Scheduling Models with Additional Features

Synchronization, Pliability and Resiliency

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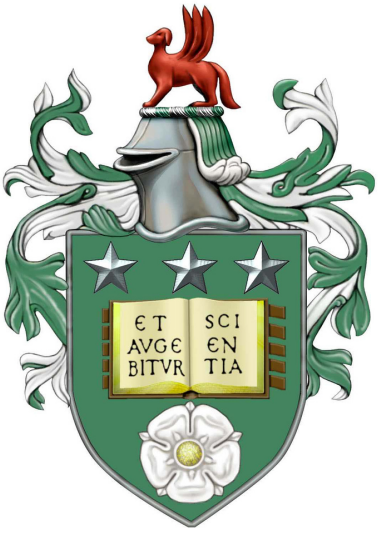

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The candidate confirms that the work submitted is his own, except where work which has formed part of a jointly authored publication has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Some parts of the work presented in Chapters 4, 5 and 6 have been published in the following articles:

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For both papers, all four authors worked on text, formulation, proof reading and literature search. In the first paper, scientific results (other than in Section 4) are mostly due to C. Weiß, though the other authors had large part in finding good presentation methods for this work. Results in Section 4 are due to S. Waldherr, N. Shakhlevich and C. Weiß.
In the second paper, results in Sections 2 and 3 are mostly due to C. Weiß. The result in Section 3 was achieved in collaboration with N. Shakhlevich. The first complete proof for the result in Section 4 was achieved by S. Waldherr and C. Weiß in collaboration. N. Shakhlevich played a large role in finding a good presentation for that work.

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Abstract

In this thesis we study three new extensions of scheduling models with both practical and theoretical relevance, namely synchronization, pliability and resiliency. Synchronization has previously been studied for flow shop scheduling and we now apply the concept to open shop models for the first time. Here, as opposed to the traditional models, operations that are processed together all have to be started at the same time. Operations that are completed are not removed from the machines until the longest operation in their group is finished.

Pliability is a new approach to model flexibility in flow shops and open shops. In scheduling with pliability, parts of the processing load of the jobs can be re-distributed between the machines in order to achieve better schedules. This is applicable, for example, if the machines represent cross-trained workers.

Resiliency is a new measure for the quality of a given solution if the input data are uncertain. A resilient solution remains better than some given bound, even if the original input data are changed. The more we can perturb the input data without the solution losing too much quality, the more resilient the solution is.

We also consider the assignment problem, as it is the traditional combinatorial optimization problem underlying many scheduling problems. Particularly, we study a version of the assignment problem with a special cost structure derived from the synchronous open shop model and obtain new structural and complexity results. Furthermore we study resiliency for the assignment problem.

The main focus of this thesis is the study of structural properties, algorithm development and complexity. For synchronous open shop we show that for a fixed number of machines the makespan can be minimized in polynomial time. All other traditional scheduling objectives are at least as hard to optimize as in the traditional open shop model.

Starting out research in pliability we focus on the most general case of the model as well as two relevant special cases. We deliver a fairly complete complexity study for all three versions of the model.

Finally, for resiliency, we investigate two different questions: 'how to compute the resiliency of a given solution?' and 'how to find a most resilient solution?'. We focus on the assignment problem and single machine scheduling to minimize the total sum of completion times and present a number of positive results for both questions. The main goal is to make a case that the concept deserves further study.

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Chapter 1

Introduction

The area of scheduling has received continuous interest from researchers for more than sixty years. Originating from industrial production and machine scheduling, today scheduling research has many different applications in public and commercial service provision. A famous description of the area as a whole is due to Pinedo [121], who stated that "Scheduling deals with the allocation of scarce resources to tasks over time. It is a decision making process with the goal of optimizing one or more objectives."

Over the years many standard models and their extensions have been studied, leading to different types of results, including exact algorithms, complexity theorems, approximation algorithms, heuristics and mathematical programming formulations. However, new interesting questions still emerge frequently. As industry is faced with new challenges, more and more often managers turn to operational research to help optimize the processes in their companies. Either the models derived from such a request may be completely new, or they may be similar to one of the well-known classical models with one or more additional, unstudied features.

In this thesis, we consider problems of the latter type, where known models are enhanced by special features. These features can arise, for example, from government regulations (e.g. health and safety), special set ups of industrial plants, measures taken by a company to improve work flow (e.g. cross-training) or measures for solution quality (e.g. energy efficiency or fairness). We investigate three of these features more closely, namely synchronization, pliability and resiliency. Our interest lies in complexity study, exact polynomial time algorithms and underlying structural results to help guide heuristic approaches in the future.

Synchronization has previously been studied for flow shop scheduling, where jobs have to be processed by a sequence of specialized machines, one after the other, in order to obtain the finished products. This is a standard setting, for example, in industrial plants. In a traditional flow shop scheduling problem machines are numbered, and each job has to be processed by the machines in order of their numbering, starting with the first machine, then the second and so on. For synchronous flow shop we additionally require that the jobs are moved from one

machine to the next one all at the same time. The main applications for synchronous flow shop models come from industrial plants with special job movement provisions. Consider a set up where machines are connected via the same transportation unit (e.g. a common conveyor belt). In that case, when the transportation unit moves forward, it moves all jobs at the same time.

We consider the extension of synchronous scheduling to open shop. Here, as for flow shop, a production process consists of several steps for each product, each step executed by a specialized machine. The difference between the traditional open shop and flow shop scheduling models is that the requirement to be processed by the machines in order of their numbering is lifted for open shop. Instead, for each job it is part of the decision process in which order it is processed by the machines. Again, for the model with synchronization job changes on the machines have to be made at the same time. Applications for synchronous open shop arise for example from health and safety regulations. If workers need to change jobs on the machines manually, then all machines may have to be stopped for workers to be able to safely enter the machine room. Unless job changes are at the same time, this procedure causes a large overhead in the production process.

The second feature, pliability, is interesting for shop scheduling environments, and we again focus on flow shop and open shop problems. The main assumption is that workers are cross-trained, and can take over some work from other workers. This helps reduce overheads that appear when solving regular shop scheduling problems. It has been noted by several authors ([37, 38, 39]) that this kind of "flexibility" can greatly improve the overall efficiency of a production environment. While several related models have been studied in the literature, pliability is a completely new feature, and we initiate research in this area by considering the most general version of pliability, as well as a couple of interesting special cases.

Lastly, resiliency is a new measure for solution quality in cases where the problem data are uncertain. Traditionally, when solving scheduling problems, it is assumed that processing times as well as other problem parameters are exactly known beforehand. In reality, often the problem parameters are not available as exact data, but rather as estimates. Furthermore, practitioners are frequently not in need of strictly optimal solutions, and are instead satisfied with solutions that are good enough, i.e. they do not exceed a certain threshold or are better than what they had before. These two observations motivate the search for resilient schedules, which are defined by the following two conditions. First, they should not exceed a given threshold of the objective function for the given parameter estimates. Second, they should not lose too much quality and continue to lie within the threshold, even if the actual problem parameters differ slightly from the original estimates.

As with pliability, resiliency has not previously been considered in literature. It is related to the well-known concepts of robustness [4] and stability [140]. Our contribution is to provide the necessary definitions to start the research in this area and to provide some initial general results, as well as an example of deeper study into a couple of specific problems and identification of promising further research questions.

A secondary area of interest in this thesis is the linear assignment problem, which plays a role in several chapters as an underlying model, both in its general version and in versions with special cost matrices. The linear assignment problem is well-studied in the literature [25]. In the two dimensional version we have to find an assignment of elements of one set to elements of another set, i.e. finding pairs. The goal is to minimize the cost of the assignment, which can be computed as the sum of the costs of each individual pair. The costs of the pairs are usually given in form of a cost matrix $W = (w_{ij})$, where entry w_{ij} denotes the cost of assigning the i-th element of set one to the j-th element of set two. A standard example of an application from scheduling is assigning tasks to workers, where the cost of a task-worker pair is the time it takes for the worker to complete the assigned task. If the tasks have to be completed one after the other, then the total time it takes to finish the product is equal to the sum of times that each worker needs for their task.

For the higher dimensional version, with d dimensions, we are given d sets and instead of finding pairs we have to find d-tuples, which include exactly one element from each of the sets. For example, when assigning final year projects in the academic sector, we need to consider projects, supervisors and students. The cost for a project-supervisor-student triple is guided by the preference of the student for the project and the expertise of the supervisor in the project area.

The complexity status of the linear assignment problem, both for two and more dimensions, has been known for many years. In general, it is solvable in $O(n^3)$ time [84, 88] for $d = 2$ dimensions and strongly NP-hard for all higher dimensions $d \geq 3$ [80, 84]. However, things are different if the cost matrices associated with the assignment problem have special structures.

A famous example is the so-called Monge condition. An $m \times n$ matrix $W = (w_{ij})$ is called a Monge matrix, if for all $1 \leq i < r \leq m$ and $1 \leq j < s \leq n$ we have

$$w_{ij} + w_{rs} \leq w_{is} + w_{rj}.$$

Monge matrices give rise to very efficient algorithms for many types of optimization problems, such as the linear assignment problem, the (Hoffman) transportation problem and the travelling salesperson problem (see the survey paper [26]). In particular, for the linear assignment problem it can be shown that if the cost matrix is Monge, then the problem can be solved in linear time [26]. The same still holds for the multi-dimensional linear assignment problem, if the above Monge condition is generalized for multi-dimensional arrays.

Our contribution in this area is the definition of a new Monge-like cost structure, which has several applications. For this structure we study the assignment problem in particular, which is shown to arise from the synchronous open shop problem. We prove that assignment problems of this type are solvable in linear time as long as the dimension d is fixed, and not part of the input. We also study resiliency for the general version of the assignment problem.

This thesis is structured as follows. In the next chapter we provide notation, definitions from the areas of graph theory, scheduling and the assignment problem needed throughout

this thesis, as well as a few traditional results related to our later research. After that, the thesis is divided into three independent parts, one for each of the three features we study, synchronization, pliability and resiliency. In addition to the general introduction in the next chapter, each part also has its own introductory section, where additional notation and related work pertaining only to the contents of the respective part is discussed.

In Part I, the main focus is on scheduling synchronous open shops. We show that for any fixed number of machines minimizing the length of the schedule is polynomially solvable. Other traditional scheduling objectives are strongly NP-hard to solve even for two machines. We also discuss an underlying assignment problem and prove a new structural property for the solution of an assignment problem with a cost matrix of Monge-like structure.

Part II is about pliability. We define the model and provide an initial complexity study for the most common objective functions. Furthermore, we suggest several promising directions to extend the model for further research.

The new concept of solution resiliency is investigated in Part III. We define resiliency for the case of a general combinatorial optimization problem and compare this definition to related concepts in the literature. Then we provide some general results and study two specific problems in greater depth: the two-dimensional linear assignment problem, as it underlies many scheduling problems, as well as scheduling on a single machine to minimize the total sum of completion times.

Conclusions and further remarks are given at the end of each part separately.

Chapter 2

Preliminaries

In this chapter we provide an overview of definitions and notation which are used in the thesis as a whole. We also discuss related work and important results pertaining to some of the notions introduced.

Due to the nature of the work presented in this thesis, there are also definitions and notation only needed in specific chapters. These are not presented here, instead they are introduced in the appropriate places inside the chapters, in which they are used. Any related work is also presented at that time. Furthermore, to keep this section concise, we omit a general introduction to combinatorial optimization and complexity theory. For such introductions see [59] or more modern texts like [84]. An introduction to algorithms is also given in [33].

We start by introducing some elements of graph theory in Section 2.1. In Section 2.2 some general notions from scheduling theory are established, with a focus on those areas which are needed later in the thesis. The specific models closer related to our work and the classical results concerning those models are discussed in more detail in Section 2.3. The linear assignment problem, which is important in several places throughout the thesis, is introduced in Section 2.4. We repeat results both for the general cases and for the versions with Monge cost structures, which are explained as well.

2.1 Necessary notation from graph theory

This section is a concise introduction of the key concepts of graph theory needed for this thesis. The main goal is to provide the notation used for graphs in this thesis. For a more thorough and detailed introduction to graphs see, e.g., [44] or for a more algorithmic focused introduction the appropriate chapters in [84].

A *graph* $G = (V, E)$ consists of a *vertex set* V and an *edge set* E, where an edge $e \in E$ is a two-elemental subset of V. If $e = \{v, w\} \in E$ for two vertices $v, w \in V$ then v and w are called *adjacent* and e is called *incident* with v and w. The *degree* $\deg_G(v)$ of a vertex $v \in V$ is the

number of edges incident with v. We may also write $\deg(v)$ if there can be no confusion about the graph in question.

A *trail* or *walk* of length k in graph G is a sequence of vertices $(v_0, v_1, v_2, \ldots, v_k)$, such that $\{v_i, v_{i+1}\} \in E$ for all $i = 0, 1, 2 \ldots, k-1$. If $v_0 = v_k$ then the trail is called *closed* or a *circuit*. A trail is called a *path* if v_i and v_j are different for each choice of $0 \leq i < j \leq k$ other than $\{i, j\} = \{0, k\}$. A path is called a *cycle* if $v_0 = v_k$, i.e. if it is closed. A graph G is called *connected* if for each pair of vertices $v, w \in G$ there exists a path (of some length) in G which starts in v and ends in w. A graph G is called *acyclic* if it does not contain a cycle.

A graph $G = (V, E)$ is called *complete* if $\{v, w\} \in E$ for all $v, w \in V$, $v \neq w$. The complete graph with n vertices is unique up to isomorphisms and is denoted by K_n. A graph $G = (V, E)$ is called *bipartite* if the vertex set V can be partitioned into two subsets V_1 and V_2, such that for each edge $\{v, w\} \in E$ we have $v \in V_1$ and $w \in V_2$. A complete bipartite graph $G = (V_1 \cup V_2, E)$ is a bipartite graph such that $\{v, w\} \in E$ for all $v \in V_1$ and $w \in V_2$. The complete bipartite graph with m vertices in one vertex set and n vertices in the other is unique up to isomorphisms and is denoted by $K_{m,n}$.

Given a graph $G = (V, E)$ then its line graph $L(G)$ is defined as the graph which has as vertex set the edge set of G, and two edges of G are adjacent vertices in $L(G)$ if they are incident with the same vertex in G, i.e. $L(G) = (E, L(E))$, where $L(E) = \{\{e_1, e_2\} \subset E : |e_1 \cap e_2| = 1\}$.

A *directed graph* or *digraph* $G = (V, E)$ is defined similarly, only that the edges additionally have a direction. In this case edges are given as ordered pairs $e = (v, w) \in E$ such that $E \subseteq V \times V$. If $e = (v, w) \in E$ then e is called an outgoing edge of v and an incoming edge of w. The in-degree $\deg_G^-(v)$ of a vertex v in an undirected graph G is the number of edges in G entering v. The out-degree $\deg_G^+(v)$ is the number of edges leaving G. Again, the degree $\deg_G(v)$ is the number of edges incident with v, both entering and leaving, i.e. $\deg_G(v) = \deg_G^+(v) + \deg_G^-(v)$.

A *directed trail* or *directed walk* of length k in a directed graph G is a sequence of vertices $(v_0, v_1, v_2, \ldots, v_k)$ such that $(v_i, v_{i+1}) \in E$ for all $i = 0, 1, 2, \ldots, k-1$. Directed paths and directed cycles defined analogously. A directed graph G is *strongly connected* if for each pair of vertices $v, w \in G$ it contains a directed path of some length that starts in v and ends in w. It is *weakly connected* if the underlying undirected graph obtained from G by dropping the directions on all edges is connected. A directed graph G is called *acyclic* if it does not contain a directed cycle.

Given a directed or undirected graph $G = (V, E)$ a trail in G is called a (directed or undirected) *Eulerian trail* or *Eulerian walk* if it uses every edge in G exactly once (it may use vertices multiple times). An Eulerian trail is called *Eulerian tour* or *Eulerian circuit* if it is closed. A undirected graph G is *Eulerian* if every vertex in G has even degree. A directed graph G is called *Eulerian* if for every vertex v in G we have $\deg_G^-(v) = \deg_G^+(v)$. It is well-known that a (strongly) connected directed or undirected graph G is Eulerian if and only if it contains an Eulerian tour (see, e.g., [84]).

A directed or undirected path/cycle in a directed or undirected graph G is called *Hamiltonian path/cycle* if it includes every vertex in G. A directed or undirected graph G is called *Hamiltonian* if it contains a Hamiltonian cycle.

2.2 General definitions and notation for scheduling

In this section we introduce basic definitions and notation from scheduling theory. We start by presenting a very general notion of a scheduling problem and a feasible schedule. After that, we introduce the three-field notation, the normal manner in which scheduling problems are denoted.

2.2.1 A general scheduling problem

For a general scheduling problem a set of n jobs $\mathcal{J} = \{J_1, J_2, \dots J_n\}$ needs to be processed on a set m machines $\mathcal{M} = \{M_1, M_2, \dots, M_m\}$. Throughout the thesis, if there is no ambiguity, we use a simplified notation $\mathcal{J} = \{1, 2, \dots, n\}$ for jobs and $\mathcal{M} = \{A, B\}$ if there are only two machines.

Each job j, $1 \leq j \leq n$, consists of $n_j \geq 1$ operations $\{O_{1j}, O_{2j}, \dots, O_{n_j j}\}$. Operation $O_{\iota j}$, $1 \leq \iota \leq n_j$, is associated with a processing time $p_{\iota j}$ and a subset $\mu_{\iota j} \subset \mathcal{M}$ of machines by which it may be processed. Then the operation $O_{\iota j}$ has to be processed on one of the machines in $\mu_{\iota j}$ for $p_{\iota j}$ time units.

Nearly everywhere in this thesis it is enough to consider one of two special cases for the number of operations per job n_j and the sets $\mu_{\iota j}$. In the first case each job j has only one operation O_{1j}, i.e. $n_j = 1$ for all jobs j and $\mu_{1j} = \mathcal{M}$. We identify job j with its single operation and denote its processing time by p_j. Examples for scheduling models of this type are introduced in Sections 2.3.1–2.3.3. In the second case each job has m operations, one operation O_{ij} for each machine M_i, with $\mu_{ij} = \{M_i\}$, i.e. operation O_{ij} has to be processed by machine M_i. In both cases, in order to reduce notation, we drop the index ι for the operations. Examples for scheduling models of this type are introduced in Sections 2.3.4 and 2.3.5. The only scheduling model used in this thesis that does not belong to one of these two cases is job shop scheduling, which is introduced in Section 2.3.6.

A schedule S is an allocation of each operation $O_{\iota j}$ to a machine $M_i \in \mu_{\iota j}$ and a time interval $I_{\iota j} = [T_{\iota j}, T_{\iota j} + p_{\iota j})$ on machine M_i. We call $C_{\iota j} = T_{\iota j} + p_{\iota j}$ the completion time of operation $O_{\iota j}$ and $C_j = \max\{C_{\iota j} | \iota = 1, \dots, n_j\}$ the completion time of job j. For a schedule to be feasible we usually require that no job is processed by two machines at the same time and no machine processes two jobs at the same time:

F1 the time intervals $I_{1j}, I_{2j}, \dots, I_{n_j j}$ belonging to the operations of job j are pairwise disjoint,

F2 if two operations $O_{\iota_1 j_1}$ and $O_{\iota_2 j_2}$ are processed on the same machine M_i then $I_{\iota_1 j_1} \cap I_{\iota_2 j_2} = \emptyset$.

Additional requirements for feasibility can arise depending on the model at hand. For example, in addition to the above, a schedule may have to respect release dates or deadlines associated with the jobs. If job j has a release date r_j, then no operation of job j may be processed before time r_j. Similar, if job j has a deadline D_j, then all operations of job j have to be processed by time D_j. If release dates and deadlines are given, the following two requirements for a feasible schedule are added:

F3 for each job j and each operation $O_{\iota j}$ we have $r_j \leq T_{\iota j}$,

F4 for each job j we have $C_j \leq D_j$.

Note that instead of deadlines, jobs can also have due dates, denoted by d_j for each job j. The difference is that feasible schedules may violate due dates (for a cost), so **F4** is not added for due dates. Due dates are of importance for several scheduling objectives, which will be explained in greater detail later on.

Also, the set of operations belonging to a job j, may be partially ordered by precedence constraints. This is the case if some operations belonging to job j need to be completed before other operations belonging to the same job j can be started. The precedence constraints are usually given in the form of an acyclic directed graph G, where $O_{\iota_1 j}$ has to be processed before $O_{\iota_2 j}$ if there is a path in G from the vertex associated with $O_{\iota_1 j}$ to the vertex associated with $O_{\iota_2 j}$. Formally, the following additional requirement needs to be satisfied by a feasible schedule:

F5 if operation $O_{\iota_1 j}$ is required to be processed before operation $O_{\iota_2 j}$, then $C_{\iota_1 j} \leq T_{\iota_2 j}$.

For this thesis, it is enough to consider precedence constraints which are represented by a directed path $O_{1j} \to O_{2j} \to O_{3j} \to \ldots \to O_{n_j j}$, as happens for flow shop scheduling and job shop scheduling (see Sections 2.3.4 and 2.3.6.

It is possible for some requirements to be newly introduced or relaxed if the investigated model calls for this. For example, for a feasible schedule in models with synchronization it is required that operations which are processed in intersecting time intervals are started at the same time (see also Chapter 3). On the other hand, requirement **F2** is relaxed for multiprocessor or batching machines, which can compute several operations simultaneously. See [17] for details and other examples.

2.2.2 The three-field notation

Scheduling models are classified using the so-called three-field notation, $\alpha|\beta|\gamma$, where α specifies the machine configuration, β specifies job characteristics and special processing requirements and γ specifies the optimization goal. The notation was first introduced in [65]. A general introduction to the three-field notation is provided in [17].

Machine configurations

The machine configurations are explained in greater detail in Section 2.3. The four most important machine configurations for this thesis are parallel identical machines, parallel uniform machines, flow shop and open shop. They are denoted by $\alpha = P$, $\alpha = Q$, $\alpha = F$ and $\alpha = O$ respectively. Additionally, there may be a number after the letter, to specify the exact number of machines, e.g. $\alpha = F2$ for two-machine flow shop. If no number is given, then we assume the number of machines is part of the input. If there is only one single machine, then this is denoted by $\alpha = 1$.

Job characteristics

The β-field specifies any non-standard characteristic of the jobs or processing requirements. We provide a few important examples here and introduce other entries when they are used.

- If release dates or deadlines are given, then this is specified by the entry r_j or D_j respectively in the β-field.
- If all jobs have the same processing time, then this is specified by the entry $p_j = p$ in the β-field. For unit processing times we write $p_j = 1$. Similar specifications may happen for release dates, deadlines or due dates.
- If *pmtn* appears in the β-field, then operations may be interrupted at any time during their processing and later restarted on the same machine, or on another machine that can process them. This may happen several times for the same operation, leading to several intervals in which the operation is processed, rather than only one. However the full processing time of the operations must still be scheduled before the operation is completed. All other requirements for a feasible schedule apply as well.
- The entry $p - batch$ in the β-field indicates that parallel batching is allowed on the machines. Parallel batching is introduced in Section 2.3.1.

Optimization goals and objective functions

Finally, the optimization goal in the γ-field usually is a combination of one or more objective functions, though for this thesis only optimization goals consisting of a single scheduling objective are considered. In general, a scheduling objective is given by a function $f(C_1, C_2, \ldots, C_n)$ dependent on the completion times $C_1, C_2, \ldots, C_n$ of the jobs. The goal is then to find a feasible schedule which minimizes the objective function f. The most important scheduling objectives for this thesis are the following:

- the makespan $C_{\max} = \max\{C_1, C_2, \ldots, C_n\}$, i.e. the completion time of the last completed job;

– the sum of completion times $\sum C_j = \sum_{j=1}^{n} C_j$;

– the maximum lateness $L_{\max} = \max_{1 \le j \le n}(C_j - d_j)$ for given due dates d_j.

Other prominent scheduling objectives are the total tardiness $\sum T_j = \sum_{j=1}^{n} T_j$ with $T_j = \max\{0, C_j - d_j\}$ and the number of late jobs $\sum U_j = \sum_{j=1}^{n} U_j$, with

$$U_j = \begin{cases} 1 & \text{if } C_j > d_j, \\ 0 & \text{otherwise.} \end{cases}$$

For each of the min-sum objectives there also exists a weighted version, with weights w_j given for each job, i.e. the weighted sum of completion times $\sum w_j C_j = \sum_{j=1}^{n} w_j C_j$, the weighted total tardiness $\sum w_j T_j = \sum_{j=1}^{n} w_j T_j$ and the weighted number of late jobs $\sum w_j U_j = \sum_{j=1}^{n} w_j U_j$.

Elementary reductions are well known for these traditional scheduling objectives, see, e.g., [17]. For example, by setting all weights $w_j = 1$ we can see that the unweighted min-sum objectives are special cases of the weighted ones. Therefore, the weighted versions are at least as hard to minimize as their respective unweighted versions. Similarly, if all due dates are equal, then minimizing the maximum lateness reduces to minimizing the makespan. Consequently, the makespan objective is a special case of the maximum lateness objective and the maximum lateness is at least as hard to minimize as the makespan. The elementary reductions for traditional scheduling objectives are presented in Fig. 2.1, see, e.g. [17].

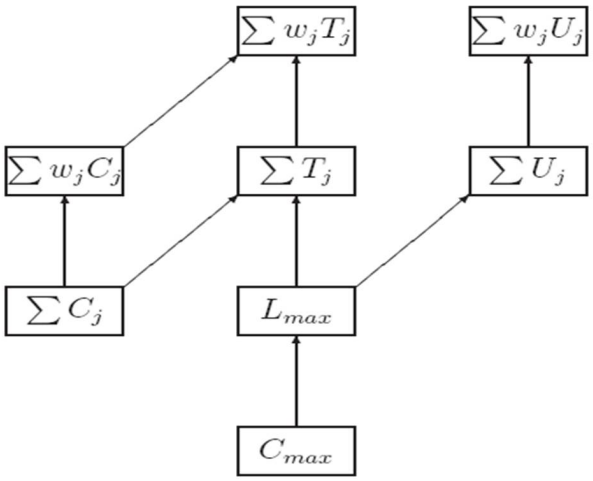


Figure 2.1: Reductions between traditional scheduling objectives

Note that all of the objectives mentioned here are non-decreasing in the completion times. An objective function $f(C_1, C_2, \ldots, C_n)$ which is non-decreasing in the completion times is called regular. In this thesis we normally deal with regular objective functions. Objectives which are not regular arise, for example, in just-in-time scheduling, see, e.g., [9]. An example of such

an objective is the combined total earliness and total tardiness, $\sum(E_j + T_j) = \sum_{j=1}^{n}(E_j + T_j)$, where $E_j = \min\{0, C_j - d_j\}$.

2.3 Scheduling models

Since this thesis is about scheduling models with additional features it is natural to start with a summary of the traditional models from which they are derived. We focus on complexity results and exact polynomial time algorithms for the most important and general problems (with the least additional requirements and assumptions in the β-field). If more specialized results are needed in places later in the thesis, they are provided there.

For a broader or more detailed introduction to scheduling models we recommend [17] or [121]. A useful overview of complexity results in form of tables is also given on the web page [19], though the tables are also partially available in [17].

2.3.1 Single machine scheduling problems

For scheduling problems with only one machine we usually assume that each job consists of only one operation. Recall that in this case, we identify with the job itself and the processing time of a job j is denoted by p_j. Below we first summarize briefly the results for classical single machine scheduling which are interesting for this thesis, then we move on to scheduling a batching machine, as it is an important related model for the later part on synchronization.

Classical single machine scheduling

Problem $1||C_{\max}$ is trivial, as each schedule without idle times is optimal. Problem $1||L_{\max}$ is solvable in $O(n \log n)$ time by scheduling jobs, without idle times, in order of non-decreasing due dates [17, 77]. This is called Jackson's rule or simply EDD-rule (earliest due date first).

Minimizing the total completion time $1||\sum C_j$ is also done $O(n \log n)$ time by scheduling jobs, without idle times, in order of non-decreasing processing times [17, 139]. This is known as Smith's rule or SPT-rule (shortest processing time first). As a generalization, problem $1||\sum w_j C_j$ is solvable in $O(n \log n)$ time by scheduling jobs in order of non-decreasing ratio $\frac{p_j}{w_j}$. This is known as Smith's ratio rule [17, 139].

Finally, problem $1||\sum U_j$ is solved $O(n \log n)$ time by Moore's algorithm [17, 115]. First schedule jobs in order of non-decreasing due dates. Then, if the job in position j is the first late job in the sequence, move the job with the largest processing time amongst all jobs in positions $1, 2, 3, \ldots, j$ to the end of the sequence. The process is repeated, again with the position of the first late job in the sequence. It is stopped when all jobs are either on time or have been moved to the end of the schedule during the algorithm. The resulting optimal schedule is in two parts, all on-time jobs in the beginning of the schedule, in order of their due dates, and all late jobs are at the end of the schedule in any order.

For the other traditional scheduling objectives $f \in \{\sum T_j, \sum w_j T_j, \sum w_j U_j\}$ problem $1||f$ is NP-hard. To be more precise, problems $1||\sum T_j$ and $1||\sum w_j U_j$ are NP-hard in the ordinary sense (see [49, 91] and [80, 99] respectively), while problem $1||\sum w_j T_j$ is strongly NP-hard [91, 102].

For this thesis the above summary suffices in terms of introduction to classical single machine problems. A more detailed collection of results, which includes additional job characteristics like release dates or precedence constraints, can be found in [17].

Scheduling a batching machine

There are two types of batching models: serial batching (s-batching) and parallel batching (p-batching). In this thesis only the latter model is of interest. For an introduction to s-batching see [17].

If parallel batching is allowed on a machine, then part of the decision process is to partition the job set into batches. All jobs of a batch are processed in parallel and the processing time of a batch is equal to the processing time of the longest job within the batch. The completion time of a job is equal to the completion time of the batch to which the job belongs. Instead of sequencing jobs, as we did for the classical problems above, we first partition jobs into batches and then sequence the batches on the machine.

There are two variants dependent on the maximum batch-size b, the number of jobs any batch may contain. In unrestricted p-batching we have $b \geq n$, i.e. a batch may contain arbitrarily many jobs. In that case the maximum batch size is not denoted in the three-field notation and the problem is denoted by $1|p-batch|f$, if f is the objective function. In restricted p-batching we have $b < n$ and no batch may contain all jobs. The problem is denoted by $1|p-batch, b<n|f$ if the maximum batch size is part of the input or $1|p-batch, b=\bar{b}|f$ if the maximum batch size is fixed to some integer $\bar{b}$. Note that in general even in the restricted model it is not required that batches are filled up to the maximum batch size and batches may contain fewer than b jobs.

In general, unrestricted p-batching is much easier than restricted p-batching. Indeed, for unrestricted p-batching only problems $1|p-batch|\sum w_j U_j$ and $1|p-batch|\sum w_j T_j$ are proven to be NP-hard (in the ordinary sense) with $1|p-batch|\sum T_j$ still open. All other traditional scheduling objectives $f \in \{C_{\max}, L_{\max}, \sum C_j, \sum w_j C_j, \sum U_j\}$ can be optimized in polynomial time; see [17] and [18] for details.

Conversely, the only polynomial algorithm known for restricted p-batching (without any additional special assumptions) is for problem $1|p-batch, b<n|C_{\max}$ [17, 18]. Here jobs are assigned to a batch in order of non-increasing processing times until the maximum batch size is reached, then a new batch is started with the next job. The problem is solvable in $O(n \log n)$ time, due to sorting.

All problems involving due dates are strongly NP-hard to solve even for maximum batch size $b = 2$, i.e. $1|p-batch, b=2|f$ is strongly NP-hard for $f \in \{L_{\max}, \sum T_j, \sum U_j, \sum w_j T_j, \sum w_j U_j\}$

[18].

Problems $1|p-batch, b<n|\sum C_j$ and $1|p-batch, b<n|\sum w_jC_j$ are open, to the best of our knowledge.

2.3.2 Scheduling problems with parallel identical machines

In this section we discuss scheduling parallel identical machines. As for single machine scheduling, each job has only one operation. Any job can be processed by any of the machines and all machines take the same amount of time p_j to process a given job j. We first summarize results for scheduling identical parallel machines without preemption, after that we turn to the preemptive case.

Parallel identical machines without preemption

Note that any NP-hardness result from single machine scheduling carries over to scheduling parallel machines. Therefore problem $P2||\sum w_jT_j$ is strongly NP-hard, due to the strong NP-hardness of the single machine problem with the same objective.

Furthermore, even for two machines, minimizing the makespan is equivalent to the PARTITION problem. Indeed, the minimum makespan of $\frac{\sum p_j}{2}$ can be achieved if and only if the set of processing times can be partitioned into two sets with the same sum of elements. Thus, problem $P2||C_{\max}$ is NP-hard in the ordinary sense [102]. Problem $P||C_{\max}$ with the number of machines m part of the input is strongly NP-hard to solve [58]. The situation is similar for the objective functions $f \in \{L_{\max}, \sum T_j, \sum U_j, \sum w_jU_j\}$ instead of $f = C_{\max}$. See [17] for a detailed set of references.

Using an extension of Smith's rule we can solve problem $P||\sum C_j$ in $O(n \log n)$ time, by sorting jobs in order of non-decreasing processing times and then repeatedly assigning the shortest unassigned job to the first free space on any machine.

Finally, problem $P2||\sum w_jC_j$ is NP-hard in the ordinary sense [21] and problem $P||\sum w_jC_j$ is strongly NP-hard [17].

Parallel identical machines with preemption

For the preemptive case, McNaughton showed that problem $P|pmtn|C_{\max}$ is solvable in $O(n)$ time [111]. First the optimal makespan is computed, then machine M_1 is filled with jobs until the optimal makespan is reached. At that point, the current job is preempted and restarted on the next machine M_2. The remaining schedule is built in a similar manner.

Problem $P|pmtn|L_{\max}$ is solvable in $O(n \log n)$ time by Sahni's algorithm [133] combined with an algorithm to compute the optimal maximum lateness for a given instance [11].

Minimizing the number of late jobs for a fixed number m of machines, i.e. problem $Pm|pmtn|\sum U_j$, is solvable in $O(n^{3(m-1)})$ time [92, 93, 98], while $P||\sum U_j$ with the number of machines m part of the input is NP-hard in the ordinary sense [94]. In the weighted case, the problem is NP-hard

in the ordinary sense even for two machines, because the single machine problem with objective function $\sum w_j U_j$ is NP-hard, both with and without preemption.

It is a well known result that for problem $P|pmtn|\sum w_j C_j$ an optimal schedule without preemption exists [111]. Consequently, problem $P|pmtn|\sum C_j$ is solvable in $O(n \log n)$ time (by the same method as in the non-preemptive case) while $P2|pmtn|\sum w_j C_j$ is NP-hard in the ordinary sense, again by [21].

Finally, problems $P2|pmtn|\sum T_j$ and $P2|pmtn|\sum w_j T_j$ are NP-hard in the ordinary sense and in the strong sense respectively, due to the NP-hardness results for problems $1||\sum T_j$ and $1||\sum w_j T_j$. Note that for the preemptive versions of the single machine problems there exist optimal schedules without preemptions.

2.3.3 Scheduling problems with parallel uniform machines

Now we turn our attention to scheduling problems with parallel uniform machines. Again, each job has only one operation and may be scheduled on any machine. The difference to the model with identical machines is that now machines may have different processing speeds. If the processing time of job j is p_j then the actual time it takes for machine M_i with speed s_i to process job j is $\frac{p_j}{s_i}$.

Note that problems with parallel identical machines can be seen as special cases of the problems with parallel uniform machines where all machine speeds are equal to 1. Thus, all NP-hardness results known for parallel identical machines carry over to parallel uniform machines.

While the exact running times of algorithms may differ, the complexity status (i.e. NP-hard or polynomially solvable) of the problems $Q||f$ and $Q|pmtn|f$ with uniform machines and a traditional scheduling objective f, is the same as that of the related problems with identical machines. Indeed, problem $Q||\sum C_j$ is solvable in polynomial time while optimizing any other traditional scheduling objective is at least NP-hard in the ordinary sense, even for two machines. Similarly, problems $Q|pmtn|f$ with $f \in \{C_{\max}, L_{\max}, \sum C_j\}$ are polynomially solvable (see [89] for the latter two objectives), while the problems $Q2|pmtn|f$ with $f \in \{\sum w_j C_j, \sum T_j, \sum w_j T_j, \sum w_j U_j\}$ are at least ordinarily NP-hard. Finally, again like for parallel identical machines problem $Qm|pmtn|\sum U_j$ is solvable in polynomial time, while problem $Q|pmtn|\sum U_j$ is NP-hard.

For this thesis we leave our introduction at that. More details and references beyond the ones already given in Section 2.3.2 can be found in [17].

2.3.4 Flow shop problems

In flow shop scheduling we consider problems with m machines and n jobs, where each job has exactly m operations and for all jobs j operation O_{ij} has to be processed by machine M_i. Furthermore, for all jobs j, before operation O_{i^*j} can be started all operations O_{ij} with $i < i^*$

have to be completed, i.e. the set of operations of each job is ordered by precedence constrains given by the graph $O_{1j} \rightarrow O_{2j} \rightarrow O_{3j} \rightarrow \ldots \rightarrow O_{mj}$. For scheduling objectives which make the introduction of voluntary idle times (idle times which are not forced by the constraints of a feasible schedule) unnecessary, such as the makespan objective, the problem is to find for each machine M_i the sequence in which the jobs are processed on machine M_i.

The most well-known classical result for flow shop scheduling is that problem $F2||C_{\max}$ is solvable in $O(n \log n)$ time by Johnson's algorithm [79]. First we partition the job set $\mathcal{J}$ into two subsets $\mathcal{J}_1 = \{j \in \mathcal{J} | p_{1j} \leq p_{2j}\}$ and $\mathcal{J}_2 = \{j \in \mathcal{J} | p_{1j} > p_{2j}\}$. Then we construct the first part of the schedule by sequencing on both machines all jobs in set $\mathcal{J}_1$ in order of non-decreasing processing time p_{1j}. Once all jobs in set $\mathcal{J}_1$ are sequenced, in the second part of the schedule we sequence on both machines all jobs in set $\mathcal{J}_2$ in order of non-increasing processing time p_{2j}.

A schedule in which the sequence of jobs is the same on each machine, like the one constructed by Johnson's algorithm is called a permutation schedule. Johnson's algorithm proves that an optimal permutation schedule exists for problem $F2||C_{\max}$. Optimal permutation schedules also exist for problem $F3||C_{\max}$ (see, e.g., [17]). For four or more machines, this property no longer holds. Counterexamples for the case with four machines are provided, for example, in [122].

In terms of complexity, Johnson's result remains the only positive one unless additional assumptions are made. Indeed, problems $F3||C_{\max}$, $F2||L_{\max}$ and $F2||\sum C_j$ are all strongly NP-hard ([60] and [102]). Consequently problem $F2||f$ is also strongly NP-hard for each traditional scheduling objective f other than the makespan.

Even allowing preemption does not improve the situation. Indeed, problem $F2|pmtn|C_{\max}$ is again solvable in $O(n \log n)$ time [63], while problems $F3|pmtn|C_{\max}$, $F2|pmtn|L_{\max}$ and $F2|pmtn|\sum C_j$ are strongly NP-hard, like their non-preemptive counterparts ([63] for the makespan, [31] for the maximum lateness and [50] for the total sum of completion times).

For now we leave it at this general introduction as it is not practical to introduce more specialized results at this time. However, flow shop plays an important role in this thesis and more results are discussed in the parts of the thesis where those results are needed.

2.3.5 Open shop problems

Open shop is similar to flow shop, in that each job has exactly m operations and for all jobs j operation O_{ij} has to be processed by machine M_i. However, the precedence constraints between operations of one job are dropped, making it part of the decision process which operation of a job is processed first, second and so on.

In general, open shop scheduling is usually slightly easier than flow shop scheduling. The famous result by Gonzalez and Sahni [62] shows that problem $O2||C_{\max}$ can be solved in linear time. In the algorithm, the two jobs with the largest operations on the two machines are treated separately, while the order of all other jobs only depends on whether $p_{1j} \leq p_{2j}$ or $p_{1j} > p_{2j}$. For a full description see also, e.g., [17].

Problems $O3||C_{max}$ and $O|n = 3|C_{\max}$ are NP-hard in the ordinary sense [62]. Minimizing the makespan becomes strongly NP-hard for problem $O||C_{\max}$, i.e. if the number of machines m is part of the input [17]. All other traditional objectives are strongly NP-hard to minimize even for two machines, as for flow shop (see [95, 96] for the maximum lateness and [1] for the total sum of completion times).

With preemption allowed, open shop again appears to be slightly easier than flow shop. Problems $O|pmtn|C_{\max}$ and $O|pmtn|L_{\max}$, with the number of machines m arbitrary, can be solved in polynomial time by linear programming [31]. The result still holds, even if release dates are given in addition. For problem $O|pmtn|C_{\max}$ there also exists an algorithm which finds an optimal solution in $O(r \min\{r, m^2\} + m \log n)$ time, where r is the number of operations with non-zero processing time, i.e. $r \leq nm$ [62]. Problem $O2|pmtn|\sum C_j$ is NP-hard in the ordinary sense [50], while problem $O3|pmtn|\sum C_j$ is strongly NP-hard [105]. Finally problem $O2|pmtn|\sum U_j$ is NP-hard in the ordinary sense [95, 96].

This is where we stop our general introduction. Extensions to open shop scheduling are a key part of this thesis and in the subsequent parts we turn our attention to more specialized results that are directly related to our work.

2.3.6 Job shop problems

We conclude our review of classical scheduling models with a brief introduction to job shop scheduling. Job shop, denoted by J in the α-field of the three-field notation, is a generalization of flow shop, where a job j can have arbitrarily many operations n_j and for each operation $O_{\iota j}$ there is some dedicated machine M_i on which it must be processed, in which case we have $\mu_{\iota j} = \{M_i\}$. Still, as in flow shop, operation $O_{\iota^* j}$ cannot be started before all operations $O_{\iota j}$, $\iota < \iota^*$, are completed. Flow shop is a special case of job shop where $n_j = m$ for all jobs j and $\mu_{ij} = \{M_i\}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

The job shop model does not play a large role in this thesis. We do not provide a summary of results as we did for the previous models and we only introduce it here as it is convenient to do so immediately after introducing the other two shop scheduling problems. Clearly all NP-hardness results from flow shop carry over. Below we give two additional results, one negative and the other positive, as examples. For a more thorough summary see [17].

First, as an additional NP-hardness result, problem $J2||C_{\max}$ is strongly NP-hard even if there are only two different processing times $p_{\iota j} \in \{1, 2\}$ for all operations $O_{\iota j}$ [101]. As a positive result, problem $J2|n_j \leq 2|C_{\max}$, where each job as at most two operations, is solvable in $O(n \log n)$ time [78].

2.4 The linear assignment problem

As the last part of this chapter we introduce the linear assignment problem, focusing on the balanced case, where n objects in one set need to be assigned to n objects in another set. The

unbalanced case, where m objects in the first set need to be assigned to n objects in the second, $m < n$, and some of the n objects are not assigned is not relevant for this thesis. We also discuss the closely related matching problems in bipartite and in general graphs. First we introduce the two-dimensional linear assignment problem in Section 2.4.1 and the matching problems in bipartite and general graphs in Section 2.4.2. Then we turn to multi-dimensional assignment problems in Section 2.4.3. Finally, in Section 2.4.4, we introduce the Monge condition and explain how the complexity of the linear assignment problem changes when Monge costs are involved.

2.4.1 The two-dimensional linear assignment problem

The assignment problem is an important combinatorial optimization problem with many applications in different areas (see, e.g., [25]). In this problem the objective is to assign n items of one set $I = \{1, \ldots, n\}$ to n items of another set $J = \{1, \ldots, n\}$, where I may represent a set of workers, jobs, transmitting devices, and J may correspond to machines, rooms, receivers, etc. In the linear assignment problem, additionally an $n \times n$ weight (cost) matrix $W = (w_{ij})$ is given, and the goal is to find an assignment with minimum total weight.

An *assignment* S may be represented by a set of n pairs, $S = \{(i^1, j^1), \ldots, (i^n, j^n)\}$ with $\{i^1, i^2, \ldots, i^n\} = \{j^1, j^2, \ldots, j^n\} = \{1, 2, \ldots, n\}$ and it is characterized by the weight

$$w(S) = \sum_{(i,j) \in S} w_{ij}.$$

An assignment S is also called a *solution* and a solution S^* with minimum weight is called an *optimal solution*.

An alternative representation of the assignment problem uses binary decision variables x_{ij} indicating the assignment of i-items to j-items:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \qquad 1 \le i \le n, \\ & \sum_{i=1}^{n} x_{ij} = 1, \qquad 1 \le j \le n, \\ & x_{ij} \in \{0, 1\}, \qquad 1 \le i, j \le n. \end{aligned} \tag{2.1}$$

Clearly, there exists a one-to-one correspondence between any solution S and the binary solution matrix $X_S = (x_{ij})$ with $x_{i^k j^k} = 1$ for every pair $(i^k, j^k) \in S$, $1 \le k \le n$. The linear assignment problem can be solved in polynomial time, e.g., by the Hungarian algorithm [88], which can be implemented to run in $O(n^3)$ time [25].

We can also represent a solution to the assignment problem as a permutation

$$\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

The solution S given by the pairs $S = \{(i^1, j^1), \ldots, (i^n, j^n)\}$ is represented by permutation π_S with $\pi_S(i^k) = j^k$, such that $S = \{(i^1, \pi_S(i^1)), (i^2, \pi_S(i^2)), \ldots, (i^n, \pi_S(i^n))\}$. Furthermore, in that case the solution matrix X_S as defined above is the permutation matrix corresponding to permutation π_S in the usual definition of permutation matrices (see, e.g., [20], p. 2).

2.4.2 Matching problems in bipartite and general graphs

The linear assignment problem is closely related to weighted matching problems in graphs. Given a graph $G = (V, E)$ a *matching* M in G is a subset of the edges, $M \subseteq E$, such that in the graph $G(M) = (V, M)$ no vertex has degree larger than 1. A matching M^* is called *maximal*, if adding any edge $e \in E \setminus M^*$ to M^* would destroy the matching property, i.e. would cause one vertex in $G(M^*)$ to have degree two. A matching M^* in G is a *maximum matching* of G, if $|M^*| = \max\{|M| : M \text{ is a matching in } G\}$. Finally, a matching M^* in G is called *perfect*, if every vertex in G is matched by M^*, i.e. in $G(M^*)$ every vertex has exactly degree 1. Observe that in that case we have $|M^*| = \frac{|V|}{2}$.

If in addition the edges of G are weighted by a weight function $w : E \to \mathbb{R}$, then the weight of a matching M in G is defined by

$$w(M) = \sum_{e \in M} w(e).$$

Given a graph $G = (V, E)$, the minimum weight perfect matching problem is to find a perfect matching of minimum weight in G [84], or to decide that no perfect matching exists.

Matching theory is a vast field on its own, see [107] for a detailed introduction. We focus on the relation of the minimum weight perfect matching problem to the assignment problem and provide two general complexity results needed in this thesis.

If $G = (V_1 \cup V_2, E)$ is a bipartite weighted graph with $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$, then the minimum weight perfect matching problem in G can be transformed into an assignment problem with weight matrix $W = (w_{ij})$ given by

$$w_{ij} = \begin{cases} w(\{v_i, v_{n+j}\}), & \text{if } e = \{v_i, v_{n+j}\} \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly, if a perfect matching exists in G then the solution to the assignment problem with weight matrix W can be transformed into a perfect matching in G with minimum weight. Conversely, if no perfect matching exists in G, then any solution to the assignment problem has infinite weight. Therefore, if G is a bipartite graph, then the minimum weight perfect matching problem can be solved in $O(n^3)$ time with the same methods as the assignment problem (see also [84]).

If G is a general graph, not necessarily bipartite, then the structure of a solution becomes much harder to grasp. According to [84], minimum weight perfect matching in general graphs is

one of the "hardest" polynomially solvable combinatorial optimization problems. We do not go into detail here and simply cite the result that the problem is still solvable in $O(n^3)$ time by a weighted version of Edmond's blossom algorithm. The algorithm was first proposed by Edmonds [51], then Gabow [55] and Lawler [90] independently found the first $O(n^3)$ implementations. The currently best theoretical time bound of $O(nm + n^2 \log n)$ was again obtained by Gabow [56]. See also [84] for details. Practical implementation of the algorithm is still an issue today, see [83].

2.4.3 The multi-dimensional linear assignment problem

The extension of the linear assignment to the multi-dimensional case gives rise to the so called *axial d-dimensional assignment problem* ($d \geq 2$): given a d-dimensional $n \times \cdots \times n$ weight array $W = (w_{i_1 \ldots i_d})$, an assignment S is a set of n d-tuples $\{(i_1^1, \ldots, i_d^1), (i_1^2, \ldots, i_d^2), \ldots, (i_1^n, \ldots, i_d^n)\}$ with $\{i_\ell^1, \ldots, i_\ell^n\} = \{1, \ldots, n\}$ for all $\ell = 1, \ldots, d$, and its weight is

$$w(S) = \sum_{(i_1, \ldots, i_d) \in S} w_{i_1 \ldots i_d}.$$

The formulation in terms of the binary decision variables $x_{i_1 \ldots i_d}$ is as follows:

$$\begin{array}{lll} \min & \displaystyle\sum_{i_1, \ldots, i_d} w_{i_1 \ldots i_d} x_{i_1 \ldots i_d} & \\ \text{s.t.} & \displaystyle\sum_{\substack{i_1, \ldots, i_d \\ \text{s.t. } i_\ell = k}} x_{i_1 \ldots i_d} = 1, & 1 \leq \ell \leq d,\ 1 \leq k \leq n, \\ & x_{i_1 i_2 \ldots i_d} \in \{0, 1\}, & 1 \leq i_1, i_2, \ldots, i_d \leq n. \end{array} \tag{2.2}$$

The associated solution array $X_S = (x_{i_1 \ldots i_d})$ has a 1-entry $x_{i_1 \ldots i_d} = 1$ for every d-tuple $(i_1, \ldots, i_d) = (i_1^k, \ldots, i_d^k) \in S$, where $1 \leq k \leq n$. Note that in any solution X_S, the number of 1-entries is n, while the remaining $n^d - n$ entries are 0.

Unlike the two-dimensional version, the d-dimensional assignment problem is strongly NP-hard for each fixed $d \geq 3$, see, e.g., [80].

In relation to matchings, the problem can be interpreted as finding minimum weight perfect matchings in hypergraphs (e.g. 3-dimensional matching, see [59]), which is strongly NP-complete.

2.4.4 The linear assignment problem with Monge costs

There exists a multitude of special cases, which have been of interest to researchers and for which the assignment problem can be solved faster than in cubic time (see, e.g., [25]). In what follows, we introduce one of these special cases that is of major interest for us, namely assignment problems with Monge costs. The Monge property indicates a special structure of the cost matrix, and has a long history of study going back to Gaspard Monge in 1781 [114].

It is well known that many combinatorial optimization problems, including the assignment problem, can be solved faster (by a greedy algorithm) if the weight matrix is a Monge matrix. We remind here the main definitions and results following the survey paper [26]. An $n \times n$ matrix $W = (w_{ij})$ is called a *Monge matrix* if for all row indices $1 \leq i < r \leq n$ and all column indices $1 \leq j < s \leq n$ the so-called Monge property is satisfied:

$$w_{ij} + w_{rs} \leq w_{is} + w_{rj}. \tag{2.3}$$

An optimal solution to the assignment problem with a Monge matrix is given by the n pairs $(1,1),(2,2),\ldots,(n,n)$. Thus the assignment problem is solvable in $O(n)$ time, if the cost matrix is Monge.

A number of typical applied scenarios with Monge and Monge-like arrays are discussed in the survey papers [24] and [26]. As an example from scheduling, problem $1||\sum C_j$ famously can be formulated as an assignment problem with Monge cost (see also [26]). Given a sequence of jobs $(j_1, j_2, \ldots, j_n)$ as a solution to problem $1||\sum C_j$, note that the total completion time for the corresponding schedule is

$$\begin{aligned}
\sum C_j &= C_{j_1} + C_{j_2} + C_{j_3} + \ldots + C_{j_{n-1}} + C_{j_n} \\
&= p_{j_1} + (p_{j_1} + p_{j_2}) + (p_{j_1} + p_{j_2} + p_{j_3}) + \ldots + (p_{j_1} + p_{j_2} + p_{j_3} + \ldots + p_{j_{n-1}}) \\
&\quad + (p_{j_1} + p_{j_2} + p_{j_3} + \ldots + p_{j_{n-1}} + p_{j_n}) \\
&= np_{j_1} + (n-1)p_{j_2} + (n-2)p_{j_3} + \ldots + 2p_{j_{n-1}} + p_{j_n}.
\end{aligned}$$

The job j_k in position k contributes its own processing time p_{j_k} to the objective $n-k+1$ times.

We can model problem $1||\sum C_j$ as assigning jobs to positions in the sequence. If a solution S to the assignment problem is given, then in the corresponding solution sequence for the scheduling problem job j is in position i if $(i,j) \in S$. The cost matrix $W = (w_{ij})$ for the assignment problem is given by

$$w_{ij} = (n - i + 1)p_j.$$

If the jobs j are numbered in non-decreasing order of processing times, then for indices $1 \leq i < r \leq n$ and $1 \leq j < s \leq n$ we have

$$w_{ij} + w_{rs} = (n-i+1)p_j + (n-r+1)p_s \leq (n-i+1)p_s + (n-r+1)p_j = w_{is} + w_{rj}$$

and the matrix W is Monge.

The Monge property can be generalized to multi-dimensional arrays. An $n \times \cdots \times n$ array $W = (w_{i_1 \ldots i_d})$ is called a (d-dimensional) Monge array if for all $i_\ell, j_\ell \in \{1,\ldots,n\}$, $\ell = 1,\ldots,d$, we have

$$w_{s_1 s_2 \ldots s_d} + w_{t_1 t_2 \ldots t_d} \leq w_{i_1 i_2 \ldots i_d} + w_{j_1 j_2 \ldots j_d}, \tag{2.4}$$

where $s_\ell = \min\{i_\ell, j_\ell\}$ and $t_\ell = \max\{i_\ell, j_\ell\}$ for $\ell = 1,\ldots,d$. An optimal solution to the

multi-dimensional assignment problem with a Monge array is given by the n d-tuples $(1,\ldots,1)$, $(2,\ldots,2)$, …, $(n,\ldots,n)$. Again, the problem is solvable in $O(n)$ time.

Note that in many texts on Monge structures (e.g. [26, 124]), results for the assignment problem are presented in their generalized form for the closely related transportation problem. In its usual, continuous form, the d-dimensional transportation problem is formulated as follows:

$$\begin{aligned}
\min \quad & \sum_{i_1,\ldots,i_d} w_{i_1\ldots i_d} x_{i_1\ldots i_d} \\
\text{s.t.} \quad & \sum_{\substack{i_1,\ldots,i_d \\ \text{s.t. } i_\ell = k}} x_{i_1\ldots i_d} = a_k^\ell, \quad 1 \le \ell \le d,\ 1 \le k \le n, \\
& x_{i_1 i_2 \ldots i_d} \ge 0, \qquad\qquad 1 \le i_1, i_2, \ldots, i_d \le n.
\end{aligned}$$

Here a_k^ℓ are given non-negative supply/demand-values satisfying $\sum_{k=1}^n a_k^1 = \sum_{k=1}^n a_k^2 = \ldots = \sum_{k=1}^n a_k^d$. The assignment problem is the special case of the integer version of the transportation problem, with $a_k^\ell = 1$ for all $k = 1,\ldots,n$ and $\ell = 1,\ldots,d$.

Part I

Synchronization

Chapter 3

Definitions, notation and related work

In this part of the thesis we present new results for scheduling on multiple machines with synchronous changes. Previous research in synchronous scheduling models was mostly done on the synchronous flow shop problem. Our focus in this part is the extension of synchronous models to the open shop problem, which turns out to be slightly easier computationally. We also deal with an underlying assignment problem with a special cost structure, that is closely related to the Monge conditions (2.3) and (2.4).

First we provide additional definitions and notation needed and discuss some previous related research in this chapter. Chapter 4 deals with the synchronous open shop problem with two machines. We show that optimizing the makespan objective can be done in polynomial time, while all other standard objectives are strongly NP-hard to optimize. The focus of Chapter 5 is a multi-dimensional assignment problem underlying the synchronous open shop problem with makespan criterion and $m > 2$ machines. We prove that the assignment problem is solvable in linear time for fixed dimension and use this to show that minimizing the makespan in m machine synchronous open shop is polynomially solvable for any fixed number m of machines. In Chapter 6 we generalize an auxiliary result for two machines from Chapter 4 to the m machine case. Finally, in Chapter 7 we provide conclusions and further research directions for the problems studied as well as some suggestions on how to extend synchronous scheduling to parallel machine models.

3.1 Introduction and definitions

Scheduling problems with synchronization arise in applications where job processing includes several stages, performed by different processing machines, and all movements of jobs between machines have to be done simultaneously. This may be caused by special requirements of job

transfers, as it happens, for example, if jobs are installed on a circular production unit which rotates to move jobs simultaneously to machines of the next stage (see [75, 143, 151]). Alternatively, there may be health and safety regulations requiring that no machine is in operation while jobs are being removed from or moved to a machine. Similar synchronization takes place in the context of switch based communication systems, where senders transmit messages to receivers in a synchronous manner, as this eliminates possible clashes for receivers (see [64, 82, 126]).

Synchronization arises naturally in assembly line systems where each assembly operation may start only after all preceding operations are completed, see [30], [46], [146], and the surveys [15] and [16]. In the context of shop scheduling models, synchronization aspects were initially studied for flow shops ([75], [143], [152]). Using previous results from [61] and [128] as well as new proofs, it has been shown that scheduling synchronous flow shop to minimize the makespan is solvable in $O(n \log n)$ time for two machines and strongly NP-hard for any fixed number $m \geq 3$ of machines. All other standard scheduling objectives are strongly NP-hard to solve, even for two machines. Interesting special cases with dominating machines have been studied in terms of complexity in [152]. Dealing with the NP-hardness of many of the investigated problems, exact methods as well as heuristics have been proposed both for the general model and for special cases, e.g in [87], [143] and [150].

The work presented in this thesis is the first study of synchronous open shops and has previously been published in [154] and [155]. We focus on complexity results and exact polynomial time algorithms. In Chapter 4 we deal with the problem from a scheduling perspective with different objectives, mostly following [155], while in Chapter 5 we study the underlying assignment problem for the makespan criterion following [154]. Note that in Chapter 5 the notation of the scheduling problem differs from our usual notation, in order to make it consistent with the notation for the assignment problem. Details are discussed there.

Formally, the open shop model with synchronization is defined as follows. As in the classical open shop, n jobs $J_1, J_2, \ldots, J_n$ have to be processed by m machines $M_1, M_2, \ldots, M_m$, $n \geq m$. Each job J_j, $1 \leq j \leq n$, consists of m operations O_{ij} for $1 \leq i \leq m$, where O_{ij} has to be processed on machine M_i without preemption for p_{ij} time units. The synchronization requirement implies that job processing is organized in synchronous cycles, with operations of the same cycle starting at the same time. Within one cycle, machines which process operations of smaller processing times have to wait until the longest operation of the cycle is finished before the next cycle can start. Thus, the length of a cycle is equal to the maximum processing time of its operations. Similar to the classical open shop model, we assume that unlimited buffer exists between the machines, i.e., jobs which are finished on one machine can wait for an arbitrary number of cycles to be scheduled on the next machine.

The goal is to assign the nm operations to the m machines in n cycles and decide a feasible starting time for each cycle (a common starting time for all its operations), such that a given objective function f is optimized. In the results discussed here we focus on traditional, regular scheduling objectives, thus in our case the starting time for each cycle can always be chosen as

the completion time of the cycle before it. Following the earlier research by [75] and [152], we denote synchronous movement of the jobs by "*synmv*"in the β-field of the traditional three-field notation. We write $O|synmv|f$ for the general synchronous open shop problem with objective function f and $Om|synmv|f$ if the number m of machines is fixed (i.e., not part of the input). While in the conclusions we extend our results to some other objective functions, our research focuses on the problems of minimizing the makespan $O|synmv|C_{\max}$, finding a feasible schedule subject to deadlines $O|synmv, C_j \leq D_j|-$ and minimizing the total completion time $O|synmv|\sum C_j$.

Usually, we assume that every cycle contains exactly m operations, one on each machine. In that case, together with the previously stated assumption $n \geq m$, exactly n cycles are needed to process all jobs. However, sometimes it is beneficial to relax the requirement for exactly m operations per cycle. Then a feasible schedule may contain incomplete cycles, with less than m operations. We denote such a relaxed model by including "*rel*"in the β-field.

This problem can be transformed to a variant of problem $O|synmv|C_{\max}$ by introducing dummy jobs, used to model idle intervals on the machines. Dummy jobs have zero-length operations on all machines, and it is allowed to assign several operations of a dummy job to the same cycle. Thus, in a feasible schedule with dummy jobs, all cycles are complete, but some of the m operations in a cycle may belong to dummy jobs. Similarly to the observation in [85] that introducing incomplete cycles in a synchronous flow shop may be beneficial even for regular objectives, we will show that a schedule for the relaxed problem $O|synmv, rel|C_{\max}$ consisting of more than n cycles may outperform a schedule for the non-relaxed problem $O|synmv|C_{\max}$ with n cycles.

3.2 Related work

Note that an important observation about synchronous flow shop problems is that for two machines the condition of synchronous movement is equivalent to the no-wait or blocking conditions studied previously (see [143, 152]). In the two machine case, in order for a schedule to satisfy the "no-wait" condition a job has to be scheduled on the second machine immediately when it is finished on the first machine. For open shop the roles of the machines may be reversed, if a job starts processing on the second machine. The "blocking" condition means that a job cannot leave the first machine until the second machine is free to process it, i.e. during that time the job remains on the first machine and that machine cannot process subsequent jobs. Again, for open shop the role of the machines may be reversed. For more details on no-wait and blocking, see the survey paper [69].

For synchronous open shop, even for two machines the equivalence with the no-wait condition does no longer hold. In fact, it is shown in [134] that the makespan criterion in two-machine no-wait open shop is strongly NP-hard to solve, while we will see in the next chapter that two-machine synchronous open shop is solvable in linear time, after pre-sorting.

On the other hand there exist several problems in previous research which are very closely related or even equivalent to (variants of) the synchronous open shop problem. In this brief review we focus on the two most important and most closely related problems, since other related problems can usually be modelled in a similar fashion. The first problem, historically, arises from scheduling satellite communication in TDMA systems (time division multiple access). It has been under study since the late 70s and still receives attention, see, e.g., [23, 32, 64, 76, 82, 126, 127]. There are many formulations and results for different versions of the problem, some closely related to synchronous open shop scheduling, while others do not appear to be very similar.

For our purposes, in an instance of the problem a directed bipartite graph $G = (V_1 \cup V_2, E)$ is given where the vertex sets V_1 and V_2 correspond to senders and receivers and the edges $E \subset V_1 \times V_2$ model transmissions. We assume $|V_1| = m$, $|V_2| = n$ with $m \leq n$. For any transmission $(i, j) \in E$ from sender $i \in V_1$ to receiver $j \in V_2$, there is given a transmission time t_{ij}. Each sender can send at most one message at any time, and each receiver can receive at most one message at any time, while independent senders/receivers can perform transmissions in parallel. Switches in traffic are made simultaneously, so that a feasible solution can be characterized by a set of periods, each of which does not involve the same sender or the same receiver more than once. Hence, each period is described by at most m pairs (i, j) denoting messages from senders $i \in V_1$ to receivers $j \in V_2$ that can be sent simultaneously. A feasible solution consisting of κ periods is a partition $E_1 \cup E_2 \cup \cdots \cup E_\kappa$ of the edge set E such that no two edges of the same set E_q are incident to the same vertex. The duration of a period $q \in \{1, \ldots, \kappa\}$ corresponds to the longest transmission of that period, i.e.

$$w(E_q) = \max\{t_{ij} \mid (i, j) \in E_q\},$$

and the total transmission time is equal to $\sum_{q=1}^{\kappa} w(E_q)$.

It is easy to see that the described version of the problem is equivalent with synchronous open shop if $E = V_1 \times V_2$ (or if this is not the case we can add a zero-weight edge for every edge missing in G). Indeed, exchanging the words "sender" and "receiver" for "machines" and "jobs", the word "message/transmission" for "operation" and the word "period" for "cycle", we obtain the definition of (relaxed) synchronous open shop. If we additionally require $\kappa = n$ then we have the non-relaxed version of synchronous open shop, with exactly n cycles.

Another problem that is closely related to both synchronous open shop and the problem from satellite communication is the max-weight edge coloring problem (MEC). In problem MEC a graph $G = (V, E)$ is given, with vertex set V and edge set E. A feasible edge coloring of G with κ colors is a partition $E_1 \cup E_2 \cup \cdots \cup E_\kappa$ of the edge set E with an assignment of a color c to every subset E_c, $1 \leq c \leq \kappa$, such that no two edges of the same color c are incident to the same vertex. Additionally, the edges $e \in E$ have weights $w(e)$, and the weight of a feasible edge coloring is defined as $\sum_{c=1}^{\kappa} w(c)$, where $w(c)$ is the maximum weight among the edges that have

color c,

$$w(c) = \max\{w(e) \mid e \in E_c\}. \tag{3.1}$$

The objective is to find an edge coloring of minimum weight. Note that when introduced in this way, the number of colors is not of importance and we are allowed to use a greater than necessary number of colors if this improves the objective value.

Clearly, synchronous open shop with m machines and n jobs and the problem from satellite communication with m senders and n receivers can be modelled as MEC on complete bipartite graphs, where each color defines a different cycle/period. Problem MEC has attracted considerable attention of researchers, see e.g., [40, 45, 108, 109, 112]. The related max-weight vertex coloring problem (MVC) was studied in [40, 43, 54]. Since the problem MEC on a graph G is equivalent to coloring the vertices of the line graph $L(G)$, any algorithm for MVC can also be applied to MEC.

The most important results we obtain from the research in satellite communication and MEC are the NP-hardness results from [43] and [126], formulated for MEC, which imply that problems $O|synmv|C_{\max}$ and $O|synmv, rel|C_{\max}$ are strongly NP-hard if both n and m are part of the input. Moreover, using improved NP-hardness results results from [40], [43], [54], [82], and [112], formulated for MEC on cubic bipartite graphs, we conclude that these two open shop problems remain strongly NP-hard even if each job has non-zero processing time on at most three machines, and if there are only three different values for non-zero processing times.

Observation 1. The problems $O|synmv|C_{\max}$ and $O|synmv, rel|C_{\max}$ are strongly NP-hard, even if each job has non-zero processing time on at most three machines, and if there are only three different values for non-zero processing times.

In the area related to satellite communication many heuristics and exact branch-and-bound methods were studied (e.g. [32, 127]), which we do not discuss in detail as our focus is on complexity study. Another string of research from that area is related to what might be called the preemptive version of the satellite communication problem. Here, a single message is allowed to be split up over several time periods, which are not necessarily in sequence, see [23] and [76]. In that case, the number of time periods is larger than n, but not because some machines are idle, as in the relaxed version, but because additional preemptions were introduced. If the number of preemptions is allowed to be arbitrary, the problem is polynomially solvable and the maximum number of time periods needed for an optimal solution is $n^2 - 2n + 2$ [23]. On the other hand, if the number of periods (increased due to preemptions) is restricted to $O(n)$, then the problem stays strongly NP-hard [23]. Note that the paper [23] focuses on the case were $m = n$, but if $m < n$ we can add additional senders without any messages to send in order to obtain the case $m = n$.

These results can be transferred to the synchronous open shop problem in the case where preempting operations is allowed. We conclude that synchronous open shop with the makespan criterion is polynomially solvable if we are allowed to preempt arbitrarily many times. Another

way to see this is by solving first problem $O|pmtn|C_{\max}$ (which is polynomially solvable, see [31, 62]) and then adding more preemptions in order to achieve synchronization of the schedule.

Interesting results on MEC in complete bipartite graphs focus mostly on approximation and inapproximability. For bipartite graphs, there is an 1.74-approximation algorithm for problem MEC [108], which can be extended, with the same ratio, to general graphs. The ratio can be reduced to $\frac{7}{6}$ if the bipartite graph has maximum degree 3 [40]. It is also shown in [40] that a ratio less than $\frac{7}{6}$ cannot be achieved for cubic bipartite graphs even if $w(e) \in \{1, 2, 3\}$ for all edges $e \in E$, unless P = NP. Again the results can be transferred immediately to synchronous open shop scheduling.

Taking into account these previous results, our contributions in the following chapters focus on two areas that have not yet been explored. Firstly, we are interested in cases were the number of machines m is fixed, especially $m = 2$, which appear to be ignored by previous research, both on satellite communication and the MEC problem. Assuming the number of machines to be predetermined rather than part of the input is a natural assumption for synchronous open shop scheduling. Secondly, we investigate optimization objectives other than the makespan. To the best of our knowledge research on satellite does not consider other standard scheduling objectives. Note also that without any additional modelling the MEC problem is ill-suited for the consideration of other objectives. For example it would have to be made clear how to group the edges as "jobs" and how to define the completion time of such a "job".

The main contribution of this part is a complete complexity study for (non-preemptive) synchronous open shop scheduling, as discussed in Section 7.1.

Chapter 4

Synchronous Open Shop Scheduling with Two Machines

This chapter is organized as follows. In Section 4.1, we consider problem $O2|synmv|C_{\max}$ and its link to the two-dimensional linear assignment problem. We establish a new structural property of an optimal solution and based on it we formulate an $O(n)$-time solution algorithm, assuming jobs are pre-sorted on each machine. Then we address problem $O2|synmv, rel|C_{\max}$ and provide a tight bound on the maximum number of cycles needed to get an optimal solution. In Section 4.2 we show that problem $O2|synmv, C_j \leq D_j|-$ is strongly NP-hard and that it is at least NP-hard in the ordinary sense even if there are only two different deadlines. This result implies NP-hardness for all due date related objectives. Finally, in Sections 4.3 and 4.4 we show that problem $O2|synmv|\sum C_j$ is strongly NP-hard.

4.1 Minimizing the makespan

In this section, we consider synchronous open shop problems with the makespan objective for two machines A and B. Note that the more general version with $m \geq 2$ machines is studied in Chapter 5, as application for the underlying assignment problem. The reason for this separation is that this section is focused on the scheduling problem, while the focus in Chapter 5 as a whole is on the assignment problem, for which synchronous open shop is only one of several applications.

In the first part of the section we use a weaker version of the underlying assignment problem from Chapter 5 to show that problem $O2|synmv|C_{\max}$ is solvable in linear time, after pre-sorting. The proof, while not as general as the one in Chapter 5, is significantly easier and the resulting structural property stronger.

The second part of the section is dedicated to the relaxed problem $O2|synmv, rel|C_{\max}$. We provide examples where for the relaxed version a schedule with incomplete cycles outperforms

a schedule were all cycles are complete (i.e. have one operation on each machine). Furthermore we show that we need at most $n+1$ cycles in an optimal schedule for the relaxed problem and provide necessary conditions that hold whenever an additional cycle is needed.

4.1.1 Problem $O2|synmv|C_{\max}$

Problem $O2|synmv|C_{\max}$ can be naturally modelled as an assignment problem (2.1). Consider two non-increasing sequences of processing times of the operations on machines M_1 and M_2, renumbering the jobs in accordance with the sequence on M_1:

$$p_{11} \geq p_{12} \geq \ldots \geq p_{1n},\ p_{2k_1} \geq p_{2k_2} \geq \ldots \geq p_{2k_n}. \tag{4.1}$$

To simplify the notation, let $(a_i)_{i=1}^n$ and $(b_j)_{j=1}^n$ be the corresponding sequences of processing times in non-increasing order. The i-th operation on M_1 with processing time a_i and the jth operation on M_2 with processing time b_j can be paired in a cycle with cycle time $\max\{a_i, b_j\}$ if these two operations are not associated with the same job. Let $\mathcal{F} = \{(1, j_1), (2, j_2), \ldots, (n, j_n)\}$ be the set of forbidden pairs: $(i, j_i) \in \mathcal{F}$ if operations O_{1i} and O_{2j_i} belong to the same job.

Using binary variables x_{ij} to indicate whether the i-th operation on M_1 and the jth operation on M_2 (in the above ordering) are paired in a cycle, the problem can be formulated as the following variant of the assignment problem:

$$\begin{array}{lll} \text{AP}_{\mathcal{F}}\text{:} & \min & \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} w_{ij}x_{ij} \\ & \text{s.t.} & \sum\limits_{i=1}^{n} x_{ij} = 1, \qquad 1 \leq j \leq n, \\ & & \sum\limits_{j=1}^{n} x_{ij} = 1, \qquad 1 \leq i \leq n, \\ & & x_{ij} \in \{0,1\}, \quad 1 \leq i,j \leq n, \\ & & x_{ij} = 0, \qquad (i,j) \in \mathcal{F}, \end{array}$$

with the cost matrix $\mathcal{W} = (w_{ij})$, where

$$w_{ij} = \max\{a_i, b_j\}, \quad 1 \leq i,j \leq n. \tag{4.2}$$

Due to the pre-defined 0-variables $x_{ij} = 0$ for forbidden pairs of indices $(i,j) \in \mathcal{F}$ it is prohibited that two operations of the same job are allocated to the same cycle.

Note that in Chapter 5 a slightly different formulation is used to model synchronous open

shop as an assignment problem:

$$\begin{array}{lll} \mathrm{AP}_\infty\colon & \min & \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} c_{ij}x_{ij} \\ & \text{s.t.} & \sum\limits_{i=1}^{n} x_{ij} = 1, \quad 1 \le j \le n, \\ & & \sum\limits_{j=1}^{n} x_{ij} = 1, \quad 1 \le i \le n, \\ & & x_{ij} \in \{0,1\}, \quad 1 \le i,j \le n, \end{array}$$

with the cost matrix $\mathcal{C} = (c_{ij})$, where for $1 \le i,j \le n$

$$c_{ij} = \begin{cases} \max\{a_i, b_j\}, & \text{if } (i,j) \notin \mathcal{F}, \\ \infty, & \text{if } (i,j) \in \mathcal{F}. \end{cases} \tag{4.3}$$

Here, for the forbidden pairs $(i,j) \in \mathcal{F}$ there are ∞-entries in the cost matrix, one in every row and every column. A feasible solution of the open shop problem and $\mathrm{AP}_\mathcal{F}$ exists if and only if the optimal solution value of AP_∞ is less than ∞.

As will be discussed in greater detail in Chapter 5 (Section 5.4) and at the end of this section, problem AP_∞ is more general than problem $\mathrm{AP}_\mathcal{F}$ in cases were the costs are not necessarily of type (4.2) and (4.3). However, due to its less general nature formulation $\mathrm{AP}_\mathcal{F}$ allows us to produce stronger results, see Theorems 3 and 7. The main advantage of formulation $\mathrm{AP}_\mathcal{F}$ is the possibility to use finite w-values for all pairs of indices, including w_{ij}'s defined for forbidden pairs $(i,j) \in \mathcal{F}$.

Example 2. Consider an example with $n = 4$ jobs and the following processing times:

j	1	2	3	4
p_{1j}	7	5	3	2
p_{2j}	3	4	6	2

The sequences (a_i) and (b_j) of processing times are of the form:

i	1	2	3	4
a_i	7	5	3	2
Job	J_1	J_2	J_3	J_4

j	1	2	3	4
b_j	6	4	3	2
Job	J_3	J_2	J_1	J_4

The forbidden pairs are $\mathcal{F} = \{(1,3),(2,2),(3,1),(4,4)\}$, the associated matrices $\mathcal{W}$ and $\mathcal{C}$ are

$$\mathcal{W} = \left(\begin{array}{c|cccc} i \setminus j & 1 & 2 & 3 & 4 \\ \hline 1 & 7 & \mathbf{7} & 7 & 7 \\ 2 & \mathbf{6} & 5 & 5 & 5 \\ 3 & 6 & 4 & 3 & \mathbf{3} \\ 4 & 6 & 4 & \mathbf{3} & 2 \end{array}\right), \quad \mathcal{C} = \left(\begin{array}{c|cccc} i \setminus j & 1 & 2 & 3 & 4 \\ \hline 1 & 7 & \mathbf{7} & \infty & 7 \\ 2 & \mathbf{6} & \infty & 5 & 5 \\ 3 & \infty & 4 & 3 & \mathbf{3} \\ 4 & 6 & 4 & \mathbf{3} & \infty \end{array}\right).$$

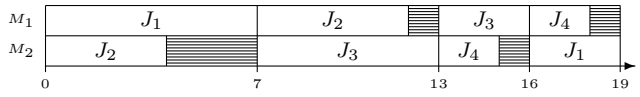


Figure 4.1: Gantt chart of an optimal schedule for Example 2

The entries in bold font in $\mathcal{W}$ and $\mathcal{C}$ correspond to the optimal solution illustrated in Fig. 4.1. Here, $x_{12} = 1$ for the pair of jobs J_1, J_2 assigned to the same cycle, and $x_{23} = x_{34} = x_{41} = 1$ for the other cycles. The makespan is $7 + 6 + 3 + 3 = 19$. ∎

It is well-known (cf. [13], [26]) that matrix $\mathcal{W} = (w_{ij})$ defined by (4.2) satisfies the Monge property (2.3), i.e., for all row indices $1 \leq i < r \leq n$ and all column indices $1 \leq j < s \leq n$ we have

$$w_{ij} + w_{rs} \leq w_{is} + w_{rj}.$$

Without the additional condition on forbidden pairs $\mathcal{F}$, a greedy algorithm finds an optimal solution $\mathbf{X} = (x_{ij})$ to the assignment problem and that solution is of the diagonal form:

$$x_{ii} = 1 \text{ for } i = 1, \ldots, n; \quad x_{ij} = 0 \text{ for } i \neq j. \tag{4.4}$$

Forbidden pairs or ∞-entries may keep the Monge property satisfied so that the greedy algorithm remains applicable, as discussed by [26] and [124]. However, if at least one of the forbidden pairs from $\mathcal{F}$ is a diagonal element, then solution (4.4) is infeasible for problem $\mathrm{AP}_{\mathcal{F}}$. A similar observation holds for problem AP_{∞} if an ∞-entry lies on the diagonal.

As demonstrated in Chapter 5, there exists an optimal solution $\mathbf{X}$ for problem AP_{∞}, which satisfies a so-called *corridor property*: the 1-entries of $\mathbf{X}$ belong to a *corridor* around the main diagonal of width 2, so that for every $x_{ij} = 1$ of an optimal solution the condition $|i - j| \leq 2$ holds. Notice that in Example 2 there are two forbidden pairs in $\mathcal{F}$ of the diagonal type, $(2, 2)$ and $(4, 4)$; the specified optimal solution satisfies the corridor property. A related term used typically in two-dimensional settings is the bandwidth (see, e.g., [35]).

The corridor property is proved in the next chapter in its generalized form for the case of the m-dimensional assignment problem with a nearly Monge array (this is an array where ∞-entries are allowed and the Monge property has to be satisfied by all finite entries). As presented in detail there, the m-dimensional version of the assignment problem models the m-machine synchronous open shop problem. It appears that for the case of $m = 2$ the structure of an optimal solution can be characterized in a more precise way, which makes it possible to develop an easier solution algorithm.

In the following, we present an alternative characterization of optimal solutions for $m = 2$ and develop an efficient algorithm for constructing an optimal solution. Note that the arguments in Chapter 5 are presented with respect to problem AP_{∞}; in this chapter our arguments are based on formulation $\mathrm{AP}_{\mathcal{F}}$ and on its relaxation $\mathrm{AP}_{\mathcal{F}=\emptyset}$, with the condition "$x_{ij} = 0$ for $(i, j) \in \mathcal{F}$" dropped.

A *block* $\mathbf{X}_h$ of size s is a square submatrix consisting of $s \times s$ elements with exactly one 1-entry in each row and each column of $\mathbf{X}_h$. We call a block *large* if it is of size $s \geq 4$, and *small* otherwise. Our main result is establishing a block-diagonal structure of an optimal solution $\mathbf{X} = (x_{ij})$,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \mathbf{0} & \dots & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{X}_{z-1} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{X}_z \end{pmatrix} \tag{4.5}$$

with blocks $\mathbf{X}_h$, $1 \leq h \leq z$, of the form

$$\begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.6}$$

around the main diagonal, and 0-entries elsewhere. Note that the submatrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is excluded from consideration.

Theorem 3. ("Small Block Property"): *There exists an optimal solution to problem* $\mathrm{AP}_{\mathcal{F}}$ *in block-diagonal structure, containing only blocks of type (4.6).*

The proof of the small block property

The general idea of the proof can be described as follows. Starting with an arbitrary optimal solution which does not satisfy the small block property, replace repeatedly each block of size $s \geq 4$ by two blocks of cumulative size s, one of which is small. The replacement is performed for the relaxed problem $\mathrm{AP}_{\mathcal{F}=\emptyset}$, ignoring forbidden pairs $\mathcal{F}$, but making sure that the cost of the new solution (possibly infeasible in terms of $\mathcal{F}$) is not larger than that of its predecessor. Additionally, we keep 0-entries on the main diagonal unchanged, so that no new blocks of size 1 are created. As a result a new solution is constructed, feasible for $\mathrm{AP}_{\mathcal{F}=\emptyset}$, of no higher cost than the original one, consisting of small blocks only. If the constructed solution is infeasible for $\mathrm{AP}_{\mathcal{F}}$, then at the next stage infeasible blocks of size 2 and 3 are replaced by feasible blocks, also without increasing the cost, achieving an optimal solution consisting of small blocks.

First we prove the possibility of block splitting (Lemma 4), and then explain how infeasible blocks can be converted into feasible ones (Lemmas 5 and 6 for blocks of size 2 and 3, respectively). It leads to the main result (Theorem 3) – the existence of an optimal solution consisting

of small blocks of type (4.6).

For a block consisting of 1-entries in rows and columns $\{j_1, j_1+1, \ldots, j_1+s-1\}$, renumber those rows and columns as $\{j_1, j_2, \ldots, j_s\}$ with $j_i = j_1 + i - 1$, $1 \le i \le s$. The cost associated with block $\mathbf{X}_h$ is defined as

$$w(\mathbf{X}_h) = \sum_{u=1}^{s}\sum_{v=1}^{s} w_{j_u j_v} x_{j_u j_v},$$

so that the total cost of solution $\mathbf{X}$ with blocks (4.5) is

$$w(\mathbf{X}) = \sum_{h=1}^{z} w(\mathbf{X}_h).$$

Lemma 4. *If an optimal solution to problem* $\mathrm{AP}_{\mathcal{F}}$ *contains a block* $\mathbf{X}_y$ *of size* $s > 3$*, defined over rows and columns* $\{j_1, j_2, \ldots, j_s\}$*, then without increasing the cost it can be replaced by two blocks, one block of size* 2 *or* 3 *defined over rows and columns* $\{j_1, j_2\}$ *or* $\{j_1, j_2, j_3\}$*, and one block defined over the remaining rows and columns. Furthermore, if a diagonal entry* $x_{j_k j_k}$ *in the initial solution is* 0*, then in the modified solution* $x_{j_k j_k}$ *is* 0 *as well.*

Proof: Given a solution, we identify the non-zero entries in columns j_1, j_2 and j_3, and denote the corresponding rows by j_a, j_b, j_c. For these indices we have

$$x_{j_a j_1} = 1, \qquad x_{j_b j_2} = 1, \qquad x_{j_c j_3} = 1. \tag{4.7}$$

Furthermore, for non-zero entries in rows j_1, j_2, and j_3, we denote the corresponding columns by j_t, j_u, j_v and have

$$x_{j_1 j_t} = 1, \qquad x_{j_2 j_u} = 1, \qquad x_{j_3 j_v} = 1. \tag{4.8}$$

The proof is presented for the case

$$x_{j_1 j_1} = x_{j_2 j_2} = x_{j_3 j_3} = 0. \tag{4.9}$$

Notice that the case $x_{j_1,j_1} = 1$ contradicts the assumption that block $\mathbf{X}_y$ is large. In the case of $x_{j_2 j_2} = 1$ we replace block $\mathbf{X}_y$ by block $\mathbf{X}'_y$ as shown in Fig. 4.2. Here the 1-entries which are subject to change are enclosed in boxes and * denotes an arbitrary entry, 0 or 1. This transformation involves 4 entries in rows $\{j_1, j_2\}$ and columns $\{j_2, j_t\}$. Notice that the marked 1-entries in the initial block $\mathbf{X}_y$ belong to a diagonal of type $\diagup$, while the marked 1-entries in the resulting block $\mathbf{X}'_y$ belong to a diagonal of type $\diagdown$, so that $w(\mathbf{X}'_y) \le w(\mathbf{X}_y)$ by the Monge property.

In the case of $x_{j_1 j_1} = x_{j_2 j_2} = 0$, $x_{j_3 j_3} = 1$, at least one of the values, a or t, is larger than 3 ($a = 3$ or $t = 3$ is not possible for $x_{j_3 j_3} = 1$; $a \le 2$ and $t \le 2$ is not possible since block $\mathbf{X}_y$ is large). If $t > 3$, then the transformation is similar to that in Fig. 4.2: it involves 4 entries in rows $\{j_1, j_3\}$ and columns $\{j_3, j_t\}$. Alternatively, if $a > 3$, then the transformation involves 4 entries in rows $\{j_3, j_a\}$ and columns $\{j_1, j_3\}$. In either case, the 1-entries in the initial solution

$\mathbf{X}_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_s
j_1	0	**0**	$\cdots$	$\boxed{1}$	$\cdots$	0
j_2	0	$\boxed{1}$	$\cdots$	**0**	$\cdots$	0
j_3	*	0				*
$\vdots$	$\vdots$	$\vdots$		$\ddots$		$\vdots$
j_s	*	0		$\cdots$		*

$\mathbf{X}'_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_s
j_1	0	$\boxed{1}$	$\cdots$	**0**	$\cdots$	0
j_2	0	**0**	$\cdots$	$\boxed{1}$	$\cdots$	0
j_3	*	0				*
$\vdots$	$\vdots$	$\vdots$		$\ddots$		$\vdots$
j_s	*	0		$\cdots$		*

Figure 4.2: Transformation of block $\mathbf{X}_y$ into $\mathbf{X}'_y$

belong to a diagonal of type $\diagup$ and to a diagonal of type $\diagdown$ after the transformation, so that the cost does not increase by the Monge property.

Thus, in the following we assume that condition (4.9) holds.

Case $t = 2$, or equivalently $x_{j_1 j_2} = 1$. This implies $b = 1$. If $a \neq u$, then the transformation from $\mathbf{X}_y$ to $\bar{\mathbf{X}}_y$ shown in Fig. 4.3 creates a small block of size 2 without increasing the cost.

$\mathbf{X}_y$	j_1	$\overset{j_t,}{j_2}$	$\cdots$	j_u	$\cdots$	j_s
j_b, j_1	0	1	$\cdots$	0	$\cdots$	0
j_2	**0**	0	$\cdots$	$\boxed{1}$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	$\boxed{1}$	0	$\cdots$	**0**	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_s	0	0	$\cdots$	0	$\cdots$	*

$\bar{\mathbf{X}}_y$	j_1	$\overset{j_t,}{j_2}$	$\cdots$	j_u	$\cdots$	j_s
j_b, j_1	0	1	$\cdots$	0	$\cdots$	0
j_2	$\boxed{1}$	0	$\cdots$	**0**	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	**0**	0	$\cdots$	$\boxed{1}$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_s	0	0	$\cdots$	0	$\cdots$	*

Figure 4.3: Transformation of block $\mathbf{X}_y$ into $\bar{\mathbf{X}}_y$

Consider the case $a = u$ and notice that we can assume $a = u > 3$. Indeed, cases $a = u = 1$ and $a = u = 2$ cannot happen as the corresponding assignment is infeasible (see Fig. 4.4 (a)

and (b)), and in case $a = u = 3$ the block is already small (see Fig. 4.4 (c)). For $a = u > 3$ the transformation illustrated in Fig. 4.3 is not applicable as it results in a new diagonal entry $x_{j_a j_a} = 1$. Instead, we perform the two transformations from $\mathbf{X}_y$ to $\widetilde{\mathbf{X}}_y$ and then to $\widetilde{\widetilde{\mathbf{X}}}_y$ shown in Fig. 4.5 and 4.6, creating eventually a small block of size 3.

(a)

	$\begin{smallmatrix} j_u \\ j_1 \end{smallmatrix}$	$\begin{smallmatrix} j_t \\ j_2 \end{smallmatrix}$
j_a, j_1	1	1
j_2	1	0

(b)

	j_1	$\begin{smallmatrix} j_t, j_u \\ j_2 \end{smallmatrix}$
j_1	0	1
j_a, j_2	1	1

(c)

	j_1	$\begin{smallmatrix} j_t \\ j_2 \end{smallmatrix}$	$\begin{smallmatrix} j_u \\ j_3 \end{smallmatrix}$
j_1	0	1	0
j_2	0	0	1
j_a, j_3	1	0	0

Figure 4.4: Cases where $t = 2$ and $a = u \leq 3$: (a) $a = u = 1$, (b) $a = u = 2$, (c) $a = u = 3$

Observe that both of the values, c and v, are different from $a = u$. Note further that we have $c \neq 1$ as $t = 2$, $c \neq 2$ as $u > 3$, and $c \neq 3$ due to (4.9). Similarly $v \neq 1$ as $a > 3$, $v \neq 2$ as $t = 2$, and $v \neq 3$ due to (4.9). Thus $c > 3$ and $v > 3$. The relationship between $a = u$ and c is immaterial, as the above transformations work in both cases, $a = u < c$ and $a = u > c$. Similarly, the relationship between $a = u$ and v is immaterial as well. Moreover, the presented transformation works for either case, $c = v$ or $c \neq v$.

$\mathbf{X}_y$	j_1	$\begin{smallmatrix} j_t, \\ j_2 \end{smallmatrix}$	j_3	$\cdots$	$\begin{smallmatrix} j_a, \\ j_u \end{smallmatrix}$	$\cdots$	j_v
j_b, j_1	0	1	0	$\cdots$	0	$\cdots$	0
j_2	0	0	**0**	$\cdots$	$\boxed{1}$	$\cdots$	0
j_3	0	0	0	$\cdots$	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_u, j_a	1	0	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_c	0	0	$\boxed{1}$	$\cdots$	**0**	$\cdots$	0

$\widetilde{\mathbf{X}}_y$	j_1	$\begin{smallmatrix} j_t, \\ j_2 \end{smallmatrix}$	j_3	$\cdots$	$\begin{smallmatrix} j_a, \\ j_u \end{smallmatrix}$	$\cdots$	j_v
j_b, j_1	0	1	0	$\cdots$	0	$\cdots$	0
j_2	0	0	$\boxed{1}$	$\cdots$	**0**	$\cdots$	0
j_3	0	0	0	$\cdots$	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_u, j_a	1	0	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_c	0	0	**0**	$\cdots$	$\boxed{1}$	$\cdots$	0

Figure 4.5: Transformation from $\mathbf{X}_y$ to $\widetilde{\mathbf{X}}_y$

Case $a = 2$ is similar to the case of $t = 2$ since the $\mathbf{X}$-matrices for these two cases are

$\widetilde{\mathbf{X}}_y$	j_1	$\substack{j_t,\\ j_2}$	j_3	$\cdots$	$\substack{j_a,\\ j_u}$	$\cdots$	j_v
j_b, j_1	0	1	0	$\cdots$	0	$\cdots$	0
j_2	0	0	1	$\cdots$	0	$\cdots$	0
j_3	**0**	0	0	$\cdots$	0	$\cdots$	$\boxed{1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_u, j_a	$\boxed{1}$	0	0	$\cdots$	0	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_c	0	0	0	$\cdots$	1	$\cdots$	0

$\widetilde{\widetilde{\mathbf{X}}}_y$	j_1	$\substack{j_t,\\ j_2}$	j_3	$\cdots$	$\substack{j_a,\\ j_u}$	$\cdots$	j_v
j_b, j_1	0	1	0	$\cdots$	0	$\cdots$	0
j_2	0	0	1	$\cdots$	0	$\cdots$	0
j_3	$\boxed{1}$	0	0	$\cdots$	0	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_u, j_a	**0**	0	0	$\cdots$	0	$\cdots$	$\boxed{1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_c	0	0	0	$\cdots$	1	$\cdots$	0

Figure 4.6: Transformation from $\widetilde{\mathbf{X}}_y$ to $\widetilde{\widetilde{\mathbf{X}}}_y$

transposes of each other. Recall that whenever the swaps are done in the case of $t = 2$, the 1-entries on a diagonal of type $\diagup$ become 0-entries, while the 0-entries on a diagonal of type $\diagdown$ become 1-entries, so that the Monge inequality (2.3) is applicable. In the case of $a = 2$, the initial 1-entries in the transpose matrix also belong to a diagonal of type $\diagup$, while the new 1-entries are created on a diagonal of type $\diagdown$.

Case $a > 2$ and $t > 2$ with $a < b$ and $t < u$. Consider the transformation from $\mathbf{X}_y$ to $\widehat{\mathbf{X}}_y$ shown in Fig. 4.7. It uses the Monge property two times, once for the entries in rows $\{j_2, j_a\}$ and columns $\{j_1, j_u\}$, and another time for the entries in rows $\{j_1, j_b\}$ and columns $\{j_2, j_t\}$.

If $a \neq u$ and $b \neq t$, then the resulting matrix $\widehat{\mathbf{X}}_y$ satisfies the conditions of the lemma and a matrix without changed diagonal entries and with a small block of size 2 is obtained.

Consider the case $a = u$ or $b = t$. By the definition of the indices a, b, t, u, according to (4.7)-(4.8), we have $a \neq b$ and $t \neq u$. The latter two conditions, combined with either $a = u$ or $b = t$, imply $a \neq t$ and $b \neq u$.

Then, after $\mathbf{X}_y$ is transformed into $\widehat{\mathbf{X}}_y$, we perform one more transformation from $\widehat{\mathbf{X}}_y$ to $\widehat{\widehat{\mathbf{X}}}_y$ shown in Fig. 4.8. Then, the resulting matrix $\widehat{\widehat{\mathbf{X}}}_y$ satisfies the conditions of the lemma.

Case $a > 2$ and $t > 2$ with $a < b$ and $t > u$. We start with an additional pre-processing step shown in Fig. 4.9 replacing $x_{1t} = x_{2u} = 1$ by 0-entries and $x_{1u} = x_{2t} = 0$ by 1-entries without increasing the cost.

In the resulting matrix $\mathbf{X}^*_y$, we interchange the notation of the columns j_t and j_u in accordance with definition (4.8) and proceed as described above for the case $t < u$. Since $a > 2$ and

$\mathbf{X}_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_u
j_1	0	**0**	$\cdots$	1	$\cdots$	0
j_2	**0**	0	$\cdots$	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	1	0	$\cdots$	0	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	1	$\cdots$	**0**	$\cdots$	0

$\widehat{\mathbf{X}}_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_u
j_1	0	1	$\cdots$	**0**	$\cdots$	0
j_2	1	0	$\cdots$	0	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	**0**	0	$\cdots$	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	**0**	$\cdots$	1	$\cdots$	0

Figure 4.7: Transformation from $\mathbf{X}_y$ to $\widehat{\mathbf{X}}_y$

$\widehat{\mathbf{X}}_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_u
j_1	0	1	$\cdots$	0	$\cdots$	0
j_2	1	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	0	0	$\cdots$	**0**	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	0	$\cdots$	1	$\cdots$	**0**

$\widehat{\widehat{\mathbf{X}}}_y$	j_1	j_2	$\cdots$	j_t	$\cdots$	j_u
j_1	0	1	$\cdots$	0	$\cdots$	0
j_2	1	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	0	0	$\cdots$	1	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	0	$\cdots$	**0**	$\cdots$	1

Figure 4.8: Transformation from $\widehat{\mathbf{X}}_y$ to $\widehat{\widehat{\mathbf{X}}}_y$

therefore $u \neq 1$, no new diagonal entry is produced in the pre-processing.

Case $a > 2$ and $t > 2$ with $a > b$. This case corresponds to the transposed of the picture in the previous case. We undertake a similar pre-processing step as before, to transform this case into one with $a < b$. ∎

$\mathbf{X}_y$	j_1	j_2	$\cdots$	j_u	$\cdots$	j_t
j_1	0	0	$\cdots$	**0**	$\cdots$	$\boxed{1}$
j_2	0	0	$\cdots$	$\boxed{1}$	$\cdots$	**0**
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	1	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	1	$\cdots$	0	$\cdots$	0

$\mathbf{X}_y^*$	j_1	j_2	$\cdots$	j_u	$\cdots$	j_t
j_1	0	0	$\cdots$	$\boxed{1}$	$\cdots$	**0**
j_2	0	0	$\cdots$	**0**	$\cdots$	$\boxed{1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_a	1	0	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
j_b	0	1	$\cdots$	0	$\cdots$	0

Figure 4.9: Transformation from $\mathbf{X}_y$ to $\mathbf{X}_y^*$

Lemma 5. *If a solution* $\mathbf{X}$ *contains an infeasible block of size* 2*, i.e.,* $x_{j_1,j_2} = x_{j_2,j_1} = 1$ *with at least one of the entries* (j_1, j_2) *or* (j_2, j_1) *belonging to* $\mathcal{F}$*, then without increasing the cost it can be replaced by two feasible blocks of size* 1*, given by* $x_{j_1,j_1} = 1$ *and* $x_{j_2,j_2} = 1$*.*

Proof: For the above transformation the cost does not increase due to the Monge property. As far as feasibility is concerned, by the definition of set $\mathcal{F}$, there is exactly one forbidden entry in each row and each column. Thus, if $(j_1, j_2) \in \mathcal{F}$, then neither (j_1, j_1) nor (j_2, j_2) are forbidden. Similar arguments hold for $(j_2, j_1) \in \mathcal{F}$. ∎

Lemma 6. *If a solution* $\mathbf{X}$ *contains an infeasible block of size* 3*, then that block can be replaced, without increasing the cost, by three feasible blocks of size* 1*, or by two feasible blocks, one of size* 1 *and another one of size* 2*.*

Proof: Let $\mathbf{X}_y$ be an infeasible block consisting of rows and columns j_1, j_2 and j_3. The proof is presented for the case

$$x_{j_1 j_1} = x_{j_3 j_3} = 0;$$

otherwise $\mathbf{X}_y$ can be decomposed into smaller blocks. Under the above assumption, $\mathbf{X}_y$ is of one of the three types $\mathbf{X}_y^{(I)}$, $\mathbf{X}_y^{(II)}$ or $\mathbf{X}_y^{(III)}$, shown in Fig. 4.10.

Notice that $\mathbf{X}_y^{(III)}$ can be replaced by $\mathbf{X}_y^{(II)}$ without increasing the cost, using the Monge property, and so we only have to deal with $\mathbf{X}_y^{(I)}$ and $\mathbf{X}_y^{(II)}$, which are symmetric. We first demonstrate that each block, $\mathbf{X}_y^{(I)}$ or $\mathbf{X}_y^{(II)}$, can be replaced by blocks $\mathbf{X}_y^{(a)}$, $\mathbf{X}_y^{(b)}$ or $\mathbf{X}_y^{(c)}$ shown in Fig. 4.11, without increasing the cost. Then we proof that at least one of those blocks is feasible.

$\mathbf{X}_y^{(I)}$	j_1	j_2	j_3
j_1	0	0	1
j_2	1	0	0
j_3	0	1	0

$\mathbf{X}_y^{(II)}$	j_1	j_2	j_3
j_1	0	1	0
j_2	0	0	1
j_3	1	0	0

$\mathbf{X}_y^{(III)}$	j_1	j_2	j_3
j_1	0	0	1
j_2	0	1	0
j_3	1	0	0

Figure 4.10: Blocks $\mathbf{X}_y^{(I)}$, $\mathbf{X}_y^{(II)}$ and $\mathbf{X}_y^{(III)}$

$\mathbf{X}_y^{(a)}$	j_1	j_2	j_3
j_1	1	0	0
j_2	0	0	1
j_3	0	1	0

$\mathbf{X}_y^{(b)}$	j_1	j_2	j_3
j_1	0	1	0
j_2	1	0	0
j_3	0	0	1

$\mathbf{X}_y^{(c)}$	j_1	j_2	j_3
j_1	1	0	0
j_2	0	1	0
j_3	0	0	1

Figure 4.11: Blocks $\mathbf{X}_y^{(a)}$, $\mathbf{X}_y^{(b)}$ and $\mathbf{X}_y^{(c)}$

The transformation of $\mathbf{X}_y^{(I)}$ into $\mathbf{X}_y^{(a)}$ or $\mathbf{X}_y^{(b)}$ involves a quadruple of 1-entries, so that the cost does not increase due to the Monge property. Transforming $\mathbf{X}_y^{(I)}$ into the diagonal solution $\mathbf{X}_y^{(c)}$ we achieve a minimum cost assignment for the Monge submatrix given by rows and columns $\{j_1, j_2, j_3\}$ (see, e.g., [26]). The same arguments hold for the transformation of $\mathbf{X}_y^{(II)}$ into $\mathbf{X}_y^{(a)}$, $\mathbf{X}_y^{(b)}$ or $\mathbf{X}_y^{(c)}$.

In the following, we deal with feasibility. If the initial infeasible block is of type $\mathbf{X}_y^{(I)}$, then at least one of the pairs (j_1, j_3), (j_2, j_1) or (j_3, j_2) is forbidden. Therefore, at least one pair (j_1, j_1) or (j_3, j_3) is feasible. If (j_1, j_1) is feasible, then block $\mathbf{X}_y^{(a)}$ or $\mathbf{X}_y^{(c)}$ is feasible. Similarly, if (j_3, j_3) is feasible, then block $\mathbf{X}_y^{(b)}$ or $\mathbf{X}_y^{(c)}$ is feasible.

Similar arguments can be used if the initial infeasible block is of type $\mathbf{X}_y^{(II)}$. ∎

Combining Lemmas 4-6 and using them repeatedly we arrive at the main result of Theorem 3. Below we present the formal proof.

Proof of Theorem 3: Consider any optimal solution. Apply Lemma 4 repeatedly until all blocks are of size 1, 2, or 3. Since all diagonal 1-entries of the new solution are also present in the original solution, those entries are feasible. Therefore, all blocks of size 1 are feasible. By Lemmas 5-6 all blocks of size 2 or 3 are either feasible or can be converted into feasible blocks without increasing the cost. Thus, the resulting solution has blocks of size 1, 2 and 3, it is feasible, and its cost is not larger than the cost of the original optimal solution.

Finally, the only small blocks that are not of type (4.6), have three 1's on the secondary diagonal, since other configurations of 0's and 1's combine blocks of type (4.6). Due to the

arguments used in the proof of Lemma 6 with respect to block $\mathbf{X}_y^{(III)}$, such blocks can also be eliminated, which concludes the proof. ∎

A linear time dynamic programming algorithm to solve problem $\mathrm{AP}_{\mathcal{F}}$

The small block property leads to an efficient $O(n)$-time dynamic programming algorithm to find an optimal solution. Here we use formulation AP_{∞} rather than $\mathrm{AP}_{\mathcal{F}}$, as infinite costs can be easily handled by recursive formulae. The algorithm enumerates optimal partial solutions, extending them repeatedly by adding blocks of size 1, 2 or 3.

Let S_i denote an optimal partial solution for a subproblem of AP_{∞} defined by the submatrix of $\mathcal{W}$ with the first i rows and i columns. If an optimal partial solution S_i is known, together with solutions S_{i-1} and S_{i-2} for smaller subproblems, then by Theorem 3 the next optimal partial solution S_{i+1} can be found by selecting one of the following three options:

- extending S_i by adding a block of size 1 with $x_{i+1,i+1} = 1$; the cost of the assignment increases by $w_{i+1,i+1}$;

- extending S_{i-1} by adding a block of size 2 with $x_{i,i+1} = x_{i+1,i} = 1$; the cost of the assignment increases by $w_{i,i+1} + w_{i+1,i}$;

- extending S_{i-2} by adding a block of size 3 with the smallest cost:

  (i) $x_{i-1,i+1} = x_{i,i-1} = x_{i+1,i} = 1$ with the cost $w_{i-1,i+1} + w_{i,i-1} + w_{i+1,i}$, or

  (ii) $x_{i-1,i} = x_{i,i+1} = x_{i+1,i-1}$ with the cost $w_{i-1,i} + w_{i,i+1} + w_{i+1,i-1}$.

Let $w(S_i)$ denote the cost of S_i. Then

$$\begin{aligned} w(S_{i+1}) \quad = \min \quad & \{w(S_i) + w_{i+1,i+1}, \\ & w(S_{i-1}) + w_{i,i+1} + w_{i+1,i}, \\ & w(S_{i-2}) + w_{i-1,i} + w_{i,i+1} + w_{i+1,i-1}, \\ & w(S_{i-2}) + w_{i-1,i+1} + w_{i,i-1} + w_{i+1,i}\}\,. \end{aligned} \tag{4.10}$$

The initial conditions are defined as follows:

$$\begin{aligned} & w(S_0) = 0, \\ & w(S_1) = w_{11}, \\ & w(S_2) = \min\{w_{11} + w_{22},\ w_{12} + w_{21}\}\,. \end{aligned}$$

Thus, $w(S_3), \ldots, w(S_n)$ are computed by (4.10) in $O(n)$ time.

Theorem 7. *Problem* $O2|synmv|C_{\max}$ *is solvable in* $O(n)$ *time, after pre-sorting.*

Further implications of the small block property

Concluding this subsection, we provide several observations about the presented results. First, the small block property for problem $O2|synmv|C_{\max}$ has implications for the assignment problem AP_∞ with costs (4.3) and for more general cost matrices. The proof of the small block property is presented for problem $AP_\mathcal{F}$. It is easy to verify that the proof is valid for an arbitrary Monge matrix $\mathcal{W}$, not necessarily of type (4.2); the important property used in the proof requires that the set $\mathcal{F}$ has no more than one forbidden pair (i, j) in every row and in every column, and that all entries of the matrix $\mathcal{W}$, including those corresponding to the forbidden pairs $\mathcal{F}$, satisfy the Monge property. Thus, the small block property and the $O(n)$-time algorithm hold for problem AP_∞ if

(i) there is no more than one ∞-entry in every row and every column of the cost matrix $\mathcal{C}$, and

(ii) matrix $\mathcal{C}$ can be transformed into a Monge matrix by modifying only the ∞-entries, keeping other entries unchanged.

Note that not every nearly Monge matrix satisfying (i) can be completed into a Monge matrix satisfying (ii); see Section 5.4 in the next chapter for further details. However, the definition (4.3) of the cost matrix $\mathcal{C}$ for the synchronous open shop allows a straightforward completion by replacing every entry $c_{ij} = \infty$ by $c_{ij} = \max\{a_i, b_j\}$. While completability is not used in the proof of the more general corridor property presented in Chapter 5, the proof of the small block property depends heavily on the fact that the matrix of the synchronous open shop problem can be completed into a Monge matrix. In particular, we use completability when we accept potentially infeasible blocks in the proof of Lemma 4 and repair them later on with the help of Lemmas 5 and 6. In the literature, the possibility of completing an incomplete Monge matrix (a matrix with unspecified entries) was explored by [41] for the traveling salesman problem. They discuss Supnick matrices, a subclass of incomplete Monge matrices, for which completability is linked with several nice structural and algorithmic properties.

Finally, we observe that while the assignment matrices arising from the multi-machine case are completable in the same way as for the two-machine case (see again the next chapter), it remains open whether this can be used to obtain an improved result for more than two machines as well. The technical difficulties of that case are beyond the scope of this thesis.

4.1.2 Problem $O2|synmv, rel|C_{\max}$

In this section, we consider the relaxed problem $O2|synmv, rel|C_{\max}$ where more than n cycles are allowed, with unallocated (idle) machines in some cycles. Recall that we model idle intervals on machines by introducing dummy jobs with zero-length operations on both machines ($a_j = b_j = 0$ for a dummy job J_j). Given a set of n actual jobs, denote an optimal makespan for problem $O2|synmv|C_{\max}$ without dummy jobs by $C^*_{\max}$, while the optimal makespan for the relaxed problem $O2|synmv, rel|C_{\max}$ with dummy jobs is denoted by $C^r_{\max}$. An example shown

in Fig. 4.12 illustrates that the introduction of dummy jobs can reduce the makespan. In that example, the input data is as follows:

j	1	2	3	4
a_j	5	5	5	1
b_j	5	5	5	1

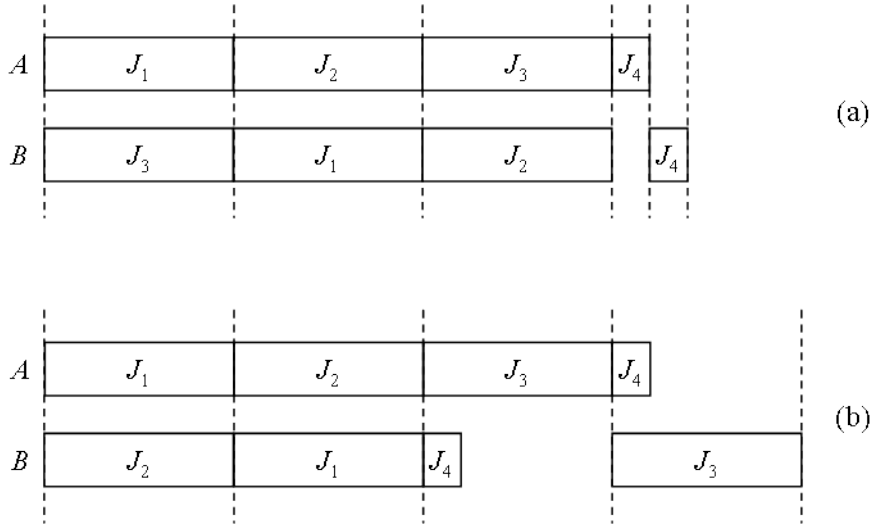


Figure 4.12: *Two open shop schedules, $C^r_{\max} < C^*_{\max}$: (a) An optimal schedule with one dummy job paired with job* 1*; (b) An optimal schedule without dummy jobs.*

Any feasible schedule for problem $O2|synmv|C_{\max}$ is also feasible for $O2|synmv, rel|C_{\max}$, which implies that $C^r_{\max} \leq C^*_{\max}$. If in the optimal schedule S^r for problem $O2|synmv, rel|C_{\max}$ schedule no dummy jobs are used, then $C^r_{\max} = C^*_{\max}$. Below we investigate properties of instances for which $C^r_{\max} < C^*_{\max}$.

Note that whenever in a schedule S for problem $O2|synmv, rel|C_{\max}$ a cycle appears consisting only of dummy operations, we can remove that cycle from the schedule and renumber all dummy operations and dummy jobs in order to end up with a schedule of the same makespan, which uses one less dummy job. Thus we can assume that in any schedule S for problem $O2|synmv, rel|C_{\max}$ there is no cycle consisting only of dummy operations.

We first show that at most one dummy job is needed in order obtain an optimal schedule for problem $O2|synmv, rel|C_{\max}$.

Lemma 8. *There exists an optimal schedule for problem $O2|synmv, dummy|C_{\max}$ with at most one dummy job used, so that the number of cycles is no more than $n + 1$.*

Proof: Suppose in an optimal schedule S^r there are two dummy jobs J_u and J_v, in addition to (possibly) other dummy jobs. Let the four cycles in which those dummy jobs appear be

$\left|{}^{J_u}_{J_{j_1}}\right|$, $\left|{}^{J_{j_2}}_{J_u}\right|$, $\left|{}^{J_v}_{J_{j_3}}\right|$ and $\left|{}^{J_{j_4}}_{J_v}\right|$, where J_{j_1}, J_{j_2}, J_{j_3} and J_{j_4} are the real jobs, not necessarily pairwise different.

If $J_{j_1} \neq J_{j_2}$, then combining the first two cycles $\left|{}^{J_u}_{J_{j_1}}\right|$, $\left|{}^{J_{j_2}}_{J_u}\right|$ into $\left|{}^{J_{j_2}}_{J_{j_1}}\right|$ and eliminating the dummy job J_u results in a schedule S' with the makespan $C'_{\max}$ such that

$$C'_{\max} - C^r_{\max} = \max\{a_{j_2}, b_{j_1}\} - (a_{j_2} + b_{j_1}) \leq 0.$$

Since S^r is optimal, so is S'.

If $J_{j_1} = J_{j_2}$, but $J_{j_3} \neq J_{j_4}$, then a similar transformation is applied to $\left|{}^{J_v}_{J_{j_3}}\right|$, $\left|{}^{J_{j_4}}_{J_v}\right|$ resulting in $\left|{}^{J_{j_4}}_{J_{j_3}}\right|$.

Consider now the case that $J_{j_1} = J_{j_2}$ and $J_{j_3} = J_{j_4}$, i.e., the four cycles are of the form $\left|{}^{J_u}_{J_{j_1}}\right|$, $\left|{}^{J_{j_1}}_{J_u}\right|$, $\left|{}^{J_v}_{J_{j_3}}\right|$, $\left|{}^{J_{j_3}}_{J_v}\right|$. We combine these four cycles into two cycles $\left|{}^{J_{j_1}}_{J_{j_3}}\right|$, $\left|{}^{J_{j_3}}_{J_{j_1}}\right|$, eliminating the dummy jobs J_u and J_v. Denoting the resulting schedule by S'' and its makespan by $C''_{\max}$ we conclude:

$$C''_{\max} - C^r_{\max} = (\max\{a_{j_1}, b_{j_3}\} + \max\{a_{j_3}, b_{j_1}\}) - (a_{j_1} + b_{j_1} + a_{j_3} + b_{j_3}) \leq 0.$$

Again, S'' is optimal due to the optimality of S^r.

This way, in any optimal schedule with at least two dummy jobs, we can reduce the number of dummy jobs by at least one without increasing the makespan. Consequently, an optimal schedule with at most one dummy job must exist. ∎

Note that this means that problem $O2|synmv, rel|C_{\max}$ can be solved in linear time (after sorting the processing times in the order given by (4.1)). Indeed, given an instance of problem $O2|synmv, rel|C_{\max}$ with n jobs, we first add dummy job J_{n+1} with processing times $a_{n+1} = b_{n+1} = 0$ in order to use the results from the last section. Then, similar to the problem without dummy jobs, we solve assignment problem $\mathrm{AP}_{\mathcal{F}}$ with the extended $(n+1) \times (n+1)$ weight matrix given by

$$w_{ij} = \max\{a_i, b_j\}, \quad 1 \leq i, j \leq n+1.$$

and set of forbidden entries $\mathcal{F} = \{(1, j_1), (2, j_2), \ldots, (n, j_n)\}$, where operations $O_{1,i}$ and O_{2,j_i} belong to the same job. Note that compared to the last section, the weight matrix is extended by one line and one column, but the set of forbidden entries remains the same as without the dummy job (we allow for the two operations of a dummy job to be placed in the same cycle, in which case the dummy job is not needed to achieve an optimal schedule).

Later, in Chapter 6, we will see that the result of Lemma 8 is generalizable, in the sense that for problem $O|synmv, rel|C_{\max}$ with m machines an optimal schedule exists in which at most $m-1$ dummy jobs are used, so that the number of cycles is at most $n+m-1$ and we show that the bound is tight for any number m of machines. Observe that in the case of two machines the example in Figure 4.12 suffices to show tightness.

For two machines, we can additionally provide strong necessary conditions for instances in

which optimal schedules for problem $O2|synmv, rel|C_{\max}$ have $n+1$ cycles, i.e. $C^r_{\max} < C^*_{\max}$.

Theorem 9. *If $C^r_{\max} < C^*_{\max}$, i.e. every optimal schedule for problem $O2|synmv, dummy|C_{\max}$ has a dummy job $J_u = J_{n+1}$ and $n+1$ cycles, then the following properties hold.*
(i) The two operations of the dummy job J_u are paired in two cycles with two operations of the same real job.
(ii) The job paired with the dummy job J_u is the one with the smallest processing time on machine A, i.e., job J_n due to numbering (4.1). That job has also the smallest processing time on machine B, i.e., $J_{k_n} = J_n$.
(iii) Job J_n, which is paired with the dummy job, satisfies

$$\begin{array}{l} a_n + b_n < a_j, \text{ for all } j < n, \\ a_n + b_n < b_j, \text{ for all } j < n. \end{array} \tag{4.11}$$

Proof: (i) Consider an optimal schedule S^r with a single dummy job J_u paired with operations of two different jobs: $\left|\begin{smallmatrix} J_u \\ J_{j_1} \end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} J_{j_2} \\ J_u \end{smallmatrix}\right|$, $J_{j_1} \neq J_{j_2}$. Then combining these two cycles into $\left|\begin{smallmatrix} J_{j_2} \\ J_{j_1} \end{smallmatrix}\right|$ and eliminating the dummy job J_u we get a schedule with no larger makespan (as in the case of schedule S' in the proof of Lemma 8), a contradiction to assumption that a dummy job is needed to achieve optimality.

(ii) Suppose S^r is an optimal schedule, $C^r_{\max} < C^*_{\max}$ and the single dummy job J_u paired with J_j, which is different from any shortest job on machine A. Taking into account numbering (4.1) the latter implies that

$$a_j > a_n. \tag{4.12}$$

Recall that due to Property (i) jobs J_u and J_j are paired in both cycles in which they appear.

We combine the three cycles $\left|\begin{smallmatrix} J_u \\ J_j \end{smallmatrix}\right|$, $\left|\begin{smallmatrix} J_j \\ J_u \end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} J_n \\ J_x \end{smallmatrix}\right|$ into $\left|\begin{smallmatrix} J_n \\ J_j \end{smallmatrix}\right|$, $\left|\begin{smallmatrix} J_j \\ J_x \end{smallmatrix}\right|$, where J_x is the job paired with J_n in schedule S^r. Denoting the resulting schedule by $\widetilde{S}$ and its makespan by $\widetilde{C}_{\max}$, we conclude that

$$\widetilde{C}_{\max} - C^r_{\max} = (\max\{a_n, b_j\} + \max\{a_j, b_x\}) - (a_j + b_j + \max\{a_n, b_x\})\,.$$

The case analysis, presented below, demonstrates that the above difference is non-positive. Then schedule $\widetilde{S}$ is optimal due to the optimality of S^d, a contradiction to the assumption that a dummy job is needed to achieve optimality.

Case 1: if $b_x \geq a_j$, then

$$\begin{aligned} \widetilde{C}_{\max} - C^r_{\max} &= (\max\{a_n, b_j\} + b_x) - (a_j + b_j + b_x) \\ &\leq (\max\{a_j, b_j\} + b_x) - (a_j + b_j + b_x) \leq 0, \end{aligned}$$

where the first equality follows from $b_x \geq a_j$ combined with (4.12).

Case 2: if $b_x < a_j$ and in addition $b_x \leq a_n$, then

$$\widetilde{C}_{\max} - C^r_{\max} = (\max\{a_n, b_j\} + a_j) - (a_j + b_j + a_n) \leq 0.$$

Case 3: if $b_x < a_j$ and $b_x > a_n$, then

$$\begin{aligned} \widetilde{C}_{\max} - C^r_{\max} &= (\max\{a_n, b_j\} + a_j) - (a_j + b_j + b_x) \\ &\leq (\max\{b_x, b_j\} + a_j) - (a_j + b_j + b_x) \leq 0. \end{aligned}$$

The second part of Property (ii), stating that the job paired with the dummy job has the smallest processing time on machine B can be proved similarly.

(iii) Let S^r be an optimal schedule and suppose in S^r dummy job J_u is paired with J_n which is a shortest job on machine A and in accordance with Property (ii) also on machine B,

$$b_n \leq b_j \quad \text{for any } j = 1, 2, \ldots, n.$$

Suppose further there exists a job J_h that does not satisfy (4.11). Without loss of generality we assume

$$a_n + b_n \geq a_h. \tag{4.13}$$

Similar to the proof of property (ii), we combine the three cycles $\left|\begin{smallmatrix} J_u \\ J_n \end{smallmatrix}\right|$, $\left|\begin{smallmatrix} J_n \\ J_u \end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} J_h \\ J_x \end{smallmatrix}\right|$ into $\left|\begin{smallmatrix} J_n \\ J_x \end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} J_h \\ J_n \end{smallmatrix}\right|$, where J_x is now the job paired with J_h in schedule S^r. For the resulting schedule $\widehat{S}$ and for the initial schedule S^r, the change in the makespan is given by

$$\widehat{C}_{\max} - C^r_{\max} = (\max\{a_n, b_x\} + \max\{a_h, b_n\}) - (a_n + b_n + \max\{a_h, b_x\}).$$

Using conditions

$$\begin{aligned} \max\{a_n, b_x\} &\leq \max\{a_h, b_x\}, \\ \max\{a_h, b_n\} &\leq a_n + b_n, \end{aligned}$$

where the first one follows from $a_n \leq a_h$, while the second one follows from (4.13), we conclude that

$$\widehat{C}_{\max} - C^r_{\max} \leq (\max\{a_h, b_x\} + \max\{a_h, b_n\}) - (a_n + b_n + \max\{a_h, b_x\}) \leq 0.$$

As before, the resulting optimality of schedule $\widehat{S}$ is in contradiction to the assumption that a dummy job is needed to achieve optimality. ∎

Notice that conditions (4.11) are necessary but not sufficient for $C^r_{\max} < C^*_{\max}$. For example, for the instance given by the table below conditions (4.11) are satisfied, but an optimal schedule without dummy jobs has a smaller makespan than an optimal schedule with a dummy job, see Fig. 4.13 (a)-(b).

j	1	2	3	4
a_j	5	5	3	1
b_j	5	5	3	1

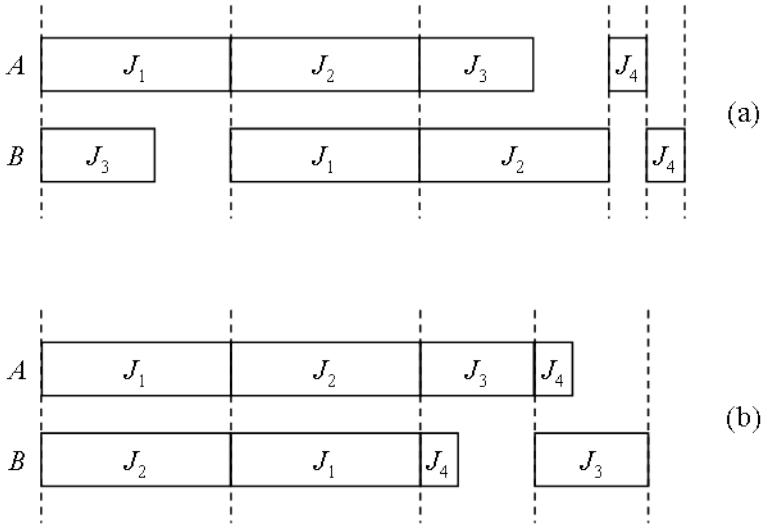


Figure 4.13: *An example illustrating that conditions (4.11) are not sufficient for introducing a dummy job: (a) an optimal schedule with one dummy job; (b) an optimal schedule without dummy jobs.*

4.2 Scheduling with deadlines

In this section, we consider problem $O|synmv, C_j \leq D_j|-$, where each job J_j, $1 \leq j \leq n$, has a given deadline D_j by which it has to be completed. We prove that finding a feasible schedule with all jobs meeting their deadlines is NP-complete in the strong sense even if there are only two machines and each job has only one non-zero processing time. Furthermore, we show that problem $O2|synmv, C_j \leq D_j, D_j \in \{D', D''\}|-$, where the set of all deadlines is limited to two values, is at least NP-complete in the ordinary sense. The proofs presented below are based on the ideas of [18] who established the complexity status of the parallel batching problem with deadlines.

Consider the 3-PARTITION problem (3-PART) known to be strongly NP-complete, cf. [59]. Given a set $Q = \{1, \ldots, 3q\}$, $q \in \mathbb{N}$, a bound $E \in \mathbb{N}$ and natural numbers e_i for every $i \in Q$, satisfying $\sum_{i \in Q} e_i = qE$ and $\frac{E}{4} < e_i < \frac{E}{2}$, can Q be partitioned into q subsets Q_k, $1 \leq k \leq q$, such that $\sum_{i \in Q_k} e_i = E$ for all k?

Based on an instance of 3-PART, we construct an instance $I(q)$ of the two-machine synchronous open shop problem $O2|synmv, C_j \leq D_j|-$ with $n = 6q^2$ jobs, q deadlines and two machines, denoted by A and B. Each job $J_{j,l}$ has two indices j and l to distinguish between

jobs of different types, $j = 1, 2, \ldots, 2q$ and $l = 1, 2, \ldots, 3q$. We introduce constants

$$T = \sum_{i=1}^{3q} i, \qquad T_j = \sum_{i=1}^{j} i, \qquad W = q^3 E.$$

For each l, $1 \le l \le 3q$, the processing times $a_{j,l}$ and $b_{j,l}$ of the jobs $J_{j,l}$ on machines A and B are defined as follows:

$$\begin{array}{ll} a_{j,l} = lW + (q-j)e_l, & b_{j,l} = 0 \text{ for } 1 \le j \le q; \\ a_{q+1,l} = 0, & b_{q+1,l} = lW + qe_l; \\ a_{j,l} = 0, & b_{j,l} = lW \text{ for } q+2 \le j \le 2q. \end{array}$$

The deadlines $D_{j,l}$ are set to

$$\begin{array}{ll} D_{j,l} = jTW + (jq^2 - T_j q + T_j)E & \text{for } 1 \le j \le q; \\ D_{j,l} = qTW + (q^3 - T_q q + T_q)E & \text{for } q+1 \le j \le 2q. \end{array}$$

Throughout the proof we use the following terms for different classes of jobs. Parameter l, $1 \le l \le 3q$, characterizes *jobs of type* l. For each value of l there are $2q$ jobs of type l, q of which have non-zero A-operations (we call these A-jobs) and the remaining q jobs have non-zero B-operations (we call these B-jobs). Among the q B-jobs of type l, there is one *long B-job of type l*, namely $J_{q+1,l}$ with processing time $lW + qe_l$, and there are $q-1$ *short B-jobs of type l*, namely $J_{q+2,l}, J_{q+3,l}, \ldots, J_{2q,l}$, each with processing time lW. Overall, there are $3q$ long B-jobs, one of each type l, $1 \le l \le 3q$, and $3q\,(q-1)$ short B-jobs, with $q-1$ short jobs of each type l. Note that, independent of l, job $J_{j,l}$ is an A-job if $1 \le j \le q$, and a B-job if $q+1 \le j \le 2q$.

With respect to the deadlines, the jobs with non-zero B-operations are indistinguishable. The jobs with non-zero A-operations have deadlines $D_{j,l}$ depending on j; we refer to those jobs as *component$_j$ A-jobs*. For each j, there are $3q$ jobs of that type.

Lemma 10. *If there exists a solution $Q_1, Q_2, \ldots, Q_q$ to an instance of 3-PART, then there exists a feasible schedule for the instance $I(q)$ of the two-machine synchronous open shop problem with q deadlines.*

Proof: We construct a schedule S^* consisting of q components $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$, each of which consists of $3q$ cycles, not counting zero-length cycles. In component Γ_k, $1 \le k \le q$, machine A processes $3q$ component$_k$ A-jobs, one job of each type l, $l = 1, 2, \ldots, 3q$. Machine B processes 3 long B-jobs and $3\,(q-1)$ short B-jobs, also one job of each type l, $l = 1, 2, \ldots, 3q$.

Within one component, every cycle combines an A-job and a B-job of the same type l, $1 \le l \le 3q$. The ordering of cycles in each component is immaterial, but component Γ_k precedes component Γ_{k+1}, $1 \le k \le q-1$. If $Q_k = \{l_1, l_2, l_3\}$ is one of the sets of the solution to 3-PART, then the three long B-jobs $J_{q+1,l_1}, J_{q+1,l_2}, J_{q+1,l_3}$ are assigned to cycle Γ_k.

Finally, there are $3q^2$ cycles of length zero. We assume that each zero-length operation is scheduled immediately after the non-zero operation of the same job. The resulting schedule S^* is shown in Fig. 4.14.

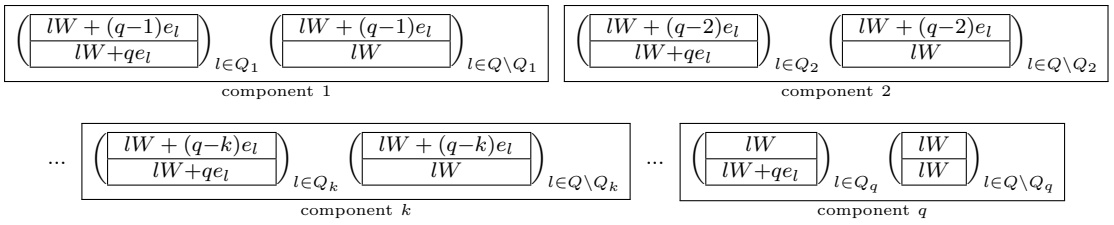


Figure 4.14: Schedule derived from a solution to 3-PART

It is easy to verify that if $Q_1, Q_2, \ldots, Q_q$ define a solution to the instance of 3-PART, then the constructed schedule S^* is feasible with all jobs meeting their deadlines. ∎

We now prove the reverse statement. The proof is structured into a series of properties where the last one is the main result of the lemma.

Lemma 11. *If there exists a feasible schedule S for the instance $I(q)$ of the synchronous open shop problem with q deadlines, then the following properties hold:*

(1) each cycle of non-zero length contains an A-job of type l and a B-job of the same type l, $l = 1, 2, \ldots, 3q$; without loss of generality we can assume that each zero length operation is scheduled in the cycle immediately after the non-zero length operation of the same job;

(2) no component$_j$ A-job is scheduled on machine A before any component$_i$ A-job, with $1 \leq i \leq j-1$; hence S is splittable into components $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$ in accordance with A-jobs;

(3) each component Γ_j, $1 \leq j \leq q$, defines a set Q_j of indices that correspond to long B-jobs scheduled in Γ_j; the resulting sets $Q_1, Q_2, \ldots, Q_q$ define a solution to the instance of 3-PART.

Proof: (1): In a feasible schedule S satisfying the first property, all cycles have a balanced workload on machines A and B: in any component Γ_k, $1 \leq k \leq q$, the cycle lengths are W, $2W$, …, $3qW$, with the value qe_l or $(q-k)e_l$ added. Thus, the total length of such a schedule is at least $q \cdot TW$. For a schedule that does not satisfy the first property, the machine load is not balanced in at least two cycles, so that the lW-part of the processing time does not coincide in these cycles. Thus, the total length of such a schedule is at least $qTW + W = qTW + q^3E$. Since $q > 1$, the latter value exceeds the largest possible deadline

$$\max_{1 \leq j \leq 2q, 1 \leq l \leq 3q} \{D_{j,l}\} = qTW + (q^3 - T_q q + T_q)E,$$

a contradiction.

Note that the above especially shows that zero length operations are only paired in cycles with other zero length operations. Therefore, we can assume without loss of generality that zero length operations are scheduled immediately after the non-zero length operations of the same job. Indeed, if this is not the case, we can change the order of cycles, and possibly the assignment of zero length operations to the zero length cycles in order to achieve the assumed structure, without changing the feasibility of the schedule.

(2): Consider a schedule S in which all component$_u$ A-jobs precede component$_{u+1}$ A-jobs for $u = 1, 2, \ldots, i-1$, but after that a sequence of component$_i$ A-jobs is interrupted by at least one component$_j$ A-job with $j > i$. Let the very last component$_i$ A-job scheduled in S be $J_{i,v}$ for some $1 \leq v \leq 3q$. Then the completion time of the cycle associated with $J_{i,v}$ is at least $iTW + W$, where TW is a lower bound on the total length of all component$_u$ A-jobs, $u = 1, 2, \ldots, i$, and W is the smallest length of a cycle that contains the violating component$_j$ A-job. Since W is large, job $J_{i,v}$ does not meet its deadline

$$D_{i,v} = iTW + (iq^2 - T_i q + T_i)E,$$

a contradiction.

The second property implies that on machine A all component$_1$ A-jobs are scheduled first, followed by all component$_2$ A-jobs, etc. Thus, the sequence of jobs on machine A defines a splitting of the schedule S into components $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$.

(3): Given a schedule S satisfying the first two properties, we first define sets $Q_1, Q_2, \ldots, Q_q$ and then show that they provide a solution to 3-PART.

Schedule S consists of components Γ_j, $1 \leq j \leq q$. In each component Γ_j machine A processes all component$_j$ A-jobs $J_{j,l}$ ($1 \leq l \leq 3q$), each of which is paired with a B-job of the same type l. Recall that a B-job $J_{q+1,l}$ of type l is long, with processing time $lW + qe_l$. All other B-jobs $J_{j,l}$, $q+2 \leq j \leq 2q$, of type l are short, with processing time lW. Considering the long B-jobs of component Γ_j, define a set Q_j of the associated indices, i.e., $l \in Q_j$ if and only if the long B-job $J_{q+1,l}$ is scheduled in component Γ_j. Denote the sum of the associated numbers in Q_j by $e(Q_j) := \sum_{l \in Q_j} e_l$.

The length of any cycle in component Γ_j is either $a_{j,l} = lW + (q-j)e_l$ if the component$_j$ A-job of type l is paired with a short B-job of type l, or $b_{q+1,l} = lW + qe_l$ if it is paired with the long B-job $J_{q+1,l}$. Then the completion time C_{Γ_j} of component Γ_j, $1 \leq j \leq q$, can be calculated as

$$\begin{aligned} C_{\Gamma_j} &= \sum_{h=1}^{j} \left(\sum_{l \in Q_h} [lW + qe_l] + \sum_{l \in Q \setminus Q_h} [lW + (q-h)e_l] \right) \\ &= \sum_{h=1}^{j} (TW + (q-h)qE + he(Q_h)), \end{aligned}$$

which for a feasible schedule S does not exceed the common deadline $D_{j,l}$ of A-jobs in component

Γ_j, $D_{j,l} = jTW + jq^2E - T_jqE + T_jE = \sum_{h=1}^{j}(TW + (q-h)qE + hE)$. Notice that the deadline of any B-job in component Γ_j is not less than $D_{j,l}$. Thus, for any j, $1 \le j \le q$ we get

$$\begin{aligned} D_{j,l} - C_{\Gamma_j} &= \sum_{h=1}^{j}(TW + (q-h)qE + hE) \\ &\quad - \sum_{h=1}^{j}(TW + (q-h)qE + he(Q_h)) \\ &= \sum_{h=1}^{j} h\,(E - e(Q_h)) \ge 0. \end{aligned} \tag{4.14}$$

If all inequalities in (4.14) hold as equalities, i.e.,

$$\sum_{h=1}^{j} h\,(E - e(Q_h)) = 0, \quad j = 1, 2, \ldots, q,$$

then it is easy to prove by induction that $E - e(Q_h) = 0$ for each $h = 1, \ldots, q$ and therefore the partition $Q_1, Q_2, \ldots, Q_q$ of Q defines a solution to 3-PART.

Assume the contrary, i.e., there is at least one strict inequality in (4.14). Then a linear combination L of inequalities (4.14) with strictly positive coefficients has to be strictly positive. Using coefficients $\frac{1}{j} - \frac{1}{j+1}$ for $j = 1, 2, \ldots, q-1$ and $\frac{1}{q}$ for $j = q$ we obtain:

$$L = \sum_{j=1}^{q-1}\left[\left(\frac{1}{j} - \frac{1}{j+1}\right)\sum_{h=1}^{j} h\,(E - e(Q_h))\right] + \frac{1}{q}\sum_{h=1}^{q} h\,(E - e(Q_h)) > 0.$$

It follows that

$$\begin{aligned} 0 < L &= \sum_{h=1}^{q-1}\left[h\,(E - e(Q_h))\sum_{j=h}^{q-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)\right] + \frac{1}{q}\sum_{h=1}^{q} h\,(E - e(Q_h)) \\ &= \sum_{h=1}^{q-1}\left[h\,(E - e(Q_h))\left(\frac{1}{h} - \frac{1}{q}\right)\right] + \frac{1}{q}\sum_{h=1}^{q} h\,(E - e(Q_h)) \\ &= \sum_{h=1}^{q}(E - e(Q_h)) = 0, \end{aligned}$$

where the last equality follows from the definition of E for an instance of 3-PART. The obtained contradiction proves the third property of the lemma. ∎

Lemmas 10 and 11 together imply the following result.

Theorem 12. *Problem $O2|synmv, C_j \le D_j|-$ is NP-complete in the strong sense, even if each job has only one non-zero operation.*

Similar arguments can be used to formulate a reduction from the PARTITION problem (PART) to the two-machine synchronous open shop problem, instead of the reduction from 3-PART. Notice that in the presented reduction from 3-PART all B-jobs have the same deadline, while A-jobs have q different deadlines, one for each component Γ_j defining a set Q_j. In the reduction from PART we only require two different deadlines D, D', one for each of the two sets corresponding to the solution to PART. Similar to the reduction from 3-PART, we define component$_1$ A-jobs with deadline D and component$_2$ A-jobs with deadline D' which define a splitting of the schedule into two components Γ_1, Γ_2. For each of the natural numbers of PART we define one long B-job and one short B-job and show that the distribution of the long jobs within the two components of the open shop schedule corresponds to a solution of PART. Omitting the details of the reduction, we state the following result.

Theorem 13. *Problem $O2|synmv, C_j \leq D_j, D_j \in \{D', D''\}|-$ with only two different deadlines is at least ordinary NP-complete, even if each job has only one non-zero operation.*

At the end of this section we note that the complexity of the relaxed versions of the problems, which allow incomplete cycles modelled via dummy jobs, remains the same as stated in Theorems 12 and 13. Indeed, Property 1 of Lemma 11 stating that each non-zero operation of some job is paired with a non-zero operation of another job, still holds for the version with dummy jobs. Therefore, in the presence of dummy jobs a schedule meeting the deadlines has the same component structure as in Lemmas 10 and 11, so that the same reduction from 3-PART (PART) works for proving that $O2|synmv, rel, C_j \leq D_j|-$ is strongly NP-complete and $O2|synmv, rel, C_j \leq D_j, D_j \in \{D', D''\}|-$ is at least ordinary NP-complete.

4.3 Minimizing the total completion time

In this section, we prove that the synchronous open shop problem with the total completion time objective is strongly NP-hard even in the case of $m = 2$ machines. The proof uses some ideas from [128] in which the NP-hardness of problem $F2|no-wait|\sum C_j$ is proved. Note that the latter problem is equivalent to the synchronous flow shop problem $F2|synmv|\sum C_j$.

For our problem $O2|synmv|\sum C_j$ we construct a reduction from the auxiliary problem AUX, which can be treated as a modification of the HAMILTONIAN PATH problem known to be NP-hard in the strong sense [59].

Consider the HAMILTONIAN PATH problem defined for an arbitrary connected graph $G' = (V', E')$ with $n-1$ vertices $V' = \{1, 2, \ldots, n-1\}$ and edge set E'. It has to be decided whether a path exists which visits every vertex exactly once. To define the auxiliary problem AUX, we introduce a directed graph $\overrightarrow{G}$ obtained from G' in two stages:

- first add to G' a universal vertex 0, i.e., a vertex connected by an edge with every other vertex; denote the resulting graph by $G = (V, E)$;

– then replace each edge of graph G by two directed arcs in opposite directions; denote the resulting directed graph by $\overrightarrow{G} = (V, \overrightarrow{E})$.

For problem AUX it has to be decided whether an Eulerian tour ϵ in $\overrightarrow{G}$ starting and ending at 0 exists where the last n vertices constitute a Hamiltonian path, ending at 0. The two problems HAMILTONIAN PATH and AUX have the same complexity status (the proof is moved to the next section, in order to not distract from the core statements of this section). An example that illustrates graphs G', G and $\overrightarrow{G}$ is shown in Fig. 4.15; a possible Eulerian tour in $\overrightarrow{G}$ is $\epsilon = (0, 1, 0, 2, 0, 3, 0, 4, 2, 4, 3, 2, \boxed{1, 2, 3, 4, 0})$, where the last $n = 5$ vertices form a Hamiltonian path.

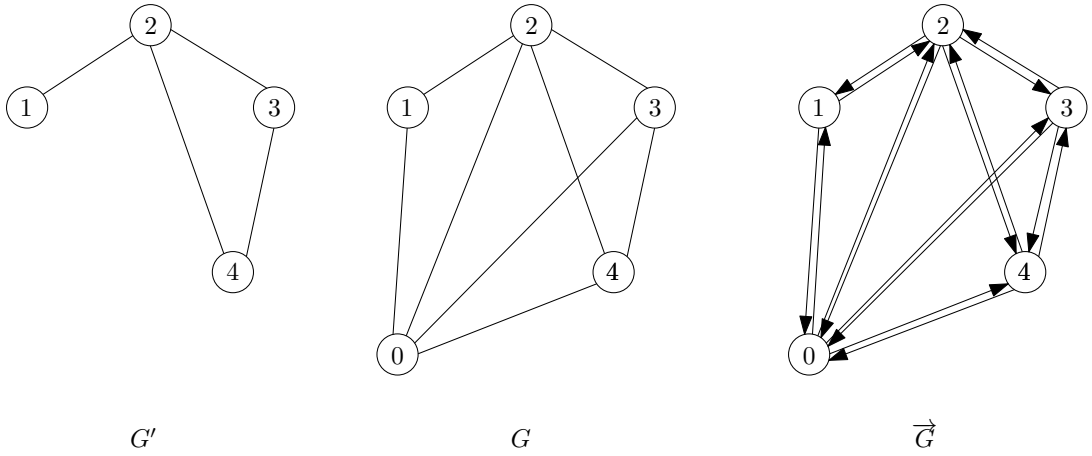


Figure 4.15: Constructing the graph $\overrightarrow{G}$ for problem AUX

Given an instance of AUX with n vertices $V = \{0, 1, \ldots, n-1\}$ and arcs $\overrightarrow{E}$, we introduce an instance of the synchronous open shop problem SO using the constants

$$\sigma = |\overrightarrow{E}|/2,\ K = 8n,\ \xi = 4K\sigma^2,\ L = 2n^9\xi\,.$$

Furthermore, for each vertex $v \in V$ let $d(v) = \deg^-(v) = \deg^+(v)$ be the in-degree $\deg^-_{\overrightarrow{G}}(v)$ (the number of arcs entering v), which here equals its out-degree $\deg^+_{\overrightarrow{G}}(v)$ (the number of arcs leaving v). Note that $\sigma = \sum_{v \in V'} d(v)/2$.

In a possible solution to AUX, if one exists, every vertex $v \in V \backslash \{0\}$ has to be visited $d(v)$ times and each arc $(v, w) \in \overrightarrow{E}$ has to be traversed exactly once. We introduce instance SO for problem $O2|synmv|\sum C_j$ For each vertex v we create $d(v)$ vertex-jobs $\mathrm{Ve}^1_v, \mathrm{Ve}^2_v, \ldots, \mathrm{Ve}^{d(v)}_v$, one for each visit of vertex v in an Eulerian tour ϵ, and for each arc (v, w) we create an arc-job Ar_{vw}. For vertex $v = 0$ we create $d(0)$ vertex-jobs, as described, and additionally one more vertex-job Ve^0_0 that corresponds to the origin of the Eulerian tour ϵ. In addition to these $2\sigma + 1$ vertex-jobs and 2σ arc-jobs, we create $2n^9 + 1$ "forcing" jobs $F_0, F_1, \ldots, F_{2n^9}$ to achieve a special structure of a target schedule. Let N be the of jobs. Their processing times are given in Table 4.1.

	$d(0)+1$ jobs for vertex $v=0$			$d(v)$ jobs for each vertex $v \in \{1,2,\ldots,n-1\}$	
job	Ve_0^0	$\mathrm{Ve}_0^1 \cdots \mathrm{Ve}_0^{d(0)-1}$	$\mathrm{Ve}_0^{d(0)}$	$\mathrm{Ve}_v^1 \cdots \mathrm{Ve}_v^{d(v)-1}$	$\mathrm{Ve}_v^{d(v)}$
time on M_1	0	$\xi + K$	$\xi + K + 1$	$\xi + K - 2v$	$\xi + K - 2v + 1$
time on M_2	ξ	ξ	L	$\xi + 2v$	$\xi + 2v$

	one job for each arc $(v,w) \in \overrightarrow{E}$	forcing jobs	
job	Ar_{vw}	F_0	$F_1 \cdots F_{2n^9}$
time on M_1	$\xi + 2v$	L	L
time on M_2	$\xi + K - 2w$	0	L

Table 4.1: Processing times of the jobs in instance SO

We call each operation with a processing time of L a "long operation" and each operation with a processing time of less than L a "short operation". Further, we refer to a job as a long job if at least one of its operations is long and as a short job if both of its operations are short.

The threshold value of the objective function is defined as $\Theta = \Theta_1 + \Theta_2$, where

$$\begin{aligned} \Theta_1 &= (8\sigma^2 + 2\sigma)\xi + 4\sigma^2 K - 2\sum_{v \in V} v d(v) + n^2, \\ \Theta_2 &= 2(n^9+1)\left(\left(n^9+1\right)L + 4\sigma\xi + 2\sigma K + n\right). \end{aligned}$$

As we show later, in a schedule with $\sum C_j \leq \Theta$, the total completion time of the short jobs is Θ_1 and the total completion time of the long jobs is Θ_2.

Theorem 14. *Problem $O2|synmv|\sum C_j$ is strongly NP-hard.*

Proof: Consider an instance AUX and the corresponding scheduling instance SO. We prove that an instance of problem AUX has a solution, if and only if the instance SO has a solution with $\sum C_j \leq \Theta$.

"⇒": Let the solution to AUX be given by an Eulerian tour $\epsilon = (v_0, v_1, \ldots, v_{2\sigma})$ starting at $v_0 = 0$ and ending at $v_{2\sigma} = 0$ such that the last n vertices form a Hamiltonian path. The solution to problem SO consists of two parts and it is constructed as follows:

- In Part 1 of the schedule, machine M_1 processes $2\sigma + 1$ vertex-jobs and 2σ arc-jobs in the order that corresponds to traversing ϵ. Machine M_2 starts with processing the forcing job F_0 in cycle 1 and then proceeds in cycles $2, 3, \ldots, 4\sigma + 1$ with the same sequence of vertex-jobs and arc-jobs as they appear in cycles $1, 2, \ldots, 4\sigma$ on machine M_1. Notice that in Part 1 all vertex- and arc-jobs are fully processed on both machines except for job $\mathrm{Ve}_0^{d(v)}$ which is processed only on M_1 in the last cycle $4\sigma + 1$.

- In Part 2 of the schedule, machine M_1 processes the forcing jobs $F_0, F_1, \ldots, F_{2n^9}$ in the order of their numbering. Machine M_2 processes in the first cycle of Part 2 (cycle $4\sigma + 2$) the vertex-job $\mathrm{Ve}_0^{d(v)}$ which is left from Part 1. Then in the remaining cycles

$4\sigma+3,\ldots,4\sigma+2+2n^9$, every job F_i $(i=1,\ldots,2n^9)$ on M_1 is paired with job F_{i+1} on M_2 if i is odd, and with job F_{i-1}, otherwise.

In Fig. 4.16 we present an example of the described schedule based on graph $\overrightarrow{G}$ of Fig. 4.15. Notice that there are $n=5$ vertices in $\overrightarrow{G}$, and parameter σ equals 8. Traversing the Eulerian tour $\epsilon=(0,1,0,2,0,3,0,4,2,4,3,2,\boxed{1,2,3,4,0})$ incurs the sequence of vertex-jobs and arc-jobs (Ve_0^0, Ar_{01}, Ve_1^1, Ar_{10}, Ve_0^1, Ar_{02}, …, Ve_1^2, Ar_{12}, Ve_2^4, Ar_{23}, Ve_3^3, Ar_{34}, Ve_3^4, Ar_{40}, Ve_0^4). There are $2\sigma+1=17$ vertex-jobs, $2\sigma=16$ arc-jobs and $2n^9+1=2\cdot 5^9+1$ jobs F_i, so that all jobs are allocated in $34+2\cdot 5^9$ cycles. The schedule is represented as a sequence of cycles, where the operations on machines M_1 and M_2 are enframed and the lengths of the corresponding operations are shown above or below. Operations of equal length in one cycle are shown as two boxes of the same length; the sizes of the boxes of different cycles are not to scale.

We demonstrate that the constructed schedule satisfies $\sum C_j=\Theta$. Observe that most cycles have equal workload on both machines, except for the n cycles that correspond to the vertex-jobs of the Hamiltonian path; in each such cycle the operation on M_1 is one unit longer than the operation on M_2.

First consider the short jobs. The initial vertex-job Ve_0^0 that corresponds to the origin $v_0=0$ of $\epsilon=(v_0,v_1,\ldots,v_{2\sigma})$ completes at time ξ. Each subsequent vertex-job that corresponds to v_i, $1\le i\le 2\sigma-n$, where we exclude the last n vertices of the Hamiltonian path, completes at time $(2i+1)\xi+iK$. Consider the next $n-1$ vertex-jobs v_i with $2\sigma-n+1\le i\le 2\sigma-1$ (excluding the very last vertex-job $\mathrm{Ve}_0^{d(0)}$ as it is a long job); every such job v_i completes at time $(2i+1)\xi+iK+(n+i-2\sigma)$.

The remaining short jobs correspond to arc-jobs. The completion time of the i-th arc-job $\mathrm{Ar}_{v_{i-1}v_i}$ is $2i\xi+iK-2v_i$ for $1\le i\le 2\sigma-n$ and $2i\xi+iK-2v_i+(n+i-2\sigma)$ for $2\sigma-n+1\le i\le 2\sigma$.

Thus, the total completion time of all short jobs sums up to

$$\begin{aligned}
&\xi+\sum_{i=1}^{2\sigma-n}[(2i+1)\xi+iK]+\sum_{i=2\sigma-n+1}^{2\sigma-1}[(2i+1)\xi+iK+(n+i-2\sigma)]\\
&\quad+\sum_{i=1}^{2\sigma-n}[2i\xi+iK-2v_i]+\sum_{i=2\sigma-n+1}^{2\sigma}[2i\xi+iK-2v_i+(n+i-2\sigma)]\\
&=\left[\xi+\sum_{i=1}^{2\sigma-1}(2i+1)\xi+\sum_{i=1}^{2\sigma}2i\xi\right]+\left[\sum_{i=1}^{2\sigma-1}iK+\sum_{i=1}^{2\sigma}iK\right]+\left[\sum_{i=1}^{n-1}i+\sum_{i=1}^{n}i\right]-2\sum_{i=1}^{2\sigma}v_i\\
&=(8\sigma^2+2\sigma)\xi+4\sigma^2K+n^2-2\sum_{v\in V}vd(v)=\Theta_1.
\end{aligned}$$

Here we have used the equality

$$\sum_{i=1}^{2\sigma}v_i=\sum_{v\in V}vd(v)$$

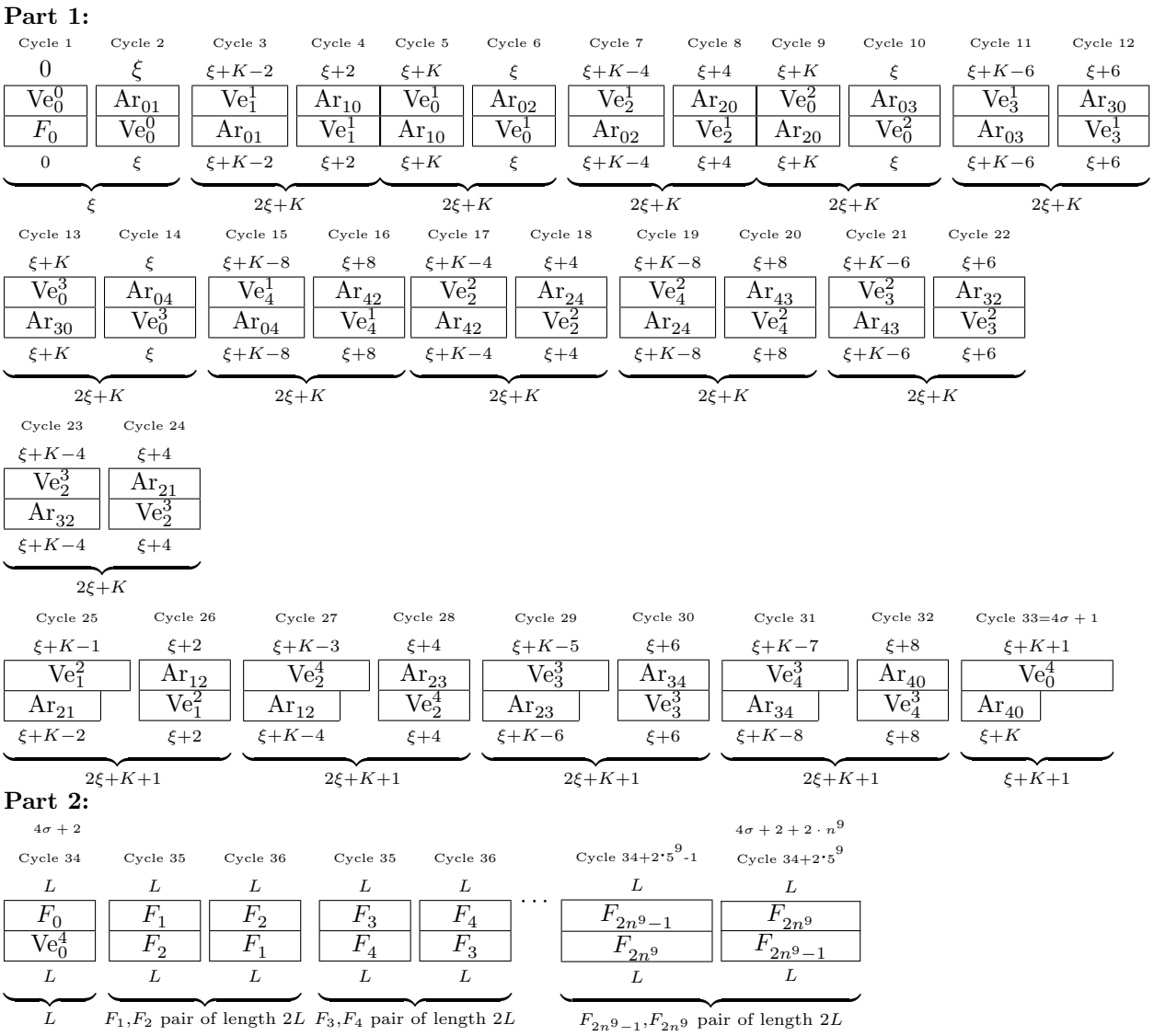


Figure 4.16: An optimal solution to SO

which holds for the Eulerian tour $\epsilon = (v_0, v_1, \ldots, v_{2\sigma})$ with $v_i \in V$.

Next, consider the completion times of the long jobs. The second operations of jobs $\mathrm{Ve}_0^{d(0)}$ and F_0 appear in cycle $4\sigma+2$; all other long operations are scheduled in cycles $4\sigma+3, \ldots, 4\sigma+2+2n^9$. There is a common part of the schedule, with cycles $1, 2, \ldots, 4\sigma+1$ that contributes to the completion time of every long job; the length of that common part is

$$\Delta = 4\sigma\xi + 2\sigma K + n.$$

Then the first two long jobs, F_0 and $\mathrm{Ve}_0^{d(0)}$, are both completed at time $\Delta + L$ and for $i = 1, \ldots, n^9$ the completion time of each pair of jobs F_{2i-1} and F_{2i} is $\Delta + (2i+1)L$. Thus, the total completion time of the long jobs sums up to

$$2(\Delta + L) + 2\sum_{i=1}^{n^9} [\Delta + (2i+1)L] \tag{4.15}$$

$$= 2\Delta(n^9+1) + 2(n^9+1)^2 L \tag{4.16}$$

$$= 2(n^9+1)[(4\sigma\xi + 2\sigma K + n) + (n^9+1)L] = \Theta_2 \tag{4.17}$$

and therefore the total completion time sums up to $\Theta = \Theta_1 + \Theta_2$.

"$\Leftarrow$": Now we prove that if an instance of AUX does not have a solution, then also SO does not have a solution with $\sum C_j \leq \Theta$. Suppose to the contrary that there exists a schedule with $\sum C_j \leq \Theta$ and let S be an optimal schedule. Such a schedule satisfies the following structural properties (the proof is again moved to the next section).

1. In each cycle in S, both operations are either short or long.
2. All long operations are scheduled in the last $2n^9+1$ cycles. This defines the splitting of schedule S into Parts 1 and 2, with cycles $1, 2, \ldots, 4\sigma+1$ and $4\sigma+2, \ldots, 4\sigma+2+2n^9$.
3. The sum of completion times of all long jobs is at least Θ_2.
4. In S, machine M_1 operates without idle times.
5. In Part 1 of S, job Ve_0^0 is processed in the first two cycles which are of the form $\begin{array}{|c|}\hline \mathrm{Ve}_0^0 \\ \hline F_0 \\ \hline \end{array}$ $\begin{array}{|c|}\hline * \\ \hline \mathrm{Ve}_0^0 \\ \hline \end{array}$, where $\boxed{\ *\ }$ represents a short operation. While the order of these two cycles is immaterial, without loss of generality we assume that $\begin{array}{|c|}\hline \mathrm{Ve}_0^0 \\ \hline F_0 \\ \hline \end{array}$ precedes $\begin{array}{|c|}\hline * \\ \hline \mathrm{Ve}_0^0 \\ \hline \end{array}$; otherwise the cycles can be swapped without changing the value of $\sum C_j$.
6. The two operations of each vertex-job and the two operations of each arc-job are processed in two consecutive cycles, first on M_1 and then on M_2.

7. In Part 1 of S, machine M_1 alternates between processing arc-jobs and vertex-jobs. Moreover, an operation of a vertex-job corresponding to v is followed by an operation of an arc-job corresponding to an arc leaving v. Similarly, an operation of an arc-job for arc (v, w) is followed by an operation of a vertex-job for vertex w.
   By Property 6, the same is true for machine M_2 in Part 1 and in the first cycle that follows it.

8. The first arc-job that appears in S corresponds to an arc leaving 0. Among the vertex-jobs, the last one is $\mathrm{Ve}_0^{d(0)}$.

Using the above properties we demonstrate that if problem AUX does not have a solution, then the value of $\sum C_j$ in the optimal schedule S exceeds Θ. Due to Property 3 it is sufficient to show that the total completion time $\sum_{j=1}^{\mu} C_j$ of all short jobs exceeds Θ_1. Let us assume that $\{1, 2, \ldots, \mu\}$ with $\mu = 4\sigma$ are the short jobs of the instance SO. This set consists of 2σ short vertex-jobs (the long vertex-job $\mathrm{Ve}_0^{d(0)}$ is excluded) and 2σ arc-jobs.

Properties 1-2 allow the splitting of S into two parts. Part 2 plays an auxiliary role. Part 1 is closely linked to problem AUX.

The sequence of arc- and vertex-jobs in Part 1 of S defines an Eulerian tour in $\overrightarrow{G}$. Indeed, all arc-jobs appear in S and by Property 7 the order of the arc- and vertex-jobs in S defines an Eulerian trail in $\overrightarrow{G}$. Since for every vertex v, its in-degree equals its out-degree, an Eulerian trail must be an Eulerian tour. Denote it by $\epsilon = (v_0, v_1, \ldots, v_{2\sigma})$. Due to Property 8 and by the assumption of Property 5 the Eulerian tour ϵ starts and ends at $v_0 = 0$.

In Fig. 4.17 we present the structure of Part 1 of schedule S, where Ve_v^* represents one of the vertex-jobs Ve_v^1, Ve_v^2,..., $\mathrm{Ve}_v^{d(v)}$, with processing time $\xi + K - 2v + 1$ or $\xi + K - 2v$ on machine M_1, depending on whether the upper index is $d(v)$ or a smaller number. Part 2 is as in the proof of " $\Rightarrow$".

Part 1:

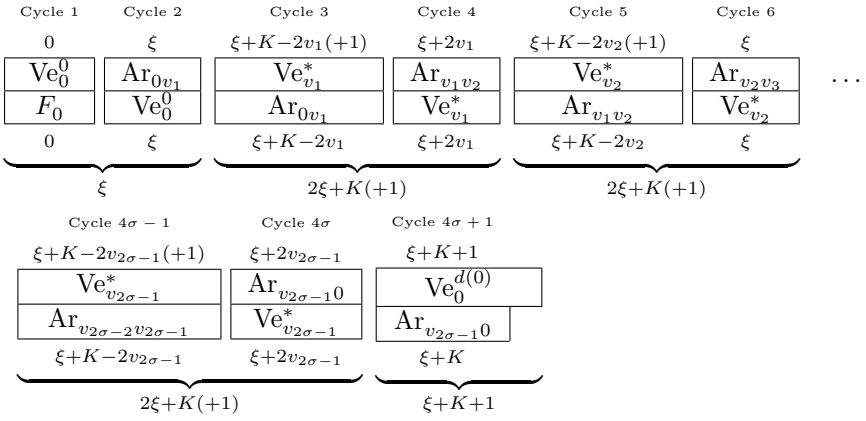


Figure 4.17: Structure of schedule S

Notice that all operations of the short jobs appear only in Part 1 of the above schedule, with one short job completing in each cycle $2, 3, \ldots, \mu+1$. In each cycle of Part 1, both operations are of the same length, except for the $n-1$ cycles where the vertex-jobs $\mathrm{Ve}_v^{d(v)}$, $v \in \{1, 2, \ldots, n-1\}$, are scheduled on machine M_1, and the final cycle of Part 1 where $\mathrm{Ve}_0^{d(0)}$ is scheduled. In these cycles the operation on M_1 is one unit longer than the operation on M_2. Let

$$\vartheta := \left\{ \mathrm{Ve}_v^{d(v)} \mid v = 1, 2, \ldots, n-1 \right\}$$

be the set of the $n-1$ jobs with one extra unit of processing. Job $\mathrm{Ve}_0^{d(0)}$ is not included in this set as its precise location is known by Property 8.

We show that for any Eulerian tour $\epsilon = (v_0, v_1, \ldots, v_{2\sigma})$, the value of $\sum_{j=1}^{\mu} C_j$ does not depend on the order of the vertices in ϵ; it only depends on the positions of the $n-1$ jobs from ϑ. In particular, we demonstrate that

$$\sum_{j=1}^{\mu} C_j = \Upsilon + \sum_{\ell=2}^{\mu+1} (\mu - \ell + 2)\, x_\ell, \tag{4.18}$$

where $\Upsilon = \Theta_1 - n^2$ is a constant, and $x_\ell \in \{0, 1\}$ indicates whether some job from ϑ is allocated to cycle ℓ or not.

The constant term Υ is a lower bound estimate for $\sum_{j=1}^{\mu} C_j$ obtained under the assumption that one additional time unit for each job from ϑ and also for job $\mathrm{Ve}_0^{d(0)}$ is ignored. If we drop "+1" from the input data of the instance SO, then both machines have equal workload in every cycle. Job Ve_0^0 contributes ξ to Υ. The job corresponding to v_i, except for job $\mathrm{Ve}_0^{d(0)}$ (which is a long job) contributes $(2i+1)\xi + iK$. The arc-job corresponding to (v_i, v_{i+1}) contributes $2i\xi + iK - 2v_{i+1}$. Thus,

$$\begin{aligned} \Upsilon &= \xi + \sum_{i=1}^{2\sigma-1} [(2i+1)\xi + iK] + \sum_{i=1}^{2\sigma} [2i\xi + iK - 2v_{i+1}] \\ &= (8\sigma^2 + 2\sigma)\xi + 4\sigma^2 K - 2\sum_{v \in V} v d(v) = \Theta_1 - n^2. \end{aligned}$$

Consider now the effect of the additional time unit on machine M_1 for each job from ϑ and for job $\mathrm{Ve}_0^{d(0)}$. If some ϑ-job is allocated to a cycle ℓ, then the additional unit of processing increases by one the completion time of every short job finishing in cycles $\ell, \ell+1, \ldots, \mu+1$, and thus contributes $(\mu+1) - \ell + 1$ to $\sum_{j=1}^{\mu} C_j$. This justifies formula (4.18).

As shown in the above template, each of the $n-1$ jobs $j \in \vartheta$ can be scheduled in any odd-numbered cycle $\ell \in \{3, 5, \ldots, \mu-1\}$. Also, by Property 8, an additional time unit appears in cycle $\mu+1$ due to the allocation of $\mathrm{Ve}_0^{d(0)}$ to machine M_1, which affects the completion time of a short job in that cycle. Thus, the minimum value of $\sum_{j=1}^{\mu} C_j$ is achieved if all $n-1$ jobs from ϑ are allocated to the latest possible odd-numbered positions, i.e., to positions $\ell = (\mu-1) - 2i$

for $i = 0, 1, \ldots, n-2$. Together with an extra "1" related to the allocation of $\mathrm{Ve}_0^{d(0)}$ to cycle $\mu + 1$, this results in

$$\sum_{\ell=2}^{\mu+1} (\mu - \ell + 2)\, x_\ell = 1 + \sum_{i=0}^{n-2} [\mu - (\mu - 1 - 2i) + 2]$$
$$= 1 + 3 + \cdots + (2n-1) = n^2,$$

so that $\sum_{j=1}^{\mu} C_j$ is equal to Θ_1 if jobs ϑ are allocated to the latest feasible positions. Due to (4.18), any other allocation of jobs ϑ, which does not involve the last $n-1$ odd-numbered positions, results in a larger value of $\sum_{j=1}^{\mu} C_j$.

By the main assumption of the part "$\Leftarrow$", AUX does not have a solution where the last n vertices form a Hamiltonian path. Therefore, the last n vertices of any Eulerian tour $\epsilon = (v_0, v_1, \ldots, v_{2\sigma})$ have at least two occurrences of the same vertex v and therefore in the associated schedule, among the last n vertex-jobs there are at least two vertex-jobs Ve_v^i, Ve_v^j associated with v. Thus, it is impossible to have $n-1$ jobs from ϑ allocated to the last $n-1$ odd-numbered cycles and to achieve the required threshold value Θ_1. ∎

At the end of this section we observe that the proof of Properties 1 – 8 can be adjusted to handle the case with dummy jobs. Indeed, in an optimal solution of the instance, even if we allow dummy jobs, dummy operations are not allowed to be paired with actual operations of non-zero length (see Property 4). We conclude therefore that the complexity status of the relaxed problem is the same as that for the standard one.

Theorem 15. *Problem $O2|synmv, rel|\sum C_j$ is strongly NP-hard.*

4.4 Details for the NP-hardness of $O2|synmv|\sum C_j$

In this section we work out the details of the NP-hardness proof for problem $O2|synmv|\sum C_j$ that were left out in Section 4.3.

We first show that the HAMILTONIAN PATH problem and its modification AUX from the previous section have the same complexity status. Recall the definitions from graph theory introduced in Section 2.1. For completeness, let us briefly repeat the construction of graph $\overrightarrow{G}$ needed for problem AUX. We start out with a connected graph $G' = (V', E')$ with $V' = \{1, 2, \ldots, n-1\}$. To construct the graph $\overrightarrow{G}$ for problem AUX, we first construct the graph $G = (V, E)$ by adding a universal vertex 0 which is connected by an edge with every vertex. In a second step, we construct a directed graph $\overrightarrow{G} = (V, \overrightarrow{E})$ by replacing each edge $\{v, w\} \in E$ by two directed edges (v, w) and (w, v), one in each direction. Note that in $\overrightarrow{G}$ we have $\deg^-_{\overrightarrow{G}}(v) = \deg^+_{\overrightarrow{G}}(v)$ for all $v \in V$ by construction, i.e. $\overrightarrow{G}$ is Eulerian.

Lemma 16. *The following statements are equivalent.*

(1) *There exists a Hamiltonian path in* G'.

(2) *There exists a Hamiltonian path in* G *ending in* 0.

(3) *There exists a directed Hamiltonian path in* $\overrightarrow{G}$ *ending in* 0.

(4) *There exists an Eulerian tour in* $\overrightarrow{G}$*, starting and ending in* 0*, such that the last* n *vertices form a Hamiltonian path.*

Proof: The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We prove that (3) $\Rightarrow$ (4). Suppose there exists a Hamiltonian path $h = (v_0, v_1, \ldots, v_{n-1})$ in $\overrightarrow{G}$ with $v_{n-1} = 0$. Then create a graph $\overrightarrow{G^*}$ by removing from $\overrightarrow{G}$ all arcs that appear in h. By the properties of vertex 0, graph $\overrightarrow{G^*}$ is still strongly connected. Furthermore, for each inner vertex $v_1, v_2, \ldots, v_{n-2}$ of h, its in-degree in $\overrightarrow{G^*}$ still equals its out-degree, as one entering and one leaving edge is removed for each of them. Lastly note that for $v_{n-1} = 0$ we have $\deg^+_{\overrightarrow{G^*}}(0) = \deg^-_{\overrightarrow{G^*}}(0) + 1$, as an arc entering 0 is removed, but not a leaving one. Similarly, for v_0, $\deg^-_{\overrightarrow{G^*}}(v_0) = \deg^+_{\overrightarrow{G^*}}(v_0) + 1$, as an arc leaving v_0 is removed, but not the one entering v_0.

We conclude that $\overrightarrow{G^*}$ contains an Eulerian trail ϵ^*, that starts in 0 and ends in v_0. Then the concatenation $\epsilon = \epsilon^* \circ h$ starts and ends in 0 and its last n vertices form a Hamiltonian path h.

It remains to prove (4) $\Rightarrow$ (1). Let ϵ be an Eulerian tour in $\overrightarrow{G}$, starting and ending in 0, such that the last n vertices form a Hamiltonian path, and let that path be $h = (v_0, v_1, ..., v_{n-1})$ with $v_{n-1} = 0$. Notice that the vertex 0 appears only once in h. Then $h^* = (v_0, v_1, ..., v_{n-2})$ is a path in G. Since h^* consists of all $n-1$ vertices of G, h^* is a Hamiltonian path in G'. ∎

Therefore G' is a yes-instance of the HAMILTONIAN PATH problem if and only if $\overrightarrow{G}$ is a yes-instance for problem AUX and the two problems have the same complexity status.

The remainder of this section deals with Properties 1-8 from the proof of Theorem 14. These properties characterize the structure of an optimal schedule S for instance SO, under the assumption that $\sum C_j \leq \Theta$ for that schedule.

Lemma 17. *An optimal schedule* S *for instance SO with* $\sum C_j \leq \Theta$ *satisfies Properties 1-8.*

Proof: Property 1: In each cycle in S, both operations are either short or long.

Assume the opposite and let s be the first cycle that contains a short and a long operation. Since the number of long operations is even, there is at least one further cycle t which contains operations of both types and in which the long operation is on a different machine than in cycle s. In the following, assume that long operations are on machine M_2 in cycle s and on machine M_1 in cycle t; the alternative case is similar. Let J_s and j_s be the jobs in cycle s, with the operation of J_s being long and the operation of j_s being short. Let J_t and j_t be the jobs in cycle t, with the operation of J_t being long and the operation of j_t being short.

Assume first that $J_s \neq J_t$ and $j_s \neq j_t$. Construct a new schedule S' by swapping the operations on M_2, see Fig. 4.18. Then, the length of cycle s in S' decreases by at least

$L-(\xi+K+1)$ while all other cycles keep their lengths unchanged. The completion time of job J_s either decreases, if its second operation is in cycle after t, or it increases by at most $(\xi+K+1)$ plus the total length of cycles $s+1,\ldots,t$. Thus, $C'_{J_s}-C_{J_s} \leq (t-s-1)L+(\xi+K+1)$. For all other jobs that finish in cycle s or later, the completion times decrease by at least $L-(\xi+K+1)$. As there are at least $t-s+1$ cycles in the tail part of the schedule, starting from cycle s, this affects at least $t-s+1$ jobs. Thus, the difference in total completion time is

$$\begin{aligned}\sum_{j\in N}(C'_j-C_j) &\leq (t-s-1)L+(\xi+K+1)+\sum_{j\in N\setminus\{J_s\}}(C'_j-C_j)\\ &\leq (t-s-1)L+(\xi+K+1)\\ &\quad -(t-s+1)(L-(\xi+K+1))\\ &< -2L+|N|(\xi+K+1),\end{aligned}$$

where $|N|$ is the number of jobs in the instance SO or equivalently the number of cycles. To show that we get an improved solution S', we prove that

$$-2L+|N|(\xi+K+1)<0 \tag{4.19}$$

using the estimate on $\sigma=|\overrightarrow{E}|/2$,

$$n\leq\sigma<n^2 \text{ (for } n>2), \tag{4.20}$$

combined with the definitions of $|N|$, ξ, L and K:

$$K = 8n, \tag{4.21}$$
$$|N| = 2n^9+4\sigma+2\leq 2n^9+4n^2+2, \tag{4.22}$$
$$\xi = 4K\sigma^2=32n\sigma^2<32n^5, \tag{4.23}$$
$$L = 2n^9\xi=2n^9\cdot 32n\sigma^2\geq 64n^{12}>576n^{10} \quad (\text{for } n>3). \tag{4.24}$$

Indeed,

$$\begin{aligned}&-2L+|N|(\xi+K+1)\leq\\ &\quad\leq -2n^9\xi+2n^9(K+1)+(4\sigma+2)(\xi+K+1)\\ &\quad< -576n^{10}+2n^9(8n+1)\\ &\qquad +(4n^2+4)(32n^5+8n+1)\\ &\quad< -576n^{10}+18n^{10}+8n^2\cdot 41n^5<0.\end{aligned}$$

Thus, swapping the operations leads to a smaller total completion time, contradicting that S has minimal total completion time.

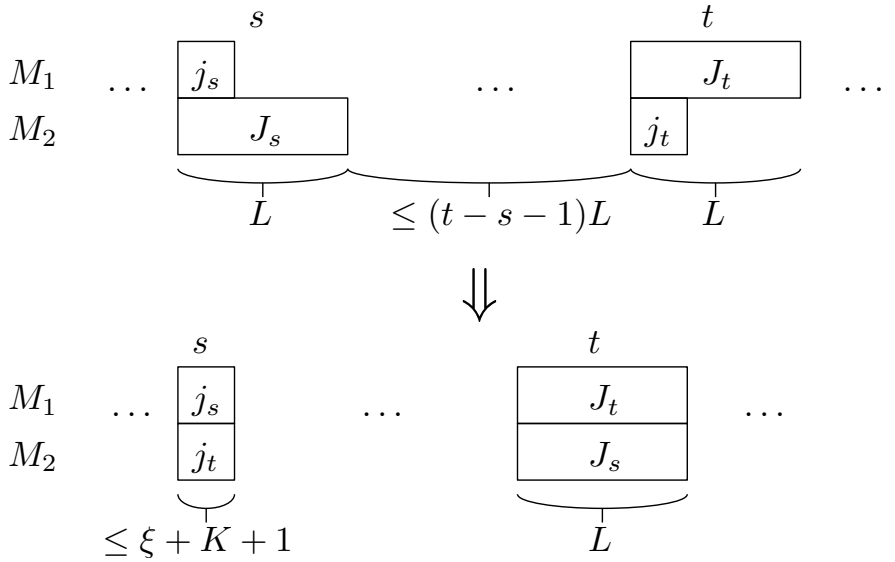


Figure 4.18: Creating a cycle of long operations and a cycle of short operations if $J_s \neq J_t$ and $j_s \neq j_t$

Consider the case $J_s = J_t$, $j_s \neq j_t$ and assume first that there are other long operations scheduled before s. The case where there is no other long operation scheduled prior to cycle s is discussed afterwards. Let r be the last cycle before s in which a long operation is scheduled, $r < s$. Since s is the first cycle that contains a short as well as a long operation, both operations in cycle r are long. In this case construct S' by exchanging in cycles r, s and t the three operations on machine M_2, as shown in Fig. 4.19(a). As a result, the completion time of job J_r either decreases, if the last operation of that job is in a cycle after cycle t, or it increases by at most $(s-r)(\xi+K+1)+(t-s)L$, where the first term represents the estimate on the length of short cycles $r+1,\ldots,s$ in S' and the second term estimates the length of cycles $s+1,\ldots,t$, which may be short or long. For all other jobs, including job J_s, that finish in cycle s or later, their completion times decrease by at least $L-(\xi+K+1)$. Note that there are at least $t-s+2$ such jobs. Thus,

$$\begin{aligned}\sum_{j\in N}(C'_j - C_j) &\leq [(s-r)(\xi+K+1)+(t-s)L] \\ &\quad -(t-s+2)(L-(\xi+K+1)) \\ &= -2L+(t-r+2)(\xi+K+1) < 0,\end{aligned}$$

where the last inequality can be proved as a slight modification of (4.19).

Assume now that there is no long operation scheduled prior to cycle s and let $u > s$ be the first cycle that contains a long operation of some job J_u on the same machine as the long

operation in cycle t. Construct S' by swapping the M_1-operations in cycles t and u and the M_2-operations in cycles s and t, see Fig. 4.19(b) for an example with $u > t$. The completion time of job J_s (or job J_u if $s < u < t$) increases by at most $|u - t|L$. For the remaining jobs scheduled in the tail part of the schedule, starting from cycle s, their completion times reduce by at least $L - (\xi + K + 1)$. Again, using the evaluation from above, this leads to a decrease in the total completion time,

$$\begin{aligned}\sum_{j \in N}(C'_j - C_j) &\leq |u - t|L - (\max\{u,t\} - s + 1)\,(L - (\xi + K + 1)) \\ &= -(\min\{u,t\} - s + 1)\,L \\ &\quad + (\max\{u,t\} - s + 1)\,(\xi + K + 1) \\ &\leq -2L + |N|\,(\xi + K + 1) \overset{(4.19)}{<} 0.\end{aligned}$$

In all previous cases a better schedule S' is constructed by grouping two short operations j_s and j_t in one cycle. Next, consider $j_s = j_t$ and first assume that there are other short operations scheduled in some cycle $r < s$. Since s is the first cycle that contains a short as well as a long operation, both operations in cycle r are short. Construct a schedule S' as illustrated in Fig. 4.20(a). Then, the completion time of job j_s decreases by at least $2(L - (\xi + K + 1))$, as both its operations were paired with long operations before and are now paired with short ones. Further, the completion times of all jobs that were completed in cycles r, $r+1$, $\ldots$, $t-1$ in S, increase by at most $(\xi + K + 1)$ and the completion times of all jobs that are completed in cycle t or later decrease by at least $L - (\xi + K + 1)$. Thus,

$$\begin{aligned}\sum_{j \in N}(C'_j - C_j) &\leq -2(L - (\xi + K + 1)) + (t - r)(\xi + K + 1) \\ &\quad - (L - (\xi + K + 1)) \\ &< -3L + (t - r + 3)(\xi + K + 1) < 0,\end{aligned}$$

where the last inequality can be proved similar to (4.19).

Consider now the case where $r > s$ and assume additionally that both operations in cycle r are short (see Fig. 4.20(b) for an example of $r > t > s$). In this case, the completion times of job j_s and all jobs that finish in cycle s or later, except for job J_s, decrease by at least $L - 2(\xi + K + 1)$. The completion time of job j_s decreases further by the total length Δ of cycles $s+1$, $\ldots$, $t-1$, while the completion time of job J_s may increase by at most $2(\xi + K + 1)$

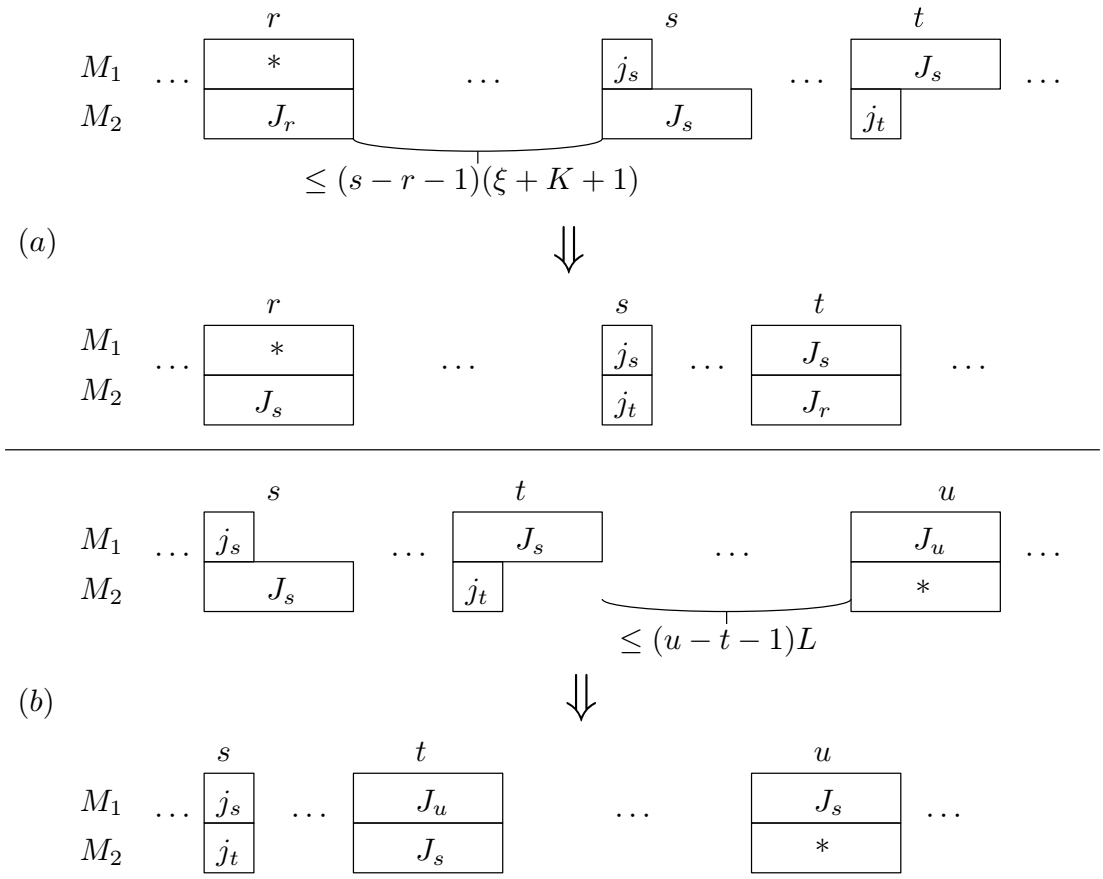


Figure 4.19: Creating a cycle of long operations and a cycle of short operations if $J_s = J_t$, $j_s \neq j_t$ and
(a) there is a long operation in cycle r, $r < s$,
(b) there is no long operation in any cycle r, $r < s$, but there is a long operation in cycle u, $u > s$

plus Δ, again leading to a decrease in the total completion time:

$$\begin{aligned}\sum_{j\in N}(C'_j - C_j) &\leq -((\max\{t,r\}-s)\cdot(L-2(\xi+K+1)) - \Delta \\ &\quad +2(\xi+K+1)+\Delta \\ &\leq -2(L-2(\xi+K+1))+2(\xi+K+1) \\ &= -2L+6(\xi+K+1) \overset{(4.19)}{<} 0,\end{aligned}$$

where the last inequality follows from (4.19) since $6 < |N| = 2n^9 + 4\sigma + 2$.

The only remaining case with $j_s = j_t$ is where no cycle r exists such that both operations in r are short, i.e., all short operations are paired with long operations. In this case, for cycle s with a short operation on M_1 and a long operation on M_2, we select a cycle t' similar to t,

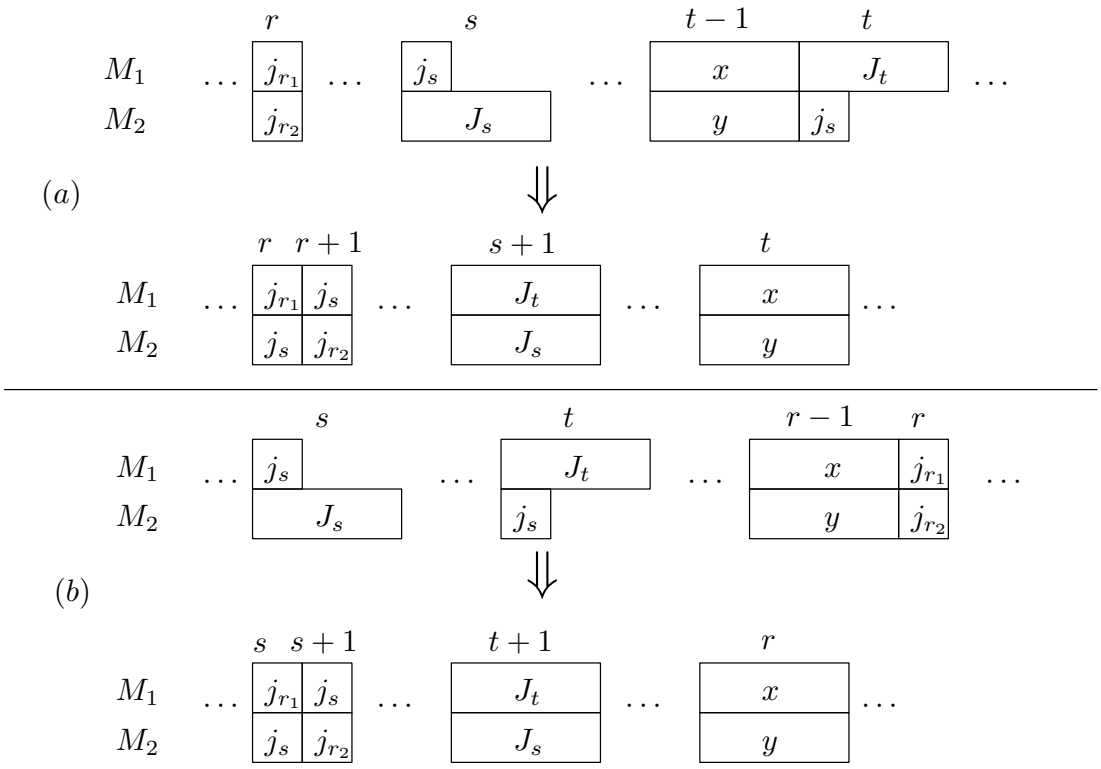


Figure 4.20: Creating a cycle of long operations and a cycle of short operations if $J_s \neq J_t$, $j_s = j_t$ and there is a cycle r with two short operations,
(a) $r < s$,
(b) $r > s$

with a short operation on M_2 and a long operation on M_1, $t' \neq t$ (such a cycle always exists for $n > 2$). Then the two short operations in cycles s and t' belong to different jobs, and the case reduces to one of the cases with $j_s \neq j_t$ considered before.

In case that $j_s = j_t$ and $J_s = J_t$ we can first swap one of the long operations with a long operation in another cycle, as described for $J_s = J_t$, and afterwards continue with moving the two short operations to the front of the schedule, as described for $j_s = j_t$, again leading to a decrease in the total completion time. Therefore, in an optimal schedule there is no cycle consisting of a long and a short operation.

Property 1 is proved. From now on we refer to cycles as short or long assuming that there are no "mixed cycles" in an optimal schedule.

Property 2: All long operations are scheduled in the last $2n^9 + 1$ cycles. This defines the splitting of schedule S into Parts 1 and 2, with cycles $1, 2, \ldots, 4\sigma+1$ and $4\sigma+2, \ldots, 4\sigma+2+2n^9$.

Let $U = (s + 1, \ldots, t)$ be the last sequence of short cycles and let s be a long cycle that precedes U. Assume first that $|U| \geq 2$. Then, as cycles in U are the last short cycles, at least $|U| - 1$ jobs finish within U (U may contain the two short operations of jobs $\mathrm{Ve}_0^{d(0)}$ and F_0 for

which their respective second, long operations may be scheduled in later cycles). Construct a schedule S' by moving the long cycle s after U, see Fig. 4.21. Then the completion times of at least $|U|-1$ jobs decrease by L while the completion times of the two jobs scheduled in cycle s in S increase by at most $|U|(\xi+K+1)$,

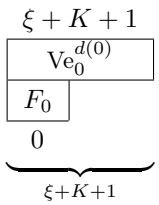
$$\begin{aligned}\sum_{j\in N}(C'_j - C_j) &\leq -(|U|-1)L + 2|U|(\xi+K+1)\\ &= L - |U|(L-2(\xi+K+1)).\end{aligned}$$

Using conditions $|U|\geq 2$ and $L-2(\xi+K+1)>0$ (which can be proved in the same way as (4.19)) we deduce

$$\sum_{j\in N}(C'_j - C_j) \leq L - 2(L-2(\xi+K+1)) = -L + 4(\xi+K+1).$$

The last expression is negative, again by the same arguments as (4.19). Thus, we get a contradiction to the optimality of S.

If $|U|=1$ and the short cycle in U is different from



(4.25)

then at least one job, different from $\mathrm{Ve}_0^{d(0)}$ and F_0, finishes in U. In this case the previous transformation reduces the completion time of at least one job by L, while the completion times of the two jobs scheduled in cycle s in S increase by at most $(\xi+K+1)$,

$$\sum_{j\in N}(C'_j - C_j) \leq -L + 2(\xi+K+1) < 0,$$

where the last inequality can be proved in the same way as (4.19).

Consider now the case with U consisting of only one short cycle of the form (4.25). Notice that such a cycle is avoided in the optimal solution presented in Fig. 4.16.

Let $\widetilde{U}$ be the last sequence of short cycles before U and assume first that $\widetilde{U}$ is preceded by some long cycles. Clearly $\widetilde{U}$ does not contain short operations of jobs F_0 and $\mathrm{Ve}_0^{d(0)}$, as they appear in U. Since no other job consists of both, long and short operations, at least $|\widetilde{U}|$ short jobs finish within $\widetilde{U}$. Then the arguments presented in the beginning of the proof of Property 2 are applicable for the set of cycles $\widetilde{U}$ used instead of U.

Lastly, consider the remaining case with U consisting of only one short cycle of the form (4.25) and there are no other short cycles in the preceding part of the schedule that follow

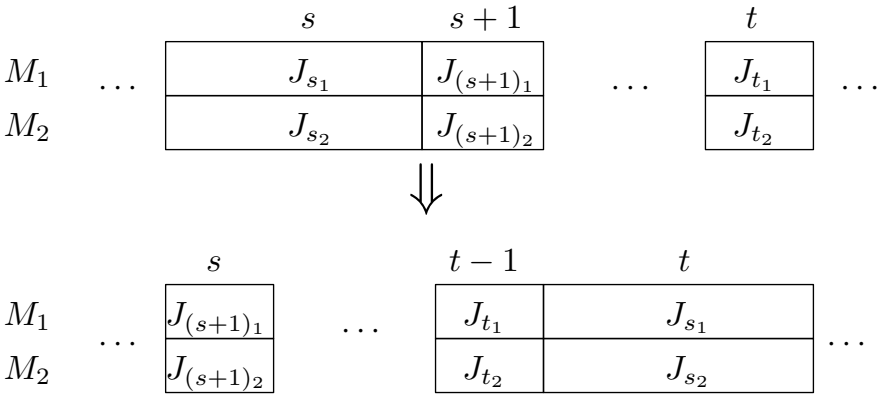


Figure 4.21: Moving long cycle s after a sequence of short cycles $U = \{s+1, \ldots, t\}$

a long cycle. Then, the first 4σ cycles are short and the cycle (4.25) appears among the last $2(n^9+1)$ cycles. Using pairwise interchange arguments it is easy to make sure that in an optimal schedule, the latter cycles are of the form shown in Fig. 4.22, where without loss of generality jobs F_i, except for F_0, are renumbered in the order they appear in schedule S on machine M_1.

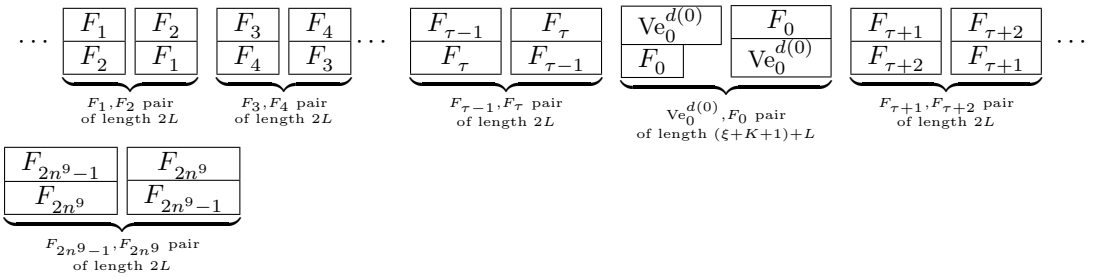


Figure 4.22: A special short cycle appearing among the last $2(n^9+1)$ cycles

In Fig. 4.22, τ is the number of jobs from the set $\{F_1, F_2, ..., F_{2n^9}\}$ completed before cycle (4.25), $1 \le \tau \le 2n^9$. Let Δ be the total length of the first 4σ short cycles. If the length of cycle (4.25) was L, then the total completion time of all long jobs would be

$$2(n^9+1)\Delta + 2L\sum_{i=1}^{n^9+1} 2i = 2(n^9+1)\left(\Delta + (n^9+2)L\right).$$

In reality, the length of cycle (4.25) is less than L by the amount $L - (\xi + K + 1)$, so that completion times of the jobs that appear after $F_{\tau-1}, F_\tau$ should be adjusted. The number of

jobs completed in the corresponding tail part of the schedule is $2(n^9+1)-\tau$, so that

$$\begin{aligned}\sum_{\text{long jobs}} C_j &= 2(n^9+1)\left(\Delta+(n^9+2)L\right)\\ &\quad -(2(n^9+1)-\tau)(L-\xi-K-1)\\ &= 2(n^9+1)\left((n^9+1)L+\Delta+\xi+K+1\right)\\ &\quad +\tau(L-\xi-K-1)\,.\end{aligned}$$

We demonstrate that the objective value for schedule S exceeds the given threshold Θ, using the estimate (4.20) together with the following conditions:

$$\begin{aligned}&\Delta \geq 4\sigma\xi+2\sigma K,\\ &L-\xi-K-1>0,\end{aligned}$$

where the first one is a lower bound on Δ calculated as the sum of processing times on the second machine of the short operations in the first 4σ cycles (which includes every operation except the zero-length operation of job F_0), while the second one can be proved in a similar way as (4.19). We have

$$\sum_{\text{long jobs}} C_j > 2(n^9+1)\left((n^9+1)L+(4\sigma\xi+2\sigma K)+\xi+K+1\right).$$

Recall that

$$\begin{aligned}\Theta_1 &< (8\sigma^2+2\sigma)\xi+4\sigma^2K+n^2,\\ \Theta_2 &= 2(n^9+1)\left(\left(n^9+1\right)L+4\sigma\xi+2\sigma K+n\right),\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{j\in N} C_j-\Theta > \sum_{\text{long jobs}} C_j-\Theta_1-\Theta_2\\ &> 2n^9\left(\xi+K+1-n\right)-\left[(8\sigma^2+2\sigma)\xi+4\sigma^2K+n^2\right]\\ &> 2n^9\left(\xi+K+1-n\right)-\left[10n^4\xi+4n^4K+n^4\right]>0,\end{aligned}$$

where the last inequality holds as $n\geq 2$. Thus, to achieve $\sum C_j \leq \Theta$ cycle (4.25) should not appear among the last $2n^9+1$ cycles. With the exchange arguments from above all other cycles containing short operations have to be scheduled prior to the long operations while the last $2n^9+1$ cycles only contain long operations.

Property 3: The sum of completion times of all long jobs is at least Θ_2.

Due to Property 2, all long operations are scheduled in the last $2n^9+1$ cycles. Again by interchange arguments, the long cycle $\begin{array}{|c|}\hline F_0\\ \hline \mathrm{Ve}_0^{d(0)}\\ \hline\end{array}$ should be the first one among all long cycles, while other long cycles should be grouped in pairs, as shown in Fig. 4.22. Following the

arguments used in the proof of part "⇒" for calculating the sum of completion times of the long jobs, it is easy to verify that calculations (4.15)-(4.16) hold in the current case as well. Instead of the precise value of Δ that leads to (4.17), now we can only substitute an estimate of Δ,

$$\Delta \geq 4\sigma\xi + 2\sigma K + n,$$

which corresponds to the total length of short operations on machine M_1. Thus, we obtain:

$$\begin{aligned} \sum_{\text{long jobs}} C_j &\overset{(4.16)}{=} 2\Delta(n^9+1) + 2\left(n^9+1\right)^2 L \qquad (4.26)\\ &\geq 2\left(4\sigma\xi + 2\sigma K + n\right)\left(n^9+1\right) + 2\left(n^9+1\right)^2 L = \Theta_2. \end{aligned}$$

Property 4: In S, machine M_1 operates without idle times.

If there is an idle time on machine M_1, then $\Delta \geq 4\sigma\xi + 2\sigma K + n + 1$ in Property 3 and thus

$$\begin{aligned} \sum_{j\in N} C_j &> \sum_{\text{long jobs}} C_j \overset{(4.26)}{=} 2\Delta(n^9+1) + 2\left(n^9+1\right)^2 L\\ &> 2\left(4\sigma\xi + 2\sigma K + n + 1\right)\left(n^9+1\right) + 2\left(n^9+1\right)^2 L\\ &= \Theta_2 + 2(n^9+1) > \Theta_2 + \Theta_1 = \Theta. \end{aligned}$$

Therefore, there should be no idle time on machine M_1 to achieve $\sum_{j\in N} C_j \leq \Theta$.

Property 5: In Part 1 of S, job Ve_0^0 is processed in the first two cycles which are of the form $\begin{array}{|c|}\hline \mathrm{Ve}_0^0 \\ \hline F_0 \\ \hline \end{array}\ \begin{array}{|c|}\hline * \\ \hline \mathrm{Ve}_0^0 \\ \hline \end{array}$, where $\boxed{\ *\ }$ represents a short operation. While the order of these two cycles is immaterial, without loss of generality we assume that $\begin{array}{|c|}\hline \mathrm{Ve}_0^0 \\ \hline F_0 \\ \hline \end{array}$ precedes $\begin{array}{|c|}\hline * \\ \hline \mathrm{Ve}_0^0 \\ \hline \end{array}$; otherwise the cycles can be swapped without changing the value of $\sum C_j$.

Since the sum of completion times of all long jobs is at least Θ_2, the remaining 4σ short jobs may only contribute a total completion time of at most Θ_1 to obtain a schedule with total completion time $\sum C_j \leq \Theta$. For all of these jobs, their operations on machine M_2 have length of at least ξ. Thus, it is not possible for $i+1$ short jobs to be completed at time $i\xi$ and we can use the lower bound $i\xi$ for the completion time $C_{[i]}$ of the i-th job:

$$C_{[i]} \geq i\xi \qquad \text{for } 1 \leq i \leq 4\sigma. \qquad (4.27)$$

This implies that

$$\sum_{\text{short jobs}} C_j \geq \xi \sum_{i=1}^{4\sigma} i = 2\sigma(4\sigma+1)\xi\,.$$

Notice that for $i = 1$ there is only one job that can be completed at time ξ, namely Ve_0^0, and this happens only if the first two cycles satisfy the statement of Property 5.

Suppose the statement of Property 5 does not hold for S. Then the above estimate needs to be adjusted by ξ since in that case the completion time of the first completed job is at least 2ξ rather than ξ. It follows that

$$\begin{aligned}\sum_{\text{short jobs}} C_j - \Theta_1 &\geq [2\sigma(4\sigma+1)\xi + \xi] - \\ &\quad - \left[(8\sigma^2+2\sigma)\xi + 4\sigma^2 K - 2\sum\nolimits_{v\in V} vd(v) + n^2\right] \\ &= \xi - 4\sigma^2 K + 2\sum\nolimits_{v\in V} vd(v) - n^2 \\ &= 2\sum\nolimits_{v\in V} vd(v) - n^2.\end{aligned}$$

Notice that $n > 2$ and by construction $d(v) \geq 2$ for each vertex v (G' is connected). This leads to $2\sum_{v\in V} vd(v) - n^2 \geq 2n(n-1) - n^2 = n^2 - 2n > 0$ for $n > 2$. Thus, the total completion time of all jobs is greater than Θ, a contradiction.

Property 6: The two operations of each vertex-job and the two operations of each arc-job are processed in two consecutive cycles, first on M_1 and then on M_2.

We have demonstrated in the proof of Property 5 that $C_{[1]} < 2\xi$ (with job Ve_0^0 defining $C_{[1]}$); otherwise the lower bound Θ_1 is violated. Similar arguments can be used to prove (by induction) that the lower bound Θ_1 is achievable only if $C_{[i]} < (i+1)\,\xi$ for $1 \leq i \leq 4\sigma$. Combining this with (4.27) we can limit our consideration to schedules satisfying

$$i\xi \leq C_{[i]} < (i+1)\,\xi \qquad \text{for } 1 \leq i \leq 4\sigma. \tag{4.28}$$

Property 6 holds for the first job Ve_0^0 due to Property 5. Let job j be the short job that is processed on machine M_1 in cycle 2. Then, as $\xi \leq p_{1j} < 2\xi$, for Ve_0^0 (4.28) is satisfied.

Suppose j does not satisfy the conditions of Property 6. Then cycle 3 consists of two short jobs k and ℓ that have not been processed yet in the preceding cycles. The situation is illustrated below.

Cycle 1	Cycle 2	Cycle 3	Cycle 4	
0	$\geq \xi$	$\geq \xi$	$\geq \xi$	
Ve_0^0	j	k	$*$	$\ldots$
F_0	Ve_0^0	ℓ	$*$	
0	ξ	$\geq \xi$	$\geq \xi$	

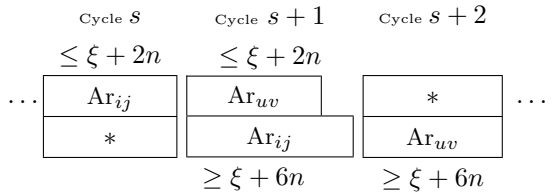
In that case no job other than Ve_0^0 can be finished in the first three cycles, so $C_{[2]}$, corresponding to some job finishing no earlier than cycle 4, is at least as big as the finishing time of cycle 4. As the processing time of any short operation other than Ve_0^0 is at least ξ, cycles $2, 3$ and 4 have a combined length of at least 3ξ as illustrated above. Thus, we have $C_{[2]} \geq 3\xi$ in violation of (4.28).

Therefore j should be processed in cycles 2 and 3, first on M_1 and then on M_2. The proof

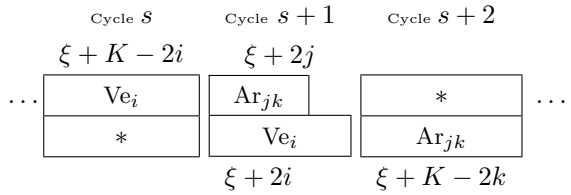
of Property 6 can be done by induction using the above arguments.

Property 7: In Part 1 of S, machine M_1 alternates between processing arc-jobs and vertex-jobs. Moreover, an operation of a vertex-job corresponding to v is followed by an operation of an arc-job corresponding to an arc leaving v. Similarly, an operation of an arc-job for arc (v, w) is followed by an operation of a vertex-job for vertex w. By Property 6, the same is true for machine M_2 in Part 1 and in the first cycle that follows it.

Note that due to the numbers and distributions of vertex- and arc-jobs, if two vertex-jobs are scheduled consecutively, then there should also be two arc-jobs scheduled consecutively. Hence we can restrict our proof to the latter case. So assume there are cycles $s, s+1, s+2$ in which two operations of arc-jobs are scheduled consecutively on the first machine in cycles $s, s+1$ (and thus their second operations are scheduled in cycles $s+1, s+2$ by Property 6). Then, because the processing times of the arc-jobs are chosen such that they are no larger than $\xi + 2n$ on machine M_1 and no smaller than $\xi + 6n$ on machine M_2, there is an idle time on machine M_1 in cycle $s+1$, as illustrated below. However, this is a contradiction to Property 4 as Machine M_1 has to operate without idle-time. Therefore this situation cannot happen, which proves the first part of Property 7.



We now show that an operation of a vertex-job corresponding to v is followed by an operation of an arc-job corresponding to an arc leaving v. Note that due to Property 6 a vertex- or arc-job processed on machine M_1 in some cycle s is processed on machine M_2 in cycle $s+1$. Assume there is an operation of vertex-job Ve_i that is succeeded by an operation of an arc-job Ar_{jk} for some $i \neq j$ in cycles $s, s+1$ on machine M_1 and $s+1, s+2$ on machine M_2.



Among all such pairs $(\mathrm{Ve}_i, \mathrm{Ar}_{jk})$ select the one with $i > j$ (notice that the case that $i < j$ for all pairs is not possible). Then there is an idle time on machine M_1 in cycle $s+1$, contradicting Property 4.

In a similar fashion it can be shown that an operation of an arc-job corresponding to an arc entering a vertex w is followed by a vertex-job corresponding to vertex w.

Property 8: The first arc-job that appears in S corresponds to an arc leaving 0. Among the

vertex-jobs, the last one is $\mathrm{Ve}_0^{d(0)}$.

Due to Property 5, the first two cycles contain the two operations of job Ve_0^0. Thus, according to Property 7, both operations of this job have to be succeeded by operations of an arc-job leaving vertex 0. Further, as shown in the proof of Property 6, the last vertex-job to be completed is $\mathrm{Ve}_0^{d(0)}$. ■

Chapter 5

The Assignment Problem with Nearly Monge Arrays and Incompatible Partner Indices

In this chapter we study the d-dimensional assignment problem in which entries of the cost array satisfy the Monge property, except for ∞-entries, which may violate it. We assume that the ∞-entries are incurred by incompatible partner indices and their number is bounded by an upper bound λ for each index. The problem has applications in synchronous open shop scheduling and satellite communication. Other related problems are studied as well. We show that the problem can be solved in linear time for fixed d and λ, and it becomes strongly NP-hard if d or λ is part of the input.

5.1 Introduction

The linear assignment problem with Monge cost matrices was introduced in Section 2.4. Recall that the d-dimensional linear assignment problem is solvable in linear time if the cost array is Monge, and that an optimal solution is given by the set of entries $\{(1,1,\ldots,1),(2,2,\ldots,2),$ $\ldots,(n,n,\ldots,n)\}$.

Several practical applications give rise to ∞-entries in the weight array W, see, e.g., [24] and [26]. The purpose of infinity entries is to model forbidden assignments, i.e., if $w_{ij}=\infty$, then i cannot be assigned to j. In this situation, $a+\infty=\infty$ for all $a\in\mathbb{R}\cup\{\infty\}$ and $a<\infty$ for all $a\in\mathbb{R}$. Depending on the position of the ∞-entries, the Monge property may still be satisfied, so that a greedy solution to the assignment and transportation problem remains optimal. For example, if in a Monge matrix all entries in the lower triangle are replaced by ∞ ($w_{ij}=\infty$ for all $i>j$) or if all entries in the upper triangle are replaced by ∞ ($w_{ij}=\infty$ for all $i<j$), then

the resulting matrix still satisfies the Monge property.

In the multi-dimensional case, transportation and assignment problems remain greedily solvable in the presence of forbidden entries, if the incurred ∞-entries do not destroy the Monge property. This is shown, e.g. in [124], which exploits the relationship between multi-dimensional Monge arrays (with and without infinities) and submodular functions. Note that the requirement of [124] that the finite entries form a sublattice implies that ∞-entries do respect the Monge property.

In general, however, an arbitrary introduction of ∞-entries may destroy the Monge property in a matrix that initially satisfied it. In [41] so-called incomplete Monge matrices are studied where some values in the matrix are not specified and the Monge property must only hold for any four specified entries $w_{ij}, w_{rs}, w_{is}, w_{rj}$. In the following we introduce a related concept where unspecified entries are replaced by infinity, which implies that these assignments are forbidden. We call a weight matrix W *nearly Monge* if all quadruples of finite entries $w_{ij}, w_{rs}, w_{is}, w_{rj}$ satisfy the Monge property (2.3), while quadruples with ∞-entries may violate it. Similarly, in the multi-dimensional case, a d-dimensional array W is called nearly Monge if condition (2.4) is satisfied for all finite entries.

In this chapter, we study an important subclass of such matrices and arrays where ∞-entries are incurred by incompatible partner indices. In the two-dimensional case, if two indices $i = i^*$, $j = j^*$ are incompatible, we call them *incompatible partners*. Then the corresponding cost is $w_{i^*j^*} = \infty$ to make the assignment (i^*, j^*) *forbidden*. We assume that a parameter $\lambda \leq n$ is given which denotes an upper bound on the number of incompatible partners for every row index $i = i^*$ and every column index $j = j^*$. This implies that there are at most λ forbidden entries in each row and in each column of the matrix W.

In the multi-dimensional assignment problem, denoted by AP(d, λ), each pair of incompatible partners $i_u = i_u^*$, $i_v = i_v^*$ incurs forbidden entries for all d-tuples of the form $(i_1, \ldots, i_u^*, \ldots, i_v^*, \ldots, i_d)$, where all indices, except i_u and i_v, take all possible values from $\{1, \ldots, n\}$. Each d-tuple that contains at least one pair of incompatible partners is forbidden; the corresponding w-value is ∞. Again we assume that we have an upper bound λ, which limits for each index $i_u = i_u^*$ the number of incompatible partners for each $v \neq u$. This means that for each value i_u^* and each $v \neq u$ at most λ values $i_{v,1}^*, i_{v,2}^*, \ldots, i_{v,\lambda}^*$ exist such that all assignments $(i_1, \ldots, i_{u-1}, i_u = i_u^*, i_{u+1}, \ldots, i_{v-1}, i_v = i_{v,\mu}^*, i_{v+1}, \ldots, i_d)$ are forbidden for $\mu = 1, 2, \ldots, \lambda$. Hence, in this case the total number of incompatible partners for $i_u = i_u^*$ is at most

$$\Omega = \lambda(d-1). \tag{5.1}$$

Example 18. As an example of problem AP$(3, 1)$ consider the nearly Monge array W with $n = 3$ where all entries are equal to 1 except for the ∞-entries. If the first pair of incompatible partners is $i_1^* = 2$, $i_2^* = 3$, then all assignments $(2, 3, i_3)$ with $i_3 \in \{1, 2, 3\}$ are forbidden, i.e. the ∞-entries are $w_{2,3,1} = w_{2,3,2} = w_{2,3,3} = \infty$. The incompatible pair $i_1^* = 1$, $i_3^* = 3$ leads to the ∞-entries $w_{1,1,3} = w_{1,2,3} = w_{1,3,3} = \infty$. The incompatible pair $i_2^* = 3$, $i_3^* = 3$ incurs

$w_{1,3,3} = w_{2,3,3} = w_{3,3,3} = \infty$. This situation corresponds to $\lambda = 1$.

An example of problem AP$(3, 2)$ with $\lambda = 2$ can be obtained from the previous example of AP$(3, 1)$ by adding more incompatible pairs. If, for example, the pair $i_1^* = 1$, $i_3^* = 2$ is added, the index $i_1^* = 1$ gets a second incompatible partner for $v = 3$. Note that if instead we add the incompatible pair $i_1^* = 1$, $i_2^* = 2$, then $\lambda = 1$ remains unchanged, as the index $i_1^* = 1$ did not have an incompatible partner for $v = 2$ before.

Applications of assignment problems with incompatible partners are typical for scenarios where items from different sets have to be matched, but there is a certain number of combinations that are forbidden. For example, in timetabling the allocation of certain classes to some classrooms or time slots has to be avoided. In scheduling problems with unit processing times and arbitrary release dates and deadlines (which can be modelled as an assignment problem), it is not allowed to allocate a job to a time slot before its release date or after its deadline. In scheduling problems with multiple machines there can be additional constraints that do not allow some job-machine pairs. In transportation scheduling, allocations of certain types of vehicles to some routes can be forbidden; in addition there may be restrictions on drivers' allocation.

Assignment problems with nearly Monge arrays defined by incompatible partners arise for example in applications related to satellite communication or in synchronous open shop scheduling. There also exists a close relation to the well-known max-weight edge coloring problem on bipartite graphs. We discuss the two applications in more detail in Section 5.2 and the relation to max-weight edge coloring in Section 5.3.

The remainder of the chapter is organized as follows. In Section 5.2 we present two applied scenarios that can be modelled as problem AP$(d, 1)$ with nearly Monge arrays. In Section 5.3 we show that problem AP(d, λ) is strongly NP-hard if one of the parameters, d or λ, is part of the input. Following that, in Section 5.4 we present some basic properties of nearly Monge matrices. In Sections 5.5 and 5.6 we study the problem with fixed d and λ, which is a typical assumption in applications. In Section 5.5 we formulate a "corridor property" that characterizes the structure of an optimal solution. It implies that 1-entries of an optimal solution array belong to a corridor of limited width around the main diagonal. Based on that property, in Section 5.6 we present an efficient algorithm that solves problem AP(d, λ) in linear time for fixed d and λ, and is fixed parameter tractable (FPT) for the parameters d and λ (for an introduction to FPT see [47], for FPT in scheduling see [113]). This has new implications for the complexity of the applications in satellite communications, synchronous open shop scheduling and max-weight edge coloring. In Section 5.7 the corridor property for other versions of the assignment problem is discussed.

5.2 Applications

In this section we show that synchronous open shop scheduling with m machines and the makespan objective as well as the equivalent problem from satellite communication can be naturally modelled as problem $\mathrm{AP}(m,\lambda)$ with a m-dimensional nearly Monge weight array and $\lambda = 1$. In both applications, there are given d n-tuples $\Gamma^\ell = \left(\gamma_1^\ell, \gamma_2^\ell, \ldots, \gamma_n^\ell\right)$, $1 \le \ell \le d$, and the associated assignment problems have a cost array W of the form

$$w_{i_1 \ldots i_d} = \max\left\{\gamma_{i_1}^1, \gamma_{i_2}^2, \ldots, \gamma_{i_d}^d\right\}. \tag{5.2}$$

If the numbers in each n-tuple Γ^ℓ are listed in non-decreasing order, then W is a Monge array [13]. This can be seen as follows. If we choose $w_{i_1 i_2 \ldots i_d}$, $w_{j_1 j_2 \ldots j_d}$, $w_{s_1 s_2 \ldots s_d}$ and $w_{t_1 t_2 \ldots t_d}$ as in the definition of the multi-dimensional Monge property (2.4), then due to the maximum-definition and the ordering at least one of the entries $w_{i_1 i_2 \ldots i_d}$ or $w_{j_1 j_2 \ldots j_d}$ is at least as large as $w_{t_1 t_2 \ldots t_d}$. Furthermore, again due to the maximum-definition and the ordering, each entry $w_{i_1 i_2 \ldots i_d}$ and $w_{j_1 j_2 \ldots j_d}$ is at least as large as $w_{s_1 s_2 \ldots s_d}$. Notice that similar argument can be made if the numbers in each n-tuple are listed in non-increasing order, as in (4.1) in the last chapter.

First consider the synchronous open shop scheduling problem. In order to bring the notation more in line with the usual notation for the assignment problem, in this chapter we denote the number of machines by d and use i as the index for numbering jobs. Thus, in an instance of the synchronous open shop problem there are given d machines $M_1, \ldots, M_d$ and n jobs, where job i consists of d operations $O_{1i}, O_{2i}, \ldots, O_{di}$.

Recall that any feasible schedule for the synchronous open shop problem $O|synmv|C_{\max}$ can be characterized by n cycles $(i_1^1, \ldots, i_d^1), (i_1^2, \ldots, i_d^2), \ldots, (i_1^n, \ldots, i_d^n)$ where each cycle $k \in \{1, \ldots, n\}$ is described by d job indices $(i_1^k, \ldots, i_d^k)$, assuming that in cycle k job i_ℓ^k is allocated to machine M_ℓ for $\ell = 1, \ldots, d$.

Similar to the two-machine case from the previous chapter, the problem of allocating jobs to d machines within n cycles can be modelled as a d-dimensional assignment problem with costs $w_{i_1 \ldots i_d}$ for d-tuples $(i_1, \ldots, i_d)$ defined by

$$w_{i_1 \ldots i_d} = \begin{cases} \max\left\{p_{1,i_1}, p_{2,i_2}, \ldots, p_{d,i_d}\right\}, & \text{if all job indices } i_1, i_2, \ldots, i_d \text{ are different,} \\ \infty, & \text{otherwise.} \end{cases}$$

Here ∞-entries prohibit the allocation of two operations of the same job to one cycle. Since for each machine-job pair (ℓ, i_ℓ) exactly one incompatible partner (ℓ', i_ℓ) exists for each other machine ℓ', $\ell \neq \ell'$, we have $\lambda = 1$.

In order to achieve the Monge property for finite entries of the array $W = (w_{i_1 \ldots i_d})$, we define for each machine M_ℓ the n-tuple $\Gamma^\ell = \left(\gamma_1^\ell, \gamma_2^\ell, \ldots, \gamma_n^\ell\right)$ by the processing times $p_{\ell i}$ of operations $O_{\ell i}$ $(1 \le i \le n)$ that have to be processed on that machine, and renumber the

γ-values so that (5.4) holds. Then the array $\hat{W}$ of weights

$$\hat{w}_{i_1\dots i_d} = \begin{cases} \max\left\{\gamma^1_{i_1}, \gamma^2_{i_2}, \dots, \gamma^d_{i_d}\right\}, & \text{if all job indices corresponding to} \\ & \gamma^1_{i_1}, \gamma^2_{i_2}, \dots, \gamma^d_{i_d} \text{ are different,} \\ \infty, & \text{otherwise,} \end{cases} \tag{5.3}$$

is of type (5.2), apart from the infinities, and therefore all finite entries satisfy the Monge property.

Turning to the problem in satellite communication, we again deviate slightly from the usual notation to be more in line with the notation for the assignment problem. We assume there $|V_1| = d$ senders, and $|V_2| = n$ receivers with $d \le n$. Transmissions from sender $\ell \in V_1$ to receiver $i \in V_2$ are denoted by $(\ell, i) \in E$, and the transmission time is $t_{\ell i}$.

Recall that a feasible solution can be characterized by a set of periods, each of which does not involve the same sender or the same receiver more than once. Similar to the synchronous open shop problem, each period can be described by d pairs $(1, i_1), \dots, (d, i_d)$ denoting that for $\ell = 1, \dots, d$, messages are sent from $\ell \in V_1$ to $i_\ell \in V_2$. For the moment we assume here that each sender sends a message in every time period; the general case, where senders are allowed to be idle in some periods, will be discussed at the end of Section 5.3. The entries of the cost array W correspond to the durations of the periods, i.e. for any selection of d receivers $i_1, i_2, \dots, i_d \in V_2$ we define

$$w_{i_1\dots i_d} = \begin{cases} \max\left\{t_{1,i_1}, t_{2,i_2}, \dots, t_{d,i_d}\right\}, & \text{if all receivers } i_1, i_2, \dots, i_d \text{ are different,} \\ \infty, & \text{otherwise.} \end{cases}$$

Note that $w_{i_1\dots i_d} = \infty$ if and only if $i_j = i_k$ for some $1 \le j \ne k \le d$, i.e., if the same receiver is activated simultaneously by two different senders j and k in one period. Thus, each index in the W-matrix has exactly one incompatible partner in each other dimension, which means that $\lambda = 1$ is an upper bound on the number of incompatible partner indices.

As before, in order to achieve the Monge property for finite entries of the array $W = (w_{i_1\dots i_d})$, we define for each sender ℓ the n-tuple $\Gamma^\ell = \left(\gamma^\ell_1, \gamma^\ell_2, \dots, \gamma^\ell_n\right)$ by the durations $t_{\ell i}$ of the messages to be sent from ℓ to all receivers $i \in V_2$, and renumber the γ-values so that

$$\gamma^\ell_1 \le \gamma^\ell_2 \le \dots \le \gamma^\ell_n. \tag{5.4}$$

Then the array $\hat{W}$ of weights

$$\hat{w}_{i_1\dots i_d} = \begin{cases} \max\left\{\gamma^1_{i_1}, \gamma^2_{i_2}, \dots, \gamma^d_{i_d}\right\}, & \text{if all messages corresponding to } \gamma^1_{i_1}, \dots, \gamma^d_{i_d} \\ & \text{have different receivers,} \\ \infty, & \text{otherwise,} \end{cases} \tag{5.5}$$

is a permutation of W of type (5.2), apart from the infinities, and therefore all finite entries

satisfy the Monge property.

Example 19. Consider an example with $d = 2$ senders, $n = 4$ receivers, and the following transmission times $t_{\ell i}$ for senders $\ell \in V_1 = \{1, 2\}$ and receivers $i \in V_2 = \{1, 2, 3, 4\}$:

$\ell \setminus i$	1	2	3	4
1	5	2	7	3
2	4	2	3	6

Then the associated matrix W is of the form

$$W = \left(\begin{array}{c|cccc} & (2,1) & (2,2) & (2,3) & (2,4) \\ \hline (1,1) & \infty & 5 & 5 & 6 \\ (1,2) & 4 & \infty & 3 & 6 \\ (1,3) & 7 & 7 & \infty & 7 \\ (1,4) & 4 & 3 & 3 & \infty \end{array}\right),$$

where pairs (ℓ, i) denote the messages. Notice that matrix W is not nearly Monge.

Consider matrix $\hat{W}$ obtained from W using $\Gamma^1 = (t_{12}, t_{14}, t_{11}, t_{13}) = (2, 3, 5, 7)$ and $\Gamma^2 = (t_{22}, t_{23}, t_{21}, t_{24}) = (2, 3, 4, 6)$:

$$\hat{W} = \left(\begin{array}{c|cccc} & (2,2) & (2,3) & (2,1) & (2,4) \\ \hline (1,2) & \infty & 3 & 4 & 6 \\ (1,4) & 3 & 3 & 4 & \infty \\ (1,1) & 5 & 5 & \infty & 6 \\ (1,3) & 7 & \infty & 7 & 7 \end{array}\right).$$

Matrix $\hat{W}$ is nearly Monge, but not Monge.

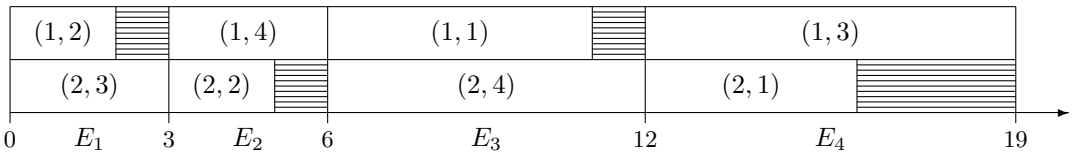


Figure 5.1: A feasible transmission schedule for Example 19

A feasible solution is given by $E_1 = \{(1, 2), (2, 3)\}$, $E_2 = \{(1, 4), (2, 2)\}$, $E_3 = \{(1, 1), (2, 4)\}$ and $E_4 = \{(1, 3), (2, 1)\}$. The corresponding schedule is shown in Fig. 5.1; its total transmission time is $3 + 3 + 6 + 7 = 19$, which can be shown to be optimal.

As described above, $\lambda = 1$ corresponds to the situation that messages to the same receiver are incompatible, i.e. cannot be sent simultaneously. In a more general setting, also messages to receivers in close proximity to each other may be incompatible, e.g., due to interference. If,

for example, for each receiver one or two 'neighbors' are affected, we get problems with $\lambda = 2$ or $\lambda = 3$.

Example 19 can be reformulated to define an instance of the synchronous open shop problem, replacing "senders" and "receivers" by "machines" and "jobs", respectively. The entries of matrices W and $\hat{W}$ remain the same. In the schedule shown in Fig. 5.1, the sets E_1, E_2, E_3 and E_4 have now the meaning of cycles, while pairs (ℓ, i) represent now operations $O_{\ell i}$. Note that this example uses the same set of jobs as Example 2 in Section 4.1, although the numbering of the jobs is different (recall that in Section 4.1 jobs were numbered by non-increasing processing times on the first machine).

In the subsequent sections we will also consider the relaxed versions of both synchronous open shop scheduling and satellite communication, where some senders or machines are allowed to stay idle so that less than d activities may occur in a cycle or period. As we already demonstrated for two-machine synchronous open shop, there are instances for which an optimal solution with idle times may outperform any solution without idle times. In the next section we provide a construction, which helps modelling those versions as problem AP(d, λ), using a concept similar to the idea of dummy jobs.

5.3 NP-hardness of problem AP(d, λ)

In this section we prove that problem AP(d, λ) with arbitrary d and fixed λ is strongly NP-hard, even for $\lambda = 1$, via a reduction from max-weight edge coloring in complete bipartite graphs. Recall that both applications described in Section 5.2 can be modelled as problem MEC on complete bipartite graphs. Furthermore, we discuss the complementary result that AP(d, λ) with arbitrary λ and fixed d is strongly NP-hard, even if $d = 3$.

While the NP-hardness of problem AP(d, λ) with arbitrary d and $\lambda = 1$ is not surprising, as in the previous section we used it to model two problems known to be strongly NP-hard, there is still value in providing a full, formal reduction. Firstly, using the same construction as in the reduction below will later allow us to model the relaxed versions of both applications from the previous section as problem AP(d, λ). Secondly, the construction provides a general way to model any application of MEC on bipartite graphs, or of MVC on the line graphs of bipartite graphs as problem AP(d, λ), instead of treating each application separate.

Theorem 20. *Problem* MEC$(K_{m,n})$ *reduces to problem* AP(d, λ) *with* $d = m$ *and* $\lambda = 1$*,*

$$\text{MEC}(K_{m,n}) \propto \text{AP}(m, 1). \tag{5.6}$$

Proof: Without loss of generality we assume that $m \leq n$. Clearly, the number of colors κ in any feasible solution to MEC$(K_{m,n})$ satisfies $n \leq \kappa \leq nm$, and the number of edges that get the same color $c \in \{1, \ldots, \kappa\}$ may vary from 1 to m.

We present the proof for $m = 2$ first and then generalize it for an arbitrary m. In order to

prove (5.6) for $m = 2$, we create an auxiliary problem $\mathrm{MEC}^{2n}(K_{2,2n})$ which corresponds to the MEC for the bipartite graph $K_{2,2n}$ with exactly $2n$ colors allowed. We show that

$$\mathrm{MEC}(K_{2,n}) \propto \mathrm{MEC}^{2n}(K_{2,2n}) \tag{5.7}$$

and

$$\mathrm{MEC}^{2n}(K_{2,2n}) \propto \mathrm{AP}(2,1). \tag{5.8}$$

Given a bipartite graph $G = (V_1 \cup V_2, E)$ for the original problem $\mathrm{MEC}(K_{2,n})$, create the extended bipartite graph $\widetilde{G} = (V_1 \cup \widetilde{V}_2, \widetilde{E})$ for the auxiliary problem with unchanged first set of vertices V_1, while the second set of vertices V_2 is extended by adding n additional vertices so that $|\widetilde{V}_2| = 2n$. The edge set $\widetilde{E}$ is extended accordingly, to include $mn = 2n$ new edges connecting all vertices from V_1 with the n auxiliary vertices from $\widetilde{V}_2$, the weights of those edges being set to 0. Clearly, any feasible solution to $\mathrm{MEC}(K_{2,n})$ that uses κ colors can be extended to a feasible solution to $\mathrm{MEC}^{2n}(K_{2,2n})$ that uses $2n$ colors without changing the objective value such that κ colors contribute to the weight function, while $2n - \kappa$ colors carry 0-weight. Conversely, if we have a solution to $\mathrm{MEC}^{2n}(K_{2,2n})$, we simply drop all auxiliary edges (having weight 0) and obtain a feasible solution with the same objective value for $\mathrm{MEC}(K_{2,n})$. Thus, reduction (5.7) holds.

In what follows we reformulate problem $\mathrm{MEC}^{2n}(K_{2,2n})$ as an assignment problem which finds for every edge $(1, \widetilde{v}_i)$ a mate $(2, \widetilde{v}_j)$ with $\{1, 2\} \subseteq V_1$ and $\{\widetilde{v}_i, \widetilde{v}_j\} \subseteq \widetilde{V}_2$, such that both edges can get the same color. Such a pair of edges in the same color contributes the amount $\max\{w(1, \widetilde{v}_i), w(2, \widetilde{v}_j)\}$ to the objective function.

First create two sequences of edges $\left((1, \widetilde{v}_1^1), (1, \widetilde{v}_1^2), \ldots, (1, \widetilde{v}_1^{2n})\right)$ and $\left((2, \widetilde{v}_2^1), (2, \widetilde{v}_2^2), \ldots, (2, \widetilde{v}_2^{2n})\right)$ each consisting of $2n$ edges listed in non-decreasing order of their weights:

$$\begin{aligned} w\left(1, \widetilde{v}_1^1\right) = w\left(1, \widetilde{v}_1^2\right) = \cdots = w\left(1, \widetilde{v}_1^n\right) \le w\left(1, \widetilde{v}_1^{n+1}\right) \le \cdots \le w\left(1, \widetilde{v}_1^{2n}\right), \\ w\left(2, \widetilde{v}_2^1\right) = w\left(2, \widetilde{v}_2^2\right) = \cdots = w\left(2, \widetilde{v}_2^n\right) \le w\left(2, \widetilde{v}_2^{n+1}\right) \le \cdots \le w\left(2, \widetilde{v}_2^{2n}\right). \end{aligned}$$

Here the first n terms of each list correspond to the edges incident to auxiliary vertices which have 0-weights. We define the matrix W with $2n$ rows and $2n$ columns as follows:

- each row i, $1 \le i \le 2n$, corresponds to the edge $e_1^i = (1, \widetilde{v}_1^i)$;
- each column j, $1 \le j \le 2n$, corresponds to the edge $e_2^j = (2, \widetilde{v}_2^j)$;
- the weight-value w_{ij} corresponds to the weight of assigning the same color to the pair of edges e_1^i and e_2^j,

$$w_{ij} = \begin{cases} \max\{w(e_1^i), w(e_2^j)\}, & \text{if } \widetilde{v}_1^i \neq \widetilde{v}_2^j, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.9}$$

The matrix with entries $\max\{w(e_1^i), w(e_2^j)\}$ is the two-dimensional case of (5.2) and therefore satisfies the Monge property. Thus, matrix W defined by (5.9), where ∞-entries correspond to

incompatible pairs of edges e_1^i, e_2^j, is a nearly Monge matrix with $\lambda = 1$ incompatible partner for every $i = i^*$ or $j = j^*$.

It is easy to see that for any feasible solution to problem $\mathrm{MEC}^{2n}(K_{2,2n})$ there exists a feasible solution to problem $\mathrm{AP}(2,1)$ with the same (finite) weight and vice versa. Thus, reduction (5.8) holds.

Consider now the case of an arbitrary $m \leq n$. Given problem $\mathrm{MEC}(K_{m,n})$, introduce the extended problem $\mathrm{MEC}^{mn}(K_{m,mn})$ by adding $(m-1)n$ auxiliary vertices to V_2 and introducing edges of weight 0 that connect them with m vertices from V_1. As before, any feasible solution to $\mathrm{MEC}(K_{m,n})$ with κ colors can be extended to a feasible solution of $\mathrm{MEC}^{mn}(K_{m,mn})$ with mn colors (and vice versa) such that κ colors contribute to the weight function, while $mn - \kappa$ colors carry 0-weight. Thus,

$$\mathrm{MEC}(K_{m,n}) \propto \mathrm{MEC}^{mn}(K_{m,mn}). \tag{5.10}$$

In order to reduce $\mathrm{MEC}^{mn}(K_{m,mn})$ to $\mathrm{AP}(m,1)$, create m sequences of edges

$$\left(\left(h, \widetilde{v}_h^1\right), \left(h, \widetilde{v}_h^2\right), \ldots, \left(h, \widetilde{v}_h^{mn}\right)\right),$$

each listed in non-decreasing order of the weights, $1 \leq h \leq m$. The array W is m-dimensional, with each dimension h corresponding to edges $e_h^i = \left(h, \widetilde{v}_h^i\right)$ incident to h, the number of entries in each dimension being mn. The weight-value $w_{i_1 \ldots i_m}$ corresponds to the weight of assigning the same color to m edges $e_1^{i_1} = (1, \widetilde{v}_1^{i_1})$, $e_2^{i_2} = (2, \widetilde{v}_2^{i_2})$, $\ldots$, $e_m^{i_m} = (m, \widetilde{v}_m^{i_m})$,

$$w_{i_1 \ldots i_m} = \begin{cases} \max\limits_{1 \leq h \leq m} \{w(e_h^{i_h})\}, & \text{if all vertices } \widetilde{v}_1^{i_1}, \widetilde{v}_2^{i_2}, \ldots, \widetilde{v}_m^{i_m} \text{ are different,} \\ \infty, & \text{otherwise.} \end{cases} \tag{5.11}$$

Again, the array with entries $\max\limits_{1 \leq h \leq m} \{w(e_h^{i_h})\}$ satisfies the Monge property, see (5.2). Therefore, the array W defined by (5.11), where ∞-entries correspond to incompatible pairs of edges, is a nearly Monge array with $\lambda = 1$ incompatible partner for every $i_h = i_h^*$. Thus, similar to the two-dimensional case, we have

$$\mathrm{MEC}^{mn}(K_{m,mn}) \propto \mathrm{AP}(m,1), \tag{5.12}$$

which together with (5.10) proves the theorem. ∎

In the proof of Theorem 20 we have developed reduction (5.7), which is useful for handling the generalized versions of the two applications, scheduling satellite transmissions and synchronous open shop problems. These two problems were introduced in Section 5.2 under the assumption that exactly d activities should be assigned in each period/cycle. Both problems were modelled as $\mathrm{AP}(d,1)$ by defining for each sender or machine ℓ, $1 \leq \ell \leq d$, the n-tuple $\Gamma^\ell = \left(\gamma_1^\ell, \gamma_2^\ell, \ldots, \gamma_n^\ell\right)$, where γ-values are in the non-decreasing order (5.4), and by setting up

the cost array $\hat{W}$ of the form:

$$\hat{w}_{i_1 \ldots i_d} = \begin{cases} \max\left\{\gamma_{i_1}^1, \gamma_{i_2}^2, \ldots, \gamma_{i_d}^d\right\}, & \text{if all activities corresponding to } \gamma_{i_1}^1, \ldots, \gamma_{i_d}^d \\ & \text{are compatible,} \\ \infty, & \text{otherwise,} \end{cases} \tag{5.13}$$

see (5.5) and (5.3). If fewer than d activities per period/cycle are allowed, then using the same idea as in reduction (5.10) together with (5.12), the two applications are modelled as AP$(d, 1)$ with n actual receivers/jobs and $(d-1)n$ auxiliary ones. We set the γ-values for the auxiliary receivers/jobs to 0. Then each extended dn-tuple Γ^ℓ is of the form

$$\Gamma^\ell = \left(\underbrace{0, 0, \ldots, 0}_{d(n-1) \text{ elements}} , \underbrace{\gamma_1^\ell, \gamma_2^\ell, \ldots, \gamma_n^\ell}_{n \text{ elements}} \right).$$

This results in the extended cost array $\widetilde{W}$ with entries of type (5.13) defined for dn receivers/jobs rather than for n. If parameters d and λ are fixed, then the $O(n)$-time algorithm of Section 5.6 solves the associated applied problems in linear time, after sorting the input data in order to achieve (5.4).

We now turn to complexity aspects of problem AP(d, λ) with one parameter fixed and another one being part of the input. If $\lambda = 0$, then the weight array of problem AP$(d, 0)$ is Monge rather than nearly Monge, and an optimal solution is of the diagonal structure [26]. For the case of $\lambda = 1$ we use Theorem 20 and the NP-hardness result known for MEC$(K_{m,n})$ from [43, 126] to make the following conclusion.

Observation 21. Problem AP$(d, 1)$, where $d \geq 2$ is part of the input, is NP-hard in the strong sense, even if there are only three distinct finite weights.

Note that the restriction related to three weight values, except for ∞'s, follows from the reduction presented in [126], which uses only three distinct weights. Interestingly, problem MEC$(K_{n,n})$ with two distinct weights is solvable in polynomial time, as shown in [43, 126].

Consider now the counterpart of AP(d, λ) with fixed d and an arbitrary λ. If $d = 2$, then problem AP$(2, \lambda)$ can be solved in $O(n^3)$ time by the Hungarian algorithm [25, 88] that finds an optimal solution to the assignment problem with an arbitrary, not necessarily Monge-like matrix. If $d = 3$, then problem AP$(3, \lambda)$ becomes NP-hard in the strong sense.

Observation 22. If λ is part of the input, then problem AP$(3, \lambda)$ is NP-hard in the strong sense, even if all finite weights are equal.

To justify Observation 22 we establish a link between problem AP$(3, \lambda)$ and the famous 3-DIMENSIONAL MATCHING problem (3-DM) known to be strongly NP-complete [59]. Recall that in 3-DM, there are given three disjoint sets X, Y and Z, with n elements in each, and a set of triples $M \subseteq (X \times Y \times Z)$; the goal is to find a perfect matching, i.e. a set $M' \subseteq M$,

such that each element from X, Y and Z appears in exactly one triple in M', see [59]. Any instance of problem AP$(3, \lambda)$ with all finite weights equal to 0 can be treated as an instance of problem 3-DM, where the set M consists of all triples excluding those with incompatible partner indices. We denote this special case of 3-DM by 3-DM(λ). Conversely, any instance of problem 3-DM(λ) can be treated as an instance of problem AP$(3, \lambda)$ where all finite weights are 0. Clearly, there exists a perfect matching for an instance of 3-DM(λ) if and only if there exists an assignment of cost 0 for the corresponding instance of AP$(3, \lambda)$. Thus,

$$\text{3-DM}(\lambda) \propto \text{AP}(3, \lambda).$$

In order to demonstrate the NP-completeness of 3-DM(λ), consider the well-known NP-completeness proof for 3-DM. It is based on the reduction 3-SAT$\propto$ 3-DM presented, e.g., in [59]. Without changing the structure of the instance of 3-DM used in the proof, one can re-define it as an instance of 3-DM(λ) with $\lambda = n - 1$, so that the reduction from [59] becomes 3-SAT$\propto$ 3-DM(λ). So as to not distract from the core problem of this chapter we do not provide such a re-definition at this point. For an interested reader, it is instead provided in Section 5.8 at the end of this chapter. All in all, we get the chain of reductions

$$\text{3-SAT} \propto \text{3-DM}(\lambda) \propto \text{AP}(3, \lambda),$$

which proves Observation 22.

The complexity results discussed in this section are summarized in Table 7.2 in the Conclusions.

5.4 Some properties of nearly Monge matrices with incompatible partner indices

In this and in the subsequent sections we consider problem AP(d, λ) with fixed d and λ. Prior to discussing algorithmic aspects, we first demonstrate in Section 5.4.1 that the existing results known for the assignment problem with ∞'s are generally inapplicable. Then in Sections 5.4.2-5.4.3 we address several typical questions that arise in the context of matrices with ∞'s or with unspecified entries.

5.4.1 Nonexistence of a Monge sequence

As explained before, the introduction of ∞-entries into a Monge matrix may destroy the Monge property. In the literature on Monge-like structures it is then suggested to verify whether a non-Monge matrix possesses a so-called Monge sequence. Such a sequence guides a greedy algorithm towards finding an optimal solution to the assignment or transportation problem [26, 74]. It considers the variables in the order they appear in a Monge sequence and assigns

the highest possible values to them without violating the constraints of the problem. Formally, a *Monge sequence* of an $n \times n$ matrix $W = (w_{ij})$ is defined as a sequence of n^2 index pairs $(i_1, j_1), \ldots, (i_{n^2}, j_{n^2})$ such that whenever (i, j) precedes both (i, s) and (r, j) in the sequence, condition

$$w_{ij} + w_{rs} \leq w_{is} + w_{rj} \tag{5.14}$$

holds (cf. [74]).

A Monge sequence, if one exists, is a powerful tool for solving problems with matrices which do not satisfy the Monge property for all quadruples. Models of this type arise for example in the scheduling context. The scheduling problems studied in [72, 73] can be modelled as transportation problems with non-Monge matrices. In some cases Monge sequences can be found efficiently; in others a Monge sequence does not exist, and this calls for a development of special tailor-made algorithms, as in the case of the generalized (weighted) version of the problem from [73].

Even for the simplest version of our problem AP(d, λ) with $d = 2$ and $\lambda = 1$ the matrices in general do not possess a Monge sequence. Consider the two applications discussed in Section 5.2 with a cost matrix of type (5.2). If the w-values are defined as

$$w_{ij} = \begin{cases} \max\{i, j\}, & \text{if } i \neq j, \\ \infty, & \text{otherwise,} \end{cases}$$

and the matrix is at least of size 3×3, then a Monge sequence does not exist. Indeed, whichever element is selected as the first element of a Monge sequence, the sequence cannot be completed to satisfy (5.14). For any selected entry there always exists an ∞-entry that is not in the same row and not in the same column, while the other two entries of the associated quadruple are finite. Clearly, condition (5.14) is violated for such a quadruple. For a more general case, a similar example can be found in [138], where finite entries of W are arbitrary.

Observation 23. In general, nearly Monge arrays with incompatible partners do not give rise to Monge sequences even for $\lambda = 1$ and $d = 2$.

5.4.2 Recognizing nearly Monge arrays

Given a weight matrix W, it is easy to verify whether the Monge property (2.3) is satisfied for finite entries. A straight-forward way to do so is to check (2.3) for all pairs of finite entries of the form (w_{ij}, w_{rs}) with $i < r$ and $j < s$, which takes no more than $O(n^4)$ checks. This method can also be generalized to recognize d-dimensional nearly Monge arrays, in which case it takes no more than $O(n^{2d})$ checks. Note that a d-dimensional nearly Monge array consists of n^d entries.

The applications discussed in Section 5.2 and the MEC problem of Section 5.3 deal with a weight array W of form (5.2) with ∞-entries introduced for incompatible partners. Even if initially W is not nearly Monge, that property can be achieved by ordering the γ-values of lists

Γ^ℓ, which is equivalent to sequencing the sets I and J of the assignment problem, or permuting the rows and columns of W. In general, however, recognizing a permuted nearly Monge array is a non-trivial task, as discussed below. Array W is a *permuted* nearly Monge array if its index sets can be permuted to make the array nearly Monge.

Observation 24. For an arbitrary λ, it is NP-complete to decide whether a given array with at most λ incompatible partners (for every index) is permuted nearly Monge, even for $d = 2$ (i.e. the matrix case).

This observation follows from a similar result in [41] formulated for a permuted incomplete Monge matrix, which is equivalent to a nearly Monge matrix with an arbitrary λ. Notice that recognizing a permuted Monge matrix can be done in $O(n^2)$ time [26], and recognizing a special incomplete Monge matrix, namely a Supnick matrix, which is a symmetric Monge matrix with unspecified diagonal entries, can be done in $O(n^2 \log n)$ time [41].

In the nearly Monge case, if the number of incompatible partners λ is fixed, recognition of permuted nearly Monge matrices remains open. If a polynomial-time algorithm could be developed, it would be beneficial to achieve a time complexity smaller than $O(n^3)$, beating the time complexity of solving a general assignment problem.

Further difficulties arise in recognition of nearly Monge arrays for higher dimensions $d \geq 3$. It is known that in the absence of ∞'s, a d-dimensional array W is a Monge array if and only if every two-dimensional submatrix is a Monge matrix [2]. This property can then be efficiently used in recognizing d-dimensional Monge arrays [130]. Unfortunately this necessary and sufficient condition no longer holds for nearly Monge arrays, even if $\lambda = 1$. Consider for example a three-dimensional array $W = (w_{ijk})$ with $n = 3, \lambda = 1$, incompatible partners $(1, 2, *)$, $(2, 1, *)$, $(1, *, 3)$, $(2, *, 2)$, $(3, *, 1)$, $(*, 1, 2)$, $(*, 2, 3)$, two 1-entries $w_{111} = w_{333} = 1$ and all remaining finite entries being 0's. The two-dimensional submatrices are listed below, and all of them are nearly Monge. However, because of $w_{111} + w_{333} = 2 > 0 = w_{131} + w_{313}$, the array W is not a nearly Monge array.

$$i = 1: \begin{pmatrix} 1 & \infty & \infty \\ \infty & \infty & \infty \\ 0 & 0 & \infty \end{pmatrix} \quad i = 2: \begin{pmatrix} \infty & \infty & \infty \\ 0 & \infty & \infty \\ 0 & \infty & 0 \end{pmatrix} \quad i = 3: \begin{pmatrix} \infty & \infty & 0 \\ \infty & 0 & \infty \\ \infty & 0 & 1 \end{pmatrix}$$

$$j = 1: \begin{pmatrix} 1 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{pmatrix} \quad j = 2: \begin{pmatrix} \infty & \infty & \infty \\ 0 & \infty & \infty \\ \infty & 0 & \infty \end{pmatrix} \quad j = 3: \begin{pmatrix} 0 & 0 & \infty \\ 0 & \infty & 0 \\ \infty & 0 & 1 \end{pmatrix}$$

$$k=1: \begin{pmatrix} 1 & \infty & 0 \\ \infty & 0 & 0 \\ \infty & \infty & \infty \end{pmatrix} \quad k=2: \begin{pmatrix} \infty & \infty & 0 \\ \infty & \infty & \infty \\ \infty & 0 & 0 \end{pmatrix} \quad k=3: \begin{pmatrix} \infty & \infty & \infty \\ \infty & \infty & 0 \\ 0 & \infty & 1 \end{pmatrix}$$

Observation 25. Even if every two-dimensional submatrix of a d-dimensional array W with $\lambda = 1$ is a nearly Monge matrix, the whole array W is not necessarily a nearly Monge array.

5.4.3 Completing nearly Monge arrays

An interesting question related to arrays with unspecified or ∞-entries is the possibility of completing them by introducing finite values in order to achieve Monge arrays. Such an approach works, for example, for incomplete matrices of Supnick type, as shown in [41], but unfortunately, it does not work for nearly Monge matrices. Consider the following nearly Monge matrix with $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 1 & \infty & 0 \\ 1 & 0 & 0 & 0 & \infty \\ 4 & 2 & \infty & 0 & 1 \\ 6 & \infty & 2 & 0 & 0 \\ \infty & 6 & 4 & 1 & 0 \end{pmatrix}.$$

Attempting to complete that matrix without permuting rows and columns, we have to satisfy the following two contradicting conditions for w_{14}:

$$\begin{aligned} w_{14} + w_{23} \geq w_{13} + w_{24}, &\quad \text{or equivalently} \quad w_{14} \geq 1, \\ w_{14} + w_{35} \leq w_{15} + w_{34}, &\quad \text{or equivalently} \quad w_{14} \leq -1. \end{aligned}$$

Attempting to permute and complete the above matrix, the only suitable permutation of rows and columns that preserves the Monge property for finite entries is the one which reverses the orders of rows and columns (notice that all quadruples with finite entries satisfy the Monge property as strict inequalities). The arguments for entry w_{14} can be re-used with respect to the entry w_{52} in the permuted matrix, justifying that completing cannot be done.

Observation 26. In general, a nearly Monge matrix with incompatible partners cannot be completed into a Monge matrix by replacing ∞-entries by finite values even for $\lambda = 1$.

If permutations of rows and columns are fixed, producing a completed Monge matrix, if one exists, can be done in polynomial time by solving the following system of linear inequalities:

$$\begin{aligned} \hat{w}_{ij} + \hat{w}_{i+1,j+1} &\leq \hat{w}_{i,j+1} + \hat{w}_{i+1,j}, && 1 \leq i,j < n, \\ \hat{w}_{ij} &= w_{ij}, && 1 \leq i,j < n, \quad w_{ij} \neq \infty, \\ \hat{w}_{ij} &\in \mathbb{R}. \end{aligned}$$

Here $\hat{w}_{ij}$ are real-valued variables representing the target values of w_{ij}. Notice that for a Monge

matrix it is sufficient to achieve the Monge property for quadruples defined by adjacent pairs of rows and columns [26].

If a feasible solution satisfying the above inequalities exists, real values are assigned to all ∞-entries, so that the resulting Monge matrix $\hat{W}$ is a completion of W. In the case of infeasibility, either the matrix W is not nearly Monge (i.e. not all quadruples of finite entries satisfy the Monge property) or the matrix W is not completable to a Monge matrix.

Notice that in the applications, which arise in scheduling satellite transmissions or synchronous open shops, nearly Monge arrays are completable if the formulae for w's (5.5) and (5.3) are modified accordingly, namely, by assigning finite values $\max\left\{\gamma^1_{i_1}, \gamma^2_{i_2}, \ldots, \gamma^d_{i_d}\right\}$ to all d-tuples $(i_1, i_2, \ldots, i_n)$, ignoring the condition that all receivers or all jobs should be different. Also, formula (3.1) for MEC $(K_{m,n})$ can be adjusted to replace ∞'s by values $\max\{w(e)|e \in E_c\}$ for any combinations of m edges with different origins in V_1, even if there are repeated vertices in V_2. Due to the sorting and the max-definition (5.2), the resulting array is Monge.

5.5 The corridor property for problem AP(d, λ)

In this section we consider the assignment problem AP(d, λ) with a d-dimensional nearly Monge weight array W and at most λ incompatible partners for every index. We show that there exists an optimal solution $\hat{S}$ such that all non-zero entries $x_{i_1,\ldots,i_d} = 1$ of the solution matrix $X_{\hat{S}}$ lie in a *corridor* of a certain width ξ around the main diagonal,

$$|i_\ell - i_1| \leq \xi \quad \text{for all } \ell = 1, \ldots, d, \tag{5.15}$$

where

$$\xi = d(d-1)\lambda. \tag{5.16}$$

We refer to the above condition as the *corridor property*. Based on it, in the next section we develop a linear-time algorithm to solve problem AP(d, λ) for fixed d and λ. Note that AP(d, λ) is strongly NP-hard, if d or λ is part of the input, see Section 5.3.

Theorem 27. *For problem* AP(d, λ) *there exists an optimal solution*

$$\hat{S} = \{(\hat{\imath}^1_1, \hat{\imath}^1_2, \ldots, \hat{\imath}^1_d), (\hat{\imath}^2_1, \hat{\imath}^2_2, \ldots, \hat{\imath}^2_d), \ldots, (\hat{\imath}^n_1, \hat{\imath}^n_2, \ldots, \hat{\imath}^n_d)\}$$

such that every d-tuple $(i_1, i_2, \ldots, i_d) = (\hat{\imath}^k_1, \hat{\imath}^k_2, \ldots, \hat{\imath}^k_d)$, $1 \leq k \leq n$, satisfies (5.15) with ξ defined by (5.16).

Proof: Starting with an optimal solution $S = \left\{(i^1_1, i^1_2, \ldots, i^1_d), \ldots, (i^n_1, i^n_2 \ldots, i^n_d)\right\}$ that violates the corridor property (5.15) we perform a series of exchange steps, each of which does not increase the cost, eventually producing a solution that satisfies (5.15). We assume that $w(S) < \infty$, i.e., none of the d-tuples in S contains a pair of incompatible partners; otherwise S can be replaced by the diagonal solution, which has the desired property and no higher cost.

For the exchange step, we take a d-tuple $(i_1, i_2, \ldots, i_d)$ of the current solution that violates (5.15), select a special companion d-tuple $(j_1, j_2, \ldots, j_d)$ from the current solution and replace that pair by $(s_1, s_2, \ldots, s_d)$ and $(t_1, t_2, \ldots, t_d)$ defined by

$$s_\ell = \min\{i_\ell, j_\ell\}, \qquad t_\ell = \max\{i_\ell, j_\ell\}, \qquad \ell = 1, \ldots, d. \tag{5.17}$$

Since W is nearly Monge, the inequality

$$w_{s_1 s_2 \ldots s_d} + w_{t_1 t_2 \ldots t_d} \leq w_{i_1 i_2 \ldots i_d} + w_{j_1 j_2 \ldots j_d}$$

holds, if none of the four entries is ∞. With an appropriate choice of the companion d-tuple $(j_1, j_2, \ldots, j_d)$, we eliminate the violation related to $(i_1, i_2, \ldots, i_d)$ without increasing the cost.

We distinguish between two types of violations:

$$\begin{aligned} &\text{Type I}(i_\ell) \text{ violation}: \quad i_\ell > i_1 + \xi \quad \text{for } i_1 \leq n - \xi - 1, \\ &\text{Type II}(i_\ell) \text{ violation}: \quad i_\ell < i_1 - \xi \quad \text{for } i_1 \geq \xi + 1, \end{aligned}$$

and use the terms "d-tuple of Type I(i_ℓ)" and "d-tuple of Type II(i_ℓ)" for violating d-tuples.

First we eliminate Type I(i_2) violations for the second index. We start with the d-tuple of Type I(i_2) with the smallest first index i_1 and then we proceed with other d-tuples of Type I(i_2) considering them in increasing order of i_1. After that we eliminate all Type II(i_2) violations, starting with the d-tuple of Type II(i_2) with the largest first index i_1, proceeding then with other d-tuples of Type II(i_2) considered in decreasing order of i_1.

Having eliminated all Type I(i_2) and Type II(i_2) violations for the second index, we proceed with violations for the third index. We handle them in the same order, fixing first Type I(i_3) violations with d-tuples of Type I(i_3) considered in increasing order of i_1, and then Type II(i_3) violations with d-tuples of Type II(i_3) considered in decreasing order of i_1. For these exchanges we demonstrate that no new violations of Type I(i_2) and Type II(i_2) for second indices are created. The same approach is then applied to the remaining indices i_4, …, i_d.

Eliminating violations of Type I(i_2). Among all d-tuples with violations of Type I(i_2), select the d-tuple $(\underline{i_1}, \underline{i_2}, \ldots, \underline{i_d})$ with the smallest first index $\underline{i_1}$, so that

$$\underline{i_2} > \underline{i_1} + \xi\,. \tag{5.18}$$

Note that the choice of $\underline{i_1}$ implies that no Type I(i_2) violations happen for $i_1 < \underline{i_1}$. For the violating d-tuple $(\underline{i_1}, \underline{i_2}, \ldots, \underline{i_d})$ select a companion d-tuple $(j_1, j_2, \ldots, j_d)$ with

$$j_1 > \underline{i_1} \;\text{ and }\; j_2 \leq \underline{i_1} + \xi \tag{5.19}$$

such that

(a) $x_{j_1 j_2 \ldots j_d} = 1$ in the current solution (recall that the associated value of $w_{j_1 j_2 \ldots j_d}$ cannot be

∞ then),

(b) the induced d-tuples $(s_1, s_2, \ldots, s_d)$, $(t_1, t_2, \ldots, t_d)$ defined by (5.17) correspond to finite entries in W, i.e. do not contain two incompatible partner indices.

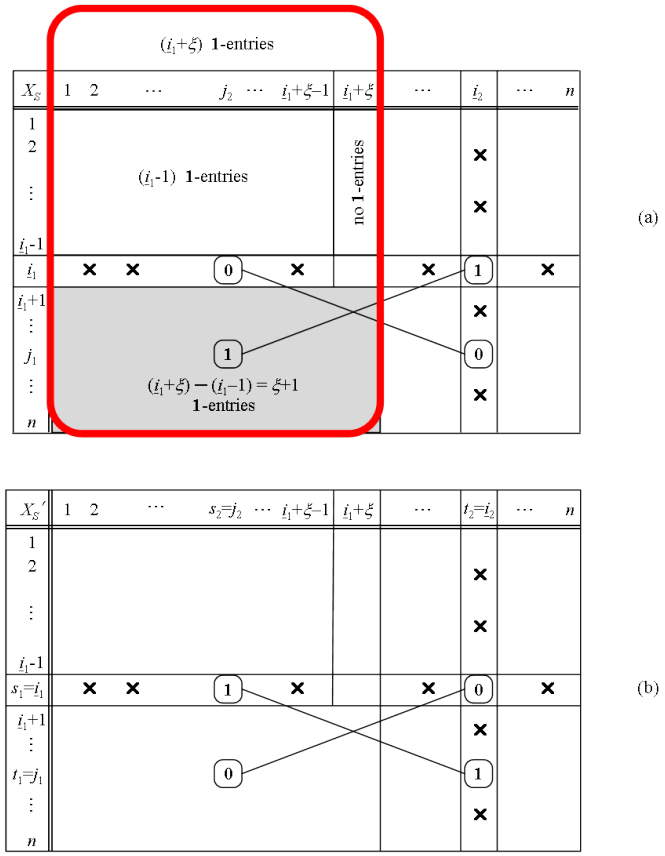


Figure 5.2: Eliminating Type I(i_2) violation
(a) Solution matrix X_S with the violating d-tuple $(\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_d)$ of Type I(i_2)
(b) Modified solution matrix X'_S with Type I(i_2) violation in row $\underline{i}_1$ eliminated

For violation (5.18) the required d-tuple $(j_1, j_2, \ldots, j_d)$ exists since, as we show below, there are $\xi + 1$ candidate 1-entries with indices j_1 and j_2 satisfying (5.19). Clearly, at least one of those $\xi + 1$ candidates has all indices compatible with the indices from $(\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_d)$. If this was not the case, i.e., if there was at least one incompatible index between $(\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_d)$ and each of the $\xi + 1$ candidate entries, then the total number of incompatible partner indices for $(\underline{i}_1, \underline{i}_2, \ldots, \underline{i}_d)$ would be equal to $\xi + 1 = d(d-1)\lambda + 1$, a contradiction to the assumption that for any index $i_u = \underline{i}_u$, $1 \le u \le d$, there are at most $\Omega = \lambda(d-1)$ incompatible partners, see (5.1).

We now justify that there are indeed $\xi+1$ candidate 1-entries with indices j_1 and j_2 satisfying (5.19). We start with the 2-dimensional case, with the binary solution matrix $X_S = (x_{i_1 i_2})$

illustrated in Fig. 5.2 (a). The symbol "×" in Fig. 5.2 is used to mark compulsory 0's in the solution matrix, caused by incompatible partner indices.

Consider the first $(\underline{i_1} + \xi)$ columns of the associated matrix X_S, marked by a rounded rectangle in Fig. 5.2 (a). Since every column of X_S contains one 1-entry, the selected part contains $(\underline{i_1} + \xi)$ 1-entries. By the assumption, there are no violations of Type I(i_2) for the first $(\underline{i_1} - 1)$ rows; therefore the first $(\underline{i_1} - 1)$ rows of the selected part contain $(\underline{i_1} - 1)$ 1-entries in columns 1, 2, ..., $\underline{i_1} + \xi - 1$, and they do not contain 1-entries in column $\underline{i_1} + \xi$. Thus, the remaining part of the selection, corresponding to rows $\underline{i_1} + 1$, ..., n and columns 1, 2, ..., $\underline{i_1} + \xi$, contains

$$\big(\underline{i_1} + \xi\big) - \big(\underline{i_1} - 1\big) = \xi + 1$$

1-entries.

It is easy to see that the same arguments are applicable for the multi-dimensional case. Fig. 5.2 can be treated as a 'projection' of the d-dimensional case into the space of the first two indices with

$$\widetilde{x}_{i_1 i_2} = \sum_{i_3,\ldots,i_d} x_{i_1 i_2 i_3 \ldots i_d}.$$

Replacing $\big(\underline{i_1}, \underline{i_2}, \ldots, \underline{i_d}\big)$, $(j_1, j_2, \ldots, j_d)$ by $(s_1, s_2, \ldots, s_d)$, $(t_1, t_2, \ldots, t_d)$ eliminates the current Type I(i_2) violation for $i_1 = \underline{i_1}$. Note that it does not matter whether the companion d-tuple $(j_1, j_2, \ldots, j_d)$ is violating or not; that d-tuple is removed from the solution as a result of the exchange step. It also does not matter whether $(s_1, s_2, s_3, \ldots, s_d) = \big(\underline{i_1}, j_2, s_3, \ldots, s_d\big)$ or $(t_1, t_2, \ldots, t_d) = \big(j_1, \underline{i_2}, t_3, \ldots, t_d\big)$ are violating: a possible violation for $(s_1, s_2, \ldots, s_d)$ can only be of Type II(i_2), as $s_2 = j_2 \le \underline{i_1} + \xi$ by (5.19), and it is repaired at a later stage; a possible violation for $(t_1, t_2, \ldots, t_d)$ may be of any type, but due to $t_1 = j_1 > \underline{i_1}$ it is also repaired at a later stage.

Proceeding with each next smallest index i_1, for which violation of Type I(i_2) occurs, all Type I(i_2) violations are repaired in a similar way, resulting in a solution where no Type I(i_2) violations remain.

Eliminating violations of Type II(i_2). Among all d-tuples with violations of Type II(i_2), select the d-tuple $\big(\overline{i_1}, \overline{i_2}, \overline{i_3}, \ldots, \overline{i_d}\big)$ with the largest first index $\overline{i_1}$, so that the corridor property is satisfied for the second index for any $i_1 > \overline{i_1}$ and the violation under consideration is of the form

$$\overline{i_2} < \overline{i_1} - \xi\,, \tag{5.20}$$

see Fig. 5.3. The required companion d-tuple $(j_1, j_2, \ldots, j_d)$ should satisfy

$$j_1 < \overline{i_1} \;\text{ and }\; \overline{i_1} - \xi \le j_2 \le \overline{i_1} + \xi\,, \tag{5.21}$$

together with properties (a)–(b) formulated for Type I(i_2) violations.

For violation (5.20) the required d-tuple $(j_1, j_2, \ldots, j_d)$ exists since, as we show below, there

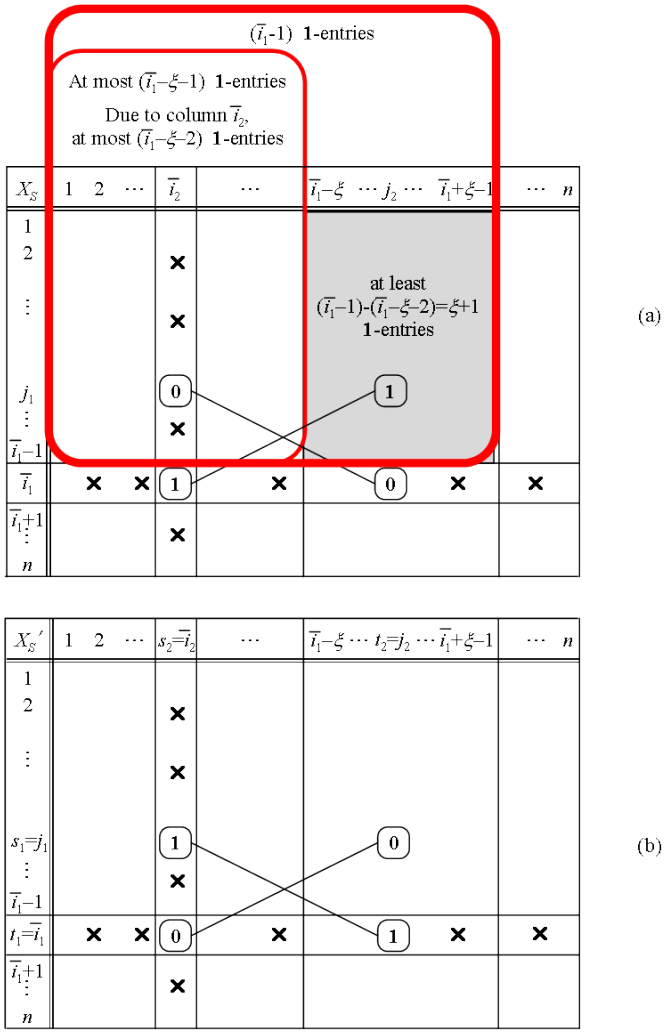


Figure 5.3: Eliminating Type II(i_2) violation
(a) Solution matrix X_S with the violating d-tuple $(\overline{i_1}, \overline{i_2}, \ldots, \overline{i_d})$ of Type II(i_2)
(b) Modified solution matrix X'_S with Type II(i_2) violation in row $\overline{i_1}$ eliminated

are $\xi + 1$ candidate 1-entries with indices j_1 and j_2 satisfying

$$j_1 < \overline{i_1} \quad \text{and} \quad \overline{i_1} - \xi \le j_2 \le \overline{i_1} + \xi - 1, \tag{5.22}$$

which is in fact a stronger condition than (5.21). Again, at least one of those $\xi + 1$ entries has all indices compatible with the indices from $\left(\overline{i_1}, \overline{i_2}, \ldots, \overline{i_d}\right)$, so that the exchange step with the induced d-tuples $(s_1, s_2, \ldots, s_d)$, $(t_1, t_2, \ldots, t_d)$ achieves its goal.

To justify that there exist $\xi + 1$ candidate 1-entries with indices j_1 and j_2 satisfying (5.22), consider first the 2-dimensional case. Since X_S does not contain Type I(i_2) violations, all

$\left(\overline{i_1}-1\right)$ 1-entries of the $\left(\overline{i_1}-1\right)$ first rows belong to the area marked by the larger rounded rectangle in Fig. 5.3 (a) with $j_2 \leq \overline{i_1}+\xi-1$. Moreover, the part of the selected area $j_2 \leq \overline{i_1}-\xi-1$, marked by the smaller rounded rectangle in Fig. 5.3 (a), contains no more than $\left(\overline{i_1}-\xi-1\right)$ 1-entries. In fact it contains no more than $\left(\overline{i_1}-\xi-2\right)$ 1-entries since the 1-entry in column $\overline{i_2}$ does not belong to the selected area. Thus, the remaining part of the selection, corresponding to rows $1, 2, \ldots, \overline{i_1}-1$ and columns $\overline{i_1}-\xi, \ldots, \overline{i_1}+\xi-1$ contains

$$\left(\overline{i_1}-1\right)-\left(\overline{i_1}-\xi-2\right)=\xi+1$$

1-entries, which all satisfy (5.22). Again it is easy to see that the same arguments hold for the d-dimensional case.

Replacing $\left(\overline{i_1}, \overline{i_2}, \ldots, \overline{i_d}\right)$, $(j_1, j_2, \ldots, j_d)$ by $(s_1, s_2, \ldots, s_d)$, $(t_1, t_2, \ldots, t_d)$ eliminates violation (5.20), so that the corridor property is now achieved for the second index for any $i_1 \geq \overline{i_1}$. Note that $(t_1, t_2, \ldots, t_d) = \left(\overline{i_1}, j_2, t_3, \ldots, t_d\right)$ satisfies the corridor property, while $(s_1, s_2, s_3, \ldots, s_d) = \left(j_1, \overline{i_2}, s_3, \ldots, s_d\right)$ may violate it. The possible violation is of Type II(i_2) with $s_1 < \overline{i_1}$, and it is repaired at a later stage.

Proceeding with each next largest index i_1, for which a violation of Type II(i_2) occurs, all Type II(i_2) violations are repaired in a similar way, resulting in a solution where no violations of any type remain for the second index.

After all violations of Type I(i_2) and Type II(i_2) are eliminated, we start eliminating violations for the third index, ensuring that no new violations are created for the second index. We use the same approach:

- first eliminate all violations of Type I(i_3), starting with a violating d-tuple with the smallest i_1, and proceeding then with other violating d-tuples considered in increasing order of i_1;

- next eliminate all violations of Type II(i_3), starting with a violating d-tuple with the largest i_1, and proceeding with other violating d-tuples considered in decreasing order of i_1.

The existence of the d-tuple $(j_1, j_2, \ldots, j_d)$ needed for the exchange step can be proven in the same way as above. We only need to demonstrate that the induced d-tuples $(s_1, s_2, \ldots, s_d)$, $(t_1, t_2, \ldots, t_d)$, defined in accordance with (5.17), do not create new violations in the previously repaired second index, while repairing violations in the third one.

Consider the exchange step based on $(i_1, i_2, \ldots, i_d)$ and $(j_1, j_2, \ldots, j_d)$ with

$$i_1 < j_1. \tag{5.23}$$

Since in this stage violations for the second index have been already repaired, we have

$$-\xi \leq i_2 - i_1 \leq \xi\,, \tag{5.24}$$

$$-\xi \leq j_2 - j_1 \leq \xi\,. \tag{5.25}$$

If $i_2 < j_2$, then the induced d-tuples are of the form $(i_1, i_2,\ s_3 \ldots, s_d)$, $(j_1, j_2,\ t_3 \ldots, t_d)$, and by (5.24)-(5.25) no new violation appears for the second index. Alternatively, if

$$i_2 > j_2, \tag{5.26}$$

then for the induced d-tuple $(i_1, j_2,\ s_3 \ldots, s_d)$ there is no violation for the second index since

$$\begin{array}{lll} j_2 - i_1 > -\xi & \text{by (5.23) and by the} & \text{first inequality from (5.25),} \\ j_2 - i_1 < \xi & \text{by (5.26) and by the} & \text{second inequality from (5.24).} \end{array}$$

For the induced d-tuple $(j_1, i_2,\ s_3 \ldots, s_d)$ there is also no violation for the second index since

$$\begin{array}{lll} i_2 - j_1 > -\xi & \text{by (5.26) and by the} & \text{first inequality from (5.25),} \\ i_2 - j_1 < \xi & \text{by (5.23) and by the} & \text{second inequality from (5.24).} \end{array}$$

It is easy to verify that if instead of (5.23) condition $i_1 > j_1$ holds, then similar arguments are applicable. Thus, repairing violations related to the third index in the described way cannot create violations related to the second one.

Using the same approach repeatedly, we eliminate violations with respect to each index i_ℓ, $\ell = 4, \ldots, d$. Each time, when eliminating violations for i_ℓ, we do not create new violations for indices $i_2, i_3, \ldots, i_{\ell-1}$. This completes the proof of Theorem 27. ∎

5.6 A linear-time algorithm for problem $\mathrm{AP}(d, \lambda)$ with fixed d and λ

In this section we consider problem $\mathrm{AP}(d, \lambda)$ with fixed d and λ and develop a linear-time algorithm for it. We start with the two-dimensional problem $\mathrm{AP}(2, \lambda)$.

By Theorem 27, we can restrict the search to solutions S such that all 1-entries appear inside the corridor, which can be considered as a combination of the main diagonal, 2λ diagonals above it and 2λ diagonals below it. An example of a solution matrix X_S that satisfies the described property is presented below for $\lambda = 1$, $d = 2$ and $n = 10$, with the corridor for 1-entries marked by $*$:

X_S	1	2	3	4	5	6	7	8	9	10
1	*	*	*	0	0	0	0	0	0	0
2	*	*	*	*	0	0	0	0	0	0
3	*	*	*	*	*	0	0	0	0	0
4	0	*	*	*	*	*	0	0	0	0
5	0	0	*	*	*	*	*	0	0	0
6	0	0	0	*	*	*	*	*	0	0
7	0	0	0	0	*	*	*	*	*	0
8	0	0	0	0	0	*	*	*	*	*
9	0	0	0	0	0	0	*	*	*	*
10	0	0	0	0	0	0	0	*	*	*

Introduce a layered network $L = (s, t, V, A)$ with the vertex set V consisting of disjoint subsets $V(0)$, $V(1)$, ..., $V(n)$, $V(n+1)$. Subsets $V(0)$ and $V(n+1)$ are single-element subsets containing the source $V(0) = \{s\}$ and the sink $V(n+1) = \{t\}$. The arcs A have end vertices belonging to two consecutive layers, $A = \bigcup_{k=0}^{n} (V(k) \times V(k+1))$. The vertices of layer $V(k)$, $1 \leq k \leq n$, characterize the partial solutions consisting of the first k rows and satisfying the corridor property.

For two vertices $v_\ell(k) \in V(k)$ and $v_m(k+1) \in V(k+1)$, let $X_\ell(k)$ and $X_m(k+1)$ be two associated partial solutions. If $X_\ell(k) \subset X_m(k+1)$, then the arc $(v_\ell(k), v_m(k+1))$ has the cost of assigning the 1-entry to a relevant position in row $k+1$; that cost is given by $w_{k+1,j}$, where j is the column with $x_{k+1,j} = 1$ in $X_m(k+1)$. Otherwise $v_\ell(k)$ and $v_m(k+1)$ are not connected by an arc. For completeness, introduce auxiliary arcs of cost 0 from every vertex of $V(n)$ to t. Clearly, an optimal solution corresponds to a shortest path in the network L.

In order to reduce the size of L, we include in $V(k)$ only the vertices corresponding to non-dominated partial solutions: if $X_{\ell'}(k)$ and $X_{\ell''}(k)$ have 1-entries assigned to the same set of columns and $w(X_{\ell'}(k)) \leq w(X_{\ell''}(k))$, where $w(X)$ represents the cost of the associated solution, then it is sufficient to include in $V(k)$ only one vertex corresponding to $X_{\ell'}(k)$. This implies that each vertex of $V(k)$ is characterized by a unique subset of columns which contain 1-entries in the first k rows. In order to estimate $|V(k)|$, we prove the following statement.

Statement 28. *Any partial solution $X_\ell(k)$ of layer k consists of*
(a) α columns $1, 2, \ldots, \alpha$, each containing one 1-entry, where

$$\alpha = \max\{k - 2\lambda, 0\},$$

(b) $n - \beta$ columns $\beta + 1, \ldots, n$, each containing 0-entries only, where

$$\beta = \min\{k + 2\lambda, n\},$$

(c) among the columns $\alpha + 1, \ldots, \beta$ there are $(k - \alpha) = \min\{2\lambda, k\}$ columns, each containing

one 1*-entry, and* $(\beta - k) = \min\{2\lambda, n - k\}$ *columns with* 0*-entries only.*

The following example with $k = 6$, $n = 10$, $\lambda = 1$ illustrates the structure of a partial solution $X_\ell(k)$:

	$X_\ell(k)$	1	2	3	4 (α)	5 ($\alpha+1$)	6 (k)	7	8 (β)	9 ($\beta+1$)	10 (n)
	1	*	*	*	0	0	0	0	0	0	0
	2	*	*	*	*	0	0	0	0	0	0
	3	*	*	*	*	*	0	0	0	0	0
	4	0	*	*	*	*	*	0	0	0	0
	5	0	0	*	*	*	*	*	0	0	0
$k \rightarrow$	6	0	0	0	*	*	*	*	*	0	0
$k+1 \rightarrow$	7	0	0	0	0	*	*	*	*	*	0

(α columns with 1-entry each; $(k-\alpha)$ columns with 1-entries; $(\beta - k)$ columns without 1-entries; no 1-entries)

Proof: Conditions (a) and (b) hold since by the corridor property there can be no 1-entry for any combination of $i \in \{k+1, \ldots, n\}$ and $j \in \{1, 2, \ldots, \alpha\}$, and also for any combination of $i \in \{1, 2, \ldots, k\}$ and $j \in \{\beta + 1, \ldots, n\}$.

Condition (c) deals with the remaining entries of $X_\ell(k)$, in rows $i \in \{1, 2, \ldots, k\}$ and columns $j \in \{\alpha + 1, \ldots, \beta\}$. By the definition of the assignment problem, the total number of 1-entries in the first k rows and n columns is k, α of which appear in columns $j \in \{1, 2, \ldots, \alpha\}$. Since no 1-entry can appear in columns $j \in \{\beta + 1, \ldots, n\}$, the number of the remaining 1-entries is $k - \alpha$. ∎

Statement 28 implies that the number of different subsets of columns which may contain 1-entries, and equivalently $|V(k)|$, depends on the selection of $(k - \alpha)$ columns out of $(\beta - \alpha)$ columns in the middle part of the solution matrix. Since $k - \alpha \leq 2\lambda$ and $\beta - \alpha \leq 4\lambda$, $|V(k)|$ can be bounded by

$$\theta = \binom{4\lambda}{2\lambda}. \tag{5.27}$$

Thus, the described approach reduces the assignment problem $\mathrm{AP}(2, \lambda)$ to the shortest path problem in the layered network L, where the number of layers is $n+2$ and in each layer there are no more than θ vertices. The network can be constructed in $O(|A|)$ time, where $|A| \leq \theta^2 (n + 1)$. An optimal solution can be found by dynamic programming in $O(|A|) = O(n)$ time as well.

Consider now the d-dimensional problem $\mathrm{AP}(d, \lambda)$ for fixed $d \geq 2$ and fixed λ. Statement 28

can be generalized by replacing 2λ by $\xi = d(d-1)\lambda$, so that

$$\begin{aligned} \alpha &= \max\{k-\xi, 0\}, \\ \beta &= \min\{k+\xi, n\}, \\ k-\alpha &= \min\{\xi, k\}, \\ \beta-k &= \min\{\xi, n-k\}. \end{aligned}$$

Any feasible d-dimensional partial solution $X_\ell(k)$ consists of k 1-entries with $i_1 \in \{1, 2, \ldots, k\}$. For each index i_z, $2 \le z \le d$, α of these 1-entries have $i_z \in \{1, 2, \ldots, \alpha\}$, $2 \le z \le d$, and the remaining $(k-\alpha)$ 1-entries have $i_z \in \{\alpha+1, \ldots, \beta\}$. Due to this, the number of nodes $|V(k)|$ in layer k depends on the selection of $(k-\alpha)$ choices for each index i_z, $2 \le z \le d$, out of $(\beta-\alpha)$ possible choices. Since $k-\alpha \le \xi$ and $\beta-\alpha \le 2\xi$, $|V(k)|$ can be bounded by Θ^{d-1}, where Θ is the amount of possible combinations for one fixed index i_z,

$$\Theta = \binom{2\xi}{\xi}. \tag{5.28}$$

There are at most $\Theta^{2(d-1)}$ arcs in-between two layers, if each vertex in one layer is connected to every vertex in the next one. Therefore the total number of arcs $|A|$ is bounded by $\Theta^{2(d-1)}(n+1)$, and this defines the time complexity of solving the associated shortest path problem by dynamic programming. Observe that a tighter estimate for $|A|$ can be derived with a more careful analysis of the arc set.

The complexity estimate $\Theta^{2(d-1)}(n+1)$ with Θ defined by (5.28) implies that problem $\mathrm{AP}(d, \lambda)$ is solvable in $O(n)$ time if d and λ are fixed, and it is fixed-parameter tractable (FPT), with parameters d and λ.

Going back to the applications discussed in Sections 5.2-5.3 we observe that they correspond to the case of $d = m$ and $\lambda = 1$, so that the described approach implies the $O(n)$ time complexity. Since modelling the two applications as $\mathrm{AP}(d, 1)$ incurs sorting the input data in non-decreasing order and extending the instance (as described in Section 5.3 after the proof of Theorem 20), we obtain the following corollary.

Corollary 29. The problems of scheduling satellite transmissions or synchronous open shops and problem $\mathrm{MEC}(K_{m,n})$ with $m = d$ are solvable in $O(n \log n)$ time if d is fixed, and they are fixed parameter tractable if d is a parameter.

We conclude this section by analyzing the maximization version of problem $\mathrm{AP}(d, \lambda)$. In that problem incompatible partners should be modelled by $-\infty$-entries in the weight array W, in order to discourage their choice. In the following, we use the notion of an inverse Monge array W: such an array satisfies the Monge property (2.4) with "$\le$" replaced by "$\ge$" [26]. Equivalently, array W is inverse Monge if and only if $-W$ is Monge.

The maximization version of problem $\mathrm{AP}(d, \lambda)$ is

(i) solvable in linear time, if d is fixed and W is an inverse nearly Monge array with incompatible partners modelled by $-\infty$;

(ii) NP-hard, if $d \geq 3$ and W is a nearly Monge array with incompatible partners modelled by $-\infty$'s.

The maximization problem of type (i) with an inverse nearly Monge weight array W is equivalent to the minimization version of problem $\mathrm{AP}(d, \lambda)$ with the nearly Monge weight array $-W$; therefore the linear-time algorithm described above is applicable.

The maximization problem of type (ii) with a nearly Monge weight array is no easier than the version of the same problem with a Monge array; the latter problem is known to be NP-hard (see [26], p. 132, or [27]). The results from [27] are also discussed in Section 5.7.1.

5.7 The corridor property for other versions of the assignment problem

The corridor property that characterizes the structure of an optimal solution and restricts the search to entries around the diagonal, also holds for other versions of the assignment problem. In the literature, a property of this type was established, for example, for the three-dimensional assignment problem with decomposable costs and for the planar 3-dimensional assignment problem. We discuss these two results in Subsections 5.7.1-5.7.2. Both problems deal with a min-sum objective. An alternative version, known as the bottleneck assignment problem, deals with a min-max objective; we generalize our results for that version of the assignment problem in Subsection 5.7.3.

5.7.1 Axial three-dimensional assignment problem with decomposable costs

The *axial three-dimensional assignment problem with decomposable costs* (3AP-DC) is defined as problem (2.2) with a cost array W given by $w_{ijk} = a_i b_j c_k$, where (a_i), (b_j) and (c_k) are three non-decreasing sequences of n positive numbers. Note that W is an inverse Monge array and therefore the maximization version of problem $3\mathrm{AP} - \mathrm{DC}$ is solved by the entries on the main diagonal [27]. We now focus on the minimization version of $3\mathrm{AP} - \mathrm{DC}$. Although W does not satisfy the Monge property and in fact $3\mathrm{AP} - \mathrm{DC}$ is NP-hard [27], still there exists an optimal solution with a corridor-like structure. As proven in [27], there exists an optimal solution $\{(i^1, j^1, k^1), \ldots, (i^n, j^n, k^n)\}$ to $3\mathrm{AP} - \mathrm{DC}$ such that every triple of indices (i^g, j^g, k^g), $1 \leq g \leq n$, satisfies

$$n + 2 \leq i^g + j^g + k^g \leq 2n + 1.$$

Below we illustrate the structure of solution matrices for the two instances of problems $\mathrm{AP}(3, 1)$ and $3\mathrm{AP} - \mathrm{DC}$ with $n = 15$. We consider one layer for each problem with $k^g = 5$, marking

feasible positions for 1-entries by $*$. For problem $\mathrm{AP}(3,1)$, those positions belong to the corridor along the main diagonal with $1 \le i \le 11$ to satisfy $|i-k| \le d\,(d-1)\,\lambda = 6$; additionally they satisfy $|i-j| \le 6$. For problem $3\mathrm{AP}-\mathrm{DC}$ positions for 1-entries form a corridor that spans along the counterdiagonal, such that $17 \le i+j+5 \le 31$.

Solution to $\mathrm{AP}(3,1)$, layer $k^g = 5$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	*	*	*	*	*	*	*	0	0	0	0	0	0	0	0
2	*	*	*	*	*	*	*	*	0	0	0	0	0	0	0
3	*	*	*	*	*	*	*	*	*	0	0	0	0	0	0
4	*	*	*	*	*	*	*	*	*	*	0	0	0	0	0
5	*	*	*	*	*	*	*	*	*	*	*	0	0	0	0
6	*	*	*	*	*	*	*	*	*	*	*	*	0	0	0
7	*	*	*	*	*	*	*	*	*	*	*	*	*	0	0
8	0	*	*	*	*	*	*	*	*	*	*	*	*	*	0
9	0	0	*	*	*	*	*	*	*	*	*	*	*	*	*
10	0	0	0	*	*	*	*	*	*	*	*	*	*	*	*
11	0	0	0	0	*	*	*	*	*	*	*	*	*	*	*
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Solution to $3\mathrm{AP}-\mathrm{DC}$, layer $k^g = 5$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*
2	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*
3	0	0	0	0	0	0	0	0	*	*	*	*	*	*	*
4	0	0	0	0	0	0	0	*	*	*	*	*	*	*	*
5	0	0	0	0	0	0	*	*	*	*	*	*	*	*	*
6	0	0	0	0	0	*	*	*	*	*	*	*	*	*	*
7	0	0	0	0	*	*	*	*	*	*	*	*	*	*	*
8	0	0	0	*	*	*	*	*	*	*	*	*	*	*	*
9	0	0	*	*	*	*	*	*	*	*	*	*	*	*	*
10	0	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	*	*	*	*	*	*	*	*	0
13	*	*	*	*	*	*	*	*	*	*	*	*	*	0	0
14	*	*	*	*	*	*	*	*	*	*	*	*	0	0	0
15	*	*	*	*	*	*	*	*	*	*	*	0	0	0	0

The width of the corridor for our model $\mathrm{AP}(d,\lambda)$ is fixed for given d and λ, which makes

it possible to organize the search efficiently using dynamic programming. The width of the corridor for 3AP − DC depends on n, so that dynamic programming would be of exponential time complexity. Recall that 3AP − DC is NP-hard, unlike the version of problem AP$(3, \lambda)$ with a fixed λ. Still the corridor property helps in the development of successful heuristics narrowing the search towards the corridor area which contains a subset of candidate triples, reduced from n^3 to $n(n^2-1)/3$, see [27].

5.7.2 Planar 3-dimensional assignment problem with a layered Monge matrix

The planar 3-dimensional assignment problem (P3AP) is another example of a model where optimal solutions have a corridor-like structure, provided each layer of the cost array is a Monge matrix, see [35]. The width of the corridor depends on the number of layers p, $p \leq n$. Formally the *p-layer planar 3-dimensional assignment problem* (p-P3AP) with an $n \times n \times p$ cost array $C = (c_{ijk})$ is defined as follows:

$$\begin{array}{lll} \min & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}x_{ijk} & \\ \text{s.t.} & \sum_{k=1}^{p} x_{ijk} \leq 1, & 1 \leq i,j \leq n, \\ & \sum_{i=1}^{n} x_{ijk} = 1, & 1 \leq j \leq n, \quad 1 \leq k \leq p, \\ & \sum_{j=1}^{n} x_{ijk} = 1, & 1 \leq i \leq n, \quad 1 \leq k \leq p, \\ & x_{ijk} \in \{0,1\}, & 1 \leq i,j \leq n, \quad 1 \leq k \leq p. \end{array} \tag{5.29}$$

Assuming that indices i and j define rows and columns, while index k defines layers, the task is to select np 1-entries $x_{ijk} = 1$ such that

(a) in each layer k, $1 \leq k \leq p$, there are n 1-entries, one in each row and one in each column, and

(b) among all layers no 1-entry appears more than once for the same pair of indices (i, j).

The problem p-P3AP is NP-hard for every fixed $p \geq 2$. However, for the version p-P3AP$_{\text{Monge}}$ of that problem, with 2-dimensional Monge matrices C^k ($c^k_{ij} = c_{ijk}$) in every layer k, $1 \leq k \leq p$, the corridor property holds, and the problem is fixed parameter tractable with respect to parameter p, using a dynamic programming algorithm, see [35]. In particular it is proven in [35] that there always exists an optimal solution $\{(i^g_k, j^g_k, k) | 1 \leq g \leq n, 1 \leq k \leq p\}$ to p-P3AP$_{\text{Monge}}$, such that

$$|i^g_k - j^g_k| \leq 2p - 2, \tag{5.30}$$

for all $1 \leq g \leq n$ and $1 \leq k \leq p$.

It is not a coincidence that the corridor condition (5.30) from [35] resembles the corridor condition (5.15) from Section 5.5. As we show below, the three-dimensional problem p-P3AP$_{\text{Monge}}$ with Monge matrices C^k, $1 \le k \le p$, reduces to the multi-dimensional problem AP(d, λ) with a nearly Monge array W.

Theorem 30. *The problem* p-P3AP$_{\text{Monge}}$ *reduces to problem* AP(d, λ) *with* $d = p + 1$ *and* $\lambda = 1$,

$$p - \text{P3AP}_{\text{Monge}} \propto \text{AP}(p + 1, 1).$$

Proof: Given an instance of p-P3AP$_{\text{Monge}}$, we define an instance of AP$(p+1, 1)$ by introducing variables $x_{ij_1\ldots j_p}$ and the array W with entries

$$w_{ij_1\ldots j_p} = \begin{cases} c^1_{i,j_1} + c^2_{i,j_2} + \cdots + c^p_{i,j_p}, & \text{if all indices } j_1, j_2, \ldots, j_p \text{ are different,} \\ \infty, & \text{otherwise.} \end{cases} \tag{5.31}$$

Here, the first index i corresponds to the row index in C, indices j_k define the column indices in layers C^k, $k = 1, 2, \ldots, p$, so that $w_{i,j_1\ldots j_p}$ is the combined cost of selecting p 1-entries with the fixed first index i as part of the solution to problem (5.29):

$$\begin{aligned} x_{i,j_1,1} = x_{i,j_2,2} = \cdots = x_{i,j_p,p} = 1 \quad &\Longleftrightarrow \quad x_{i,j_1\ldots j_p} = 1, \\ \sum_{k=1}^{p} c_{i,j_k,k} x_{i,j_k,k} \quad &= \quad w_{i,j_1,\ldots,j_p} x_{i,j_1,\ldots,j_p}. \end{aligned}$$

The ∞-entries, which arise for the duplicating j-indices, prohibit assignments of the form $x_{i,j_u,u} = x_{i,j_v,v} = 1$ if $j_u = j_v$.

Every feasible solution to p-P3AP$_{\text{Monge}}$ satisfies properties (a)–(b); thus it defines a finite solution to AP$(p + 1, 1)$ with the same value of the objective. On the other hand, every finite solution to AP$(p+1, 1)$ defines a feasible solution to p-P3AP$_{\text{Monge}}$ satisfying (a)–(b), also with the same value of the objective; an ∞-solution to AP$(p + 1, 1)$ translates into an infeasible solution to p-P3AP$_{\text{Monge}}$.

It is easy to verify that the finite values of W satisfy the Monge property, so that W is a nearly Monge matrix with at most one incompatible partner for any index $j_u = j^*_u$ in every dimension corresponding to index j_v, $v \ne u$. Moreover, there are no incompatible partners for the first index i. ∎

The corridor property (5.30) derived in [35] for problem p-P3AP$_{\text{Monge}}$ can be reformulated in terms of the associated problem AP$(p + 1, 1)$ with variables $x_{i,j_1\ldots j_p}$ as

$$|j_\ell - i| \le 2p - 2 \quad \text{for } 1 \le \ell \le p.$$

Notice that the corridor property established in Section 5.5 for problem AP(d, λ) with $d = p+1$

and $\lambda = 1$ is weaker,

$$|j_\ell - i| \leq d(d-1)\lambda = (p+1)p \quad \text{for } 1 \leq \ell \leq p.$$

This discrepancy happens because the finite entries of array W defined by (5.31) satisfy not only the multi-dimensional Monge property of the form

$$w_{s\ s_1 \dots s_p} + w_{t\ t_1 \dots t_p} \leq w_{i\ j_1 \dots j_p} + w_{i'\ j'_1 \dots j'_p}, \tag{5.32}$$

where

$$\begin{aligned} s &= \min\{i, i'\}, \quad s_\ell = \min\{j_\ell, j'_\ell\}, \quad \ell = 1, \dots, p, \\ t &= \max\{i, i'\}, \quad t_\ell = \max\{j_\ell, j'_\ell\}, \quad \ell = 1, \dots, p, \end{aligned}$$

but also the condition similar to the 2-dimensional Monge property, namely for $i \leq i'$ and every $\ell = 1, \dots, p$,

$$w_{ij_1 \dots j_{\ell-1} \boldsymbol{s_\ell} j_{\ell+1} \dots j_p} + w_{i'j'_1 \dots j'_{\ell-1} \boldsymbol{t_\ell} j'_{\ell+1} \dots j'_p} \leq w_{ij_1 \dots j_{\ell-1} \boldsymbol{j_\ell} j_{\ell+1} \dots j_p} + w_{i'j'_1 \dots j'_{\ell-1} \boldsymbol{j'_\ell} j'_{\ell+1} \dots j'_p}, \tag{5.33}$$

where the changes affect the indices in bold, the remaining indices staying the same. Note that (5.33) implies (5.32), but not the other way around. The latter can be illustrated by the example with the 3-dimensional array $W^{\max}$ defined by $w^{\max}_{ijk} = \max\{i, j, k\}$, which satisfies (5.32) but not (5.33). Thus, property (5.33) is stronger than property (5.32).

In what follows we show that the proof of Theorem 27 can be adjusted to derive a corridor width of $\xi^* = 2p\lambda$ for problem $\mathrm{AP}(p+1, \lambda)$ with a weight array W satisfying property (5.33). We then use ξ^* in order to derive a corridor width of $\xi^{**} = 2p-2$ if W is defined by (5.31), thus matching the result from [35] for problem p-P3AP$_{\mathrm{Monge}}$. This justifies our earlier statement that the discrepancy of the two results is due to the stronger property (5.33) and illustrates the capabilities of the proof technique used for Theorem 27.

To prove the corridor width $\xi^* = 2p\lambda$, first consider a typical exchange step from the proof of Theorem 27 which deals with a d-tuple $(i, j_1, \dots, j_{\ell-1}, \boldsymbol{j_\ell}, j_{\ell+1}, \dots, j_p)$ with violating $\boldsymbol{j_\ell}$. For the exchange, we select a companion d-tuple $\left(i', j'_1, \dots, j'_{\ell-1}, \boldsymbol{j'_\ell}, j'_{\ell+1}, \dots, j'_p\right)$ in order to generate two induced d-tuples without increasing the value of the objective. As shown in the proof, the corridor of width ξ contains $\xi + 1$ candidates for a companion d-tuple. The value $\xi = d(d-1)\lambda$ used in Theorem 27, together with the upper bound (5.1) on the number of incompatible partners, guarantees that at least one of the candidates is suitable for the exchange step, so that both induced d-tuples do not have incompatible indices. Recall that the exchange step in the proof of Theorem 27 affects all indices.

Condition (5.33) makes it possible to define simpler exchanges in comparison with those used in the proof of Theorem 27. If a d-tuple $(i, j_1, \dots, \boldsymbol{j_\ell}, \dots, j_p)$ is of Type I($\boldsymbol{j_\ell}$), then the companion d-tuple $(i', j'_1, \dots, \boldsymbol{j'_\ell}, \dots, j'_p)$ should satisfy $i' > i$ and $\boldsymbol{j'_\ell} < \boldsymbol{j_\ell}$. In the exchange step, the d-tuples $(i, j_1, \dots, j_{\ell-1}, \boldsymbol{j_\ell}, j_{\ell+1}, \dots, j_p)$ and $\left(i', j'_1, \dots, j'_{\ell-1}, \boldsymbol{j'_\ell}, j'_{\ell+1}, \dots, j'_p\right)$ corresponding to the

right-hand side of (5.33), are replaced by the two induced d-tuples $\left(i, j_1, \ldots, j_{\ell-1}, \boldsymbol{j'_\ell}, j_{\ell+1}, \ldots, j_p\right)$ and $\left(i', j'_1, \ldots, j'_{\ell-1}, \boldsymbol{j}_\ell, j'_{\ell+1}, \ldots, j'_p\right)$, where only two indices $\boldsymbol{j}_\ell$ and $\boldsymbol{j'_\ell}$ are exchanged. The induced d-tuples do not contain incompatible indices if $\boldsymbol{j_\ell}$ is compatible with $\{i', j'_1, \ldots, j'_{\ell-1}, j'_{\ell+1}, \ldots, j'_p\}$ and $\boldsymbol{j'_\ell}$ is compatible with $\{i, j_1, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_p\}$. Since there are at most λp incompatible partners for $\boldsymbol{j_\ell}$ and the first set of p indices, and at most λp incompatible partners for $\boldsymbol{j'_\ell}$ and the second set of p indices, the total number of companion d-tuples for $(i, j_1, \ldots, j_{\ell-1}, \boldsymbol{j}_\ell, j_{\ell+1}, \ldots, j_p)$, which can lead to incompatibilities, is bounded by $2\lambda p$. Choosing the corridor width of $\xi^* = 2\lambda p$ guarantees the existence of $\xi^* + 1 = 2\lambda p + 1$ candidates for a companion d-tuple in the desirable area and therefore the existence of at least one suitable candidate among them. It is easy to make sure that similar arguments are applicable for violations of Type II($\boldsymbol{j_\ell}$).

Now consider a weight array W defined by (5.31). For that type of array there are no incompatible partners for first indices i, and the arguments presented in the previous paragraph can be adjusted: there are at most $\lambda\,(p-1)$ incompatible partners for $\boldsymbol{j_\ell}$ and at most $\lambda\,(p-1)$ incompatible partners for $\boldsymbol{j'_\ell}$. This implies that instead of $\xi^* = 2p\lambda$ we can consider the corridor width $\xi^{**} = 2(p-1)\lambda$. Substituting $\lambda = 1$ which holds for (5.31) we get the required estimate $\xi^{**} = 2p - 2$.

5.7.3 Bottleneck assignment problem with a bottleneck nearly Monge matrix and its generalizations

Traditional research on optimization problems with Monge matrices examines first problems with min-sum objectives and then explores the counterparts with min-max objectives. In particular, the results known for the min-sum versions of the transportation and assignment problems with Monge matrices can be transferred to the *bottleneck* versions of those problems, in which "+" is replaced by "max" in the objective function (2.2) and in the definition of the Monge property (2.4), see [26]. For the bottleneck assignment problem, the goal is to minimize $\max\{w_{i_1\ldots i_d} | x_{i_1\ldots i_d} = 1\}$ subject to the constraints from (2.2), while the cost array W satisfies the bottleneck Monge property:

$$\max\{w_{s_1 s_2 \ldots s_d}, w_{t_1 t_2 \ldots t_d}\} \leq \max\{w_{i_1 i_2 \ldots i_d}, w_{j_1 j_2 \ldots j_d}\}. \tag{5.34}$$

Here $s_\ell = \min\{i_\ell, j_\ell\}$ and $t_\ell = \max\{i_\ell, j_\ell\}$ for $\ell = 1, \ldots, d$. Then an optimal solution to the bottleneck assignment is given by the same n d-tuples $(1, \ldots, 1)$, $(2, \ldots, 2)$, …, $(n, \ldots, n)$, as for the case of the linear assignment problem.

Adapting our definition of a nearly Monge matrix to the bottleneck case, we call an array W which contains $\infty-$entries *bottleneck nearly Monge* if condition (5.34) is satisfied for all finite entries. It is easy to verify that the proof of the corridor property presented in Section 5.5 can be modified accordingly, without affecting the width of the corridor, so that the bottleneck assignment problem with a bottleneck nearly Monge cost array can be solved in linear time by

the adapted dynamic programming approach of Section 5.6.

The next step in extending the applicability of our results is to consider the *algebraic assignment problem* with an underlying totally ordered commutative semigroup $(H, \oplus, \preceq)$, see [26]. This problem can be considered as a generalization of both the linear and the bottleneck assignment problems. It is known that if in the algebraic assignment problem the cost array is *algebraic Monge* (i.e., condition (2.4) holds with "+" replaced by "$\oplus$" and "$\leq$" replaced by "$\preceq$"), then an optimal solution is of the same diagonal shape as in the case of linear assignment and bottleneck assignment.

In the context of our results, we consider the extension of a cost array with ∞-entries. Note that the semigroup $(H, \oplus, \preceq)$ can be naturally modified to include an ∞-element if such an element does not already exist. Consider $(H^*, \oplus^*, \preceq^*)$ given by the set $H^* := H \cup \{\infty\}$, the extension $\oplus^*$ of $\oplus$ such that $a \oplus^* \infty = \infty \oplus^* a = \infty$ for all $a \in H^*$ and the extension $\preceq^*$ of $\preceq$ such that $a \preceq^* \infty$ for all $a \in H^*$. Then $(H^*, \oplus^*, \preceq^*)$ is a totally ordered commutative semi-group. We define an *algebraic nearly Monge* array as before: it is required that the algebraic Monge property should be satisfied for finite entries only. The arguments of Sections 5.5-5.6 can be adapted accordingly; notice that the proof of the corridor property only uses the properties of $(\mathbb{R} \cup \{\infty\}, +, \leq)$ that it shares with semi-groups of the form $(H^*, \oplus^*, \preceq^*)$. Thus, the proposed methodology is applicable to the algebraic assignment problem as well.

5.8 NP-completeness of 3-DM with incompatible partner indices

In this section we demonstrate that reduction

$$3\text{−SAT} \propto 3\text{−DM}$$

presented in [59], p. 50–53, can be re-stated as

$$3\text{−SAT} \propto 3\text{−DM}(\lambda).$$

Here 3-SAT and 3-DM denote the traditional 3-SATISFIABILITY and 3-DIMENSIONAL MATCHING problems, while 3-DM(λ) denotes the version of 3-DM introduced in Section 5.3. It corresponds to a special case of the 3-dimensional assignment problem AP$(3, \lambda)$ with at most λ incompatible partners for any index and any other dimension.

In 3-SAT, there are given a set of variables and a set of clauses, with each clause containing at most 3 variables. The question is whether there exists a truth assignment for variables that satisfies all clauses.

In 3-DM, there are given 3 disjoint sets X, Y and Z, with $|X| = |Y| = |Z|$, and a set of triples $M \subseteq (X \times Y \times Z)$. The question is whether there exists a perfect matching, i.e., a set

$M' \subseteq M$, such that each element of X, Y and Z appears in exactly one triple in M'.

In the special case 3-DM(λ) of 3-DM, the set M is defined as the set of all possible triples $\{(x, y, z)\}$ with $x \in X$, $y \in Y$, $z \in Z$, except for forbidden triples F,

$$M = (X \times Y \times Z) \backslash F, \tag{5.35}$$

where the set F is defined by specifying pairs of incompatible partner indices,

$$F = \{(x_\alpha, y_\beta, *)\} \cup \{(x_\gamma, *, z_\delta)\} \cup \{(*, y_\nu, z_\mu)\}.$$

The place holder $*$ denotes an arbitrary element of the corresponding set. Parameter λ in 3-DM(λ) is the maximum number of incompatible partners, which any element from X, Y or Z, may have in another set.

Consider the reduction 3-SAT$\propto$ 3-DM from [59]. The instance of 3-DM constructed there is based on the instance of 3-SAT with n variables $u_1, u_2, \ldots, u_n$ and m clauses $c_1, c_2, \ldots, c_m$. The number of elements in each set X, Y and Z in the instance of 3-DM is $2nm$. To define these sets, four types of elements are introduced:

- elements $u_i[j]$ and $\bar{u}_i[j]$ for each $1 \le i \le n$ and $1 \le j \le m$;
- elements $a_i[j]$ and $b_i[j]$ for each $1 \le i \le n$ and $1 \le j \le m$;
- elements $s_1[j]$ and $s_2[j]$ for each $1 \le j \le m$;
- elements $g_1[k]$ and $g_2[k]$ for each $1 \le k \le (n-1)m$.

The three sets X, Y and Z are composed of the above elements in the following way:

- $X = \{u_i[j], \bar{u}_i[j] \mid 1 \le i \le n,\ 1 \le j \le m\}$, so that $|X| = 2nm$;
- $Y = \{a_i[j] \mid 1 \le i \le n,\ 1 \le j \le m\} \cup \{s_1[j] \mid 1 \le j \le m\} \cup \{g_1[k] \mid 1 \le k \le (n-1)m\}$, so that $|Y| = nm + m + (n-1)m = 2nm$;
- $Z = \{b_i[j] \mid 1 \le i \le n, 1 \le j \le m\} \cup \{s_2[j] \mid 1 \le j \le m\} \cup \{g_2[k] \mid 1 \le k \le (n-1)m\}$, so that $|Z| = nm + m + (n-1)m = 2nm$.

The set of triples M consists of

(I) triples $(\bar{u}_i[j], a_i[j], b_i[j])$ and $(u_i[j], a_i[j+1], b_i[j])$ for all $1 \le i \le n$ and $1 \le j \le m$, where the addition $j+1$ is computed modulo m;

(II) triples $(u_i[j], s_1[j], s_2[j])$, if u_i is a variable in clause c_j, and triples $(\bar{u}_i[j], s_1[j], s_2[j])$, if $\bar{u}_i$ is a variable in clause c_j;

(III) triples $(u_i[j], g_1[k], g_2[k])$ and $(\bar{u}_i[j], g_1[k], g_2[k])$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq (n-1)m$.

It is proven in [59] that a solution to the constructed instance of 3-DM exists if and only if there exists a solution to the instance of 3-SAT. We demonstrate that the above instance of 3-DM can be re-stated as an instance of 3-DM(λ). To this end we define the set F and show that $(X \times Y \times Z) \backslash F$ coincides with M defined as (I)-(III).

Consider the following set F of templates for incompatible partners:

- $(\bar{u}_i[j], a_p[q], *)$ and $(\bar{u}_i[j], *, b_p[q])$ for all $i \neq p$ or $j \neq q$;
- $(u_i[j], a_p[q], *)$ for all $i \neq p$ or $j+1 \neq q$, where the addition $j+1$ is computed modulo m;
- $(u_i[j], *, b_p[q])$ for all $i \neq p$ or $j \neq q$;
- $(u_i[j], s_1[q], *)$ and $(u_i[j], *, s_2[q])$ for all $j \neq q$ or $u_i \notin c_j$;
- $(\bar{u}_i[j], s_1[q], *)$ and $(\bar{u}_i[j], *, s_2[q])$ for all $j \neq q$ or $\bar{u}_i \notin c_j$;
- $(*, a_i[j], s_2[q])$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq q \leq m$;
- $(*, s_1[q], b_i[j])$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq q \leq m$;
- $(*, g_1[k], g_2[k'])$ for all $k \neq k'$;
- $(*, g_1[k], b_i[j])$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq (n-1)m$;
- $(*, g_1[k], s_2[j])$ for all $1 \leq k \leq (n-1)m$ and $1 \leq j \leq m$;
- $(*, a_i[j], g_2[k])$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq (n-1)m$;
- $(*, s_1[j], g_2[k])$ for all $1 \leq j \leq m$ and $1 \leq k \leq (n-1)m$.

In order to prove that F defined above provides a complete characterization of M in terms of (5.35), we show that for any choice of element $x \in X$, a triple $e = (x, y, z)$ with some $y \in Y$ and $z \in Z$ belongs to M if and only if none of the $(x, y, *)$, $(x, *, y)$ or $(*, y, z)$ belongs to F. We restrict our consideration to the case of $x = \bar{u}_i[j]$ for some $1 \leq i \leq n$, $1 \leq j \leq m$; the case of $x = u_i[j]$ is similar, and these are the only two possible choices for x.

($\Rightarrow$) Let $e \in M$. Then in accordance with (I)-(III), there are three possibilities for e, namely $(\bar{u}_i[j], a_i[j], b_i[j])$, $(\bar{u}_i[j], s_1[j], s_2[j])$ with $\bar{u}_i$ being a variable of c_j, or $(\bar{u}_i[j], g_1[k], g_2[k])$. Comparing each of these three triples with the templates of F we conclude that none of them includes a pair of incompatible partners.

($\Leftarrow$) Let $e = (\bar{u}_i[j], y, z)$ be a triple that does not contain a pair of incompatible partners defined by F. We perform a case analysis, dependent on the type of the second entry y,

demonstrating that $e \in M$. Recall that Y consists of elements of types $a_p[q]$, $s_1[q]$, $g_1[k]$, while Z consists of elements of types $b_p[q]$, $s_2[q]$, $g_2[k]$.

Suppose y is of type $a_p[q]$. Then, as $(\bar{u}_i[j], a_p[q], *)$ is incompatible unless $i = p$ and $j = q$, we have $e = (\bar{u}_i[j], a_i[j], z)$. Furthermore, as $(*, a_i[j], s_2[q])$ and $(*, a_i[j], g_2[k])$ are both incompatible for all $1 \leq q \leq m$ and $1 \leq k \leq (n-1)m$, then e must be of the form $(\bar{u}_i[j], a_i[j], b_p[q])$. Moreover, as $(\bar{u}_i[j], *, b_p[q])$ is incompatible unless $i = p$ and $j = q$, we have $e = (\bar{u}_i[j], a_i[j], b_i[j])$. Thus e is a triple of type (I) in M.

Suppose y is of type $s_1[q]$. Since $(\bar{u}_i[j], s_1[q], *)$ is incompatible unless $j = q$ and $\bar{u}_i \in c_j$, we have $e = (\bar{u}_i[j], s_1[j], z)$. Furthermore, since both $(*, s_1[j], b_p[q])$ and $(*, s_1[j], g_2[k])$ are incompatible for all $1 \leq p \leq n$, $1 \leq q \leq m$ and $1 \leq k \leq (n-1)m$, then e must be of the form $(\bar{u}_i[j], s_1[j], s_2[q])$. Finally, $(\bar{u}_i[j], *, s_2[q])$ is incompatible unless $j = q$ and $\bar{u}_i \in c_j$. Thus $e = (\bar{u}_i[j], s_1[j], s_2[j])$ and $\bar{u}_i \in c_j$. Thus e is a triple of type (II) in M.

Finally, suppose y is of type $g_1[k]$. Since both $(*, g_1[k], b_i[j])$ and $(*, g_1[k], s_2[q])$ are incompatible for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq q \leq m$, then e must be of the form $(\bar{u}_i[j], g_1[k], g_2[k'])$. Since $(*, g_1[k], g_2[k'])$ is incompatible unless $k = k'$, we have $e = (\bar{u}_i[j], g_1[k], g_2[k])$. Thus e is a triple of type (III) in M.

We conclude by estimating the maximum number λ of incompatible partners for any index. Recall that there are exactly $2nm$ elements in each set X, Y and Z and every element is in at least one triple from M. Therefore no element can have $2nm$ incompatible partners in another set. On the other hand, the element $g_1[1] \in Y$ has $2nm - 1$ incompatible partners $z \in Z$, namely all elements in Z other than $g_2[1]$. It follows that $\lambda = 2nm - 1$.

Chapter 6

The relaxed problem $O|synmv, rel|C_{\max}$

In this brief chapter, we extend the consideration of the relaxed problem $O|synmv, rel|C_{\max}$ to instances with more than two machines. Recall that in Section 4.1.2 we showed that in the two machine case it can be beneficial to introduce dummy jobs and allow more than the minimum number n of cycles. We also proved that for an optimal solution at most one additional cycle is needed and gave necessary conditions for an additional cycle to be needed. The main result of this chapter is a tight upper bound on the number of cycles in an optimal schedule for an arbitrary number m of machines. First we provide a new example that introducing dummy jobs can reduce the makespan of an optimal schedule, this time for $m = 3$ machines.

Example 31. Consider an example with $m = 3$ machines, $n = 5$ jobs and the following processing times:

j	1	2	3	4	5
p_{1j}	3	2	4	3	1
p_{2j}	5	3	2	3	1
p_{3j}	4	5	1	4	1

In the upper part of Fig. 6.1 an optimal schedule for problem $O3|synmv|C_{\max}$ with $n = 5$ cycles and a makespan of 18 is shown. For the relaxed problem $O3|synmv, rel|C_{\max}$ adding a single dummy job J_6 leads to an improved schedule with 6 cycles and makespan 17 (see the lower part of Fig. 6.1).

Recall from Section 5.3 that the maximum total number of cycles of non-zero length is nm, which occurs if each of the nm "actual" operations is scheduled in an individual cycle. Then, in each of these nm cycles one actual operation and $m-1$ dummy operations are processed. As was already observed in the last chapter, in order achieve an optimal schedule it is therefore sufficient to include $n(m-1)$ dummy jobs, each dummy job consisting of m zero-length operations. This

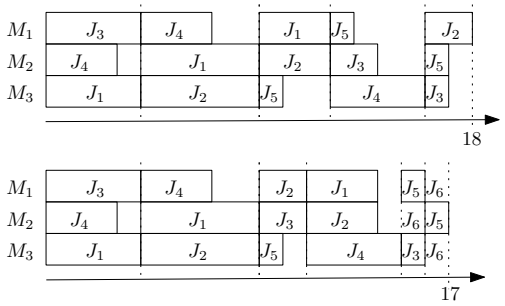


Figure 6.1: An optimal schedule for $O3|synmv|C_{\max}$ and an improved schedule for $O3|synmv, rel|C_{\max}$ (with dummy job J_6)

implies that problem $Om|synmv, rel|C_{\max}$ for a fixed number m of machines can be solved in linear time after presorting, by the algorithm from the last chapter.

Clearly, for algorithmic purposes it is desirable to have the number of added dummy jobs as small as possible. As discussed in [153], for the synchronous flow shop problem $F|synmv, rel|C_{\max}$, instances exist where for an optimal solution $(n-1)(m-2)$ dummy jobs are needed. In the following we show that for the open shop problem $O|synmv, rel|C_{\max}$ at most $m-1$ dummy jobs are needed to obtain an optimal solution.

Theorem 32. *There exists an optimal solution to problem $O|synmv, rel|C_{\max}$ with at most $m-1$ dummy jobs, so that the number of cycles is at most $n+m-1$.*

Proof: Let S be an optimal schedule with ξ dummy jobs, $\xi \geq m$. We construct another schedule $\widetilde{S}$ with $C_{\max}(\widetilde{S}) \leq C_{\max}(S)$ and $\xi - 1$ dummy jobs. Then the proof follows by induction. Notice that it is allowed to assign several operations of the same dummy job to any cycle.

Case 1: If there exists a cycle I' which consists solely of dummy operations of the same job $J_d \in \{J_{n+1}, J_{n+2}, \ldots, J_{n+\xi}\}$, then that dummy job can be eliminated and $\widetilde{S}$ is found.

Case 2: If there exists a cycle I' which consists solely of dummy operations, some of which belong to different dummy jobs, then we can achieve Case 1 by selecting a dummy job J_d arbitrarily and swapping its operations from outside I' with the dummy operations in I'. The resulting schedule is feasible and has the same makespan.

Case 3: Suppose no cycle in S consists purely of dummy operations. Let I' be the shortest cycle and let ν be the number of actual operations in I', $1 \leq \nu \leq m$. We demonstrate that each actual operation processed in I' can be swapped with a dummy operation from another cycle. Consider an actual operation O_{ij} in cycle I' with machine M_i processing job J_j. Select another cycle I'' (its existence is demonstrated below) such that it does not involve an operation of J_j and has a dummy operation on M_i. Swap operations on M_i in I' and I'', reducing the number of actual operations in I' by 1. Clearly, after the swap both cycles are feasible, because introducing a dummy operation into I' cannot cause a conflict, and because no operation of J_j was processed in I'' before the swap. After the swap, both cycles I' and I'' have either the

same length as before or cycle I' becomes shorter. Performing the described swaps for each actual operation O_{ij} in cycle I', we arrive at Case 1 or 2.
A cycle I'' exists since

- there are at least ξ cycles with a dummy operation on M_i ($\xi \geq m$) and those cycles are different from I';
- there are exactly $m-1$ cycles with J_j processed on a machine that differs from M_i, and those cycles are different from I'. ∎

Note that Theorem 32 generalizes the result from Section 4.1.2 that at most one dummy job is needed for the problem with two machines. We assume that the necessary conditions proved in Section 4.1.2 may also be generalizable. However, for the m machine case, there would have to be one set of conditions for every number of dummy jobs that could be introduced, i.e. $m-1$ different sets of conditions. Thus a generalization would become relatively complicated even for small numbers m of machines. Since additionally, due to the linear time algorithm the introduction of $m-1$ dummy jobs does not substantially increase the running time and so the necessary conditions, while structurally interesting, are not particularly important for algorithmic purposes, we decided not to attempt a generalization here.

Instead, we continue by demonstrating that the bound $m-1$ is tight for any choice of m (for $m=2$ an example where at least one dummy job is needed was already provided in Section 4.1.2).

Example 33. Consider an instance of problem $O|synmv, rel|C_{\max}$ with m machines, $n = m+1$ jobs and processing times $p_{ij} = 2m$ for $i = 1, \ldots, m, j = 1, \ldots, n-1$, and $p_{in} = 1$ for $i = 1, \ldots, m$.

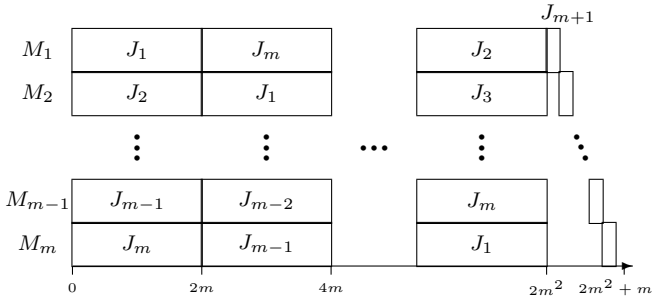


Figure 6.2: An optimal schedule with $2m$ cycles, m of which are complete and m are incomplete

An optimal schedule consists of m complete cycles of length $2m$ each, containing operations of the jobs $\{J_1, J_2, \ldots, J_m\}$ only, and m incomplete cycles with the single actual job J_{m+1} grouped with $m-1$ dummy jobs, see Fig. 6.2. The optimal makespan is $C^{\text{opt}}_{\max} = 2m^2 + m$. In any schedule with less than $m-1$ dummy jobs, at least one operation of job J_{m+1} is grouped with another operation of an actual job, the length of such a cycle being $2m$. Thus, a schedule

with less that $m-1$ dummy jobs consists of at least $m+1$ cycles of length $2m$, so that the makespan is at least $2m(m+1) > C_{\max}^{\text{opt}}$.

Notice that since this thesis mainly focuses on scheduling aspects, we have presented Theorem 32 in the scheduling language for the sake of consistency and self-containment. Knowing that $O|synmv, rel|C_{\max}$ is equivalent to the max-weight edge coloring problem on the complete bipartite graph $K_{m,n}$, we conclude this chapter by linking Theorem 32 to the results known in the area of max-weight coloring. It is known that an optimal max-weight edge coloring in an edge-weighted graph G can always be obtained using at most $2\Delta - 1$ colors, where Δ is the maximum vertex degree of G, see for example [40, 43]. This bound is worse than the bound given in Theorem 32, as for a complete bipartite graph $G = K_{m,n}$ with $m < n$ we have $\Delta = n$, and therefore $2\Delta - 1 = n + n - 1 > n + m - 1$. However, for the vertex coloring version of max-weight coloring on a vertex-weighted graph G, it is shown in [43] that an optimal max-weight vertex coloring can be obtained using at most $\Delta + 1$ colors. Note that the max-weight edge coloring problem on a graph H can be seen as the max-weight vertex coloring problem on the line graph $G = L(H)$ of H. Then, since the line graph of $K_{m,n}$ has maximum degree $\Delta = n + m - 2$, the bound $\Delta + 1$ on the number of colors needed yields $n + m - 1$, which is equal to the maximum number of cycles stated in Theorem 32.

Chapter 7

Conclusions and further research

7.1 Synchronous open shop scheduling

In this part of the thesis the main focus was on synchronous open shop scheduling problems. The results are summarized in Table 7.1. Note that the polynomial-time results in lines 2 and 3 do not include pre-sorting of all jobs.

Problem	Complexity	References
$O\|synmv\|C_{\max}$	str. NP-h.	Section 3.2 and [40, 43, 126]
$Om\|synmv\|C_{\max}$	$O(n)$	Section 5.6
$O2\|synmv\|C_{\max}$	$O(n)$	Section 4.1
$O2\|synmv, C_j \leq D_j\|-$	str. NP-c.	Section 4.2
$O2\|synmv, C_j \leq D_j,$ $D_j \in \{D', D''\}\|-$	NP-c.	Section 4.2
$O2\|synmv\|\sum C_j$	str. NP-h.	Section 4.3

Table 7.1: Summary of the results for synchronous open shop scheduling

All results from Table 7.1 also hold for the relaxed versions of the scheduling problems, in which cycles may consist of less than m jobs.

For problem $O2|synmv|C_{\max}$ we showed that it can be modelled as the assignment problem with an underlying Monge cost matrix and forbidden partners. We proved a new structural property, namely the small block property and used it to formulate a much easier and faster solution algorithm than the standard assignment algorithms, see, e.g., [88].

Similarly, for $Om|synmv|C_{\max}$ with $m > 2$, we showed that it can be modelled as an m-dimensional assignment problem with a nearly Monge cost matrix. Using a slightly more complicated structural property, we proved that the problem is solvable in linear time (after presorting), like its two-machine counter part. However the linear constant in the running time grows very fast and appears to become impractical even for relatively small numbers of

machines. Our result for two machines gives hope that this general result for fixed m may also be improved and highlights possible approaches for such an improvement.

The NP-completeness results of Section 4.2 imply that if instead of hard deadlines D_j soft due dates d_j are given (which are desirable to be met, but can be violated), then the corresponding problems $O|synmv|f$ with the traditional regular due date related objectives f such as the maximum lateness $L_{\max} = \max_{1\leq j\leq n}\{C_j - d_j\}$, the number of late jobs $\sum_{j=1}^{n} U_j$, or the total tardiness $\sum_{j=1}^{n} T_j$ are NP-hard, even if there are only two values of the due dates, $d_j \in \{d, d'\}$. The corresponding problems become strongly NP-hard in the case of arbitrary due dates d_j.

Also, due to the symmetry known for problems with due dates d_j and those with release dates r_j, we conclude that problem $O2|synmv, r_j|C_{\max}$ is strongly NP-hard and remains at least ordinary NP-hard if there are only two different values of release dates for the jobs.

In Section 4.3 we show that $O2|synmv|\sum C_j$ and its relaxed version are strongly NP-hard. Thus, due to the reducibility between scheduling problems with different objectives, the open shop problem with synchronization is NP-hard for any traditional scheduling objective function, except for $C_{\max}$.

Overall the synchronized version of the open shop problem appears to be no harder than the classical version, with two additional positive results for it: 1) $Om|synmv|C_{\max}$ is polynomially solvable for any fixed m while $Om||C_{\max}$ is NP-hard for $m \geq 3$ [62]; 2) $O|synmv, n = n'|C_{\max}$ is polynomially solvable for any fixed number of jobs n' (due to the symmetry of jobs and machines), while $O|n = n'|C_{\max}$ is NP-hard for $n' \geq 3$. Moreover, in a solution to $O|synmv|C_{\max}$ with $n \leq m$ all jobs have the same completion time, so that an optimal schedule for $C_{\max}$ is also optimal for any other non-decreasing objective f. It follows that we can solve problem $O|synmv, n = n'|f$, with a fixed number of jobs n', for any such objective f.

Finally, comparing the open shop and flow shop model with synchronization, we also observe that the open shop problems are no harder. Furthermore, we obtain a positive result for $Om|synmv|C_{\max}$ with an arbitrary fixed number of machines m, while its flow shop counterpart $Fm|synmv|C_{\max}$ is strongly NP-hard for $m \geq 3$, see [152].

7.2 Assignment problem with a nearly Monge matrix

In addition to our results on the synchronous open shop problem, Chapter 5 presents a complexity study of the d-dimensional assignment problem with a nearly Monge array. It serves as the underlying model for synchronous open shop scheduling with the makespan objective, as well as for the related problem in satellite communication and the famous MEC problem for a complete bipartite graph. A summary of the results is presented in Table 7.2.

In particular we study the version of problem $\mathrm{AP}(d, \lambda)$ where the dimension d and the number of incompatible partners λ are fixed, which are natural assumptions for applications. For that special case we prove an important structural property that guides the search for an

Problem	Parameters		Complexity	Reference
AP$(2,\lambda)$	$d=2$,	λ arbitrary	$O(n^3)$	[25, 88] (2-dim. assignment with an arb. cost matrix)
AP$(3,\lambda)$	$d=3$,	λ arbitrary	str. NP-hard	[59] and Observation 22
AP$(d,0)$	d arbitrary,	$\lambda=0$	$O(n)$	[26] (assignment problem with a Monge array)
AP$(d,1)$	d arbitrary,	$\lambda=1$	str. NP-hard	[43, 126] and Observation 21
AP(d,λ)	d fixed,	λ fixed	$O(n)$	Section 5.6

Table 7.2: The summary of complexity results for problem $AP(d,\lambda)$

optimal solution. It allows us to limit the consideration to entries that lie inside a corridor of total width $2d(d-1)\lambda+1$ around the diagonal, $d(d-1)\lambda$ to each side of the diagonal and the diagonal itself. As we discuss in Section 5.7, the result can be extended to more general types of the assignment problem with Monge-like matrices.

Even though Table 7.2 already gives a rather complete picture, there are a couple of questions left open related directly to the assignment problem with a nearly Monge array. First note that we were unable to show that the corridor width of $2d(d-1)\lambda+1$ is tight. It is easy to check that for the small block property in Section 4.1 blocks of size 3 are needed, and therefore the minimum corridor width must be $4(d-1)\lambda+1$ (for $d=2$ and $\lambda=1$ the corridor must include entries up to two lines or columns away from the diagonal). Unfortunately, we have been unable to find examples where a corridor of size larger than $4(d-1)\lambda+1$ is needed to find an optimal solution.

Conversely, it seems hard to reduce the gap from the other side, by proving a corridor property with a reduced corridor width. Indeed, there are examples for dimension $d=3$, where starting with a particularly bad optimal solution it is impossible to use exchanges similar to the ones in our proof in order to transform it into a solution inside a corridor of width less than

$2d(d-1)\lambda+1$. One such example is given by the $7\times 7\times 7$ array W, with costs

$$w_{ijk}=\begin{cases}0, & \text{if triple } (i,j,k) \text{ does not contain incompatible partners,}\\ \infty, & \text{otherwise,}\end{cases}$$

the set of incompatible partners $\{(1,1,*)$, $(1,*,2)$, $(*,1,2)$, $(*,3,1)$, $(2,2,*)$, $(*,2,3)$, $(3,3,*)$, $(3,*,3)$, $(4,4,*)$, $(*,4,4)$, $(*,5,5)$, $(5,6,*)$, $(7,5,*)$, $(7,*,5)$, $(*,6,6)$, $(6,7,*)$, $(6,*,7)$, $(*,7,7)\}$ and the starting solution S given by the set of triples $\{(1,4,7)$, $(2,1,1)$, $(3,2,2)$, $(4,3,3)$, $(5,5,4)$, $(6,6,5)$, $(7,7,6)\}$.

Clearly S is optimal, as it does not contain a triple with incompatible partners. Furthermore, S satisfies the corridor property with the corridor width proven in this thesis, since $7-1=6=d(d-1)\lambda$ with $d=3$ and $\lambda=1$. For any smaller corridor width greater than zero, the only violating assignment entry is $(1,4,7)$, but observe that any attempt to remove entry $(1,4,7)$ from the solution with an exchange step similar to the one in our prove leads to an entry containing two incompatible partners in the new solution. What is more, any exchange step between two entries of solution S leads to an entry containing incompatible partners, so even first moving some of the other entries around before attempting to remove entry $(1,4,7)$ does not succeed. Therefore in order to achieve an reduced corridor width, if that is possible, a different proof technique would be needed, where exchanges are not needed in the first place or particularly bad starting solutions are avoided.

On the other hand, there are special cases where the tight corridor width of $4(d-1)\lambda+1$ can be achieved. One such example is the problem discussed in Section 5.7.2, where the structural property of the matrix is stronger than nearly Monge.

The other open question related to the assignment problem concerns the complexity of $\mathrm{AP}(d,\lambda)$ with d fixed and λ part of the input. In our argument for NP-hardness of $\mathrm{AP}(d,\lambda)$ with d fixed and λ part of the input we use $\lambda=n-1$. However, note that in order to guarantee a solution to the assignment problem with value less than ∞ exist, it is necessary and sufficient to restrict the size of λ such that

$$\lambda\leq\frac{n}{2(d-1)}.$$

Sufficiency can be seen either by repeated application of the Theorem of Hall (see, e.g., [71, 84, 107]) or by proving the corridor property again with the exchange step defined in Section 5.7.2. While that easier exchange step does not guarantee optimality for arbitrary nearly Monge arrays, it suffices to guarantee feasibility and the resulting corridor is of width $2(d-1)\lambda$ to either side of the diagonal. Then the given bound for λ guarantees that the array is large enough to contain a corridor of the size necessary to find a feasible solution.

For necessity consider for example the $n\times n$ matrix $W=(w_{ij})$, for any $n>1$, given by

$$w_{ij}=\begin{cases}0 & \text{if } i>\frac{n}{2}+1 \text{ or } j>\frac{n}{2}+1,\\ \infty & \text{otherwise.}\end{cases}$$

Clearly W is a nearly Monge matrix with $\frac{n}{2} < \lambda \leq \frac{n}{2} + 1$, but each feasible assignment contains at least one entry (i, j) with $1 \leq i, j \leq \frac{n}{2} + 1$, an ∞-entry. Examples for higher dimensions can be constructed analogously. Thus, $\lambda \leq \frac{n}{2(d-1)}$ is, in general, necessary to guarantee that a feasible solution exists.

It would be interesting to consider the complexity for AP(d, λ) with d fixed and $\lambda \leq \frac{n}{2(d-1)}$ arbitrary. Notice that since the upper bound of λ still depends on n, this is not the same as assuming λ is fixed, and thus none of the results in Table 7.2 are applicable to this case. While we expect the problem to be strongly NP-hard even for $\lambda \leq \frac{n}{2(d-1)}$, the question remains open.

Turning to other problems, a natural extension of the current research is related to the transportation problem TP$(2, \lambda)$ with a nearly Monge matrix incurred by incompatible supply/demands pairs, that can be considered as a generalization of AP$(2, \lambda)$. In the absence of ∞-entries, the transportation problem with a square Monge matrix is solvable by a greedy algorithm in $O(n)$ time [74]. For a general square matrix the problem can be solved in $O(n^3 \log n)$ time by Orlin's algorithm [117]. It would be interesting to see if a faster than standard algorithm can be achieved for TP$(2, \lambda)$. The following example shows that the corridor property, as presented in Chapter 5, is not quite relevant for TP$(2, \lambda)$. Consider an instance with supplies $a_1 = n-1$, $a_2 = \ldots = a_n = 1$, demands $b_1 = n-1$, $b_2 = \ldots = b_n = 1$ and a Monge cost matrix W. For this instance the greedy algorithm produces an optimal diagonal solution. However, introducing one incompatible pair $(1, 1)$, or equivalently $w_{11} = \infty$, makes the previous optimal solution infeasible. For the modified problem an optimal solution is defined by the first row and the first column, apart from the top left-most entry with ∞-cost. Thus, the corridor that characterizes a possible deviation from the previous optimal solution without ∞'s, is as large as the whole matrix.

Further extensions of our study can be related to the travelling salesman problem TSP$(2, \lambda)$ with a nearly Monge matrix and at most λ forbidden partners per vertex. For a Monge matrix without ∞-entries, an optimal solution is a pyramidal tour which can be found in $O(n)$ time as shown in [120]. Clearly, if we allow an arbitrary number of infinities in the matrix, then finding a finite solution is as hard as finding a Hamiltonian circuit in an arbitrary graph, and therefore strongly NP-hard [80]. On the other hand, for Supnick matrices, which can be viewed as a subclass of nearly Monge matrices with $\lambda = 1$, the TSP is always solved by the pyramidal tour $(1, 3, 5, \ldots, n, \ldots, 6, 4, 2)$, see [144]. An interesting question related to TSP$(2, \lambda)$ with infinities is how far an optimal solution may deviate from the pyramidal tour if λ is a fixed parameter. Pyramidal tours with step-backs, as introduced in [53], might be a good starting point for that study.

Design of approximation algorithms for problems with nearly Monge matrices is another important research direction. With respect to AP(d, λ) with arbitrary d, the closest problem is MEC, in which the cost matrix is of type (5.2). A range of approximation algorithms for the latter problem is proposed in [40, 43, 54, 108, 109]. These ideas might be helpful to find approximation results for AP(d, λ), which is a generalization of MEC, as the cost values may

be arbitrary, unrelated to the max-formula (5.2).

7.3 Further research for synchronous scheduling models

Considering that for synchronization to be of interest the machine environment needs to consist of at least two machines, several extensions of synchronization to other scheduling models are possible. In shop scheduling, the natural next step would be to investigate job shop models. Since job shop is at least as hard as flow shop, the only problem where a positive result may be possible is $J2|synmv|C_{\max}$ and indeed that problem is still open to the best of our knowledge. Recall that $J2||C_{\max}$ is strongly NP-hard [101] and that $J2|n_j \leq 2|C_{\max}$ where each job has at most two operations is solvable in $O(n \log n)$ time [78]. Similar results might be expectable for synchronous job shop scheduling.

Taking applications into account, the most promising direction appears to be the research of synchronous parallel machine models. Here, possible areas of application come from distributed computing, where jobs can be processed in parallel on different machines for some time, but then, before the next step can be started all results from the previous computations have to be collected.

First consider problems with identical machines, $P|synmv|f$, with m machines and n jobs. It is easy to observe that problem $P|synmv|f$ is equivalent to $1|p{-}batch, b = m|f$, i.e. scheduling a single p-batching machine with maximum batch size $b = m$. As p-batching is well-studied, we can immediately conclude that if $m < n$, then $P|synmv|C_{\max}$ is solvable in $O(n \log n)$ time and that $P2|synmv|L_{\max}$ is strongly NP-hard [18]. Problem $P2|synmv|\sum C_j$ is still open and equivalent to the long standing open problem $1|p - batch, b = 2|\sum C_j$.

On the other hand, for $m \geq n$ problem $P|snymv|f$ is polynomially solvable for all traditional scheduling objectives other than $\sum T_j$, $\sum w_j T_j$ and $\sum w_j U_j$. For $\sum T_j$ the problem is open, similar to its p-batching counter part, while it is NP-hard in the ordinary sense for $\sum w_j T_j$ and $\sum w_j U_j$, see, e.g., [17].

Now we move to synchronous uniform machines, a possibly more realistic scenario with the applications we consider. Note that scheduling synchronous uniform machines would be equivalent with a form of p-batching were different jobs within a batch are processed at different speeds. Clearly, all NP-hardness results from the identical machine case carry over to the uniform machine case.

The only positive result that can be obtained with relative ease is that the two-machine problem $Q2|synmv|C_{\max}$ is solvable in cubic time. To see this, note that in a feasible schedule two jobs are partnered in each cycle (though it may be interesting to consider a relaxed version, similar to the relaxed version for synchronous open shop). If two machines M_1 and M_2 with speeds s_1 and s_2 are given, $s_2 \geq s_1$, we can assume that in each cycle the job with the larger processing requirement to be processed by the faster machine M_2.

Computing for each pair of jobs j_1, j_2, $p_{j_2} \geq p_{j_1}$, the time it would take to process a cycle

which consists of j_1 and j_2, we obtain the weight matrix W given by

$$w_{j_1 j_2} = w_{j_2 j_1} = \begin{cases} \max\left\{\frac{p_{j_1}}{s_1}, \frac{p_{j_2}}{s_2}\right\}, & \text{if } j_1 \neq j_2, \\ \frac{p_{j_1}}{s_2}, & \text{otherwise.} \end{cases}$$

Similar to modelling synchronous open shop as the assignment problem, it is easy to see that we can model problem $Q2|synmv|C_{\max}$ as a general matching problem on a complete graph K_n, with weights given by W. Recall that that the matching problem in general graphs can be solved in cubic time (see [55, 90]).

Notice that by sorting jobs in order of non-increasing processing times, we can once more turn W into a (symmetric) Monge matrix. Unfortunately, little is known about general matching with Monge cost matrices other than the results for general cost matrices. Observe, however, that problem $Q2|synmv|C_{\max}$ becomes solvable in linear time if an assignment of jobs to machines is given. In that case, pairing jobs in $\frac{n}{2}$ cycles, if the assignment of jobs to machines is balanced, becomes a linear assignment problem with a Monge cost matrix which is a submatrix of W.

For an attempt to reduce the running time $O(n^3)$ for the general matching problem in Monge matrices, a good approach might be to study the results for the unbalanced linear assignment problem in Monge matrices. Recently, it was proven that the unbalanced linear assignment problem with an $n \times m$ Monge cost matrix ($n > m$) is solvable in $O(nm)$ time [147]. A major part of that result is to show that when running the assignment algorithm using augmenting paths, it is sufficient to consider those paths which, simply put, do not contain crossing edges (for details see [147]). A similar idea may be possible for the general matching algorithm.

Lastly, we want to note that an analogous construction as for $Q2|synmv|C_{\max}$ works for problem $Q|synmv|C_{\max}$ with $m \geq 2$ machines, leading to an m-dimensional general matching problem. Again, the cost arrays resulting from such a construction can be transformed into Monge arrays, so the traditional NP-hardness results for multi-dimensional matching are not sufficient to show NP-hardness of, e.g., $Q3|synmv|C_{\max}$. The complexity status of the multi-dimensional general matching problem with a Monge cost array is open, to the best of our knowledge.

Part II

Pliability

Chapter 8

Definitions, notation and related work

In this part of the thesis we study flow shop and open shop scheduling with pliable jobs. As opposed to the traditional models, processing times of each operation are not given in advance, but are part of the decision process and subject to certain constraints. For each job j there is given a job processing time p_j for the job as a whole, which has to be split into m operations, such that all constraints are met. The full definition can be found in Section 8.1.

We study the complexity of three different models of pliability. The work presented in this part is an introduction of scheduling models with pliability. The results presented here are not only interesting in their own right, but also provide a broad basis for future investigation of more complicated, extended models as well as of heuristics for the models found to be NP-hard here.

8.1 Introduction and definitions

In traditional flow shop and open shop models (see Sections 2.3.4 and 2.3.5), n jobs of the set $\mathcal{J} = \{1, 2, \ldots, n\}$ are processed by m machines M_i, $1 \leq i \leq m$. Each job j, $1 \leq j \leq n$, consists of m operations, one operation on each machine M_i, with processing times p_{ij}, $1 \leq i \leq m$. A job cannot be processed by two machines at the same time and a machine cannot process two jobs simultaneously. In the flow shop model, all jobs have the same machine order, while in the open shop model the machine order is not fixed and can be chosen arbitrarily. The goal is to select an order of operations on each machine and for open shops additionally the order of operations for each job, so that a given objective function f depending on job completion times is minimized.

Both models, flow shop and open shop, have a long history of study, see Sections 2.3.4 and 2.3.5 or, e.g., [17]. Over the last 60 years, the classical versions were extended to handle addi-

tional features of practical importance, such as processing with preemption and lot streaming among others. Recall that in models with preemption an operation can be cut into an arbitrary number of pieces which are then processed independently. The model with lot streaming allows dividing operations into sublots which can then be treated as new operations (cf. [29], [145]). In the preemptive case, pieces of the same job processed by two machines cannot overlap over time, while in the case of lot streaming overlapping may happen if the pieces belong to different sublots. In both cases, operation lengths p_{ij} are given in advance for all mn operations, and these amounts of work have to be completed in full even if splitting happens.

Other concepts related to operation splitting and relocation, which have appeared more recently, deal with flexible operations and operation redistribution ([67] and [22] respectively). In the first model, jobs typically consist of more than m operations, some of which are fixed and have to be processed by dedicated machines while others are flexible and need to be assigned to one of the appropriate machines. In the second model, operations may be preempted and operation parts can be moved to neighboring machines (the previous or next machine in line), if these machines are equipped to handle them. Again, in both models, operation lengths p_{ij} are given for all operations.

In this part of the thesis we study a different way of splitting jobs, where operation lengths are not given in advance, but need to be selected. To distinguish our model from those studied previously, we introduce the notion of *pliability* for it. Formally, a pliable job j is given by its total processing time p_j, which has to be split among the m machines. Operation lengths x_{ij} have to be determined as part of the decision making process. The combined length of all operations of job j on all machines has to match the required total processing requirement of job j:

$$\sum_{i=1}^{m} x_{ij} = p_j, \quad 1 \leq j \leq n.$$

We study three versions of models with pliable jobs.

(i) *Unrestricted pliability*: the jobs can be arbitrarily split among the machines, so that

$$0 \leq x_{ij} \leq p_j, \quad 1 \leq i \leq m, \;\; 1 \leq j \leq n.$$

(ii) *Restricted pliability with a common lower bound*: since in some applications jobs cannot be split into arbitrarily small pieces, a common compulsory minimum amount of work $\underline{p}$ for all operations is given, and conditions

$$\underline{p} \leq x_{ij} \leq p_j, \quad 1 \leq i \leq m, \;\; 1 \leq j \leq n$$

have to be satisfied.

(iii) *Restricted pliability*: individual lower and upper bounds $\underline{p}_{ij}$, $\overline{p}_{ij}$ are given for all opera-

tions, and it is required that

$$\underline{p}_{ij} \leq x_{ij} \leq \overline{p}_{ij}, \quad 1 \leq i \leq m, \;\; 1 \leq j \leq n.$$

Notice that for a feasible instance we must have

$$\sum_{i=1}^{m} \underline{p}_{ij} \leq p_j \leq \sum_{i=1}^{m} \overline{p}_{ij} \;\; \text{for all } j = 1, \ldots, n.$$

Clearly model (i) is a special case of model (ii) with $\underline{p} = 0$, and model (ii) is a special case of model (iii) with $\underline{p}_{ij} = \underline{p}$, $\overline{p}_{ij} = p_j$. The classical flow shop and open shop problems are special cases of model (iii) with $\underline{p}_{ij} = \overline{p}_{ij} = p_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Note that the term "job splitting" is also used in parallel machine models, where it describes the possibility of jobs being split into sub-jobs and two sub-jobs being processed at the same time, even if they belong to the same job, see [136]. However, in spite of the similar terminology (in the model with pliability jobs are split into operations), the model with pliability and the model with "job splitting" have very little relation.

Extending the standard three field notation ($\alpha|\beta|\gamma$ notation), we denote the flow shop and open shop models with unrestricted pliability of type (i) by $F|plbl|\gamma$ and $O|plbl|\gamma$. In the presence of additional restrictions of models (ii)-(iii), the given bounds are indicated in the second field as $plbl(\underline{p})$ and $plbl(\underline{p}_{ij}, \overline{p}_{ij})$, respectively.

8.2 Related work

Models with pliability arise, for example, in scenarios where intermediate actions are needed, such as quality control, pre-processing, post-processing, or setup operations, and these actions can be performed by either of two consecutive machines. Alternatively, operators specializing on serving particular machines may be able to perform not only the main operations they are trained for, but also additional operations on adjacent machines, thus reducing possible delays and idle times in the system. In manufacturing applications, not only the operators can be flexible, but machines as well if they are designed to perform operations of different types. Various examples of flexible machines (such as CNC machines and machines producing printed circuit boards) are reviewed in [34].

Comparing the pliability model with the relevant models studied in the literature, we summarize below the main common points and the points of difference.

The pliability model is most closely related to models with (a) flexible operations and (b) operation redistribution; the models with preemption and lot streaming are less relevant as they usually allow dividing operations into an arbitrary large or small number of pieces, while in the pliability model the number of job parts is exactly m.

– Models (a) and (b) are studied in the context of flow-shop scheduling; the pliability

model deals with flow shops and open shops. It allows further generalization for job shops, although this is beyond the scope of this thesis.

– All models, including the pliability one, deal with flexible allocation of operations or their parts. The level of flexibility is slightly different in the three models. In model (a), each flexible operation has to be allocated to one machine in full. In model (b), operations can be split at any point of time; still there is a limitation on the machine choice for the allocation: only an adjacent machine in the flow shop chain of machines can be selected. In the pliability model, every machine must get at least the minimum workload associated with job j (namely, 0, $\underline{p}$ or $\underline{p}_{ij}$, depending on the model type), the remaining workload can be allocated to any machine M_i, but without exceeding the maximum amount $\overline{p}_{ij}$, if given.

– In models (a) and (b), durations p_{ij} are given for all operations; for the pliability model only the total job lengths p_j are given.

The models (a) and (b) and the pliability model are closest related for flow shop with two machines. Indeed, in that case the version of model (a), where flexible operations can be preempted and restarted on the next machine, coincides with a special case of model (b) and the pliability model of type (iii). Suppose an instance of such a preemptive version of model (a) with two machines is given and assume there are n jobs with processing times p_{1j}, p_{2j} for the operations on machines M_1 and M_2 and p_{3j} for the flexible operation. Then we can transform it into an instance of the pliability model of type (iii) with n jobs where processing times p_j, lower bounds $\underline{p}_{ij}$ and upper bounds $\overline{p}_{ij}$ are given by

– $\underline{p}_{ij} = p_{ij}$ for $i = 1, 2$ and all jobs j,

– $p_j = p_{1j} + p_{2j} + p_{3j}$ for all jobs j and

– $\overline{p}_{ij} = p_j$ for $i = 1, 2$ and all jobs j.

The preemptive version of model (a) has not been considered in the literature, although it is observed in [104] that this case would be an interesting extension of the model.

The flow shop problem (a) with flexible operations is NP-hard for the makespan objective even in its simplest setting, when 3-operation jobs with a flexible middle operation have to be processed by 2 machines, see [67]. It remains NP-hard even if the job sequence is fixed [104]. Therefore the study of the models with flexible operations focuses on approximability results [67], pseudopolynomial-time algorithms [104], construction heuristics and local search methods [131, 132]. The models often incorporate special features such as limitations on the buffer capacities used for handling jobs in-between the machines, requirements to optimize workstation utilization or throughput rate, etc. The main special case of the flexible model, for which efficient algorithms have been developed, is the one with identical jobs, see [34], [68].

The flow shop problem with redistribution is less studied, compared to the model with flexible operations. To the best of our knowledge, there is only one paper on this model [22]. The authors focus first on establishing a mathematical programming formulation, then propose some heuristics and evaluate them empirically. Complexity aspects of the problem are not discussed in detail.

While less strongly related models do not contribute additional insights towards our complexity study, it is still worth pointing out that flexibility in scheduling is also extensively studied in terms. A flow shop model with workforce flexibility is studied in, e.g., [37, 38, 39], where the type of flexibility leads to a model with controllable processing times (assigning more workers to a machine shortens processing times on that machine and lengthens them on a machine with fewer workers).

Another area where flexibility is naturally studied is for assembly lines, where n operations of a production process, called tasks, have to be assigned to m machines, called stations, in order to optimize some objective, see the survey papers [12, 15, 16, 135]. Traditionally, the assignment of tasks to stations in the assembly line does not change while the line is running. In a slightly newer, dynamic version, workers can take over work from their neighbors or can decide to leave some of their work undone to give it over to the next in line, according to pre-set rules. Different versions of the dynamic model were studied, for example, in [5], [6], [110] and [118].

Our study continues the line of research on flow shop models with flexibility and relocation, and extends it to open shop counterparts. The main outcome is a complexity classification of the three types of the pliability model and efficient solution algorithms for some special cases. The model was derived in collaboration with S. Knust, N. V. Shakhlevich and S. Waldherr. The results presented below have been achieved either solely by the author or with his collaboration.

Chapter 9

Shop scheduling problems with pliable jobs

In this chapter we present our work on flow shop and open shop scheduling with pliable jobs. The chapter is organized as follows. In Section 9.1 we provide general results that serve as the basis for subsequent sections. In Sections 9.2, 9.3 and 9.4 we handle problems with min-max objectives, of type (i), (ii) and (iii), respectively. Section 9.5 studies problems with min-sum objectives. Concluding remarks and suggestions for further research are provided in a separate chapter after this one.

Throughout this chapter we assume that $n \geq m$; we discuss the alternative case in the Conclusions.

9.1 General properties and reductions

In this section we establish a link between the pliability model and the classical flow shop and open shop with and without preemption and propose general methods for different types of pliability models.

9.1.1 Pliability of type (i)

In order to address problems $O|plbl|f$ and $F|plbl|f$, it is often useful to relax the requirement of dedicated machines typical for open shops and flow shops and to consider identical parallel machines instead. The pliability condition, that allows splitting jobs into operations, can then be interpreted as processing with preemption. If the resulting problem $P|pmtn|f$ has an optimal solution of "open shop type" or "flow shop type" (O-*type* or F-*type* schedule for short), then such a schedule is also optimal for the original problem.

Formally, in an O-type schedule each machine processes exactly one piece of each job.

Clearly, such a schedule for $P|pmtn|f$ is also a feasible schedule for $O|plbl|f$. Moreover, if all jobs have the same release time, then the condition that each machine processes exactly one piece of each job can be relaxed, such that each machine processes at most one piece of each job, i.e. no job is preempted and then restarted on the same machine. Indeed, in that case a feasible open shop schedule can be achieved by adding zero-length operations at the beginning of a schedule for all missing open shop operations. This approach does no longer work if jobs have different release times.

In an F-type schedule, each machine processes at most one part of each job (as in an O-type schedule); additionally jobs visit the machines in a flow shop like manner, moving from machine M_i to M_{i+1}, $1 \leq i \leq m-1$. Again, missing operations can be modelled as zero-length operations; however they might need to appear in the middle of the schedule if a missing operation has to be inserted on one of the machines M_i with $2 \leq i \leq m$. Note that in the case of permutation schedules, where all machines process the jobs in the same order, the notion of an F-type schedule coincides with the notion of a Permutation Flow Shop-like schedule, introduced in [123].

For a scheduling problem $\alpha|\beta|\gamma$, let $\mathcal{S}(\alpha|\beta|\gamma)$ denote the set of its feasible solutions. Since any solution to $F|plbl|f$ is feasible for $O|plbl|f$, and in its turn any solution to $O|plbl|f$ is feasible for $P|pmtn|f$, we conclude:

$$\mathcal{S}(F|plbl|f) \subseteq \mathcal{S}(O|plbl|f) \subseteq \mathcal{S}(P|pmtn|f). \tag{9.1}$$

In our study we revise known algorithms and NP-hardness results for problem $P|pmtn|f$ with the focus on optimal schedules of O- or F-type. Clearly, if an O- or F-type schedule exists that achieves the same objective value as an optimal schedule for problem $P|pmtn|f$, then it is also optimal for the corresponding open shop or flow shop problem with pliable jobs.

Prot et al. demonstrate in [123] the existence of an optimal F-type schedule for problem $P|pmtn|f$ for very general non-decreasing objective functions f, including those traditionally studied in scheduling theory. Clearly, if a schedule is optimal for $P|pmtn|f$ and $F|pmtn|f$, then it is also optimal for $O|pmtn|f$ due to the inclosed structure of solution regions (9.1). Therefore, the following statement holds.

Theorem 34. *If the objective function f is of the form $\sum_{j\in N} f_j(C_j)$ or $\max_{j\in N}\{f_j(C_j)\}$, where $f_j(C_j)$ are non-decreasing functions, then there exists a common optimal schedule for problems $F|plbl|f$, $O|plbl|f$ and $P|pmtn|f$.*

Theorem 34 implies several complexity results for problems with pliable jobs. Consider first the case when a particular version of problem $P|pmtn|f$ is NP-hard. Then the corresponding versions of $F|plbl|f$ and $O|plbl|f$ are also NP-hard, unless P = NP. Otherwise a polynomial-time algorithm for solving one of the latter problems would also solve problem $P|pmtn|f$ in polynomial time.

Consider now the case when a particular version of problem $P|pmtn|f$ is solvable in poly-

nomial time. Then the solution to $P|pmtn|f$ provides a template for solving the corresponding version of problem $F|plbl|f$ via linear programming (if the objective function f allows for linear programming), as shown in [123]. In that solution, we fix the order of job completion times and re-distribute the job parts to achieve the flow-shop order. Since the results in [123] justify that the resulting solution has the same value of f, the resulting F-type schedule is also optimal for the open shop problem $O|plbl|f$.

Theorem 35 summarizes the complexity results obtained by combining the observations from the last two paragraphs with the traditional complexity results for parallel machine scheduling (see, e.g., [17, 97] or Section 2.3.2).

Theorem 35. *Problems $F2|plbl|f$ and $O2|plbl|f$ with $f \in \{\sum w_jC_j, \sum T_j, \sum w_jU_j\}$ are NP-hard in the ordinary sense, and they are NP-hard in the strong sense if $f = \sum w_jT_j$.*
Problems $F|plbl|f$ and $O|plbl|f$ with $f = \sum U_j$ are NP-hard in the ordinary sense and strongly NP-hard with $f = \sum w_jC_j$;
they are solvable in polynomial time via linear programming if $f \in \{C_{\max}, L_{\max}, \sum C_j\}$.

Thus, the results from [123] link the pliability models with $P|pmtn|f$, allowing the transfer of results from one area into another. However, special properties of the problems under study often allow us to develop faster algorithms. We present them in Sections 9.2.1, 9.2.2 and 9.5.1.

9.1.2 Pliability of type (ii)

In this section we focus on the flow shop pliability model $F|plbl(\underline{p})|f$ establishing two important properties for it. Unfortunately these properties cannot be easily generalized for $O|plbl(\underline{p})|f$. They also do not hold for the more general pliability model of type (iii).

In Section 9.1.2 we prove that for an arbitrary objective f, there exists an optimal permutation schedule. Note that the same property for the special case (i) was proved in [123] in a different way, making use of the fact that jobs can be cut down in arbitrarily small pieces. That technique is not appropriate for the pliability model of type (ii) with a common lower bound $\underline{p}$.

In Section 9.1.2 we show how problem $F|plbl(\underline{p})|f$ can be decomposed into two subproblems: one subproblem of type (i) and another one, with equal processing times, of type (ii). This decomposition is the main methodology we adopt for solving the problems with $f \in \{C_{\max}, L_{\max}, \sum C_j\}$ in Sections 9.3.1, 9.3.3 and 9.5.2.

The existence of an optimal permutation schedule

To prove the main result, consider first an auxiliary property related to so-called *adjacent jobs swaps*. By that property, the order of any two jobs u and v, which are adjacent in a permutation schedule, can be reversed without making changes to the rest of the schedule. However, to achieve this, the lengths of operations of jobs u and v may be redistributed, if necessary.

Lemma 36. *Given a permutation schedule S for problem $F|plbl(\underline{p})|f$ in which job v is scheduled immediately after job u, there exists a permutation schedule S' with u scheduled immediately after v, while all remaining jobs are scheduled in the same time slots as in S. For schedule S',*

$$\begin{aligned} &C'_j = C_j, \quad j \in N\backslash\{u,v\}, \\ &C'_u \le C_v, \\ &C'_v \le C_v. \end{aligned}$$

Notice that on the last machine M_m, the completion time of u in S' is no larger than the completion time of v in S, see Fig. 9.1 for illustration. Here, the grey boxes represent the fixed parts of schedules S and S' where the jobs $N\backslash\{u,v\}$ are processed.

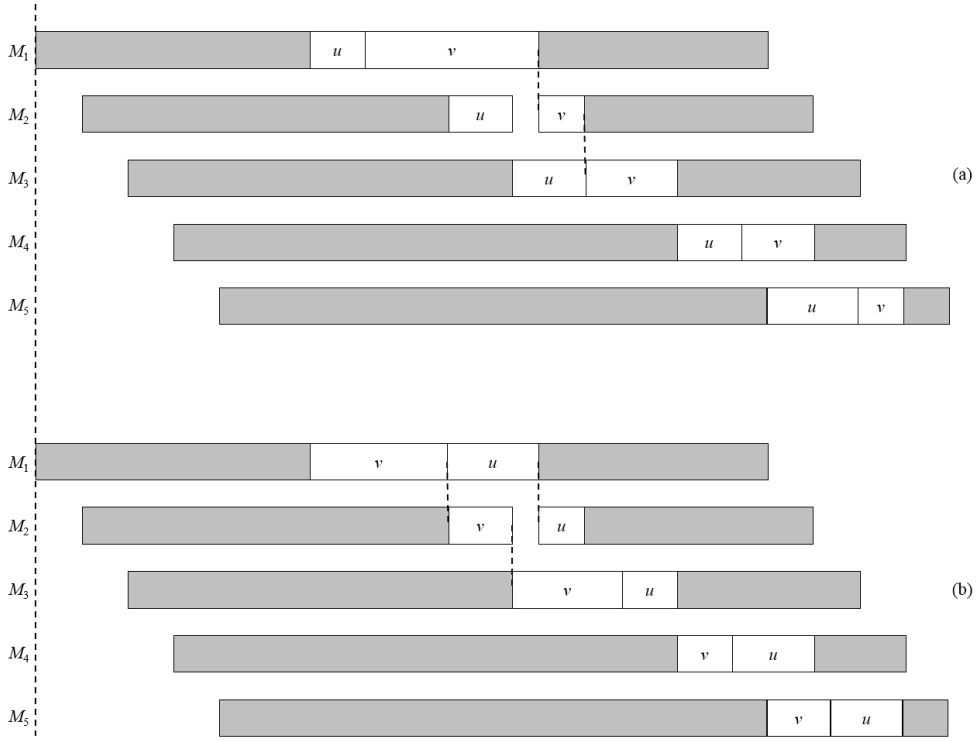


Figure 9.1: Adjacent jobs swap: (a) schedule S; (b) schedule S'

Lemma 36 is proved in Section 9.6. The proof uses the fact that there is a common lower bound $\underline{p}$ for the processing times of all operations, and it cannot be generalized for the model of type (iii).

Theorem 37. *Given an arbitrary schedule S for problem $F|plbl(\underline{p})|f$, there exists a permutation schedule S' in which every job has the same completion time on machine M_m as in S.*

Proof: The proof is done by induction on the number of machines m. For $m = 1$ the statement is obvious. Consider $m \ge 2$, assuming that the statement of the theorem holds for $m - 1$

machines.

Let S be a non-permutation schedule on m machines. We split S into two sub-schedules $S(M_1, \ldots, M_{m-1})$ and $S(M_m)$ defined over the corresponding machine sets. For the instance of the problem defined by $S(M_1, \ldots, M_{m-1})$, the induction hypothesis holds: there exists a permutation schedule $S'(M_1, \ldots, M_{m-1})$ such that each job has the same completion time on machine M_{m-1} as in $S(M_1, \ldots, M_{m-1})$.

We extend $S'(M_1, \ldots, M_{m-1})$ by adding the final part $S(M_m)$ of the original schedule S, creating a complete schedule $\hat{S}$ for m machines. Note that schedule $\hat{S}$ is feasible since by the induction hypothesis each job completes on machine M_{m-1} at the same time in $S(M_1, \ldots, M_{m-1})$ and in $S'(M_1, \ldots, M_{m-1})$.

If $\hat{S}$ is a permutation schedule, then no further action is needed. Otherwise consider $S'(M_1, \ldots, M_{m-1})$ and apply a sequence of adjacent jobs swaps that leads eventually to the same job order as in $S(M_m)$. We demonstrate that each swap on the first $m-1$ machines does not affect operations in $S(M_m)$.

Assume u is scheduled immediately before v in $S'(M_1, \ldots, M_{m-1})$, but somewhere after v in $S(M_m)$. Then after swapping u and v on the first $m-1$ machines, the completion time of job v on machine M_{m-1} becomes smaller, and hence job v is not postponed on M_m. On the other hand, the completion time of job u on M_{m-1} is no larger than the completion time of job v before the swap. This means that u finishes on M_{m-1} before v starts on M_m, and therefore before u starts on M_m.

Performing at most $O(n^2)$ swaps in the part $S'(M_1, \ldots, M_{m-1})$, we get a permutation schedule on m machines without changing the completion times on machine M_m. ∎

It is worth noting that in the proof of Theorem 37, the schedule transformations keep the operations on the last machine unchanged. This implies that an optimal permutation schedule exists for any objective function depending on job completion times, monotone or non-monotone. Note also that Theorem 37 does not hold for the model of type (iii): it is known that for problem $F||C_{\max}$ with more than three machines there exist instances for which only non-permutation schedules are optimal (see, e.g., [122]). Recall that $F||C_{\max}$ is a special case of the pliability model of type (iii).

Decomposing type (ii) problems into two subproblems

Given an instance I of problem $F|plbl(\underline{p})|f$ of type (ii), introduce two auxiliary instances: instance I^e of type (ii) with *equal* processing times $p_j^e = m\underline{p}$ for all jobs $j \in \mathcal{J}$ and with the same lower bound value $\underline{p}$ as in the original instance I, and instance I^d of type (i) with *diminished* processing times $p_j^d = p_j - m\underline{p}$ and zero lower bounds. Notice that $p_j = p_j^e + p_j^d$.

Let S^d and S^e be solutions to instances I^d and I^e which satisfy the following conditions:

1. S^d and S^e are permutation schedules with the same job sequence $(1, 2, \ldots, n)$.

2. S^e has a staircase structure, uniquely defined by completion times $C^e_{ij} = (i+j-1)\,\underline{p}$ of its operations (i,j), with $j \in \mathcal{J}$ and $i \in \{1,\ldots,m\}$ where machine M_i is idle in the time interval $\big[0,(k-1)\,\underline{p}\big]$.

3. In S^d, every machine operates without idle times from time 0 until all assigned operations are completed; some operations in S^d may be of length zero.

Note that optimal schedules S^d fullfilling the third condition are constructed for problems $F|plbl|f$ with $f \in \{C_{\max}, L_{\max}, \sum C_j\}$ in Sections 9.2.1, 9.2.2 and 9.5.1 respectively.

Solutions S^d and S^e satisfying the above properties can be easily combined into a permutation schedule S_* for the original instance I, as illustrated in Fig. 9.2. That figure is produced for $F|plbl(\underline{p})|C_{\max}$ and discussed in more detail in Section 9.3.1. For objectives that differ from $C_{\max}$, the structure of S^d may be different, with machines having different workloads.

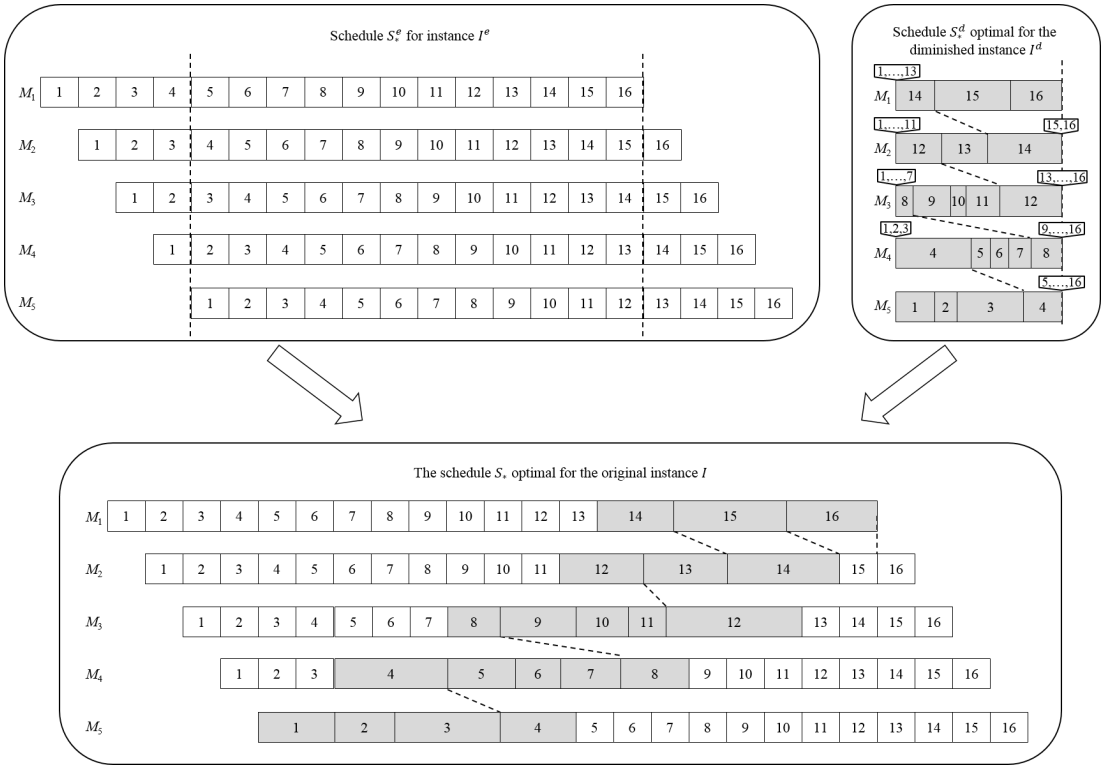


Figure 9.2: Schedules S^e_* and S^d_* for instances I^e and I^d, and the combined schedule S_* optimal for instance I

Theorem 38. *Let S^d and S^e be feasible schedules for instances I^d and I^e satisfying conditions 1-3. There exists a combined schedule S for the original instance I with*

$$C_{ij} = C^d_{ij} + (i+j-1)\,\underline{p}, \tag{9.2}$$

where C^d_{ij} and $C^e_{ij} = (i+j-1)\,\underline{p}$ are completion times of operations (i,j) in S^d and S^e. Conversely, if in a permutation schedule S for instance I with the job order $(1,2,\ldots,n)$ there are no idle times except for time intervals $\left[0,(i-1)\,\underline{p}\right]$ (as in Condition 2 for schedule S^e), then S can be decomposed into two schedules S^e and S^d such that relation (9.2) holds.

Proof: Consider the disjunctive graph representation of the permutation schedule in Fig. 9.3. For each operation of job j on machine M_i there is a node (i,j) with a weight equal to the processing time of the corresponding operation, namely x^d_{ij} for S^d and $\underline{p}$ for S^e. For each schedule, the completion time of any operation (i,j) is calculated as the length of the longest path from the source node $(1,1)$ to node (i,j). Combining S^d and S^e means that all node weights in the graph for S^d are increased by the same amount $\underline{p}$. Since any path from $(1,1)$ to (i,j) includes exactly $i+j-1$ nodes, the structure of the longest path does not change. Its length increases by $(i+j-1)\underline{p}$ and thus for the completion times of the combined schedule, (9.2) holds.

Similar arguments justify the reverse statement on decomposing S into S^e and S^d. ∎

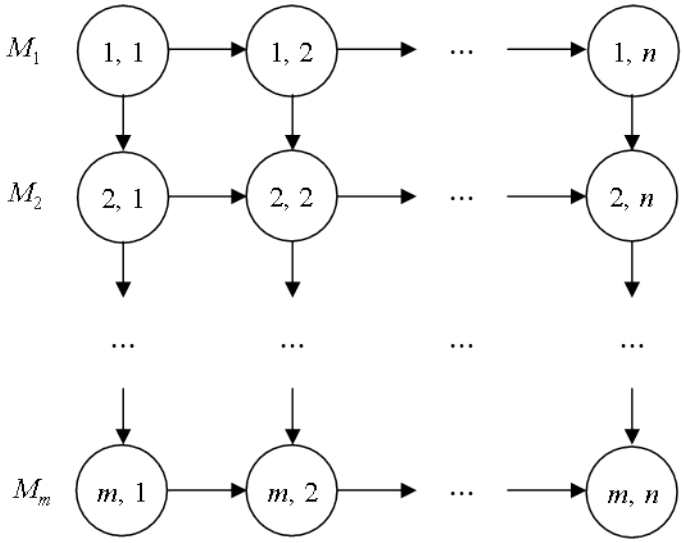


Figure 9.3: The disjunctive graph representation of schedule S^d

9.1.3 Pliability of type (iii)

We first formulate reductions which hold for the pliability problems and traditional scheduling problems $F||f$ and $O||f$.

Theorem 39. *For the flow shop and open shop problems, the following reductions hold:*

$$\alpha||f \propto \alpha|plbl(\underline{p}_{ij},\overline{p}_{ij})|f, \tag{9.3}$$

$$\alpha|plbl|f \propto \alpha|plbl(\underline{p})|f \propto \alpha|plbl(\underline{p}_{ij}, \overline{p}_{ij})|f, \tag{9.4}$$

where $\alpha \in \{F, O\}$.

Here, reduction (9.3) follows from the fact that the pliability problems $F|plbl(plbl(\underline{p}_{ij}, \overline{p}_{ij}))|f$ and $O|plbl(plbl(\underline{p}_{ij}, \overline{p}_{ij}))|f$ with $\underline{p}_{ij} = \overline{p}_{ij}$, $1 \le i \le m$, $1 \le j \le n$, coincide with the traditional flow shop and open shop problems. The chain of reductions (9.4) reflects the fact that pliability model (i) is a special case of model (ii), which in its turn is a special case of model (iii).

From (9.3) it follows that the NP-hardness results known for $F||f$ and $O||f$ are also valid for the pliability problems of type (iii). In particular, problem $O3|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is NP-hard in the ordinary sense, while problems $F3|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$, $O|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ and $\alpha 2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|f$ with $\alpha \in \{F, O\}$ and $f \in \{L_{\max}, \sum C_j\}$ are NP-hard in the strong sense.

Using (9.4), it is possible to extend the NP-hardness results for $f \in \{\sum w_j C_j, \sum T_j, \sum w_j T_j, \sum U_j, \sum w_j U_j\}$, discussed in Section 9.1.1 in relation to type (i) problems $F|plbl|f$ and $O|plbl|f$, to the pliability problems of types (ii) and (iii). Note that for the problems of type (iii) these results are dominated by those obtained through reduction (9.3).

9.2 Type (i) problems with minmax criteria

In this section we apply the methodology presented in Section 9.1.1 to pliability problems of type (i). We then use the obtained results to solve the more general model of type (ii) in Section 9.3.

9.2.1 Type (i) problems $F|plbl|C_{\max}$ and $O|plbl|C_{\max}$

Consider the relaxed problem $P|pmtn|C_{\max}$. An optimal schedule can be constructed in $O(n)$ time by McNaughton's wrap-around algorithm, see [111], achieving the optimum makespan value C_*,

$$C_* = \max\left\{\frac{1}{m}p(\mathcal{J}), \max_{j\in\mathcal{J}}\{p_j\}\right\}, \tag{9.5}$$

where $p(\mathcal{J}) = \sum_{j=1}^{n} p_j$. In order to force McNaughton's algorithm to produce a solution of F- and O-type, suitable for problems $F|plbl|C_{\max}$ and $O|plbl|C_{\max}$, we consider the jobs in the order of their numbering and allocate them in the time window $[0, C_*]$ first on machine M_m, then on M_{m-1}, etc., until the remaining jobs are allocated on machine M_1. Notice that machine M_m is always fully occupied in the interval $[0, C_*]$, while the other machines might only be partly occupied in that interval, if $C_* = p_q$ for some $q \in \mathcal{J}$. The order in which the machines are considered, gives an easy way for introducing zero-length operations, as illustrated in Fig. 9.4. The resulting schedule satisfies the requirements of F- and O-type schedules, has the minimum makespan C_*, and therefore it is optimal for both problems, $F|plbl|C_{\max}$ and $O|plbl|C_{\max}$.

Theorem 40. *Problems $F|plbl|C_{\max}$ and $O|plbl|C_{\max}$ are solvable in $O(n)$ time.*

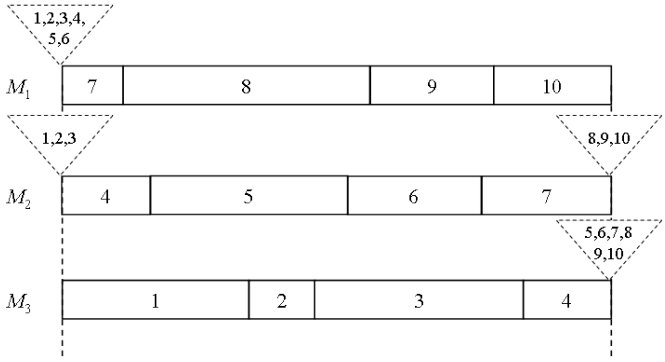


Figure 9.4: Modifying an optimal schedule for $P|pmtn|C_{\max}$ into F-type and O-type schedules by adding zero-length operations

9.2.2 Type (i) problems $F|plbl|L_{\max}$ and $O|plbl|L_{\max}$

As in the previous section, consider first the relaxed problem $P|pmtn|L_{\max}$. The $O(n \log n+mn)$ algorithm described in [11] finds an optimal solution which is of F-type. Thus, it also solves the problem $F|plbl|L_{\max}$.

Theorem 41. *Problem $F|plbl|L_{\max}$ is solvable in $O(n \log n + mn)$ time.*

Interestingly, the term mn in the complexity estimate cannot be reduced, since there are instances for which $\Omega(nm)$ non-zero operations are needed for an F-type schedule.

For example, consider an instance with m machines, n jobs and

$$\begin{aligned} p_j &= j, \quad d_j = j \quad \text{for } 1 \le j \le m, \\ p_j &= m \quad d_j = j \quad \text{for } m+1 \le j \le n. \end{aligned}$$

Note that in a feasible schedule with $L_{\max} = 0$ job 1 has to finish on machine M_m at time 1. Furthermore all jobs $1 \le j \le m$ have to start processing at time 0 and have to be continuously processed until their due dates. The latter means that all machines are fully loaded between times 0 and 1.

Suppose that $n \ge 2m$. Since jobs have to traverse machines in the order of the machine numbering, no job other than 1 can be processed on machine M_m starting at time 0. Indeed, such a job would have to be stopped and moved to another machine at time 1, when job 1 has to finish on machine M_m (possibly with a zero-length operation). However, moving such a job to another machine is impossible, because it has already been processed by machine M_m and therefore cannot move back. Thus, machine M_m processes job 1 in time interval $[0, 1]$.

Similar arguments can be made for jobs $2, 3, \ldots, m$ to see that in time interval $[0, 1]$ job $j \le m$ is processed by machine M_{m-j+1}. Now, since job $m + 1$ has to start processing on machine M_1 at the latest at time 1, clearly job m has to be stopped and moved to the next machine. Job m needs to be continuously processed, so job $m-1$ has to be stopped on machine

M_2 and moved to the next machine. Continuing this line of argument, we can see that in time interval $[1, 2]$ job j, $2 \leq j \leq m+1$, is processed by machine M_{m-j+2}. Note that since job $m+1$ is started only at time 1, has processing time m and due date $m+1$, it has to be continuously processed. See Fig. 9.5 for an example with four machines and eight jobs.

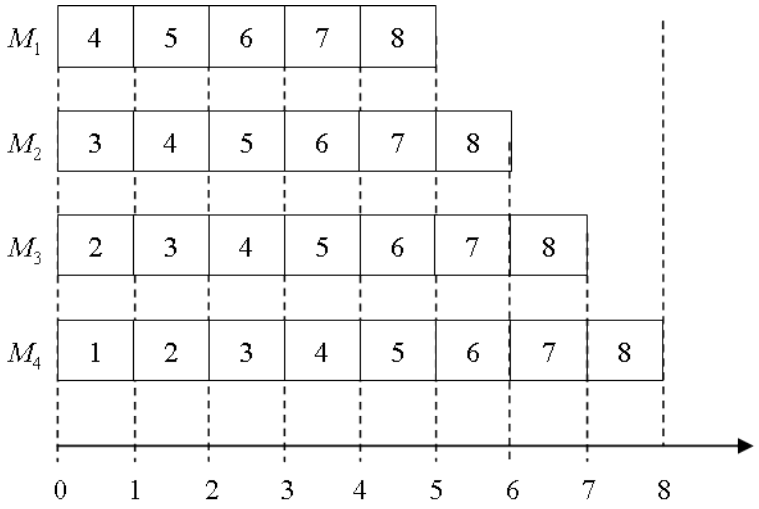


Figure 9.5: An optimal solution to an instance of $F|plbl|L_{\max}$ with $\Omega(mn)$ non-zero operations

Then we can use the same argument again for jobs $3, 4, \ldots, m+2$ and time interval $[2, 3]$. More generally, we can see inductively that jobs $j^*, j^*+1, \ldots, j^*+m-1$ are processed in time interval $[j^*-1, j^*]$ by machines M_{m-j+j^*}, $j^* \leq j \leq j^*+m-1$. The only time the inductive argument cannot be continued is at the end of the schedule, when not enough jobs are left to continue enforcing this staircase type structure. For example, in the schedule shown in Fig. 9.5, job 8 could instead be processed wholly by machine M_1, with zero-length operations on all other machines, since no job needs to start at time 5. Consequently, job 7 could remain on machine M_2 between times 4 and 7 and so on.

However, even if this is the case we still end up with a schedule where all jobs apart from the first $m-1$ and the last $m-1$ have m non-zero operations. The first job has $m-1$ zero-length operations the second job $m-2$ and so on. Similarly, the last job may potentially have $m-1$ operations, the second to last $m-2$ and so on. Thus, we have at least

$$Q = mn - 2(1 + 2 + \ldots + m - 1) = mn - m(m-1) = m(n-m+1)$$

operations. Notice that for $n \geq 2m$ we have

$$Q = m(n-m+1) \geq m\frac{n}{2} = \Omega(mn).$$

Therefore, for problem $F|plbl|L_{\max}$ instances exist where $\Omega(mn)$ non-zero operations are

needed. Note that this result is in contrast to the situation for problem $F|plbl|C_{\max}$ with the makespan objective, where even for F-type schedules we could keep the property that there are at most two non-zero operations for each job.

Turning to open shop, although Theorem 41 can be extended to handle problem $O|plbl|L_{\max}$, we present here a faster approach for $O|plbl|L_{\max}$. First, find the optimal value L_* of $L_{\max}$ for the relaxed problem $P|pmtn|L_{\max}$ in $O(n \log n)$ time, using, for example, the closed form expression for L_* from [11]. After that adjust the due dates to $d'_j = d_j + L_*$, treat them as deadlines and find a feasible schedule for $P|pmtn|L_{\max}$. The fastest algorithm of time complexity $O(n \log(nm))$ is due to [133]. With the assumption $n > m$, the time complexity of Sahni's algorithm reduces to $O(n \log n)$ (note that a trivial optimal solution, with one job per machine, exists if $n \leq m$). It is a property of Sahni's algorithm that the resulting parallel machine schedule has at most one preemption per job, and a preempted job is not restarted on the same machine. Therefore the schedule is of O-type, if zero-length operations are added at the beginning of the schedule.

Theorem 42. *Problem $O|plbl|L_{\max}$ is solvable in $O(n \log n)$ time.*

9.3 Type (ii) problems with minmax criteria

In this section we illustrate the methodology of Section 9.1.2 for the pliability problems of type (ii) by solving problems $F|plbl|C_{\max}$ and $F|plbl|L_{\max}$. We discuss the difficulties encountered for the open shop counterparts of these problems at the end of the section.

9.3.1 Type (ii) problem $F|plbl(\underline{p})|C_{\max}$

By Theorem 37 we limit our consideration to the class of permutation schedules, and use the decomposition technique described in Section 9.1.2. Given an instance I of problem $F|plbl|C_{\max}$, introduce instances I^d and I^e.

Let S^d be a permutation schedule for instance I^d (without idle times other than at the end of the schedule), and let S^e be a solution to I^e. Both solutions S^d and S^e are illustrated in Fig. 9.2. Let S be the schedule for the original instance I obtained by combining S^d and S^e. By Theorem 38

$$C_{\max}(S) = C_{mn}(S) = C_{mn}(S^d) + (m + n - 1)\,\underline{p} = C_{\max}(S^d) + (m + n - 1)\,\underline{p}. \tag{9.6}$$

Thus, if $C_{\max}(S^d)$ achieves its minimum value, then $C_{\max}(S)$ is minimum as well.

Following the approach from Section 9.2.1, construct an optimal schedule S^d_* with McNaughton's algorithm [111], using an arbitrary job permutation. Note that, by construction, schedule S^d_* is of permutation type. Without loss of generality we assume that the jobs are sequenced in the order of their numbering, and the same job order is used in an optimal solution S^e_* to I^e.

Consider the combined schedule S_*, obtained as a merger of S_*^d and S_*^e. Due to (9.6), S_* is an optimal schedule among all permutation schedules, and due to Theorem 37 it is globally optimal among all schedules.

The most time consuming step in the described approach is the merging of S_*^d and S_*^e. Its time complexity is $O(nm)$, and it defines the overall time complexity for constructing a complete optimal schedule for $F|plbl(\underline{p})|C_{\max}$.

Alternatively, if it is sufficient to specify the formulae for starting times of all operations and their lengths, then the following method can be used.

Let (i, j) denote an operation of job j on machine M_i. Denote the starting times of that operation in schedules S_*^e and S_*^d by $t_{ij}(S_*^e)$, $t_{ij}(S_*^d)$, and operation lengths by $p_{ij}(S_*^e)$, $p_{ij}(S_*^d)$. For schedule S_*^e all starting times are defined by a common formula

$$t_{ij}(S_*^e) = (i + j - 2)\,\underline{p},$$

and all operations are of equal lengths,

$$p_{ij}(S_*^e) = \underline{p},$$

see Fig. 9.2.

Schedule S_*^d contains at most $(m + n - 1)$ non-zero operations and it is specified by $O(n)$ formulae for their starting times and operation lengths. Let $C_*(M_i)$ denote the completion time of the last operation on machine M_i in schedule S_*^d, $1 \le i \le m$.

Consider the combined schedule S_*. We distinguish between three types of operations.

– *Initial* operations are those operations of length $\underline{p}$, which are obtained as a result of the merger with the zero-length operations of the initial part of S_*^d. Their starting times are the same as in schedule S_*^e, i.e. $t_{ij}(S_*) = t_{ij}(S_*^e)$.

– *Middle* operations are those, which are obtained as a result of the merger with the non-zero operations of the middle part of S_*^d (they are represented as shaded boxes in Fig. 9.2). Their starting times are equal to the starting times of S_*^d increased by $(i + j - 2)\,\underline{p}$, i.e. $t_{ij}(S_*) = t_{ij}(S_*^d) + (i + j - 2)\,\underline{p}$.

– *Final* operations are those operations of length $\underline{p}$, which are obtained as a result of the merger with the zero-length operations of the last part of S_*^d. Their starting times are equal to their starting times in schedule S_*^e increased by $C_*(M_i)$, i.e. $t_{ij}(S_*) = t_{ij}(S_*^e) + C_*(M_i)$.

Thus we conclude:

$$\begin{aligned} t_{ij}(S_*) &= \begin{cases} (i+j-2)\,\underline{p}, & \text{if } (i,j) \text{ is one of the initial operations,} \\ t_{ij}(S_*^d) + (i+j-2)\,\underline{p}, & \text{if } (i,j) \text{ is one of the middle operations,} \\ (i+j-2)\,\underline{p} + C_*(M_i), & \text{if } (i,j) \text{ is one of the final operations,} \end{cases} \\ p_{ij}(S_*) &= p_{ij}(S_*^e) + p_{ij}(S_*^d). \end{aligned} \tag{9.7}$$

Theorem 43. *An optimal schedule for problem $F|plbl(\underline{p})|C_{\max}$ of type (ii) can be specified by at most $m+n-1$ formulae for the starting times of all operations and for their lengths, where each formula is computable in $O(1)$ time.*

Thus, while it appears harder to solve the model of type (ii) compared to the model of type (i), the additional computational costs of solving $F|plbl(\underline{p})|C_{\max}$ are related to the calculation and output of completion times for nm operations rather than the structural building of the schedule.

Finally, note that with the arguments used in this section together with (9.2) from Section 9.1.2 and the results from Section 9.2.1, we can compute an optimal makespan for an instance of $F|plbl(\underline{p})|C_{\max}$ as the maximum of two lower bounds, similar to (9.5). The makespan of the optimal schedule S_*^d for the diminished instance I^d is given by (9.5), adjusted to account the $m\underline{p}$ units of processing time missing from each job, i.e.

$$C_{\max}(S_*^d) = \max\left\{ \frac{1}{m}\left(p(\mathcal{J}) - mn\underline{p}\right), \max_{j\in\mathcal{J}}\left\{p_j - m\underline{p}\right\} \right\}. \tag{9.8}$$

Using (9.8) together with (9.2), we obtain two possible lower bounds for the makespan of problem $F|plbl(\underline{p})|C_{\max}$. If the optimal makespan of the diminished instance is given by the average machine load, the optimal combined schedule S_* we obtain for the original instance has makespan

$$\frac{1}{m}\left(p(\mathcal{J}) - mn\underline{p}\right) + (m+n-1)\underline{p} = \frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p}.$$

Otherwise, the optimal combined schedule S_* has makespan

$$\max_{j\in\mathcal{J}}\left\{p_j - m\underline{p}\right\} + (m+n-1)\underline{p} = \max_{j\in\mathcal{J}}\left\{p_j\right\} + (n-1)\underline{p}.$$

With these arguments we obtain the following corollary.

Corollary 44. Given an instance of problem $F|plbl(\underline{p})|C_{\max}$, the optimal makespan is given by

$$C_* = \max\left\{ \frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p}, \max_{j\in\mathcal{J}}\left\{p_j\right\} + (n-1)\underline{p} \right\}. \tag{9.9}$$

9.3.2 Type (ii) problem $O|plbl(\underline{p})|C_{\max}$

The open shop problem with equal lower bounds of type (ii) and the makespan objective appears to be much harder to handle than the flow shop version. The main difficulty is that, obviously, the simple structure of a permutation schedule, like we have in the flow shop version, is not available for open shop. Furthermore, it is difficult to find a way of computing, exactly, the makespan of an optimal schedule, as we did for flow shop, see Corollary 44.

The general problem $O|plbl(\underline{p})|C_{\max}$ is still open. For two machines, it can be solved in $O(n)$ time. First we split each job such that it has equal processing times on both machines. An open shop problem with this type of processing times, i.e. only dependent on the jobs, is called a proportionate open shop.

For proportionate open shop, the algorithm in [62] solves the problem optimally in $O(n)$ time and can be described as follows. We renumber the jobs such that 1 is the longest job. Then on machine M_1 we schedule jobs in order of their numbering, and on machine M_2 we schedule jobs in the sequence $(2, 3, 4, \ldots, n-1, n, 1)$.

Note that the value of the makespan is equal to the minimum makespan value given by (9.5) for the associated preemptive parallel machine problem (see [62]). Since the lower bound for the makespan of the preemptive parallel machine problem with the same job data is also a lower bound for the makespan of problem $O|plbl(\underline{p})|C_{\max}$, problem $O2|plbl(\underline{p})|C_{\max}$ is solved optimally. We do not state this result as a theorem here, since is is dominated by the linear time result for the open shop problem with two machines, arbitrary upper and lower bounds of type (iii) and the makespan objective, demonstrated in Section 9.4.2.

The same idea does not work for $m \geq 3$ machines, for two reasons. First, proportionate open shop with three or more machines is NP-hard to solve, like a general open shop [48, 106]. Note that in [48] the roles of jobs and machines are exchanged and that in [106] NP-hardness is proven for the more general case of so-called ordered open shops, but an instance of proportionate open shop is used in the actual construction.

The second reason why the idea we used for two machines cannot be generalized, is due to the fact that the lower bound (9.5) for preemptive parallel machines cannot always be reached as the makespan of problem $O|plbl(\underline{p})|C_{\max}$. Consider for example an instance with $m = 3$ machines, equal lower bound $\underline{p} = 1$ and processing times

Job	1	2	3	4	5	6	7	8
p_j	10.5	3	3	3	3	3	3	3

. (9.10)

Clearly, the optimal makespan for the corresponding parallel machine problem $P3|pmtn|C_{\max}$ with the same job data is 10.5, which is equal to both the largest processing time and the average machine load. Since the splitting of jobs $2, 3, \ldots, 8$ is fixed (they must have processing time 1 on each machine), that value can only be achieved for the open shop problem if job 1 is split such that it has processing time 3.5 on each machine. Furthermore, in order to achieve a

makespan of 10.5 job 1 has to be scheduled continuously between times 0 and 10.5 and no idle time can appear in the schedule.

However, it is not possible to satisfy all three conditions at the same time. Without loss of generality assume that job 1 is processed by the machines in order of their numbering, first by M_1, then by M_2 and finally by M_3. If job 1 is scheduled continuously and split such that it has processing time 3.5 on each machine, then it is started on machine M_2 at time 3.5. Since only three minimum operations fit there into the time window between 0 and 3.5, this leaves an idle time of 0.5 on machine M_2 before time 3.5. Thus, it is not possible to achieve a schedule with makespan 10.5. For an example of such a schedule see Fig. 9.6, where the dotted line represents the optimal makespan for parallel machines and the solid line represents the makespan 11.

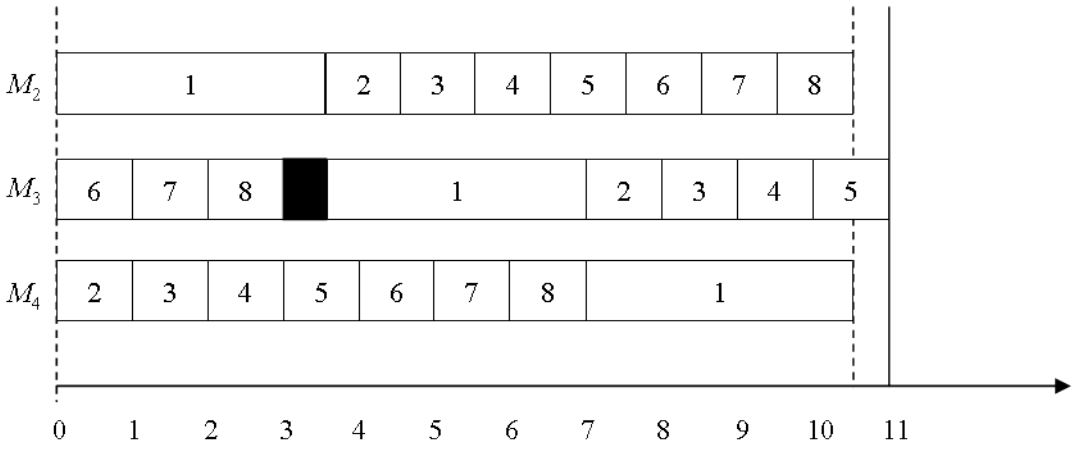


Figure 9.6: A schedule for the instance with processing times given by (9.10) in which
– job 1 is split such that it has processing time 3.5 on each machine and
– job 1 is scheduled continuously between in $[0, 10.5]$

Note that the optimal makespan for the example instance is 10.75. It can be achieved by starting with the schedule shown in Fig. 9.6 and moving 0.25 units of processing load of job 1 from machine M_2 to machine M_3. In the obtained schedule job 1 is not scheduled continuously (it is interrupted for 0.25 time units between machines M_2 and M_3) and job 1 is also not split evenly across the machines. What is more, we can use similar arguments to the ones above to show that if all jobs are split evenly across the machines, then any feasible schedule has at least makespan 11. Thus even solving the proportionate open shop instance optimally does not yield an optimal schedule for problem $O|plbl(\underline{p})|C_{\max}$.

It is easy to see that the above example can be generalized for lower bounds $\underline{p} \neq 1$ in which case job 1 has processing time $9\underline{p} + \frac{3}{2}\underline{p}$ and all other jobs have processing time $3\underline{p}$. It can also be generalized, with some more work for problems with more machines, in which case job 1 has processing time $p_1 = m^2\underline{p} + \frac{m}{m-1}\underline{p}$ and all other jobs have processing time $m\underline{p}$. The number of jobs must also be increased to $m^2 - m + 2$ (including job 1), such that the average machine

load is equal to

$$m^2\underline{p} + \underline{p} + \frac{1}{m-1}\underline{p} = p_1.$$

For a positive partial result, consider problem $O|plbl(\underline{p})|C_{\max}$ with the set of jobs $\mathcal{J}$, such that

$$\frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p} \geq \max_{j\in\mathcal{J}}\{p_j\} + (n-1)\underline{p}.$$

In that case, by Corollary 44, the flow shop problem $F|plbl(\underline{p})|C_{\max}$ with the same job data has makespan $\frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p}$ and in an optimal schedule the processing load is equal on all machines. We can adjust the flow shop schedule by moving the last part with minimum operations to the front, $m-1$ minimum operations on machine M_m, $m-2$ on machine M_{m-1} and in general $i-1$ on machine M_i. As an example, in the schedule shown in Fig. 9.2 we would move the operations of job 16 on machine M_2, 15, 16 on machine M_3, 14, 15, 16 on machine M_4 and 13, 14, 15, 16 on machine M_5. No conflicts arise, since $n \geq m$, and the makespan of the obtained schedule is

$$\frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p} - (m-1)\underline{p} = \frac{1}{m}p(\mathcal{J}),$$

the optimal makespan for the parallel machine problem. Clearly, since we achieve the optimal makespan for the parallel machine problem, the schedule is optimal for problem $O|plbl(\underline{p})|C_{\max}$.

Corollary 45. Problem $O|plbl(\underline{p})|C_{\max}$ can be solved in $O(mn)$ time if the optimal makespan of the corresponding flow shop problem $F|plbl(\underline{p})|C_{\max}$ is equal to

$$\frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p}.$$

The problem remains open for the case where

$$\max_{j\in\mathcal{J}}\{p_j\} + (n-1)\underline{p} > \frac{1}{m}p(\mathcal{J}) + (m-1)\underline{p}.$$

In order to find good approximate solutions it might also be useful to split the jobs such that the resulting traditional open shop problem is of proportionate type and then use approximation algorithms for proportionate open shop. A recent result in [116] shows that a schedule exists for proportionate open shop, which has a makespan at most $(2-\frac{1}{m})$ times larger than the lower bound for the corresponding preemptive parallel machine problem (9.5). Furthermore, such a schedule is computable in polynomial time. It is also claimed in [137] that an algorithm exists which finds a schedule for proportionate open shop with makespan at most $1+\frac{1}{m}$ times larger than the lower bound (9.5), but a full paper has not yet appeared to the best of our knowledge.

9.3.3 Type (ii) problem $F|plbl(\underline{p})|L_{\max}$

Let I denote an arbitrary instance of problem $F|plbl(\underline{p})|L_{\max}$. We again restrict ourselves to permutation schedules and use the methodology introduced in Section 9.1.2. First we show that

there exists an optimal schedule for problem $F|plbl(\underline{p})|L_{\max}$ in which jobs are in earliest due date order (EDD). Notice that the above property holds for model (i) with zero lower bounds $\underline{p} = 0$; this follows from the structure of an optimal schedule constructed by the algorithm from [11].

Lemma 46. *For an instance I of problem $F|plbl(\underline{p})|L_{\max}$ there exists an optimal permutation schedule in which jobs are in EDD order.*

Proof: By Theorem 37 there exists an optimal permutation schedule S for instance I. Suppose that in S, the jobs are not in EDD order, i.e., there are two jobs u, v with $d_u > d_v$, but job u is scheduled before job v. Using Lemma 36 we swap jobs u and v so that in the resulting schedule S', u and v are in EDD order and

$$\begin{aligned} &C_j(S') = C_j(S), \quad j \in N \setminus \{u, v\}, \\ &C_u(S') \leq C_v(S), \\ &C_v(S') \leq C_v(S). \end{aligned}$$

The lateness of all jobs other than u stays the same or decreases, while $L_u(S')$ satisfies:

$$L_u(S') = C_u(S') - d_u < C_u(S') - d_v \leq C_v(S) - d_v \leq L_{\max}(S).$$

Thus starting with S and repeating adjacent jobs swaps we obtain an optimal permutation schedule S_* that is in EDD order. ∎

Given an instance I of $F|plbl(\underline{p})|L_{\max}$, renumber the jobs so that $d_1 \leq d_2 \leq \cdots \leq d_n$. Define the two instances:

$$\begin{aligned} I^e: \quad & p_j^e = m\underline{p}, \\ & d_j^e = (j + m - 1)\,\underline{p}, \end{aligned} \qquad \begin{aligned} I^d: \quad & p_j^d = p_j - m\underline{p}, \\ & d_j^d = d_j - (j + m - 1)\,\underline{p}. \end{aligned}$$

Notice that the data for the original instance I satisfy

$$p_j = p_j^e + p_j^d, \; d_j = d_j^e + d_j^d.$$

In the class of permutation schedules with the fixed job sequence $(1, 2, \ldots, n)$, the optimal schedule S_*^e for instance I^e is the same as in the top left Gantt chart in Fig. 9.2 and the optimal value of $L_{\max}$ is $L_*^e = 0$.

Let S^d be a permutation schedule for instance I^d with jobs in EDD order (without idle times other than at the end of the schedule) and let S be the schedule for the original instance I that is obtained by combining S^d with S_*^e. By Theorem 38 we have

$$L_j(S) = C_{mj} - d_j = C_{mj}^d + (j + m - 1)\underline{p} - d_j = C_{mj}^d - d_j^d = L_j(S^d), \tag{9.11}$$

so that $L_{\max}(S) = L_{\max}(S^d)$.

Now consider an optimal schedule S_*^d for instance I^d constructed by the algorithm from [11], see Section 9.2.2. As discussed before, it is the property of Baptiste's algorithm that the resulting schedule is of F-type and it corresponds to the permutation schedule with the EDD job sequence $(1, 2, \ldots, n)$. Therefore the job permutations in both optimal schedules S_*^d and S_*^e are equal. Combining S_*^d and S_*^e delivers a schedule S_* for instance I, which is optimal among all permutation schedules, in which jobs are in EDD order, due to (9.11). Theorem 38 and Lemma 46 justify that for problem $F|plbl(\underline{p})|L_{\max}$ there exists an optimal permutation schedule with jobs in EDD order. This means that S_* is an optimal schedule (among all schedules) for the original instance I. Moreover, the optimal $L_{\max}$-value for instance I is equal to the optimal $L_{\max}$-value for instance I^d.

The most time-consuming step is the algorithm from [11], which takes $O(n \log n + mn)$ time, dominating the time needed to renumber the jobs in EDD order and the time for combining the two schedules.

Theorem 47. *Problem $F|plbl(\underline{p})|L_{\max}$ is solvable in $O(n \log n + mn)$ time.*

9.4 Type (iii) problems with the makespan objective and $m = 2$

In this section we consider the versions of the flow shop and open shop problems with the makespan objective and individual lower and upper bounds on operation processing times. The flow shop problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$, discussed in Section 9.4.1, is NP-hard, while its open shop counterpart $O2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is solvable in linear time, as shown in Section 9.4.2.

As before, to simplify notation for the two-machine case we adopt traditional notation A and B for machines M_1 and M_2, $\underline{a}_j$ and $\overline{a}_j$ for the lower and upper bounds of the A-operations, $\underline{b}_j$ and $\overline{b}_j$ for the lower and upper bounds of the B-operations. The objective is to find operation lengths a_j and b_j for A- and B-operations of every job j, $1 \leq j \leq n$, which define how the total processing time p_j is split among the machines,

$$a_j + b_j = p_j,$$

so that the given boundaries are satisfied:

$$\underline{a}_j \leq a_j \leq \overline{a}_j \text{ and } \underline{b}_j \leq b_j \leq \overline{b}_j.$$

9.4.1 Problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$

In this section we prove NP-hardness of problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$. Note that the NP-hardness proof for the discrete version (with three operations per job) given in [67] in the context of flexible operations, is not applicable to the model with pliable jobs. Indeed, in the

proof in [67] the problem is reduced to a parallel machine problem by setting the processing times of the first and the second operation (which have to be processed on M_1 and M_2 respectively) to zero. Then jobs consist only of the remaining third operations, which are flexible and can be scheduled on either of the machines. Such an instance of the problem from [67] can be interpreted as an instance of parallel machine scheduling without preemption.

The same approach does not work to prove the NP-hardness of problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$. We would end up with an instance where $\underline{p}_{ij} = 0$ and $\overline{p}_{ij} = p_j$ - the model of type (i), which is already proved to be polynomially solvable.

Using a different idea, below we prove the NP-hardness of problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ via a reduction directly from the PARTITION problem.

Theorem 48. *Problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is NP-hard.*

Proof: Consider an instance of PARTITION with integers $e_1, \ldots, e_n$ and $\sum_{j=1}^{n} e_j = 2E$. The objective is to decide whether a set $N_1 \subset \{1, 2, \ldots, n\}$ exists with $\sum_{i \in N_1} e_i = E$. We construct an instance of the flow shop problem with jobs $\mathcal{J} = \{1, 2, \ldots, n+1\}$ and processing times of the form:

j	1	2	$\cdots$	n	$n+1$
p_j	e_1	e_2	$\cdots$	e_n	$2E$
$[\underline{a}_j, \overline{a}_j]$	$[0, e_1]$	$[0, e_2]$	$\cdots$	$[0, e_n]$	$[E, E]$
$[\underline{b}_j, \overline{b}_j]$	$[0, e_1]$	$[0, e_2]$	$\cdots$	$[0, e_n]$	$[E, E]$

which implies that the jobs $\{1, 2, \ldots, n\}$ can be arbitrarily split among the two machines, while the job $n+1$ has fixed processing times on both machines. In the following we show that PARTITION has a solution if and only if a flow shop schedule of makespan $C_{\max} = 2E$ exists.

"$\Rightarrow$": Let N_1 with $\sum_{i \in N_1} e_i = E$ be a solution to PARTITION and define $N_2 = \{1, 2, \ldots, n\} \setminus N_1$. Then in an optimal schedule with $C_{\max} = 2E$ the jobs are in the same order on both machines: first the jobs of the set N_1, then job $n+1$ and finally the jobs of the set N_2. The operation lengths of the jobs in N_1 are chosen so that $a_j = 0$, $b_j = e_j$, and operation lengths of the jobs in N_2 are chosen in the opposite way: $a_j = e_j$, $b_j = 0$. The corresponding schedule is illustrated in Fig. 9.7.

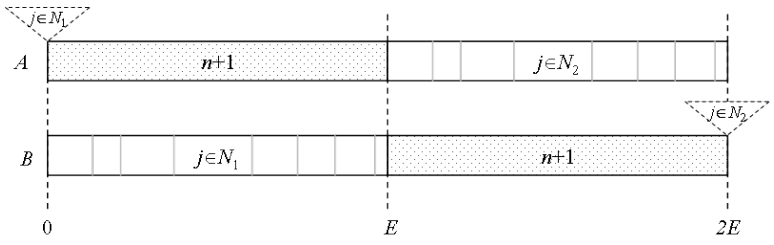


Figure 9.7: An optimal solution to the instance of the flow shop problem

"$\Leftarrow$": It is known that for the two-machine flow shop an optimal permutation schedule exists. Let S be such a schedule and its makespan be $2E$. The fixed splitting of job $n+1$ leaves

time window $[E, 2E]$ on machine A and time window $[0, E]$ on machine B, where operations of jobs $\{1, \ldots, n\}$ can be processed. Denoting the subset of jobs that precede $n+1$ by $N_1 \subset \mathcal{J}$, we obtain a solution to PARTITION. ∎

Note that, other than in the discrete version with flexible operations (see [104]), our problem does not remain NP-hard if we assume that the job-sequence is fixed. Indeed even for an arbitrary number of machines m and a more general objective function f, assuming a fixed job-sequence on all machines, we can adjust the linear program obtained in [123] in order to find an optimal distribution of processing times to the machines. Let $x_{i,j}$ denote the processing time of job j on machine M_i, $S_{i,j}$ the starting time of job j on machine M_i and C_j the completion time of job j. Then, after re-numbering the jobs in the order in which they appear in the given sequence, the following LP computes an optimal distribution of processing times to the machines, as well as a starting time for each operation:

$$
\begin{array}{lllll}
\text{minimize} & f(C_1, C_2, \ldots, C_n) & & & \\
\text{subject to} & \sum_{i=1}^{m} x_{i,j} = p_j, & \forall j = 1, 2, \ldots, n, & & \\
& s_{i,j} + x_{i,j} \leq s_{i+1,j}, & \forall i = 1, 2, \ldots, m-1 & \forall j = 1, 2, \ldots, n, & \\
& s_{i,j} + x_{i,j} \leq s_{i,j+1}, & \forall i = 1, 2, \ldots, m, & \forall j = 1, 2, \ldots, n-1, & \\
& s_{m,j} + x_{m,j} = C_j, & \forall j = 1, 2, \ldots, n, & & \\
& \underline{p}_{i,j} \leq x_{i,j} \leq \overline{p}_{i,j}, & \forall i = 1, 2, \ldots, m, & \forall j = 1, 2, \ldots, n, & \\
& x_{i,j}, s_{i,j}, C_j \geq 0 & \forall i = 1, 2, \ldots, m, & \forall j = 1, 2, \ldots, n. &
\end{array}
$$

Here, the objective function f depending on the completion times needs to be separable and continuous piecewise linear, see [123].

9.4.2 Problem $O2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$

Introduce notation $p(\mathcal{J}) = \sum_{j \in \mathcal{J}} p_j$ for the cumulative processing time of all jobs. If the splitting $p_j = a_j + b_j$ is determined for all jobs $j \in \mathcal{J}$, then a schedule that minimizes the makespan can be found by the well-known $O(n)$-time algorithm from [62], and the optimal value of the makespan is given by

$$C_{\max} = \max \left\{ \sum_{j \in \mathcal{J}} a_j, \ \sum_{j \in \mathcal{J}} b_j, \ p_q \right\},$$

where q is a longest job,

$$p_q = \max \{ a_j + b_j | j \in \mathcal{J} \}. \tag{9.12}$$

In what follows we formulate three LP problems $LP(A)$, $LP(B)$ and $LP(q)$ which characterize schedules with the makespan corresponding to each of the three lower bounds, the load $\sum_{j \in \mathcal{J}} a_j$ of machine A, the load $\sum_{j \in \mathcal{J}} b_j$ of machine B, and the processing time p_q of the

longest job q. Notice that some of the problems may be infeasible. An optimal solution is selected among the solutions to these three problems as the one delivering the smallest makespan value.

Consider first problem $LP(A)$ that characterizes the class of schedules with $C_{\max} = a(N) = p_q + \Delta$, where $\Delta \geq 0$:

$$\begin{array}{lll} LP(A): & \text{minimize} & \Delta \\ & \text{subject to} & \sum_{j\in\mathcal{J}} a_j = p_q + \Delta, \\ & & \sum_{j\in\mathcal{J}} b_j \leq p_q + \Delta, \\ & & a_j + b_j = p_j, \qquad j \in \mathcal{J}, \\ & & \underline{a}_j \leq a_j \leq \overline{a}_j, \qquad j \in \mathcal{J}, \\ & & \underline{b}_j \leq b_j \leq \overline{b}_j, \qquad j \in \mathcal{J}, \\ & & \Delta \geq 0. \end{array}$$

From the first and the third constraints we derive expressions for Δ and b_j:

$$\begin{aligned} \Delta &= \sum_{j\in\mathcal{J}} a_j - p_q, \\ b_j &= p_j - a_j, \quad j \in \mathcal{J}. \end{aligned} \tag{9.13}$$

Using them, we re-write $LP(A)$ as follows:

$$\begin{array}{lll} LP'(A): & \text{minimize} & \sum\limits_{j\in\mathcal{J}} a_j \\ & \text{subject to} & \sum\limits_{j\in\mathcal{J}} a_j \geq \max\left\{\frac{1}{2}p(\mathcal{J}), p_q\right\}, \\ & & \ell_j \leq a_j \leq u_j, \qquad j \in N, \end{array}$$

where

$$\ell_j = \max\left\{\underline{a}_j,\ p_j - \overline{b}_j\right\}, \quad u_j = \min\left\{\overline{a}_j,\ p_j - \underline{b}_j\right\}. \tag{9.14}$$

The resulting problem is the knapsack problem with continuos variables a_j, $j \in N$, solvable in $O(n)$ time by the algorithm in [10].

Problem $LP(B)$ is formulated similarly for the class of schedules with $C_{\max} = b(\mathcal{J}) \geq p_q$; it is also solvable in $O(n)$ time.

Consider now problem $LP(q)$ that considers the class of schedules with $C_{\max} = p_q$. There is no objective function to minimize, since $C_{\max} = p_q$ is a constant and does not depend on the splitting of the jobs. Thus, we need to find a feasible solution with respect to the following

constraints:

$$\begin{aligned} LP(q): \quad & \sum_{j\in\mathcal{J}\setminus\{q\}} a_j \le b_q, \\ & \sum_{j\in\mathcal{J}\setminus\{q\}} b_j \le a_q, \\ & a_j + b_j = p_j, \qquad j\in\mathcal{J}, \\ & \underline{a}_j \le a_j \le \overline{a}_j, \qquad j\in\mathcal{J}, \\ & \underline{b}_j \le b_j \le \overline{b}_j, \qquad j\in\mathcal{J}. \end{aligned}$$

Using expression $b_j = p_j - a_j$ for $j \in \mathcal{J}$ we obtain:

$$\begin{aligned} LP'(q): \quad & \sum_{j\in\mathcal{J}} a_j \le p_q, \\ & \sum_{j\in\mathcal{J}} a_j \ge p(\mathcal{J}) - p_q, \\ & \ell_j \le a_j \le u_j, \qquad j\in N. \end{aligned}$$

Here ℓ_j and u_j are given by (9.14). The latter problem can be solved in $O(n)$ time by performing the following steps.

1. Compute $a_* = \sum_{j\in\mathcal{J}} \ell_j$, the smallest value of $\sum_{j\in\mathcal{J}} a_j$.

2. If a_* satisfies both main conditions, i.e., $p(N) - p_q \le a_* \le p_q$, then stop - a feasible solution is found.

3. If $a_* > p_q$, then stop - problem $LP'(q)$ is infeasible.

4. If $a_* < p(\mathcal{J}) - p_q$, then solve the LP problem:

$$\begin{aligned} \max \quad & \sum_{j\in\mathcal{J}} a_j \\ \text{s.t.} \quad & \sum_{j\in\mathcal{J}} a_j \le p_q, \\ & \ell_j \le a_j \le u_j, \quad j\in\mathcal{J}, \end{aligned} \tag{9.15}$$

   and verify whether for the found solution the required condition $\sum_{j\in\mathcal{J}} a_j \ge p(\mathcal{J}) - p_q$ is satisfied. Problem (9.15) is again the knapsack problem with continuos variables a_j, $j \in \mathcal{J}$, solvable in $O(n)$ time.

Since each of the problems $LP(A)$, $LP(B)$ and $LP(q)$ can be solved in $O(n)$ time, we make the following conclusion.

Theorem 49. *Problem* $O2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ *is solvable in* $O(n)$ *time.*

9.5 Type (i) and type (ii) problems with min-sum criteria

In this section we consider the pliability problems with min-sum objectives. For pliability of type (iii) these problems are strongly NP-hard even in the two-machine case due to Theorem 39 and because problems $F2||f$ and $O2||f$ are strongly NP-hard for all traditional min-sum objectives f (see Sections 2.3.4 and 2.3.5 or [17]). For this reason, we study problems of type (i) and (ii), with the main focus on $f = \sum C_j$.

9.5.1 Type (i) problems $F|plbl|\sum C_j$ and $O|plbl|\sum C_j$

Recall that problem $F|plbl|\sum C_j$ can be solved in polynomial time via LP, see Theorem 35. In this section we provide a faster algorithm, based on a solution algorithm for problem $Q|pmtn|\sum C_j$, where we construct a schedule with a staircase structure.

After renumbering jobs in non-decreasing order of processing times, the schedule is constructed as follows. Job 1 has zero processing time on machines $M_1, M_2, \ldots, M_{m-1}$ and its full processing time is assigned to machine M_m, starting at time 0. Job 2 has zero processing time on machines $M_1, M_2, \ldots, M_{m-2}$, and its processing time on machine M_{m-1} is equal to p_1. The remainder of its processing time is scheduled on machine M_m. Continuing in this manner, we obtain a stair case structure, see Fig. 9.8.

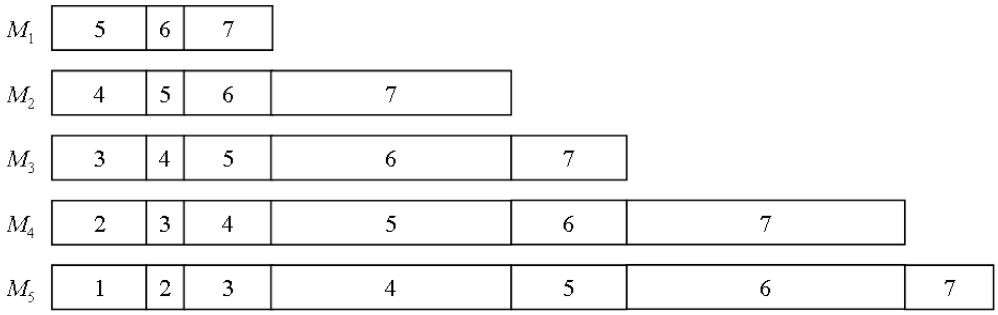


Figure 9.8: A schedule for problem $F|plbl|\sum C_j$ in staircase form

For the remainder of this subsection, denote by S the schedule constructed as described above. The optimality of S for problem $F|plbl|\sum C_j$ can be seen by following the arguments used in [17] for $Q|pmtn|\sum C_j$, but we also prove it below for completeness.

Clearly, it is sufficient to show that the constructed schedule is optimal for $P|pmtn|\sum C_j$, due to (9.1). Note that for problem $P|pmtn|\sum C_j$ an optimal schedule without preemption exists and that its objective value is

$$\sum_{j=1}^{n} \left\lceil \frac{n-j+1}{m} \right\rceil p_j. \tag{9.16}$$

For the completion times in schedule S we have $C_1 = p_1$, $C_2 = p_2$, ..., $C_m = p_m$ and $C_j = p_j + C_{j-m}$ for $j > m$. Summing up these completion times, we obtain (9.16), which is the optimal value. Therefore, schedule S is optimal for $P|pmtn|\sum C_j$ and also for $F|plbl|\sum C_j$, see Fig. 9.9.

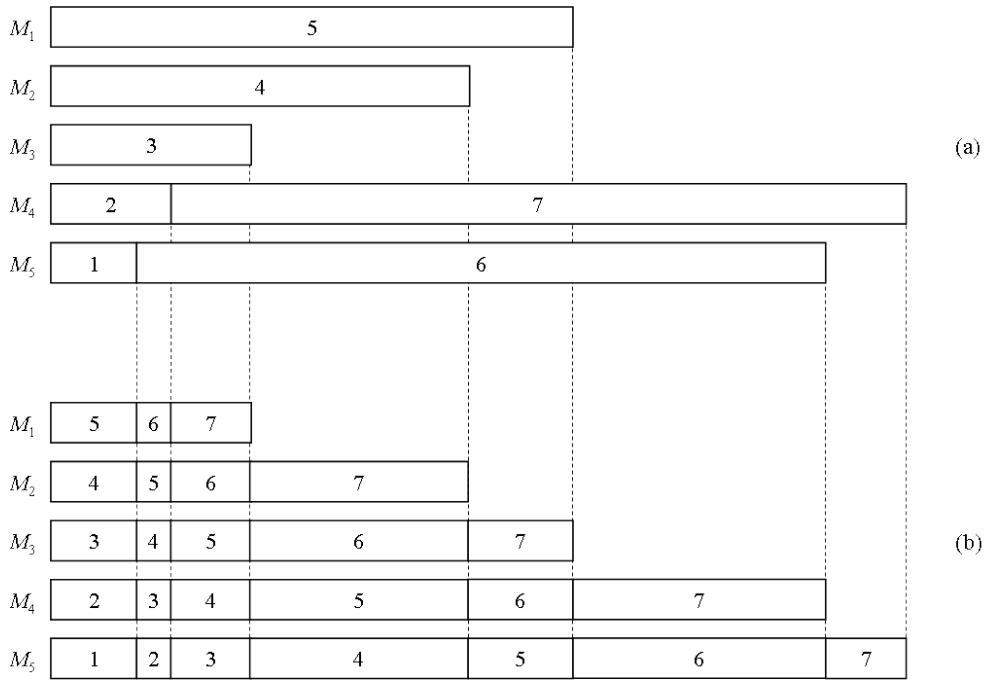


Figure 9.9: Two equivalent schedules optimal for (a) $P|pmtn|\sum C_j$ and $P||\sum C_j$ (b) $F|plbl|\sum C_j$

Notice that it takes $O(n \log n)$ time to sort the jobs, and $O(nm)$ time to schedule (and output) $O(m)$ operations for each job. Thus the total time complexity is $O(nm + n \log n)$.

Alternatively, we can adopt an approach with formulae, similar to Section 9.2.1. For this, we first show below that it takes only $O(n \log n)$ time to compute the starting time and length of each synchronous interval in the schedule. Indeed, denote by ℓ_ι the length of interval ι and by s_ι its starting time, for $\iota = 1, 2, \ldots, n$. The starting times and lengths of the initial m intervals can be computed by setting $s_1 = 0, \ell_1 = p_1$ and then using the recursive formulas

$$s_{\iota+1} = s_\iota + \ell_\iota$$

and

$$\ell_{\iota+1} = p_{\iota+1} - s_{\iota+1}.$$

For $\iota \geq m$ we have

$$s_{\iota+1} = s_\iota + \ell_\iota$$

and

$$\ell_{\iota+1} = p_{\iota+1} - (s_{\iota+1} - s_{\iota-m+2}).$$

To see that the latter holds, note that the job $\iota+1$ that finishes on machine M_m in interval $\iota+1$ and therefore determines the length of that interval, starts on machine M_1 in interval $\iota-m+2$ and is continuously processed in intervals $\iota-m+2, \iota-m+3, \ldots, \iota+1$. Thus after the first $m-1$ intervals processing time $\ell_{\iota+1}$ remains to be scheduled on the last machine.

Clearly, after the jobs are sorted, values s_ι and ℓ_ι can be computed in $O(n+m)$ time, and as we may assume $n \geq m$ (otherwise some upstream machines have only zero-length operations), we obtain a total time complexity of $O(n \log n)$.

Having computed the starting time and length of each interval, we can now specify formulae to compute the starting times and lengths of the operations of each job in $O(1)$. For ease of notation, we define $s_\iota = 0$ and $\ell_\iota = 0$ if $\iota \leq 0$. Then the starting time of operation O_{ij} of job j on machine M_i is given by s_{i+j-m} and its length by

$$x_{ij} = \ell_{i+j-m}.$$

Theorem 50. *Problem $F|plbl|\sum C_j$ is solvable in $O(n \log n + mn)$ time providing a full schedule, or in $O(n \log n)$ time, by providing formulae for the starting times and lengths of all operations, which can be computed in $O(1)$ time.*

To solve problem $O|plbl|\sum C_j$ with unrestricted pliability of type (i), we start with an optimal schedule S for $P||\sum C_j$, which is also optimal for $P|pmtn|\sum C_j$ (see [17]). Then we transform S into an O-type schedule for $O|plbl|\sum C_j$ by adding zero-length operations in the beginning of S such that every job has an operation on every machine. Clearly, the resulting schedule is still optimal for $P|pmtn|\sum C_j$ and therefore it is also optimal for $O|plbl|\sum C_j$.

Theorem 51. *Problem $O|plbl|\sum C_j$ is solvable in $O(n \log n)$ time.*

9.5.2 Type (ii) problem $F|plbl(\underline{p})|\sum C_j$

To solve problem $F|plbl(\underline{p})|\sum C_j$, we again use the methodology from Section 9.1.2 to show that we can focus on the diminished instance only. Then we use the result from Section 9.5.1 to construct an optimal schedule for the diminished instance.

Remember that due to Theorem 37 an optimal permutation schedule exists to solve problem $F|plbl(\underline{p})|\sum C_j$. We split instance I of problem $F|plbl(\underline{p})|\sum C_j$ into I^d and I^e, similar to the construction in Section 9.1.2 (no due dates appear). Then Theorem 38 holds for permutation schedules for instances I, I^d and I^e.

Note that for instance I^d, if a given permutation schedule S^d has idle times other than at the end of the schedule, then those can be removed either by moving operations to the left or by moving processing load to downstream machines, without increasing the objective. Thus we can restrict our consideration to permutation schedules S^d for instance I^d that have no idle

times other than at the end of the schedule (where some machines finished their workload while others are still processing).

Given a permutation schedule S^d for the diminished instance I^d and a schedule S^e for instance I^e using the same job permutation as S^d, let S be the corresponding schedule for the original instance I. By Theorem 38 we have

$$\sum_{j=1}^{n} C_j(S) = \sum_{j=1}^{n} C_j(S^d) + \sum_{j=1}^{n}((j+m-1)\,\underline{p}) = \sum_{j=1}^{n} C_j(S^d) + \sum_{j=1}^{n} C_j(S^e).$$

As the objective value $\sum_{j=1}^{n} C_j(S^e)$ of schedule S^e for instance I^e is the same for any permutation schedule S^e (without additional idle times), clearly $\sum_{j=1}^{n} C_j(S)$ is minimum if and only if $\sum_{j=1}^{n} C_j(S^d)$ is minimum.

Then using similar arguments as in Sections 9.3.1 and 9.3.3 we see that the optimal schedule for problem $F|plbl|\sum C_j$ constructed in Section 9.5.1 can be extended to an optimal solution for $F|plbl(\underline{p})|\sum C_j$, using Theorem 38.

To obtain the time complexity of problem $F|plbl(\underline{p})|\sum C_j$, observe that sorting jobs and finding an optimal solution for instance I^d takes at most $O(n\log n + mn)$ time and that combining schedules S^d and S^e takes $O(nm)$ time.

Theorem 52. *Problem $F|plbl(\underline{p})|\sum C_j$ is solvable in $O(n\log n + mn)$ time.*

It is again possible to solve the problem faster if it is sufficient to provide formulae for the starting times and lengths of all operations. Note that the first $m-1$ intervals are all of length $\ell_\iota = \underline{p}$, with starting time $s_\iota = (\iota-1)\underline{p}$. Then for $\iota \geq m-1$ we have

$$s_{\iota+1} = s_\iota + \ell_\iota$$

and

$$\ell_{\iota+1} = p_{\iota-m+2} - (s_{\iota+1} - s_{\iota-m+2}),$$

similar to the formulae from Section 9.5.1 for the model of type (i), only with $p_{\iota-m+2}$ in the second formula, rather than $p_{\iota+1}$, as now job $\iota-m+2$ finishes in interval $\iota+1$.

Once the starting times and lengths of all intervals ι, $1 \leq \iota \leq m+n-1$, are computed, the starting time of operation O_{ij} of job j on machine M_i is given by s_{i+j-1} and its length is given by $x_{ij} = \ell_{i+j-1}$.

Theorem 53. *Problem $F|plbl(\underline{p})|\sum C_j$ is solvable in $O(n\log n)$ time by specifying formulae which can be computed in $O(1)$ time.*

9.5.3 Type (ii) problem $O|plbl(\underline{p})|\sum C_j$

In this subsection we focus on problem $O|plbl(\underline{p})|\sum C_j$, with restricted pliability of type (ii). We show that for this problem, we can construct a schedule that has the same objective value

as an optimal schedule for problem $P|pmtn|\sum C_j$. Clearly, such a schedule must be optimal also for $O|plbl(\underline{p})|\sum C_j$.

First we introduce the notion of a Latin square, see, e.g., [25]. A Latin square is an $m \times m$ array filled with m different numbers such that each number appears exactly once in every row and every column. In an instance with m machines, we construct the schedule as a sequence of $m \times m$ Latin squares of operations of m jobs, where the rows of the Latin squares represent the different machines and the columns represent the order of operations on the machines.

Let the jobs be numbered in order of non-decreasing processing times. On each machine, the first and the last operation of each Latin square may be longer, while all $m-2$ operations in the middle columns of the Latin square are of minimum size $\underline{p}$. The first Latin square is constructed for jobs $1, 2, \ldots, m$, the second is constructed for jobs $m+1, m+2, \ldots, 2m$, etc. such that the k-th Latin square is constructed for jobs $(k-1)m+1, (k-1)m+2, \ldots, km$. If the number of jobs is not a multiple of the number of machines, then in the last Latin square some jobs are missing.

It is sufficient to consider only two Latin squares, one tying into the other, and construct the schedule alternating between the two chosen Latin squares. Any combination of Latin squares L_1 and L_2 such that the first column of L_1 is equal to the last column of L_2 and the last column of L_1 is equal to the first column of L_2 would work. In what follows we use the two Latin squares of the form

$$L_1 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & \ldots & m-1 & m \\ \hline m & 1 & 2 & \ldots & m-2 & m-1 \\ \hline m-1 & m & 1 & \ldots & m-3 & m-2 \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline 3 & 4 & 5 & \ldots & 1 & 2 \\ \hline 2 & 3 & 4 & \ldots & m & 1 \\ \hline \end{array},$$

and

$$L_2 = \begin{array}{|c|c|c|c|c|c|} \hline m & m-1 & m-2 & \ldots & 2 & 1 \\ \hline m-1 & m-2 & m-3 & \ldots & 1 & m \\ \hline m-2 & m-3 & m-4 & \ldots & m & m-1 \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline 2 & 1 & m & \ldots & 4 & 3 \\ \hline 1 & m & m-1 & \ldots & 3 & 2 \\ \hline \end{array}$$

Note that L_2 uses the same columns as L_1 in reversed order.

When constructing the schedule, we first assign all minimum length operations of jobs $1, 2 \ldots, m$ to the machines in the structure given by L_1, i.e. such that the sequence of operations on machine M_i is equal to the i-th line of L_1. In order to make sure the full processing time of each job is scheduled, we lengthen the last operation of each job (the operations represented

by the last column of the Latin square). Thus, jobs $1, \ldots, m$ have $m-1$ minimum operations in the beginning of the schedule, and then one operation of length $p_j - (m-1)\underline{p}$ on the last machine on which they appear, see Fig. 9.10.

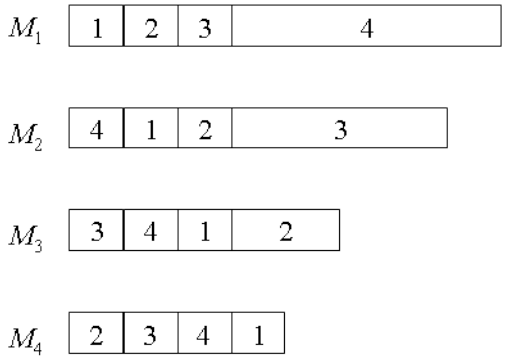


Figure 9.10: The start of a schedule for problem $O|plbl(\underline{p})|\sum C_j$

Now we schedule all minimum length operations of the next m jobs, jobs $m+1, m+2, \ldots, 2m$, in the structure given by L_2, starting at time p_m, when the last operation of the previous Latin square finishes. Note that this means that the last operation of job j, $1 \leq j \leq m$, is scheduled on the same machine as the first operation of job $m+j$, see Fig. 9.11.

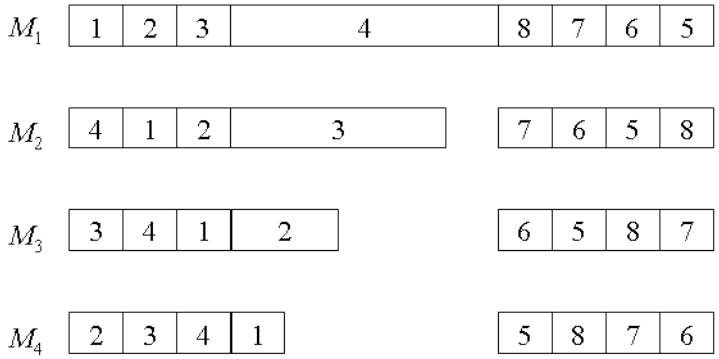


Figure 9.11: Adding the second Latin square of operations

Again, as the second step we lengthen the operations of the Latin square in order to make sure all processing time is scheduled. This time, where possible, we first lengthen the operations in the first column (the first operations of jobs $m+1$, $m+2$, ..., $2m$), such that there are no idle times left on any machine. This is possible, because the jobs are numbered according to non-decreasing processing times, and therefore $p_j \geq p_m$ for all jobs $j > m$. In a second step, we lengthen the operations in the last column (as we did for the first Latin square) to schedule all remaining processing time, see Fig. 9.12.

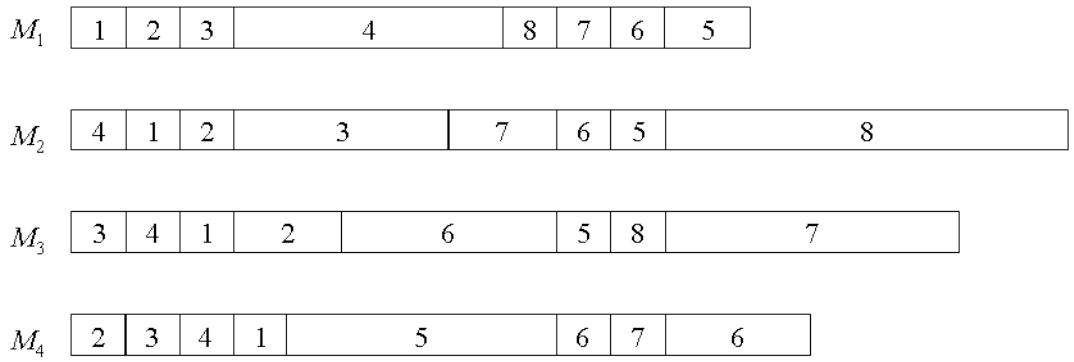


Figure 9.12: A complete schedule for problem $O|plbl(\underline{p})|\sum C_j$ with 8 jobs

If we continue to construct the schedule in this manner, it is easy to see that we end up with completion times $C_1 = p_1$, $C_2 = p_2$, ..., $C_m = p_m$ and $C_j = p_j + C_{j-m}$ for $j > m$. Again we meet the lower bound for $P|pmtn|\sum C_j$ and thus the constructed schedule is optimal.

The schedule can be constructed in such a way that only two different, pre-set Latin squares are needed, one tying into the other as described above. For each job, only the lengths of the first and the last operations need to be computed, all other operations are of minimum length. For jobs j, $j \leq m$, the first operation is of minimum length $\underline{p}$ and the last operation is of length $p_j - (m-1)\underline{p}$. If instead $(k-1)m < j \leq km$ for some $k \geq 2$ and job j is processed within the k-th Latin square, then the first operation of job j has length $C_{(k-1)m} - C_{j-m}$ and the last operation of job j has length $p_j - (C_{(k-1)m} - C_{j-m} + (m-2)\underline{p})$.

The completion times of all job can be computed recursively in $O(n)$ using the formula for the corresponding parallel machine problem, after sorting the jobs in SPT order. Once the completion times of all jobs are known, computing the lengths of the first and the last operations of all jobs with the formulas above again takes $O(n)$ time, $O(1)$ time for each individual operation.

The schedule as a whole can be computed in $O(n \log n + mn)$ time, including the sorting of the jobs, and the computation and output of starting and completion times for mn non-zero operations.

Theorem 54. *Problem $O|plbl(\underline{p})|\sum C_j$ is solvable in $O(n \log n + mn)$ time.*

9.5.4 Other problems with min-sum criteria

In this section we give a brief overview of other traditional min-sum criteria, namely weighted sum of completion times, number of late jobs, weighted number of late jobs, total tardiness and weighted total tardiness. Recall that by Theorem 35 for any such objective f problems $O|plbl|f$ and $F|plbl|f$ with the number of machines part of the input are at least ordinarily NP-hard.

Weighted sum of completion times

By Theorem 35 problems $O2|plbl|\sum w_jC_j$ and $F2|plbl|\sum w_jC_j$ are NP-hard in the ordinary sense and problems $O|plbl|\sum w_jC_j$ and $F|plbl|\sum w_jC_j$ are strongly NP-hard.

Problem $Om|plbl|\sum w_jC_j$ is pseudo-polynomially solvable in $O(mn\left(\sum p_j\right)^{m-1})$ time with the known algorithm for $Pm|pmtn|\Sigma w_jC_j$ (see [97]). Using the same algorithm in combination with the results from [123], we can also solve $Fm|plbl|\sum w_jC_j$ in $T(n,m)+O(mn\left(\sum p_j\right)^{m-1})$ time, where $T(n,m)$ is the time it takes to solve the LP from [123], see Section 9.1.1.

It is difficult to generalize these results to the model with a common lower bound of type (ii). An approach using the methodology from Section 9.1.2, where we first split an instance I into an diminished instance I^d and an instance I^e where all processing times are equal to $m\underline{p}$ and then combine two optimal permutation schedules with the same job order, does not work. In Section 9.5.2 we could use such an approach, since the objective value of a permutation schedule for instance I^e was independent of the permutation used. However, this is no longer the case if weights are introduced. Therefore the optimal permutation schedule for instance I^e may not have the same job order as an optimal permutation schedule for the diminished instance I^d, and the two schedules cannot be combined.

For a slightly different reason the approach from Section 9.5.3 also cannot be generalized in an obvious way. The problem here is that the sequence of jobs for an optimal solution of weighted sum of completion times is not necessarily in order of non-decreasing processing times. Thus it can no longer be guaranteed that operations can be lengthened to make sure no idle times appear in the schedule.

Number of late jobs

Problem $P|pmtn|\sum U_j$ is NP-hard in the ordinary sense (see, e.g., [17, 97]) and thus the same is true for problems $O|plbl|\sum U_j$ and $F|plbl|\sum U_j$ by Theorem 35. We do not investigate the possibility of a pseudo-polynomial time algorithm since the existence of such an algorithm for $P|pmtn|\sum U_j$ is still open, to the best of our knowledge, and the answer to this question is beyond the scope of this thesis.

On the other hand problem $Pm|pmtn|\sum U_j$ with a fixed number of machines is polynomially solvable and the algorithm produces an O-type schedule (up to zero-length operations in the beginning of the schedule) in $O(n^{3(m-1)})$ time (see [97]). Hence, $Om|plbl|\sum U_j$ can be solved in $O(n^{3(m-1)})$ time.

In order to solve problem $Fm|plbl|\sum U_j$ we need a slightly different approach than for problem $Fm|plbl|\sum w_jC_j$ in the previous section, since $\sum U_j$ is not continuous, and therefore the LP from [123] cannot be used directly to transform a schedule for problem $Pm|pmtn|\sum U_j$ into an F-type schedule. Recall, however, that the structural result in [123] still shows that an optimal F-type schedule exists for problem $Fm|plbl|\sum U_j$ (see also Theorem 34). Note further that there exists an optimal schedule in which all jobs, which are on time, finish before any late

job finishes (this is also shown in Lemma 55 for the more general model of type (ii)). Thus we can consider the on time jobs separately from the late jobs.

We solve problem $Fm|plbl|\sum U_j$ in two steps. First we use the known algorithm for problem $Pm|pmtn|\sum U_j$ in order to identify the optimal set $N_1 \subseteq \mathcal{J}$ of on time jobs in $O(n^{3(m-1)})$ time [97].

Now consider only the job in N_1. Due to the structural result in [123] there exists an F-type schedule where all jobs in N_1 are on time. Treating the due dates of the on time jobs as deadlines, we can find such an F-type schedule S using the algorithm in [11], see also Section 9.2.2. This is the same as finding a schedule of maximum lateness $L_{\max} \leq 0$ for the sub-instance given by the set of jobs N_1.

We extend S by scheduling all late jobs $j \in \mathcal{J} \setminus N_1$ at the end of S in any order, such that the conditions for feasibility and F-type are fulfilled. In total, problem $Fm|plbl|\sum U_j$ can be solved in $O(n^{3(m-1)} + mn + n\log n) = O(n^{3(m-1)})$ time, assuming $m \geq 2$ and $n \geq m$.

For problem $O|plbl(\underline{p})|\sum U_j$ of type (ii), it is unclear whether the result from the model with unrestricted pliability can be generalized. Indeed, for open shop even the easier problem with the makespan objective is still unsolved, as discussed earlier in Section 9.3.2.

In contrast to that a generalization is again possible for the flow shop version. First we extend the observation we made for the structure of an optimal schedule of problem $Fm|plbl|\sum U_j$ to the model with equal but non-zero lower bounds.

Lemma 55. *For problem $F|plbl(\underline{p})|\sum U_j$ there exists an optimal permutation schedule, in which first all on time jobs are scheduled in non-decreasing order of due-dates, and then all late jobs.*

Proof: An optimal permutation schedule exists by Theorem 37. If it is not of the desired form, we use adjacent jobs swaps (Lemma 36) to reorder the jobs accordingly. We can see that no additional late jobs are created with arguments similar to the arguments used for Lemma 46. ∎

Thus, if the set of on time jobs $N_1 \subseteq \mathcal{J}$ is known, then we can solve problem $Fm|plbl(\underline{p})|L_{\max}$ for the jobs in N_1, obtaining a schedule where all these jobs are on time. Then in a second step we add the late jobs to the end of the schedule, in any order.

In order to show that problem $Fm|plbl(\underline{p})|\sum U_j$ is solvable in $O(n^{3(m-1)})$ time, we prove the following theorem.

Lemma 56. *Let I be an instance of problem $F|plbl(\underline{p})|\sum U_j$ and let I^d be the diminished instance as defined in Section 9.3.3. Then, given a subset $N_1 \subseteq \mathcal{J}$ of the set of jobs, there exists a schedule S for instance I in which all jobs of set N_1 are on time, if and only if there exists a schedule S^d for instance I^d in which all jobs corresponding to the jobs in N_1 are on time.*

Proof: Construct an instance I_{sub} of problem $F|plbl(\underline{p})|L_{\max}$ with job set N_1. Denote the corresponding diminished instance by I^d_{sub}.

Note first that there exists a schedule for instance I in which all jobs in set N_1 are on time if and only if there exists a schedule for instance I_{sub} with $L_{\max} \leq 0$. A similar observation can be made for instances I^d and I^d_{sub}.

Furthermore, since the diminished instance I^d_{sub} of I_{sub} is constructed similarly to the construction in Section 9.3.3, the optimal $L_{\max}$-values for I_{sub} and I^d_{sub} are equal, see (9.11). Thus, a schedule with $L_{\max} \leq 0$ exists for instance I_{sub} if and only if one exists for I^d_{sub}.

Therefore, the following four statements are equivalent:

1. there exists a schedule S for instance I in which all jobs in set N_1 are on time,
2. there exists a schedule S_{sub} for instance I_{sub} with $L_{\max} \leq 0$,
3. there exists a schedule S^d_{sub} for instance I^d_{sub} with $L_{\max} \leq 0$,
4. there exists a schedule S^d for instance I^d in which all jobs in set N_1 are on time.

The equivalence of the first and fourth statement proves the lemma. ∎

Using Lemmas 55 and 56 we can solve problem $Fm|plbl(\underline{p})|\sum U_j$ in the following way. Given instance I construct the diminished instance I^d as defined in Section 9.3.3. Use the algorithm for problem $Fm|plbl|\sum U_j$ to solve instance I^d optimally, obtaining a maximum set of on time jobs N_1^d. By Lemma 56, the set N_1 of jobs corresponding to the jobs in N_1^d is a maximum set of on time jobs for instance I.

As the second step, solve problem $Fm|plbl(\underline{p})|L_{\max}$ for the instance with job set N_1, obtaining a schedule with $L_{\max} \leq 0$. Finally, extend the schedule to a schedule for instance I by adding at the end, feasibly, all jobs in set $\mathcal{J} \setminus N_1$.

The first step, solving $Fm|plbl|\sum U_j$ for the diminished instance I^d, takes $O(n^{3(m-1)})$ time. After that, solving problem $Fm|plbl(\underline{p})|L_{\max}$ takes no longer than $O(n \log n + mn)$ time and adding the jobs in set $\mathcal{J} \setminus N_1$ at the end of the schedule takes no longer than $O(mn)$ time. Therefore the total runtime of the algorithm is $O(n^{3(m-1)})$.

Theorem 57. *For all $m \geq 2$ problem $Fm|plbl(\underline{p})|\sum U_j$ is solvable in $O(n^{3(m-1)})$ time.*

Weighted number of late jobs, total tardiness and weighted total tardiness

It is well known that problems $1|pmtn|\sum T_j$ and $1|pmtn|\sum w_jU_j$ are NP-hard in the ordinary sense and that problem $1|pmtn|\sum w_jT_j$ is strongly NP-hard (see, e.g., [17], [97]). Therefore, by Theorem 35, problems $F2|plbl|\sum T_j$, $O2|plbl|\sum T_j$, $F2|plbl|\sum w_jU_j$ and $O2|plbl|\sum w_jU_j$ are NP-hard in the ordinary sense and problems $F2|plbl|\sum w_jT_j$ and $O2|plbl|\sum w_jT_j$ are strongly NP-hard.

It is open whether $P2|pmtn|\sum T_j$ is solvable in pseudo-polynomial time [17, 97], so we do not attempt to answer the question for strong NP-hardness for problems $O2|plbl|\sum T_j$ and $F2|plbl|\sum T_j$ here. On the other hand, problem $Pm|pmtn|\sum w_jU_j$ is solvable in pseudo-polynomial time, more precisely in $O\left(n^{3m-5}\left(\sum w_i\right)^2\right)$ time for $m \geq 3$ and $O\left(n^2\left(\sum w_i\right)\right)$ for

$m = 2$. Again, the algorithms produce an O-type solution up to zero-length operations, see [97, 98]. Thus problem $Om|plbl|\sum w_jU_j$ is pseudo-polynomially solvable with the same time complexity.

For problems $Fm|plbl|\sum w_jU_j$ and $Fm|plbl(\underline{p})|\sum w_jU_j$ we can use the same idea as in the previous section: first solve the related parallel machine problem to find out the optimal set N_1 of on time jobs (after creating the diminished instance in the case of problem $Fm|plbl(\underline{p})|\sum w_jU_j$), then solve problem $Fm|plbl(\underline{p})|L_{\max}$ or $Fm|plbl(\underline{p})|L_{\max}$ respectively to obtain a schedule in which all jobs of set N_1 are on time. Consequently, problems $Fm|plbl|\sum w_jU_j$ and $Fm|plbl(\underline{p})|\sum w_jU_j$ are pseudo-polynomially solvable with the same time complexity as the associated preemptive parallel machine problem.

9.6 Proof of Lemma 36

Consider problem $F|plbl(\underline{p})|f$ and two permutation schedules S and S', as described in the formulation of Lemma 36. These two schedules differ in allocation of jobs u and v, while the remaining jobs are kept unchanged.

Given schedule S, we perform the following preprocessing. Introduce time windows $[\ell_i, r_i]$ on every machine M_i, $1 \le i \le m$, available for processing $\{u, v\}$, where ℓ_i is the completion time of the last operation on machine M_i that precedes job u in S and r_i corresponds to the starting time of the first operation on machine M_i that follows job v in S. Having fixed windows $[\ell_i, r_i]$, we exclude from consideration jobs $\mathcal{J}\backslash\{u, v\}$. An example of schedule S is presented in Fig. 9.13, where we mark compulsory parts of jobs u, v, each of length $\underline{p}$, by dotted boxes.

Time windows $[\ell_i, r_i]$ can be narrowed down to $[\ell'_i, r'_i] \subseteq [\ell_i, r_i]$ excluding subintervals which remain idle in any feasible schedule:

$$\begin{array}{ll} \ell'_1 = \ell_1, & \ell'_i = \max\left\{\ell_i, \ell'_{i-1} + \underline{p}\right\}, \quad i = 2, 3, \ldots, m, \\ r'_m = r_m, & r'_i = \min\left\{r_i, r'_{i+1} - \underline{p}\right\}, \quad i = m-1, m-2, \ldots, 1. \end{array} \tag{9.17}$$

In Fig. 9.13, the adjustment of r_1 excludes from consideration idle interval $[r'_1, r_1]$, which cannot be used on machine M_1, while the adjustment of ℓ_2 excludes idle interval $[\ell_2, \ell'_2]$, which cannot be used on machine M_2, whichever job order is fixed, u preceding v on all machines or another way around.

Additionally, schedule S can be decomposed into subschedules defined over machine sets $\{M_1, \ldots, M_{i-1}\}$ and $\{M_i, \ldots, M_m\}$, if

$$\ell'_i + \underline{p} \ge r'_{i-1} \tag{9.18}$$

for some $2 \le i \le m$; see Fig. 9.13 with $i = 4$ and machine sets $\{M_1, M_2, M_3\}$ and $\{M_4, M_5\}$. Here condition (9.18) ensures that for the fixed job order, with u preceding v, allocating the compulsory part of job v in $\left[r'_{i-1} - \underline{p}, r'_{i-1}\right]$ on machine M_{i-1} and allocating the compulsory

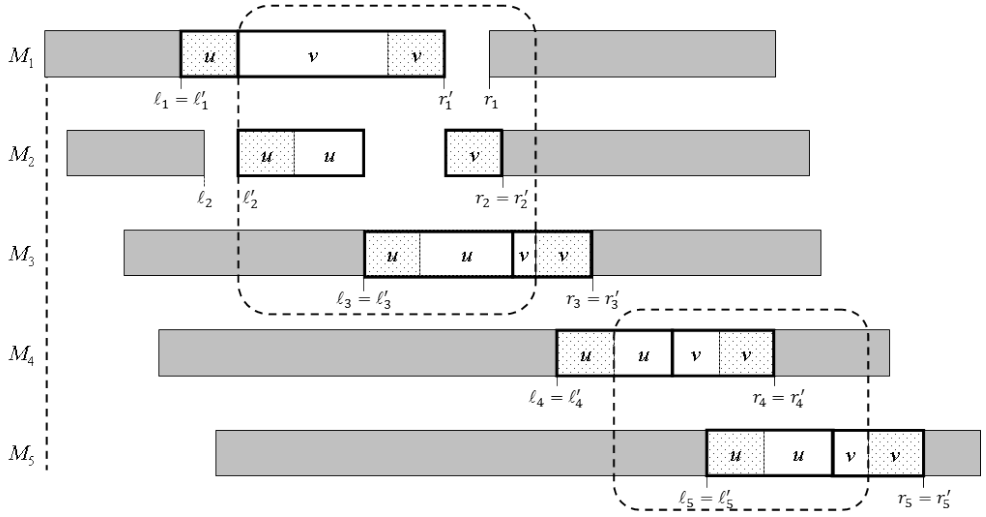


Figure 9.13: Pre-processing:
- introducing intervals $[\ell'_i, r'_i] \subseteq [\ell_i, r_i]$ for each machine M_i, $1 \le i \le m$;
- decomposing schedule S into two subschedules with machines $\{M_1, M_2, M_3\}$ and $\{M_4, M_5\}$ under the condition $\ell'_4 + \underline{p} \ge r'_3$

part of job u in $\left[\ell'_i, \ell'_i + \underline{p}\right]$ on machine M_i leaves the remaining intervals on $\{M_1, \ldots, M_{i-1}\}$ non-overlapping with those on machines $\{M_i, \ldots, M_m\}$. The same is true for the opposite job order, with v preceding u. Thus it is appropriate to prove Lemma 36 for two subsets of machines considered separately. To define the subinstances based on schedule S, the processing time p_u of job u is split into two values: the total processing time of job u in schedule S on the first subset of machines $\{M_1, \ldots, M_{i-1}\}$ and that on the second subset of machines $\{M_i, \ldots, M_m\}$. The processing time p_v of job v is split accordingly.

In what follows we assume that condition (9.18) does not hold for any $2 \le i \le m$, i.e.

$$\ell'_k + \underline{p} < r'_{k-1} \text{ for any } 2 \le k \le m. \tag{9.19}$$

Additionally, due to the definition (9.17), the adjusted time windows $[\ell'_i, r'_i]$ satisfy

$$\begin{aligned} \ell'_{i-1} + \underline{p} &\le \ell'_i, \\ r'_{i-1} + \underline{p} &\le r'_i, \end{aligned} \tag{9.20}$$

for any machine M_i, $2 \le i \le m$. To prove Lemma 36, we first formulate the necessary and sufficient conditions for the existence of a feasible schedule where the sequence of jobs u and v can be arbitrary, but it is the same for all machines. This result immediately justifies Lemma 36: since the initial permutation schedule S, with u preceding v, is feasible, the necessary and sufficient conditions hold, and if the conditions hold, then jobs u and v can be scheduled feasibly in the corresponding time windows $[\ell'_i, r'_i]$ with job v preceding job u, producing schedule S'.

Lemma 58. *There exists a feasible schedule for scheduling jobs u and v in the same order on machines M_i, $1 \leq k \leq m$, within time windows $[\ell'_i, r'_i]$ satisfying (9.19) and (9.20), if and only the following three conditions hold:*

(A) $r'_i - \ell'_i \geq 2\underline{p}$, $1 \leq i \leq m$;

(B) $p_u + \underline{p} \leq r'_m - \ell'_1$ *and* $p_v + \underline{p} \leq r'_m - \ell'_1$;

(C) $p_u + p_v \leq T_1 + 2T_2$, *where T_1 is the total time within $[\ell'_1, r'_m]$ when exactly one of the machines $M_1, M_2, \ldots, M_m$ is available, and T_2 is the total time within the same interval when at least two machines are available.*

Proof of Lemma 58: Without loss of generality we consider a fixed job order with u preceding v.

"$\Rightarrow$": We show that if one of the conditions is violated, then a feasible schedule cannot be achieved. Indeed, if (A) does not hold for some machine M_i, then the two operations of jobs u and v of the minimum length $\underline{p}$ cannot be scheduled.

If the first inequality from (B) does not hold, then the whole interval $[\ell'_1, r'_m]$ cannot fit the full processing amount p_u of job u, distributed over m machines, together with the minimum amount $\underline{p}$ of job v allocated to M_m. Similarly, if the second inequality from (B) does not hold, then $[\ell'_1, r'_m]$ cannot fit the full processing amount p_v of job v, distributed over m machines, together with the minimum amount $\underline{p}$ of job u allocated to M_1.

Finally, if (C) does not hold, then the relaxed problem, with m identical parallel machines processing two jobs with preemption, does not have a feasible solution.

"$\Leftarrow$": Suppose now that (A), (B), (C), (9.20) and (9.19) hold. We formulate an algorithm that finds a feasible schedule with u preceding v on every machine.

The algorithms starts with splitting $[\ell'_1, r'_m]$ into intervals of two types, I_1 and I_2: within any I_1-interval exactly one of the machines $M_1, M_2, \ldots, M_m$ is available and within any I_2-interval at least two machines are available. Note that parameters T_1 and T_2 that appear in condition (C) of the lemma correspond to the combined lengths of I_1-intervals and I_2-intervals, respectively. Moreover, by (9.19), any time point within $[\ell'_1, r'_m]$ falls into one of these two categories, so that

$$T_1 + T_2 = r'_m - \ell'_1. \tag{9.21}$$

Since the set of available intervals starts with an I_1-type interval of size at least $\underline{p}$ and finishes with an I_1-type interval, also of size at least $\underline{p}$,

$$T_1 \geq 2\underline{p}. \tag{9.22}$$

Perform (temporarily) an additional adjustment of the problem instance, increasing p_u, p_v to the maximum combined value

$$p'_u + p'_v = T_1 + 2T_2, \tag{9.23}$$

and making sure that p'_u and p'_v satisfy the following two additional inequalities:

$$p'_u \geq \underline{p} + T_2, \tag{9.24}$$
$$p'_v \geq \underline{p} + T_2. \tag{9.25}$$

The extra amount of processing $(p'_u + p'_v) - (p_u + p_v)$ will be removed from a feasible schedule after it is constructed. To demonstrate the existence of p'_u, p'_v satisfying (9.23)-(9.25), assume without loss of generality that the initial processing times satisfy

$$p_u \geq p_v \tag{9.26}$$

and consider the following possible set of new processing times:

$$\begin{array}{lll} p'_u = p_u, & p'_v = T_1 + 2T_2 - p_u, & \text{if } p_u \geq \underline{p} + T_2, \\ p'_u = \underline{p} + T_2, & p'_v = T_1 + T_2 - \underline{p}, & \text{if } p_u < \underline{p} + T_2. \end{array}$$

Notice that

- $p'_u \geq p_u$, i.e., the processing time of job u does not decrease;
- condition $p'_v \geq p_v$ holds due to (C) in the first case and due to (9.22) and (9.26) in the second case, i.e., the processing time of job v does not decrease;
- (9.23) and (9.24) are satisfied in both cases;
- the validity of (9.25) in the first case follows from (9.21) combined with the first inequality from (B), and its validity in the second case follows from (9.22).

The algorithm for allocating jobs u and v which satisfy (9.23)-(9.25) starts with intervals of type I_2, making sure that each of the jobs u and v is continuously processed in these intervals. Denote by $p_{ij}(I_2)$ the amount of processing of job $j \in \{u, v\}$ allocated to machine M_i in an I_2-type interval and define:

$$\begin{array}{ll} p_{1u}(I_2) = 0, & p_{1v}(I_2) = r'_1 - \ell'_2, \\ p_{iu}(I_2) = \min\left\{r'_{i-1}, \ell'_{i+1}\right\} - \ell'_i, & p_{iv}(I_2) = r'_i - \max\left\{r'_{i-1}, \ell'_{i+1}\right\}, \quad i = 2, \ldots, m-1, \\ p_{mu}(I_2) = r'_{m-1} - \ell'_m, & p_{mv}(I_2) = 0. \end{array}$$

An illustrative example is presented in Fig. 9.14, where I_1- and I_2-intervals are marked by single-line and double-line rounded frames. It is easy to verify that the following properties hold for the given partial allocation:

- machines M_1 and M_m process only one job, u or v;
- every remaining machine M_i, $2 \leq i \leq m-1$, processes job u first and job v next;

– processing times of all operations assigned to intervals of type I_2, other than $p_{1u}(I_2)$ and $p_{mv}(I_2)$, are no less than $\underline{p}$, due to (9.19) and (9.20);

– the total allocated amount for each job, u and v, is T_2, so that each job is processed at any time within all intervals of type I_2;

– the time intervals with u-operations do not overlap; the same is true for v-operations.

Note that if in an interval of type I_2 more than two machines are available, then two of them receive jobs u and v for processing, while the remaining available machines are left idle; consider, e.g., an idle time interval $[\ell'_3, r'_1]$ on machine M_2 in Fig. 9.14.

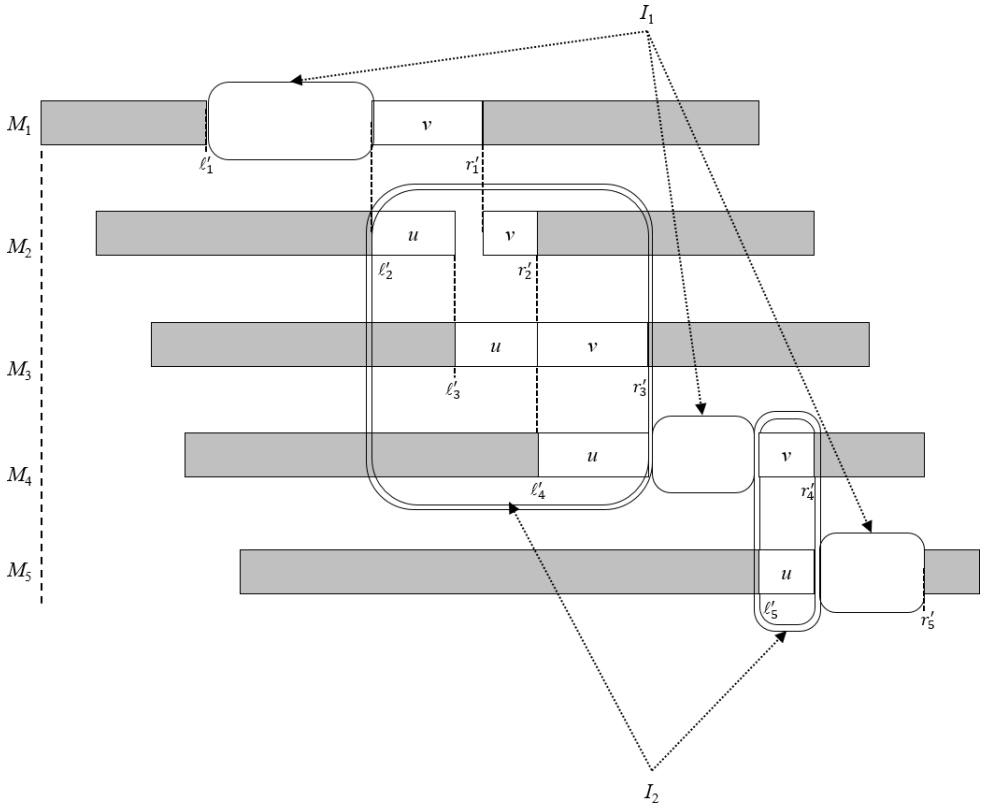


Figure 9.14: Allocation of jobs u and v into intervals I_2

In the next step we proceed with intervals I_1. First allocate the minimum quantity $\underline{p}$ to machines M_1 and M_m:

$$p_{1u}(I_1) = \underline{p}, \quad p_{mv}(I_1) = \underline{p},$$

which is achievable since (9.20) holds. Thus each of the jobs, u and v, gets $\underline{p} + T_2$ units of processing. The remaining amounts for each job are allocated arbitrarily to the free subintervals of I_1, possibly lengthening some of the previously allocated operations.

In the resulting schedule all intervals of type I_1 and I_2 are used as much as possible, with exactly one job processed at any time within I_1 and two jobs processed at any time within I_2. Thus the total processing time of jobs u and v is $T_1 + 2T_2$ and both jobs are scheduled in full.

If processing times have been extended to achieve (9.23), then the constructed schedule should be modified by reducing the operations with lengths larger than $\underline{p}$, without violating this lower bound. Note that in a feasible instance for problem $F|plbl(\underline{p})|f$, each job $j \in N$ satisfies $p_j \geq m\underline{p}$.

The described algorithm constructs a feasible schedule with u preceding v, if conditions (A)-(C) and (9.20)-(9.19) hold. Since these conditions are symmetric with respect to p_u and p_v, the proposed algorithm can be used to construct a feasible schedule with v preceding u, as stated in Lemma 36. ∎

Chapter 10

Conclusions and further research

The results we obtained for flow shop and open shop scheduling with pliable jobs are summarized in Tables 10.1-10.2. For comparison, we provide the related results for traditional models, with and without preemption (for references see [17] and/or Chapter 2.3 of this thesis). We do not perform the comparison with the lot streaming model, as that model is less related to the model with pliable jobs. Note that the complexity result for problem $F|plbl(\underline{p})|C_{\max}$ is given under the assumption that the trivial part of the schedule, which consists only of minimum length operations, does not need to be computed.

Problem:	trad. model, no pmtn	trad. model, pmtn	pliability model		
			plbl (i)	$plbl(\underline{p})$ (ii)	$plbl(\underline{p}_{ij}, \overline{p}_{ij})$ (iii)
$O2\|\|C_{\max}$	$O(n)$	$O(n)$	$O(n)$ Th. 40,49	$O(n)$ Th. 49	$O(n)$ Th. 49
$F2\|\|C_{\max}$	$O(n \log n)$	$O(n \log n)$	$O(n)$ Th. 40	$O(n)$ Th. 43	NP-h. Th. 48
$Om\|\|C_{\max}$	NP-h. $(m \geq 3)$,	$O(n^2)$	$O(n)$ Th. 40	open	NP-h.$(m \geq 3)$, Sect. 9.1.3
$Fm\|\|C_{\max}$	sNP-h. $(m \geq 3)$	sNP-h. $(m \geq 3)$	$O(n)$ Th. 40	$O(n)$ Th. 43	sNP-h.$(m \geq 3)$ Sect. 9.1.3
$O\|\|C_{\max}$	sNP-h.	$O(n^2m^2)$	$O(n)$ Th. 40	open	sNP-h., Sect. 9.1.3
$F\|\|C_{\max}$	sNP-h.	sNP-h.	$O(n)$ Th. 40	$O(n)$ Th. 43	sNP-h. Sect. 9.1.3
$O\|\|L_{\max}$	sNP-h. $(m \geq 2)$	LP	$O(n \log n)$ Th. 42	open	sNP-h.$(m \geq 2)$ Sect. 9.1.3
$F\|\|L_{\max}$	" " "	sNP-h. $(m \geq 2)$	$O(n \log n + mn)$ Th. 42	$O(n \log n + mn)$ Th. 47	" " "

Table 10.1: Open shop and flow shop problems with pliable jobs and minmax objectives

Problem:	trad. model, no pmtn	trad. model, pmtn	pliability model		
			plbl (i)	*plbl*($\underline{p}$) (ii)	*plbl*($\underline{p}_{ij}, \overline{p}_{ij}$) (iii)
$O\|\|\Sigma C_j$	sNP-h. ($m \geq 2$)	NP-h. ($m \geq 2$)	$O(n \log n)$ Th. 51	$O(n \log n + nm)$ Th. 54	sNP-h.($m \geq 2$) Sect. 9.1.3
$F\|\|\Sigma C_j$	" " "	sNP-h. ($m \geq 2$)	$O(n \log n + nm)$ Th. 50	$O(n \log n + nm)$ Th. 52	" " "
$Om\|\|\Sigma w_j C_j$	sNP-h. ($m \geq 2$)	sNP-h. ($m \geq 2$)	both NP-h. (pseudo-	both NP-h. (open whether	sNP-h.($m \geq 2$) Sect. 9.1.3
$Fm\|\|\Sigma w_j C_j$	" " "	" " "	poly. solv.) Sect. 9.5.4	strongly) Sect. 9.5.4	" " "
$O\|\|\Sigma w_j C_j$	sNP-h.	sNP-h.	sNP-h. Th. 35	sNP-h. Th. 39	sNP-h. Sect. 9.1.3
$F\|\|\Sigma w_j C_j$	" " "	" " "	" " "	" " "	" " "
$Om\|\|\Sigma U_j$	sNP-h. ($m \geq 2$)	NP-h. ($m \geq 2$)	$O\left(n^{3(m-1)}\right)$ Sect. 9.5.4	open	sNP-h.($m \geq 2$) Sect. 9.1.3
$Fm\|\|\Sigma U_j$	" " "	sNP-h ($m \geq 2$)	$O\left(n^{3(m-1)}\right)$ Sect. 9.5.4	$O\left(n^{3(m-1)}\right)$ Th. 57	" " "
$O\|\|\Sigma U_j$	sNP-h.	NP-h.	NP-h. Th. 35	NP-h. Th. 39	sNP-h. Sect. 9.1.3
$F\|\|\Sigma U_j$	" " "	sNP-h.	" " "	" " "	" " "

Table 10.2: Open shop and flow shop problems with pliable jobs and minsum objectives

10.1 General observations

As a rule, the problems with unrestricted pliability of type (i) appear to be no harder than their counterparts for the traditional model, with or without preemption, and are often solvable by faster algorithms than the traditional ones. This is due to the strong relation to the parallel machine with preemption discussed in Section 9.1.1.

Contrarily, due to the reductions in Theorem 39, the problems with restricted pliability of type (iii) are no easier than their traditional counterparts. Additionally, the type (iii) problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is NP-hard while its classical counterpart $F2||C_{\max}$ is solvable in $O(n \log n)$ time (see [79]).

Comparing open shop and flow shop models, we observe that for pliability of type (i) open shop models have the same time complexity as flow shop models for all traditional objective functions f , other than $L_{\max}$ and $\sum C_j$, where they are slightly easier. On the other hand, surprisingly, for pliability of type (ii) open shop models cause considerable problems and we were forced to leave them open for the problems $O|plbl(\underline{p})|C_{\max}$, $O|plbl(\underline{p})|L_{\max}$ and $Om|plbl(\underline{p})|\sum U_j$. Note that corresponding flow shop problems are polynomially solvable. Finally, for pliability of type (iii), open shop models and flow shop models have mostly the same complexity status. The only major deviation from this rule is that problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is NP-hard while problem $O2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ is solvable in linear time.

10.2 Pliability of type (i) and (ii) with $n < m$

We performed the complexity analysis for the case $n \geq m$. If the number of machines m is larger than the number of jobs n, $n < m$, then many of the algorithms for the models with pliability of type (i) and (ii) can be simplified. For the open shop models of type (i) we can schedule the full processing time of each job on a different machine, with zero-length operations on all other machines in the beginning of the schedule. Since in the described schedule the completion time of each job is equal to its processing time, such a schedule is optimal for every regular objective function.

For the flow shop models of type (i) we can use the same idea, only that we have to schedule jobs such that the shortest job is fully processed by the last machine, the second shortest by the second to last machine and so on. Zero-length operations can be added in such a way that the schedule is of permutation type, with jobs in SPT order on each machine. Job j has $m-j$ zero-length operations on machines $M_1, M_2, \ldots, M_{m-j}$, followed by an operation of length p_j on machine M_{m-j+1} and finally $j-1$ zero-length operations on machines $M_{m-j+1}, M_{m-j+2}, \ldots, M_m$, finishing at time p_j on machine M_m. Again, in the described schedule the completion time of each job is equal to its processing time.

For models with pliability of type (ii) we adjust the above schedules to respect the requirement that all operations have to be at least of length $\underline{p}$. For the open shop schedule, we schedule $m-1$ operations of length $\underline{p}$ for each job, then schedule the remaining processing time fully on one machine. Since $m > n$ and therefore $m - 1 \geq n$, the first part of the schedule, with the minimum length operations, takes $(m-1)\underline{p}$ time. It can be thought of as $m-1$ synchronous cycles, each of length $\underline{p}$ (see Chapter 3). Each job is scheduled in every cycle, but machines may be idle in one or several cycles. After that, a job j has $p_j - (m-1)\underline{p}$ processing time left, which is all scheduled on the one machine that it has not yet visited. Consequently, each job j is processed continuously from time 0, finishing at time p_j. As before, the described schedule is optimal for every regular objective.

Lastly, for flow shop models of type (ii), the situation is slightly more complicated. Note first that Theorem 37 is independent of the relation between n and m and thus optimal permutation schedules exist. Assume a permutation of the jobs is given and jobs are numbered in order of that permutation. We construct a permutation schedule with jobs in the given permutation on all machines by scheduling $m-1$ minimum length operations of the first job, $m-2$ of the second, and so on, similar to the beginning part of the schedule in Fig. 9.2. Job j has $m-j$ minimum length operations scheduled in that part. Then, in the middle part of the schedule each job has one operation of length $p_j - (m-1)\underline{p}$ scheduled on machine M_{m-j+1}. Then, the remaining minimum length operations of each job are scheduled on the downstream machines, starting as early as possible without violating the given permutation or any of the feasibility constraints.

In the constructed schedule, job 1 finishes at time p_1, job 2 at time $\max\{p_1 + \underline{p}, p_2 + \underline{p}\}$, job

3 at time $\max\{p_1 + 2\underline{p}, p_2 + 2\underline{p}, p_3 + 2\underline{p}\}$, and in general job j, $1 \leq j \leq n$, at time

$$C_j = \max\{p_u + (j-1)\underline{p} | u \leq j\}. \tag{10.1}$$

Clearly, if the permutation of jobs on each machine is equal to the given permutation (the numbering of jobs), then no job can finish earlier and the splitting of the jobs is optimal for the given permutation and any regular objective.

Thus, the scheduling problem reduces to finding an optimal permutation of the jobs. For the makespan objective we can choose any permutation, as the completion time of the last job is always $\max\{p_j | j \in \mathcal{J}\} + (n-1)\underline{p}$. For the total sum of completion times, jobs should be scheduled in SPT order, so that each job j finishes at time $p_j + (j-1)\underline{p}$. For the maximum lateness objective the jobs should be sequenced in EDD order, by the same arguments as in Lemma 46. The optimal permutation to minimize the number of late jobs can be found using the same method as for the single machine problem [17, 115]. Starting with the jobs numbered in EDD order, if job j_0 is the first late job, move the job $j \leq j_0$ with the largest processing time to the end of the schedule (as a late job) and renumber jobs accordingly. Proceed in this manner until all jobs that were not moved are on time.

For the other traditional scheduling objectives, finding an optimal permutation appears to be harder. In general, the problem of finding an optimal permutation can be treated as a sequencing problem with completion times given by (10.1). Note that the completion times given by (10.1) are normally smaller than they would be for a single machine sequencing problem. However, it is unclear whether this makes it any easier or harder to obtain an optimal sequence.

To conclude this section, if $n < m$, we are still able solve all problems that we can solve for the case $n \geq m$. Moreover, the structure of an optimal solution becomes much simpler in the case where $n < m$, as usually only one machine processes more than the minimum amount of processing time. Open shop problems of type (i) can be solved in $O(n)$ time for any traditional objective function, and flow shop problems of type (i) in $O(n \log n)$ time ($O(n)$ time for the makespan objective).

For models of type (ii), if the trivial parts of the schedules with only minimum length operations, are obtained by some form of pre-processing (e.g. pre-set templates), then the open shop problem with any traditional objective function can again be solved in $O(n)$ time. Furthermore note that in the usual case, $n \geq m$, we were unable to fully solve the open shop problem with pliability of type (ii), even for the makespan objective. The flow shop model of type (ii) can be solved in $O(n)$ time for the makespan objective, and $O(n \log n)$ time for objectives $L_{\max}$, $\sum C_j$ and $\sum U_j$. Thus all problems solvable for the usual case, $n \geq m$, are also solvable for the case $n \leq m$. Due to the very easy structure of an optimal schedule, it also appears to be possible to specify the schedules in terms of formulae, similar to the approach in Sections 9.3.1, 9.5.1 and 9.5.2.

10.3 Further research

There are many possible directions for future research in pliability, in addition to the open problems already discussed in this and the previous chapters. The most obvious direction is to apply pliability to job shop and mixed shop scheduling. Mixed shop is a generalization of both open and job shop. Practical motivation for this comes again from flexibility in manufacturing, as for the open shop and flow shop models. It would be interesting to see how the results for pliability of type (i) and (ii) generalize if jobs have several operations on each machine.

Since flow shop with pliability of type (iii) is at least NP-hard in the ordinary sense for the makespan objective and in the strong sense for all other traditional objectives, even for two machines, it would be also be useful to study heuristics. If we restrict the search space to permutation schedules (which are not necessarily optimal for the models of type (iii)), then for a given permutation the LP stated at the end of Section 9.4.1 can be used to find an optimal job splitting. A heuristic could search for a good permutation, repeatedly applying the LP.

There are several extensions of the models with pliability that are worth studying further. First of all, the natural next step from the model of type (ii) would be to investigate models of pliability with only job dependent or only machine dependent lower bounds, i.e. $\underline{p}_{ij} = \underline{p}_j$ or $\underline{p}_{ij} = \underline{p}_i$. In the case of job dependent lower bounds it is easy to see that the NP-hardness result from Theorem 48 still holds. Indeed, in the reduction we used to prove that theorem, the lower bounds are only dependent on the jobs (that is 0 for jobs $1, 2, \ldots n$ and E for job $n+1$), while upper bounds are assumed to be equal to p_j for each job j. Therefore problem $F2|plbl(\underline{p}_{ij} = \underline{p}_j)|C_{\max}$ is at least NP-hard in the ordinary sense.

For the model with machine dependent lower bounds we appear to have a better chance for positive results. While we do not go into detail here, initial study suggests that problem $F2|plbl(\underline{p}_{ij} = \underline{p}_i)|C_{\max}$ is solvable in polynomial time, by generalizing the two machine version of the algorithm for the problem $F2|plbl(\underline{p})|C_{\max}$ with pliability of type (ii). As opposed to the model of type (ii), it seems that the permutation of jobs in an optimal schedule is no longer arbitrary. Problems with other objectives or larger number of machines might be polynomially solvable as well, but this is left for future study. A promising first step might be to attempt a generalization of Theorem 37. There seems to be no obvious reason why the proofs used in Sections 9.1.2 and 9.6 should not work for the model with machine dependent lower bounds.

Another type of pliability that seems interesting from a practical point of view can be defined as follows. Suppose "ideal" operation lengths p_{ij} are given for each operation O_{ij}, such that $p_j = \sum_{i=1}^{m} p_{ij}$. Given a parameter Δ we define the upper and lower bounds on the operation lengths for the model with pliable jobs as

$$\begin{aligned} \underline{p}_{ij} &= p_{ij} - \Delta \\ \text{and} \quad \overline{p}_{ij} &= p_{ij} + \Delta \end{aligned} \tag{10.2}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Similar to pliability of type (iii), all NP-hardness results from traditional open and flow shop problems carry over, since $\Delta = 0$ models flow shop and open shop without pliability. However, to prove the NP-hardness of problem $F2|plbl(\underline{p}_{ij}, \overline{p}_{ij})|C_{\max}$ we constructed an instance where one job has fixed processing time on both machines (see the proof of Theorem 48). Since for restricted pliability of the type (10.2) either $\Delta = 0$ and all jobs have fixed processing times on both machines or $\Delta > 0$ and no job has fixed processing time on any machine, this proof is no longer applicable. In the former case the problem becomes the traditional flow shop problem $F2||C_{\max}$ and is solvable in $O(n \log n)$ time by Johnson's algorithm [79]. It may well be that for restricted pliability of this type flow shop with two machines and the makespan objective is polynomially solvable for any $\Delta \geq 0$. However, at the time this thesis is written the problem is still open.

Finally, interesting extensions of models with pliability arise when pliability is not (or not only) restricted by lower and upper bounds, but by an additional cost of some kind. This cost may come in form of resources, e.g. additional power needed on the machine that takes over more than just its own workload. For an easy example of such a model consider the model of pliability discussed in the previous paragraph, where we can make a payment to increase parameter Δ, i.e. make our system more flexible. In other scenarios it might happen that performing a part of the operation on a machine not specialized for it takes more time. In that case the processing time "stretches" by some factor when we move the operation to the new machine. For many of these models with some kind of cost, we can adjust the NP-hardness proofs from the last chapter to show that even with no additional "hard" upper and lower bounds for operation lengths given, they are at least as hard as the models with restricted pliability of type (iii). Still, these models appear to be of major interest for applications and in our opinion would be worth studying further.

Part III

Resiliency

Chapter 11

Definitions, notation and related work

In this part of the thesis we propose a new approach for finding solutions to optimization problems resilient to parameter changes. Introducing the resiliency concept for a general combinatorial optimization problem, we consider the area of scheduling and the famous assignment problem as examples to illustrate the new methodology, its applicability to specific domains and implications. Unless otherwise stated, throughout this part we assume that we deal with minimization problems. However, similar considerations can be made for maximization problems.

As opposed to the previous two parts, the research presented here is not a closed piece of work. Instead, it is a snap shot of work still ongoing. The main goal is to make the case that resiliency is a concept worth further study.

11.1 Introduction

Currently the main notions used when dealing with uncertain parameters are solution stability, sensitivity and robustness. While these concepts provide a solid theoretical justification for decision making under uncertainty, they mainly focus on the closeness to an optimal solution and their applicability is often limited to a relatively small class of problems. This is discussed in more detail in Section 11.2

A natural approach to dealing with uncertainty is to produce a promising solution in advance based on parameter estimates in order to make necessary preparations. In scheduling, for example, it may be beneficial to prepare the system for job processing, to install required tools and to make the set-ups. The preferred solution should keep its quality even if the actual values of parameters differ from the original estimates. Investigating the change tolerance of solutions is also important for scenarios where a large set of similar problems needs to be solved. In that

case it may be possible to solve only one of the problems and then reuse the solution, if the solution is tolerant enough to changes in the input data (see, e.g., [141]).

Based on these considerations, we introduce the concept of solution resilience. Initially we assume that the scale of uncertainty is similar for all parameters; later on we generalize the concept to handle the case when some parameters can be more or less uncertain than others.

Consider a combinatorial optimization problem of finding an optimal solution S^* in the set of feasible solutions $\mathcal{S}$ that minimizes a given objective function $f(S, \eta)$, where η is the vector of problem parameters:

$$f(S^*, \eta) = \min \{ f(S, \eta) \,|S \in \mathcal{S} \}. \tag{11.1}$$

Let $f\left(S^0, \eta\right)$ denote the objective value for some fixed solution S^0 and fixed parameters $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$. If under some scenario the actual values of the parameters are $\eta' = \eta + \Delta$, where $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ is the deviation vector, the solution may be still acceptable if the objective value does not exceed the desirable upper bound: $f\left(S^0, \eta + \Delta\right) \leq B$.

Definition 59. Given fixed values η for problem parameters and an upper bound B on the value of the objective function, a solution S^0 is *B-feasible* if

$$f\left(S^0, \eta\right) \leq B.$$

If vector η represents the estimates of problem parameters, then the *B-bounded region of solution* S^0 is the set of parameters $\eta + \Delta$ such that solution S^0 is *B-feasible*:

$$f\left(S^0, \eta + \Delta\right) \leq B.$$

Notice that the components of the deviation vector $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ are not necessarily non-negative.

For different solutions, their B-bounded regions may differ substantially, with big variation for some parameters and limited variation for others. Under an assumption that the scale of uncertainty is similar for all parameters, a fair measure for comparison is the size $\|\Delta\|$ of the deviation vector measured in terms of some norm ℓ_q, $q \geq 1$:

$$\begin{aligned} \|\Delta\|_q &= \left(\sum_{j=1}^{n} |\Delta_j|^q \right)^{1/q}, \\ \|\Delta\|_\infty &= \max_{1 \leq j \leq n} \{ |\Delta_j| \}. \end{aligned}$$

We also use the notation $\|\cdot\|$ to mean an arbitrary (but fixed) ℓ_q-norm. Then the merit of solution S^0 can be measured as the maximum radius ρ that guarantees the B-feasibility for all possible parameter changes with $\|\Delta\| \leq \rho$.

Definition 60. A *ball* with center η and radius ρ is the set of values $\eta + \Delta$ such that

$$\|\Delta\| \leq \rho.$$

The *resiliency ball* of a B-feasible solution S^0 is a ball of the largest radius $\rho(S^0, \eta, B)$ within the B-bounded region of S^0:

$$\rho(S^0, \eta, B) = \max\left\{\xi | f\left(S^0, \eta + \Delta\right) \leq B \text{ for all } \|\Delta\| \leq \xi\right\}. \tag{11.2}$$

The radius $\rho(S^0, \eta, B)$ is called the resiliency radius of S^0. If S^0 is not B-feasible, then we define $\rho(S^0, \eta, B) = -1$.

We also define the notion of a worst-case deviation for a given B-feasible solution S^0.

Definition 61. Given a B-feasible solution S^0, deviation Δ is called a *worst-case deviation* for S^0 if $\|\Delta\| = \rho(S^0, \eta, B)$ and $f(S^0, \eta + \Delta) = B$.

Note that for different norms ℓ_q the shape of a ball varies and that the resiliency radius and the worst-case deviation of a B-feasible solution is often dependent on the chosen norm.

To illustrate Definitions 59 and 60, consider an instance of problem $1||\sum C_j$. The job set consists of two jobs with processing time estimates $p_1 = 3$, $p_2 = 4$, and the upper bound on the objective value is $B = 14$. In solution S^0 job 1 is processed first followed by job 2. Fig. 11.1 illustrates the B-bounded region, represented as a triangle, and three resiliency balls defined for metrics ℓ_1, ℓ_2 and ℓ_∞. The deviations marked as red dots are worst-case deviations for the respective norms. Note that the formulae for calculating the resiliency radii and worst-case deviations are presented in Section 12.3.1.

In the presence of uncertainty, when parameter values may differ from estimates η, it is reasonable to give preference to the most change-tolerant solution among those which are B-feasible. The B-feasible solution S^* with the largest resiliency radius guarantees that for the largest size deviations solution S^* remains B-feasible.

Definition 62. Within the set of B-feasible solutions, S^* is the *most resilient* one if

$$\rho(S^*, \eta, B) = \max_S \left\{\rho\left(S, \eta, B\right)\right\}.$$

The radius $\rho_{\max} = \rho(S^*, \eta, B)$ of the solution S^* is called an *optimum radius* for the given upper bound B.

The problem of finding the most resilient solution amongst the solutions of a given combinatorial optimization problem $\mathcal{P}$, given an upper bound B and an estimated vector of input parameters η, is called the resiliency version of problem $\mathcal{P}$.

Above we defined solution resilience under the assumption that the scale of uncertainty is similar for all input parameters. At the end of this introduction we define a slightly more

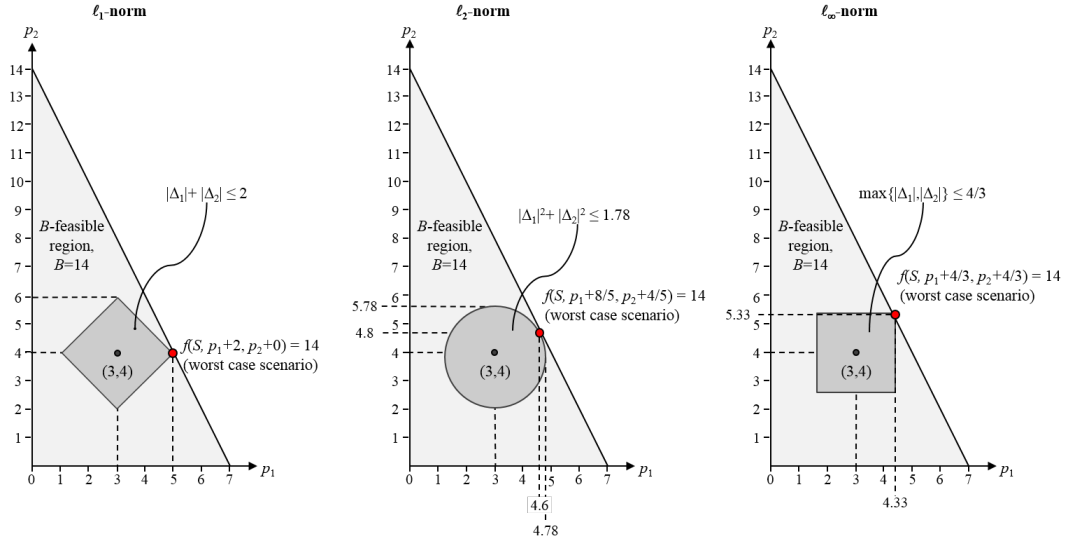


Figure 11.1: The B-feasible region and resiliency balls for an instance of problem $1||\sum C_j$ with $p_1 = 3$, $p_2 = 4$, $B = 14$, and solution $S^0 = (1, 2)$

general version of resiliency. In this version, different input parameters can have different levels of uncertainty. In addition to the vector of estimated input parameters η we are also given a vector of fluctuation factors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where the fluctuation factor $\alpha_j \in \mathbb{Q}_{\geq 0}$ models the uncertainty of estimate η_j. Given a deviation vector Δ the actual value of the input parameters is

$$\eta_j + \alpha_j \Delta_j, 1 \leq j \leq n, \tag{11.3}$$

rather than $\eta_j + \Delta_j$ as in the version with the same level of uncertainty for all parameters from the beginning.

The generalised version of (11.2) is

$$\rho_\alpha(S^0, \eta, B) = \max\left\{\xi | f\left(S^0, \eta + \alpha\Delta\right) \leq B \text{ for all } \|\Delta\| \leq \xi\right\}, \tag{11.4}$$

where $\alpha\Delta$ is shorthand for $(\alpha_1\Delta_1, \alpha_2\Delta_2, \ldots, \alpha_n\Delta_n)$. Usually there is no confusion about the vector α we work with and in order to reduce notation we drop α from the left-hand side of (11.4). The definition of a worst-case deviation is extended in a similar way.

It is important to note that scaling all fluctuation factors α_j by the same factor ϖ does not change the identity of the most resilient solution. Indeed, $f(S^0, \eta+(\varpi\alpha)\Delta) = f(S^0, \eta+\alpha(\varpi\Delta))$, and therefore scaling all fluctuation factors by a common factor ϖ will in turn change the resiliency radii of all B-feasible solutions S^0 by a factor $\frac{1}{\varpi}$. Thus, as it is easier for algorithmic purposes, when computing most resilient solutions we can always assume that all fluctuation factors are integer.

Observe that if $\alpha_j = 0$ then parameter η_j is not subject to uncertainty and instead it is

fixed. On the other hand, if α_j is very large, then η_j can vary greatly, and even deviation vectors with small lengths can lead to huge changes in the actual input-value, if the deviation is concentrated in the component Δ_j.

Clearly the model without fluctuation factors we introduced first is a special case of the general model, where all fluctuation factors α_j are equal to 1. Another interesting special case of the general model is the model where uncertainty is a binary property, i.e. an input parameter is either certain ($\alpha_j = 0$) or it is uncertain ($\alpha_j = 1$).

In the following chapters we study three specific versions of solution resilience, namely the general version with arbitrary fluctuation factors α_j and the two special cases previously mentioned, one with the same level of uncertainty for all input parameters, $\alpha_j = 1$ for all j, the other with binary uncertainty of the form $\alpha_j \in \{0, 1\}$. Given estimates η_j, fluctuation factors α_j and upper bound B we are interested in two problems which arise from the definitions above:

1. computing the resiliency radius of a given solution S^0 and

2. finding a most resilient solution.

While the first problem is usually easier and needed to solve the second, it is still helpful to discuss them separately.

The norms ℓ_1 and ℓ_∞ are of special interest for this type of research. The norm ℓ_1 measures the sum of deviations of all parameters, while the norm ℓ_∞ measures the maximum deviation of all parameters. Thus, similar to the literature on stability (see, e.g., [28, 52, 140, 142]) we focus our initial investigation on these two norms, though we generalise our results to other ℓ_q-norms whenever this is easily possible.

In the next Section we discuss the work and literature related to resilience. After that, as in the previous parts, we present our results in Chapter 12 and finally conclusions in Chapter 13. The concept of resiliency was derived in collaboration with E. Gurevsky and N. V. Shakhlevich. The results presented in the next chapter and the conclusions drawn in the one thereafter are primarily the work of the author, although in a few cases the initial ideas or improved presentation have been achieved in collaboration, again with E. Gurevsky and N. V. Shakhlevich.

11.2 Related work

As was mentioned before, to the best of our knowledge resiliency has not been studied before, either under that name or another. However, there are many concepts dealing with uncertain data, some of them closely related to resiliency, others less closely. In this review we focus on stability and robustness (in the sense of min-max and min-max regret problems) for combinatorial optimization problems, which appear to be the concepts most closely related to our study. Sensitivity, which is mentioned in the introduction, is closely related to stability and only interesting for our work in so far as many of the results in that area can be viewed as results

for stability with only one uncertain parameter. Greenberg [66] provides a famous annotated bibliography for the study of sensitivity, stability and related issues.

We mostly focus on pointing out similarities and differences in the models, since it appears that results from the literature on stability and robustness are not easily transferable to resiliency.

11.2.1 Stability

There are many different approaches and definitions for stability analysis. Most of the early results are for mathematical programming, especially LP and ILP [66]. Since the 90s (see [140], earlier in Russia) researchers have shown increased interest in applying the concept directly to combinatorial optimization problems, without ILP as a go-between. Some of the definitions used in different areas are more closely related to resiliency, while others do not seem as important for our purposes. This review is focused on a definition of stability very close to that of resiliency and an explanation of the similarities and differences between stability and resiliency. Other variants of stability are introduced in [140]. In order to better be able to compare the notions we use similar notation to the one used in the introduction for resiliency.

In a very broad sense, resiliency can be viewed as a special type of stability. Very generally, if S^0 is a solution to a combinatorial optimization problem $\mathcal{P}$ with input data η, and Pr is a property of solutions of problem $\mathcal{P}$ (for example "optimality"), then S^0 is called stable with regard to property Pr, if there exists an $\rho > 0$ such that S^0 has property Pr for all input data $\eta + \Delta$, with $\|\Delta\| \leq \rho$ for some norm $\|\cdot\|$ [52]. Similar to the resiliency radius, the stability radius of solution S^0 with regard to property Pr is defined as the largest such value ρ.

It is easy to see that resiliency is the same as stability with regard to property B-feasibility. However, we have found no source that considers B-feasibility as the property for which to study stability in previous research. In most resources the investigation instead seems to focus on optimality as the property in question. This is so common, that usually stability is defined for optimality only (e.g. in [140]).

Alternatively, some authors consider the property ε-optimality, where a solution is ε-optimal if it is a $(1+\varepsilon)$-approximation of an optimal solution (see, e.g., [28]). Note that optimality is a special case of ε-optimality, with $\varepsilon = 0$. Stability is then defined in the same way as before, only that property Pr in the definition is "ε-optimality" instead of "optimality".

In what follows, we use the words stable, stability radius and stability with the understanding that we mean them with regard to the property of optimality. Note that in this case only optimal solutions for the original data can be stable. We use the term stability with regard to ε-optimality, if we consider that property instead. The terminology of resiliency is used as defined in Section 11.1.

Research in the area of stability is mostly focused on computing the stability radius of an optimal solution, the equivalent of the first of the two research questions stated for resiliency. Observe that if there exist two different optimal solutions then usually very small perturbations

of input data are sufficient to make one of them lose their optimality. Indeed, for a large class of problems it can be shown that if there is more than one optimal solution, then all of them have stability radius 0 [140]. This means that in the case where more than one optimal solution exists, usually all solutions are unstable. Therefore, the question of finding a most stable solution does often not make sense in this context.

Note that if there is more than one optimal solution, then the question of stability instead can be changed to "how much can we change the input parameters, such that the set of optimal solutions does not need to be extended?" [140]. This question does not necessarily lead to the same definition of stability as we gave above, unless there exists only one optimal solution [140]. Since it deals with stability of a set of solutions asking for the most stable solution does again not make a lot of sense. For the same reason, this version of stability is not as closely related to resiliency as the version defined above and we do not consider it further.

Stability with regard to ε-optimality is the version of stability most closely related to resiliency that we were able to find. Note however, that it is still not the same as resiliency. Indeed, the property of being ε-optimal is dependent on the value of an optimal solution for the actual input parameters $\eta + \Delta$, while the bound B in the definition of resiliency is a fixed value, not dependent on the actual input parameters.

Observe that for ε-optimality with $\varepsilon > 0$ it might make sense to ask for a most stable solution, as the definition of ε-optimality allows for several stable ε-optimal at the same time. Still, we have found no indication that the question has been considered.

Thus, the question for a most stable solution does either not make sense or is not considered in the three versions of stability presented above. In contrast to that, our results in the next chapter show that it is not only relevant to compute the resiliency radius of a non-optimal solution, but also that dependent on the bound B a most resilient solution might not necessarily be optimal for the original input estimates η.

There are two other important differences between stability for optimal and ε-optimal solutions and resiliency. Firstly, fluctuation factors in the general sense are not considered in the stability literature, even though some results consider the case where some input parameters are uncertain and other input parameters fixed [28]. However, our results in Section 12.2.1 show that the consideration of different norms and of arbitrary fluctuation factors does not necessarily make the task of computing the resiliency radius much harder.

Secondly, and more importantly, note that the stability radius of a solution S^0 is dependent on S^0 and on other solutions which may become optimal for some perturbation of the parameters and which are not known in advance. This makes computing the stability radius a challenging task which for many combinatorial optimization problems is at least as hard as solving the original problem even if an optimal solution to the original problem is given. This is shown in [125] for the case where only one input parameter is uncertain. A similar result was also shown in [148] for the case where input parameters may change but have to remain positive. This means for many NP-hard problems it is NP-hard to compute the stability radius of an optimal

solution, even if the solution is given.

The complimentary result is also available. Consider the more general case of stability for ε-optimal solutions and combinatorial optimization problems where a solution can be represented by a vector x with 0/1-entries. Suppose further the original combinatorial optimization problem is solvable in polynomial time. It is shown in [28] that if the objective function is either of type $\min \sum c_i x_i$ or $\min \max\{c_i x_i\}$ then the stability radius of a given ε-optimal solution under the ℓ_∞-norm can be computed in polynomial time. Here, c_i is a vector of uncertain cost-parameters and x_i is a vector with 0/1-entries representing the given solution.

On the other hand the resiliency radius of a solution S^0 is only dependent on solution S^0 and the given bound B, which is a constant when computing a specific resiliency radius. The task of computing the resiliency radius therefore seems to be much easier to handle. In Section 12.2.1 we show how to compute the resiliency radius of, amongst other problems, the TSP problem (travelling sales person) in polynomial time. In contrast to this, computing the stability radius of a given optimal solution to the TSP problem is at least as hard as solving the TSP problem itself, i.e. strongly NP-hard (see, e.g., [84]).

For these reasons, even though the two concepts look very similar in their definitions, they are distinctly different from each other not only in terms of achievable results but also in terms of the research questions studied. Indeed, the largest part of the results we present later on focuses on finding most resilient solutions, while the question of a most stable solution does not appear to be considered. Thus in spite of the close relation of the two concepts, we were unable to find results in the literature on stability that carry over to resiliency.

For a broader review of stability and related areas see, e.g., [140, 142]. In the area of scheduling [70] provides a thorough review and new results for sensitivity analysis. They point out several challenges and difficulties for this type of analysis in scheduling problems, where in addition to the discrete structure a solution can have a temporal structure. After that they provide many results, either new or extensions of previous results to other types of problems. They also consider the search for a solution of minimum sensitivity, that is an optimal solutions which remains optimal for the largest amount of change in a given single parameter. However, this is still not the same as a most resilient solution, since first optimality is again the focus and only one parameter is assumed to be uncertain.

11.2.2 Robustness

Robust discrete optimization in the sense of min-max and min-max regret versions of combinatorial optimization problems is a newer research field than stability and sensitivity analysis. See [86] for an introduction and survey and [4] for a more recent survey. Like for stability there are several other meanings attached to the words "robust optimization", see, e.g., [14].

As in the last section, we focus this review on the similarities and differences of robustness and resiliency, rather than providing a collection of results, which are usually not transferable anyway. Using again notation similar to the one we used in the definition for resiliency, for

better comparability, there are two different types of uncertainty considered for min-max and min-max regret problems

In the discrete scenario case, instead of one vector of input parameters η a set $H = \{\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(m)}\}$ of m different possible vectors of input parameters, also called scenarios, is given. In the interval scenario case, for each entry η_j of the vector of input parameters an interval of possible values $\eta_j \in \mathcal{I}_j = \left[\underline{\eta}_j, \overline{\eta}_j\right]$ is given. The set H of all scenarios is given as the Cartesian product of the n different intervals, i.e. for input vector η we have $\eta \in H = \prod_{j=1}^{n} \mathcal{I}_j$.

Then the min-max version of a combinatorial optimization problem with set of solutions $\mathcal{S}$ and objective function $f(S, \eta)$ is given by

$$\min_{S \in \mathcal{S}} \max_{\eta \in H} f(S, \eta).$$

The goal is to find a solution S^* such that the maximum objective value over all possible scenarios is minimized.

The min-max regret version of a combinatorial optimization problem with set of solutions $\mathcal{S}$ and objective function $f(S, \eta)$ is given by

$$\min_{S \in \mathcal{S}} \max_{\eta \in H} (f(S, \eta) - f^*(\eta)),$$

where $f^*(\eta)$ is the objective value of an optimal solution to scenario η. The term $f(S, \eta) - f^*(\eta) = R(S, \eta)$ is also called the regret of solution S under scenario η. Here, the goal is to find a solution S^* such that the maximum regret over all scenarios is minimized. We refer to [4] for further definitions and for the notation usually used in the area of robustness.

As opposed to stability, literature in this area is usually not concerned with measuring the robustness of a given optimal solution with regard to some scenario. Instead the goal is most often to find a robust solution, in the sense that it solves the min-max or min-max regret version of a problem. Still, worst case scenarios for a given solution S^0 play a large role in solving these problems, see [4].

Therefore, this model of robustness is more interesting to our second research problem of finding most resilient solutions, than it is to our first, finding the resiliency radius of a given solution. The interval scenario case, due to its continuity, appears somewhat more closely related to resiliency than the discrete scenario case. However, note that considering resiliency, a worst case deviation for a solution depends on a upper bound B which is "used up" by all entries in the deviation vector at the same time. On the other hand for many problems in the interval scenario case a worst case scenario is given by vector η of input parameters that uses only extreme values $\underline{\eta}_j$ or $\overline{\eta}_j$ and seems much more straight forward to compute [4].

It is worth pointing out that for many minimization problems the interval scenario min-max version can be solved as fast as the problem itself, by solving the original problem for input parameters $\eta_{\max} = (\overline{\eta}_1, \overline{\eta}_2, \ldots, \overline{\eta}_n)$, see, e.g., [4]. This result is not transferable to the resiliency version of a problem. Indeed, even without fluctuation factors a worst case input vector $\eta + \Delta$

may be very different from the input vector of a worst case interval scenario. Recall that often a worst case interval scenario is given by a vector that only uses extreme values. In our results below we show that for norm ℓ_∞ it is often sufficient to consider a worst case deviation Δ for which all entries are the same, while for norm ℓ_1 we show that there are problems for which a worst case deviation vector Δ has only one non-zero entry.

Interval scenario min-max regret versions are often harder to solve than interval min-max versions. Since below we investigate the assignment problem and problem $1||\sum C_j$ for resiliency, we refer for comparison to the result from [3] that the interval min-max regret assignment problem is strongly NP-hard. Furthermore, the interval min-max regret version of problem $1||\sum C_j$, which can be seen as a special case of the assignment problem, see Section 2.4.4, is NP-hard if the intervals are arbitrary but polynomially solvable if all intervals have the same center [100].

For the discrete scenario case, the min-max and min-max regret assignment problem are strongly NP-hard if the number of scenarios is part of the input [3] and at least NP-hard in the ordinary sense even if there are only two scenarios [42, 86]. The discrete scenario min-max and min-max regret versions of problem $1||\sum C_j$ are NP-hard in the ordinary sense even for only two scenarios [36, 86].

For further complexity results see, e.g., the summaries in [4]. Note that for many famous polynomially solvable combinatorial optimization problems the min-max and min-max regret versions (apart from interval min-max) are at least NP-hard.

Chapter 12

Resiliency for Combinatorial Optimization Problems with Uncertain Data

This Chapter is structured as follows. In Section 12.1 we provide general properties of solution resilience, which apply to all combinatorial optimization problems. After that we focus on specific problems, the assignment problem in Section 12.2 and problem $1||\sum C_j$ in Section 12.3. These two problems have been chosen since they tie quite nicely into each other. Other scheduling problems and generalizations of the results below are under consideration and some are discussed in the conclusions and further work sections in Chapter 13.

As mentioned in the introduction, these results represent a snap shot of ongoing work. They are presented here to show that especially in the area of computational complexity, resiliency has some advantages over the related concepts reviewed in Section 11.2 and that it is worthy of further consideration.

12.1 General Properties

In this section we discuss general aspects of complexity and other properties that the problems studied in the later sections have in common.

12.1.1 Complexity Aspects

Consider two decision problems related to the original combinatorial optimization problem (11.1) and to the problem of finding a solution with the maximum radius of the resiliency ball:

$\mathrm{DP}_{\mathrm{original}}(B)$: Does there exist solution S such that $f(S) \leq B$?
$\mathrm{DP}_{\mathrm{radius}}(B, \xi)$: Does there exist solution S such that $\rho(S, \eta, B) \geq \xi$?

Statement 63. *If $\xi = 0$, then the complexity status of the decision problems $\mathrm{DP}_{\mathrm{original}}(B)$ and $\mathrm{DP}_{\mathrm{radius}}(B, 0)$ is the same for any type of the norm and for arbitrary choice of fluctuation factors α_j.*

Indeed, if there exists a polynomial-time algorithm for $\mathrm{DP}_{\mathrm{radius}}(B, 0)$, then it also provides an answer to $\mathrm{DP}_{\mathrm{original}}(B)$. Conversely, if there exists a polynomial-time algorithm for $\mathrm{DP}_{\mathrm{original}}(B)$, then it also solves $\mathrm{DP}_{\mathrm{radius}}(B, 0)$.

On the other hand, if one of the DP-problems is NP-complete, then the other one is NP-complete, since a yes-answer for one problem implies a yes-answer for another one. Statement 63 means that for an NP-hard combinatorial optimization problem, the problem of finding a solution with the maximum radius within the B-feasible region cannot be polynomial solvable, whichever norm ℓ_q or ℓ_∞ is used.

Statement 64. *If problem $\mathrm{DP}_{\mathrm{original}}(B)$ is NP-complete, then $\mathrm{DP}_{\mathrm{radius}}(B, \xi)$ is also NP-complete for any type of the norm. Similarly, for any (strongly) NP-hard combinatorial optimization problem, the resiliency version of the same problem is also (strongly) NP-hard.*

Note that the above statement about NP-hardness holds even for the special case with all $\alpha_j = 1$, so it also holds for the more general cases.

12.1.2 Properties of objective functions that limit the search for worst-case deviations

Many combinatorial optimization problems have objective functions with some form of monotonicity (non-decreasing or non-increasing) in the input parameters. Let $\mathcal{P} = (\eta, \mathcal{S}, f)$ be a combinatorial optimization problem with vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ of input parameters, set of feasible solutions $\mathcal{S}$ and objective function f. Then f is called non-decreasing in the input parameters, if for all solutions $S \in \mathcal{S}$ and for any vector $\eta' = (\eta'_1, \eta'_2, \ldots, \eta'_n)$, with $\eta_j \leq \eta'_j$ for all $1 \leq j \leq n$, we have $f(S, \eta) \leq f(S, \eta')$.

Analogously, f is called non-increasing in the input parameters, if for all solutions $S \in \mathcal{S}$ and for any vector $\eta' = (\eta'_1, \eta'_2, \ldots, \eta'_n)$, with $\eta_j \leq \eta'_j$ for all $1 \leq j \leq n$, we have $f(S, \eta) \geq f(S, \eta')$.

Examples for problems with an objective function non-decreasing in the input parameters are the traditional assignment problem (where the input parameters are the weights) and many scheduling problems in which the input consists of processing times and weights. Note that for

scheduling problems this is not the same as assuming the objective function is regular. There are many problems with non-regular objective functions for which the objective function is still non-decreasing in the input parameters (see, for example, Table 12.1).

Examples for combinatorial optimization problems with objective functions non-increasing in the input parameters are many scheduling problems with due-dates, if the due-dates are seen as input parameters.

Clearly, for any combinatorial optimization problem with an objective function non-decreasing in the input parameters, the following statement holds.

Statement 65. *Let $\mathcal{P}$ be a combinatorial optimization problem with with an objective function f non-decreasing in the input-parameters. Then for any solution S of $\mathcal{P}$ there exists a worst-case deviation for S in which all entries in the deviation vector Δ are non-negative.*

Observe that the statement still holds if we only assume the objective function to be non-decreasing in the uncertain input parameters. A similar statement can be formulated for combinatorial optimization problems with an objective function non-increasing in the input parameters, and with all entries in Δ non-positive.

For more general cases, assume that for the objective function f the entries of the input vector η could be resorted, such that we can write η as a string of sub-vectors $\eta = (\eta^+, \widetilde{\eta}, \eta^-)$, with f non-decreasing in the entries of η^+ and f non-increasing in the entries of η^-. We can make the following combined statement.

Statement 66. *Let $\mathcal{P}$ be a combinatorial optimization problem with f, η, η^+, and η^- as decribed above. Then for any feasible solution S of $\mathcal{P}$ there exists a worst-case deviation in which all deviations for parameters η^+ are non-negative, and all deviations for parameters η^- are non-positive.*

As we put special focus on scheduling in this thesis, we reformulate Statement 66 in terms of scheduling problems.

Statement 67. *Let $\mathcal{P}$ be a scheduling problem with processing times p_j, weights w_j and due-dates d_j and an objective function non-decreasing in the p_j and w_j and non-increasing in the d_j. Then for any feasible solution S of $\mathcal{P}$ there exists a worst-case deviation for S in which all deviations for processing times and weights are non-negative, and all deviations for due-dates are non-positive.*

Note that if due-dates are given but not uncertain, we can again assume that all deviations are non-negative. We reformulated Statement 66 for the scheduling parameters most commonly used in this thesis. For other scheduling parameters, like release dates, additional assumptions have to be made, but it might well be possible for many scheduling problems to extend Statement 67 many other typical scheduling parameters.

12.1.3 Problems with the ℓ_∞-norm

In the case of the ℓ_∞-norm it is often easier to calculate the resiliency radius ρ and a worst-case deviation. If the objective function is non-decreasing or non-increasing in the input parameters, then the worst-case deviation vector is characterized by a common deviation for all parameters. Below we state the result for an objective function non-decreasing in the input parameters.

Statement 68. *Let $\mathcal{S}$ be a set of solutions of a combinatorial optimization problem and let its objective function f be non-decreasing in the input parameters η. Let ξ be a non-negative real value. Then for any solution $S^0 \in \mathcal{S}$, under the ℓ_∞ norm the deviation vector Δ given by*

$$\Delta_j = \xi, \qquad 1 \leq j \leq n,$$

is a worst-case deviation amongst all deviations of size at most ξ, i.e. $f(S^0, \eta+\bar{\Delta}) \leq f(S^0, \eta+\Delta)$ for all deviations $\bar{\Delta}$ with $\|\bar{\Delta}\|_\infty \leq \xi$.

Instead of using (11.4) to calculate the resiliency radius of a given solution S^0 we can use the following mathematical programming formulation:

$$\begin{array}{ll} \max & \xi \\ \text{s.t.} & \Delta_j = \xi, \quad 1 \leq j \leq n, \\ & f\left(S^0, \eta + \alpha\Delta\right) \leq B, \\ & \xi \geq 0 \end{array} \tag{12.1}$$

Proof: Let Δ be the deviation given by $\Delta_j = \xi$ for all j and let Δ^* be an arbitrary worst-case deviation amongst all deviations of size at most ξ. Note that Δ has size ξ by the definition of the ℓ_∞-norm and that $f(S^0, \eta + \Delta^*) = f(S^0, \eta + \Delta)$ because $\Delta_j \geq \Delta_j^*$ for all j and because f is non-decreasing in the input parameters. Thus Δ is also a worst-case deviation amongst all deviations of size at most ξ.

In order to verify the second part observe that the linear programming formulation maximizes the size of a worst-case deviation vector. ∎

It is easy to see that the analogous result with a negative common deviation holds in the case where the objective function is non-increasing in the input parameters.

12.2 The Assignment Problem with Uncertain Costs

Consider the linear assignment problem

$$\begin{aligned} \text{AP}: \quad \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x_{ij} & \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, & 1 \le j \le n, \\ & \sum_{j=1}^{n} x_{ij} = 1, & 1 \le i \le n, \\ & x_{ij} \in \{0,1\}, & 1 \le i \le n,\ 1 \le j \le n, \end{aligned}$$

see (2.1) in Section 2.4. Recall that any solution S^0 that satisfies the constraints can be given as a set of n pairs of indices (i,j) such that $x^0_{ij} = 1$ for $(i,j) \in S^0$ and $x^0_{ij} = 0$ for the remaining (i,j)-pairs. The objective value for S^0 is

$$f\left(S^0, W\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x^0_{ij}.$$

In the uncertain version of the problem the values w_{ij} represent the estimates of the costs, $1 \le i \le n$, $1 \le j \le n$. Together with fluctuation factors α_{ij} and deviations Δ_{ij} the actual cost matrix is of the form

$$w'_{ij} = w_{ij} + \alpha_{ij}\Delta_{ij}. \tag{12.2}$$

The assignment problem is of interest for two reasons. First, as one of the most famous combinatorial optimization problems it is a good problem with which to start investigating resiliency. Secondly, regarding to our main area of interest, namely scheduling, the assignment problem can be used to model many different scheduling problems, for examples see Table 12.1.

We first demonstrate how to compute resiliency radii and find most resilient solutions for the case with arbitrary fluctuation factors in Section 12.2.1. Then we continue by giving improved algorithms to find most resilient solutions for the special case with $\alpha_{ij} \in \{0,1\}$ in Section 12.2.2. Finally, we compare the results for the min-sum version of the assignment problem to the min-max version of the assignment problem, the so-called bottleneck assignment problem in Section 12.2.3.

12.2.1 The assignment problem with arbitrary fluctuation factors

In this subsection we deal with the model with arbitrary fluctuation factors and perturbations of type (12.2). We first show how to construct a worst-case deviation for a given schedule for any ℓ_q-norm, and then show how to solve the problem of finding a most resilient assignment

Problem	w_{ij}	Reference
$1\|\|\sum C_j$	$w_{ij} = (n-i+1)\,p_j$	[17, 26]
$1\|\|\sum \|C_i - C_j\|$	$w_{ij} = (i-1)\,(n-i+1)\,p_j$	[81]
$1\|\|\sum \|W_i - W_j\|$ (waiting times)	$w_{ij} = i\,(n-i)\,p_j$	[8]
$1\|d_j = d\|\sum(\gamma_1 E_j + \gamma_2 T_j)$ (given $d \geq \sum_{j=1}^{n} p_j$)	$w_{ij} = \min\{\gamma_1\,(i-1)\,, \gamma_2\,(n-i+1)\}\,p_j$	[119]
$1\|d_j = d\|\sum(\gamma_1 E_j + \gamma_2 T_j + \gamma_3 d_j + \gamma_4 \tau)$ (d, τ are decision variables that define a common due window $[d, d+\tau]$)	$w_{ij} = \min\{\gamma_1(i-1) + \gamma_3 n, \gamma_2\,(n-i+1) + \gamma_3, \gamma_4 n\}\,p_j$	[103]
$1\|r_j, p_j = 1\|\sum f_j(C_j)$ ($f_j(C_j)$ is a monotone, non-decreasing function of C_j)	$w_{ij} = f_j(t_i+1)$, if $t_i \geq r_j$, $w_{ij} = \infty$, otherwise (with t_i the n earliest possible starting times, $t_1 = r_1$, $t_{i+1} = \max\{r_{i+1}, t_i+1\}$, $i \geq 1$)*	[17]
$Q\|p_j = 1\|\sum f_j(C_j)$ ($f_j(C_j)$ is a monotone, non-decreasing function of C_j)	$w_{ij} = f_j(t_i)$ (with t_i the n earliest possible finishing times)	[17]

$*$ with jobs numbered in order of non-decreasing release dates, $r_1 \leq r_2 \leq \ldots \leq r_n$

Table 12.1: Examples of scheduling problems that can be modelled as assignment problems

under the norm ℓ_1. We also show that the problem is (weakly) polynomially solvable under norm ℓ_∞.

Constructing worst-case deviations

For the perturbed costs of type (12.2) we have

$$f\left(S^0, W + \alpha\Delta\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_{ij} + \alpha_{ij}\Delta_{ij}\right)x_{ij}^0 = f\left(S^0, W\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\Delta_{ij}x_{ij}^0 \leq B.$$

When computing the worst-case deviation and the resiliency radius of a given solution S^0 we restrict ourselves to B-feasible solutions S^0 such that $\alpha_{ij} > 0$ for at least one $(i,j) \in S^0$. If S^0 is not B-feasible, which can be tested in $O(n)$ time, then its resiliency radius is equal to -1 by definition. Furthermore, if S^0 is B-feasible and $\alpha_{ij} = 0$ for all $(i,j) \in S^0$, then clearly $f(S^0, W + \alpha\Delta) = f(S^0, W) \leq B$ for all deviation vectors Δ and the resiliency radius of S^0 is equal to ∞.

So given estimated weights w_{ij}, fluctuation factors α_{ij} and the upper bound B, let S^0 be a B-feasible solution such that $\alpha_{ij} > 0$ for at least one $(i,j) \in S^0$. In order to derive the formula for the radius and worst-case deviation of S^0 under norm ℓ_q, consider the following

mathematical programming problem:

$$\begin{aligned} \text{KP}: \quad \min \quad & \sum_{(i,j)\in S^0} (\Delta_{ij})^q \\ \text{s.t.} \quad & \sum_{(i,j)\in S^0} \alpha_{ij}\Delta_{ij} \geq B', \\ & \Delta_{ij} \geq 0, \qquad\qquad (i,j) \in S^0, \end{aligned}$$

where B' is a constant,

$$B' = B - f\left(S^0, W\right). \tag{12.3}$$

Since the optimum is achieved when the main constraint holds as equality, we get the continuous knapsack problem, with (possibly) a non-linear objective function, depending on the type of the norm.

Lemma 69. *Given estimated weights w_{ij}, fluctuation factors α_{ij}, an upper bound B and a B-feasible solution S^0 such that $\alpha_{i_0 j_0} > 0$ for some $(i_0, j_0) \in S^0$, then for any q, $1 \leq q < \infty$, formulation KP computes a worst-case deviation of S^0 under the norm ℓ_q.*

Proof: Let Δ^* be a solution to formulation KP. As was observed before, the optimum for formulation KP is achieved when the main constraint holds as equality, i.e. $f(S^0, W + \alpha\Delta^*) = B$. We are left to prove that $\rho(S^0, W, B) = \|\Delta^*\|_q$. By definition of KP, for any Δ with $\|\Delta\|_q \leq \|\Delta^*\|_q$, clearly we have $f(S^0, W + \alpha\Delta) \leq B$. Therefore $\rho(S^0, W, B) \geq \|\Delta^*\|_q$.

Furthermore, for any $\epsilon > 0$ the deviation vector given by

$$\Delta^{\epsilon}_{ij} = \begin{cases} \Delta^*_{ij} + \epsilon, & \text{for } i = i_0, j = j_0, \\ \Delta^*_{ij}, & \text{otherwise,} \end{cases}$$

leads to $f(S^0, W + \alpha\Delta^{\epsilon}) > B$. Since for any chosen deviation length $\xi > \|\Delta^*\|_q$, there exists an $\epsilon > 0$ such that $\xi > \|\Delta^{\epsilon}\|_q$, this means that $\rho(S^0, W, B) \leq \|\Delta^*\|_q$. ∎

Note that we cannot generalize formulation KP to arbitrary combinatorial optimization problems. Indeed, if the objective function f has constant parts for some solutions the last part of the above proof, with the Δ^{ϵ}, does no longer work. As an example for such an objective function consider the standard scheduling goal of minimizing the number of late jobs, $f = \sum U_j$.

Now we use formulation KP to derive formulas for the worst-case deviation and the resiliency radius of the given solution S^0. It should be pointed out that the formulas below are also applicable to problems for which the solution regions are subsets of the solution region of the assignment problem. Recall that the solution region of the assignment problem can be seen as the set of all $n \times n$ permutation matrices, or equivalently of all permutations of n-elemental sets (see Section 2.4). Then most famously the TSP (travelling sales person) problem can be seen as an assignment problem for which additionally only permutations consisting of exactly one cycle are allowed as solutions. While due to Statement 64 the resiliency version of the TSP

is strongly NP-hard, the below formulas can still be used to compute the resiliency radius of a given solution S^0 (to the TSP problem).

Case ℓ_1. In this case $q = 1$ and the knapsack problem KP has a linear objective. Then the greedy algorithm finds an optimal solution with $\Delta_{ij} = 0$ for all (i,j)-pairs, except for the one which corresponds to the largest α_{ij}. Let $\alpha_{uv} = \max\left\{\alpha_{ij} | \, (i,j) \in S^0\right\}$. If there are several elements with the maximum α-value, select (u,v) arbitrarily. The optimal solution is given by

$$\begin{array}{l} \Delta_{uv} = \frac{B'}{\alpha_{uv}}\,, \\ \Delta_{ij} = 0 \qquad \text{for } (i,j) \in S^0 \backslash \left\{(u,v)\right\}, \end{array}$$

and the radius for S^0 is

$$\rho(S^0, w, B) = \frac{B'}{\alpha_{uv}}\,. \tag{12.4}$$

Case $\ell_q, 1 < q < \infty$. If $q = 2$ (which represents the Euclidean norm) or a larger integer, then problem KP can be solved using the Lagrangean multipliers method (see, e.g., [129]), which results in

$$\begin{aligned} \Delta_{ij} &= \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)}} \times \alpha_{ij}^{1/(q-1)}\,, \qquad (i,j) \in S^0, \\ \rho(S^0, w, B) &= \left(\sum_{(i,j)\in S^0} \Delta_{ij}^q\right)^{1/q} = \left(\left(\frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)}}\right)^q \times \sum_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)}\right)^{\frac{1}{q}} \\ &= \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)}} \left(\sum_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)}\right)^{\frac{1}{q}}. \end{aligned} \tag{12.5}$$

For the special case of the Euclidean norm ($q = 2$),

$$\begin{aligned} \Delta_{ij} &= \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}^2} \times \alpha_{ij}\,, \qquad (i,j) \in S^0, \\ \rho(S^0, w, B) &= \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}^2} \left(\sum_{(i,j)\in S^0} \alpha_{ij}^2\right)^{1/2} = \frac{B'}{\left(\sum\limits_{(i,j)\in S^0} \alpha_{ij}^2\right)^{1/2}}\,. \end{aligned}$$

Case ℓ_∞. For the ℓ_∞ norm, the results of Statement 68 hold, and we can use formulation

(12.1) instead of KP:

$$\begin{array}{ll} \max & \xi \\ \text{s.t.} & \sum\limits_{(i,j)\in S^0} (w_{ij} + \alpha_{ij}\xi) \leq B, \\ & \xi \geq 0, \end{array}$$

or equivalently

$$\begin{array}{ll} \max & \xi \\ \text{s.t.} & \xi \leq \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}}, \\ & \xi \geq 0, \end{array}$$

where B' is given by (12.3). Thus the common deviation for all uncertain parameters (including those which do not contribute to S^0) is

$$\Delta_{ij} = \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}}, \qquad \text{for all } 1 \leq i,j \leq n, \tag{12.6}$$

and the maximum resiliency radius is the same value:

$$\rho(S^0, w, B) = \frac{B'}{\sum\limits_{(i,j)\in S^0} \alpha_{ij}}. \tag{12.7}$$

For all three types of formulas the quotients are well-defined, as $\alpha_{ij} > 0$ for at least one $(i,j) \in S^0$ (recall that $\alpha_{ij} \geq 0$ by definition of fluctuation factors).

In the special case where $\alpha_{ij} = 1$ for all (i,j), $1 \leq i,j \leq n$, a worst-case deviation for all norms is given by

$$\Delta_{ij} \quad = \quad \frac{B'}{n}, \qquad (i,j) \in S^0.$$

The resulting resiliency radius in the ℓ_q-norm is

$$\rho(S^0, w, B) = \frac{B'}{n^{\frac{q-1}{q}}},$$

for $q < \infty$ and

$$\rho(S^0, w, B) = \frac{B'}{n},$$

for $q = \infty$. Thus the resiliency radius is only dependent on $B' = B - f(S^0, W)$, and the most resilient solution is the optimal solution to the original assignment problem with weights W.

Note that in the case with arbitrary fluctuation factors it is no longer necessarily true that the optimal solution of the assignment problem with weights W is the most resilient solution,

even if there are only two different fluctuation factors. Indeed, as an example under the ℓ_1 norm consider the 2×2 matrices

$$W = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \quad \alpha = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and upper bound $B = 10$. Clearly, the optimal solution of the assignment problem with weight matrix W is given by $S^{\text{opt}} = \{(1,1),(2,2)\}$ with estimated weight 2 and resiliency radius 4, while a most resilient solution is given by $S^{\text{res}} = \{(1,2),(2,1)\}$ with estimated weight 5 and resiliency radius 5.

It is easy to check that the same instance works as an example for any other norm as well. Also, if we subtract -1 from each entry in α, the example still works for the special case where $\alpha_{ij} \in \{0,1\}$. In that case the resiliency radius of S^{res} is equal to ∞.

Finally observe that the most resilient solution may also depend on the norm. For this consider the slightly changed example

$$W = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

with upper bound $B = 9$. Then under the ℓ_1-norm solution S^{opt} has resiliency radius $7/2 = 3,5$ and S^{res} has resiliency radius $4/1 = 4$. Thus S^{res} is again the most resilient solution under the ℓ_1-norm. Conversely, under the ℓ_∞-norm S^{opt} has resiliency radius $7/3 = 2 + \frac{1}{3}$ while S^{res} as resiliency radius $4/2 = 2$ and so S^{opt} is the most resilient solution under the ℓ_∞-norm.

Finding the most resilient assignment under the ℓ_1-norm

Given estimates $W = (w_{ij})$, fluctuation factors $\alpha = (\alpha_{ij})$ and an upper bound B, we now provide a method to find a solution S^* with maximum resiliency radius in the ℓ_1-norm in $O(n^5)$ time. To do so, we show that the solution can be obtained as the solution of one of at most $O(n^2)$ linear assignment problems.

Note that for norm ℓ_1 the resiliency radius of any solution S^0 is dependent only on the value $B' = B - f(S^0, p)$ and the value $\alpha_{uv} = \max\{\alpha_{ij} | (i,j) \in S^0\}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n^2}$ be an ordering of the values α_{ij} such that $\alpha_k \leq \alpha_{k+1}$ for all $1 \leq k \leq n^2 - 1$. Let $W^k = (w_{ij}^{(k)})$, with

$$w_{ij}^{(k)} = \begin{cases} w_{ij}, & \text{if } \alpha_{ij} \leq \alpha_k, \\ \infty, & \text{otherwise.} \end{cases} \tag{12.8}$$

Observe that there are n^2 cost matrices W^k and by construction for a given k, $1 \leq k \leq n^2$, the maximum α_{ij} value amongst all finite $w_{ij}^{(k)}$ is α_k. We solve the assignment problem for each of the weight matrices W^k and obtain n^2 solutions S^k with resiliency radii ρ^k. Then choose the solution S^{k^*} with largest resiliency radius amongst all the solutions S^k as candidate for the solution with the overall maximum resiliency radius. We show that S^{k^*} has the maximum

resiliency radius amongst all solutions (not necessarily of type S^k).

Lemma 70. *The solution S^{k^*} chosen as described above is a most resilient solution.*

Proof: For a proof by contradiction suppose this is not the case, and let S' be a solution of maximum resiliency radius $\rho' > \rho^{k^*}$. Note that $\rho' > \rho^{k^*} \geq \rho^k$ for all $1 \leq k \leq n^2$. Let $\alpha' = \max\{\alpha_{ij} | (i,j) \in S'\}$. Clearly, $\alpha' = \alpha_k$ for some $1 \leq k \leq n^2$, so choose k' such that $\alpha' = \alpha_{k'}$.

Due to (12.4) in the worst-case for solution S' exactly one weight is increased by $\alpha_{k'}\rho'$ and in the worst-case for solution $S^{k'}$ exactly one weight is increased by $\alpha_{k'}\rho^{k'}$. We have

$$\begin{aligned} B &= \sum_{(i,j)\in S'} w_{ij} + \alpha_{k'}\rho' \\ &\geq \sum_{(i,j)\in S^{k'}} w_{ij} + \alpha_{k'}\rho' \\ &> \sum_{(i,j)\in S^{k'}} w_{ij} + \alpha_{k'}\rho^{k'} \\ &= B. \end{aligned}$$

Here the inequality in the second line is due to the optimality of $S^{k'}$ for the assignment problem with weights $W^{k'}$ and the inequality in the third line is due to $\rho' > \rho^{k'}$ by choice of S'. As we obtained a contradiction, no solution with resiliency radius greater than ρ^{k^*} can exist and S^{k^*} is a solution with maximum resiliency radius. ∎

From Lemma 70 it follows immediately that the proposed method to find a most resilient solution is correct. In general, solving $O(n^2)$ assignment problems takes no longer than $O(n^5)$ time ($O(n^3)$ time for each individual assignment, see [25, 88]). This is the part of the processing which dominates the time complexity, as computing the resiliency radii can be done in $O(n)$ time for each solution, using formula (12.4).

Theorem 71. *The resiliency version of the assignment problem under the ℓ_1-norm with arbitrary fluctuation factors α_{ij} is solvable in $O(n^5)$ time.*

In reality we do not always have to solve n^2 assignment problems. If, for example, $\alpha_k = \alpha_{k+1}$ for some k then also $W^k = W^{k+1}$ and $S^k = S^{k+1}$, so only one of the matrices W^k and W^{k+1} needs to be considered. A special case of this is described in Section 12.2.2. Also, in a preprocessing step we can solve the bottleneck assignment problem for matrix α in order to find the minimum $k^{\min}$ such that a assignment of finite weight exists in $W^{k^{\min}}$. Then instead of starting with weight matrix W^1 it is sufficient to only consider weight matrices W^k with $k \geq k^{\min}$. Still, in the worst-case we have to solve $O(n^2)$ assignment problems.

Finding the most resilient assignment under the ℓ_∞-norm

Due to Statement 68 and also due to (12.7) there exists a worst-case deviation for any solution S^0 such that all entries of the deviation matrix Δ_{ij} are equal to a common value ξ. Furthermore, the assignment problem for given weights

$$w'_{ij} = w_{ij} + \alpha_{ij}\xi \tag{12.9}$$

can be solved in $O(n^3)$ time. We use this in order to construct a weakly polynomial algorithm, using binary search for the value ξ^* of the radius of a most resilient solution.

First we make sure that no assignment of infinite resiliency radius exist, by solving the assignment problem with weights

$$w_{ij}^{(0)} = \begin{cases} w_{ij}, & \text{if } \alpha_{ij} = 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{12.10}$$

similar to our approach for the ℓ_1-norm. Clearly, if there exists a solution to the assignment problem with weights given by (12.10) that has objective value no larger than B, then it is a solution to the original problem with infinite resiliency radius. Otherwise we can assume that for any B-feasible solution S^0 we have $\alpha_{ij} > 0$ for at least one $(i, j) \in S^0$.

The binary search is conducted as follows. For a given value of ξ, we solve the assignment problem with weights (12.9). Let $\Gamma(\xi)$ be the value of an optimal solution. Then, if $\Gamma(\xi) > B$, we have $\xi^* < \xi$, if $\Gamma(\xi) < B$ we have $\xi^* > \xi$ and if $\Gamma(\xi) = B$ we have $\xi^* = \xi$. For the last part, we use that for any B-feasible solution S^0 we have $\alpha_{ij} > 0$ for at least one $(i, j) \in S^0$ (since there are no B-feasible solutions with infinite resiliency radius).

Recall that for the purpose of finding a most resilient solution we can assume that all α_{ij} are integer. Consequently, for $\xi = B$ all weights w'_{ij} with non-zero α_{ij}-value are equal to B. In that case, any solution with at least one $\alpha_{ij} > 0$ has weight B or larger and exceeds weight B if ξ is increased further. Thus, since we are in the case where for any B-feasible solutionS^0 we have $\alpha_{ij} > 0$ for at least one $(i, j) \in S^0$, $\xi = B$ is an upper bound for the search space for ξ^*. As a lower bound for the search space we use $\xi = 0$.

Lemma 72. *Given estimated weights w_{ij}, fluctuation factors α_{ij} and an upper bound B the resiliency version of the assignment problem under the ℓ_∞-norm is solvable in time $O(n^3 \log \frac{B}{\varepsilon})$, where ε is the minimum difference between two candidate values for the maximum resiliency radius.*

Proof: The binary search procedure described above takes $O(n^3 \log \frac{B}{\varepsilon})$ time to find an optimal solution. ∎

Note that the search space is discrete, the number of possible candidates for ξ^* is bounded by $n!$ (there is only one resiliency radius for each assignment). Therefore the value ε cannot be arbitrarily small. In fact, as the resiliency radius of a given solution can be computed as a

Problem	Parameter	Upper bound of the search space
Minimum weight spanning tree	weights	B
Shortest path	weights	B
$1\|\|\sum C_j$	processing times	B
$1\|\|\sum w_j C_j$	processing times weights	$B/\min\{w_j \mid j \text{ is a job}\}$ $B/\min\{p_j \mid j \text{ is a job}\}$
$P\|\|C_{\max}$	processing times	B
$F\|\|C_{\max}$	processing times	B
$O\|\|C_{\max}$	processing times	B

Table 12.2: Upper bounds of the search space for different examples of combinatorial optimization problems

formula linear in the w_{ij}, α_{ij} and B, the value ε can also be lower bounded by such a formula. Therefore the binary search finds an optimal solution in (weakly) polynomial time.

Theorem 73. *The resiliency version of the assignment problem under the ℓ_∞-norm is solvable by binary search in (weakly) polynomial time.*

It should be remarked that the result from this section is generalizable to other combinatorial optimization problems. However, note that $\xi = B$ is not always an appropriate upper bound for the search space. In those cases different upper bounds may have to be found, or, dependent on the objective function f an easy to compute upper bound may not exist. That said, the bounding of the search space seems to be the only major obstacle in the way to generalizing this result and for many combinatorial optimization problems an upper bound can be easily found, for examples see Table 12.2.

Finding the most resilient assignment under the ℓ_q-norm

Regrettably, neither the result for the ℓ_1-norm nor the result for the ℓ_∞-norm can be easily generalized for other ℓ_q-norms. In the results for the ℓ_1 and ℓ_∞-norms, we could use that many entries in the worst-case deviation matrix were equal. However, for general fluctuation factors α_{ij} the worst-case deviations for ℓ_q-norms other than $q = 1$ and $q = \infty$ are much more complicated than that. In fact, given a solution S^0 all relevant entries in a worst-case deviation matrix Δ, i.e. those Δ_{ij} with $(i, j) \in S^0$, can be pairwise different for arbitrary fluctuation factors and norms ℓ_q, $1 < q < \infty$.

The complexity status of these problems is left open for future research.

12.2.2 Assignment problem with binary fluctuation factors $\alpha_{ij} \in \{0, 1\}$

We now consider the version of the assignment problem, where only some of the weights are uncertain, while the others are fixed. In the first two subsections we provide methods to find most resilient assignments under the ℓ_1 and ℓ_∞-norms. In the third subsection we provide some explanations why finding resilient solutions for general ℓ_q-norms appears to be more difficult.

Norm ℓ_1

For the ℓ_1-norm, we can find a solution with maximum resiliency radius as fast as we can solve the assignment problem itself. Following the same idea as in Section 12.2.1 for the case with arbitrary fluctuation factors α_{ij}, note that if there are only two different values for α_{ij}, then we only have two different cost matrices W^k. Thus we only have to solve 2 different assignment problems, instead of $O(n^2)$ many as in the general case. A constant number of assignment problems can be solved in $O(n^3)$ time (see again [25, 88]).

It was already shown in Section 12.1.1 that the resiliency version of a problem cannot be solved faster than the problem itself.

Norm ℓ_∞

We now provide an algorithm to find a solution with maximum resiliency radius in the ℓ_∞-norm. As opposed to the algorithm from Section 12.1.3, the one we discuss in this problem is strongly polynomial with runtime $O(n^4)$.

Consider the resiliency version of the assignment problem with weight matrix W and upper bound B. First we repeat and introduce a couple of useful notions. For a given B-feasible solution S denote its resiliency radius $\rho(S, W, B)$ by $\xi^{(S)}$. Recall that due to (12.7), under the norm ℓ_∞ we can assume that all entries of worst-case deviation matrix for solution S are equal $\xi^{(S)}$. Then

$$B = \sum_{(i,j)\in S} \left(w_{ij} + \alpha_{ij}\xi^{(S)} \right), \tag{12.11}$$

if $\alpha_{ij} = 1$ for at least one pair $(i, j) \in S$, otherwise $\xi^{(S)} = \infty$.

Furthermore, for any solution S denote by $h(S)$ the number of pairs $(i, j) \in S$ such that $\alpha_{ij} = 1$,

$$h(S) = |\{(i, j) \in S \mid \alpha_{ij} = 1\}|.$$

If S is B-feasible and $h(S) > 0$, using notation $\xi^{(S)}$ for the resiliency radius of S we can simplify (12.7) to

$$\xi^{(S)} = \frac{B - W(S)}{h(S)}, \tag{12.12}$$

where

$$W(S) = \sum_{(i,j)\in S} w_{ij}$$

for weight matrix W and a given solution S.

Denote by AP(W) the assignment problem with weight matrix W. In order to check for solutions with infinite resiliency radius, i.e. B-feasible solutions S with $h(S) = 0$, we construct weight matrix W^∞ with weights

$$w_{ij}^{(\infty)} = \begin{cases} w_{ij}, & \text{if } \alpha_{ij} = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly, a solution S with $h(S) = 0$ is B-feasible for problem AP(W^∞) if and only if it is B-feasible for AP(W). Furthermore, for AP(W^∞) any solution S with $h(S) > 0$ is not B-feasible. Thus an optimal solution to the assignment problem AP(W^∞) has infinite resiliency radius, if a solution with infinite resiliency radius exists.

The following algorithm finds a solution S^* with maximum resiliency radius.

ALGORITHM
Input: weight matrix $W^{(0)} = W$, fluctuation factors $\alpha_{ij} \in \{0, 1\}$ and upper bound B.
Output: A solution S^* with maximum resiliency radius.

1. Solve problem AP($W^{(\infty)}$) and let $S^{(\infty)}$ be the optimal solution.
   IF $W(S^{(\infty)}) \leq B$ then STOP: $S^{(\infty)}$ is a solution with infinite resiliency radius, set $S^* = S^{(\infty)}$.

2. Solve problem AP($W^{(0)}$) and let $S^{(0)}$ be the optimal solution.
   IF $W^{(0)}(S^{(0)}) > B$ then STOP: no B-feasible solution exists.
   ELSE let $\xi^{(0)}$ be the resiliency radius of $S^{(0)}$.

3. Set $k = 0$.
   WHILE $h(S^{(k)}) > 1$ DO

   (i) Set $k = k + 1$.

   (ii) Define weight matrix $W^{(k)}$ with weights

   $$w_{ij}^{(k)} = w_{ij} + \alpha_{ij}\xi^{(k-1)}, \qquad 1 \leq i, j \leq n.$$

   (iii) Solve problem AP($W^{(k)}$) and let $S^{(k)}$ be the optimal solution.

   (iv) IF $W^{(k)}(S^{(k)}) = B$ THEN STOP: $S^{(k-1)}$ with resiliency radius $\xi^{(k-1)}$ is a solution with maximum resiliency radius, set $S^* = S^{(k-1)}$.
   ELSE let $\xi^{(k)}$ be the resiliency radius of $S^{(k)}$.

ENDWHILE

4. $S^{(k)}$ with resiliency radius $\xi^{(k)}$ is a solution with maximum resiliency radius, set $S^* = S^{(k)}$.

END ALGORITHM

In order to show that the algorithm is correct, we prove the following lemma.

Lemma 74. *1. For all $k \geq 1$, if the algorithm computes $\xi^{(k)}$, then $\xi^{(k-1)} < \xi^{(k)}$.*

2. The algorithm stops before $k > n - 1$.

3. The algorithm is correct.

Proof of 1.: Due to the definition of $W^{(k)}$ and of $\xi^{(k-1)}$ we have $W^{(k)}(S^{(k-1)}) = B$. Furthermore, $W^{(k)}(S^{(k)}) \leq B$, due to the optimality of $S^{(k)}$. As $\xi^{(k)}$ is not computed in step (iv) of the while-loop if $W^{(k)}(S^{(k)}) = B$, that means $W^{(k)}(S^{(k)}) < B$. Then due to the definition of $W^{(k)}(S^{(k)})$ we have

$$\sum_{(i,j)\in S^{(k)}} \left(w_{ij} + \alpha_{ij}\xi^{(k-1)}\right) < B \tag{12.13}$$

and by (12.11) also

$$\sum_{(i,j)\in S^{(k)}} \left(w_{ij} + \alpha_{ij}\xi^{(k)}\right) = B. \tag{12.14}$$

Subtracting the (12.13) from (12.14), after simplification we obtain

$$\xi^{(k)} - \xi^{(k-1)} > 0.$$

Proof of 2.: We only deal with the case where the algorithm completes the while-loop until a solution S with $h(S) \leq 1$ is found, as that is the case where k becomes largest. So it is sufficient to show that for some $k^* \leq n - 1$ we have $h(S^{(k^*)}) \leq 1$. Since $h(S^{(0)}) \leq n$, we can show equivalently that for all $k \geq 1$, if $\xi^{(k)}$ is computed in step (iv) of the while-loop, then $h(S^{(k)}) < h(S^{(k-1)})$. This means after at most $n - 1$ repeats of the while-loop, we arrive at a solution S with $h(S) \leq 1$ (or the algorithm stops earlier, because a value $\xi^{(k)}$ is not computed in step (iv)).

Similar to the proof of the first statement, note that $W^{(k)}(S^{(k-1)}) = B$ and that if $\xi^{(k)}$ is computed, then

$$W^{(k)}(S^{(k)}) = \sum_{(i,j)\in S^{(k)}} \left(w_{ij} + \alpha_{ij}\xi^{(k-1)}\right) < B.$$

Observe also that for any solution S we can rewrite

$$W^{(k)}(S) = W^{(0)}(S) + h(S)\xi^{(k-1)}. \tag{12.15}$$

We consider the cases $k = 1$ and $k \geq 2$ separately. For $k = 1$, since $S^{(0)}$ is an optimal solution for AP($W^{(0)}$), we have $W^{(0)}(S^{(1)}) \geq W^{(0)}(S^{(0)})$. Also note that

$$W^{(0)}(S^{(1)}) + h(S^{(1)})\xi^{(0)} = W^{(1)}(S^{(1)}) < W^{(1)}(S^{(0)}) = W^{(0)}(S^{(0)}) + h(S^{(0)})\xi^{(0)},$$

where the first and last equality are due to (12.15), and the middle inequality is due $W^{(1)}(S^{(0)}) = B$ and $W^{(1)}(S^{(1)}) < B$, as described in the paragraph above. This implies that $h(S^{(1)})\xi^{(0)} < h(S^{(0)})\xi^{(0)}$ and therefore $h(S^{(1)}) < h(S^{(0)})$, which is what we want to prove.

So now consider the case $k \geq 2$. Recall that $S^{(k-1)}$ is an optimal solution to AP($W^{(k-1)}$). Thus

$$W^{(0)}(S^{(k-1)}) + h(S^{(k-1)})\xi^{(k-2)} \leq W^{(0)}(S^{(k)}) + h(S^{(k)})\xi^{(k-2)},$$

but with arguments similar to the proof of the first statement we also have

$$W^{(0)}(S^{(k-1)}) + h(S^{(k-1)})\xi^{(k-1)} > W^{(0)}(S^{(k)}) + h(S^{(k)})\xi^{(k-1)}.$$

Putting both inequalities together, we obtain

$$\begin{aligned} & W^{(0)}(S^{(k-1)}) + h(S^{(k-1)})\xi^{(k-2)} - W^{(0)}(S^{(k)}) - h(S^{(k)})\xi^{(k-2)} \\ < \; & W^{(0)}(S^{(k-1)}) + h(S^{(k-1)})\xi^{(k-1)} - W^{(0)}(S^{(k)}) - h(S^{(k)})\xi^{(k-1)} \end{aligned}$$

and we simplify to

$$h(S^{(k)})(\xi^{(k-1)} - \xi^{(k-2)}) < h(S^{(k-1)})(\xi^{(k-1)} - \xi^{(k-2)}).$$

Since $\xi^{(k-1)} > \xi^{(k-2)}$ by the first statement, this means that $h(S^{(k)}) < h(S^{(k-1)})$, which finishes the proof.

Proof of 3.: The algorithm can stop in step 1, step 2, step (iv) in the WHILE-loop and in step 4.

Clearly, if the algorithm returns a solution S^* with $\xi^{(S^*)} = \infty$ obtained in step 1, that solution has infinite resiliency radius and therefore is optimal. Otherwise, as discussed when we introduced assignment problem AP(W^∞), no B-feasible solution S with $h(S) = 0$ exists.

If the algorithm stops after step 2 because $W^{(0)}(S^{(0)}) > B$ then no B-feasible exists.

If the algorithm stops in step (iv) of the WHILE-loop, then we have $W^{(k)}(S^{(k)}) = B$. We show that in that case, $S^{(k-1)}$ is already a solution of maximum resiliency radius. Since after checking in step 1 of the algorithm we already know that no B-feasible solution exists which uses only fixed weights, it is sufficient to show that all solutions, which use at least one uncertain weight, cannot have a resiliency radius larger than $\xi^{(k-1)}$.

So let S' be a solution to the assignment problem which uses at least one uncertain weight.

As $S^{(k)}$ is an optimal solution to AP($W^{(k)}$) and $W^{(k)}(S^{(k)}) = B$, that means

$$W^{(k)}(S') = W^{(0)}(S') + h(S')\xi^{(k-1)} \geq B.$$

Then, due to $h(S') \geq 1$, for any value $\xi' > \xi^{(k-1)}$ we have

$$W^{(0)}(S') + h(S')\xi' > B.$$

Therefore the resiliency radius of S' is at most $\xi^{(k-1)}$, which was what we wanted to show.

Finally, assume the algorithm stops in step 4 and returns a solution $S^* = S^{(k^*)}$ for some $k^* \leq n-1$. Then we have $h(S^{(k^*)}) = 1$, due to the exit condition of the while-loop and since no B-feasible solution S with $h(S) = 0$ exists (otherwise the algorithm would have stopped after step 1).

We show that solution $S^{(k^*)}$ with resiliency radius $\xi^{(k^*)}$ is optimal. Note that $S^{(k^*)}$ is constructed as the optimal solution to assignment problem AP($W^{(k^*)}$) with weights

$$w_{ij}^{(k^*)} = w_{ij}^{(0)} + \alpha_{ij}\xi^{(k^*-1)}, \qquad 1 \leq i, j \leq n.$$

Towards contradiction, suppose there is a solution S' with resiliency radius $\xi' > \xi^{(k^*)}$. We show that then $W^{(k^*)}(S') < W^{(k^*)}(S^{(k^*)})$, in contradiction to the optimality of $S^{(k^*)}$ for assignment problem AP($W^{(k^*)}$).

Using again (12.15), we write

$$W^{(k^*)}(S') = W^{(0)}(S') + h(S')\xi^{(k^*-1)}$$

and because $h(S^{(k^*)}) = 1$, we have

$$W^{(k^*)}(S^{(k^*)}) = W^{(0)}(S^{(k^*)}) + \xi^{(k^*-1)}.$$

Furthermore, by definition of the resiliency radius we have

$$W^{(0)}(S') + h(S')\xi' = B = W^{(0)}(S^{(k^*)}) + \xi^{(k^*)}.$$

Together with $h(S') \geq 1 = h(S^{(k^*)})$ and $\xi' > \xi^{(k^*)} > \xi^{(k^*-1)}$ (the last inequality is due to the

first statement), we obtain

$$\begin{aligned} W^{(k^*)}(S') &= W^{(0)}(S') + h(S')\xi^{(k^*-1)} \\ &= W^{(0)}(S') + h(S')\xi^{(k^*-1)} + h(S')\xi' - h(S')\xi' \\ &= B + h(S')\xi^{(k^*-1)} - h(S')\xi' \\ &= B - h(S')(\xi' - \xi^{(k^*-1)}) \\ &\leq B - (\xi' - \xi^{(k^*-1)}) \\ &< B - (\xi^{(k^*)} - \xi^{(k^*-1)}) \\ &= B - \xi^{(k^*)} + \xi^{(k^*-1)} \\ &= W^{(0)}(S^{(k^*)}) + \xi^{(k^*-1)} \\ &= W^{(k^*)}(S^{k^*}), \end{aligned}$$

in contradiction to the optimality of $S^{(k^*)}$ for problem AP($W^{(k^*)}$).

Therefore, solution S' cannot exist and S^{k^*} has maximum resiliency radius $\xi^{(k^*)}$. ∎

Theorem 75. *The algorithm is correct and takes at most $O(n^4)$ time.*

Proof: The correctness is proved to Lemma 74. As the algorithm stops before $k > n-1$ at most $O(n)$ assignment problems are solved, each taking at most $O(n^3)$ time. Thus the algorithm can be implemented with a runtime of $O(n^4)$. ∎

Note that for the problem with general α_{ij} a similar approach could be implemented. However, it would not lead to a polynomial, but a pseudo-polynomial algorithm. Indeed, in every step instead of $h(S)$ we reduce the sum of the α_{ij} used in solution S^k by at least 1 leading to a runtime of $O(n^3\Theta)$, where Θ is the weight of a maximum weight assignment with weight matrix $\alpha = (\alpha_{ij})$ (recall that for the purposes of finding a most resilient solution we can assume all α_{ij} to be integer).

Norm ℓ_q

Again, the arguments from above do not work for general ℓ_q-norms, $1 < q < \infty$. The problem here is that with decreasing number $h(S)$ of uncertain weights used in S, the actual deviation for each uncertain weight has to increase in order to obtain the same radius. Thus the technique from the last section is not applicable.

The problem may be NP-hard, but we were unable to find a proof for this. Similar to the more general version with arbitrary fluctuation factors, the question remains open for now.

12.2.3 The bottleneck assignment problem

At the end of our investigation of the assignment problem, we want to compare our findings for the min-sum version of the assignment problem to the situation for the min-max version of the

assignment problem, also known as bottleneck assignment problem. The bottleneck assignment problem is defined as

$$\begin{aligned} \text{AP}: \quad \min \quad & \max_{1\le i,j\le n} w_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, \qquad 1 \le j \le n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \qquad 1 \le i \le n, \\ & x_{ij} \in \{0,1\}, \qquad 1 \le i \le n,\ 1 \le j \le n. \end{aligned}$$

Considering uncertainty, let $W = (w_{ij})$ be the estimates for the weights and α_{ij} be the fluctuation factors. Then for a given deviation matrix $\Delta = (\Delta_{ij})$, the actual weights are of the form

$$w'_{ij} = w_{ij} + \alpha_{ij}\Delta_{ij},$$

similar to (12.2).

The bottleneck assignment seems to be much easier to handle than the usual min-sum version. We first show that under any norm ℓ_q, $q \in \mathbb{N}_{>0} \cup \{\infty\}$, worst-case deviations with only one non-zero entry exist. Then we use this to construct an algorithm to find a most resilient solution to the bottleneck assignment problem by solving another bottleneck assignment problem.

Lemma 76. *Let W and α be the estimated weights and the fluctuation factors for the resiliency version of the bottleneck assignment problem as above. Let S^0 be a given solution and Δ be any deviation matrix. Then there exists a deviation matrix $\widetilde{\Delta}$, such that*

1. *$\|\widetilde{\Delta}\| \le \|\Delta\|$, where $\|\cdot\|$ may be any ℓ_q-norm,*
2. *$f(S^0, \widetilde{\Delta}) = f(S^0, \Delta)$ and*
3. *matrix $\widetilde{\Delta}$ has only one non-zero entry.*

Proof: Given the starting deviation Δ, denote by $W' = (w'_{ij})$ the actual costs, similar to (12.2). Let $(i^*, j^*) \in S^0$ such that $w'_{i^*j^*} = \max\{w'_{ij} | (i,j) \in S^0\}$, or equivalently $f(S^0, W') = w'_{i^*j^*}$. If all entries of Δ, other than $\Delta_{i^*j^*}$, are 0-entries, then set $\widetilde{\Delta} = \Delta$ and we are done.

Otherwise define deviation matrix $\widetilde{\Delta}$ by

$$\widetilde{\Delta}_{ij} = \begin{cases} \Delta_{ij}, & \text{if } i = i^*, j = j^* \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f(S^0, W + \alpha\widetilde{\Delta}) = f(S^0, W') = w'_{i^*j^*}$ and that $\|\widetilde{\Delta}\| \le \|\Delta\|$, by construction. Furthermore, clearly $\widetilde{\Delta}$ has at most one non-zero entry, namely $\widetilde{\Delta}_{i^*j^*}$. ∎

Lemma 76 immediately implies that for any solution S^0 and any upper bound B a worst-case deviation with at most one non-zero entry exists. The next lemma deals with computing

that worst-case deviation.

Lemma 77. *Let W and α be as in Lemma 76. Let B be a given upper bound and let S^0 be a B-feasible solution. For each $(i,j) \in S^0$ let*

$$\xi_{ij} = \begin{cases} \frac{B-w_{ij}}{\alpha_{ij}}, & \text{if } \alpha_{ij} > 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{12.16}$$

Choose $(i^*, j^*) \in S^0$ such that $\xi_{i^*j^*} = \min\{\xi_{ij} | (i,j) \in S^0\}$, with ties broken arbitrarily, and define deviation matrix Δ by

$$\Delta_{ij} = \begin{cases} \xi_{ij}, & \text{if } i = i^*, j = j^* \\ 0, & \text{otherwise.} \end{cases}$$

Then $\rho(S^0, W, B) = \|\Delta\| = \xi_{i^*j^*}$ and Δ is a worst-case deviation of S^0 under any ℓ_q-norm.

Proof: If $\|\Delta\| = \infty$, i.e. $\alpha_{ij} = 0$ for all $(i,j) \in S^0$, then there is nothing to prove. Otherwise, due to Lemma 76, under any ℓ_q-norm, there exists a worst-case deviation with at most one non-zero entry. Note that amongst all deviations with at most one non-zero entry, Δ is the one with the smallest non-zero entry such that the upper bound B is reached. Any further increase in value $\Delta_{i^*j^*}$ would increase the objective value of solution S^0 over bound B, as $\alpha_{i^*j^*} > 0$, and therefore Δ is a worst-case deviation. The formula for the resiliency radius $\rho(S^0, W, B)$ follows immediately. ∎

From Lemma 77, observe that for any ℓ_q-norm, given an upper bound B and a B-feasible solution S^0, we have

$$\rho(S^0, W, B) = \|\Delta\| = \xi_{i^*j^*} = \min\left\{\xi_{ij} | (i,j) \in S^0\right\},$$

with ξ_{ij} defined as in (12.16). Thus, a most resilient solution to the bottleneck assignment problem, is a B-feasible solution S^*, which maximises

$$\min\left\{\xi_{ij} | (i,j) \in S^*\right\}.$$

We construct a new weight matrix $\Psi = (\psi_{ij})$, given by

$$\psi_{ij} = \begin{cases} \xi_{ij}, & \text{if } B - w_{ij} \geq 0, \\ -1, & \text{otherwise.} \end{cases} \tag{12.17}$$

Note that if $\psi_{\hat{i}\hat{j}} = -1$ for fixed $\hat{i}$ and $\hat{j}$ then any solution S with $(\hat{i}, \hat{j}) \in S$ cannot be B-feasible, as $w_{\hat{i}\hat{j}} > B$.

Using weights Ψ we solve the following maximization version of the bottleneck assignment

problem:

$$\begin{array}{lll} \text{AP}: \max & \min\limits_{1\leq i,j\leq n} \psi_{ij}x_{ij} & \\ \text{s.t.} & \sum\limits_{i=1}^{n} x_{ij} = 1, & 1 \leq j \leq n, \\ & \sum\limits_{j=1}^{n} x_{ij} = 1, & 1 \leq i \leq n, \\ & x_{ij} \in \{0,1\}, & 1 \leq i \leq n,\ 1 \leq j \leq n. \end{array} \tag{12.18}$$

By construction any assignment S with non-negative objective value with respect to problem (12.18) is also a B-feasible solution with respect to the original problem with weights W, due to the definition of weights Ψ. Conversely, if S is an assignment with negative objective value with respect to problem (12.18), then S is not B-feasible with respect to the original problem, again due to the definition of Ψ.

Let S^* be an optimal solution to (12.18). If the objective value of S^* with respect to the new problem (12.18) is non-negative, then S^* is a most resilient solution for the resiliency version with estimated weights W, fluctuation factors α and upper bound B. Otherwise, S^* is not B-feasible for the original problem, due to the our observations in the previous paragraph, and in fact no B-feasible solution exists for the original problem.

Theorem 78. *A most resilient solution S^* for the bottleneck assignment problem with estimated weights w_{ij}, fluctuation factors α_{ij} and an upper bound B can be found by solving the bottleneck assignment problem defined by (12.18) with weight matrix Ψ defined by (12.17).*

We can construct matrix Ψ in $O(n^2)$ time. After that pre-processing step is done, the bottleneck assignment problem (12.18) can be solved in $O(n^2\sqrt{n \log n})$ time with the algorithm by [57]. In total, this method to solve the resiliency version of the bottleneck assignment problem (under any ℓ_q norm) takes $O(n^2\sqrt{n \log n})$ time.

Interestingly, the result from this section corresponds well to the results for min-max and min-max regret versions of the bottleneck assignment problem, which also can be solved by solving the bottleneck assignment problem on a special selected weight matrix. Furthermore, this method can be generalized to the min-max and min-max regret versions of other bottleneck combinatorial optimization problems. See [4] and [7] for details. It would be interesting to see if a generalization of the above approach to the resiliency versions of other bottleneck problems is also possible.

12.3 Problem $1||\sum C_j$ with uncertain cost

Next we study problem $1||\sum C_j$. Note that the resiliency version of $1||C_{\max}$ is trivial, as every schedule (without idle times) is a most resilient schedule, similar to the original problem where every schedule (without idle times) has the same objective value. Also, a worst-case deviation can always be achieved by increasing only the processing time of the job with the largest fluctuation factor α_j.

We start by briefly repeating the most important notation and facts. For problem $1||\sum C_j$ there are given n jobs with processing times p_j, $1 \leq j \leq n$, and the goal is to minimize the total sum of completion times. It is sufficient to restrict the solution region to all permutations of jobs, representing schedules without idle times. Consider a schedule S^0 given as n pairs (i, j), $1 \leq i, j \leq n$, indicating assignment of job j to position i. Recall that problem $1||\sum C_j$ can be modelled as the assignment problem AP with variables

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \in S^0, \\ 0, & \text{otherwise,} \end{cases}$$

and costs w_{ij} for assigning job j to position i defined as

$$w_{ij} = (n - i + 1)p_j,$$

see Section 2.4.4. The input parameters, which now may be subject to uncertainty, are the job processing times p_j.

As in the previous section, we consider first resiliency in its most general type. We discuss the correspondence between the fluctuation factors of the scheduling problem and the fluctuation factors of the associated assignment problem. Then we show how to construct worst-case deviations for the case with arbitrary fluctuation factors α_j, using the results from Section 12.2.

After that we turn to the problem of finding most resilient solutions, again first applying the results from Section 12.2, where possible. Note that similar ideas work for other scheduling problems in Table 12.1. We also provide a more efficient algorithm than we gave for the assignment problem, to find a most resilient solution under the ℓ_∞-norm. Moreover, given estimated processing times p_j and fluctuation factors α_j, the new algorithm allows us to find the function $\rho_\infty(B)$, which calculates the optimal resiliency radius dependent on the upper bound B.

Finally we investigate the version with fluctuation factors $\alpha_j \in \{0, 1\}$ and provide an algorithm to find most resilient solutions for norm ℓ_1. As for the algorithm for norm ℓ_∞, given estimated processing times p_j and fluctuation factors α_j, this algorithm is also able to find the function $\rho_1(B)$, which calculates the optimal resiliency radius dependent on the upper bound B.

12.3.1 The case with arbitrary fluctuation factors α_j for the job processing times

In the presence of uncertain data, the perturbed job processing times of the form

$$p' = p + \alpha\Delta$$

result in the following weights for the associated assignment problem:

$$w'_{ij} = (n-i+1)\,(p_j + \alpha_j \Delta_j) = w_{ij} + (n-i+1)\alpha_j \Delta_j.$$

Given fluctuation factors α_j for the scheduling problem, the corresponding fluctuation factors α_{ij} for the associated assignment problem are defined by

$$\alpha_{ij} = (n-i+1)\alpha_j, \qquad \text{for } \; 1 \leq i,j \leq n. \tag{12.19}$$

Note that even in the case where all fluctuation factors for α_j for the scheduling problem are equal to 1, for assignment problem we still have fluctuation factors

$$\alpha_{ij} = n-i+1, \qquad \text{for } \; 1 \leq i,j \leq n.$$

Thus the case with equal fluctuation factors for the scheduling problem translates into a case with n different fluctuation factors for the assignment problem. This can also happen in other instances where we transform one problem into another in order to solve it, see for example the problems in Table 12.1. It is an additional motivation to consider fluctuation factors in the way we do, as then the factors arising from the transformation of one problem into another can often be dealt with as part of the fluctuation factors.

Computing radii and worst-case deviations for given solutions

We want to apply the results from Section 12.2.1. In order to do so we deal with fluctuation factors α_{ij}, defined by (12.19), for the weights w_{ij} in the associated assignment problem. Furthermore, to make use of the results for the knapsack problem as we did in Section 12.2, we introduce additional conditions $\Delta_{ij}^{\mathrm{AP}} = \Delta_j$ for $1 \leq i,j \leq n$, where $\Delta_{ij}^{\mathrm{AP}}$, represent variables in the knapsack problem KP.

Observe that the worst-case deviations we construct are worst-cases for the scheduling problem, but not for the associated assignment problem. Indeed, in Section 12.2 we implicitly assume that a worst-case deviation leaves all weights which are not used in the given solution unchanged. It is easy to check that any deviation, which changes weights not used in a given solution to the assignment problem is not a worst-case deviation. In contrast to this, if we change the length of a job j, we also change all weights associated with j in the assignment matrix, even though only one of those weights is used in any given solution.

In spite of the last observation, we can still use the formulae derived in Section 12.2.1 as they do not take into account changes happening to the weights not used by a given solution. As before, we assume that S^0 is a B-feasible solution and that $\alpha_j > 0$ for at least one job j. Note that if $\alpha_{j_0} > 0$ for a fixed job j_0, then $\alpha_{ij_0} > 0$ for all $1 \leq i \leq n$.

Case ℓ_1. Applying (12.4) for an arbitrary solution S^0, let $(u,v) \in S^0$ be the job-position pair

such that $\alpha_{uv} = \max\{\alpha_{ij}|(i,j) \in S^0\}$. Then the radius of S^0 is

$$\rho(S^0, p, B) = \frac{B'}{\alpha_{uv}} = \frac{B'}{(n-u+1)\alpha_v}\,. \tag{12.20}$$

A worst-case deviation can be achieved by increasing the length of job v by $\rho(S^0, p, B)$,

$$\begin{aligned} \Delta_v &= \tfrac{B'}{(n-u+1)\alpha_v}\,, \\ \Delta_j &= 0, \qquad \text{for all } j \neq v. \end{aligned}$$

Case $\ell_q,\ 1 < q < \infty$. In order to apply (12.5), first one needs to calculate the constant

$$H_q = \sum_{(i,j)\in S^0} \alpha_{ij}^{q/(q-1)} = \sum_{(i,j)\in S^0} ((n-i+1)\alpha_j)^{q/(q-1)}\,,$$

Then

$$\rho(S^0, p, B) = \frac{B'}{H_q^{(q-1)/q}}\,. \tag{12.21}$$

In order to compute a worst-case deviation, let $i\,[j]$ be the position of job j in solution S^0, i.e. $(i(j), j) \in S^0$ for each job j. Then a worst-case deviation for solution S^0 is given by

$$\Delta_j = \frac{B'}{H_q} \times ((n - i\,[j] + 1)\alpha_j)^{1/(q-1)}\,, \quad 1 \le j \le n.$$

Case ℓ_∞. In order to apply (12.7), calculate $H_\infty = \sum_{(i,j)\in S^0} \alpha_{ij} = \sum_{(i,j)\in S^0} ((n-i+1)\alpha_j)$. Then the radius is

$$\rho(S^0, p, B) = \frac{B'}{H_\infty} \tag{12.22}$$

and a worst-case deviation is given by

$$\Delta_j = \rho(S^0, p, B) = \frac{B'}{H_\infty}\,, \quad 1 \le j \le n.$$

Going back to the case where all α_j are equal, $\alpha_j = 1$, note that in that case the auxiliary terms H_q and H_∞ do not depend on the choice of the given solution. Similarly, the value $\alpha_{uv} = \max\{\alpha_{ij}|(i,j) \in S^0\}$ used for the ℓ_1-norm is defined by the job in the first position, $\alpha_{uv} = n$, and equal for all given solutions.

Therefore the radius for any given solution S^0 under any ℓ_q-norm depends only on the value $B' = B - f(S^0, p)$. The larger B' for the given solution, the larger its radius. It is easy to see that the solution with the largest value B' is the optimal solution of the original problem. Thus, in the case where all α_j are equal, the optimal solution for the scheduling problem, i.e. the one with jobs ordered in the SPT-order, is also the most resilient one, similarly to the assignment problem with equal fluctuation factors. Again this is no longer necessarily the case if the α_j

are not equal.

Finding most resilient solutions for norms ℓ_q, $q < \infty$

For the case of the ℓ_1-norm, clearly we can apply the $O(n^5)$ time algorithm we developed for the assignment problem in Section 12.2.1. A better time complexity, using the special structure of our scheduling problem and its related assignment problem may well be possible. Below, in Section we discuss such an algorithm for the special case with binary fluctuation factors, $\alpha_j \in \{0, 1\}$. However, for the general case our search for a better algorithm has been unsuccessful and the question remains open for now.

Similar to the results to the assignment problem, no polynomial time algorithm could be found for norms ℓ_q, $1 < q < \infty$, and indeed, the time complexity of that problem remains open as well.

Finding most resilient solutions for norm ℓ_∞

Now we consider the ℓ_∞-norm, where we can obtain a new positive result, which was not possible for the assignment problem. Not only can we solve the problem of finding a most resilient solution for the ℓ_∞-norm in $O(n^2 \log n)$ time, but we can in fact calculate the largest resiliency radius $\rho_\infty(B)$ and its corresponding schedule as a function of the upper bound $B \geq 0$. It is sufficient to consider the case where B is large enough such that at least one B-feasible solution exists, otherwise $\rho_\infty(B) = -1$ by definition.

An optimal solution for problem $1||\sum C_j$ is a sequence of jobs given by a known precedence rule, in this case the SPT-order (shortest processing time first). Note that it is observed, e.g. in [70], that an optimal schedule for problem $1||\sum C_j$ stays optimal under perturbation of the processing times, if and only if the SPT-order of jobs does not change. Recall also that due to Statement 68, for a given solution S^0 and a given upper bound B there exists a worst-case deviation Δ, such that all entries in Δ are equal. So we can restrict our consideration to deviations Δ with $\Delta_j = \xi$ for all j. In that case we have actual processing times

$$p'_j = p_j + \alpha_j \xi.$$

Using these two properties, we show how to identify $O(n^2)$ candidate schedules for most resilient solutions and then compute the function $\rho_\infty(B)$ as the upper envelope of $O(n^2)$ lines arising from those candidate schedules.

We first show that starting with $\xi = 0$ and increasing it continuously, the SPT-order of the actual processing times p'_j changes at most $O(n^2)$ times. Let S^0 be the SPT-schedule for the original processing time estimates p_j and let jobs be numbered in the order in which they appear in S^0. Furthermore let $B^{(0)}$ be the objective value of schedule S^0 for the original processing time estimates p_j. It is easy to see that we can additionally assume that in S^0 ties between the p_j are, where possible, broken by scheduling first the job with the smaller α_j value. Then there

is some fixed value $\bar{\xi}$ for which two given jobs $j_1 < j_2$ change their SPT-ordering for the actual processing times p'_j if and only if $p_{j_1} < p_{j_2}$ and $\alpha_{j_1} > \alpha_{j_2}$. At value $\bar{\xi}$ the processing times of jobs j_1 and j_2 are equal, i.e.

$$p_{j_1} + \alpha_{j_1}\bar{\xi} = p_{j_2} + \alpha_{j_2}\bar{\xi}$$

or

$$\bar{\xi} = \frac{p_{j_2} - p_{j_1}}{\alpha_{j_1} - \alpha_{j_2}}.$$

Since $\alpha_{j_1} > \alpha_{j_2}$ the quotient is always well-defined and since $p_{j_1} < p_{j_2}$ we have $\bar{\xi} > 0$, i.e. the order does not change for very small values $\bar{\xi}$. Observe that without breaking ties in the ordering of the p_j as described above above, we only get conditions $p_{j_1} \leq p_{j_2}$ and $\alpha_{j_1} > \alpha_{j_2}$ and $\bar{\xi} = 0$ is possible.

Conditions $p_{j_1} < p_{j_2}$ and $\alpha_{j_1} > \alpha_{j_2}$ imply that there are at most n such changes for each job, leading to a total of at most n^2 changes in the overall SPT-ordering as ξ increases. These $O(n^2)$ different SPT-orderings for the actual processing times p'_j depending on ξ are the previously mentioned candidate schedules.

In order to compute those schedules, we first compute the $O(n^2)$ values of ξ at which changes to the order of the actual processing times happen. For two jobs $j_1 < j_2$ with $p_{j_1} < p_{j_2}$ and $\alpha_{j_1} > \alpha_{j_2}$ let

$$\xi_{(j_1,j_2)} = \frac{p_{j_2} - p_{j_1}}{\alpha_{j_1} - \alpha_{j_2}}.$$

We re-order the values $\xi_{(j_1,j_2)}$ in non-decreasing order, obtaining a sequence $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(\kappa)}$, for some integer $\kappa \leq n^2$. In order to make the following arguments easier, we assume that the values $\xi^{(\iota)}$ are pairwise different from each other. At the end of the section we will discuss what happens if this is not the case and show that the same results still apply.

Denote by S^ι the SPT-schedule for the actual processing times $p_j^{(\iota)} = p_j + \alpha_j\xi^{(\iota)}$. Again, ties in the ordering of the actual processing times should be broken, where possible, by scheduling first the job with the smaller α_j value. Then clearly S^ι is an SPT-schedule for all values of actual processing times $p'_j = p_j + \alpha_j\xi$, with $\xi^{(\iota)} \leq \xi < \xi^{(\iota+1)}$.

Note that when transforming schedule S^ι into $S^{\iota+1}$ only two jobs swap their position, because the values $\xi^{(\iota)}$ are pairwise different. Moreover, these two jobs are neighboring jobs, since otherwise another job would swap positions with at least one of them, which again cannot happen because the $\xi^{(\iota)}$ are pairwise different. Therefore knowing S^ι, the schedule $S^{\iota+1}$ can be obtained in $O(1)$ time.

For each schedule S^ι we also compute value $B^{(\iota)}$ as the objective value of schedule S^ι assuming the processing times are $p_j^{(\iota)} = p_j + \alpha_j\xi^{(\iota)}$. Note that for a given schedule S^ι its objective value increases linearly in ξ with the factor

$$\lambda^{(\iota)} = \sum_{j=1}^{n} \left((n - i_j(S^\iota) + 1)\alpha_j\right),$$

where $i_j(S^\iota)$ is the position of job j in S^ι. Observe that the change from $\lambda^{(\iota)}$ to $\lambda^{(\iota+1)}$ can again be computed in $O(1)$, as only the position of two neighboring jobs is swapped. If job j_1 and j_2 with $p_{j_1} < p_{j_2}$ swap positions from schedule S^ι to schedule $S^{\iota+1}$ then

$$\lambda^{(\iota+1)} = \lambda^{(\iota)} + \alpha_{j_2} - \alpha_{j_1}.$$

Then we can compute

$$B^{(\iota+1)} = B^{(\iota)} + \lambda^{(\iota)}(\xi^{(\iota+1)} - \xi^{(\iota)}).$$

As the values $\xi^{(\iota)}$ are pairwise different, the same is true for the values $B^{(\iota)}$.

In order to save space and time, it is sufficient to only commit the values $\xi^{(\iota)}$, $B^{(\iota)}$ and $\lambda^{(\iota)}$ to memory. Each schedule S^ι can be found in $O(n \log n)$ time for a given deviation ξ, if necessary.

Lemma 79. *We can compute values $\xi^{(\iota)}$, $B^{(\iota)}$ and $\lambda^{(\iota)}$ in $O(n^2 \log n)$ time.*

Proof: Clearly computing values $\xi^{(\iota)}$ takes at most $O(n^2)$ time and subsequent sorting for n^2 values takes $O(n^2 \log n)$ time.

After finding schedule S^0 in $O(n \log n)$ time and value $B^{(0)}$ in $O(n)$ time as pre-processing steps, we can recursively obtain the schedules S^ι. Note that we do not produce schedules S^ι, instead we obtain S^ι by manipulation of the already existing sequence in $O(1)$ time for each ι. Then, for schedule S^ι we compute and save the values $B^{(\iota)}$ and $\lambda^{(\iota)}$ again in $O(1)$ time. As there are $O(n^2)$ different schedules, the process of computing all values $B^{(\iota)}$ and $\lambda^{(\iota)}$ takes $O(n^2)$ time.

The total processing time is dominated by sorting the values $\xi^{(\iota)}$, which takes $O(n^2 \log n)$ time. ∎

Note that if schedules S^ι are all saved separately, then $O(n^2)$ permutations of n numbers have to be saved, which takes at least $O(n^3)$ time and space.

We now show that we actually chose the right set of candidate schedules, i.e. that given any upper bound B there is $\iota \in \{0, 1, 2, \ldots, \kappa\}$ such that S^ι is a most resilient solution for bound B. Note that we can again assume $B \geq B^0$, such that some B-feasible solution exists.

Lemma 80. *Let an upper bound $B \geq B^0$ be given and choose ι such that $B^{(\iota)} \leq B < B^{(\iota+1)}$ or $\iota = \kappa$ if $B \geq B^{(\kappa)}$. Then S^ι is a most resilient solution for the upper bound B.*

Proof: Let $\widehat{\xi} = \rho(S^\iota, p, B)$ be the resiliency radius of S^ι for the upper bound B. Clearly, we have $\xi^{(\iota)} \leq \widehat{\xi} < \xi^{(\iota+1)}$ or $\widehat{\xi} \geq \xi^{(\kappa)}$ if $\iota = \kappa$, by definition of $B^{(\iota)}$. Assume there is a solution S^* with a larger resiliency radius $\widetilde{\xi}$.

Without loss of generality we can assume that not all α_j are 0 (otherwise there is no uncertainty). For a solution S let $\widehat{F}(S)$ be the objective value of S assuming the processing times are $\widehat{p_j} = p_j + \alpha_j \widehat{\xi}$ and define $\widetilde{F}(S)$ analogously. Then

$$\widehat{F}(S^*) < \widetilde{F}(S^*) = B = \widehat{F}(S^\iota),$$

a contradiction, as S^ι is, by definition, an optimal schedule for processing times $\widehat{p_j} = p_j + \alpha_j\widehat{\xi}$, namely the one with jobs in SPT-order. ∎

We can now calculate the function $\rho_\infty(B)$, the maximum resiliency radius dependent on B, as the piecewise linear function given by the points $(B^{(\iota)}, \xi^{(\iota)})$ with the slope $\frac{1}{\lambda^{(\iota)}}$ between points $((B^{(\iota)}, \xi^{(\iota)})$ and $(B^{(\iota+1)}, \xi^{(\iota+1)})$. It can be viewed as the upper envelope of the $O(n^2)$ lines given by the formulas

$$\xi^{(\iota)} - B^{(\iota)}\frac{1}{\lambda^{(\iota)}} + \frac{1}{\lambda^{(\iota)}}B.$$

The algorithm by [149] computes the upper envelope of k lines in $O(k \log k)$ time, leading to a total complexity of $O(n^2 \log n)$ time to compute function $\rho_\infty(B)$.

Theorem 81. *The function $\rho_\infty(B)$ which for every upper bound B calculates the maximum resiliency radius under the ℓ_∞-norm can be computed in $O(n^2 \log n)$ time as the upper envelope of $O(n^2)$ lines.*

Now return back to our assumption that the values $\xi^{(\iota)}$ are pairwise different. Note that some of the $O(1)$ computations will take more time if there are equalities amongst the $\xi^{(\iota)}$, as several computations have to be done in the same step. However, if there are k equal values $\xi^{(\iota)}, \xi^{(\iota+1)}, \ldots, \xi^{(\iota+k-1)}$, then then computing the new schedule $S^\iota = S^{\iota+1} = \ldots S^{\iota+k-1}$ and the values $B^{(\iota)} = B^{(\iota+1)} = \ldots = B^{(\iota+k-1)}$ still only involves inverting the order of $O(k)$ jobs, which can be done in $O(k)$ time. The total time taken to compute all schedules (or respectively all values $B^{(\iota)}$) is still bounded by the number of values $\xi^{(\iota)}$, which is bounded by $O(n^2)$. Thus the complexity analysis from above still holds.

The basic ideas presented in this section on how to obtain a set of candidate schedules for the most resilient solution also work for other sequencing problems with known precedence rules, such as EDD (earliest due date first) for the maximum lateness objective, Smith's rule for the weighted total sum of completion times or Johnson's rule for problem $F2||C_{\max}$. However, note that the time analysis might differ for other problems and we might not necessarily be able to obtain the function $\rho_\infty(B)$ as easily as we did here.

12.3.2 The case with binary fluctuation factors $\alpha_j \in \{0, 1\}$

In this section we show that if fluctuation factors are restricted to $\alpha_j \in \{0, 1\}$ then we can calculate the largest resiliency radius $\rho_1(B)$ under the ℓ_1-norm as the function of the bound B. Similar to the result of Theorem 81 the function ρ_1 is a piecewise linear function, given as the upper envelope of a set lines. Again, those lines are given by candidate schedules, this time $O(n)$ many, rather than $O(n^2)$.

Recall that in the case where fluctuation factors are restricted to the values 0 and 1, a job j is called uncertain if $\alpha_j = 1$ and job j is called fixed if $\alpha_j = 0$. Let $\mathcal{J}' = \{j'_1, j'_2, \ldots, j'_\nu\}$ and $\mathcal{J}'' = \{j''_1, j''_2, \ldots, j''_\mu\}$ be the subsets of uncertain and fixed jobs respectively, $\mathcal{J} = \mathcal{J}' \cup \mathcal{J}''$,

with the jobs in each set numbered in the SPT order:

$$\begin{aligned} p_{j'_1} &\leq p_{j'_2} \leq \cdots \leq p_{j'_\nu}, \\ p_{j''_1} &\leq p_{j''_2} \leq \cdots \leq p_{j''_\mu}. \end{aligned}$$

We assume that neither of the two sets are empty. Otherwise, if set $\mathcal{J}'$ is empty, all jobs are fixed and we are left with the traditional scheduling problem. If set $\mathcal{J}''$ is empty, then all jobs have fluctuation factor 1 and the most resilient solution (for any B) is the solution where jobs are in order of non-decreasing processing time estimates p_j. Furthermore, we assume that B is large enough such that a B-feasible solution exists. Otherwise the largest resiliency radius is -1 by definition.

Due to formula (12.20) and since $\alpha_j \in \{0, 1\}$, for any B-feasible schedule S there exists a worst case deviation Δ where only the processing time of the earliest uncertain job in S is increased. If the earliest uncertain job in S is at position i, then the resiliency radius of S is given by

$$\rho_1(S, p, B) = \frac{B'}{n - i + 1} = \frac{B - \sum C_j(S)}{n - i + 1}. \tag{12.23}$$

Thus it only depends on the objective value of schedule S and the position of the earliest uncertain job. It decreases, as the objective value increases, but increases, as the position of the earliest late job increases. Consequently, we obtain the following lemma.

Lemma 82. *In any B-feasible schedule S^* with the largest resiliency radius, the jobs from $\mathcal{J}'$, if considered in isolation from $\mathcal{J}''$, appear in the non-decreasing order of p_j. The analogous statement holds for the jobs from $\mathcal{J}''$.*

Proof: Suppose S is a schedule for which the statement of the lemma does not hold. Construct schedule S' where jobs in $\mathcal{J}'$ take up the same positions as in S but are in SPT order. Clearly, the objective value of S' is smaller than that of S, thus if S is B-feasible, then so is S'. Furthermore the position of the earliest uncertain job in S' is the same as in S, since the job sets $\mathcal{J}'$, and $\mathcal{J}''$ occupy the same positions in the schedule as before. Therefore, by (12.23), the resiliency radius of S' is larger than that of S, and S cannot be a most resilient solution.

Analogously, we proof the same statement for $\mathcal{J}''$. ∎

Similar to Lemma 82 we can prove the next lemma.

Lemma 83. *Let S^* be a B-feasible schedule with the largest resiliency radius, and let i be the position of the earliest uncertain job in S^*. Then the jobs in positions $1, 2, \ldots, i-1$ considered independently are in SPT-order. Similarly, the jobs in positions $i+1, i+2, \ldots, n$ considered independently are in SPT order.*

Proof: The first statement is clear, as all jobs in positions $1, 2, \ldots, i-1$ are fixed, due to the definition of i, and therefore in SPT order by Lemma 82.

Suppose the second statement does not hold and the jobs in positions $i+1, i+2, \ldots, n$ considered independently are not in SPT order. Then consider schedule S where the same set of jobs occupies positions $i+1, i+2, \ldots, n$ as in S^*, but they are in SPT order. Similar to the proof of Lemma 82 the objective value of schedule S is smaller than that of schedule S^* and the position of the earliest uncertain job remains the same. Therefore, by (12.23), schedule S has a larger resiliency radius than schedule S^*, in contradiction to the assumption that S^* is a schedule with the largest resiliency radius. ∎

From Lemmas 82 and 83 the structure of a candidate schedule for the most resilient schedule is given uniquely by the position of the first uncertain job. Since the latest possible position for the first uncertain job is $\mu+1$, if all μ fixed jobs are at the start of the schedule, there are $\mu+1 \leq n$ candidate schedules.

Theorem 84. *For norm ℓ_1, the candidate schedule $S^{(i)}$ where the earliest uncertain job is in position i, $1 \leq i \leq \mu+1$, is of the form:*

- *fixed jobs $j_1'', j_2'', \ldots, j_{i-1}'' \in \mathcal{J}''$ are scheduled in SPT order in positions $1, 2, \ldots, i-1$,*
- *the shortest uncertain job j_1' is inserted in position i,*
- *the remaining jobs $\{j_2', j_3', \ldots, j_\nu'\} \cup \{j_i'', j_{i+1}'', \ldots, j_\mu''\}$, certain and uncertain, are scheduled in SPT order in positions $i+1, i+2, \ldots, n$.*

Proof: The theorem follows immediately from Lemmas 82 and 83. ∎

Therefore, for norm ℓ_1, there are at most n different job sequences $S^{(1)}, S^{(2)}, \ldots, S^{(\mu+1)}$, each of which corresponds to a certain insertion position of j_1'. In fact, with the same arguments we used in the proofs of the two lemmas we can see that there is no point in inserting job j_1' before its place in the overall SPT order, as it does not lead to a most resilient schedule. Therefore, assuming jobs are numbered in order of non-decreasing processing times, we only need to consider insertion positions i with $j_1' \leq i \leq \mu+1$ and the corresponding schedules $S^{(j_1')}, S^{(j_1'+1)}, \ldots, S^{(\mu+1)}$.

For schedule $S^{(i)}$, given a vector p of fixed processing time estimates, the function $\rho_1(S^{(i)}, p, B)$ is a linear function given by

$$\rho_1(S^{(i)}, p, B) = -\frac{\sum C_j(S^{(i)})}{n-i+1} + \frac{1}{n-i+1}B,$$

if $B \geq \sum C_j(S^{(i)})$. Define for all $B \geq 0$ the linear function

$$\rho_{S^{(i)}}(B) = -\frac{\sum C_j(S^{(i)})}{n-i+1} + \frac{1}{n-i+1}B,$$

the linear extension of $\rho_1(S^{(i)}, p, B)$.

Lemma 85. *The set of functions $\rho_{S^{(i)}}(B)$ for all schedules $S^{(i)}$ can be computed in $O(n)$ time.*

Proof: Since function $\rho_{S^{(i)}}(B)$ is given by the position index i and objective value $\sum C_j(S^{(i)})$, it is sufficient to show that all objective values $\sum C_j(S^{(i)})$ can be computed in $O(n)$ time.

Note that schedule $S^{(j'_1)}$ can be computed in $O(n \log n)$ time and $\sum C_j(S^{(j'_1)})$ can be computed in $O(n)$ time. Schedule $S^{(i+1)}$ can be obtained from schedule $S^{(i)}$ by moving the i-th fixed job j''_i from position j''_i to position i. All jobs between position i and $j''_i - 1$, which are the first $j''_i - i$ uncertain jobs jobs $j'_1, j'_2, \ldots, j'_{j''_i - i}$, are moved one position to the back of the schedule.

Therefore, we can recursively obtain objective value $\sum C_j(S^{(i+1)})$ from $\sum C_j(S^{(i)})$ using

$$\sum C_j(S^{(i+1)}) = \sum C_j(S^{(i)}) + (n - i + 1 - (n - j''_i + 1))p_{j''_i} - \sum_{k=1}^{j_{i''}-i} p_{j'_k}.$$

If in each step we remember the current value of the term

$$\sum_{k=1}^{j_{i''}-i} p_{j'_k},$$

then each uncertain job only needs to be added once to the term, while computing the objective value of all schedules $S^{(i)}$. Furthermore each fixed job is moved at most once while computing the objective value for all schedules $S^{(i)}$. Therefore, recursively computing the objective values for all schedules $S^{(i)}$ takes at most $O(\mu + \nu) = O(n)$ time. ∎

The complete function $\rho_1(B)$, which for any given B such that at least one B-feasible solution exists, calculates the maximum resiliency radius, is then given by the upper envelope of the $O(n)$ lines $\rho_{S^{(i)}}(B)$. If no B-feasible solution exists, the maximum resiliency radius is -1. Since the upper envelope of k functions can be found in $O(k \log k)$ time by the algorithm due to [149], the problem of computing $\rho_1(B)$ is solvable in $O(n \log n)$ time.

We summarize the main results of this section in the theorem below.

Theorem 86. *If fluctuation factors α_j are restricted to $\alpha_j \in \{0, 1\}$, then the resiliency version of problem $1||\sum C_j$ under the ℓ_1-norm is solvable in $O(n \log n)$ time.*

Furthermore, the function $\rho_1(B)$ which, for given processing time estimates p, calculates the maximum resiliency radius under the ℓ_1-norm as a function of the bound B can be computed in $O(n \log n)$ time, as the upper envelope of $O(n)$ lines.

12.3.3 Problem $P||\sum C_j$

Note that in order to generalise the results from Sections 12.3.1 and 12.3.2 to the problem with m machines, we have to adjust the positional factors in the definition of the associated assignment problem. Instead of $w_{ij} = (n - i + 1)p_j$ we define

$$w_{ij} = \left\lceil \frac{n - i + 1}{m} \right\rceil p_j.$$

Consequently, in the definitions for the version with uncertainty, with fluctuation factors for α_j for processing time estimates p_j and a given deviation Δ, the actual weights are

$$\begin{aligned} w'_{ij} &= \left\lceil \frac{n-i+1}{m} \right\rceil (p_j + \alpha_j \Delta_j) \\ &= \left\lceil \frac{n-i+1}{m} \right\rceil p_j + \left\lceil \frac{n-i+1}{m} \right\rceil \alpha_j \Delta_j \\ &= w_{ij} + \left\lceil \frac{n-i+1}{m} \right\rceil \alpha_j \Delta_j. \end{aligned}$$

Thus, we define the fluctuation factors for the associated assignment problem as

$$\alpha_{ij} = \left\lceil \frac{n-i+1}{m} \right\rceil \alpha_j.$$

Note that problem $P||\sum C_j$ can still be treated as a sequencing problem, using the following transformation from a sequence of jobs to a schedule. Given a sequence of jobs ϕ, suppose the position of job j is $\phi(j) = km + r$ for $1 \leq r \leq m$. Then we schedule job j as the $k+1$-st job on machine r. The transformation from a schedule back to a sequences is done analogously. In that case, scheduling jobs in the SPT-order is still an optimal solution for the traditional problem.

It is easy to see that the same arguments as in Sections 12.3.1 and 12.3.2. work to generalise the results from those sections.

Chapter 13

Conclusions and Further research

In this chapter we present conclusions and suggest some directions for future research. We also point out some challenges that have not been mentioned explicitly before. Since open questions for the specific problems we investigated have already been pointed out in the appropriate sections of Chapter 12, we do not repeat them here. Instead, most of our remarks are either about generalization of the obtained results or general insights in the new concept of resiliency.

Conclusions about resiliency in general and some challenges are discussed in Section 13.1. In Section 13.2 we deal with the possibility of generalizing some of our results, especially for scheduling. Finally, Section 13.3 is about some broader ideas for possible future research.

13.1 Advantages and challenges of the resiliency concept

Our general results and the two problems we studied more closely indicate strongly that resiliency is a viable concept for the measurement of solution quality under uncertainty. Note that resiliency allows to measure the quality of any given solution via the resiliency radius, as well as algorithmic search for a "best" solution via the concept of most resilient solutions. In this way, resiliency lies in between stability analysis and robust discrete optimization, with connections to both of these concepts. Considering mostly the ℓ_1 and ℓ_∞-norm, we achieved positive results for both problems we considered, computing the resiliency radius of a given solution and finding most resilient solutions.

Compared to stability, the resiliency radius for the two problems under investigation is no harder to compute than the stability radius of a solution. This remains true, even if we allow for different input parameters to have different scale of uncertainty, i.e. for arbitrary fluctuation factors. Additionally, we can compute the resiliency radius of any given solution S^0, not necessarily optimal or ε-optimal. Furthermore, it appears that the resiliency radius

may well be computable in a similar manner for problems that are not polynomially solvable (how to do so for the TSP problem was explained in Section 12.2.1). As we pointed out in Section 11.2, usually computing the stability radius of a solution is at least as hard as solving the original problem [125, 148].

Compared to robustness, our examples suggest that positive complexity results seem very much possible for the resiliency versions of polynomially solvable combinatorial optimization problems. Recall that for robustness many problems other than interval min-max versions seem to be computationally hard [4].

However, there are also several challenges. We detail two possible disadvantages of resiliency here. The first is the question of practical application. While we have given good theoretical indication that resiliency is a useful concept, and it appears easy to apply in practice, the practical usefulness remains to be seen. The applications for stability and sensitivity analysis (see, e.g. [142]), are helpful to see some practical use for resiliency radii, but especially the practical interest of a most resilient solution has yet to be studied.

One facet of this, which resiliency has in common with stability and robustness, is the reliance on worst case deviations/scenarios in order to determine the quality of a solution. For resiliency worst case deviations often take very specific forms, where all entries in the deviation vector are equal, or a deviation vector has only one non-zero entry.

In many practical applications this is questionable, as the worst case deviation/scenario might only happen with a very low probability, as has been observed for robustness in [4]. There are several possible solutions mentioned in [4], which may help to refine the definition of resiliency to become more practical. In Section 13.3 we suggest several ideas how to deal with this issue.

Another challenge arises because the definition of resiliency implicitly assumes that a solution S for input parameters η still "makes sense" for the perturbed input parameters $\eta + \Delta$. This assumption is appropriate for the assignment problem and similar problems, where solutions can be represented as vectors x with 0/1-entries, the input parameters are weights and the objective function is of the form $\sum_{j=1}^{n} \eta_j x_j$ or $\max_{j=1}^{n} \eta_j x_j$. It is also appropriate for scheduling problems where a schedule is uniquely specified by a sequence of jobs on each machine, such as problems $1||f$ with f a traditional scheduling objective or $F2||C_{\max}$.

On the other hand it is unclear how to handle more complicated problems, where a solution cannot be represented in a simple combinatorial structure. In scheduling, for example, preemption represents such a challenge. If the length of an operation increases due to perturbation, it is unclear which piece of the operation, if preempted, should be lengthened. Another example are scheduling objectives with an earliness component, where an optimal schedule may have idle times and a simple representation as a sequence of jobs is no longer sufficient. Here, as the processing time of an operation becomes longer, it may be unclear if the additional processing load should be scheduled in front of the original operation (if there is idle time somewhere in front of the operation) or after the original operation (pushing other operations back) or both.

If problems of such a type should be considered in the future, either the definition of resiliency has to be refined, or these and possibly other questions need to be part of the research. For example, in the case of preemption it may be appropriate in an applied scenario to declare rules on which job pieces to change in case of deviation from the predicted processing time values. Different rules may lead to different resiliency radii and the search for a best set of rules may be an interesting challenge.

It may also happen for some problems that the solution S simply becomes infeasible for the perturbed input data, i.e. is no longer part of the solution set. However, this is less of an issue, since we can simply define the weight of an infeasible solution to be equal to ∞ and thus still measure the resiliency radius of S.

All in all, despite these two and more challenges that might emerge along the way, we believe that resiliency is an exciting and useful concept for further study. It lies in between two very well established concepts in combinatorial optimization and unites advantages of both areas. We believe there is the potential for many interesting results.

13.2 Generalization and transfer of our results to other problems

We consider first possible extensions of the results for the assignment problem to general $0-1$ combinatorial optimization problems (defined below). After that we turn to extensions of the additional results obtained for problem $1||\sum C_j$ that are not reliant on those for the assignment problem.

13.2.1 Resiliency for $0-1$ combinatorial optimization problems

It was mentioned previously that the assignment problem is a special case of a combinatorial optimization problem where the input parameters η_j are weights, a solution can be represented as a vector x with 0/1-entries and the objective is to minimize a function $f = \sum_{j=1}^{n} \eta_j x_j$. These problems are often times referred to as $0-1$ combinatorial optimization problems, see, e.g., [4, 28].

Many well-known and important problems can be described in that way, such as shortest path, minimum spanning tree and minimum weight perfect matching on a given fixed graph G. It seems very likely that the results we achieved for the assignment problem, both for computing resiliency radii and for finding most resilient solutions, can be generalized to all problems of this type, or at least a subclass.

I order to compute resiliency radii for $0-1$ combinatorial optimization problems, note that we can reproduce formulation KP that was used to compute a worst case deviation for the assignment problem (see Section 12.2.1). Given a solution vector x^0 define solution $S^0 = \{j|x_j = 1\}$, similar to the notation for the assignment problem. Then a worst case deviation of

S^0 can be computed by formulation

$$\begin{aligned} \text{KP}^*\text{:} \quad \min \quad & \sum_{j \in S^0} (\Delta_j)^q \\ \text{s.t.} \quad & \sum_{j \in S^0} \alpha_j \Delta_j \geq B', \\ & \Delta_j \geq 0, \qquad\qquad j = 1, 2, \ldots, n. \end{aligned}$$

Again B' is given by the difference of the given bound B and the original objective value of the solution given by vector x, i.e. $B' = B - \sum_{j \in S^0} \eta_j x_j$.

The result from Lemma 69 can be extended as well. Thus, it should be possible to derive formulas of a similar type as we did for the assignment problem.

For finding most resilient solutions, at least the result from Theorem 71 appears generalizable. Using a similar approach, we solve the original problem $O(n)$ times, where n is the number of input parameters. This way, we obtain $O(n)$ candidate solutions and pick the one with the largest resiliency radius. The total process takes $O(nT(n))$ time, where $T(n)$ is the time it takes to solve the original problem. Note that for the assignment problem we used the usual notation of a weight matrix (rather than a vector with n^2 entries) and therefore the number of input parameters is n^2.

Details for the generalization of these and other results still have to be worked out. Similarly, it would be interesting to see if the approach used for the bottleneck assignment problem can be generalized to the bottleneck version of a general $0-1$ combinatorial optimization problem, where the objective is to minimize function $\max_{j=1}^{n} \eta_j x_j$ instead of $\sum_{j=1}^{n} \eta_j x_j$.

13.2.2 Resiliency for list scheduling problems

In sections 12.3.1 and 12.3.2 we obtain results for the resiliency version of problem $1||\sum C_j$, based on the knowledge that an optimal schedule is given by the SPT-order. A scheduling problem which can be solved by sequencing jobs in a known way is sometimes called a list scheduling problem, see, e.g., [70]. Prominent examples are problems $1||L_{\max}$, $1||\sum w_j C_j$ and $F2||C_{\max}$.

We solve the resiliency version of problem $1||\sum C_j$ under the ℓ_∞ norm by obtaining $O(n^2)$ candidate schedules, one for each time the SPT-order changes. Then for a given bound B we pick the most resilient of these $O(n^2)$ schedules.

A similar approach should work to find most resilient solutions for other list scheduling problems. For example we would obtain one candidate schedule for each possible order of due dates for problem $1||L_{\max}$. Again, details have to be worked out, but seem rather straight forward.

However, finding the function $\rho_\infty(B)$ (see Theorem 81) for an arbitrary list scheduling problem can be more tricky. Note that for different objective functions the schedules do not necessarily give rise to a linear function, as they did for problem $1||\sum C_j$. Indeed, for problem

$1||L_{\max}$ the function is may only be piecewise linear as the objective value of a given schedule stays the same for certain deviations (if only one job is very late, small changes for other jobs do not affect the maximum lateness). This issue has to be investigated further.

It is also unclear how to compute the resiliency radius for any given list scheduling problem. The method from the assignment problem we used to solve this issue for problem $1||\sum C_j$ cannot be immediately extended to other list scheduling problems. Note that neither $1||L_{\max}$ nor $1||\sum w_j C_j$ can be modelled as the assignment problem.

13.2.3 Resiliency for other scheduling problems

Naturally, it would also be interesting to see which of our results, if any, extend to other scheduling problems, not necessarily of list scheduling type. Most interesting amongst those might be problem $1||\sum U_j$, the first general single machine problem not of list scheduling type, and $O2||C_{\max}$, which is prominently featured in other parts of this thesis. Note however, that computing resiliency radii might also be of interest and solvable for NP-hard single machine problems $1||\sum T_j$, $1||\sum w_j T_j$ and $1||\sum w_j U_j$.

Unfortunately none of the results we achieved seem easily generalizable to other scheduling problems, not of list scheduling type. For problem $O2||C_{\max}$ the resiliency radius of a given schedule may be easily computable for norm ℓ_1, where it seems likely that the length of only one operation needs to be increased for a worst case deviation. It might be sufficient to increase on each machine the length of the operation with the largest fluctuation factor and then pick whichever of the two deviations leads to a violation of the bound B earlier. This approach seems to work for schedules without idle times (other than in the end of the schedule). Schedules with idle times are most likely more problematic. Note that idle times in the middle of the schedule may arise in spite of the objective being regular.

For problem $1||\sum U_j$ and the ℓ_1-norm the situation seems to be more complicated. It is easy to come up with examples where more than one processing time or due date has to be changed in a worst case deviation. Indeed, consider the example with two jobs and input data

j	p_j	$\alpha_j^{(p)}$	d_j	$\alpha_j^{(d)}$
1	2	1	3	0
2	2	3	8	0

,

where $\alpha_j^{(p)}$ is the fluctuation factor of the processing time of job j and $\alpha_j^{(d)}$ is the fluctuation factor for the due date of job j. Consider solution S^0 with job sequence $(1, 2)$ and upper bound $B = 1$ on the objective. Due to the piecewise constant objective function, when the upper bound B is reached there is still some leeway for deviation before it is violated.

In order to violate bound B both jobs have to be late. Clearly, the only way for job 1 to be late is to increase its processing time by 1. However, for the second job it is better to increase its own processing time, due to the larger fluctuation factor. Thus, in a worst case deviation,

both processing times need to be increased. Similarly, note that if only due dates are subject to uncertainty, then clearly both due dates need to be decreased in order to make both jobs late.

Turning to norm ℓ_∞, recall that all entries in a worst case deviation vector are equal for problem $O2||C_{\max}$ due Statement 68. For problem $1||\sum U_j$, it is easy to generalize Statement 68 in order to see that the entries in the deviation vector are of equal absolute value, positive for the processing times and negative for the due dates. For both problems, $1||\sum U_j$ and $O2||C_{\max}$, a way to compute the resiliency radius of a given schedule may be found by careful analysis of such deviations vectors. Here, it needs to be investigated how the value of a solution changes when all processing times increase simultaneously (and for $1||\sum U_j$ due dates decrease simultaneously).

13.3 Future research – a broader view

In this section we suggest some directions for further research that are not directly related to the results we achieved previously. We focus on two areas, one an extension of the definition for resiliency in order to deal with some of the challenges discussed in Section 13.1, the other deals with the application of resiliency to linear programming.

13.3.1 Interval based resiliency

As was observed in Section 13.1, one major drawback of the resiliency concept is the reliance on worst case deviations which may not be very likely to happen. One way of dealing with this problem is to give not only an estimate η_j and a fluctuation factor α_j for each input parameter, but also an interval of likely deviations $\mathcal{I}_j = \left[\underline{\eta}_j, \overline{\eta}_j\right]$, similar to the definitions in interval min-max and min-max regret problems [4]. Then all entries Δ_j in the deviation vector must be such that $\eta_j + \alpha_j \Delta_j \in \mathcal{I}_j$. Clearly, this means

$$\Delta_j \in \left[\frac{\underline{\eta}_j - \eta_j}{\alpha_j}, \frac{\overline{\eta}_j - \eta_j}{\alpha_j}\right].$$

The intervals may be chosen such that no deviations outside of the intervals are possible. If a certain amount of risk is acceptable, the intervals can also be chosen tighter, such that deviations outside the interval are possible, but only with a small chance.

Observe that for the results we achieved up until now, it is necessary that the deviation vector is not restricted in such a way. For small values of bound B many results may still hold, as the deviations do not become very large, however, for larger values of B this is no longer true. What is more, it is no longer the case that for the ℓ_1-norm it is often sufficient to consider deviations with only one non-zero entry, since the allowed range of deviation may not be large enough.

Similarly, for the ℓ_∞ norm, not all entries in a deviation vector are equal. However, Statement 68 appears to be generalizable in the following way. Given a largest deviation ξ, a worst

case deviation vector is of the form $\Delta_j = \min\{\xi, \overline{\Delta}_j\}$ for objective functions non-decreasing in the input parameters, where

$$\overline{\Delta}_j = \frac{\overline{\eta}_j - \eta_j}{\alpha_j}.$$

Generalizations for objective function non-increasing in the input parameters are possible in an analogous way.

In order to compute resiliency radii for the ℓ_∞-norm and an objective function non-decreasing in the input parameters, it may be possible to proceed in the following way. First, increase all entries of the deviation vector to the smallest value $\overline{\Delta}_j$. If upper bound B is violated, then a worst case deviation can be computed ignoring the intervals. Otherwise, continue and increase all entries in the deviation vector that still can be increased to the next smallest value $\overline{\Delta}_j$. Those which cannot be increased are fixed at the previous value $\overline{\Delta}_j$. Again, test if the upper bound B is violated. If yes, compute a worst case deviation with some values fixed at the smallest $\overline{\Delta}_j$ and all other entries of the deviation vector all equal to some ξ between the smallest and the second smallest value of $\overline{\Delta}_j$. If the upper bound B is still not violated, continue in the same way.

Such an approach should work for the assignment problem with arbitrary fluctuation factors and thus also for the associated scheduling problems. For the assignment problem this suggests that $O(n \log n)$ time is needed to compute the resiliency radius, due to the sorting of values $\overline{\Delta}_j$. More careful analysis is needed to verify this result.

For other problems computing the resiliency radius may not be as straight forward, depending on how difficult it is to compute the resiliency radius under the ℓ_∞-norm without additional intervals.

13.3.2 Resiliency for linear programming

A lot of research in stability and sensitivity has been done for linear and integer linear programming, see, e.g., [66]. Robust mathematical programming, which in the definition of scenarios is very similar to robust combinatorial optimization as we introduced above, is also an important and prolific, albeit more recent field [14]. It seems natural to consider resiliency for linear programming as well.

Consider a linear program

$$\begin{array}{lllll} \min & cx & & \\ \text{s.t.} & Ax & \geq & b, \\ & x & \geq & 0, \end{array}$$

for a constraint matrix A, a cost vector c and a right-hand side b. In order to be able to deal with infeasible solutions, define the objective value of an infeasible solution to be equal to ∞. Given a bound $B < \infty$ this means that infeasible solutions are not B-feasible.

We first deal with the case where only the cost vector c is subject to uncertainty. Given fluctuation factors α_j for cost component c_j, upper bound B and a B-feasible solution x^0, we

can compute a worst case deviation of x^0 under the ℓ_q-norm via formulation

$$\begin{array}{lll} \mathrm{F}_0: & \min & \sum_{j=1}^n \Delta_j^q \\ & \text{s.t.} & \sum_{j=1}^n \left((\alpha_j x_j^0)\Delta_j\right) \geq B', \\ & & \Delta_j \geq 0. \end{array}$$

Here, similar to formulation KP from Section 12.2.1, $B' = B - \sum_{j=1}^n c_j x_j^0$. Note that due to $x_j^0 \geq 0$ by definition of the original linear program, we can assume $\Delta_j \geq 0$ without loss of generality. This also means that for norm ℓ_∞ again a worst case deviation exists where all entries in the deviation vector are equal. Defining new fluctuation factors $\alpha'_j = \alpha_j x_j^0$ we should be able to adjust the formulas from Section 12.2.1 in order to compute the worst case deviation and resiliency radius of solution x^0.

Finding a most resilient solution on the other hand is not as straight forward. An approach similar to the one in Section 12.2.1 for the ℓ_1-norm does not seem promising, as the x_j are continuous variables, rather than boolean. Currently we do not know what approaches may be promising.

Now consider the case where the constraint matrix $A = (a_{ij})$ and the right-hand side b may be uncertain. Assume there are m constraints, i.e. $A \in \mathbb{R}^{m \times n}$, and let α_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n+1$, be the fluctuation factors such that the actual values of the parameters given deviations Δ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are

$$\begin{array}{lcll} a'_{ij} & = & a_{ij} + \alpha_{ij}\Delta_{ij}\,, & i = 1, \ldots, m, \qquad j = 1, \ldots, n, \\ b'_i & = & b_i + \alpha_{i(n+1)}\Delta_{i(n+1)}\,, & i = 1, \ldots, m. \end{array}$$

Then given a B-feasible solution x_0, we can obtain a worst case deviation under the ℓ_q norm as the smallest of m deviations, given by formulations F_i, $i = 1, \ldots, m$, with

$$\begin{array}{lll} \mathrm{F}_i: & \min & \sum_{j=1}^{n+1} \Delta_{ij}^q \\ & \text{s.t.} & \sum_{j=1}^n \left(a_{ij}x_j^0 - (\alpha_{ij}x_j^0)\Delta_{ij}\right) \leq b_i + \alpha_{i(n+1)}\Delta_{i(n+1)}\,, \\ & & \Delta_{ij} \geq 0. \end{array}$$

Note that there exists a worst case deviation matrix where all values Δ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are non-positive and entries $\Delta_{i(n+1)}$, $i = 1, \ldots, m$, are non-negative. Therefore by inverting the sign in the term

$$\sum_{j=1}^n \left(a_{ij}x_j^0 - (\alpha_{ij}x_j^0)\Delta_{ij}\right)$$

we can again assume $\Delta_{ij} \geq 0$ without loss of generality to make the mathematical programming

formulation easier to handle. Using $b_i^{\text{red}} = b_i - \sum_{j=1}^{n} a_{ij}x_j^0$ we can simplify formulation F_i to

$$\begin{array}{lll} \mathrm{F}_i': & \min & \sum_{j=1}^{n+1} \Delta_{ij}^q \\ & \text{s.t.} & -\alpha_{i(n+1)}\Delta_{i(n+1)} - \sum_{j=1}^{n} \left((\alpha_{ij}x_j^0)\Delta_{ij}\right) \leq b_i^{\text{red}}, \\ & & \Delta_{ij} \geq 0. \end{array}$$

Since in a feasible solution $Ax \geq b$ and therefore b_i^{red} is negative, we can invert all signs and obtain

$$\begin{array}{lll} \mathrm{F}_i'': & \min & \sum_{j=1}^{n+1} \Delta_{ij}^q \\ & \text{s.t.} & \alpha_{i(n+1)}\Delta_{i(n+1)} + \sum_{j=1}^{n} \left((\alpha_{ij}x_j^0)\Delta_{ij}\right) \geq -b_i^{\text{red}}, \\ & & \Delta_{ij} \geq 0. \end{array}$$

Again the formulation is similar to formulation KP from Section 12.2.1, if we introduce new fluctuation factors

$$\alpha'_{ij} = \begin{cases} \alpha_{ij}x_j^0 & \text{if } j \leq n, \\ \alpha_{ij} & \text{if } j = n+1. \end{cases}$$

The formulas derived in Section 12.2.1 should be adjustable to compute a worst case deviation, keeping in mind that in the end the sign for all Δ_{ij} with $j \leq n$ has to be once more inverted to obtain a non-positive entry. We obtain m candidate deviations, one for each constraint, and pick the one with the smallest norm as the worst case deviation for solution x^0.

Note that optimal solutions usually have resiliency radius 0 as they meet at least one constraint with equality. Thus most resilient solutions in this case are very likely not optimal, unless the bound B is chosen as the optimal objective value for the original linear program. Again, it is open how one would approach finding most resilient solutions.

It is easy to see that if both the cost vector and the constraints are subject to uncertainty, then a worst case deviation can be found as the smallest of $m + 1$ deviations, one for each constraint and one for the objective function given by formulations F_i, $i = 0, 1, \ldots, m$.

Finishing this section, note that details of the above ideas have to be checked more carefully, although they appear to be valid. Further extensions of resiliency to mathematical programming, such as quadratic programming, should be considered. Interestingly, the same formulations as above seem to work for integer linear programs, as the entries of solution x^0 only appear as fixed parts of the fluctuation factors, not as variables. This suggests extending resiliency radii to areas of mathematical programming that are usually hard to solve might also be of interest in order to obtain an easier to compute measure for solution quality under uncertainty (e.g. for solutions obtained via heuristics).

Chapter 14

Final remarks

In this thesis we studied three new extensions of scheduling models with practical and theoretical relevance. Technical conclusions were already drawn in the appropriate parts of the thesis, so below we only summarize what are, in our opinion, the most key results of the work presented on these pages.

Synchronization for open shop problems was studied for the first time in this thesis. We established crucial links of the synchronous open shop problem to a problem arising from scheduling satellite communications and to the max-weight edge coloring problem. After that we showed that all three problems can be modelled as assignment problems with Monge like cost arrays, a link which to the best of our knowledge has not previously been observed in the literature. Using that link, we proved that problem $Om|snymv|C_{\max}$ is solvable in linear time for each fixed m (after pre-sorting the jobs) and also obtained a new class of polynomially solvable special cases for the problem of scheduling satellite communication and for the max-weight edge coloring problem. Furthermore, we showed that problem $O2|synmv|f$ is strongly NP-hard for each other traditional scheduling objective f and thus provided a complete complexity study for synchronous open shop.

We also introduced the new concept of pliability for flow shop and open shop problems, to model situations where parts of the work are not tied to a specific machine and there is some flexibility of assigning processing load to the machines. In the most general case, the pliability model is as hard to handle as traditional flow shop and open shop problems. However, we showed that there are many solvable special cases, which are useful both on their own and as a basis for future study of heuristics and more complicated versions of the pliability model.

Finally, we proposed the new concept of solution resiliency for combinatorial optimization problems with uncertain input data. Resiliency is a natural way of measuring the sturdiness of a solution against perturbations in the input data. It is closely related to the famous fields of sensitivity and stability analysis, but to the best of our knowledge has not previously received any attention. We provided the necessary definitions and showed how the new concept ties in

with previous research. Following that we presented a number of preliminary positive results both for the assignment problem and for scheduling problem $1||\sum C_j$ to show the advantages of resiliency compared to related concepts. It was also discussed how to generalize these results to other problems including linear programming. We hope that we could convince the reader that the concept of resiliency is as a whole a very interesting one and deserves further study.

At the very end, we want to thank the reader for their attention and interest. We hope that you found what you were looking for when opening this thesis.

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