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ON THE IMPLEMENTATION AND FORMULATION OF THE ELECTROMAGNETIC SURFACE INTEGRAL EQUATIONS

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Name of the dissertation

On the Implementation and Formulation of the Electromagnetic Surface Integral Equations

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Supervisor	Prof. Jukka Sarvas, Helsinki University of Technology
(Instructor)	Prof. Jukka Sarvas

Abstract

Surface integral equation method is one of the most popular numerical methods in the computational electromagnetics. In this thesis three problem areas in the implementation and formulation of the frequency domain surface integral equation methods are studied.

First, a recursive technique is developed to evaluate the singular integrals arising from the electromagnetic surface integral equation methods. The technique is based on singularity subtraction method in which the most singular terms are extracted and evaluated analytically. A similar recursive algorithm is also developed for higher order basis functions. The technique is efficient and easy to apply for different surface integral equation formulations and polynomial basis functions.

The second problem is how to model the electromagnetic fields in complex structures. A procedure is developed for this junction problem to model the fields separately in each region and to properly enforce the electromagnetic boundary conditions on the interfaces of the regions. The developed method is very simple and independent of the surface integral equation formulation and thus makes it easy to apply to different formulations.

The third and most important problem is the choice of the electromagnetic surface integral equation formulation. Various traditional type of formulations are developed in this thesis and their behavior is studied especially with respect to the iterative methods.

The most significant part of this thesis is the new type of a surface integral equation formulation and new techniques developed for it. The main idea in this new formulation is to use the surface charge densities as unknowns in addition to the traditional surface current densities. The formulation does not have problems with the low frequencies, it is well-balanced and the convergence of the iterative methods is fast for a very wide frequency range. The new formulation seems to be the first well-conditioned, truly broadband formulation that can be used from static to high frequencies in a general case of composite metallic and dielectric structures.

The relation between the new formulation and the Picard's extended Maxwell system is also studied.

Keywords electromagnetic surface integral equations, singular integrals, low frequency problem

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Tekijä Matti Taskinen

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Sähkömagneettisten pintaintegraaliyhtälöiden implementoinnista ja formuloinnista

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Vastaväittäjä(t)	Prof. Levent Gürel, Bilkent University, Turkki
Työn valvoja	Prof. Jukka Sarvas, Teknillinen korkeakoulu
(Työn ohjaaja)	Prof. Jukka Sarvas

Tiivistelmä

Pintaintegraaliyhtälömenetelmä on yksi suosituimmista laskennallisen sähkömagnetiikan numeerisista menetelmistä. Tässä väitöstyössä tarkastellaan taajuusalueen pintaintegraaliyhtälömenetelmien implementoinnin ja formuloinnin kolmea ongelma-aluetta.

Ensiksi kehitetään rekursiivinen menetelmä sähkömagnetiikan pintaintegraaliyhtälöiden yhteydessä tarvittavien singulaaristen integraalien laskemiseksi. Menetelmä perustuu singulaarisuuden vähennysmenetelmään, jossa kaikkein singulaarisimmat termit vähennetään ja lasketaan erikseen analyyttisesti. Samankaltainen rekursiivinen algoritmi kehitetään myös korkeamman asteen kantafunktioille. Menetelmä on tehokas ja helppo soveltaa erilaisille pintaintegraaliyhtälöille ja polynomaalisille kantafunktioille.

Toinen ongelma-alue on sähkömagneettisten kenttien mallintaminen pintaintegraaliyhtälömenetelmillä monimutkaisissa rakenteissa. Tähän nk. liitosongelmaan kehitetään menettelytapa, jossa kentät mallinnetaan erikseen jokaisessa alueessa ja sähkömagneettiset reunaehdot pakotetaan oikella tavalla alueiden rajapinnoilla. Kehitetty menetelmä on hyvin yksinkertainen ja riippumaton käytetystä pintaintegraaliyhtälön muodosta ja on siten helppo soveltaa erilaisiin pintaintegraaliyhtälöformulointeihin.

Kolmas ja tärkein ongelma-alue on sähkömagneettisen pintaintegraaliyhtälöformulaation valinta. Väitöskirjassa kehitetään lukuisia perinteisen muotoisia formulointeja ja niiden käyttäytymistä tarkastellaan erityisesti iteratiivisten menetelmien yhteydessä.

Väitöskirjan tärkein osa on uudentyyppinen pintaintegraaliyhtälömuoto ja sitä varten kehitetyt uudet menettelytavat. Uuden formuloinnin pääidea on käyttää pintavaraustiheyksiä tuntemattomina perinteisten pintavirtatiheyksien lisäksi. Formulaatiolla ei ole ongelmia matalilla taajuuksilla, se on hyvin vakaa ja iteratiivisten menetelmien suppeneminen on nopeaa laajalla taajuusalueella. Uusi pintaintegraaliyhtälömuoto näyttää olevan ensimmäinen hyvin määritelty, laajan taajuusalueen formulaatio, jota voidaan käyttää statiikasta korkeille taajuuksille yleisissä yhdistettyjen metallisten ja dielektristen kappaleiden tapauksissa.

Väitöskirjan yhteenveto-osassa tarkastellaan myös uuden pintaintegraaliyhtälömuodon ja Picardin laajennetun Maxwell-järjestelmän välistä suhdetta.

Asiasanat sähkömagnetiikan pintaintegraaliyhtälöt, singulaariset integraalit, matalan taajuuden ongelma

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Genius is 1% inspiration and 99% debugging.

— Thomas A. Edison

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I wish to thank my colleagues Pasi Ylä-Oijala and Seppo Järvenpää without whom this thesis would have never happened. The close research collaboration has been very fruitful and rewarding over the years.

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For the financial support I wish to thank the Graduate School of Applied Electromagnetics.

List of Publications

[P1] Pasi Ylä-Oijala and Matti Taskinen. Calculation of CFIE impedance matrix elements with RWG and n×RWG functions. *IEEE Trans. Antennas Propagat.*, 51(8):1837–1846, August 2003.

[P2] Seppo Järvenpää, Matti Taskinen, and Pasi Ylä-Oijala. Singularity extraction technique for integral equation methods with higher order basis functions on plane triangles and tetrahedra. *Int. J. Numer. Meth. Engng*, 58:1149–1165, October 2003.

[P3] Seppo Järvenpää, Matti Taskinen, and Pasi Ylä-Oijala. Singularity subtraction technique for high-order polynomial vector basis functions on planar triangles. *IEEE Trans. Antennas Propagat.*, 54(1):42–49, January 2006.

[P4] Pasi Ylä-Oijala, Matti Taskinen, and Jukka Sarvas. Surface integral equation method for general composite metallic and dielectric structures with junctions. *PIER*, 52:81–108, 2005.

[P5] Pasi Ylä-Oijala and Matti Taskinen. Application of combined field integral equation for electromagnetic scattering by dielectric and composite objects. *IEEE Trans. Antennas Propagat.*, 53(3):1168–1173, March 2005.

[P6] Pasi Ylä-Oijala and Matti Taskinen. Well-conditioned Müller formulation for electromagnetic scattering by dielectric objects. *IEEE Trans. Antennas Propagat.*, 53(10):3316–3323, October 2005.

[P7] Pasi Ylä-Oijala, Matti Taskinen, and Seppo Järvenpää. Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods. *Radio Science*, 40(6):RS6002 1–19, November 2005.

[P8] Matti Taskinen and Pasi Ylä-Oijala. Current and charge integral equation formulation. *IEEE Trans. Antennas Propagat.*, 54(1):58–67, January 2006.

Contribution of the Author

The research in this thesis is mainly a result of the close collaboration between the author, Pasi Ylä-Oijala and Seppo Järvenpää. The aim of the research is to develop a frequency domain electromagnetic simulation program for complex composite structures with the surface integral equation method. The computer software is build upon a combination of a higher level programming environment for fast software development and a low level programming language for fast computation of the heavy calculations. The work consists of a wide range of different tasks and skills.

The author has been both developing the theory, i.e. throwing to the ideas, solving the problems and studying the mathematical backgrounds, and also implementing the ideas, i.e. developing the routines, data structures and numerical algorithms, and testing and verifying the programs.

The article [P1] is based on author's Master's thesis and the programs of this article are build upon routines developed and implemented by the author.

Articles [P2] and [P3] are based on the same recursive method and the key ideas for the recursive algorithms of the higher order basis functions are developed by the author in collaboration with the first author.

The main ideas of the article [P4] are developed by the author in collaboration with the other authors. The author has developed and implemented the key data structures and routines.

The author developed the new surface integral equation formulations for articles [P5], [P6] and [P7] in collaboration with the first author. The programs of these articles are build upon routines developed and implemented by the author.

The article [P8] is almost a solely work of the author. The new formulation presented in this article is developed and implemented by the author.

The author has participated in the writing process of the first seven articles. The article [P8] was entirely written by the author.

The Picard's system part is new in the summary of this thesis and not included in any of the articles of the thesis.

Contents

1 Introduction

Surface integral equation method is one of the most popular numerical methods in the electromagnetic scattering analysis of metallic, homogeneous dielectric and composite objects. It is used in many important applications e.g., in the radar technology, antenna design and microwave engineering.

In the frequency domain surface integral equation method, the electromagnetic fields are simulated with a fixed frequency, i.e. the fields are assumed to be time harmonic. The electromagnetic fields are dictated by the time harmonic Maxwell's equations in which the electric and magnetic fields are tied together with partial differential equations. In homogeneous regions the fields inside the region can be expressed as surface integrals of the surface fields and so, to simulate the fields in the region numerically, only the surface fields need to be modeled. Traditionally in the surface integral equation method these surface fields are modeled with the surface currents densities. The fields of neighboring regions are related to each other by the electromagnetic boundary conditions on the interfaces of the regions.

The surface integral equations are usually implemented by discretizing the region interfaces with triangular or quadrilateral meshes and by expressing the surface currents with vector valued basis functions. The equations are often discretized by the Galerkin procedure, in which the equations are tested with the same basis functions.

In this thesis three problem areas in the implementation and formulations of the electromagnetic surface integral equations are studied.

The first problem area is the evaluation of the singular integrals arising from the surface integral equations. These integrals can be calculated numerically, for example, with so called singularity cancellation methods, in which the singularity is cancelled by a proper transformation of the coordinates. Widely applied techniques are e.g., Duffy's method [1] and polar coordinate transformation [2]. These techniques, however, have accuracy problems with non-regular elements.

Another possibility is the singular subtraction (or extraction) technique ([3], [4] and [5]) in which the singularity is extracted and integrated analytically. The same singular function is typically shared by the integrals of the neighboring basis functions or different operator terms in the surface integral equation formulation. With the singularity extraction technique, significant time savings can be obtained, because similar integrals need to be evaluated only once. In the original singularity extraction, only the most singular term was extracted. The remaining function is, however, not necessarily smooth enough for numerical integration.

In article [P1] the singularity subtraction method was further extended by subtracting more than one singular term and a recursive technique was developed to evaluate the singular integrals of arbitrary order. In [P2] and [P3] the technique was further extended to higher order basis functions.

The second problem area in the electromagnetic surface integral equation methods is how to model the fields in complex structures. The surface fields need to be modeled with proper basis functions, the surface integral equations need to be tested properly on the surfaces and the electromagnetic boundary conditions need to be enforced properly on the interfaces.

In the case of a simple interface of two dielectric regions, the modeling is quite straightforward process. The problem becomes more complicated in the intersections, or "junctions" of more than two regions. Some authors ([6], [7] and [8]) first formulate the surface integral equations on the surfaces and then use half basis functions to expand the currents and to test the equations at the junctions. This, however, unnecessarily complicates the modeling procedure. Other authors combine the currents of regions of the junctions into so called multiplets [9], but the testing is then more problematic.

In article [P4] this junction problem was studied and a procedure was developed to model and test the surface fields separately in each region and properly enforce the electromagnetic boundary conditions between the regions in a computationally simple way. The procedure does not depend on the surface integral equation formulation and thus makes it easy to apply to different formulations.

The third and the most important problem area is the choice of the surface integral equation formulation. There are many different properties that a good electromagnetic surface integral equation formulation should satisfy. The formulation should always give the correct and accurate results. A good surface equation formulation should also manage large variety of different materials and shapes of the objects. Materials may vary from free space to highly insulating, absorbing or conducting objects, and the objects may have sharp edges or resonant cavities or shapes. The discretized equation of the formulation should also have good numerical properties. The system should be solvable with the finite precision of the computers. This can usually be detected by the (singular value) condition number of the discretized system matrix.

In the case of metallic objects the fundamental integral equations are the electric field (EFIE), the magnetic field (MFIE) and the combined field (CFIE) integral equations [10]. For dielectric objects, the most popular formulations are the PMCHWT [11] and the Müller [12] formulations.

In article [P5], a new CFIE type formulation was presented to be used as a general formulation for composite objects of dielectric and metallic regions. The CFIE formulation removes the problems with the internal resonances of the metallic objects and with proper weighting coefficients, leads to better conditioned systems and faster converging iterative solutions.

Most of the traditional electromagnetic surface integral equations suffer from the so called low frequency problems. A good formulation should not have problems with low frequencies, because the low frequency solutions are typically smoother.

In [P6] a well-conditioned Müller type formulation was developed for the case of homogeneous dielectric objects. The Müller type formulations are not infected by the low frequency problem in simple dielectric cases.

The discretized electromagnetic surface integral equations typically lead to very large dense matrices and usually iterative methods are needed to solve the system. A good formulation should also have good convergence properties which depend on the spectral (or eigenvalue) distribution of the system matrix. The low frequency problem of the traditional formulations slows down the convergence of the iterative methods.

In [P7] different traditional formulations and those developed by the authors were studied with iterative methods. It was found that the second kind of integral equation formulations with properly scaled identity operators give the best overall convergences.

The traditional formulations can be enhanced by proper coupling coefficients [13], [14] and the low frequency problem can be partially removed by the loop-star and loop-tree basis functions [15], [16]. Also, certain preconditioning methods [17], [18] can be used to improve the condition of the system. These analytical preconditioners are, however, currently developed only for the metallic objects. The best way to improve the situation is a better choice of the surface integral equation formulation.

In [P8] a novel well-conditioned surface integral equation formulation, Current and Charge Integral Equation (CCIE), is presented for general case of composite metallic and dielectric structures. The new formulation does not suffer from the low frequency breakdown, is well balanced and converges rapid for a very wide frequency range. This new formulation and techniques developed in [P8] are the most important and essential parts of this thesis.

The new formulation is shown to be related to the Picard's extended Maxwell system [19], [20]. This elliptic, first order system has a correct static behavior which is not the case with the Maxwell's system used with the traditional electromagnetic surface integral equations.

2 Electromagnetic Surface Integral Equations

In the frequency domain surface integral equation method the modeled objects are assumed to be combinations of homogeneous regions. The time harmonic electromagnetic fields are modeled inside each homogeneous region as surface integrals of the surface field values.

In this work the regions are assumed to be isotropic, either "dielectric" or "metallic". In isotropic dielectric regions the electromagnetic parameters, electric permittivity ε and magnetic permeability μ are scalar valued. If the region is conductive, the electric permittivity ε may contain the (finite) conductivity σ as the imaginary part,

$$\varepsilon = \varepsilon_0 \, \varepsilon_r \, + i \, \frac{\sigma}{\omega},$$

where $i = \sqrt{-1}$ and $\omega = 2\pi f$ is the angular frequency. The metallic regions may be either perfectly electric conducting (PEC) or perfectly magnetic conducting (PMC).

In each region, the fields are independently modeled with the time harmonic Maxwell's equations. The fields in different regions are related to the fields of the neighboring regions by the electromagnetic boundary conditions.

Electromagnetic surface integral equations are derived from the surface integral representations of the electromagnetic fields. In the traditional formulations the surface current densities on the region interfaces are used as the unknowns and a given primary field is the source term.

2.1 Time Harmonic Maxwell's Equations

In a homogeneous, isotropic region having no sources, the time harmonic ($e^{-i\omega t}$) Maxwell's equations tie the electric ($\mathbf{E}$) and the magnetic ($\mathbf{H}$) field values together as

$$\begin{cases} \nabla \times \mathbf{E} = \quad i\omega\mu \, \mathbf{H}, \\ \nabla \times \mathbf{H} = -i\omega\varepsilon \, \mathbf{E}, \\ \nabla \cdot \mathbf{E} \quad = 0, \\ \nabla \cdot \mathbf{H} \quad = 0. \end{cases} \tag{1}$$

If the frequency is zero ($\omega = 0$), the fields are decoupled.

2.1.1 Normalized Fields and Maxwell's Equations

The electromagnetic material parameters, the electric permittivity ε and the magnetic permeability μ are typically of very different scales. For example, in the free space they are

$$\varepsilon_0 \, = 8.85418782 \cdot 10^{-12} \frac{As}{Vm},$$
$$\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}.$$

A modified set of Maxwell's equations can be formed by using normalized electromagnetic fields [21],

$$\begin{aligned} \tilde{\mathbf{E}} &= \sqrt{\varepsilon}\,\mathbf{E}, \\ \tilde{\mathbf{H}} &= \sqrt{\mu}\,\mathbf{H}, \end{aligned} \tag{2}$$

where the normalized electric ($\tilde{\mathbf{E}}$) and magnetic ($\tilde{\mathbf{H}}$) fields are of the same scale, since

$$\tilde{\mathbf{E}} = \sqrt{\varepsilon}\,\mathbf{E} \approx \sqrt{\mu}\,\mathbf{H} = \tilde{\mathbf{H}}.$$

Using these normalized fields, a better balanced, normalized Maxwell's equations are

$$\begin{cases} \nabla \times \tilde{\mathbf{E}} = \quad ik\,\tilde{\mathbf{H}}, \\ \nabla \times \tilde{\mathbf{H}} = -ik\,\tilde{\mathbf{E}}, \\ \nabla \cdot \tilde{\mathbf{E}} \;\; = 0, \\ \nabla \cdot \tilde{\mathbf{H}} \;\; = 0, \end{cases} \tag{3}$$

where $k = \omega\sqrt{\varepsilon\mu}$, the wave number, is the only material parameter.

2.1.2 Boundary Conditions

The electromagnetic fields in separate regions are not independent of each other. The fields are tied together by the boundary conditions which consist of the continuity relations on dielectric interfaces and of the vanishing field components on metallic surfaces.

On the interface of two dielectric regions the tangential components of the electric and magnetic fields are continuous across the interface,

$$\begin{aligned} \mathbf{E}_{tan}^{(1)} &= \mathbf{E}_{tan}^{(2)}, \\ \mathbf{H}_{tan}^{(1)} &= \mathbf{H}_{tan}^{(2)}. \end{aligned} \tag{4}$$

Similarly, the normal components of the electric ($\mathbf{D} = \varepsilon\mathbf{E}$) and the magnetic ($\mathbf{B} = \mu\mathbf{H}$) flux densities are continuous across the interface

$$\begin{aligned} D_n^{(1)} = \varepsilon_1 E_n^{(1)} &= -\varepsilon_2 E_n^{(2)} = -D_n^{(2)}, \\ B_n^{(1)} = \mu_1 H_n^{(1)} &= -\mu_2 H_n^{(2)} = -B_n^{(2)}, \end{aligned} \tag{5}$$

where the direction of surface normal is assumed to point into each region (inner normal) and leads to the change of the sign above.

Neither the tangential nor the normal components of the normalized fields (2) are directly continuous across the interfaces, but

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon_1}}\,\tilde{\mathbf{E}}_{tan}^{(1)} &= \mathbf{E}_{tan}^{(1)} = \mathbf{E}_{tan}^{(2)} &&= \frac{1}{\sqrt{\varepsilon_2}}\,\tilde{\mathbf{E}}_{tan}^{(2)}, \\ \frac{1}{\sqrt{\mu_1}}\,\tilde{\mathbf{H}}_{tan}^{(1)} &= \mathbf{H}_{tan}^{(1)} = \mathbf{H}_{tan}^{(2)} &&= \frac{1}{\sqrt{\mu_2}}\,\tilde{\mathbf{H}}_{tan}^{(2)}, \\ \sqrt{\varepsilon_1}\,\tilde{E}_n^{(1)} &= D_n^{(1)} = -D_n^{(2)} &&= -\sqrt{\varepsilon_2}\,\tilde{E}_n^{(2)}, \\ \sqrt{\mu_1}\,\tilde{H}_n^{(1)} &= B_n^{(1)} = -B_n^{(2)} &&= -\sqrt{\mu_2}\,\tilde{H}_n^{(2)}. \end{aligned} \tag{6}$$

On perfectly electric conducting (PEC) surfaces the tangential components of the electric field $\mathbf{E}_{tan}$ (and $\tilde{\mathbf{E}}_{tan}$) and the normal components of the magnetic field H_n (and $\tilde{H}_n$) vanish, and the remaining field components are independent on opposite sides of the surface.

Similarly, on perfectly magnetic conducting (PMC) surfaces the tangential components of the magnetic field $\mathbf{H}_{tan}$ (and $\tilde{\mathbf{H}}_{tan}$) and the normal components of the electric field E_n (and $\tilde{E}_n$) vanish and the remaining field components are independent on opposite sides of the surface.

2.2 Surface Integral Representations of the Electromagnetic Fields

In each separate region the electromagnetic fields can be expressed as surface integrals of the equivalent electric ($\mathbf{J}$) and magnetic ($\mathbf{M}$) surface current densities

$$\begin{aligned} \mathbf{J} &= \mathbf{n} \times \mathbf{H}, \\ \mathbf{M} &= \mathbf{n} \times \mathbf{E}, \end{aligned} \tag{7}$$

where $\mathbf{n}$ is the inner unit normal of the surface and the positive sign convention of the magnetic current density $\mathbf{M}$ follows the choice in [P8]. In all the other articles of this thesis the more common negative sign is used in $\mathbf{M}$.

Note that the "electric" surface current density $\mathbf{J}$ is related to the rotated tangential components of the magnetic field and similarly "magnetic" current density $\mathbf{M}$ equals to the rotated tangential components of the electric field.

The time harmonic (total) electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ can be expressed on a surface of a region as

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^p - \frac{1}{i\omega\varepsilon}\left(\mathcal{D} - \frac{1}{2}\mathbf{n}\nabla\cdot\right)(\mathbf{J}) + \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{M}), \\ \mathbf{H} &= \mathbf{H}^p + \frac{1}{i\omega\mu}\left(\mathcal{D} - \frac{1}{2}\mathbf{n}\nabla\cdot\right)(\mathbf{M}) + \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{J}), \end{aligned} \tag{8}$$

where $\mathbf{E}^p$ and $\mathbf{H}^p$ are the primary fields. $\mathcal{D}$ and $\mathcal{K}$ are surface integral operators

$$\begin{aligned} \mathcal{D}(f) &= \left(\nabla\nabla\cdot + k^2\right)\mathcal{S}(f), \\ \mathcal{K}(f) &= \nabla \times \mathcal{S}(f), \\ \mathcal{S}(f)(\mathbf{r}) &= \int G(\mathbf{r},\mathbf{r}')\, f(\mathbf{r}')\, ds(\mathbf{r}'), \end{aligned}$$

where $G(\mathbf{r},\mathbf{r}') = \dfrac{1}{4\pi}\dfrac{e^{ikR}}{R}$, $R = |\mathbf{r}-\mathbf{r}'|$ is the free space Green's function.

The surface integral representations (8) include four terms related to the residual terms of the singular principal value integrals. These terms play crucial roles when considering proper surface integral equation formulations of the second kind with identity operators.

2.2.1 Divergences of Currents and Normal Components of Fields

The $\nabla\nabla\cdot$ part of the $\mathcal{D}$ operator can be further modified into

$$\nabla\nabla\cdot\mathcal{S}(\mathbf{J}) = \nabla\int G(\mathbf{r},\mathbf{r}')\,\nabla^s_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}')\,ds(\mathbf{r}') = \nabla\mathcal{S}\left(\nabla\cdot\mathbf{J}\right) = \mathcal{N}\left(\nabla\cdot\mathbf{J}\right),$$

where

$$\mathcal{N} = \nabla\mathcal{S}$$

and the integral representations (8) of the fields are

$$\begin{aligned}\mathbf{E} &= \mathbf{E}^p - \frac{1}{i\omega\varepsilon}\left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\nabla\cdot\mathbf{J}) \;+ i\omega\mu\,\mathcal{S}(\mathbf{J}) \;+ \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\mathbf{M}),\\ \mathbf{H} &= \mathbf{H}^p + \frac{1}{i\omega\mu}\left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\nabla\cdot\mathbf{M}) - i\omega\varepsilon\,\mathcal{S}(\mathbf{M}) + \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\mathbf{J}).\end{aligned} \tag{9}$$

Outside of the sources, the divergences of the surface current densities and the normal components of the fields are related as

$$\begin{aligned}\nabla\cdot\mathbf{J} &= -\mathbf{n}\cdot\nabla\times\mathbf{H} = \;\; i\omega\varepsilon\,\mathbf{n}\cdot\mathbf{E} \;\; = \;\; i\omega\varepsilon\,E_n,\\ \nabla\cdot\mathbf{M} &= -\mathbf{n}\cdot\nabla\times\mathbf{E} \;= -i\omega\mu\,\mathbf{n}\cdot\mathbf{H} \;= -i\omega\mu\,H_n,\end{aligned}$$

and the surface integral representations (9) can be expressed as

$$\begin{aligned}\mathbf{E} &= \mathbf{E}^p - \left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(E_n) + i\omega\mu\,\mathcal{S}(\mathbf{J}) \;+ \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\mathbf{M}),\\ \mathbf{H} &= \mathbf{H}^p - \left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(H_n) - i\omega\varepsilon\,\mathcal{S}(\mathbf{M}) + \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\mathbf{J}).\end{aligned} \tag{10}$$

2.2.2 Normalized Surface Integral Representations

The electromagnetic field surface integral representations (9) and (10) can be expressed with the normalized fields (2) as

$$\begin{aligned}\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}^p - \frac{1}{ik}\left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\nabla\cdot\tilde{\mathbf{J}}) \;+ ik\,\mathcal{S}(\tilde{\mathbf{J}}) \;+ \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\tilde{\mathbf{M}}),\\ \tilde{\mathbf{H}} &= \tilde{\mathbf{H}}^p + \frac{1}{ik}\left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\nabla\cdot\tilde{\mathbf{M}}) - ik\,\mathcal{S}(\tilde{\mathbf{M}}) + \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\tilde{\mathbf{J}})\end{aligned} \tag{11}$$

and

$$\begin{aligned}\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}^p - \left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\tilde{E}_n) + ik\,\mathcal{S}(\tilde{\mathbf{J}}) \;+ \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\tilde{\mathbf{M}}),\\ \tilde{\mathbf{H}} &= \tilde{\mathbf{H}}^p - \left(\mathcal{N}-\frac{1}{2}\mathbf{n}\right)(\tilde{H}_n) - ik\,\mathcal{S}(\tilde{\mathbf{M}}) + \left(\mathcal{K}-\frac{1}{2}\mathbf{n}\times\right)(\tilde{\mathbf{J}}),\end{aligned} \tag{12}$$

where

$$\tilde{\mathbf{J}} = \mathbf{n}\times\tilde{\mathbf{H}} = \sqrt{\mu}\,\mathbf{J} \quad \text{and} \quad \tilde{\mathbf{M}} = \mathbf{n}\times\tilde{\mathbf{E}} = \sqrt{\varepsilon}\,\mathbf{M}$$

are the normalized surface current densities. As with the normalized Maxwell's equations, the only material parameter present in the normalized surface integral representations is the wave number k.

2.3 Traditional Surface Integral Equations

The surface integral representations of the electromagnetic fields are used to derive surface integral equations having the current densities $\mathbf{J}$ and $\mathbf{M}$ as unknowns and the primary fields $\mathbf{E}^p$ and $\mathbf{H}^p$ as known "right hand sides" of the equations. Traditionally, three different surface integral equation formulations can be derived from the original surface integral representations (8) by considering the tangential field components,

$$\begin{pmatrix} \mathbf{H}^p_{tan} \\ \mathbf{E}^p_{tan} \end{pmatrix} = \begin{pmatrix} -\frac{1}{i\omega\mu}\,\mathcal{D}_{tan} & -\frac{1}{2}\mathbf{n}\times - \mathcal{K}_{tan} \\ -\frac{1}{2}\mathbf{n}\times - \mathcal{K}_{tan} & \frac{1}{i\omega\varepsilon}\,\mathcal{D}_{tan} \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix} \tag{13}$$

or the rotated tangential components,

$$\begin{pmatrix} \mathbf{n}\times\mathbf{E}^p \\ \mathbf{n}\times\mathbf{H}^p \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{I}}{2} - \mathbf{n}\times\mathcal{K} & \frac{1}{i\omega\varepsilon}\,\mathbf{n}\times\mathcal{D} \\ -\frac{1}{i\omega\mu}\,\mathbf{n}\times\mathcal{D} & \frac{\mathcal{I}}{2} - \mathbf{n}\times\mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}, \tag{14}$$

or by combining the tangential and rotated tangential components into a CFIE type of equation (JMCFIE in [P5]),

$$\begin{pmatrix} \mathbf{n}\times\mathbf{E}^p - \eta\,\mathbf{H}^p_{tan} \\ \mathbf{n}\times\mathbf{H}^p + \frac{1}{\eta}\,\mathbf{E}^p_{tan} \end{pmatrix} = \begin{pmatrix} \mathcal{L} & \eta\,\mathbf{n}\times\mathcal{L} \\ -\frac{1}{\eta}\,\mathbf{n}\times\mathcal{L} & \mathcal{L} \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}, \tag{15}$$

where $\mathcal{I}$ is the identity operator, $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ and

$$\mathcal{L} = \frac{\mathcal{I}}{2} - \mathbf{n}\times\mathcal{K} + \frac{1}{ik}\,\mathcal{D}_{tan}.$$

In a simple PEC metallic case the magnetic current density $\mathbf{M}$ disappears because of the boundary conditions and the first equation (13) is usually reduced to the EFIE formulation containing only the $\mathcal{D}$ operator whereas the second equation (14) is usually reduced to a MFIE type of formulation with the $\mathcal{K}$ operator. The third equation (15) is reduced to the traditional CFIE formulation.

In dielectric case, the first equation leads to the well known PMCHWT formulation [11].

The last two equations have an identity operator $\mathcal{I}/2$ in the diagonal and so they are integral equations of the second kind and, therefore, usually lead to better behaving implementations.

In all cases, the unknown current densities $\mathbf{J}$ and $\mathbf{M}$ are typically expanded with RWG basis functions [22] on discretized region interfaces and the equations are discretized with the Galerkin procedure by testing with the same basis functions.

Three problem areas arise from the surface integral equations. The integral operators in the surface integral equation are singular, the discretized surface integral equations of separate regions need to be combined according to the electromagnetic boundary conditions and the most important problem is the choice of the surface integral equation formulation.

3 Singular Integrals

The discretized electromagnetic surface integral equations usually involve evaluations of integrals that are singular when the supports of the testing and basis functions overlap or touch each other. When discretized with Galerkin procedure, the singular integrals of the discretized surface integral equations (13), (14) and (15) can all be evaluated from integrals

$$\int\limits_{T_m} N_m(\mathbf{r}) \int\limits_{T_n} g(\mathbf{r},\mathbf{r}')\, N_n(\mathbf{r}')\, ds(\mathbf{r})\, ds(\mathbf{r}'),$$

where g is either the free space Green's function G, its normal derivative or surface gradient,

$$g(\mathbf{r},\mathbf{r}') = \begin{cases} G(\mathbf{r},\mathbf{r}') \\ \dfrac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}_{\mathbf{r}'}} \\ \nabla^s_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') \end{cases} .$$

The double surface integration is usually over a pair of elements T_m and T_n and the functions N_m and N_n are typically either constants or the linear nodal shape functions.

In the articles of this thesis, the evaluation of these singular integrals is based on the singularity extraction technique [3]. The extraction (or subtraction) of the singular terms is extended in [23] with recursively evaluated integrals of arbitrary order and these formulas are applied to the CFIE formulation in [P1]. The method is further generalized for higher order basis functions in [P2] and [P3].

3.1 Singularity Subtraction Technique

The singularity subtraction is based on the Taylor series of the exponential function,

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots,$$

so the free space Green's function $G(\mathbf{r},\mathbf{r}')$ has a series expansion

$$4\pi\, G(\mathbf{r},\mathbf{r}') = \frac{e^{aR}}{R} = \sum_{j=0}^{\infty} \frac{a^j R^{j-1}}{j!} = \frac{1}{R} + a + \frac{a^2 R}{2} + \frac{a^3 R^2}{6} + \frac{a^4 R^3}{24} + \ldots, \tag{16}$$

where $a = ik$. The idea is to subtract some of the first, non-smooth terms from the Green's function,

$$\int \frac{e^{aR}}{R} N(\mathbf{r}')\, ds(\mathbf{r}') = \int \left(\frac{e^{aR}}{R} - \sum_{j=0}^{N} \frac{a^{2j} R^{2j-1}}{(2j)!} \right) N(\mathbf{r}')\, ds(\mathbf{r}') + \sum_{j=0}^{N} \frac{a^{2j}}{(2j)!} \int R^{2j-1} N(\mathbf{r}')\, ds(\mathbf{r}') \tag{17}$$

and to calculate the extracted integrals analytically. The remaining integral is relatively smooth and can be calculated numerically.

The analytical integrals of the odd powers R^n can be evaluated recursively from the lower order surface and edge integrals. The recursion is ended to surface integral of R^{-3} and edge integral of R^{-1} where the former can be calculated from the solid angle of the element.

The smooth, even power terms could also be calculated analytically. The recursion of the even terms is ended to surface and edge integrals of R^0, i.e. the area and edge lengths of the element.

The integrals with derivatives of the Green's function are integrated by subtracting derivatives of the series expansion terms and integrating the extracted terms analytically using the same integrals of R^n as in (17).

Singular integrals containing scalar or vector valued linear basis functions are integrated analytically by expressing the integrands with surface gradients of R^n. For example, an integral containing RWG basis functions [22] can be calculated with

$$\int_T R^n(\mathbf{r}' - p_j)\, ds(\mathbf{r}') = \frac{1}{n+2} \int_T \nabla^s_{\mathbf{r}'} R^n\, ds(\mathbf{r}') + (\rho - p_j) \int_T R^n\, ds(\mathbf{r}'),$$

where ρ is the projection of the point $\mathbf{r}$ into the triangle T plane and p_j is the free vertex of the triangle opposite to the edge of the RWG basis function.

3.2 Singularity Subtraction for Higher Order Basis Functions

The singularity subtraction is extended to higher order basis functions in [P2]. The higher order basis functions are expressed with polynomials of local (u, v) coordinates. One power of u (or v) at the time is expressed with the surface gradient of R^n and the integral is translated into surface and edge integrals having lower orders of u (or v). This recursion is performed until the remaining integrals do not anymore contain polynomials of u or v and can, therefore, be calculated from the same integrals as in (17).

The translation from the higher order basis function to a polynomial of the local (u, v) coordinates is a bit complicated process and in order to simplify the calculation of the actual integrals containing the higher order basis functions, the recursion was further improved in [P3]. Instead of using polynomials of the local coordinates, the higher order basis functions are expressed as polynomials of the linear nodal shape functions.

The linear nodal shape functions can be expressed with RWG bases as

$$N_j(\mathbf{r}') = 1 - \frac{(\mathbf{r}' - p_j) \cdot \mathbf{m}_{\mathbf{r}'}}{h_j}$$

and, therefore, the integrals can be translated similarly using surface gradients of R^n and recursion can be applied to lower the powers of the shape functions.

4 Junction Problem: Modeling Complex Structures

In the case of composite objects containing dielectric and metallic regions, the electromagnetic fields can still be modeled separately inside each region, but when combining the surface integral equations of separate regions, the enforcing of the boundary conditions on the region interfaces is not a straightforward process. This is known as the junction problem.

The fourth article [P4] describes how the boundary conditions should be applied by properly orienting the RWG basis functions which are used to model the current densities on regions.

4.1 Oriented RWG Basis Functions

An RWG basis function [22] is build from two adjacent triangles with a common edge and it models the flow of the current component normal to the edge continuously from one triangle to the other. Furthermore, on the basis edge, the RWG basis functions assigned to the other edges do not have components normal to the edge, so the RWG bases are semi-orthogonal on the edges.

Because the currents are 90 degree rotated tangential components of the fields (7), the RWG basis function models the flow of the field component parallel to the edge continuously from one triangle to the other.

If the edge is shared by more than one RWG basis functions, it is important to orient the RWG bases to the same orientation around the edge in order to model the same parallel orientation of the field.

Because the tangential field components are continuous across the dielectric interfaces (4) and the field component parallel to the edge is tangential to all of the triangles attached to the edge, all the oriented RWG basis functions of the edge are modeling the same field values and, therefore, they all share the same (unknown) coefficient of the field.

4.2 Metallic Junctions

If any of the triangles connected to the edge is on a PEC (or PMC) surface, the tangential component of the electric (magnetic) field is always zero also on the edge and, therefore, all the magnetic (electric) currents have to be removed from these edges.

Because the non-zero components of the fields are independent on opposite sides of the metallic surface, the oriented RWG basis functions around an edge share the same unknown coefficient only between two metallic triangles. The currents inside a closed metallic region do not interact with the outer world and they can simply be removed.

4.3 Enforcing the Boundary Conditions

The boundary conditions are now straightforward to enforce. When using the oriented RWG basis functions, the dielectric continuity relations (4) are enforced simply by using

the same unknown coefficients. This can be done by adding the columns of the discretized local system matrices together and in order to get the same number of discritized equations and unknowns, the corresponding rows of the matrices are added together, too.

The metallic boundary conditions are enforced simply by ignoring certain currents on edges having metallic triangles.

As a consequence of the simplicity of the enforcing of the boundary conditions, different surface integral equation formulations can be easily applied.

5 Well-Conditioned Surface Integral Equation Formulations

The choice of the surface integral equation formulation used for the solving the unknown surface current densities is a more difficult problem. Even using the traditional surface integral equations (13), (14) and (15), there are infinite number of ways to choose different formulations. The equations can be multiplied by arbitrary coefficients and several formulations can be build from the linear combinations of the equations.

In article [P5], a new CFIE type formulation was presented for general composite objects. This formulation does not have problems with internal resonances of metallic objects. In article [P7] different formulations were studied to be used with iterative solvers and in article [P6] Müller type formulations were developed to overcome the low frequency problem in the case of simple homogeneous objects.

5.1 Internal Resonances

In the case of PEC (or PMC) objects the magnetic (electric) current densities are removed from the system and there remains only one unknown current $\mathbf{J}$ (or $\mathbf{M}$). It is well known that in order to solve the system for all the frequencies, a CFIE type of the surface integral equation formulation is needed because of possible internal resonances.

In article [P5], a new CFIE type formulation, JMCFIE (15), was presented to be used as a general formulation for composite objects having combinations of dielectric and PEC or PMC metallic regions. Two different CFIE type equations are used, one for the electric current density $\mathbf{J}$ and another for the magnetic current density $\mathbf{M}$. In the simple PEC cases, this formulation reduces to the traditional CFIE formulation.

The iterative convergence of the JMCFIE formulation was also studied and the best convergence was achieved by choosing coefficients of the equations so that the diagonal blocks of the system matrix were identical.

5.2 Formulations for Iterative Methods

The convergence of the iteration methods depends heavily on the surface integral equation formulation. The behavior of different formulations was studied in article [P7].

In all the articles of this thesis, the surface integral equations are discretized by the Galerkin method in which the equations are tested with the same functions that are used in expanding the unknown current densities. In the Galerkin method, the basis functions are assumed to span not only the space of the current densities but also the image space of the surface integral equation formulation. This can be achieved by using formulations that map the current densities back to themselves.

For a good convergence, the discretized system matrix should be diagonally dominant. Integral equation formulations of the second kind with identity operators in the diagonals are typically diagonally dominant and if the coefficients of the identity operators are scaled to ones (or 1/2 actually), the formulation is also suitable for the Galerkin method. The most optimal formulation would be a Fredholm integral operator of the second kind with identity and compact operators.

Numerically, the convergence of a typical iterative method depends on the eigenvalue distribution of the discretized system matrix. If the eigenvalues are distributed into finite number of clusters away from the origin, the convergence is faster. In the most optimal case the eigenvalues are on the positive real axis in which case the formulation leads to a positive definite matrix.

In [P7] the rotated tangential (14) and the JMCFIE (15) formulations are found to have the best overall convergences. Both are integral equations of the second kind and the identity operators are properly scaled.

5.3 Low Frequency Problem

The traditional electromagnetic surface integral equation formulations do not work correctly when the frequency is low. This can be seen from the surface integral representations of the fields (9),

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^p - \frac{1}{i\omega\varepsilon}\left(\mathcal{N} - \frac{1}{2}\mathbf{n}\right)(\nabla\cdot\mathbf{J}) \quad + i\omega\mu\,\mathcal{S}(\mathbf{J}) \;+ \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{M}), \\ \mathbf{H} &= \mathbf{H}^p + \frac{1}{i\omega\mu}\left(\mathcal{N} - \frac{1}{2}\mathbf{n}\right)(\nabla\cdot\mathbf{M}) - i\omega\varepsilon\,\mathcal{S}(\mathbf{M}) + \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{J}). \end{aligned} \tag{18}$$

When the frequency gets small ($\omega \to 0$), the divergences of the currents $\nabla\cdot\mathbf{J}$ and $\nabla\cdot\mathbf{M}$ become dominant in the equations because $1/\omega \to \infty$ and the contributions of the current densities $\mathbf{J}$ and $\mathbf{M}$ disappear. Therefore, at low frequencies in the general case $\mathbf{J}$ and $\mathbf{M}$ cannot be directly reconstructed from the integral equations derived from these integral representations.

Although the low frequency problem seems not to be a major problem if the modeled object is moderate large compared to the wave length, the low frequency problem can, however, occur locally, if the object for example has small details that must be modeled with small elements.

Another important problem is that the usual numerical procedure for modeling the objects more accurately by making the discretization of the surface denser, leads to a system with a higher condition number because of the low frequency problem with the smaller elements.

There are a couple of formulations that do not suffer from the low frequency problem. In the case of PEC objects, the rotated tangential component formulation (14) reduces to the n $\times$ MFIE formulation which does not include the hypersingular $\mathcal{N}$ operators and thus the behavior at low frequencies is better. Another well-conditioned formulation is the Müller type formulation for simple dielectric objects.

5.3.1 Müller Formulations

In article [P6] a surface integral equation formulation of the Müller type [12] is presented for homogeneous dielectric objects. The low frequency problem can be removed by properly choosing the coefficients so that the frequency dependence of the $\mathcal{N}$ operator is changed in (18).

In the case of a dielectric object, the combined system matrix of the rotated tangential formulations (14) is of the form

$$\begin{pmatrix} (\varepsilon_1+\varepsilon_2)\dfrac{\mathcal{I}}{2} - \mathbf{n}\times(\varepsilon_1\mathcal{K}_1 - \varepsilon_2\mathcal{K}_2) & \dfrac{1}{i\omega}\,\mathbf{n}\times(\mathcal{D}_1-\mathcal{D}_2) \\ -\dfrac{1}{i\omega}\,\mathbf{n}\times(\mathcal{D}_1-\mathcal{D}_2) & (\mu_1+\mu_2)\dfrac{\mathcal{I}}{2} - \mathbf{n}\times(\mu_1\mathcal{K}_1 - \mu_2\mathcal{K}_2) \end{pmatrix}, \tag{19}$$

where the electric equations are multiplied with ε_j, the magnetic equations with μ_j and the inner normals are combined ($\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$).

The off-diagonal terms containing the difference of the $\mathcal{D}$ operators are

$$\frac{1}{i\omega}\left(\mathcal{D}_1 - \mathcal{D}_2\right) = \frac{1}{i\omega}\nabla\nabla\cdot\left(\mathcal{S}_1 - \mathcal{S}_2\right) + \frac{k_1^2 - k_2^2}{i\omega}\left(\mathcal{S}_1 - \mathcal{S}_2\right),$$

where the first difference of the $\mathcal{S}$ operators is

$$\nabla\nabla\cdot\left(\mathcal{S}_1 - \mathcal{S}_2\right)(\mathbf{J}) = \int \left(\nabla G_1 - \nabla G_2\right)(\mathbf{r},\mathbf{r}')\,\nabla^s_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}')\,ds(\mathbf{r}').$$

The difference of the gradients of the two Green's functions can be expressed with the series expansion (16) as

$$\nabla G_1 - \nabla G_2 = \frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{j-1}{j!}R^{j-3}\left((ik_1)^j - (ik_2)^j\right)(\mathbf{r}-\mathbf{r}'),$$

where the two first terms vanish and, therefore, the frequency behavior of the off-diagonal terms actually is

$$\frac{1}{i\omega}\left(\mathcal{D}_1 - \mathcal{D}_2\right) \sim O(\omega),$$

and not $\sim O(1/\omega)$.

Because of this cancellation of the most singular terms of the hypersingular operator $\mathcal{D}$, the divergences are not anymore dominant at low frequencies and therefore the Müller type formulations are not infected by the low frequency problem.

In article [P7] the Müller formulation is further improved by multiplying the electric equations with $\varepsilon_j/(\varepsilon_1 + \varepsilon_2)$ and the magnetic equations with $\mu_j/(\mu_1 + \mu_2)$. This way all the coefficients of the identity operators in the diagonals of (19) are equal to one and the convergence of the iterative methods is better.

However, in dielectric junctions of more than two regions, the cancellation of the singular terms is not complete and, therefore, the Müller type formulation works at low frequencies only in the simple cases of two dielectric regions.

6 Current and Charge Formulations

In article [P8] a new extended electromagnetic surface integral equation formulation is presented. The new formulation, Current and Charge Integral Equation (CCIE) formulation is not infected by the low frequency problem of the traditional formulations.

The low frequency problem can be prevented by using the surface integral representations (10) in which the divergences of the currents are replaced by the normal components of the fields and the frequency dependence of the $\mathcal{N}$ operator is changed,

$$\begin{cases} \mathbf{E} = \mathbf{E}^p - \left(\mathcal{N} - \frac{1}{2}\mathbf{n}\right)(E_n) + i\omega\mu\,\mathcal{S}(\mathbf{J}) + \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{M}), \\ \mathbf{H} = \mathbf{H}^p - \left(\mathcal{N} - \frac{1}{2}\mathbf{n}\right)(H_n) - i\omega\varepsilon\,\mathcal{S}(\mathbf{M}) + \left(\mathcal{K} - \frac{1}{2}\mathbf{n}\times\right)(\mathbf{J}). \end{cases} \tag{20}$$

The main idea is to use these normal field components as independent unknowns in addition to the traditional (rotated) tangential components. The integral representations (20) have altogether four different residual terms of the singular principal value integrals and these can be used to build an integral equation of the second kind with four unknowns and four identity operators.

6.1 Extended System of Currents and Charges

Because the continuity relations of the normal components of the fields (5) are related to the flux densities $\mathbf{D}$ and $\mathbf{B}$, the new unknowns are chosen to be the normal components of these flux densities, i.e. the electric (d) and magnetic (b) surface charge densities

$$\begin{aligned} d = \mathbf{n}\cdot\mathbf{D} = \varepsilon\,\mathbf{n}\cdot\mathbf{E} &= \frac{1}{i\omega}\nabla\cdot\mathbf{J}, \\ b = \mathbf{n}\cdot\mathbf{B} = \mu\,\mathbf{n}\cdot\mathbf{H} &= -\frac{1}{i\omega}\nabla\cdot\mathbf{M}. \end{aligned} \tag{21}$$

Two additional equations can be obtained from the normal components of (20) and together with the rotated tangential component equations, an integral equation formulation of the second kind can be constructed as

$$\begin{pmatrix} \varepsilon\,\mathbf{n}\cdot\mathbf{E}^p \\ \mathbf{n}\times\mathbf{E}^p \\ \mathbf{n}\times\mathbf{H}^p \\ \mu\,\mathbf{n}\cdot\mathbf{H}^p \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{I}}{2}+\mathbf{n}\cdot\mathcal{N} & -\varepsilon\,\mathbf{n}\cdot\mathcal{K} & -i\omega\varepsilon\mu\,\mathbf{n}\cdot\mathcal{S} & 0 \\ \frac{1}{\varepsilon}\,\mathbf{n}\times\mathcal{N} & \frac{\mathcal{I}}{2}-\mathbf{n}\times\mathcal{K} & -i\omega\mu\,\mathbf{n}\times\mathcal{S} & 0 \\ 0 & i\omega\varepsilon\,\mathbf{n}\times\mathcal{S} & \frac{\mathcal{I}}{2}-\mathbf{n}\times\mathcal{K} & \frac{1}{\mu}\,\mathbf{n}\times\mathcal{N} \\ 0 & i\omega\varepsilon\mu\,\mathbf{n}\cdot\mathcal{S} & -\mu\,\mathbf{n}\cdot\mathcal{K} & \frac{\mathcal{I}}{2}+\mathbf{n}\cdot\mathcal{N} \end{pmatrix} \begin{pmatrix} d \\ \mathbf{M} \\ \mathbf{J} \\ b \end{pmatrix}. \tag{22}$$

6.2 Normalized Extended System

The new extended system of currents and charges (22) is quite ill-balanced. Different source terms and off-diagonal terms have coefficients of very different scales and quantities. A better balanced system can be constructed from the normalized surface integral

representations (12) and by using the normalized charge densities

$$\tilde{d} = \mathbf{n} \cdot \tilde{\mathbf{E}} = \frac{d}{\sqrt{\varepsilon}} \quad \text{and} \quad \tilde{b} = \mathbf{n} \cdot \tilde{\mathbf{H}} = \frac{b}{\sqrt{\mu}}.$$

With the normalized unknowns and sources, the new formulation is

$$\begin{pmatrix} \mathbf{n} \cdot \tilde{\mathbf{E}}^p \\ \mathbf{n} \times \tilde{\mathbf{E}}^p \\ \mathbf{n} \times \tilde{\mathbf{H}}^p \\ \mathbf{n} \cdot \tilde{\mathbf{H}}^p \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{I}}{2} + \mathbf{n} \cdot \mathcal{N} & -\mathbf{n} \cdot \mathcal{K} & -ik\, \mathbf{n} \cdot \mathcal{S} & 0 \\ \mathbf{n} \times \mathcal{N} & \frac{\mathcal{I}}{2} - \mathbf{n} \times \mathcal{K} & -ik\, \mathbf{n} \times \mathcal{S} & 0 \\ 0 & ik\, \mathbf{n} \times \mathcal{S} & \frac{\mathcal{I}}{2} - \mathbf{n} \times \mathcal{K} & \mathbf{n} \times \mathcal{N} \\ 0 & ik\, \mathbf{n} \cdot \mathcal{S} & -\mathbf{n} \cdot \mathcal{K} & \frac{\mathcal{I}}{2} + \mathbf{n} \cdot \mathcal{N} \end{pmatrix} \begin{pmatrix} \tilde{d} \\ \tilde{\mathbf{M}} \\ \tilde{\mathbf{J}} \\ \tilde{b} \end{pmatrix}. \tag{23}$$

From this form, it is easy to see that at zero frequency ($k \to 0$) the electric and magnetic fields are decoupled as they should be.

6.3 Scaled Boundary Conditions

Since none of the components of the normalized fields are directly continuous across the region interfaces (6), enforcing of the boundary conditions is more complicated with the normalized equations.

To enforce the continuity and at the same time maintain balance of the system, the normalized unknowns are further scaled by dividing them with carefully chosen relative coefficients $s_j^U = \pm c_j^U / C_U$,

$$c_j^d = \frac{1}{\sqrt{\varepsilon_j}}, \quad c_j^M = \sqrt{\varepsilon_j}, \quad c_j^J = \sqrt{\mu_j}, \quad c_j^b = \frac{1}{\sqrt{\mu_j}}, \quad \text{and} \quad C_U = \sqrt{\sum_{j=1}^{m} \left(c_j^U\right)^2},$$

where m is the number of the regions sharing the same unknown. If there is only one region, the relative constant is one.

This means that instead of the original normalized continuity relations (6), scaled boundary conditions are applied,

$$\begin{aligned} C_d\sqrt{\varepsilon_1}\tilde{E}_n^{(1)} &= C_d D_n^{(1)} &&= -C_d D_n^{(2)} = -C_d\sqrt{\varepsilon_2}\tilde{E}_n^{(2)}, \\ \frac{C_M}{\sqrt{\varepsilon_1}}\,\tilde{\mathbf{E}}_{tan}^{(1)} &= C_M \mathbf{E}_{tan}^{(1)} = C_M \mathbf{E}_{tan}^{(2)} &&= \frac{C_M}{\sqrt{\varepsilon_2}}\,\tilde{\mathbf{E}}_{tan}^{(2)}, \\ \frac{C_J}{\sqrt{\mu_1}}\,\tilde{\mathbf{H}}_{tan}^{(1)} &= C_J \mathbf{H}_{tan}^{(1)} = C_J \mathbf{H}_{tan}^{(2)} &&= \frac{C_J}{\sqrt{\mu_2}}\,\tilde{\mathbf{H}}_{tan}^{(2)}, \\ C_b\sqrt{\mu_1}\tilde{H}_n^{(1)} &= C_b B_n^{(1)} &&= -C_b B_n^{(2)} = -C_b\sqrt{\mu_2}\tilde{H}_n^{(2)}, \end{aligned} \tag{24}$$

where the constant scaling coefficients C_d, C_M, C_J and C_b may vary for each edge (currents) or element (charges) of the system as long as all the currents and charges sharing the same unknown have the same coefficients.

The sign of the coefficient is positive for the oriented basis functions of the currents but for the charges the sign needs to be opposite on the opposite sides of the interface and can be fixed, for example, to be positive on the region of the lower index.

For symmetry, also the source terms and the equations are scaled and the system is

$$\begin{pmatrix} s^d\,\mathbf{n}\cdot\tilde{\mathbf{E}}^p \\ s^M\,\mathbf{n}\times\tilde{\mathbf{E}}^p \\ s^J\,\mathbf{n}\times\tilde{\mathbf{H}}^p \\ s^b\,\mathbf{n}\cdot\tilde{\mathbf{H}}^p \end{pmatrix} = \mathbf{S}\,\mathbf{Z}\,\mathbf{S} \begin{pmatrix} \tilde{d}/s^d \\ \tilde{\mathbf{M}}/s^M \\ \tilde{\mathbf{J}}/s^J \\ \tilde{b}/s^b \end{pmatrix}, \tag{25}$$

where $\mathbf{Z}$ is the same matrix as in (23) and $\mathbf{S}$ is a diagonal matrix containing $[s^d, s^M, s^J, s^b]$.

The normalized and scaled unknowns are continuous across the interfaces and the scaling coefficients are chosen so that the identity operators in the diagonal are without extra coefficients in the combined system matrix, since

$$\sum_{j=1}^{m} \left(s_j^U\right)^2 \frac{\mathcal{I}}{2} \;=\; \frac{\mathcal{I}}{2}\frac{1}{(C_U)^2}\sum_{j=1}^{m}\frac{1}{\left(c_j^U\right)^2} \;=\; \frac{\mathcal{I}}{2}\frac{\sum_{j=1}^{m}\frac{1}{\left(c_j^U\right)^2}}{\sum_{j=1}^{m}\frac{1}{\left(c_j^U\right)^2}} \;=\; \frac{\mathcal{I}}{2}. \tag{26}$$

6.4 Removing Linear Dependences

The new normalized and scaled formulation (25) has two linear dependence problems. The currents and charges are mutually linearly dependent (21) and the charges are internally linearly depended.

In both cases additional restrictions are needed in order to be able to solve the system numerically in all cases. Usually adding new equations makes the system to have more equations than unknowns and thus it is overdetermined and the system has to be converted into a square system somehow.

In both cases of the linear dependence of the new formulation the overdetermined system is squared by picking up adequate number of linear combinations of the equations so that every equation is used at least once.

6.4.1 Linear Dependence between Currents and Charges

The surface currents and charges are linearly dependent by (21). The linear dependence can be removed with additional equations formed from normalized equations

$$\begin{aligned} ik\,\tilde{d} &= \nabla\cdot\tilde{\mathbf{J}} \\ -ik\,\tilde{b} &= \nabla\cdot\tilde{\mathbf{M}} \end{aligned}, \tag{27}$$

by applying integral operator $\mathcal{S}$,

$$\begin{aligned} ik\,\mathcal{S}(\tilde{d}) &= \mathcal{S}(\nabla\cdot\tilde{\mathbf{J}}) \;\; = \nabla\cdot\mathcal{S}(\tilde{\mathbf{J}}) \\ -ik\,\mathcal{S}(\tilde{b}) &= \mathcal{S}(\nabla\cdot\tilde{\mathbf{M}}) = \nabla\cdot\mathcal{S}(\tilde{\mathbf{M}}) \end{aligned}. \tag{28}$$

Because of the good balance in the system (23), these new equations can be added directly into the first and last equations thus creating a square system with linear combination of the equations of the overdetermined system,

$$\begin{pmatrix} s^d\,\tilde{\mathbf{E}}_n^p \\ s^M\,\tilde{\mathbf{E}}_r^p \\ s^J\,\tilde{\mathbf{H}}_r^p \\ s^b\,\tilde{\mathbf{H}}_n^p \end{pmatrix} = \mathbf{S} \begin{pmatrix} \frac{\mathcal{I}}{2}-ik\,\mathcal{S}+\mathcal{N}_n & -\mathcal{K}_n & -ik\,\mathcal{S}_n+\nabla\cdot\mathcal{S} & 0 \\ \mathcal{N}_r & \frac{\mathcal{I}}{2}-\mathcal{K}_r & -ik\,\mathcal{S}_r & 0 \\ 0 & ik\,\mathcal{S}_r & \frac{\mathcal{I}}{2}-\mathcal{K}_r & \mathcal{N}_r \\ 0 & ik\,\mathcal{S}_n-\nabla\cdot\mathcal{S} & -\mathcal{K}_n & \frac{\mathcal{I}}{2}-ik\,\mathcal{S}+\mathcal{N}_n \end{pmatrix} \mathbf{S} \begin{pmatrix} \tilde{d}/s^d \\ \tilde{\mathbf{M}}/s^M \\ \tilde{\mathbf{J}}/s^J \\ \tilde{b}/s^b \end{pmatrix},$$

where $\mathbf{F}_n = \mathbf{n}\cdot\mathbf{F}$ and $\mathbf{F}_r = \mathbf{n}\times\mathbf{F}$.

It is, however, not so clear that the equality of (27) $\Rightarrow$ (28) holds in the opposite direction. This and also the choice of the sign of the diagonal $ik\,\mathcal{S}$ terms is motivated by the Picard's extended system presented in section 7.

6.4.2 Linear Dependence in Charges

Since $\nabla\cdot\mathbf{E} = 0$ ($\mathbf{H}$ and $\tilde{b}$ similarly), integrating the normalized terms over the volume of a closed region gives

$$0 = \int \nabla\cdot\tilde{\mathbf{E}}\,dv = \int \mathbf{n}\cdot\tilde{\mathbf{E}}\,ds = \int \tilde{d}\,ds = \sum_j d_j A_j = \mathbf{A}^T\mathbf{d},$$

where the charge density $\tilde{d}$ is assumed to be discretized with piecewise constant functions so that $\mathbf{d} = [d_1,\ldots,d_N]^T$ are the unknown coefficients and $\mathbf{A} = [A_1, A_2, \ldots, A_N]^T$ are the element areas.

One of the coefficients can be expressed as weighted sum of the others,

$$0 = \sum_j d_j A_j \quad\Rightarrow\quad d_N = -\frac{1}{A_N}\sum_{j=1}^{N-1} d_j A_j,$$

so the unknowns are linearly dependent and this must be considered as an additional restriction (or equation) the system should satisfy. Otherwise, the system is ill-posed and one of the eigenvalues can be zero or very small.

In order to get a square system, the additional equation $\mathbf{A}^T\mathbf{d} = 0$ can be combined with every original row of the $\tilde{d}$ equations in a symmetric form $c\mathbf{A}\mathbf{A}^T$, where row j is multiplied with A_j and c is a constant coefficient. A properly scaled coefficient c is calculated with the deflation method, [24], [25], and [26].

The deflation method can be used to move the small eigenvalue caused by the linear dependency. For the iterative methods, the best place for any eigenvalue would be among the other eigenvalues. The mean of the eigenvalues can be calculated using the diagonal

terms of the matrix. Namely, the trace of a matrix, the sum of the diagonal elements, is equal to the sum of the eigenvalues of the matrix.

The system matrix is modified for every closed surface by adding equations of normalized area vectors $\mathbf{a} = \mathbf{A}/|\mathbf{A}|$ as

$$\mathbf{Z} + (\lambda_{new} - \lambda_{old})\, \mathbf{a}\, \mathbf{a}^T. \tag{29}$$

where $\lambda_{old} = \mathbf{a}^T \mathbf{Z} \mathbf{a}$ and $\lambda_{new} = trace(\mathbf{Z})/size(\mathbf{Z})$.

6.5 Current and Charge Integral Equation Formulations

The new Current and Charge Integral Equation (CCIE) formulation is now ready and reads as

$$\begin{pmatrix} s^d\, \tilde{\mathbf{E}}_n^p \\ s^M\, \tilde{\mathbf{E}}_r^p \\ s^J\, \tilde{\mathbf{H}}_r^p \\ s^b\, \tilde{\mathbf{H}}_n^p \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathcal{M} + \mathcal{D}^d & -\mathcal{K}_n & -ik\,\mathcal{S}_n + \nabla\cdot\mathcal{S} & 0 \\ \mathcal{N}_r & \frac{\mathcal{I}}{2} - \mathcal{K}_r & -ik\,\mathcal{S}_r & 0 \\ 0 & ik\,\mathcal{S}_r & \frac{\mathcal{I}}{2} - \mathcal{K}_r & \mathcal{N}_r \\ 0 & ik\,\mathcal{S}_n - \nabla\cdot\mathcal{S} & -\mathcal{K}_n & \mathcal{M} + \mathcal{D}^b \end{pmatrix} \mathbf{S} \begin{pmatrix} \tilde{d}/s^d \\ \tilde{\mathbf{M}}/s^M \\ \tilde{\mathbf{J}}/s^J \\ \tilde{b}/s^b \end{pmatrix} \tag{30}$$

and it's CFIE (15) form, Combined Current and Charge Integral Equation (CCCIE) formulation is

$$\begin{pmatrix} s^d\, \tilde{\mathbf{E}}_n^p \\ s^M \left(\tilde{\mathbf{E}}_r^p - \tilde{\mathbf{H}}_{tan}^p\right) \\ s^J \left(\tilde{\mathbf{H}}_r^p + \tilde{\mathbf{E}}_{tan}^p\right) \\ s^b\, \tilde{\mathbf{H}}_n^p \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathcal{M} + \mathcal{D}^d & -\mathcal{K}_n & -ik\,\mathcal{S}_n + \nabla\cdot\mathcal{S} & 0 \\ \mathcal{N}_r & \mathcal{L} & \mathcal{L}_r & -\mathcal{N}_{tan} \\ \mathcal{N}_{tan} & -\mathcal{L}_r & \mathcal{L} & \mathcal{N}_r \\ 0 & ik\,\mathcal{S}_n - \nabla\cdot\mathcal{S} & -\mathcal{K}_n & \mathcal{M} + \mathcal{D}^b \end{pmatrix} \mathbf{S} \begin{pmatrix} \tilde{d}/s^d \\ \tilde{\mathbf{M}}/s^M \\ \tilde{\mathbf{J}}/s^J \\ \tilde{b}/s^b \end{pmatrix}, \tag{31}$$

where

$$\mathcal{M} = \frac{\mathcal{I}}{2} - ik\,\mathcal{S} + \mathbf{n}\cdot\mathcal{N}, \tag{32}$$

$$\mathcal{L} = \frac{\mathcal{I}}{2} - ik\,\mathcal{S}_{tan} - \mathbf{n}\times\mathcal{K}, \tag{33}$$

and $\mathcal{D}^d$ and $\mathcal{D}^b$ contain the deflation terms.

The CCCIE formulation can be used as a general formulation for composite objects having PEC or PMC regions and problems with the internal resonances.

Both of the formulations have identity operators in the diagonal and therefore they are integral equations of the second kind. The CCCIE formulation has $ik\,\mathcal{S}$ terms in all of the diagonals and so for higher frequencies it may be more diagonally dominant than the CCIE formulation.

7 Picard's Extended Maxwell System

The low frequency problem is more deeply hidden in the Maxwell's system. The traditional electromagnetic surface integral equations are usually derived from the curl equations of the Maxwell's system. If considering only the curl equations of the normalized Maxwell's system (3),

$$(\mathbb{M} + ik)\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix} = 0, \qquad \mathbb{M} = \begin{pmatrix}0 & \nabla\times\\ -\nabla\times & 0\end{pmatrix}, \tag{34}$$

the divergence equations are still build into the system when the frequency is not zero. If the frequency is zero, the divergence restrictions are lost and the static behavior is not correct. Also the null space of the $\mathbb{M}$ operator, i.e. the space of the functions mapped to zero is always infinite dimensional.

In [19] and [20] (see also [27], [28] and [29]) (Rainer) Picard presented an extended system

$$(\mathbb{P}(\nabla) + ik)\begin{pmatrix}\Psi\\ \tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\\ \Phi\end{pmatrix} = 0, \qquad \mathbb{P}(\nabla) = \begin{pmatrix}0 & 0 & \nabla\cdot & 0\\ 0 & 0 & \nabla\times & \nabla\\ \nabla & -\nabla\times & 0 & 0\\ 0 & \nabla\cdot & 0 & 0\end{pmatrix}, \tag{35}$$

having two additional scalar unknowns, Ψ and Φ. The solutions of the Maxwell's system (34) and this Picard's extended Maxwell system (35) (or Picard's system) are the same if these scalar functions are assumed to be identically zeros,

$$\Psi \equiv 0 \equiv \Phi.$$

The similar system in [28] is a modified version of (35) so that these scalar functions have formal definitions. Here these functions are neglected.

All the four original Maxwell's equations (3) are included in the system (35) and so the static behavior is correct. The system also contains additional operators combining the linear acoustics and Maxwell's equations into a unified system [27].

The linear acoustics and Maxwell's equations are mutually orthogonal in a way that the Picard's system (35) is elliptic and the dimension of the null space of the $\mathbb{P}(\nabla)$ operator is finite and does not depend on the material parameters [19].

Furthermore,

$$\begin{aligned}\mathbb{P}(\nabla)^T &= \mathbb{P}(\nabla),\\ \mathbb{P}(\nabla)^2 &= \Delta,\\ (\mathbb{P}(\nabla) + ik)(\mathbb{P}(\nabla) - ik) &= \Delta + k^2.\end{aligned}$$

7.1 Surface Integral Representation of the Picard's System

On the surface of a region, the surface integral representation of the Picard's system (35) is (see [29])

$$\mathbf{U} \;=\; \mathbf{U}^p + \left(\frac{\mathcal{I}}{2} - \left(\mathbb{P}(\nabla) - ik\right) \mathcal{S}\, \mathbb{P}(\mathbf{n}) \right) \mathbf{U}, \tag{36}$$

where

$$\mathbf{U} \;=\; \begin{pmatrix} 0 \\ \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbb{P}(\mathbf{n}) \;=\; \begin{pmatrix} 0 & 0 & \mathbf{n}\cdot & 0 \\ 0 & 0 & \mathbf{n}\times & \mathbf{n} \\ \mathbf{n} & -\mathbf{n}\times & 0 & 0 \\ 0 & \mathbf{n}\cdot & 0 & 0 \end{pmatrix}.$$

This integral representation has only first order derivatives of the single layer integral operator $\mathcal{S}$, whereas the original representations (8) of the Maxwell's system have second order derivatives in the hypersingular $\mathcal{D}$ operator.

The integral representation (36) can be expressed with the normalized currents and charges as

$$\mathbf{U} \;=\; \mathbf{U}^p + \frac{\mathbf{U}}{2} - \left(\mathbb{P}(\nabla) - ik\right) \mathcal{S} \begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix}. \tag{37}$$

7.2 Picard's System and Current and Charge Formulations

As with the surface integral representations of the Maxwell's system, there are infinite number of ways to derive surface integral equations from the surface integral representation of the Picard's system (37). The integral representation can be projected into a surface integral equation of the currents and charges for example by operating with $\mathbb{P}(\mathbf{n})$,

$$\begin{aligned} \mathbb{P}(\mathbf{n})\mathbf{U}^p \;=\; \begin{pmatrix} \mathbf{n}\cdot\tilde{\mathbf{H}}^p \\ \mathbf{n}\times\tilde{\mathbf{H}}^p \\ -\mathbf{n}\times\tilde{\mathbf{E}}^p \\ \mathbf{n}\cdot\tilde{\mathbf{E}}^p \end{pmatrix} \;&=\; \left(\frac{\mathcal{I}}{2} + \mathbb{P}(\mathbf{n}) \left(\mathbb{P}(\nabla) - ik\right) \mathcal{S} \right) \begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix} \\ &=\; \begin{pmatrix} \frac{\mathcal{I}}{2} + \mathcal{N}_n & -\mathcal{K}_n & -ik\,\mathcal{S}_n & 0 \\ \mathcal{N}_r & \frac{\mathcal{I}}{2} + \mathbf{n}\nabla\cdot\mathcal{S} - \mathcal{K}_r & -ik\,\mathcal{S}_r & -ik\,\mathbf{n}\,\mathcal{S} \\ -ik\,\mathbf{n}\,\mathcal{S} & ik\,\mathcal{S}_r & \frac{\mathcal{I}}{2} + \mathbf{n}\nabla\cdot\mathcal{S} - \mathcal{K}_r & -\mathcal{N}_r \\ 0 & -ik\,\mathcal{S}_n & \mathcal{K}_n & \frac{\mathcal{I}}{2} + \mathcal{N}_n \end{pmatrix} \begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix}. \end{aligned}$$

If the signs of the $\tilde{\mathbf{M}}$ rows and columns are changed,

$$\begin{pmatrix} \mathbf{n}\cdot\tilde{\mathbf{H}}^p \\ \mathbf{n}\times\tilde{\mathbf{H}}^p \\ \mathbf{n}\times\tilde{\mathbf{E}}^p \\ \mathbf{n}\cdot\tilde{\mathbf{E}}^p \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{I}}{2}+\mathcal{N}_n & -\mathcal{K}_n & ik\,\mathcal{S}_n & 0 \\ \mathcal{N}_r & \frac{\mathcal{I}}{2}+\mathbf{n}\nabla\cdot\mathcal{S}-\mathcal{K}_r & ik\,\mathcal{S}_r & -ik\,\mathbf{n}\,\mathcal{S} \\ ik\,\mathbf{n}\,\mathcal{S} & -ik\,\mathcal{S}_r & \frac{\mathcal{I}}{2}+\mathbf{n}\nabla\cdot\mathcal{S}-\mathcal{K}_r & \mathcal{N}_r \\ 0 & -ik\,\mathcal{S}_n & -\mathcal{K}_n & \frac{\mathcal{I}}{2}+\mathcal{N}_n \end{pmatrix} \begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ \tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix},$$

it can be seen that this is almost the same as the CCIE formulation (30) with reverse order of the equations and unknowns. The operators of the linear dependence removal between currents and charges are, however, as normal components in the tangential equations.

Instead of $\mathbb{P}(\mathbf{n})$, a proper transformation from the surface integral representation of the Picard's system (37) into the CCIE formulation is obtained by the operator

$$\mathbb{N} = \begin{pmatrix} 1 & 0 & \mathbf{n}\cdot & 0 \\ 0 & 0 & \mathbf{n}\times & 0 \\ 0 & -\mathbf{n}\times & 0 & 0 \\ 0 & \mathbf{n}\cdot & 0 & 1 \end{pmatrix}.$$

In this case,

$$\begin{aligned} \mathbb{N}\mathbf{U}^p = \begin{pmatrix} \mathbf{n}\cdot\tilde{\mathbf{H}}^p \\ \mathbf{n}\times\tilde{\mathbf{H}}^p \\ -\mathbf{n}\times\tilde{\mathbf{E}}^p \\ \mathbf{n}\cdot\tilde{\mathbf{E}}^p \end{pmatrix} &= \left(\frac{\mathcal{I}}{2}+\mathbb{N}\left(\mathbb{P}(\nabla)-ik\right)\mathcal{S}\right)\begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M} & -\mathcal{K}_n & -ik\,\mathcal{S}_n+\nabla\cdot\mathcal{S} & 0 \\ \mathcal{N}_r & \frac{\mathcal{I}}{2}-\mathcal{K}_r & -ik\,\mathcal{S}_r & 0 \\ 0 & ik\,\mathcal{S}_r & \frac{\mathcal{I}}{2}-\mathcal{K}_r & -\mathcal{N}_r \\ 0 & -ik\,\mathcal{S}_n+\nabla\cdot\mathcal{S} & \mathcal{K}_n & \mathcal{M} \end{pmatrix}\begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix}, \end{aligned}$$

where $\mathcal{M}$ is the same operator as in (32). This surface integral equation is the same as the CCIE formulation (30) if the signs of the $\tilde{\mathbf{M}}$ rows and columns are changed. The $\mathbb{N}$ operator is, however, not an actual projection, since $\mathbb{N}^2 \neq \mathbb{N}$.

An actual projection ($\mathbb{C}^2 = \mathbb{C}$) would be

$$\mathbb{C} = \begin{pmatrix} 1 & 0 & \mathbf{n}\cdot & 0 \\ 0 & -\frac{1}{2}\mathbf{n}\times\mathbf{n}\times & \frac{1}{2}\mathbf{n}\times & 0 \\ 0 & -\frac{1}{2}\mathbf{n}\times & -\frac{1}{2}\mathbf{n}\times\mathbf{n}\times & 0 \\ 0 & \mathbf{n}\cdot & 0 & 1 \end{pmatrix}$$

and the surface integral equation obtained from this projection is

$$\mathbb{C}\mathbf{U}^p = \begin{pmatrix} \mathbf{n}\cdot\tilde{\mathbf{H}}^p \\ \frac{1}{2}\left(\tilde{\mathbf{E}}^p_{tan} + \tilde{\mathbf{H}}^p_r\right) \\ \frac{1}{2}\left(\tilde{\mathbf{H}}^p_{tan} - \tilde{\mathbf{E}}^p_r\right) \\ \mathbf{n}\cdot\tilde{\mathbf{E}}^p \end{pmatrix} = \mathbb{C}\left(\frac{\mathcal{I}}{2} + (\mathbb{P}(\nabla) - ik)\,\mathcal{S}\right)\begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{M} & -\mathcal{K}_n & -ik\,\mathcal{S}_n + \nabla\cdot\mathcal{S} & 0 \\ \frac{1}{2}\mathcal{N}_r & \frac{1}{2}\mathcal{L} & \frac{1}{2}\mathcal{L}_r & \frac{1}{2}\mathcal{N}_{tan} \\ \frac{1}{2}\mathcal{N}_{tan} & -\frac{1}{2}\mathcal{L}_r & \frac{1}{2}\mathcal{L} & -\frac{1}{2}\mathcal{N}_r \\ 0 & -ik\,\mathcal{S}_n + \nabla\cdot\mathcal{S} & \mathcal{K}_n & \mathcal{M} \end{pmatrix}\begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ -\tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix},$$

where $\mathcal{M}$ and $\mathcal{L}$ are the same operators as in (32) and (33).

If the signs of the $\tilde{\mathbf{M}}$ rows and columns above are changed and the tangential equations are multiplied with 2,

$$\begin{pmatrix} \mathbf{n}\cdot\tilde{\mathbf{H}}^p \\ \tilde{\mathbf{H}}^p_r + \tilde{\mathbf{E}}^p_{tan} \\ \tilde{\mathbf{E}}^p_r - \tilde{\mathbf{H}}^p_{tan} \\ \mathbf{n}\cdot\tilde{\mathbf{E}}^p \end{pmatrix} = \begin{pmatrix} \mathcal{M} & -\mathcal{K}_n & ik\,\mathcal{S}_n - \nabla\cdot\mathcal{S} & 0 \\ \mathcal{N}_r & \mathcal{L} & -\mathcal{L}_r & \mathcal{N}_{tan} \\ -\mathcal{N}_{tan} & \mathcal{L}_r & \mathcal{L} & \mathcal{N}_r \\ 0 & -ik\,\mathcal{S}_n + \nabla\cdot\mathcal{S} & -\mathcal{K}_n & \mathcal{M} \end{pmatrix}\begin{pmatrix} \tilde{b} \\ \tilde{\mathbf{J}} \\ \tilde{\mathbf{M}} \\ \tilde{d} \end{pmatrix},$$

and this is the same as the CCCIE formulation (31) with reverse order of the equations and unknowns.

So, although developed independently, the current and charge formulations are closely related to the Picard's extended Maxwell system (35). The Picard's system transforms naturally into surface integral equations of currents and charges where the linear dependence between them is taken into account with integral operators.

8 Some Numerical Examples of Large Systems

The new surface integral equation formulations CCIE (30) and CCCIE (31) have been implemented with Galerkin testing procedure for triangular discretization. The normalized and scaled surface current densities $\tilde{\mathbf{J}}/s^J$ and $\tilde{\mathbf{M}}/s^M$ are expanded with RWG basis function and the normalized and scaled surface charge densities $\tilde{d}/s^d$ and $\tilde{b}/s^b$ with piecewise constant functions.

In all the examples below CCCIE formulation is used, a simple diagonal preconditioning is applied and the equations are solved iteratively with residual tolerance 10^{-2} using the Generalized Minimum Residual Method (GMRES).

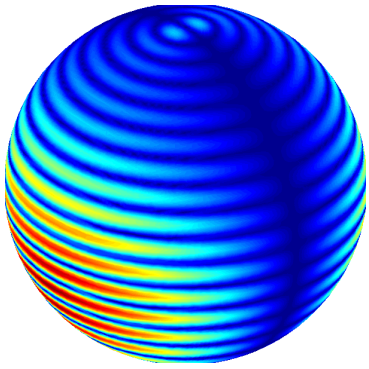
The matrix-vector product of the iteration is calculated the "slow way" by integrating over all the element pairs without acceleration methods. The calculation is divided row-wise (testing) and computed in parallel with 25-40 computers.

8.1 Spheres

In article [P8] three simple sphere models are chosen to verify and compare the new formulation: (PEC) metallic, dielectric and hybrid metallic-dielectric. The radius of the sphere is r = 4.7713 cm so at 1 GHz the wave length of the outer region is the same as the circumference of the sphere (or $k\,r = 1$). The spheres are illuminated by a x-polarized plane wave propagating in the z-direction.

The spheres are discretized with about 100 000 unknowns and the convergence rate of the iteration is studied for some selected frequencies:

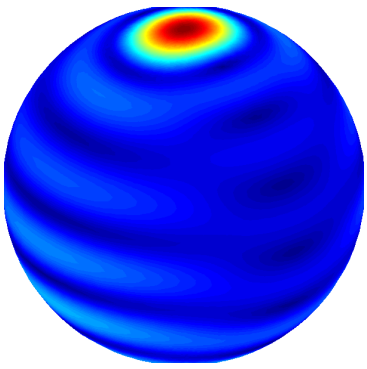
Metallic



Triangles: 40 500
Unknowns: 101 250
Matrix: 152 GB

Freq.	1 Hz	1 GHz	20 GHz
Iter.	5	7	13

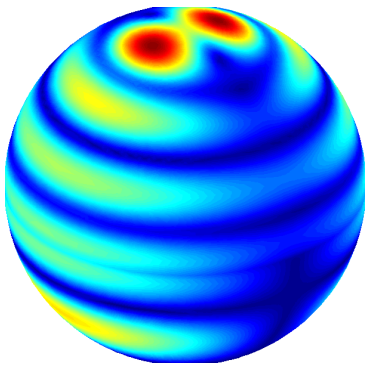
Dielectric



20 480
102 400
156 GB

1 Hz	1 GHz	7 GHz
5	9	39

Hybrid

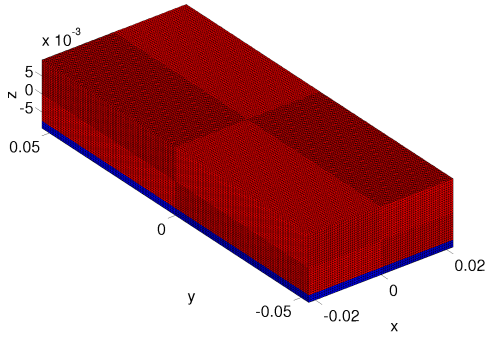


31 264
110 380
182 GB

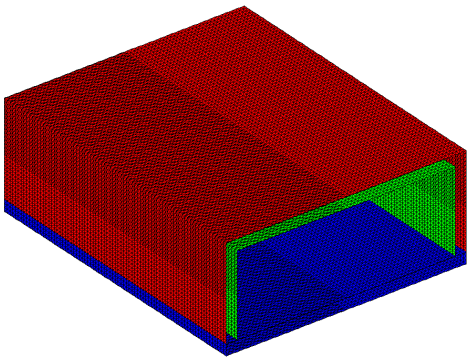
1 Hz	1 GHz	6 GHz
9	13	37

8.2 Plastic Cover with Metallic Back Plate

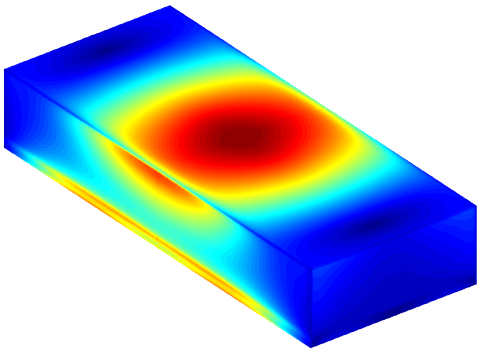
As a more complex example, a hollow plastic cover on top of a PEC back plate was modeled. The size of the box is 106 × 44 × 19 mm. The cover and the metallic back plate are 2 mm thick and the relative permittivity of the cover is 2. The frequency is 4 GHz and the source is a x-polarized electric dipole inside the cover.



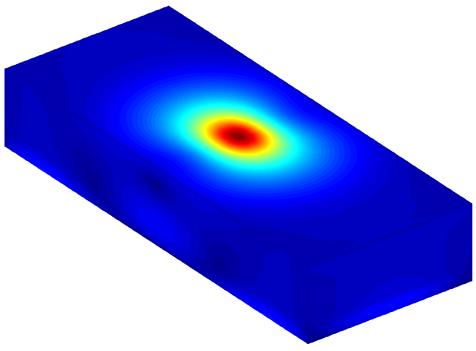
real(E)



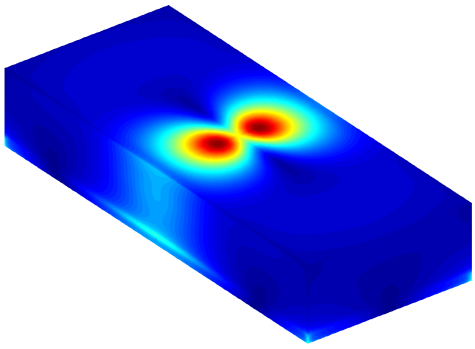
real(H)

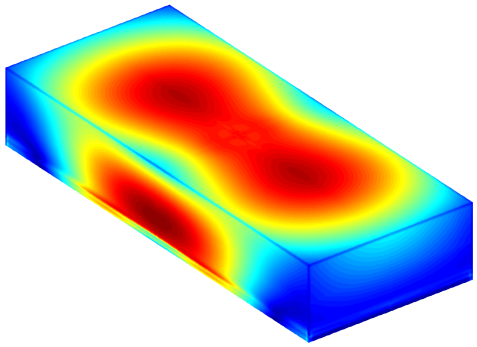


imag(E)



imag(H)





Convergence of the iteration is studied with three different discretization of about 60 000, 200 000 and 500 000 unknowns:

Triangles	14 552	57 232	129 504
Unknowns	59 824	235 936	534 260
Matrix	53.3 GB	829 GB	4253 GB
Iterations	21	21	21

As it can be seen, increasing the number of unknowns almost tenfold does not increase the number of iterations.

9 Conclusions

In this thesis problems with implementation and formulation of the electromagnetic surface integral equations were studied. The research can be divided into three main categories.

First, a technique for evaluation of the singular integrals arising from the electromagnetic surface integral equations was developed. The singular integrals of various degrees can be evaluated by recursively computed analytical integrals. The technique was further extended to the higher order basis functions.

The second problem, the modeling of the electromagnetic fields and enforcing the boundary condition in complex structures, was solved by properly oriented basis functions. The developed procedure can model arbitrary complex structures, is simple to implement and does not depend on the formulation.

The third and most important problem is the choice of the electromagnetic surface integral equation formulation. A new CFIE type formulation was developed as a general formulation for composite objects of dielectric and metallic regions. In the case of simple dielectric regions, a well-conditioned Müller formulation was developed that doesn't have problems with the low frequencies. The behavior of different formulations with iterative methods was studied. It was found that the second kind of integral equation formulations with properly scaled identity operators give the best overall convergences.

The most important part of this thesis is the new surface integral equation formulation (CCIE) and the new techniques developed for it. The new formulation does not have problems with the low frequencies and the combined field (CFIE) form of it (CCCIE) is also immune to the internal resonances of the metallic regions and can be used as a general surface integral equation formulation for composite objects. The new formulations have low condition numbers and the convergence of the iterative methods is fast for a very wide frequency range.

Both the surface current and charge densities are unknowns in the new formulations. The balance of the system is achieved using normalized field quantities, and the continuity of the fields across the domain interfaces is handled with carefully chosen scaling factors so that both the continuity and the balance are maintained, and at the same time the identity operators are properly scaled.

Linear dependence between the currents and charges is taken into account with an integral operator, and the linear dependence in charges is removed with the deflation method. The deflation method was found to be very stable and efficient and it can be used for similar linear dependency problems.

The new formulation was found to be related to the Picard's extended Maxwell system. This elliptic, first order system has a correct static behavior which is not the case with the Maxwell's system used with the traditional electromagnetic surface integral equations.

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