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Dissertation for the Degree of Doctor of Science and Technology to be presented with due permission of the Department of Engineering Physics and Mathematics, for public examination and debate in Auditorium C at Helsinki University of Technology (Espoo, Finland) on the 16th of March 2007, at 12 o'clock noon.

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Abstract: *This work studies geometrical properties of electromagnetic wave propagation. The work starts by studying geometrical properties of electromagnetic* Gaussian beams *in inhomogeneous anisotropic media. These are asymptotical solutions to Maxwell's equations that have a very characteristic feature. Namely, at each time instant the entire energy of the solution is concentrated around one point in space. When time moves forward, a Gaussian beam propagates along a curve. In recent work by A. P. Kachalov, Gaussian beams have been studied from a geometrical point of view. Under suitable conditions on the media, Gaussian beams propagate along geodesics. Furthermore, the shape of a Gaussian beam is determined by a complex tensor Riccati equation. The first paper of this dissertation provides a partial classification of media where Gaussian beams geometrize. The second paper shows that the real part of a solution to the aforementioned Riccati equation is essentially the shape operator for the phase front for the Gaussian beam. An important phenomena for electromagnetic Gaussian beams is that their propagation depend on their polarization. The last paper studies this phenomena from a very general point of view in arbitrary media. It also studies a connection between contact geometry and electromagnetism.*

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Swedish abstract: *Detta arbete behandlar geometriska egenskaper hos elektromagnetisk vågutbredning. Arbetet börjar med att studera egenskaper hos* Gaussiska strålar *i anisotropa och ohomogena medier. En Gaussisk stråle är en asymptotisk lösning till Maxwell's ekvationer som utbreds längs en kurva. (Vid varje tidpunkt är hela energin koncentrerad kring en punkt på kurvan.) I artiklar av A.P. Kachalov har elektromagnetiska Gaussiska strålar studerats från en geometrisk synvinkel. Med vissa antaganden på mediet framskrider Gaussiska strålar längs geodeser i en Riemannisk geometri och en olineär komplex Riccati ekvation bestämmer Gaussiska strålens form. Den första artikeln i denna avhandling ger en partiell karakterisering av medier där Gaussiska strålar utbreds med hjälp av Riemannisk geometri. Den andra artikeln visar att den reella delen av en lösning till den komplexa Riccati ekvationen är form-operatorn till fas-fronten för en Gaussisk stråle. Den sista artikeln studerar polarisering i elektromagnetism med hjälp av en dekomposition i Fourier rymden. Denna artikel studerar också en länk mellan elektromagnetism och kontakt-geometri.*

Organization

This dissertation is divided into five parts: the introduction, three papers, and one errata:

[**I**] M. F. Dahl, *Electromagnetic Gaussian beams and Riemannian geometry*, Progress In Electromagnetics Research, Vol. 60, pp. 265-291, 2006.

[**II**] M. F. Dahl, *Geometric interpretation of the complex Riccati equation*, Journal of Nonlinear Mathematical Physics, Vol. 14, No. 1, pp. 95-111, 2007

[**III**] M. F. Dahl, *Contact geometry in electromagnetism*, Progress In Electromagnetics Research, Vol. 46, pp. 77-104, 2004.

[**E**] Errata for [**I**].

All manuscripts have been prepared by myself. Paper [**III**] is based on my master's thesis, *Contact and symplectic geometry in electromagnetism* [Dah02]. Paper [**I**] is based on my Licentiate thesis, *Electromagnetic Gaussian beams in Riemann-Finsler geometry* [Dah06].

[**III**] represents my own research. The topic of classifying media in [**I**] was suggested by the supervisor of this work, Professor Erkki Somersalo. However, the work was carried out by myself. The main result of [**II**], Theorem 4.5, was formulated by my instructor Doctor Kirsi Peltonen, and proven by myself.

Contents

1 Acknowledgement

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2 Introduction

The aim of this work is to study geometric properties of electromagnetic wave propagation. However, let us begin with a brief description of inverse problems, which form an important motivation for this work.

There are many examples where one would like to obtain information about an object's interior in a non-invasive way. Natural examples appear in quality control, geophysical surveying, medical imaging, mine detection, industrial processes, and archeology. Mathematically, such problems are known as *inverse problems*. They are solved by first measuring how an object reacts to some physical phenomena, and then fitting this data to some physical model that depends on the material properties in the interior of the object. Examples of such physical phenomena can be impedance, acoustic waves, electromagnetic waves, and seismic waves. For example, using X-rays one can obtain a $2D$ projection of an object. By mathematically combining many such projections from different angles, one can obtain information about the object's $3D$ structure [SKJ$^+$03]. This process is known as *X-ray tomography*.

For inverse problems, an important class of measurements are *boundary travel-time measurements*; one sends in a signal at one point on the boundary and records when the signal reaches any other point on the boundary. From such measurements, one can theoretically determine how long it takes for a signal to travel between any two points on the boundary. A natural inverse problem is then as follows: Can one reconstruct the material properties inside the object from the boundary travel-time information? This reconstruction, when possible, is known as *travel-time tomography*.

Mathematically, the travel-time tomography problem is known as the *boundary rigidity problem*. For a Riemannian manifold M with boundary it reads as follows. Does the geodesic distance restricted to the boundary of M determine the metric tensor inside M? A first answer is no. If $\Phi\colon M \to M$ is a diffeomorphism that preserves the boundary, and $\Phi^* g$ is the pullback of metric tensor g, then manifolds (M, g) and $(M, \Phi^* g)$ share the same geodesic distance on their boundaries. In consequence, the boundary rigidity problem is only relevant up to a boundary-preserving diffeomorphism. A recent result of L. Pestov and G. Uhlmann [PG05] states that compact, 2-dimensional, and simple Riemannian manifolds are boundary rigid (up to the aforementioned obstruction). We shall not here give the definition of a simple manifold. However, on a simple manifold, there are no conjugate points, and any two points can be joined by a unique geodesic. A recent review of various rigidity results for Riemannian manifolds can be found in [SU05].

In view of inverse problems, it is motivated to study the travel-time metric defined by physical wave propagation. Say, how is such a metric defined? What does it depend on? What are its mathematical properties? And, when is it a Riemannian metric? A very successful mathematical tool for studying these types of questions are *Gaussian beams*. These are asymptotic solutions to hyperbolic equations (that is, wave-type equations) that have a very characteristic feature. Namely, at each time instant their entire energy is concentrated around one point in space. This means that a Gaussian beam propagates along a curve. An important result is that (under suitable assumptions) this curve is a geodesic in a Riemannian geometry. Moreover, this geometry is completely determined by the media. Thus, in some sense, Gaussian beams geometrize wave propagation; the study of propagating of Gaussian beams reduces to the study of Riemannian geometry. A very important problem, which is much less understood, is how to decompose an arbitrary field (or its wave front) into Gaussian beams. We shall not deal with this question in this work. However, such a decomposition would give a very rapid way for calculating the travel-time between two antennas. Namely, to propagate a Gaussian beam one only have to solve the geodesic equation. Since it is an ordinary differential equation, it is much faster to solve than the original partial differential equation, say, Maxwell's equations in the time domain. In mathematics, the decomposition-problem is not relevant. If one has complete control of the boundary of an object, one can generate one Gaussian beam at a time [KL04]. A historical perspective on Gaussian beams can be found in [BP73, Pop02, Ral82].

This work studies the geometry of electromagnetic wave propagation by studying geometric properties of electromagnetic Gaussian beams. A fundamental paper on this topic is [Kac05]. It shows that in anisotropic media, propagation of electromagnetic Gaussian beams is determined by *two* Hamiltonian functions. Moreover, if these Hamiltonians are smooth, 1-homogeneous, and strongly convex, they induce two Finsler geometries, and equations governing propagation of Gaussian beams can be written using these geometries. Here, *Finsler geometry* is a natural generalization of Riemannian geometry where the norm does not need to be induced by an inner product [She01b, She01a, BCS00]. Let us point out that the reason one needs two Hamiltonians is due to polarization. For example, a plane wave can be right- and left-hand polarized, and, in anisotropic media, these may propagate along different paths. Therefore one needs two Hamiltonians; one for "right hand polarized" Gaussian beams, and one for "left hand polarized" Gaussian beams. For comparison, one needs only one Hamiltonian in acoustics, where there is no polarization.

This dissertation consists of three papers [**I**], [**II**], and [**III**]. Of these, [**I**] form a natural continuation of [Kac05] described above. Namely, [**I**] studies the conditions on the Hamiltonians in terms of the anisotropic media parameters. The main result of [**I**] is a partial classification of media, where Gaussian beams geometrize. The somewhat surprising result is that if the permittivity and permeability matrices are simultaneously diagonalizable, then there are no non-Riemannian Finsler geometries in the Gaussian beam framework. A general background on the topic of geometrization of electromagnetics is given in Section 3. A short introduction of Gaussian beams is given in Section 4, and the result of [**I**] is presented in Section 4.2.

Paper [**II**] studies geometric properties of the *complex tensor Riccati equation* associated with Gaussian beams. This equation determines the Hessian of the phase function for a Gaussian beam. The main result of

[**II**] is that the real part of such a solution is essentially the shape operator of the phase front. Section 4.3 gives an outline of this result.

Unlike acoustics, wave propagation in electromagnetics depends on polarization. This can be seen both from the theory of Gaussian beams, or from the way plane waves scatter. One difficulty, however, is that there does not seem to exist a general mathematical definition of polarization for an arbitrary electromagnetic wave. For example, the concepts of linearly, elliptically, and circularly polarized waves are only defined for plane waves. The main result of [**III**] is a very general functional analytic way of viewing polarization. This result is outlined in Section 5. Paper [**III**] also establishes a relation between contact geometry and electromagnetism.

3 Maxwell's equations in differential forms

To study the geometry in electromagnetism it is necessary to formulate Maxwell's equations in the same setting as differential geometry, that is, using differential forms on a smooth manifold. There are at least two main advantages of this formalism. First, it makes many powerful results and constructions from differential geometry available. For example, in [**I**] and [**II**] it would be possible to write down the curvature tensor as a function of the permittivity and permeability matrices. However, this yields quite complicated expressions [SZWZ97]. Secondly, manifolds provide a natural framework for studying global objects; that is, objects that do not depend on their expressions in local coordinates. Due to the natural obstruction for boundary rigidity (see the introduction), this global view is well suited for studying inverse problems. Manifolds also provide a natural framework for physics; a physical phenomena should not depend on the mathematical coordinates in use. Let us emphasize that in this work, we only study the geometrization of electromagnetism in 3-space. We also assume that everything is smooth.

Using traditional vector analysis Maxwell's equations read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

Here, the physical electromagnetic wave is represented using four vector fields; electric fields $\mathbf{E}, \mathbf{D}$ and magnetic fields $\mathbf{H}, \mathbf{B}$. Thus the wave is represented by 16 scalar functions. In addition to vector fields, other formalisms are dyads [Lin92], quaternions [Lou01], tensors [Pos62], and differential forms [BH96, Lin04]. With differential forms, Maxwell's equations read

$$dE = -\frac{\partial B}{\partial t}, \tag{5}$$

$$dH = \frac{\partial D}{\partial t} + J, \tag{6}$$

$$dD = 0, \tag{7}$$

$$dB = \rho. \tag{8}$$

Here d is the exterior derivative, E and H are 1-forms representing the electric and magnetic fields, and D and B are 2-forms representing electric and magnetic fluxes, respectively. Also, J is the source current 2-form, and ρ is the charge concentration 3-form [WSA97].

An important observation is that equations (5)-(8) only involve the exterior derivative d. This means that the equations do not depend on local coordinates, so Maxwell's equations can be formulated on any smooth 3-manifold. In particular, Maxwell's equations do not depend on an inner product. Hence the equations do not involve such notions as angle, length, area, or volume.

To solve Maxwell's equations one need to complement these with *constitutive equations.* These equations reduce the number of free variables in Maxwell's equations, say, by expressing (D, B) as a function of (E, H). Physically, the constitutive equations encode properties of the electromagnetic media, and in general these might be very complicated. For example, they might include non-linear effects, hysteresis, or time-dispersion. Let us here assume that the media is anisotropic. Then the constitutive equations read

$$\begin{aligned}\mathbf{D} &= \varepsilon \cdot \mathbf{E}, \\ \mathbf{B} &= \mu \cdot \mathbf{H},\end{aligned}$$

where ε and μ are real, symmetric, and positive definite 3×3 matrices that describe the permittivity and permeability of the media. We assume that they may depend on location, but not on time nor on frequency.

Unlike Maxwell's equations, the constitutive equations do not directly translate into differential forms. The problem is that E is a 1-form, and D is a 2-form. Hence ε and μ are operators that map 1-forms into 2-forms. Since ε and μ are symmetric and positive definite, these operators can be realized using two *Hodge operators* $*_\varepsilon$, $*_\mu$ induced by two suitable Riemannian metrics [Bos01, Dah06, KLS06]. The constitutive equations then reads

$$\begin{aligned}D &= *_\varepsilon E, \\ B &= *_\mu H.\end{aligned}$$

In contrast to Maxwell's equations, the constitutive equations are completely metrical; they depend only on geometry. One metric describes electric anisotropy, and one describes magnetic anisotropy. In the variational formulation of electrostatics and magnetostatics, these metrics play an important role [BH96]. Unfortunately, however, these metrics do not seem to be related to wave propagation. One sought feature, for instance, would be that geodesics would describe the path traversed by a ray of light. However, since such a path depends on polarization, and since the above metrics do not take polarization into account, the geodesics do not have such properties. As an example, in isotropic media, the Riemannian metrics are

$$\begin{aligned}g^\varepsilon_{ij} &= \frac{1}{\varepsilon^2}\delta_{ij}, \\ g^\mu_{ij} &= \frac{1}{\mu^2}\delta_{ij}.\end{aligned}$$

4 Gaussian beams

In this section we first give a short mathematical introduction to electromagnetic Gaussian beams on a manifold. Section 4.1 describes the geometrization result in [Kac05]. Section 4.2 describes the result of [**I**], and Section 4.3 describes the result of [**II**]. Detailed expositions on Gaussian beams that emphasize their geometrical natural are [Dah06, Kac02, Kac04, Kac05, KKL01, KL04] and [**I**]. See also [Pop02, Ral82].

To define an electromagnetic Gaussian beam, suppose M is a smooth 3-manifold representing physical space. On M, let us consider an electric field of the form

$$E(x,t) \quad = \quad \mathrm{Re}\left\{E_0(x,t)\exp\left(iP\theta(x,t)\right)\right\}, \quad (x,t)\in M\times I. \tag{9}$$

Here $P>0$ is a large constant, I is an open interval representing time, E_0 is a complex 1-form, and the function $\theta\colon M\times I\to\mathbb{C}$ is the *phase function* for E. The advantage of the above representation is that qualitatively E_0 and θ contain different type of information. One can think of equation (9) as a separation of variables. The 1-form E_0 completely determines how E is polarized. To understand the role of θ, let us write

$$\exp\left(iP\theta(x,t)\right) = \exp(iP\,\mathrm{Re}\,\theta)\exp(-P\,\mathrm{Im}\,\theta).$$

As $P>0$ is large, $\mathrm{Re}\,\theta$ describes high frequency oscillations of E. Thus $\mathrm{Re}\,\theta$ describe how the field propagates. On the other hand, $\mathrm{Im}\,\theta$ influences the amplitude of E. In order for E to be stable if one takes $P\to\infty$, let us assume that $\mathrm{Im}\,\theta\geq 0$.

Plugging E into Maxwell's equation yields the *Hamilton-Jacobi equation* for θ,

$$\frac{\partial\theta}{\partial t} = h(d\theta).$$

Here h is a suitable *Hamiltonian function* $T^*M\to\mathbb{R}$. In fact, in electromagnetism there are two possible Hamiltonian functions. We shall not give their precise definition here. However, let us make two observations. Since propagation of electromagnetic waves depend on their polarization, one needs two Hamiltonians. Essentially, one Hamiltonian describes how "right hand polarized" waves propagate, and the other describe how "left hand polarized" waves propagate. The second observation is that the Hamiltonian functions are completely determined by the electromagnetic media. Thus, once the media is known, one can solve the phase function, which, in turn, describes how the wave propagates. This shows how the Hamilton-Jacobi equation is closely related to travel-time problem.

To define a Gaussian beam, let $c\colon I\to M$ be a smooth curve. (This will be the curve the Gaussian beam propagates along.) Furthermore, suppose c is covered with local coordinates x^i, and suppose that

$$\phi\colon I\to\mathbb{C},\quad p\colon I\to\mathbb{C}^3,\quad H\colon I\to\mathbb{C}^{3\times 3}$$

are the first three coefficients in the Taylor expansion of θ evaluated on $c(t)$. That is,

$$\begin{aligned} \phi(t) &= \theta(c(t), t), \\ p_j(t) &= \frac{\partial \theta}{\partial x^j}(c(t), t), \\ H_{jk}(t) &= \frac{\partial^2 \theta}{\partial x^j \, \partial x^k}(c(t), t). \end{aligned}$$

Then E is a *Gaussian beam* on c provided that for all $t \in I$,

1. $p(t) = (p_i(t))_i$ is non-zero,
2. $\phi(t)$ and $p(t)$ are real,
3. the imaginary part of $H(t) = (H_{ij}(t))_{ij}$ is positive definite.

One can show that these conditions do not depend on local coordinates, and

$$\theta(x, t) \;=\; \phi_0(t) + p_i(t) z^i + \frac{1}{2} H_{ij}(t) z^i z^j + o(|z|^3),$$

where $z^i = z^i(x, t) = x^i - c^i(t)$. In consequence,

$$|\exp(iP\theta(x, t))| \approx \exp\left(-\frac{P}{2} z^i \operatorname{Im} H_{ij} z^j\right). \tag{10}$$

In other words, at time t, the energy of E is completely concentrated around $c(t)$, and $\operatorname{Im} H$ describes the shape of the field.

Let us point out that E_0 in equation (9) is typically replaced by a finite sum $\sum_{k=0}^{N} \frac{E_k(x,t)}{(iP)^k}$, where $E_0, E_1, \dots$ are complex 1-forms [Kac05, KO90]. Then one can derive an equation for E_0 called the *transport equation.* Furthermore, from E_i, one can solve E_{i+1}. For electromagnetic Gaussian beams, these equations are studied in [Kac04]. By studying these equations one can determine the amplitude behaviour of Gaussian beams.

4.1 Geometrization of Gaussian beams

To determine the shape and the propagation of a Gaussian beam, it suffices to know ϕ, c, p, and H. For these, one can derive ordinary differential equations. Expanding both sides of the Hamilton-Jacobi equation with respect to z, and identifying the first three z^i-terms, gives the following equations:

1. ϕ is constant.
2. (c, p) is a solution to the Hamilton-equations with Hamiltonian h.
3. H is a solution to a complex matrix Riccati equation ([**I**], equation (19)).

All of these equations are ordinary differential equations. This means that they are much more well behaved than the original non-linear partial differential equation. One disadvantage, however, is that the equations are not geometrical. Although, the second equation can be formulated using symplectic geometry, a problem with the third equation is that H is not even a tensor. However, suppose Hamiltonian h is smooth, strongly convex, and 1-homogeneous. That is:

1. h is smooth on $T^*M \setminus \{0\}$.
2. h is a strongly convex function, that is, $\left(\frac{\partial^2 h^2}{\partial \xi_j \partial \xi_k}\right)_{jk}$ is positive definite matrix.
3. h is 1-homogeneous: If $(x, \xi) \in T^*M$, then

$$h(x, \lambda\xi) = \lambda h(x, \xi), \quad \lambda > 0.$$

Then equations for c, p, and H geometrize. For electromagnetic Gaussian beams, this was first proven in [Kac05] (see also [Dah06, Kac02, KKL01]). In this case, h induces a *Legendre transformation* [Dah06, She01b] defined as

$$\begin{aligned} \mathscr{L}\colon T^*M \setminus \{0\} &\to TM \setminus \{0\} \\ (x, \xi) &\mapsto \left(x, \left(\frac{\partial^2 h^2}{\partial \xi_j \partial \xi_k}(x, \xi)\xi_k\right)_j\right), \end{aligned} \tag{11}$$

and $\mathscr{L}$ induces a *Finsler norm* $F\colon TM \to \mathbb{R}$ by $F = h \circ \mathscr{L}^{-1}$ [BCS00, She01b]. Then, using F, equations for c, p, H can be written as:

1. c is a geodesic with respect to F.
2. $(c, p) = \mathscr{L}^{-1}(\dot{c})$.
3. Suppose $\Lambda(t) = (\Gamma^m_{ij} p_m)_{ij}$, where Γ^m_{ij} are the coefficients of the *Chern-Rund connection* [Dah06, She01b], and

$$G_{ij} = H_{ij} - \Lambda_{ij}. \tag{12}$$

   Then $G = G_{ij} dx^i \otimes dx^j$ is a tensor on c (see [**I**], Appendix B), and it is determined by the *complex tensor Riccati equation*

$$D_{\dot{c}} G + GCG - R = 0.$$

   Here $D_{\dot{c}} G$ is the covariant derivative of G along c, and $C = C^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ and $R = R_{ij} dx^i \otimes dx^j$ are 2-tensors on c depending on F (see [Dah06]).

The above provides a geometric and coordinate independent formulation for propagation of Gaussian beams. Namely, the propagation of Gaussian beams is determined by the phase function θ, and the above equations determine the first three terms in the Taylor expansion of θ.

4.2 Classification of media

The geometrization-result of [Kac05] shows that if the Hamiltonian functions satisfy suitable conditions, then electromagnetic Gaussian beams propagate using Finsler geometry. Since the Hamiltonians are determined by the media, this poses a natural question. Namely, in what kind of media do Gaussian beams geometrize? This is the topic of [**I**]. The somewhat surprising result of [**I**] is that if the permittivity and permeability matrices are simultaneously diagonalizable, then there are no non-Riemannian Finsler geometries in the Gaussian beam framework.

Mathematically, the Hamiltonians are defined as functions that parametrize the positive eigenvalues of a 6×6 matrix (see [**I**]). Due to their implicit definition, it does not seem to be possible to check the conditions on the Hamiltonians for the most general media. Therefore it is motivated to study the Hamiltonians in special classes of media, where they can be solved explicitly. In [**I**], this is done for media where ε and μ can be simultaneously diagonalized. That is, if U is a coordinate chart for M, then there exists a smooth mapping that to each $x \in U$ assigns a rotation matrix R such that

$$\varepsilon = R^{-1} \cdot \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \cdot R, \quad \mu = R^{-1} \cdot \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \cdot R, \tag{13}$$

where $\varepsilon_i > 0$, $\mu_i > 0$ are eigenvalues of ε and μ. We also assume that ε_i, μ_i are smooth functions on U. Three examples of such media are:

1. Isotropic media.
2. Either μ or ε is isotropic: Say, if μ is proportional to the identity matrix, one can pointwise diagonalize μ.
3. ε and μ are proportional: This class is mostly of theoretical interest. For example, in [KLS06], an electromagnetic inverse problem was solved with this assumption on the media.

The motivation for condition (13) is that in this media one can solve the Hamiltonian functions explicitly. The actual expressions are somewhat involved, but given in [**I**], Section 4.1. In [**I**], the classification of media is accomplished by introducing three functions that describe the complexity of the media. These are called the Δ_{ij}-symbols. For $i, j = 1, 2, 3$ and $i < j$, these are defined as

$$\Delta_{ij} = \frac{1}{\varepsilon_i \mu_j} - \frac{1}{\varepsilon_j \mu_i}.$$

The motivation for studying these symbols is that the Hamiltonians behave qualitatively differently depending on how many Δ_{ij}-symbols vanish.

One can prove that if two Δ_{ij}-symbols vanish at a point, then the third symbol also vanishes. In consequence, electromagnetic media divide into 3 classes: (1) all Δ_{ij}-symbols are zero, (2) one Δ_{ij}-symbol is zero, or (3) no Δ_{ij}-symbol is zero. For simplicity, let us assume that the class does not depend on location in U.

Class 1: All Δ_{ij}-symbols are zero

This is the most simple class of media, where Gaussian beams geometrize. A special case of this media is isotropic media, where $\varepsilon_i = \varepsilon$ and $\mu_i = \mu$ for $i = 1, 2, 3$. The induced Riemannian metrics are then

$$g_{\pm ij} = \varepsilon\mu\delta_{ij}.$$

Two alternative characterizations of this media are: the two Hamiltonians coincide, or ε and μ are proportional. Physically, the first characterization means that propagation does not depend on polarization.

Class 2: At least one Δ_{ij}-symbol is zero

One can prove that if the Hamiltonians are smooth and convex, then one Δ_{ij}-symbol must necessarily vanish. The converse of this is also true (see [**E**]); if $\Delta_{ij} = 0$ in U for some i, j, then Gaussian beams geometrize in U.

As an example, suppose that the media is of the form

$$\varepsilon = R^{-1} \cdot \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_2 \end{pmatrix} \cdot R, \quad \mu = \mu_0 I$$

for some smooth function $\mu_0 > 0$. Then $\Delta_{23} = 0$, and Gaussian beams geometrize.

Class 3: All Δ_{ij}-symbols are non-zero

In this class, the Hamiltonians are always non-smooth, and non-convex. Therefore their study does not seem to be motivated using differential geometry. This is illustrated in Example 5.3 in [**I**].

4.3 Geometric interpretation of $\operatorname{Re} G$

From equations (10) and (12) it follows that $\operatorname{Im} G$ describes how a Gaussian beam decays in different directions of space. The main result of [**II**] is Theorem 4.5. It shows that $\operatorname{Re} G$ is essentially the shape operator of the phase front for the Gaussian beam. This is a natural result in two ways. First, a similar result is known for Gaussian beams in $\mathbb{R}^3$ [Č01]. However, in the setting of $\mathbb{R}^3$, G is not a tensor, and the shape operator is calculated with respect to the Euclidean metric. Second, in classical differential geometry, the shape operator corresponding to a family of surfaces determined by a distance function satisfies a real tensor Riccati equation [Gra90, She01b]. The main idea of the proof of Theorem 4.5 in [**II**] is to write the tensor Riccati equation using Fermi coordinates adapted to the underlying geodesic [BU81].

5 Polarization in electromagnetics

From electromagnetic Gaussian beams it is clearly seen that the electromagnetic travel-time depends on polarization. This fact motivates the last paper [**III**], which studies the role of polarization in electromagnetism from a very general point of view. A main result of [**III**] is that Maxwell's equations decompose into two Maxwell's equations formulated on different function spaces. Essentially this means that every electromagnetic field is a sum of two electromagnetic fields that propagate independently of each other, but with different polarizations.

In [**I**], this decomposition of Maxwell's equations is accomplished by using a *helicity decomposition* for vector fields on $\mathbb{R}^3$. Essentially this decomposition is a refinement of Helmholtz' decomposition [Bla93]. It takes a vector field $\mathbf{F}$ on $\mathbb{R}^3$ and decomposes it into three components: $\mathbf{F}_+, \mathbf{F}_-, \mathbf{F}_0$. Vector fields $\mathbf{F}_\pm$ are in some sense generalizations of right and left hand polarized waves, and the third component $\mathbf{F}_0$ has zero curl. In [**III**] it is also shown how the helicity decomposition is related to both the Bohren decomposition and to contact geometry.

It should be emphasized that the helicity decomposition is well known in fluid mechanics [CM98, Mac95, Mac98, Mos71, Tur00, Wal92]. The decomposition has also been studied in electromagnetics [Hil94, Mos71].

5.1 Helicity decomposition of Maxwell's equations

Let L^2 be vector fields on $\mathbb{R}^3$ with square integrable components, and let

$$L^2_{\text{curl}} = \{\mathbf{F} \in L^2 : \nabla \times \mathbf{F} \in L^2\}.$$

Then we define *helicity* $\mathscr{H} \colon L^2_{\text{curl}} \to \mathbb{R}$ as

$$\mathscr{H}(\mathbf{F}) = \int_{\mathbb{R}^3} \mathbf{F} \cdot \nabla \times \mathbf{F} \,\mathrm{d}\,\mathbf{x}, \quad \mathbf{F} \in L^2_{\text{curl}}.$$

Helicity appears also in the study of fluid mechanics and plasma physics. There, however, the definition of helicity is slightly different: $\nabla \times \mathbf{F}$ is replaced by $(\nabla\times)^{-1}(\mathbf{F})$, and helicity is then only defined for divergence-free vector fields [AK98]. The advantage of this (fluid mechanical) helicity is that it can be interpreted in terms of the linking number in knot theory, and the asymptotic Hopf invariant [AK98]. However, the above definition of helicity seems more motivated in electromagnetics. First, quantity $\mathbf{F} \cdot \nabla \times \mathbf{F}$ is locally defined, and second, it has a contact geometric interpretation (see Section 5.2).

Helicity can be seen as a global measure of polarization. To see this, let us consider the quantity $\mathbf{F} \cdot \nabla \times \mathbf{F}$ for an arbitrary plane wave $\mathbf{F}$. In this case, $\mathbf{F} \cdot \nabla \times \mathbf{F}$ is constant with respect to both time and location. That is, even if $\mathbf{F}$ may oscillates in both intensity and/or amplitude, this oscillation is not seen in $\mathbf{F} \cdot \nabla \times \mathbf{F}$. Furthermore,

1. if $\mathbf{F}$ is a linearly polarized plane wave, then $\mathbf{F} \cdot \nabla \times \mathbf{F} = 0$,
2. if $\mathbf{F}_+$ and $\mathbf{F}_-$ are left and right hand circularly polarized plane waves, then $\mathbf{F}_\pm \cdot \nabla \times \mathbf{F}_\pm$ are equal, but with opposite signs.

In other words, the sign of $\mathbf{F} \cdot \nabla \times \mathbf{F}$ acts as an indicator of handed rotation for $\mathbf{F}$. Furthermore, this quantity is not only defined for plane waves, but for arbitrary vector fields in L^2_{curl}. (Of course, plane waves are not in L^2_{curl}, so strictly speaking, helicity is not defined for a plane wave. However, the above analysis shows what type of information helicity measures. This issue could also be addressed by replacing $\mathbb{R}^3$ with a 3-torus [ASKK99].)

An important property of $\mathbf{F} \cdot \nabla \times \mathbf{F} \, \mathrm{d}\, x$ is that it is a differential-topological quantity. If α is the 1-form corresponding to $\mathbf{F}$ under the usual identification given by the Euclidean metric, then $\mathbf{F} \cdot \nabla \times \mathbf{F} \, \mathrm{d}\, x$ corresponds to $\alpha \wedge d\alpha$ [**III**]. This means that helicity is independent of both metric and local coordinates.

The following theorem is proven in [Dah02].

Theorem 5.1 (Helicity decomposition). *Suppose $\boldsymbol{F} \in L^2_{curl}$. Then $\boldsymbol{F}$ has the decomposition*

$$\boldsymbol{F} = \boldsymbol{F}_0 \; + \; \boldsymbol{F}_+ \; + \; \boldsymbol{F}_-,$$

where $\boldsymbol{F}_0, \boldsymbol{F}_\pm \in L^2_{curl}$ and

$$\begin{aligned} \boldsymbol{F} \;&\mapsto\; \boldsymbol{F}_\lambda \textit{ is linear for } \lambda \in \{0, \pm\}, && (14)\\ (\nabla \times \boldsymbol{F})_\lambda \;&=\; \nabla \times (\boldsymbol{F}_\lambda), \quad \lambda \in \{0, \pm\}, && (15)\\ \nabla \times \boldsymbol{F}_0 \;&=\; 0, && (16)\\ \nabla \cdot \boldsymbol{F}_\pm \;&=\; 0, && (17)\\ \mathscr{H}(\boldsymbol{F}_0) \;&=\; 0, && (18)\\ \mathscr{H}(\boldsymbol{F}_+) \;&\geq\; 0, \textit{ with equality if and only if } \boldsymbol{F}_+ = 0, && (19)\\ \mathscr{H}(\boldsymbol{F}_-) \;&\leq\; 0, \textit{ with equality if and only if } \boldsymbol{F}_- = 0. && (20) \end{aligned}$$

The proof is based on giving explicit expressions for $\mathbf{F}_0, \mathbf{F}_+, \mathbf{F}_-$ in Fourier space. See Definition 3.1 in [**III**].

Equations (16)-(17) show that the helicity decomposition is a generalization of the Helmholtz' theorem, or Hodge decomposition on $\mathbb{R}^3$. Properties (19)-(20) show that $\mathbf{F}_+$ and $\mathbf{F}_-$ are in some sense oppositely oriented vector fields. For example, under a space inversion, $\mathbf{F}_+$ and $\mathbf{F}_-$ exchange places in the decomposition. These properties also show that on suitable subspaces of L^2_{curl}, helicity is non-degenerate and, in fact, a norm. It also holds that the decomposition commutes with rotations, scalings, and dilations of space [Dah02].

Using properties (14), (15), and (17), and assuming that time derivatives commutes with the helicity decomposition, we can decompose Maxwell's equations. The $+$-components of the first two Maxwell equations 1-2 are

$$\begin{aligned} \nabla \times \mathbf{E}_+ \;&=\; -\frac{\partial \mathbf{B}_+}{\partial t}, && (21)\\ \nabla \times \mathbf{H}_+ \;&=\; \frac{\partial \mathbf{D}_+}{\partial t} + \mathbf{J}_+, && (22) \end{aligned}$$

the $-$-components are

$$\begin{aligned} \nabla \times \mathbf{E}_- &= -\frac{\partial \mathbf{B}_-}{\partial t}, && (23) \\ \nabla \times \mathbf{H}_- &= \frac{\partial \mathbf{D}_-}{\partial t} + \mathbf{J}_-, && (24) \end{aligned}$$

and the 0-components are

$$\begin{aligned} \frac{\partial \mathbf{B}_0}{\partial t} &= 0, && (25) \\ \frac{\partial \mathbf{D}_0}{\partial t} &= -\mathbf{J}_0, && (26) \\ \nabla \cdot \mathbf{D}_0 &= \rho, && (27) \\ \nabla \cdot \mathbf{B}_0 &= 0. && (28) \end{aligned}$$

Thus, up to the assumption of the fields, the above equations form a completely equivalent formulation for Maxwell's equations.

One can prove that each decomposed field only depends on one real function $\mathbb{R}^3 \to \mathbb{R}$ [**III**]. This means that the electric field $\mathbf{E}$ can be represented either using Cartesian coordinate functions E_x, E_y, E_z, or using functions $e_+, e_-, e_0 \colon \mathbb{R}^3 \to \mathbb{R}$ representing $\mathbf{E}_+$, $\mathbf{E}_-$, $\mathbf{E}_0$, respectively. The advantage with the latter representation is that functions e_+, e_-, e_0 have some physical interpretation. Say, if e_+ is non-zero, then we can deduce that $\mathbf{E}$ has a $\mathbf{E}_+$ component. The same is not true in the Cartesian representation.

Another advantage of the decomposed equations is that they are completely decoupled [Mos71]. The $+$-equations involve only $+$-fields, the $-$-equations involve only $-$-fields, and the 0-equations involve only 0-fields. However, this is only valid for fields $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{D}$. Once the constitutive equations are introduced, these will couple the fields to each other. Moreover, the $\pm$-equations in the decomposed Maxwell's equations are structurally identical with the original equations. This means that the decomposed $\pm$-fields propagate as physical fields. A disadvantage of the decomposition is that it does not preserve the support of vector fields. This is particularly problematic for the sources.

5.2 Bohren decomposition

The *Bohren decomposition* is a very useful tool in electromagnetics and it has been studied in numerous references (see for example [BH83, Lak94, Lin92, LSTV94, SSTA01]). In its simplest form, it takes a solution to Helmholtz's equation

$$\nabla \times (\nabla \times \mathbf{E}) = k^2 \mathbf{E}$$

and decomposes $\mathbf{E}$ as $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, where

$$\mathbf{E}_\pm = \frac{1}{2}\left(\mathbf{E} \pm \frac{1}{k}\nabla \times \mathbf{E}\right).$$

The decomposed fields then satisfy

$$\nabla \times \mathbf{E}_\pm = \pm k \mathbf{E}_\pm.$$

For example, if we apply the Bohren decomposition to a linearly polarized wave, the decomposed fields $\mathbf{E}_\pm$ will be circularly polarized waves, but with opposite orientations. A central result in [**III**] is that the helicity decomposition (at least formally) coincides with the Bohren decomposition.

A vector field $\mathbf{F}$ is a *Beltrami field* if $\nabla \times \mathbf{F} = f\mathbf{F}$ for real function f [EG00]. Thus $\mathbf{E}_\pm$ are both Beltrami fields. A characteristic feature for such fields is a constant twisting behaviour much like circularly polarized waves. For example, for $\mathbf{E}_\pm$ we have

$$\mathbf{E}_\pm \cdot \nabla \times \mathbf{E}_\pm = \pm k \mathbf{E}_\pm \cdot \mathbf{E}_\pm.$$

Hence, as long as $\mathbf{E}_\pm$ do not vanish, they exhibit "handed rotation" as described in Section 5.1.

A Beltrami field $\mathbf{F}$ with $\nabla \times \mathbf{F} = f\mathbf{F}$ is *rotational* if f is nowhere zero. An important property of non-vanishing rotational Beltrami fields is that they are essentially in a one-to-one correspondence with contact structures [EG00]. (On $\mathbb{R}^3$ a *contact structure* is induced by a vector field $\mathbf{F}$ that satisfies $\mathbf{F} \cdot \nabla \times \mathbf{F} \neq 0$ everywhere.) Thus, a motivation for studying the Bohren decomposition is that it gives a method for generating contact structures from solutions to Helmholtz' equation. Examples and figures of such contact structures are given in [**III**]. Unfortunately, properties (19)-(20) in the helicity decomposition do not seem to imply that $\pm \mathbf{F}_\pm \cdot \nabla \times \mathbf{F}_\pm \geq 0$ holds pointwise. If this would hold, $\mathbf{F}$ would induce two contact structures as long as $\mathbf{F}_\pm$ do not vanish. However, if $\mathscr{H}(\mathbf{F}_+) > 0$, then $\mathbf{F}_+ \cdot \nabla \times \mathbf{F}_+ > 0$ on some open set, and on this set $\mathbf{F}_+$ induces a contact structure.

5.3 Other models for polarization

There are also other ways to describe polarization. Examples are the *Stokes parameters* for a plane wave [Jac99] and the *polarization vector* for a time harmonic solution [Lin92]. Electromagnetic polarization and propagation has also been studied using microlocal analysis [Den92].

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