stand-alone model although it does not preserve the security even in the simple post-processing context described in Section 7.3. Nevertheless, we can use Theorem 2 to construct universally composable protocols that remain secure in all contexts, where the restriction (S1) is satisfied. More precisely, let $\varrho_0\langle\cdot\rangle$ be a wrapper context that eliminates the possibility of synchronisation errors. For example, a participant in the context $\varrho_0\langle\cdot\rangle$ starts with π when all participants of π have sent a message "I am ready to start π" and waits until all participants of π send "I have finished π" after the completion. Then a modified protocol $\hat{\pi} = \varrho_0\langle\pi\rangle$ is universally composable, as $\varrho\langle\hat{\pi}\rangle = \varrho\langle\varrho_0\langle\pi\rangle\rangle = \varrho_1\langle\pi\rangle$ and thus the protocol π in the context $\varrho_1\langle\cdot\rangle$ satisfies the restrictions (S1)–(S2) by the construction. To be precise, the adversary learns when a participant starts and ends the protocol, but this is usually irrelevant in practice.

Although external synchronisation is always possible, the resulting network delays can cause significant performance penalties. Hence, it makes sense to consider protocols with built-in synchronisation. Such protocols must remain secure even if some protocol messages are received before the protocol execution and assure that final outputs are not released back to the context too early. Note that the proof of Theorem 2 generalises and we can show that a protocol is universally composable if there exists a non-rewinding interface $\mathcal{I}\langle\cdot\rangle$ such that

$$\left|\Pr\left[\mathcal{G}_{\mathrm{real}}^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_{\mathrm{real}}^{\mathcal{I}\langle\mathcal{A}\rangle} = 1\right]\right| \leq \varepsilon \tag{7.22}$$

for any context $\varrho\langle\cdot\rangle$ without side-computations. In other words, we must first prove that there exists an interface $\mathcal{I}\langle\cdot\rangle$ for the stand-alone model and then show that the interface also works for other contexts or alternatively provide some extension to $\mathcal{I}\langle\cdot\rangle$ that handles the corresponding synchronisation errors.

Optimality of assumptions. Obviously, we cannot relax the restriction (S1) without losing all security guarantees. Similarly, a protocol π is universally composable only if there exists a non-rewinding interface $\mathcal{I}\langle\cdot\rangle$ such that the protocol is secure in the strong stand-alone model w.r.t. this interface $\mathcal{I}\langle\cdot\rangle$. For proof, consider a strong stand-alone setting, where the participants $\mathcal{P}_1, \ldots, \mathcal{P}_n$ execute the protocol π and a non-corruptible bystander $\mathcal{P}_{n+1}$ that interprets the input ϕ_{n+1} as a code and assists the universal adversary $\mathcal{A}_{\mathrm{u}}$. More precisely, the adversary $\mathcal{A}_{\mathrm{u}}$ acts as a wire between the challenger and $\mathcal{P}_{n+1}$, i.e., $\mathcal{A}_{\mathrm{u}}$ forwards all messages sent by the challenger to $\mathcal{P}_{n+1}$ and all commands sent by $\mathcal{P}_{n+1}$ to the challenger. Since the strong stand-alone setting itself is a computational context, there must exist a universal ideal world adversary $\mathcal{A}_{\mathrm{u}}^{\circ}$. Note that $\mathcal{A}_{\mathrm{u}}^{\circ}$ is a non-rewinding interface $\mathcal{I}\langle\cdot\rangle$ that translates communication between $\mathcal{P}_{n+1}$ and the ideal world challenger. Since $\mathcal{P}_{n+1}$ can interpret the code of any stand-alone adversary $\mathcal{A}$ with a linear slowdown, the requirement is optimal up to a constant factor overhead in running times. To be precise, the claim holds only for the synchronised and asynchronous model of computations, since the participant $\mathcal{P}_{n+1}$ cannot stop the clock for his computations.

7.7 TRUSTED SETUP AND UNIVERSAL COMPOSABILITY

The design of universally composable protocols is a challenging task, since the corresponding protocols must have a black-box non-rewinding interface $\mathcal{I}\langle\cdot\rangle$. Such requirements diminish possible differences between the adversary and the interface. In fact, the interface $\mathcal{I}\langle\cdot\rangle$ can be viewed as an attack strategy that allows input extraction and gives partial control over the outputs. Hence, the asymmetry of powers between the adversary and the interface exists only if the adversarial structure itself is asymmetric or there is a significant discrepancy in the running times. In particular, note that classical multi-party protocols with honest majority indeed have non-rewinding interfaces, see for example [Bea91b]. Hence, it is straightforward to show that these protocols are indeed universally composable [Can01, Can00b]. On the other hand, no non-trivial protocol with honest minority can be universally composable unless the interface has a significantly higher computational demands than the adversary [CKL03].

There are two alternatives how to achieve universal composability in the case of honest minority. First, we can allow a significant discrepancy $t_{\mathrm{re}} \ll t_{\mathrm{id}}$ between the running times of the real and the ideal world adversaries. However, such a choice has several important drawbacks. As the correspondence between real and ideal time-success profiles is less tight, the protocol might provide a way to speed up some computations. Consequently, such protocols are guaranteed to preserve only information-theoretical properties of the corresponding ideal implementations. The large discrepancy $t_{\mathrm{re}} \ll t_{\mathrm{id}}$ also introduces complications in the security proofs. As a substitution $\pi \mapsto \pi^{\circ}$ in a security proof yields a t_{id}-time adversary $\mathcal{A}^{\circ}$, a protocol that uses π as a sub-protocol must be secure against t_{id}-time adversaries. In particular, we cannot use repetitive substitutions to analyse a context with many instances of the protocol π.

The latter is a serious drawback, since useful sub-protocols are commonly used in many places. Therefore, one needs a stronger version of universal composability that allows to replace many instances of a protocol by a single reduction step, i.e., we must analyse concurrent executions directly. For example, many authors first prove the security of a protocol in the world, where a trusted third party carries out zero-knowledge proofs, and later replace all ideal implementations with a protocol that is known to be secure in such concurrent settings [PR03, Pas04, BS05]. Another more modular alternative was proposed in [PS04] where authors assumed that the interface $\mathcal{I}\langle\cdot\rangle$ is generally efficient except for few well-described functions $\mathcal{O}_1, \ldots, \mathcal{O}_s$. Hence, a running time can be viewed as a vector $\boldsymbol{t} = (t_0, t_1, \ldots, t_s)$, where t_0 is the number of elementary steps without oracle calls and t_i is the number of calls to the oracle $\mathcal{O}_i$. As the corresponding time bounds $\boldsymbol{t}_{\mathrm{re}}$ and $\boldsymbol{t}_{\mathrm{id}}$ are more structured than before, the corresponding reduction scheme might withstand many iterations.

A trusted setup is another and often a more practical way to achieve universal composability in the case of honest minority. Such a setup phase naturally introduces necessary asymmetry into the adversarial structure. The trusted dealer $\mathcal{P}_{\mathrm{ts}}$ is non-corruptible in the real world, whereas the interface $\mathcal{I}\langle\cdot\rangle$ always corrupts $\mathcal{P}_{\mathrm{ts}}$ in the ideal world. To be precise, the dealer $\mathcal{P}_{\mathrm{ts}}$ is not present in the ideal world at all and thus the interface $\mathcal{I}\langle\cdot\rangle$ must simulate its presence, i.e., the interface $\mathcal{I}\langle\cdot\rangle$ has full control over $\mathcal{P}_{\mathrm{ts}}$ by the construction. Such an asymmetry makes the design of universally composable protocols pretty straightforward. Moreover, the

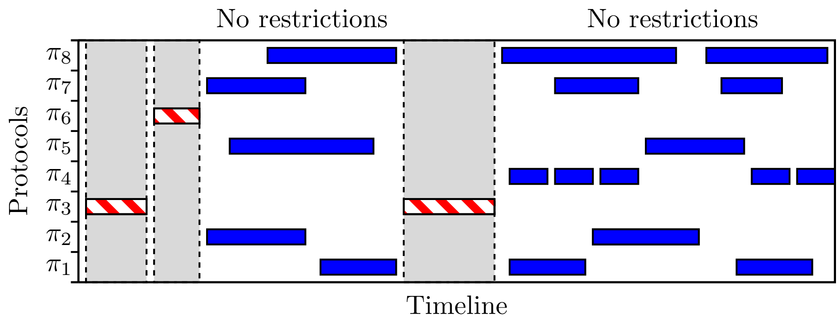


Figure 7.7: Scheduling restrictions posed by modular design. Sub-protocols π_3 and π_6 are run in isolation, since they implement trusted setup procedures.

corresponding computational overhead compared to stand-alone setting is minimal compared to the protocols with semi-inefficient simulators. Thus, we can systematically use protocols with trusted setup to achieve modularity in security proofs without losing efficiency and round complexity.

The corresponding methodology consists of three steps. First, we must design universally composable protocols for all necessary sub-tasks. At this stage our main design goal is to achieve universal composability with minimal overhead. The corresponding trusted setup phase can be as complicated as needed and the question whether such setup scenario is plausible in practice is irrelevant. In the second step, we use these sub-protocols to construct a round optimal solution. The corresponding security proof is straightforward, since all protocols are universally composable. As a final design step, we replace all trusted setup procedures with standard protocols that are secure only in the strong stand-alone security model. The resulting protocol is both efficient and secure in the strong stand-alone model and has a modular security proof.

To be precise, security in the strong stand-alone model is an understatement, as we can apply Theorem 1 in any computational context. Consequently, the compound protocol remains secure in all computational contexts, where the restrictions (S1)–(S3) and (A1)–(A2) are satisfied during the execution of sub-protocols that implement trusted setup phases, see Fig. 7.7.

However, note that a naive application of these design principles can lead to sub-optimal solutions. As all non-trivial protocols require trusted setup, the resulting protocol can contain an enormous pre-processing phase, i.e., we must still run a large portion of the protocol in isolation. Hence, it is advantageous to reduce the number of setup phases either with bootstrapping or sharing a single setup phase between many protocols. The idea behind bootstrapping is obvious. Imagine that there exists a universally composable protocol π^*_{ts} with a setup phase π°_{ts} that implements more than one setup phase. Then we can iteratively run π^*_{ts} to create enough implementations of the setup phase π°_{ts} as needed. In particular, Canetti and Rabin have shown that common reference string model can be bootstrapped [CR03]. As any functionality can be implemented in the common reference string model [CLOS02], we can always construct a protocol with a short pre-processing stage that creates an initial common reference string. On the other hand, these results show only the feasibility in the polynomial model and may be completely sub-optimal in practice.

Moreover, note that bootstrapping is inherently less efficient than sharing a

```
$\mathcal{P}_1$
    Receive $(\mathsf{pk}, \mathsf{sk})$ from $\mathcal{P}_{\mathrm{ts}}$
    Execute $\pi_1$:
        Send $c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(x_1)$ to $\mathcal{P}_2$
    Execute $\pi_2$:
        Receive $c_2$ from $\mathcal{P}_2$
        **return** $\mathsf{Dec}_{\mathsf{sk}}(c_2) \oplus x_1$

$\mathcal{P}_2$
    Receive $\mathsf{pk}$ from $\mathcal{P}_{\mathrm{ts}}$
    Execute $\pi_1$:
        Receive $c_1$ from $\mathcal{P}_1$
    Execute $\pi_2$:
        Send $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(x_2)$ to $\mathcal{P}_1$
        **return** $0$
```

Figure 7.8: Protocols π_1 and π_2 share a single setup phase and thus fail.

single setup phase $\pi_{\mathrm{ts}}^{\circ}$ between different protocols $\pi_1, \ldots, \pi_s$. However, the gain in efficiency does not come without additional hassle. Namely, we must show that a corresponding compound protocol $\varrho\langle\pi_1, \ldots, \pi_s\rangle$ remains universally composable even if they share the same setup phase $\pi_{\mathrm{ts}}^{\circ}$. Finding such proofs can be a quite complicated and error-prone task, since the messages from different protocols are now correlated and the latter increases the amount of possible attack patterns. As a trivial example, note that one-time pad is a universally composable implementation of encryption functionality as long as the encryption key is chosen uniformly. On the other hand, two-time pad (as a double execution of one-time pad with the same setup phase) is known to be insecure. In other words, universally composable protocols may become insecure if we share the same setup phase between many protocols.

Before going into the technical details, let us first clarify how one should interpret security guarantees in the presence of trusted setup. The latter is not so obvious as it seems. We use a sub-protocol π_2 in Fig. 7.8 as an illustrative example. First, note that the protocol π_2 uses asymmetric encryption and the trusted dealer $\mathcal{P}_{\mathrm{ts}}$ establishes an appropriate key pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$. Second, fix an ideal implementation π_2°, where a trusted third party $\mathcal{T}$ uses the inputs x_1 and x_2 to compute the outputs $y_1 = x_1 \oplus x_2$ and $y_2 = 0$. Then it is straightforward to show that π_2 is universally composable against static adversaries. The corresponding interfaces for malicious participants $\mathcal{P}_1$ and $\mathcal{P}_2$ given below

```
$\mathcal{I}_1\langle\mathcal{P}_1\rangle$
    $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$
    Send $x_1$ to $\mathcal{T}$.
    Receive $y_1$ from $\mathcal{T}$.
    Send $c_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(y_1 \oplus x_1)$ to $\mathcal{P}_1$.
    Output whatever $\mathcal{P}_1$ outputs.

$\mathcal{I}_2\langle\mathcal{P}_2\rangle$
    $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$
    Send $\mathsf{pk}$ to $\mathcal{P}_2$.
    Receive $c_2$ from $\mathcal{P}_2$.
    Send $\hat{x}_2 \leftarrow \mathsf{Dec}_{\mathsf{sk}}(c_2)$ to $\mathcal{T}$.
    Output whatever $\mathcal{P}_2$ outputs.
```

provide a perfect correspondence between the real and ideal world outputs. Note that we can easily find an adversarial strategy $\mathcal{A}$ such that $\mathcal{P}_2$ does not know the submitted input $\hat{x}_2 \leftarrow \mathsf{Dec}_{\mathsf{sk}}(c_2)$. Still, we can construct the corresponding generic attack, where $\mathcal{P}_2$ first runs $\mathcal{I}_2\langle\mathcal{A}\rangle$ in order to extract $\hat{x}_2$ and then sends $\hat{c}_2 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\hat{x}_2)$ to $\mathcal{P}_1$ and continues the execution of $\mathcal{I}_2\langle\mathcal{A}\rangle$. If we consider the averaged performance over all runs of the setup procedure, then both attacks induce the same output distribution. However, if we consider a fixed public key pk, then the distributions can have significant differences.

More formally, let ω_{ts} denote the randomness of $\mathcal{P}_{\mathrm{ts}}$ and let $\mathsf{Adv}(\mathcal{A}|\omega_{\mathrm{ts}})$ be the corresponding success profile. Then the correspondence $\mathcal{A} \mapsto \mathcal{I}\langle\mathcal{A}\rangle$ provides estimates only on the average-case success $\mathsf{Adv}(\mathcal{A})$. As the success profiles may have large variance, the average-case bounds are insufficient for estimating $\mathsf{Adv}(\mathcal{A}|\omega_{\text{ts}})$. Note that security guarantees w.r.t. a fixed setup run are often more natural in practice. For example, most applications use a single instance of an hash function, such as SHA-1 or WHIRLPOOL, and the security of a public-key infrastructure depends on a single fixed master key. Hence, it makes sense to also consider the universal composability w.r.t. a fixed setup run. Of course, such a setting is meaningful only in the framework of *subjective security* discussed in Section 6.5. In this setting, the interface $\mathcal{I}\langle\cdot\rangle$ cannot maliciously corrupt $\mathcal{P}_{\text{ts}}$, or otherwise the correspondence $\mathcal{A} \mapsto \mathcal{I}\langle\mathcal{A}\rangle$ does not preserve closeness between $\mathsf{Adv}(\mathcal{A}|\omega_{\text{ts}})$ and $\mathsf{Adv}(\mathcal{A}^\circ|\omega_{\text{ts}})$. Consequently, the interface $\mathcal{I}\langle\cdot\rangle$ can use only the same set of parameters that $\mathcal{A}$ obtains from $\mathcal{P}_{\text{ts}}$, i.e., the trusted setup assures only the correctness and availability of setup parameters.

In the following, we consider only the classical objective setting although the same holds also for the subjective setting. First, it is straightforward to verify that the protocol π_1 in Fig. 7.8 is also a universally composable implementation of "do nothing" functionality $\pi_1^\circ \equiv \perp$. However, the compound protocol of π_1 and π_2 with shared setup in Fig. 7.8 is insecure, since a malicious $\mathcal{P}_2$ can send c_1 back to $\mathcal{P}_2$ and thus always force the output $y_1 = x_1 \oplus x_1 = 0$. Nevertheless, it is easy to prove that concurrent composition $\varrho\langle\pi_1, \pi_1\rangle$ with the shared setup π_{ts}° is universally composable and the same is also true for $\varrho\langle\pi_2, \pi_2\rangle$. Hence, no simulation strategy $\mathcal{I}\langle\cdot\rangle$ for itself makes sharing the setup π_{ts}° impossible and the failure of $\varrho\langle\pi_1, \pi_2\rangle$ must be caused by more subtle reasons.

More formally, assume that the protocols $\pi_1, \dots, \pi_v$ share the same trusted setup phase π_{ts}°. Then any computational context implicitly defines an inner context $\varrho_{\text{ts}}\langle\pi_1, \dots, \pi_v\rangle$ that executes only $\pi_1, \dots, \pi_v$ with the inputs provided by the outer context $\varrho\langle\cdot\rangle$. Since the inner context $\varrho_{\text{ts}}\langle\pi_1, \dots, \pi_v\rangle$ itself can be viewed as a compound protocol,[6] we must just prove that $\varrho_{\text{ts}}\langle\pi_1, \dots, \pi_v\rangle$ is universally composable. In particular, note that the final interface $\mathcal{I}\langle\cdot\rangle$ for the compound protocol $\varrho_{\text{ts}}\langle\pi_1, \dots, \pi_v\rangle$ may use a modified setup procedure π_{ts} in the ideal world and thus the corresponding security proof has three phases. First, we must prove that substituting π_{ts}° with π_{ts} does not change the outcome in the real world. In other words, for any valid outer context $\varrho\langle\cdot\rangle$

$$|\Pr[\mathcal{G}_{\text{real}}^{\mathcal{A}} = 1] - \Pr[(\mathcal{G}_{\text{real}}^{*})^{\mathcal{A}} = 1]| \le \varepsilon_1 \ , \tag{7.23}$$

where $\mathcal{G}_{\text{real}}$ and $\mathcal{G}_{\text{real}}^{*}$ are the corresponding security games. Secondly, note that direct access to the process π_{ts} may reveal secret information and thus it may be easier to carry out some computational operations $\mathcal{O}_1, \dots, \mathcal{O}_s$. Hence, we should again view time bounds as vectors $\boldsymbol{t} = (t_0, \dots, t_s)$. In the simplest case, all parameters sent out by π_{ts}° and π_{ts} are public and no honest participant can carry out these expensive operations $\mathcal{O}_1, \dots, \mathcal{O}_s$. In more complicated cases, the setup procedure sends out some private information and the honest participants can also carry out some of these operations $\mathcal{O}_1, \dots, \mathcal{O}_s$. As a result, a protocol π_i may become insecure in contexts $\varrho_{\text{ts}}\langle\cdot\rangle$, where such private information is transferred from one node to another. Moreover, the adversary may use honest parties with extra abilities to carry out such operations.

[6] So far we have always assumed that all arguments must be specified in the beginning of the protocol but nothing changes if we allow a gradual demand-based release of arguments.

We can safely and systematically model these aspects by rewriting all protocols $\pi_1, \ldots, \pi_v$ in terms of oracle calls. As a result, we obtain an equivalent computational model, where the protocol π_{ts} broadcasts only public parameters and all running times are expressed as vectors. Since the corresponding reduction schemata quantify time complexity more precisely, it is possible to replace protocols in the inner context $\varrho_{\mathrm{ts}}\langle\pi_1, \ldots, \pi_v\rangle$ one by one. Finally, when there are no protocols left, we can complete the proof by removing π_{ts}.

Let us return to the protocol depicted in Fig. 7.8. In this example, we have to count decryption operations as well. Observe that the protocol π_1 is universally composable only if the context and the real world adversary do not use decryption operations. As the context contains a single decryption operation, we cannot replace π_1 with the ideal implementation π_1°. Although we can replace π_2 with the ideal implementation π_2°, the resulting adversary does an extra decryption operation and thus we still cannot replace π_1 with π_1°.

Finally, we note that the corresponding security proofs are quite streamlined in all settings, where trusted setup is used to create only public parameters psp. These security proofs do not use vectors as time bounds, but we still have to take into account the fact that the context can choose the inputs $\boldsymbol{x}$ of the protocol based on psp. Another particularly useful shared setup model is the public-key infrastructure model. However, the latter generally requires precise vectorised time bounds for the reasons shown above. In fact, the presented methodology was largely motivated by the author's own research [GLLM04, LLM05, LLM06, LL07]. These articles study a certain restricted class of protocols that utilise homomorphic encryption. In the final article [LL07], we used a slightly simplified methodology to prove that all these protocols can be constructed by carefully combining three elementary protocols that are universally composable even if they share the corresponding setup phase. Such results are important, as they can significantly simplify practical protocol design and can show what kind of open problems may have significant practical consequences.

7.8 PUBLIC INPUTS AND UNIVERSAL COMPOSABILITY

Recall that no protocol without trusted setup can remain universally composable and preserve privacy of inputs in the presence of honest minority [CKL03]. Superficially, it seems to imply that no useful protocol with honest minority can be universally composable in the standard model. However, such an impression is misleading, since some important practical functionalities indeed reveal all inputs. For example, common primitives for authentication, such as message authentication, tamper-resist storage and public broadcast, do reveal all inputs. Consequently, a study of universally composable protocols with public inputs is not a mere theoretical exercise, but also has practical consequences.

More formally, we say that the inputs of a protocol π are *public* if all participants first submit their inputs to the ideal world adversary and then continue interaction with the trusted third party as usual. A closer inspection reveals that all universally composable protocols with honest minority must have public inputs and thus the definition is optimal, see the results in [CKL03]. Moreover, protocols with public inputs cannot generally preserve universal composability and implement a non-deterministic functionality, see again [CKL03] for precise reasoning. For less technical argumentation, recall that any functionality can be implemented in the trusted setup model [CLOS02]. Hence, a trusted setup procedure itself cannot have universally composable implementation in the presence of honest minority, otherwise every function would have a universally composable implementation. The claim follows, as trusted setup procedures have no inputs and are thus protocols with public inputs.

Obviously, a protocol π with public inputs reveals all outputs when the ideal functionality is deterministic. As a result, we can guarantee only the correctness of outputs. Secondly, note that an interface $\mathcal{I}_{\mathrm{sim}}\langle\cdot\rangle$ can straightforwardly simulate the execution of the protocol π to the real world adversary $\mathcal{A}$, as all inputs of honest nodes are available to $\mathcal{I}_{\mathrm{sim}}\langle\cdot\rangle$ in the ideal world.[7] Now the correspondence between real and ideal world is complete as soon as we can extract the inputs of corrupted nodes from the traffic between $\mathcal{A}$ and $\mathcal{I}_{\mathrm{sim}}\langle\cdot\rangle$.

More formally, let $\mathsf{Extr}(\cdot)$ be a t_{ex}-time function that takes in the traffic between $\mathcal{A}$ and $\mathcal{I}_{\mathrm{sim}}\langle\cdot\rangle$ and let $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ be the corresponding guessed input. Then the input extraction fails for a single round functionality if $x_i \neq \hat{x}_i$ or $y_i = f_i(\hat{\boldsymbol{x}})$ for uncorrupted participants $\mathcal{P}_i$. For a multi-round functionality, $\mathsf{Extr}(\cdot)$ must produce guesses for each round and $\mathsf{Extr}(\cdot)$ fails if it fails for any round. A protocol π is $(t_{\mathrm{re}}, t_{\mathrm{ex}}, t_{\mathrm{cnt}}, \varepsilon)$-extractable if the input extraction fails with the probability at most ε. The following theorem describes when a protocol with honest minority is universally composable.

Theorem 3. *There exists a constant $c > 0$ such that a $(t_{\mathrm{re}}, t_{\mathrm{ex}}, t_{\mathrm{cnt}}, \varepsilon)$-extractable protocol π with public inputs is $(t_{\mathrm{re}}, c \cdot t_{\mathrm{re}}, t_{\mathrm{cnt}}, \varepsilon)$-universally composable.*

Proof sketch. Clearly, the simulating interface $\mathcal{I}_{\mathrm{sim}}\langle\cdot\rangle$ together with extraction function $\mathsf{Extr}(\cdot)$ is the necessary interface $\mathcal{I}\langle\cdot\rangle$. The claim follows, as the outputs of π and π° differ only if the extraction fails. □

Further comments. There is a strange duality between Theorem 1 and 3. Namely, the proofs of these theorems handle the information flow between a protocol π and side-computations σ in a similar fashion. In the proof of Theorem 1, we corrupt all non-participants of the protocol π and thus manage to reduce the security to the stand-alone setting. More precisely, we first learn the inputs of non-participants to incorporate the process σ into the ideal world adversary and finally replace the outputs of non-participants to assure the correspondence between the real and the ideal world. In the proof of Theorem 3, we do exactly the opposite. That is, we first learn inputs of honest participants to simulate π in the ideal world, and then use the input extraction to assure correspondence between the real and the ideal world. The resulting security guarantees are stronger only because $\mathsf{Extr}(\cdot)$ is a non-rewinding algorithm.

Message authentication. Message authentication protocols form the most prominent class of protocols with public inputs. Recall that the main aim of these protocols is to transfer inputs between the nodes $\mathcal{P}_1, \ldots, \mathcal{P}_n$ by sending messages through a malicious courier node $\mathcal{P}_{\mathrm{nw}}$. Consequently, the input extraction itself is trivial, we just have to follow the protocol instructions. Moreover, the input extraction fails only if the adversary succeeds in deception. As a result,

[7]To be precise, this claim does not hold for exotic computational models, where missing real world messages alter the message order in the ideal world.

most of the authentication protocols are automatically universally composable, as soon as they satisfy classical security requirements. As an example, consider unilateral and cross-authentication protocols between honest $\mathcal{P}_1$ and $\mathcal{P}_2$. Let x_1 and x_2 be the corresponding inputs, then both participants should learn x_1 in the unilateral and (x_1, x_2) in the cross-authentication protocol. Commonly, the security of a protocol is defined through a game in the asynchronous model of computation, where the adversary provides inputs to both parties and succeeds in deception if the outputs differ from ideal implementation. For unilateral authentication protocols, the adversary succeeds if $\mathcal{P}_2$ reaches accepting state and outputs $\hat{x}_1 \neq x_1$. For cross-authentication protocols, the adversary succeeds if either $\mathcal{P}_1$ or $\mathcal{P}_2$ reaches accepting state and outputs $(\hat{x}_1, \hat{x}_2) \neq (x_1, x_2)$.

Any computational context $\varrho\langle\cdot\rangle$ together with an adversary $\mathcal{A}$ gives a rise to a new compound adversary $\mathcal{A}^*$ that plays the original security game. Moreover, the extraction function indeed fails only if the compound adversary $\mathcal{A}^*$ succeeds in deception. Hence, classical message authentication protocols are automatically universally composable. The result holds even if we allow the adversary to corrupt participants, but then we have to redefine what a deception failure means. Analogous results hold also for group authentication protocols.

Note that authentication protocols also have a trusted setup phase, where the secret keys are created and transferred to all participants. Thus, a parallel execution of several message authentication protocols that share the same setup phase may be completely insecure. One possible solution is to use bootstrapping. For example, it is straightforward to use pseudorandom functions to generate many computationally independent session keys from a single master key. Secondly, we can study the extractability of a compound protocol $\varrho_{\mathrm{ts}}\langle\pi_1, \ldots, \pi_s\rangle$ with shared setup π°_{ts} directly. This approach is commonly known as a Bellare-Rogaway model for message authentication, see [BR93a, BR95]. In the corresponding model, deception is again defined so that adversary succeeds in deception if the corresponding input extraction procedure fails. As a result, a concurrent composition of authentication protocols is universally composable as soon as it is secure in the Bellare-Rogaway model.

In recent years, cryptographers have also investigated manual authentication protocols [Vau05, PV06a, PV06b, LN06, LP08]. These protocols do not have a complex trusted setup phase. All of them can be implemented either in the standard or in the common reference string model. The authenticity of messages is achieved by sending short authenticated strings. As a result, one can easily show that a concurrent execution of these protocols is still universally composable even if they share the setup phase. However, the latter is still insufficient for security, since short authenticated strings are sent without tags that identify the corresponding protocol instances. Consequently, we must additionally assure that the adversary cannot switch short authenticated strings between the protocols, otherwise the corresponding synchronisation errors may invalidate the composability guarantees. Thus, we must either use further analysis to determine the maximal achievable damage caused by clever message switching as in [Vau05, PV06a, PV06b], or alternatively add further usage restrictions that eliminate the possibility of such attacks as in [LN06, LP08].

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