Day convolution for monoidal bicategories

Alexander S. Corner
School of Mathematics and Statistics
University of Sheffield

A thesis submitted for the degree of
Doctor of Philosophy

August 2016

Abstract

Ends and coends, as described in [Kel05], can be described as objects which are universal amongst extranatural transformations [EK66b]. We describe a categorification of this idea, extrapseudonatural transformations, in such a way that bicodescent objects are the objects which are universal amongst such transformations. We recast familiar results about coends in this new setting, providing analogous results for bicodescent objects. In particular we prove a Fubini theorem for bicodescent objects.

The free cocompletion of a category $\mathcal{C}$ is given by its category of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$. If $\mathcal{C}$ is also monoidal then its category of presheaves can be provided with a monoidal structure via the convolution product of Day [Day70]. This monoidal structure describes $[\mathcal{C}^{op}, \mathbf{Set}]$ as the free *monoidal* cocompletion of $\mathcal{C}$. Day's more general statement, in the $\mathcal{V}$-enriched setting, is that if $\mathcal{C}$ is a promonoidal $\mathcal{V}$-category then $[\mathcal{C}^{op}, \mathcal{V}]$ possesses a monoidal structure via the convolution product. We define promonoidal bicategories and go on to show that if $\mathcal{A}$ is a promonoidal bicategory then the bicategory of pseudofunctors $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ is a monoidal bicategory.

Acknowledgements

First I would like to thank my supervisor Nick Gurski, who has helped guide me from the definition of a category all the way into the wonderful, and often confusing, world of higher category theory. Nick has been a great supervisor and a great friend to have had through all of this who has introduced me to many new things, mathematical and otherwise.

I have learnt a lot of things from many people, both those in Sheffield and those I've had the pleasure of spending time with at conferences. The category theory group in Sheffield helped me find my way through this project, in particular Tom Athorne, Jonathan Elliott, Ben Fuller, Rhiannon Griffiths, Ed Prior, and Simon Willerton. I should also thank the many visitors to Sheffield, including John Bourke, Lukas Buhné, Benjamín Alarcón Heredia, and Daniel Schäppi. Emily Riehl and the participants of the Kan Extension Seminar certainly deserve a place here too.

Special thanks must go to Thomas Cottrell, for much needed guidance and numerous conversations on so many topics, as well as Eugenia Cheng for inspiring me to learn about category theory in the first place.

I would also like to thank my parents, Dale and Dawn, my sister, Sophie, and all the rest of my family for being so supportive in everything over the last few years. My friends, new and old, in Sheffield and other far flung places likewise deserve my thanks; especially Ned.

This work was supported by an EPRSC studentship.

Contents

Contents v

Introduction 1

1 Background Theory 7
- 1.1 Ends and Coends . . . 7
- 1.2 Free Cocompletion . . . 10
- 1.3 Promonoidal Categories . . . 12
- 1.4 Day convolution . . . 14
- 1.5 Bicategories and Bicolimits . . . 16
- 1.6 Monoidal Bicategories . . . 20

2 Extrapseudonaturality and Bicodescent Objects 25
- 2.1 Extrapseudonatural Transformations . . . 25
- 2.2 Composition Lemmas . . . 30
- 2.3 Bicodescent Objects . . . 35
- 2.4 Fubini for codescent objects . . . 48
- 2.5 Example . . . 54

3 Free Cocompletions for Bicategories 57
- 3.1 Copowers in bicategories . . . 57
- 3.2 Bicategorical cocompletion . . . 58

4 Promonoidal Bicategories 65
- 4.1 Promonoidal Bicategories . . . 65
- 4.2 Monoidal bicategories as promonoidal bicategories . . . 72

5 Day Convolution for Monoidal Bicategories 77
- 5.1 Day Convolution . . . 77
- 5.2 Monoidal bicocompletion . . . 84
- 5.3 Biclosed structure on presheaves . . . 86
- 5.4 Further Remarks . . . 87

Appendix: String Diagrams 89

References 97

Introduction

The notion of monoidal category, originally defined by Mac Lane [ML63] as a category with a multiplication, abstracts the idea of a tensor product to a structure borne by a category. Familiar examples are the tensor product of vector spaces on the category $\mathbf{Vect}_k$, as well the cartesian product for any category with finite products, and disjoint union on **Set**. A monoidal category consists of a category $\mathcal{C}$ with functors $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, $I\colon \mathbf{1}\to\mathcal{C}$, and natural isomorphisms

$$\begin{aligned}\alpha_{abc}\colon (a\otimes b)\otimes c &\to a\otimes(b\otimes c),\\ \lambda_a\colon I\otimes a &\to a,\\ \rho_a\colon a\otimes I &\to a.\end{aligned}$$

The functor I serves to specify a unit object in $\mathcal{C}$, also denoted I. These isomorphisms are required to be coherent in the sense that the following diagrams commute in $\mathcal{C}$.

$$\begin{tikzcd}
& (a\otimes(b\otimes c))\otimes d \arrow[rr, "\alpha_{a(b\otimes c)d}"] & & a\otimes((b\otimes c)\otimes d) \arrow[dr, "1\times\alpha_{bcd}"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\alpha_{abc}\times 1"] \arrow[drr, "\alpha_{(a\otimes b)cd}"] & & & & a\otimes(b\otimes(c\otimes d)) \\
& & (a\otimes b)\otimes(c\otimes d) \arrow[urr, "\alpha_{ab(c\otimes d)}"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
(a\otimes I)\otimes b \arrow[rr, "\alpha_{aIb}"] \arrow[dr, "\rho_a 1"'] & & a\otimes(I\otimes b) \arrow[dl, "1\lambda_b"] \\
& a\otimes b &
\end{tikzcd}$$

Mac Lane's original definition included two more unit axioms involving the associations $(Ia)b$ and $(ab)I$. Kelly [Kel64] went on to show that these further axioms follow from the two written above.

Mac Lane followed up his definition with a proof of coherence for monoidal categories, though a more standard reference is the instance in [ML98]. The version presented was of the form 'every diagram of coherence cells commutes'. This means that a composite of instances of α, λ, and ρ between any two associations are in fact equal. Coherence theorems often take a different form saying that a weak instance of a structure is equivalent to a stricter version. In the case of monoidal categories it was shown by Joyal and Street [JS93] that every monoidal category, as above, is monoidally equivalent to a monoidal category in which all of the coherence isomorphisms are identities.

Day's promonoidal categories [Day70], [Day71] are obtained when we take the definition of a monoidal category and replace every instance of a functor with a profunctor. For example, rather than a tensor product $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ there would be a profunctor $P\colon \mathcal{C}^{op}\times\mathcal{C}\times\mathcal{C}\to\mathcal{V}$. Similarly, instead of a functor $I\colon \mathbf{1}\to\mathcal{C}$ there would be a profunctor $J\colon \mathcal{C}^{op}\to\mathcal{V}$. Each monoidal structure on a category provides a promonoidal structure for that same category. Day's definition of promonoidal category was in order to define a well-behaved monoidal structure on the presheaves of a category. If $\mathcal{C}$ is a promonoidal category then it extends to a monoidal structure on presheaves in $\widehat{\mathcal{C}} = \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{C}^{op},\mathcal{V})$. Furthermore, if $\mathcal{C}$ is a symmetric monoidal category then the resulting monoidal structure on $\widehat{\mathcal{C}}$ from the convolution product is also symmetric. Any promonoidal structure on $\mathcal{C}$ then extends to a biclosed monoidal structure on presheaves $\widehat{\mathcal{C}}$.

The definition of promonoidal category uses profunctors, the composition of which is defined using coends [Kel05], [ML98]. Coends are also used to define the convolution product on presheaves. These are colimits associated with functors $F\colon \mathcal{C}^{op}\times\mathcal{C}\to\mathcal{V}$, where $\mathcal{V}$ denotes a suitable enriching category. The analogous limit is called an end. Fixing $c\in\mathcal{C}$ gives two functors $F(c,-)\colon \mathcal{C}\to\mathcal{V}$, $F(-,c)\colon \mathcal{C}^{op}\to\mathcal{V}$ which we think of as a right and left action of F. The coend associated to F is the universal way of making these actions agree, whereas the end of F is the universal object where these actions agree already. The most common example of an end is the $\mathcal{V}$-object of natural transformations between two $\mathcal{V}$-functors. If $G,H\colon \mathcal{A}\to\mathcal{B}$ are two $\mathcal{V}$-functors then there is an isomorphism between the end of the $\mathcal{V}$-functor $\mathcal{B}(G-,H-)$ and $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A},\mathcal{B})(G,H)$.

Coends are often described as a coequalizer of a particular diagram involving the previously mentioned left and right actions of functors $F\colon \mathcal{C}^{op}\times\mathcal{C}\to\mathcal{V}$. An equivalent formulation is to define them as universal objects amongst extranatural transformations [EK66b]. An extranatural transformation is a way of slightly tweaking the notion of natural transformation. In a similar way to above, they are defined between functors $F\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$, $G\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ and act as a mediator between the left and right actions of each functor. For example, rather than requiring the usual naturality squares to commute, one of the axioms for an extranatural transformation $\alpha\colon F \overset{..}{\Rightarrow} G$ requires the commutativity of the following diagram.

$$\begin{tikzcd}
F(b,c) \arrow[r, "F(b,f)"] \arrow[d, "F(f,c)"'] & F(b,b) \arrow[d, "\alpha_{bd}"] \\
F(c,c) \arrow[r, "\alpha_{cd}"'] & G(d,d)
\end{tikzcd}$$

The category $\widehat{\mathcal{C}}$ often appears for another reason, in that it is the free cocompletion of $\mathcal{C}$ [MM94]. This is the category obtained by freely adjoining colimits in order to make $\mathcal{C}$ cocomplete. This is a destructive process in that colimits that $\mathcal{C}$ already possessed are often not colimits after freely adding those needed to make it cocomplete. If $\mathcal{C}$ is a monoidal category then the convolution monoidal product on $\widehat{\mathcal{C}}$ has a further universal property - it is the free monoidal cocompletion of $\mathcal{C}$ [IK86]. A monoidally cocomplete category is one for which each of the endofunctors $b \otimes -\colon \mathcal{C} \to \mathcal{C}$, $- \otimes c\colon \mathcal{C} \to \mathcal{C}$ preserves colimits. The way in which this occurs is typical of a pseudodistributive law for pseudomonads [Kel74], [Mar99], [MW08], [MW12], [CHP13]. Apart from issues of size [AR94], there is a pseudomonad on **Cat** which describes free cocompletion of categories. There is then a pseudodistributive law [TP06] between the free symmetric monoidal category pseudomonad and the free cocompletion pseudomonad, both on **Cat**, which makes use of Day's convolution product. With regard to the issue of size, one could adapt the idea of relative monads [ACU10], [ACU14] to a notion of relative pseudomonad [FGHW16].

Moving up a dimension we can generalise the idea of a category to not only have objects and morphisms but also 2-cells between morphisms (1-cells), the prototypical example being **Cat** consisting of (small) categories, functors, and natural transformations. In a 2-category the composition of 1-cells is strictly associative and unital. A 2-category is an instance of an enriched category in the case that the base category is **Cat**. This is to say that each hom-object $\mathcal{A}(a,b)$ is a category. Since it is a **Cat**-category then 1-cell composition is still strictly associative and unital. If we weaken this requirement for composition on 1-cells then we discover the notion of a bicategory, originally defined by Bénabou [Bén67]. One way of considering bicategories is as a many-object version of a monoidal category. In a bicategory we have objects, 1-cells (morphisms), and 2-cells.

Here, however, the composition of 1-cells is not associative or unital, being so only up to coherent isomorphism, similar to the associations on objects in monoidal categories above.

In a similar way to the coherence theorem for bicategories, Mac Lane and Paré [MP85] describe a coherence theorem for bicategories of the 'all diagrams commute' variety. The bicategorical Yoneda lemma [Str80] gives another form of coherence for bicategories, stating that each bicategory $\mathcal{A}$ is biequivalent to a strict 2-category, a sub-2-category of $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$.

Whilst it is simple to define strict n-categories, simply as n-**Cat**-enriched categories, giving hands-on definitions for weaker structures becomes much more difficult as the dimension increases. From Bénabou's bicategories [Bén67] as a weak 2-category the next step up was the definition of weak 3-categories, or tricategories, given form by Gordon, Power, and Street [GPS95]. Whereas coherence for bicategories shows that each bicategory is biequivalent to a strict 2-category, the same is not true for tricategories. Each tricategory is only triequivalent to a type of semi-strict 3-category know as a **Gray**-category [Gra74], [Gra76], [GPS95]. More generally n-categories have been specified in one of two ways, algebraic or non-algebraic. A survey of the numerous proposed definitions is given by Leinster [Lei02], while Cottrell gives a comparison of the algebraic and non-algebraic viewpoints [Cot13]. For the specific case of tricategories Gurski showed how to specify a fully algebraic definition [Gur13a]. This thesis deals with monoidal bicategories. In the same sense that monoidal categories are considered as degenerate bicategories, we can consider monoidal bicategories to be degenerate tricategories [CG11].

Our goal is to generalise Day's convolution structure to the setting of monoidal bicategories, as well as considering the free cocompletion of a bicategory under bicolimits. The setting becomes more complicated since we will be working in higher dimensions, where it becomes harder to keep track of coherence requirements and perform algebraic manipulations.

In order to more efficiently describe the constructions involved in setting up a higher analogue of Day convolution we go on to characterise codescent objects as universal objects amongst a weakened 2-dimensional generalisation of extranatural transformations. The analogous limit, the descent object, originally appeared in an article of Street [Str76] before being given a formal definition in [Str87]. These colimits can be seen as a 2-dimensional generalisation of coends and are defined in reference to pseudofunctors $P \colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$. A slightly weaker notion of codescent object that we consider is the bicodescent object. These objects are a type of bicolimit, having both a 1-dimensional and 2-dimensional universal property but requiring only existence but not uniqueness of 1-cells in the 1-dimensional part.

We now outline the structure of this thesis.

- In Chapter 1 we discuss the necessary background material needed to contextualize and motivate the later results. This chapter can be seen as an outline for the rest of the thesis in that each section will be generalized to fit in a higher-dimensional context. We recall the ideas of ends/coends and their description as universal objects amongst extranatural transformations before introducing promonoidal categories and the resulting Day convolution product on categories of presheaves. Finally we discuss bicategorical colimits and monoidal bicategories, along with a brief survey of free cocompletions.

- Chapter 2 introduces a generalised notion of extranatural transformation, themselves obtained by bending the rules for natural transformations. To fit into the higher-dimensional context that we wish to consider, the definition of extrapseudonatural transformation is a categorified version of extranatural transformation. We replace the axioms with isomorphisms and provide suitable axioms governing how the data fits together. Familiar composition lemmas are reproved in this generalised setting before embarking on the definition of bicodescent objects as universal objects amongst extrapseudonatural transformations. The chapter culminates in a Fubini-like interchange theorem for bicodescent objects.

- In Chapter 3 we concern ourselves with free bicocompletions of bicategories, by which we mean the free cocompletion of a given bicategory $\mathcal{A}$ under bicolimits. This is seen to be the bicategory of pseudofunctors $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$. We essentially retread the familiar ground of free cocompletions for categories, this time using bidescent and bicodescent objects in place of the usual ends and coends.

- Chapter 4 introduces the notion of a promonoidal bicategory, a generalisation of Day's original definition. We base the definition on the idea that a promonoidal bicategory should be a pseudomonoid in the monoidal tricategory 2-**Prof** [Chi15]. We do not fully investigate this connection, though it proves useful in trying to understand the definition of a promonoidal bicategory. The setting of our definition is also quite specific. Whereas Day defined promonoidal categories over any base symmetric monoidal closed category $\mathcal{V}$, our definition is only specified for the base bicategory **Cat**. It is also at this point that we rely heavily on string diagrams to depict the large axioms for this new definition. This is largely for presentational purposes but we do make remarks on the manipulations of such diagrams.

- The convolution tensor product for presheaves on bicategories is defined in Chapter 5. As with previous material it is superficially similar to Day's original ideas but being a categorified notion of such we have more intricate structure to deal with. We go on to prove that a promonoidal structure on a bicategory $\mathcal{A}$ extends to a monoidal structure on the bicategory $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$. In this chapter we also give some brief remarks about $\widehat{\mathcal{A}}$ being the free monoidal bicocompletion of $\mathcal{A}$, as well as some comments about the monoidal structure on $\widehat{\mathcal{A}}$ being biclosed.

Chapter 1

Background Theory

We begin by introducing the reader to much of the background material needed to motivate the later chapters. We briefly recall the ideas behind ends and coends, in particular their characterisations as universal objects amongst extranatural transformations. We go on to discuss free cocompletion and introduce Day's promonoidal categories and convolution product. The basics of bicategories and their monoidal variants are then considered. Of particular note in this chapter is that we introduce the string diagram notation which will be used to present results later in the thesis - this is introduced in the definition of a monoidal bicategory.

1.1 Ends and Coends

In this first section we recall the definition of extranatural transformation and the characterisation of coends as universal objects amongst these transformations. For the reader unfamiliar with such notions, a comprehensive overview can be found in the literature [ML98], [Bor94], [Kel05]. We will later go on to give a categorified notion of both extranatural transformations as well as ends/coends. In the rest of this chapter all notions regarding ends, coends, and profunctors can be considered from a $\mathcal{V}$-enriched perspective, *à la* [Kel05], for a given closed monoidal category $\mathcal{V}$. Our later generalisations involving profunctors between bicategories will have a fixed setting in the monoidal bicategory **Cat** though we expect a more thorough treatment could readdress this and recast the definitions and results in the setting of other suitable monoidal bicategories.

Definition 1.1.1. Let $F\colon \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{D}$ and $G\colon \mathcal{A} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors. An *extranatural transformation* $\alpha\colon F \stackrel{..}{\Rightarrow} G$ has components $\alpha_{abc}\colon F(a,b,b) \to G(a,c,c)$ such that

- for each $b \in \mathcal{B}$, $c \in \mathcal{C}$, $\alpha_{-bc}\colon F(-,b,b) \Rightarrow G(-,c,c)$ is a natural transformation;
- for each $g\colon b \to b'$ in $\mathcal{B}$, the following diagram commutes;

$$\begin{tikzcd}
F(a,b',b) \arrow[r, "F_{agb}"] \arrow[d, "F_{ab'g}"'] & F(a,b,b) \arrow[d, "\alpha_{abc}"] \\
F(a,b',b') \arrow[r, "\alpha_{ab'c}"'] & G(a,c,c)
\end{tikzcd}$$

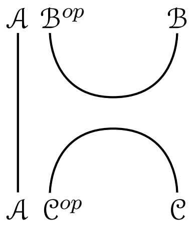
- for each $h\colon c \to c'$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{abc'}"] \arrow[d, "\alpha_{abc}"'] & G(a,c',c') \arrow[d, "G_{ahc'}"] \\
G(a,c,c) \arrow[r, "G_{ach}"'] & G(a,c,c')
\end{tikzcd}$$

Given a natural transformation between functors $F, G\colon \mathcal{A} \to \mathcal{B}$, we can think of the naturality in $\mathcal{A}$ in terms of string diagrams, simply a string from $\mathcal{A}$ to itself.

Usually string diagrams for natural/extranatural transformations would involve more labels but we will not be making much use of these, only introducing them to later motivate the ways that different transformations can be composed.

With extranaturality we bend the rules for naturality and this can be seen in the string diagrams representing extranatural transformations. Given an extranatural transformation between $F\colon \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{D}$ and $G\colon \mathcal{A} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ we depict extranaturality as follows.



Eilenberg and Kelly [EK66b] showed that composition of extranatural transformations is governed by composition of such string diagrams. If string diagrams corresponding to transformations can be composed vertically, without any closed loops, then the transformations corresponding to the diagrams can be composed in the same manner. In Chapter 2 we prove analogous results for the categorified version of extranatural transformations.

Definition 1.1.2. Let $F\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ be a functor. The *coend* of F consists of an object

$$\int^{a} F(a,a)$$

in $\mathcal{B}$, along with an extranatural transformation $i\colon F \overset{..}{\Rightarrow} \int^a F(a,a)$ which is universal in the following sense. Given any other object $b \in \mathcal{B}$ along with an extranatural transformation $j\colon F \overset{..}{\Rightarrow} b$, there exists a unique morphism $t\colon \int^a F_{aa} \to b$ such that $t \cdot i_a = j_a$ for all objects $a \in \mathcal{A}$.

In the above definition, the codomain of the extranatural transformation is $\int^a F(a,a)$ by which we mean the constant functor at that object.

Ends are similarly defined, with the end of a functor $F\colon \mathcal{A}^{op} \times \mathcal{A} \to B$ being an object $\int_a F(a,a)$ along with a universal extranatural transformation $s\colon \int_a F(a,a) \overset{..}{\Rightarrow} F$.

Remark 1.1.3. We can define ends and coends as types of limits and colimits, respectively. We will briefly look at this for coends. Let $F\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ be a functor and consider the following diagram.

$$\begin{tikzcd}
\coprod\limits_{a \in ob\mathcal{A}} F(a,a) & \coprod\limits_{a \xrightarrow{f} b} F(b,a) \arrow[l, shift right, "l"'] \arrow[l, shift left, "r"]
\end{tikzcd}$$

The functor l is determined by the functors

$$\{l_f = P(f,1)\colon P(b,a) \to P(a,a)\}_{f\colon a \to b}$$

whilst r is determined by the functors

$$\{r_f = P(1,f)\colon P(b,a) \to P(a,a)\}_{f\colon a \to b}.$$

The coend of a functor F, as above, is then given by the coequalizer of this diagram and so coends can be considered as a kind of colimit.

Remark 1.1.4. Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$. We can define a functor $\bar{F}\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ by $\bar{F}(c', c) = Fc$, simply muting the contravariant variable. In doing so we can consider the end of $\bar{F}$ which, when the definitions have been compared, can be seen to be the limit of the functor F, i.e., $\lim F \cong \int_c Fc$. We can similarly describe the colimit of F using coends.

Examples abound when it comes to ends and coends. For given $\mathcal{V}$-categories $\mathcal{C}$, $\mathcal{D}$, the functor $\mathcal{V}$-category $[\mathcal{C}, \mathcal{D}]$ is the $\mathcal{V}$-category whose objects are $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{D}$ and for which the hom-objects are given by the ends

$$[\mathcal{C}, \mathcal{D}](F, G) = \int_c \mathcal{D}(Fc, Gc)$$

in $\mathcal{V}$. Kan extensions can be expressed using coends as well. We will make much use of ends/coends and, in later sections, their 2-dimensional analogues - codescent objects.

1.2 Free Cocompletion

Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category and let $\mathcal{C}$ be a small $\mathcal{V}$-category. We will describe the category $\widehat{\mathcal{C}} = [\mathcal{C}^{op}, \mathcal{V}]$, of presheaves on $\mathcal{C}$, as the free cocompletion of the $\mathcal{V}$-category $\mathcal{C}$. In what follows we will often omit the prefix $\mathcal{V}$- when discussing $\mathcal{V}$-categories, $\mathcal{V}$-functors, and the like, simply referring to them as categories, functors, and so on, as long as there is no room for confusion. We do not provide much in the way of proof, only highlighting certain ideas that will reappear in our generalisation of this setting. The enriched case is discussed in [Kel05], with the case $\mathcal{V} = \mathbf{Set}$ considered in [MM94].

We begin by discussing the situation in the setting where $\mathcal{V} = \mathbf{Set}$. Useful general results will be specified in a more general setting for a given $\mathcal{V}$ as described above. Given a small category $\mathcal{C}$, the Yoneda embedding $y\colon \mathcal{C} \to \widehat{\mathcal{C}} = \mathbf{Cat}(\mathcal{C}^{op}, \mathbf{Set})$ describes the free cocompletion of $\mathcal{C}$ in the following sense. Given a cocomplete category $\mathcal{D}$ there is a pseudonatural equivalence of categories

$$\mathbf{Cocomp}(\widehat{\mathcal{C}}, \mathcal{D}) \simeq \mathbf{Cat}(\mathcal{C}, \mathcal{D})$$

where $\mathbf{Cocomp}(\widehat{\mathcal{C}}, \mathcal{D})$ is the category of colimit preserving functors.

We also end up with the following situation. Given a cocomplete category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$, there is a colimit preserving functor $\widehat{F}\colon \widehat{\mathcal{C}} \to \mathcal{D}$ and a natural isomorphism $\widehat{F} \cdot y \cong F$. Furthermore, $\widehat{F}$ is unique up to isomorphism. Each $\widehat{F}$ is left adjoint to to a functor

$$\mathcal{D}(F-, -)\colon \mathcal{D} \to \widehat{\mathcal{C}}$$

which takes an object $d \in \mathcal{D}$ to the presheaf $\mathcal{D}(F-, d)\colon \mathcal{C}^{op} \to \mathbf{Set}$. That is to say there is an adjunction

$$\mathcal{D}(\widehat{F}(G), d) \cong \widehat{\mathcal{C}}(G, \mathcal{D}(F-, d)).$$

The description of $\widehat{F}$ is actually that of a Kan extension, namely the left Kan extension $\mathrm{Lan}_y F$ of F along the Yoneda embedding y.

This fits into a larger picture which describes a pseudofunctor

$$\widehat{\ } : \mathbf{Cat} \to \mathbf{Cocomp}$$

where **Cocomp** is the category of cocomplete categories and colimit preserving functors between them. It would be convenient to place $\widehat{\ } : \mathbf{Cat} \to \mathbf{Cocomp}$ itself into the setting of an adjunction but here we run into size issues. While we can consider the free cocompletion of categories $\mathcal{C}$ which are not small, where we then have to consider the category of small presheaves on $\mathcal{C}$, the category **Cat** in the domain of $\widehat{\ }$ is a large category containing all small categories. The act of freely adjoining colimits has the potential to turn a small category into a large category and so **Cocomp** contains cocomplete categories which are large. The right adjoint in this setup, the inclusion of cocomplete categories into categories, then has a size issue in that if we intend to forget the cocompleteness of a large category in **Cocomp** there is nowhere for it to live in **Cat**, which is inhabited only by small categories. This matter of size is considered in a number of places [FGHW16], [AR94], [TP06]

We now state a number of useful results which we later go on to generalise in the setting of bicategories.

Lemma 1.2.1 (Co-Yoneda Lemma)**.** *Every presheaf $X \colon \mathcal{C}^{op} \to \mathcal{V}$ is the colimit of representable functors as*

$$X \cong \int^{c} X(c) \otimes y(c).$$

It is important here to know that $\mathcal{V}$ is symmetric closed and that this implies that the tensor product preserves colimits in each variable. The above lemma is useful because it allows us to give an explicit description of $\widehat{F}$ by considering what properties we want of it. First we consider that we want isomorphisms

$$F(c) \cong \widehat{F}(\mathcal{C}(-, c))$$

for each object $c \in \mathcal{C}$.

Since we want $\widehat{F}$ to preserve colimits, including coends, then we require

$$\begin{aligned}\widehat{F}(X) &\cong \widehat{F}\left(\int^{c\in\mathcal{C}} X(c)\otimes\mathcal{C}(-,c)\right)\\ &\cong \int^{c\in\mathcal{C}} \widehat{F}(X(c)\otimes\mathcal{C}(-,c))\\ &\cong \int^{c\in\mathcal{C}} X(c)\otimes\widehat{F}(\mathcal{C}(-,c))\\ &\cong \int^{c\in\mathcal{C}} X(c)\otimes F(c).\end{aligned}$$

This final step then describes $\widehat{F}$ for all presheaves on $\mathcal{C}$.

Proposition 1.2.2. *Let $\mathcal{C}$ be a small $\mathcal{V}$-category. The functor $\mathcal{V}$-category $\widehat{\mathcal{C}}$ is the free $\mathcal{V}$-cocompletion of $\mathcal{C}$.*

In the above proposition, the free $\mathcal{V}$-cocompletion refers to the free cocompletion of $\mathcal{C}$ under $\mathcal{V}$-weighted colimits [Kel05], [Bor94]. In the case that $\mathcal{V} = \mathbf{Set}$, this is the free cocompletion that we originally considered.

1.3 Promonoidal Categories

We now recall Day's definition of promonoidal category and briefly describe their relation to monoidal categories. Since we later on do not investigate Day convolution using a general monoidal bicategory, staying firmly with **Cat**, we limit our exposition here to the case where $\mathcal{V} = \mathbf{Set}$. The definition of a promonoidal category can be obtained by replacing functors in the definition of monoidal category with profunctors.

Definition 1.3.1. A *profunctor* $F\colon A \nrightarrow B$ is a functor $F\colon \mathcal{B}^{op}\times\mathcal{A}\to\mathcal{V}$.

There is a symmetric monoidal bicategory **Prof** consisting of categories, profunctors, and natural transformations [DS97]. A promonoidal category is then a pseudomonoid object in **Prof**. Unpacking this statement gives the definition below.

In the following definition we will write $P(a,b,c)$ as P_{abc} in order to save space. Similarly, we will write $\mathcal{A}(a,b)$ as $\mathcal{A}_{ab}$. The cartesian product is omitted in favour of a $\cdot$, while each expression with a repeated variable is intended to mean a coend. For example,

$$P_{xbc}\cdot P_{yxd}\cdot P_{aye} = \int^{x,y} P_{xbc}\times P_{yxd}\times P_{aye}.$$

We will also encounter Yoneda isomorphisms on a functor P which has three inputs. If for example, we fix a and b in order to give a functor P_{ab-} with one input then we will write the Yoneda isomorphism as $\mathbf{y}_3$ in order to make clear which variable is under the coend.

Definition 1.3.2. A *promonoidal category* over $\mathcal{V}$ consists of:

- A category $\mathcal{C}$;
- A functor $P\colon \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \to \mathcal{V}$;
- A functor $J\colon \mathcal{C}^{op} \to \mathcal{V}$
- A natural isomorphism $\mathbf{a}$ in $\mathbf{Cat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathcal{V})$, with components
$$\mathbf{a}_{abcd}\colon \int^{x} P(x,b,c) \times P(a,x,d) \to \int^{x} P(x,c,d) \times P(a,b,x);$$
- Natural isomorphisms $\mathbf{l}$ and $\mathbf{r}$ in $\mathbf{Cat}(\mathcal{A}^{op} \times \mathcal{A}, \mathcal{V})$, with components
$$\mathbf{l}_{ab}\colon \int^{x} Jx \times P(a,x,b) \to \mathcal{A}(a,b),$$
$$\mathbf{r}_{ab}\colon \int^{x} Jx \times P(a,b,x) \to \mathcal{A}(a,b).$$

These are subject to the following axioms.

PC1 given objects $a,b,c,d,e \in \mathcal{A}$, the following diagram commutes;

$$\begin{tikzcd}
P_{xbc}\cdot P_{yxd}\cdot P_{aye} \arrow[r, "\mathbf{a}1"] \arrow[d, "1\mathbf{a}"'] & P_{xcd}\cdot P_{ybx}\cdot P_{aye} \arrow[dr, "1\mathbf{a}"] & \\
P_{xbc}\cdot P_{yde}\cdot P_{axy} \arrow[d, "\simeq"'] & & P_{xcd}\cdot P_{yxe}\cdot P_{aby} \arrow[dd, "\mathbf{a}1"] \\
P_{yde}\cdot P_{xbc}\cdot P_{axy} \arrow[dr, "1\mathbf{a}"'] & & \\
& P_{yde}\cdot P_{xcy}\cdot P_{abx} \arrow[r, equal, "id"'] & P_{xde}\cdot P_{ycx}\cdot P_{aby}
\end{tikzcd}$$

PC2 given objects $a, b, c \in \mathcal{A}$, the following diagram commutes.

$$\begin{tikzcd}
& & Jy \cdot P_{xby} \cdot P_{axc} \arrow[r, "\mathbf{1a}"] & Jy \cdot P_{xyc} \cdot P_{abx} \arrow[dr, "\mathbf{l1}"] & & \\
& \mathcal{A}_{xb} \cdot P_{axc} \arrow[ur, "\mathbf{r}^{-1}1"] & & & \mathcal{A}_{xc} \cdot P_{abx} \arrow[dr, "\mathbf{y}_3"] & \\
P_{abc} \arrow[ur, "\mathbf{y}_2^{-1}"] \arrow[rrrrr, "id"'] & & & & & P_{abc}
\end{tikzcd}$$

Each monoidal category $(\mathcal{C}, \otimes, I, \alpha, l, r)$ describes a promonoidal category where $P(a, b, c) = \mathcal{A}(a, b \otimes c)$ and $Ja = \mathcal{A}(a, I)$.

1.4 Day convolution

When $\mathcal{C}$ is also monoidal then we can equip $\widehat{\mathcal{C}}$ with a monoidal structure given by the convolution tensor product of Day [Day71], which realises $\widehat{\mathcal{C}}$ as the free monoidal cocompletion of $\mathcal{C}$. The convolution of two presheaves $F, G \colon \mathcal{C}^{op} \to \mathcal{V}$ is given by the coend formula:

$$\int^{c} Fc \otimes \int^{d} Gd \otimes \mathcal{C}(-, c \otimes d) \cong \int^{c,d} Fc \otimes Gd \otimes \mathcal{C}(-, c \otimes d).$$

The unit for this monoidal structure is the hom-functor $\mathcal{C}(-, I)$, the image of the unit for the monoidal structure in $\mathcal{C}$.

Lemma 1.4.1. *The hom-functor $y(I)$ is a unit for the convolution tensor product on $\widehat{\mathcal{C}}$.*

Proof. The convolution of a presheaf F with $y(I)$ is given by

$$\begin{aligned}
F \star y(I) &\cong \int^{c,d \in \mathcal{C}} Fc \otimes \mathcal{C}(d, I) \otimes \mathcal{C}(-, c \otimes d) \\
&\cong \int^{c \in \mathcal{C}} Fc \otimes \mathcal{C}(-, c \otimes I) \\
&\cong \int^{c \in \mathcal{C}} Fc \otimes \mathcal{C}(-, c) \\
&\cong F.
\end{aligned}$$

□

The associator involves a sequence of isomorphisms between coends utilising the Fubini theorem for coends, the co-Yoneda lemma, and the associator in $\mathcal{C}$. The following sequence of isomorphisms shows the unravelling of the coends describing $(F \star G) \star H$, which can then be similarly shown to be isomorphic to $F \star (G \star H)$.

$$\begin{aligned}
(F \star G) \star H &\cong \int^{a,b \in \mathcal{C}} (F \star G)a \otimes Hb \otimes \mathcal{C}(-, a \otimes b) \\
&\cong \int^{a,b,c,d \in \mathcal{C}} Fc \otimes Gd \otimes \mathcal{C}(a, c \otimes d) \otimes Hb \otimes \mathcal{C}(-, a \otimes b) \\
&\cong \int^{b,c,d \in \mathcal{C}} Fc \otimes Gd \otimes Hb \otimes \int^{c \in \mathcal{C}} \mathcal{C}(a, c \otimes d) \otimes \mathcal{C}(-, a \otimes b) \\
&\cong \int^{b,c,d \in \mathcal{C}} Fc \otimes Gd \otimes Hb \otimes \mathcal{C}(-, (c \otimes d) \otimes b) \\
&\cong \int^{b,c,d \in \mathcal{C}} Fc \otimes Gd \otimes Hb \otimes \mathcal{C}(-, c \otimes (d \otimes b))
\end{aligned}$$

When we approach the generalisation of Day convolution to monoidal bicategories, we will start with a promonoidal structure before equipping presheaves with a monoidal structure. In the above discussion we started with a monoidal category $\mathcal{C}$ and used the promonoidal structure given by $P(a, b, c) = \mathcal{C}(a, b \otimes c)$. We can instead assume only that $\mathcal{C}$ carries a promonoidal structure $(\mathcal{C}, P, J, \mathbf{a}, \mathbf{l}, \mathbf{r})$ in which case we define the convolution product of two presheaves $F, G \colon \mathcal{C}^{op} \to \mathcal{V}$ to be

$$F \star G = \int^{c,d} Fc \otimes Gd \otimes P(-, c, d).$$

The associator for the monoidal structure on $\mathcal{C}$ is then described by the composite

$$\begin{aligned}
(F \star G) \star H &\cong \int^{a,b,c,d} Fa \otimes Gb \otimes P(c, a, b) \otimes Hd \otimes P(-, c, d) \\
&\cong \int^{a,b,c,d} Fa \otimes Gb \otimes Hd \otimes P(c, a, b) \otimes P(-, c, d) \\
&\cong \int^{a,b,c,d} Fa \otimes Gb \otimes Hd \otimes P(c, b, d) \otimes P(-, a, c) \\
&\cong F \star (G \star H).
\end{aligned}$$

The monoidal category axioms are checked using commutative diagrams similar to those in Section 4.2 with the omission of the invertible 2-cells, instead using identities. Unravelling this definition in the case that $\mathcal{C}$ is a monoidal category leads to the previous associator.

The following theorem is the main result that we partly generalise for monoidal bicategories.

Theorem 1.4.2 (Day)**.** *Let $\mathcal{C}$ be a promonoidal category. Then $\widehat{\mathcal{C}}$ is a monoidal category admitting a biclosed structure.*

Free cocompletions and Day convolution go hand in hand. If $\mathcal{C}$ is a monoidal category, then $\widehat{\mathcal{C}}$ is a monoidal category with respect to the convolution product, which characterises the free monoidal cocompletion of $\mathcal{C}$, as described by Im and Kelly [IK86].

Definition 1.4.3. Let $\mathcal{C}$ be a monoidal category. We say that $\mathcal{C}$ is *monoidally cocomplete* if $\mathcal{C}$ is cocomplete and the functors

$$a \otimes - \colon \mathcal{C} \to \mathcal{C}, - \otimes b \colon \mathcal{C} \to \mathcal{C}$$

preserve colimits.

Remark 1.4.4. The conditions of the previous definition are satisfied when $\mathcal{C}$ is a biclosed monoidal category. In this case, each of the functors $a \otimes -$ and $- \otimes b$ have right adjoints, hence they preserve colimits.

Proposition 1.4.5. *Let $\mathcal{C}$ be a monoidal 2-category. Then $[\mathcal{C}^{op}, \mathbf{Cat}]$ is the free monoidal cocompletion of $\mathcal{C}$.*

This is a rather strict result. The cocompletion of $\mathcal{C}$ as a **Cat**-enriched category is the cocompletion under **Cat**-weighted limits. These are strict 2-colimits in which cocones are required to commute strictly, unlike the bicolimits we have previously discussed.

1.5 Bicategories and Bicolimits

Bicategories live in the world of 2-dimensional category theory. Unlike a strict 2-category, the composition of 1-cells in a bicategory is not strictly associative. For any three composable 1-cells h, g, f there is an invertible 2-cell $a_{hgf} \colon (hg)f \cong h(gf)$ and similarly for any 1-cell f there are invertible 2-cells $l_f \colon 1f \cong f$,$r_f \colon f1 \cong f$, all of which are governed by axioms akin to those of a monoidal category - a monoidal category can be thought of as a single object version of a bicategory. The standard reference is [Bén67] for which definitions can be found for bicategories (Section 2), wherein the composition of 2-cells within a bicategory, along with the interchange law, is described in an illuminating geometric fashion.

Definition 1.5.1. Let $\mathcal{A}$ be a bicategory. The *opposite bicategory* $\mathcal{A}^{op}$ is the bicategory $\mathcal{A}$ in which the source and target of each 1-cell is reversed. For the coherence cells, $l = r^{op}$, $r = l^{op}$, and $a^{op}_{hgf} = a^{-1}_{hgf}$.

The main notion of morphism between bicategories that we will consider is that of a pseudofunctor, defined as a *orphism of bicategories* by Bénabou [Bén67][Section 4]. A pseudofunctor $P \colon \mathcal{A} \to \mathcal{B}$ between bicategories is similar to a 2-functor between 2-categories but does not preserve composition or identities on the nose. Instead there are coherent isomorphisms

$$\phi^P_{g,f} \colon PgPf \Rightarrow P(gf)$$

and

$$\phi^P_a \colon P(id_a) \Rightarrow id_{Pa}$$

in $\mathcal{B}$.

Going up a dimension from the usual world of categories affords an extra degree of freedom. This extra versatility allows us to define a range of limit and colimit notions, not all of which satisfy the usual strict universal properties from 1-dimensional category theory. The colimits that we will consider will be the type often referred to as bicolimits. These bicolimits have both a 1-dimensional and a 2-dimensional universal property. The 1-dimensional property, in dealing with 1-cells from a bicategory, only specifies the existence of mediating 1-cells and does not require any form of uniqueness. Since the top dimension in a bicategory involves strict composition of 2-cells the 2-dimensional property does require uniqueness. Unlike many other types of 2-dimensional limits one can can consider in 2-categories and bicategories, bicolimits cannot be defined using weighted colimits.

A final remark on bicolimits is to mention the way that they are objects characterised by an equivalence of categories.

Definition 1.5.2. Let $P \colon \mathcal{A} \to \mathcal{B}$ be a pseudofunctor. The *bicolimit* of P, if it exists, is an object $\mathrm{bic}P$ in $\mathcal{B}$ for which there is an equivalence of categories

$$\mathcal{B}(\mathrm{bic}P, b) \simeq \mathbf{Bicat}(\mathcal{A}, \mathcal{B})(P, \Delta_b)$$

where $b \in \mathcal{B}$ and Δ_b is the constant pseudofunctor at b. Furthermore, this equivalence is natural in b.

Since the above definition refers to an equivalence the 1-dimensional property will only concern existence of certain 1-cells, without any indication of uniqueness. However, the 2-dimensional property does govern uniqueness in induced 2-cells. We unfold this definition as a remark below and will mostly refer to this explicit description instead.

Remark 1.5.3. Let $P\colon \mathcal{A} \to \mathcal{B}$ be a pseudofunctor between bicategories. The *bicolimit* of P consists of an object $\mathrm{bic}P$ of $\mathcal{B}$, a family of 1-cells $\{i_a\colon Pa \to \mathrm{bic}P\}_{a\in\mathcal{A}}$, and a family of invertible 2-cells

$$\begin{tikzcd}
Pa \arrow[dr, "i_a"] \arrow[dd, "Pf"'] & \\
\arrow[r, phantom, "i_f\Downarrow"] & \mathrm{bic}P \\
Pb \arrow[ur, "i_b"'] &
\end{tikzcd}$$

indexed by the 1-cells of $\mathcal{A}$.

Given another object X of $\mathcal{B}$, with similar families of 1-cells j_a and invertible 2-cells j_f, there exists a 1-cell $h\colon \mathrm{bic}P \to X$ and an invertible 2-cell

$$\begin{tikzcd}
& \mathrm{bic}P \arrow[dd, "h"] \\
Pa \arrow[ur, "i_a"] \arrow[dr, "j_a"'] \arrow[r, phantom, "\Downarrow h_a"] & {} \\
& X
\end{tikzcd}$$

such that

$$\begin{tikzcd}
Pa \arrow[r, "i_a"] \arrow[d, "Pf"'] \arrow[dr, phantom, "\Downarrow i_f", pos=0.3] & \mathrm{bic}P \arrow[d, "h"] \\
Pb \arrow[r, "j_b"'] \arrow[ur, "i_b" description] \arrow[ur, phantom, "h_b\Downarrow", pos=0.7, shift right=3] & X
\end{tikzcd}
=
\begin{tikzcd}
Pa \arrow[r, "i_a"] \arrow[d, "Pf"'] \arrow[dr, "j_a" description] \arrow[dr, phantom, "h_a\Downarrow", pos=0.3, shift left=3] & \mathrm{bic}P \arrow[d, "h"] \\
Pb \arrow[r, "j_b"'] \arrow[ur, phantom, "\Downarrow j_f", pos=0.3] & X.
\end{tikzcd}$$

Furthermore, given two 1-cells $h, k\colon \mathrm{bic}P \to X$ and 2-cells $\gamma_a\colon h \cdot i_a \Rightarrow k \cdot i_a$ satisfying

$$\begin{tikzcd}
Pa \arrow[rr, "i_a"] \arrow[d, "Pf"'] \arrow[dr, "i_a"] & & \mathrm{bic}P \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow\gamma_a"] & \\
Pb \arrow[r, "i_b"'] \arrow[u, phantom, "\Downarrow i_f", shift left=4] & \mathrm{bic}P \arrow[rr, "k"'] & {} & X
\end{tikzcd}
=
\begin{tikzcd}
Pa \arrow[r, "i_a"] \arrow[d, "Pf"'] \arrow[dr, phantom, "\Downarrow i_f", pos=0.3] & \mathrm{bic}P \arrow[rr, "h"] \arrow[dr, phantom, "\Downarrow\gamma_b"] & & X \\
Pb \arrow[ur, "i_b" description] \arrow[rr, "i_b"'] & & \mathrm{bic}P \arrow[ur, "k"'] &
\end{tikzcd}$$

there is a unique 2-cell $\gamma\colon h \Rightarrow k$ such that $\gamma * 1_{i_a} = \gamma_a$ for all $a \in \mathcal{A}$.

Furthermore, given two composable 1-cells $f\colon a \to b$ and $g\colon b \to c$ in $\mathcal{A}$ there is an equality of pasting diagrams as follows.

$$\begin{tikzcd}
 & Pa \arrow[d, "Pf"'] \arrow[dr, "i_a"] \arrow[dd, bend right=60, "P(gf)"'] & & & & Pa \arrow[dr, "i_a"] \arrow[dd, bend right=60, "P(gf)"'] & \\
\cong & Pb \arrow[d, "Pg"'] \arrow[r, "i_b" description] & \mathrm{bic}P & = & & & \mathrm{bic}P \\
 & Pc \arrow[ur, "i_c"'] & & & & Pc \arrow[ur, "i_c"'] &
\arrow[from=1-2, to=2-3, phantom, "i_f \Swarrow"]
\arrow[from=2-2, to=3-2, phantom, shift left=6, "i_g \Swarrow"]
\arrow[from=2-6, to=2-7, phantom, "i_{gf} \Swarrow"]
\end{tikzcd}$$

This equality follows from the pseudonaturality axioms. Similarly there is an equality of pasting diagrams involving i_{id_a} expressing the fact that

$$1_{i_a} * \phi_a^P \cdot i_{id_a} = r_{i_a}.$$

The properties of a bicolimit ensure that any other object satisfying such properties will be adjointly equivalent in the following sense, following [Gur12].

Definition 1.5.4. Let $\mathcal{A}$ be a bicategory. Let $f\colon a \to b$ and $g\colon b \to a$ be 1-cells in $\mathcal{A}$. An *adjunction* $f \dashv g$ consists of 2-cells $\eta\colon 1_a \Rightarrow gf$ and $\epsilon\colon fg \Rightarrow 1_b$ satisfying the triangle identities. We will refer to (f, g) as an *adjoint pair*, saying that f is left adjoint to g and that g is right adjoint to f.

Definition 1.5.5. We say that an adjoint pair f and g form an adjoint equivalence if the unit and counit maps of the previous definition are invertible.

We point out Remark 1.2 of [Gur12], noting that an adjoint pair in the bicategory **Cat** reduces to the usual definition of an adjunction between functors. When specifying the adjoint of a 1-cell f we will write it as $f^\bullet$.

Remark 1.5.6. Section 1.2 of [Gur13a] gives a thorough discussion of 2-cells between adjoint pairs. If $f, g\colon a \to b$ are both left adjoints in an adjoint pair in a bicategory $\mathcal{B}$ then there is a bijection between 2-cells $\alpha\colon f \Rightarrow g$ and $\alpha^\bullet\colon g^\bullet \Rightarrow f^\bullet$. This can be seen as an instance of Lemma 1.6 therein.

Remark 1.5.7. Let $\mathcal{A}$ be a bicategory. There is a bicategory $\mathcal{A}_{\mathrm{adj}}$ which has the same 0-cells and adjoint pairs as 1-cells. A 2-cell $(f, f^\bullet, \epsilon_f, \eta_f) \Rightarrow (g, g^\bullet, \epsilon_g, \eta_g)$ consists of a pair of 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon f^\bullet \Rightarrow g^\bullet$ satisfying a suitable condition. According to [CF07] each of these pairs is invertible, with $\alpha^\bullet = \beta^{-1}$. Furthermore, if we consider the sub-bicategory $\mathcal{A}_{\text{inv}}$ of $\mathcal{A}$ with only invertible 2-cells then $\mathcal{A}_{\text{adj}}$ is itself a sub-bicategory of $\mathcal{A}_{\text{inv}}$ determined by the left adjoints.

From now on we will often refer to the bicategory of pseudofunctors between two given bicategories. If $\mathcal{A}$ and $\mathcal{B}$ are bicategories then there is a bicategory

$$\mathbf{Bicat}(\mathcal{A}, \mathcal{B})$$

which has pseudofunctors $F\colon \mathcal{A} \to \mathcal{B}$ as 0-cells, pseudonatural transformations $\alpha\colon F \Rightarrow G$ as 1-cells, and modifications $\Gamma\colon \alpha \Rrightarrow \beta$ for 2-cells [GG09]. Definitions of pseudonatural transformations and modifications can be found in [BG88], wherein pseudonatural transformations are referred to as pseudotransformations.

For completeness we mention some further facts about bicategories. Given two bicategories $\mathcal{A}$ and $\mathcal{B}$ we describe the *product bicategory* $\mathcal{A} \times \mathcal{B}$ as follows. The 0-cells are given by pairs (a, b) where $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are both 0-cells. Similarly the hom-categories are defined as

$$(\mathcal{A} \times \mathcal{B})\,((a, b), (c, d)) = \mathcal{A}(a, c) \times \mathcal{B}(b, d).$$

Street [Str80] gives us the following lemma, an analogue of the familiar hom-tensor adjunction, for bicategories and pseudofunctors.

Lemma 1.5.8 (Street)**.** *There are biequivalences*

$$\mathbf{Bicat}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathbf{Bicat}(\mathcal{A}, \mathbf{Bicat}(\mathcal{B}, \mathcal{C})).$$

1.6 Monoidal Bicategories

We will describe monoidal bicategories as one-object tricategories, following the algebraic definition of Gurski [Gur13a]. We will also use this opportunity to introduce the string diagram notation that we will use later in defining promonoidal bicategories. The reader wishing to see axioms for monoidal bicategories presented as pasting diagrams may wish to consider the aforementioned book of Gurski.

Definition 1.6.1. A *monoidal bicategory* $\mathcal{A}$ consists of the following data.

- A bicategory $\mathcal{A}$;
- A pseudofunctor $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$;
- A pseudofunctor $I\colon \mathbf{1} \to \mathcal{A}$, where $\mathbf{1}$ is the unit bicategory;

- An adjoint equivalence

$$\begin{tikzcd}
\mathcal{A}^3 \arrow[r, "\otimes\times1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\mathbf{a}"] & \mathcal{A}^2 \arrow[d, "\otimes"] \\
\mathcal{A}^2 \arrow[r, "\otimes"'] & \mathcal{A}
\end{tikzcd}$$

  in $\mathbf{Bicat}(\mathcal{A}^3, \mathcal{A})$;

- Adjoint equivalences

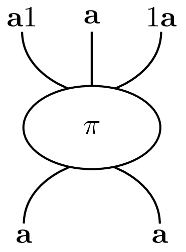
$$\begin{tikzcd}
 & \mathcal{A}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathbf{l}"] & & & \mathcal{A}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow\mathbf{r}"] & \\
\mathcal{A} \arrow[ur, "I\times1"] \arrow[rr, "1"'] & {} & \mathcal{A} & \mathcal{A} \arrow[ur, "1\times I"] \arrow[rr, "1"'] & {} & \mathcal{A}
\end{tikzcd}$$

  in $\mathbf{Bicat}(\mathcal{A}, \mathcal{A})$;

- An invertible modification

$$\begin{tikzcd}
& \mathcal{A}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[ddl, "1\times1\times\otimes"'] \arrow[ddr, "1\times\otimes\times1"] & & \mathcal{A}^3 \arrow[ddr, "\otimes\times1"] & & & & \mathcal{A}^4 \arrow[rr, "\otimes\times1\times1"] \arrow[ddl, "1\times1\times\otimes"'] & & \mathcal{A}^3 \arrow[ddl, "1\times\otimes"'] \arrow[ddr, "\otimes\times1"] \\
& & & \Downarrow\mathbf{a}\times1 & & & & & \cong \\
\mathcal{A}^3 \arrow[ddr, "1\times\otimes"'] & 1\times\mathbf{a}\,\Swarrow & \mathcal{A}^3 \arrow[rr, "\otimes\times1"] \arrow[ddl, "1\times\otimes"] & & \mathcal{A}^2 \arrow[ddl, "\otimes"] & \overset{\pi}{\Rrightarrow} & \mathcal{A}^3 \arrow[rr, "\otimes\times1"] \arrow[ddr, "1\times\otimes"'] & & \mathcal{A}^2 \arrow[ddr, "\otimes"] & \mathbf{a}\,\Swarrow & \mathcal{A}^2 \arrow[ddl, "\otimes"] \\
& & & \Downarrow\mathbf{a} & & & & & \Downarrow\mathbf{a} \\
& \mathcal{A}^2 \arrow[rr, "\otimes"'] & & \mathcal{A} & & & & \mathcal{A}^2 \arrow[rr, "\otimes"'] & & \mathcal{A}
\end{tikzcd}$$

  in $\mathbf{Bicat}(\mathcal{A}^4, \mathcal{A})$, which we will depict as follows;



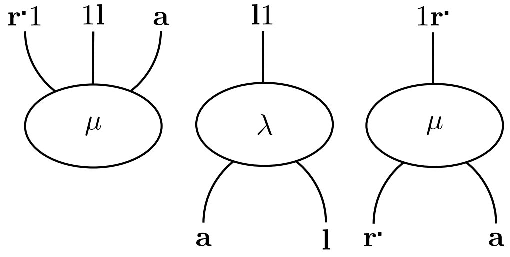
- Invertible modifications

$$\begin{tikzcd}
\mathcal{A}^2 \arrow[rrd, bend left, "1"] \arrow[rd, "1\times I\times 1"] \arrow[rdd, bend right, "1"'] & & & & \mathcal{A}^2 \arrow[rrd, bend left, "1"] \arrow[rdd, bend right, "1"'] & & \\
\arrow[r, phantom, "\Downarrow 1\times\mathbf{l}"] & \mathcal{A}^3 \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[u, phantom, "\Downarrow \mathbf{r}^{\boldsymbol{\cdot}}\times 1"] \arrow[rd, phantom, "\Downarrow \mathbf{a}"] & \mathcal{A}^2 \arrow[d, "\otimes"] & \overset{\mu}{\Rrightarrow} & & \Downarrow 1 & \mathcal{A}^2 \arrow[d, "\otimes"] \\
& \mathcal{A}^2 \arrow[r, "\otimes"'] & \mathcal{A} & & & \mathcal{A}^2 \arrow[r, "\otimes"'] & \mathcal{A}
\end{tikzcd}$$

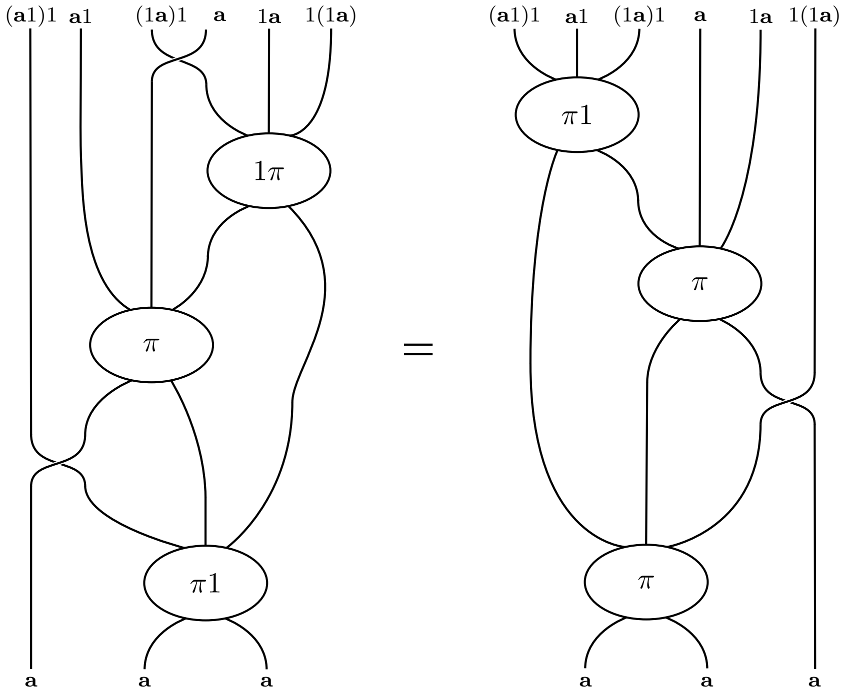
$$\begin{tikzcd}
& \mathcal{A}^3 \arrow[rd, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow \mathbf{l}\times 1"] & & & & \mathcal{A}^3 \arrow[rd, "\otimes\times 1"] \arrow[dd, "1\times\otimes"] & \\
\mathcal{A}^2 \arrow[ru, "I\times 1\times 1"] \arrow[rr, "1"] \arrow[dd, "\otimes"'] & & \mathcal{A}^2 \arrow[dd, "\otimes"] & \overset{\lambda}{\Rrightarrow} & \mathcal{A}^2 \arrow[ru, "I\times 1\times 1"] \arrow[dd, "\otimes"'] & \cong \quad \Downarrow \mathbf{a} & \mathcal{A}^2 \arrow[dd, "\otimes"] \\
& \cong & & & & \mathcal{A}^2 \arrow[rd, "\otimes"] \arrow[d, phantom, "\Downarrow \mathbf{l}"] & \\
\mathcal{A} \arrow[rr, "1"'] & & \mathcal{A} & & \mathcal{A} \arrow[ru, "I\times 1"] \arrow[rr, "1"'] & & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "1"] & & \mathcal{A} & & \mathcal{A} \arrow[rr, "1"] \arrow[rd, "1\times I"] & \Downarrow \mathbf{r} & \mathcal{A} \\
& \cong & & \overset{\rho}{\Rrightarrow} & \cong & \mathcal{A}^2 \arrow[ru, "\otimes"] & \Downarrow \mathbf{a} \\
\mathcal{A}^2 \arrow[uu, "\otimes"] \arrow[rr, "1"] \arrow[rd, "1\times 1\times I"'] & \Downarrow 1\times\mathbf{r}^{\boldsymbol{\cdot}} & \mathcal{A}^2 \arrow[uu, "\otimes"'] & & \mathcal{A}^2 \arrow[uu, "\otimes"] \arrow[rd, "1\times 1\times I"'] & & \mathcal{A}^2 \arrow[uu, "\otimes"'] \\
& \mathcal{A}^3 \arrow[ru, "1\times\otimes"'] & & & & \mathcal{A}^3 \arrow[uu, "\otimes\times 1"] \arrow[ru, "1\times\otimes"'] &
\end{tikzcd}$$

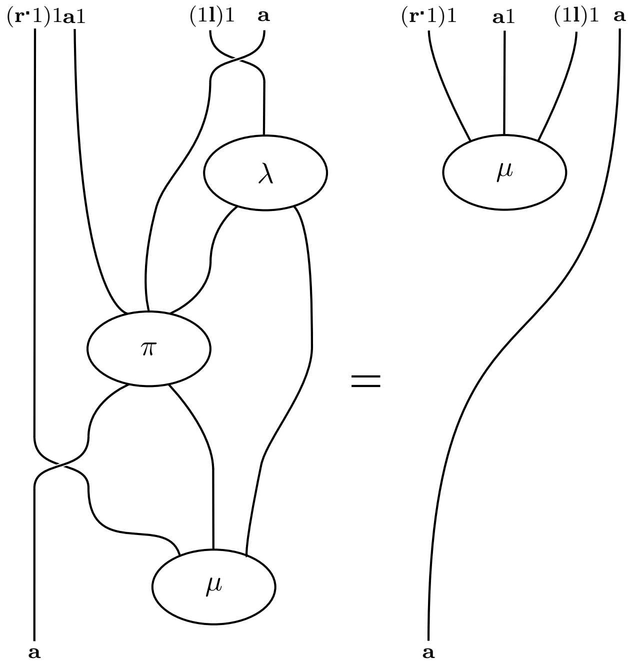
in **Bicat**$(\mathcal{A}^2, \mathcal{A})$, which we will depict, respectively, as follows.



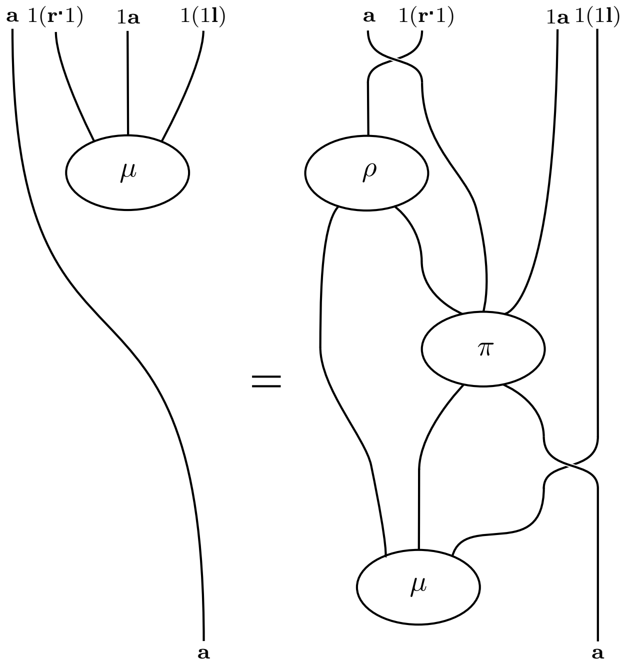
MB1 The following string diagram equation holds in $\mathcal{A}$, where the braids are naturality isomorphisms for **a**.



MB2 The following string diagram equation holds in $\mathcal{A}$, where the braids are either naturality isomorphisms for **a** or coherence cells in $\mathcal{A}$.



MB3 The following string diagram equation holds in $\mathcal{A}$.



In the above string diagrams there are various braidings that occur. In this depiction it does not matter which string is above in a braid, we simply depict the crossing for clarity.

Chapter 2

Extrapseudonaturality and Bicodescent Objects

In this chapter we will define the notion of extrapseudonatural transformation, a weak 2-dimensional generalisation of extranatural transformations - a similar generalisation is seen in the thesis of Lawler [Law15]. We could generalise dinatural transformations in an analogous way, though we do not investigate that here. We will begin by defining extrapseudonatural transformations before proving a host of useful lemmas which we will use to investigate the connection with bicodescent objects. This will culminate in a Fubini theorem for bicodescent objects.

2.1 Extrapseudonatural Transformations

Extranatural transformations were first defined by Eilenberg and Kelly [EK66b] for use in their subsequent article on closed categories [EK66a].

Definition 2.1.1. Let $P\colon \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{D}$ and $Q\colon \mathcal{A} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be pseudofunctors. An *extrapseudonatural transformation* $\beta\colon P \ddot{\Rightarrow} Q$ consists of

- for each $b \in \mathcal{B}$, $c \in \mathcal{C}$, a pseudonatural transformation $\beta_{-bc}\colon P(-,b,b) \Rightarrow Q(-,c,c)$;
- for each $g\colon b \to b'$ in $\mathcal{B}$, an invertible 2-cell in $\mathcal{D}$

$$\begin{tikzcd}
P(a,b',b) \arrow[r, "P_{1g1}"] \arrow[d, "P_{11g}"'] \arrow[dr, phantom, "\Downarrow\beta_{agc}"] & P(a,b,b) \arrow[d, "\beta_{abc}"] \\
P(a,b',b') \arrow[r, "\beta_{ab'c}"'] & Q(a,c,c);
\end{tikzcd}$$

- for each $h\colon c \to c'$ in $\mathcal{C}$, an invertible 2-cell in $\mathcal{D}$

$$\begin{tikzcd}
P(a,b,b) \arrow[r, "\beta_{abc'}"] \arrow[d, "\beta_{abc}"'] \arrow[dr, phantom, "\Downarrow\beta_{abh}"] & Q(a,c',c') \arrow[d, "Q_{1h1}"] \\
Q(a,c,c) \arrow[r, "Q_{11h}"'] & Q(a,c,c').
\end{tikzcd}$$

These are required to satisfy the following axioms.

- EP1 Given $f\colon b \to b'$ and $g\colon b' \to b''$ in $\mathcal{B}$, there is an equality of pasting diagrams

$$\begin{tikzcd}
& P_{ab''b} \arrow[dl, "P_{11g}"'] \arrow[dr, "P_{1f1}"] \arrow[ddl, "P_{11(gf)}" near end] \arrow[ddr, "P_{1(gf)1}"' near end] & \\
P_{ab''b'} \arrow[d, "P_{11f}"'] \arrow[r, phantom, "\cong" near start] & \Swarrow\beta_{a(gf)c} & P_{ab'b} \arrow[d, "P_{1g1}"] \arrow[l, phantom, "\cong" near start] \\
P_{ab''b''} \arrow[dr, "\beta_{ab''c}"'] & & P_{abb} \arrow[dl, "\beta_{abc}"] \\
& Q_{acc} &
\end{tikzcd}
=
\begin{tikzcd}
& & P_{ab''b} \arrow[dll, "P_{11g}"'] \arrow[drr, "P_{1f1}"] & & \\
P_{ab''b'} \arrow[drr, "P_{1f1}"] \arrow[dddd, "P_{11f}"'] & & \cong & & P_{ab'b} \arrow[dll, "P_{11g}"'] \arrow[dddd, "P_{1g1}"] \\
& & P_{ab'b'} \arrow[ddd, "\beta_{ab'c}"] & & \\
& \Swarrow\beta_{afc} & & \Swarrow\beta_{agc} & \\
& & & & \\
P_{ab''b''} \arrow[drr, "\beta_{ab''c}"'] & & & & P_{abb} \arrow[dll, "\beta_{abc}"] \\
& & Q_{acc} & &
\end{tikzcd}$$

for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

- EP2 Given $h\colon c \to c'$ and $i\colon c' \to c'$ in $\mathcal{C}$, there is an equality of pasting diagrams

$$\begin{tikzcd}
& P_{abb} \arrow[dl, "\beta_{abc}"'] \arrow[dr, "\beta_{abc''}"] & \\
Q_{acc} \arrow[d, "Q_{11h}"'] \arrow[ddr, "Q_{11(ih)}" near start] & \Swarrow\beta_{ab(ih)} & Q_{ac''c''} \arrow[d, "Q_{1i1}"] \arrow[ddl, "Q_{1(ih)1}"' near start] \\
Q_{acc'} \arrow[dr, "Q_{11i}"'] \arrow[r, phantom, "\cong" near start] & & Q_{ac'c''} \arrow[dl, "Q_{1h1}"] \arrow[l, phantom, "\cong" near start] \\
& Q_{acc''} &
\end{tikzcd}
=
\begin{tikzcd}
& & P_{abb} \arrow[dll, "\beta_{abc}"'] \arrow[drr, "\beta_{abc''}"] \arrow[dd, "\beta_{abc'}"] & & \\
Q_{acc} \arrow[ddd, "Q_{11h}"'] & \Swarrow\beta_{abh} & & \Swarrow\beta_{abi} & Q_{ac''c''} \arrow[ddd, "Q_{1i1}"] \\
& & Q_{ac'c'} \arrow[dll, "Q_{1h1}"'] \arrow[drr, "Q_{11i}"] & & \\
& & & & \\
Q_{acc'} \arrow[drr, "Q_{11i}"'] & & \cong & & Q_{ac'c''} \arrow[dll, "Q_{1h1}"] \\
& & Q_{acc''} & &
\end{tikzcd}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

- EP3 For each $f\colon a \to a'$ in $\mathcal{A}$ and $g\colon b \to b'$ in $\mathcal{B}$ there is an equality of pasting diagrams

$$\begin{tikzcd}
& P_{ab'b} \arrow[dl, "P_{1g1}"'] \arrow[dr, "P_{f11}"] \arrow[dd, "P_{11g}" description] & \\
P_{ab'b'} \arrow[dd, "\beta_{ab'c}"'] & & P_{a'b'b} \arrow[dd, "P_{11g}"] \\
& P_{abb} \arrow[dl, "\beta_{abc}"'] \arrow[dr, "P_{f11}"] & \\
Q_{acc} \arrow[dr, "Q_{f11}"'] & & P_{a'bb} \arrow[dl, "\beta_{a'bc}"] \\
& Q_{a'cc} &
\arrow[from=2-1, to=3-2, phantom, "\Swarrow \beta_{agc}"]
\arrow[from=1-2, to=4-3, phantom, "\cong"]
\arrow[from=4-1, to=4-3, phantom, "\Swarrow \beta_{fb'c}"]
\end{tikzcd}
=
\begin{tikzcd}
& P_{ab'b} \arrow[dl, "P_{1g1}"'] \arrow[dr, "P_{f11}"] & \\
P_{ab'b'} \arrow[dr, "P_{f11}"'] \arrow[dd, "\beta_{ab'c}"'] & \cong & P_{a'b'b} \arrow[dl, "P_{1g1}"] \arrow[dd, "P_{11g}"] \\
& P_{a'b'b'} \arrow[dd, "\beta_{a'b'c}" description] & \\
Q_{acc} \arrow[dr, "Q_{f11}"'] & & P_{a'bb} \arrow[dl, "\beta_{a'bc}"] \\
& Q_{a'cc} &
\arrow[from=2-1, to=5-2, phantom, "\Swarrow \beta_{fbc}"]
\arrow[from=2-3, to=5-2, phantom, "\Swarrow \beta_{a'gc}"]
\end{tikzcd}$$

for all $c \in \mathcal{C}$.

- EP4 For each $f\colon a \to a'$ in $\mathcal{A}$ and $h\colon c \to c'$ in $\mathcal{C}$ there is an equality of pasting diagrams

$$\begin{tikzcd}
& P_{abb} \arrow[dl, "\beta_{abc}"'] \arrow[dr, "P_{f11}"] & \\
Q_{acc} \arrow[dr, "Q_{f11}"'] \arrow[dd, "Q_{11h}"'] & & P_{a'bb} \arrow[dl, "\beta_{a'bc}"] \arrow[dd, "\beta_{a'bc'}"] \\
& Q_{a'cc} \arrow[dd, "Q_{11h}" description] & \\
Q_{acc'} \arrow[dr, "Q_{f11}"'] & & Q_{a'c'c'} \arrow[dl, "Q_{1h1}"] \\
& Q_{a'c'c} &
\arrow[from=1-2, to=3-2, phantom, "\Swarrow \beta_{fbc}"]
\arrow[from=2-1, to=5-2, phantom, "\cong"]
\arrow[from=2-3, to=5-2, phantom, "\Swarrow \beta_{a'bh}"]
\end{tikzcd}
=
\begin{tikzcd}
& P_{abb} \arrow[dl, "\beta_{abc}"'] \arrow[dr, "P_{f11}"] \arrow[dd, "\beta_{abc}" description] & \\
Q_{acc} \arrow[dd, "Q_{11h}"'] & & P_{a'bb} \arrow[dd, "\beta_{a'bc'}"] \\
& Q_{ac'c'} \arrow[dl, "Q_{1h1}"'] \arrow[dr, "Q_{f11}"] & \\
Q_{acc'} \arrow[dr, "Q_{f11}"'] & \cong & Q_{a'c'c'} \arrow[dl, "Q_{1h1}"] \\
& Q_{a'c'c} &
\arrow[from=2-1, to=3-2, phantom, "\Swarrow \beta_{abh}"]
\arrow[from=3-2, to=2-3, phantom, "\Swarrow \beta_{fbc'}"]
\end{tikzcd}$$

for all $b \in \mathcal{B}$.

- EP5 For each $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$,

$$\beta_{a1_bc} = id_{\beta_{abc} \cdot P_{abb}}, \beta_{ab1_c} = id_{Q_{acc} \cdot \beta_{abc}}.$$

- EP6 Given $g, g' \colon b \to b'$ and $\gamma \colon g \Rightarrow g'$ in $\mathcal{B}$, there is an equality of pasting diagrams

$$\begin{tikzcd}
P_{ab'b} \arrow[r, bend left=60, "P_{agb}"] \arrow[r, "P_{ag'b}"'] \arrow[r, phantom, bend left=30, "\Downarrow P_{a\gamma b}"] \arrow[d, "P_{ab'g'}"'] \arrow[dr, phantom, "\Downarrow \beta_{ag'c}"] & P_{abb} \arrow[d, "\beta_{abc}"] \\
P_{ab'b'} \arrow[r, "\beta_{ab'c}"'] & Q_{acc}
\end{tikzcd}
=
\begin{tikzcd}
P_{ab'b} \arrow[r, "P_{agb}"] \arrow[d, "P_{ab'g}"] \arrow[d, bend right=60, "P_{ab'g'}"'] \arrow[d, phantom, bend right=30, "\overset{P_{ab'\gamma}}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow \beta_{agc}"] & P_{abb} \arrow[d, "\beta_{abc}"] \\
P_{ab'b'} \arrow[r, "\beta_{ab'c}"'] & Q_{acc}
\end{tikzcd}$$

  for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

- EP7 Given $h, h' \colon c \to c'$ and $\delta \colon h \Rightarrow h'$ in $\mathcal{C}$, there is an equality of pasting diagrams

$$\begin{tikzcd}
P_{abb} \arrow[r, "\beta_{abc'}"] \arrow[d, "\beta_{abc}"'] \arrow[dr, phantom, "\Downarrow \beta_{abh}"] & Q_{ac'c'} \arrow[d, "Q_{ahc'}"] \\
Q_{acc} \arrow[r, "Q_{ach}"] \arrow[r, bend right=60, "Q_{ah'c}"'] \arrow[r, phantom, bend right=30, "\Downarrow Q_{ac\delta}"] & Q_{acc'}
\end{tikzcd}
=
\begin{tikzcd}
P_{abb} \arrow[r, "\beta_{abc'}"] \arrow[d, "\beta_{abc}"'] \arrow[dr, phantom, "\Downarrow \beta_{abh'}"] & Q_{ac'c'} \arrow[d, "Q_{ah'c'}"'] \arrow[d, bend left=60, "Qahc'"] \arrow[d, phantom, bend left=30, "\overset{Q_{a\delta c'}}{\Leftarrow}"] \\
Q_{acc} \arrow[r, "Q_{ach'}"'] & Q_{acc'}
\end{tikzcd}$$

  for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Remark 2.1.2. There are a large number of axioms in the definition of an extrapseudonatural transformation. In practice we will find that only a subset of these will need to be checked. For example, if Q in the above definition was a constant pseudofunctor then each 2-cell β_{abh} is in fact an identity $id_Q \cdot id_{\beta_{ab\cdot}}$. This would mean that EP2, EP4, EP7, and the second part of EP5 would all hold automatically.

In the one dimensional case a transformation is extranatural in the pair (a, b) if and only if it is extranatural in a and b separately. For extrapseudonatural transformations, however, this is no longer true. Being extrapseudonatural in a and b separately only implies extrapseudonaturality in (a, b) under the conditions of the following lemma, though the converse still holds.

Lemma 2.1.3. *Let* $P \colon \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *be a pseudofunctor and let* $X \in \mathcal{C}$*. Suppose, respectively, that for fixed* $a \in \mathcal{A}$ *and for fixed* $b \in \mathcal{B}$*,* $\gamma_{a-} \colon P(a, -, a, -) \ddot{\Rightarrow} X$ *and* $\gamma_{-b} \colon P(-, b, -, b) \ddot{\Rightarrow} X$ *are extrapseudonatural transformations such that* $(\gamma_{a-})_b =$

$(\gamma_{-b})_a$. *If there is an equality of pasting diagrams*

$$\begin{tikzcd}
P_{a'b'ab} \arrow[rrrd, bend left, "P_{fgab}"] \arrow[rrd, "P_{a'gab}"] \arrow[rdd, "P_{a'b'fb}"] \arrow[rdddd, bend right, "P_{a'b'fg}"'] & & \cong & \\
& \cong & P_{a'bab} \arrow[r, "P_{fbab}"] \arrow[dd, "P_{a'bfb}"] & P_{abab} \arrow[ddd, "\gamma_{ab}"] \\
\cong & P_{a'b'a'b} \arrow[rd, "P_{a'ga'b}"] \arrow[dd, "P_{a'b'a'g}"] & & \Downarrow\gamma_{fb} \\
& & P_{a'ba'b} \arrow[rd, "\gamma_{a'b}"] \arrow[ld, phantom, "\Downarrow\gamma_{a'g}"] & \\
& P_{a'b'a'b'} \arrow[rr, "\gamma_{a'b'}"'] & & X
\end{tikzcd}
=
\begin{tikzcd}
P_{a'b'ab} \arrow[rrrd, bend left, "P_{fgab}"] \arrow[rrd, "P_{fb'ab}"] \arrow[rdd, "P_{a'b'ag}"] \arrow[rdddd, bend right, "P_{a'b'fg}"'] & & \cong & \\
& \cong & P_{ab'ab} \arrow[r, "P_{agab}"] \arrow[dd, "P_{ab'ag}"] & P_{abab} \arrow[ddd, "\gamma_{ab}"] \\
\cong & P_{a'b'ab'} \arrow[rd, "P_{fb'ab'}"] \arrow[dd, "P_{a'b'fb'}"] & & \Downarrow\gamma_{ag} \\
& & P_{ab'ab'} \arrow[rd, "\gamma_{ab'}"] \arrow[ld, phantom, "\Downarrow\gamma_{fb'}"] & \\
& P_{a'b'a'b'} \arrow[rr, "\gamma_{a'b'}"'] & & X
\end{tikzcd}$$

then these 2*-cells constitute an extrapseudonatural transformation* $\gamma\colon P \overset{..}{\Rightarrow} X$.

Proof. All of the axioms to check extrapseudonaturality for the above 2-cells are satisfied as a result of the corresponding axioms for the individual transformations, naturality of coherence cells for P and of those in $\mathcal{A}$ and $\mathcal{B}$, as well as the equality of 2-cells stated above which is needed for axioms EP1 and EP2. □

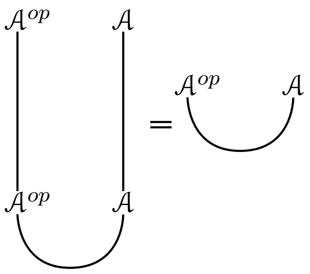
Lemma 2.1.4. *Let* $P\colon \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *be a pseudofunctor and let* $X \in \mathcal{C}$. *Suppose that* $\gamma\colon P \overset{..}{\Rightarrow} X$ *is an extrapseudonatural transformation. If* $a \in \mathcal{A}$ *is fixed then there is an extrapseudonatural transformation* $\gamma_{a-}\colon P(a, -a, -) \overset{..}{\Rightarrow} X$. *Similarly, if* $b \in \mathcal{B}$ *is fixed then there is an extrapseudonatural transformation* $\gamma_{-b}\colon P(-, b, -, b) \overset{..}{\Rightarrow} X$.

Proof. Since the pseudofunctor in the codomain is constant at the object X we need only check that EP1, EP3, EP6, and the first part of EP5 hold. The axiom EP3 is a consequence of the pseudofunctor axioms for P, the first part of EP5 holds for γ_{a-} holds via the axiom EP5 for γ, while EP6 holds for γ_{a-} also holds via EP6 for γ. To show that EP1 holds for γ_{a-} is more involved but fairly simple, resulting from instances of EP1 and EP6 for γ. We must show that two diagrams are equal, one featuring $\gamma_{1_a,hg}$ and the other featuring both $\gamma_{1_a,g}$ and $\gamma_{1_a,h}$. By EP1 for γ the diagram featuring $\gamma_{1_a,g}$ and $\gamma_{1_a,h}$ is equal to one featuring $\gamma_{1_a 1_a,hg}$. At this point we use EP6 to show that this diagram is equal to the diagram featuring $\gamma_{1_a,hg}$. Hence γ_{a-} is an extrapseudonatural transformation. A similar argument holds for γ_{-b}. □

2.2 Composition Lemmas

The article of Eilenberg and Kelly [EK66b] in which extranatural transformations are defined also investigates the ways in which they can be composed. We now present generalisations of the simplest forms of these arguments for extrapseudonatural transformations. In many cases from this point onwards we deal with extrapseudonatural transformations into or out of constant pseudofunctors on an object of a bicategory. We denote the constant pseudofunctor on an object H either by $\Delta_H \colon \mathbf{1}^{op} \times \mathbf{1} \to \mathcal{C}$ or, most commonly, simply by H.

We can picture the first of the composition lemmas in the string diagram format mentioned in Chapter 1. We have a pseudonatural transformation $\beta \colon F \Rightarrow G$ where $F, G \colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{C}$, along with an extrapseudonatural transformation $\gamma \colon G \overset{..}{\Rightarrow} H$, for some object $H \in \mathcal{C}$. We define a composite which results in an extrapseudonatural transformation $F \overset{..}{\Rightarrow} H$. Graphically this looks like the following diagram, sometimes referred to as 'stalactites' due to the shape.



Lemma 2.2.1. *Let F, $G \colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{C}$ be pseudofunctors and let $H \in \mathcal{C}$. Suppose that $\beta \colon F \Rightarrow G$ is a pseudonatural transformation and that $\gamma \colon G \overset{..}{\Rightarrow} H$ is an extrapseudonatural transformation. Then there is an extrapseudonatural transformation from F to H given by composites of the cells constituting β and γ.*

Proof. The 1-cells of the extrapseudonatural transformation are given by the composites $\delta_a = \gamma_a \cdot \beta_{aa}$. Given a 1-cell $f \colon a \to b$ in $\mathcal{A}$, we give 2-cells $\delta_f \colon \delta_a \cdot F(f, 1) \Rightarrow \delta_b \cdot F(1, f)$

by the following diagram.

$$\begin{tikzcd}
 & & F(b,a) \arrow[dll, "F(1,f)"'] \arrow[drr, "F(f,1)"] \arrow[dd, "\beta_{ba}"] & & \\
F(b,b) \arrow[dd, "\beta_{bb}"'] & \Swarrow \beta^{-1}_{1f} & & \Swarrow \beta_{f1} & F(a,a) \arrow[dd, "\beta_{aa}"] \\
 & & G(a,a) \arrow[dll, "G(1,f)"'] \arrow[drr, "G(f,1)"] & & \\
G(b,b) \arrow[drr, "\gamma_b"'] & & \Swarrow \gamma_f & & G(b,a) \arrow[dll, "\gamma_a"] \\
 & & H & &
\end{tikzcd}$$

As per Remark 2.1.2, Δ_H is a constant pseudofunctor and so the other 2-cells required are all identities.

The axioms EP2-5 are simple to check, whilst EP1 requires a sequence of involved pasting diagrams. We will consider the initial and final pasting diagrams in the sequence and describe the steps required to complete the proof. The left-hand diagram of axiom EP1 is given, in this instance, by the following pasting diagram.

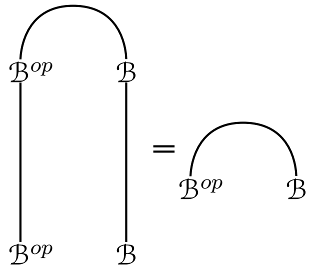
$$\begin{tikzcd}
F(c,a) \arrow[r, "F(g,1)"] \arrow[d, "F(1,f)"'] \arrow[dr, phantom, "\cong"] & F(b,a) \arrow[r, "F(f,1)"] \arrow[d, "F(1,f)"] \arrow[dr, "\beta_{ba}"] & F(a,a) \arrow[drr, "\beta_{aa}"] \arrow[d, phantom, "\Downarrow\beta_{f1}"] & & \\
F(c,b) \arrow[r, "F(g,1)"] \arrow[d, "F(1,g)"'] \arrow[dr, "\beta_{cb}"] & F(b,b) \arrow[r, phantom, "\Downarrow\beta^{-1}_{1f}"] \arrow[d, phantom, "\Downarrow\beta_{g1}"] \arrow[dr, "\beta_{bb}"] & G(b,a) \arrow[rr, "G(f,1)"] \arrow[d, "G(1,f)"] \arrow[dr, phantom, "\Downarrow\gamma_f"] & & G(a,a) \arrow[dd, "\gamma_a"] \\
F(c,c) \arrow[r, phantom, "\Downarrow\beta^{-1}_{1g}"] \arrow[dr, "\beta_{cc}"'] & G(c,b) \arrow[r, "G(g,1)"] \arrow[d, "G(1,g)"] \arrow[dr, phantom, "\Downarrow\gamma_g"] & G(b,b) \arrow[drr, "\gamma_b"] & & \\
 & G(c,c) \arrow[rrr, "\gamma_c"'] & & & H
\end{tikzcd}$$

The right-hand diagram of EP1 is given by the following diagram.

$$\begin{tikzcd}
F(c,a) \arrow[rrr, "F(g{,}1)"] \arrow[drrrr, "F(gf{,}1)" description] \arrow[dddrrr, "\beta_{ca}" description] \arrow[ddd, "F(1{,}f)"'] \arrow[ddddrr, "F(1{,}gf)" description] & & & F(b,a) \arrow[dr, "F(f{,}1)"] & \\
& & & \cong & F(a,a) \arrow[dd, "\beta_{aa}"] \\
& & & \Downarrow\beta_{gf,1} & \\
F(c,b) \arrow[drr, "F(1{,}g)"'] & \cong & \Downarrow\beta^{-1}_{1,gf} & G(c,a) \arrow[r, "G(gf{,}1)" description] \arrow[d, "G(1{,}gf)" description] \arrow[dr, phantom, "\Downarrow\gamma_{gf}"] & G(a,a) \arrow[d, "\gamma_a"] \\
& & F(c,c) \arrow[r, "\beta_{cc}"'] & G(c,c) \arrow[r, "\gamma_c"'] & H
\end{tikzcd}$$

The coherence cells in the top left of the first diagram allow us to use the composition axiom for β to replace $\beta_{g1} * 1_{F(1,f)}$ and $1_{F(g,1)} * \beta_{1f}^{-1}$ with the corresponding components of β and coherence cells for G on the composites $G(g,1) \cdot G(1,f)$ and $G(1,f) \cdot G(g,1)$. Naturality of β on 2-cells, specifically the unitors in $\mathcal{A}^{op} \times \mathcal{A}$, gives a new diagram where $\beta_{g,f}$ meets its inverse, leaving appropriate coherence cells in the square adjoining γ_g and γ_f to apply $EP1$ for γ. This leaves a diagram with γ_{gf} in the lower right corner, at which point we use instances of the composition axiom for β followed by its naturality on unitors again to yield the final diagram. The rest of the axioms are simple to check. $\square$

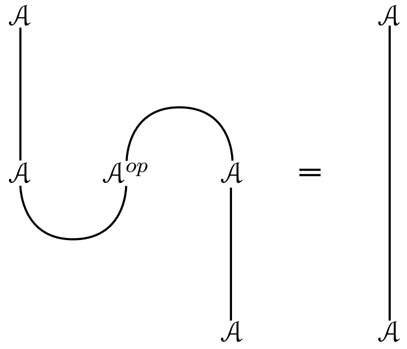
The following lemma is the opposite of the previous lemma, having an extrapseudonatural transformation out of a constant pseudofunctor. Graphically the lemma corresponds to the following 'stalagmite' diagram.



Lemma 2.2.2. *Let $F \in \mathcal{A}$ and let G, $H\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be pseuodfunctors. Suppose that $\beta\colon F \overset{..}{\Rightarrow} G$ is an extrapseudonatural transformation and that $\gamma\colon G \Rightarrow H$ is a pseudonatural transformation. Then there is an extrapseudonatural transformation from F to H given by composites of the cells constituting β and γ.*

Proof. This proof is analogous to that of the previous lemma, only this time the axiom EP2 is the involved part. $\square$

The final composition lemma corresponds to a graphical ‘yanking’ of strings.



Lemma 2.2.3. *Let* $F, H\colon \mathcal{A} \to \mathcal{C}$ *and* $G\colon \mathcal{A} \times \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ *be pseudofunctors. Let* $a, b \in \mathcal{A}$*. Suppose that*

$$\beta_b\colon Fb \overset{\cdot\cdot}{\Rightarrow} G(-,-,b), \gamma_{a-}\colon G(a,-,-) \overset{\cdot\cdot}{\Rightarrow} Ha$$

are extrapseudonatural transformations and that

$$\beta_{a-}\colon F \Rightarrow G(a,a,-), \gamma_{-b}\colon G(-,b,b) \Rightarrow H$$

are pseudonatural transformations such that $(\beta_{a-})_b = (\beta_{-b})_a$ *and* $(\gamma_{a-})_b = (\gamma_{-b})_a$*. Then there is a pseudonatural transformation from* F *to* H *given by composites of the cells constituting* β *and* γ*.*

Proof. We need to construct a pseudonatural transformation from F to H. The 1-cell components are given by $\delta_a = \gamma_{aa} \cdot \beta_{aa}$, whilst the 2-cell component, δ_f, for some

$f\colon a \to b$ in $\mathcal{A}$ is given by the pasting diagram below.

$$\begin{tikzcd}
Fa \arrow[rrrr, "Ff"] \arrow[dd, "\beta_{aa}"'] \arrow[drr, "\beta_{ba}"] & & & & Fb \arrow[dd, "\beta_{bb}"] \\
& \Downarrow\beta^{-1}_{fa} & G(b,b,a) \arrow[dd, "{G(1,f,1)}"] \arrow[drr, "{G(1,1,f)}"] & \Downarrow\beta_{bf} & \\
G(a,a,a) \arrow[dd, "\gamma_{aa}"'] \arrow[drr, "{G(f,1,1)}"] & & & & G(b,b,b) \arrow[dd, "\gamma_{bb}"] \\
& & G(b,a,a) \arrow[drr, "\gamma_{ba}"] & \Downarrow\gamma^{-1}_{bf} & \\
Ha \arrow[rrrr, "Hf"'] & \Downarrow\gamma_{fa} & & & Hb
\end{tikzcd}$$

Similar to the proof of the previous lemmas, the proof of the composition axiom relies on a sequence of pasting diagrams. The first diagram in the sequence is given by the component δ_{gf} with the coherence cells for F and G applied on the top and bottom.

$$\begin{tikzcd}
& & Fb \arrow[drr, "Fg"] \arrow[d, phantom, "\Downarrow\phi^{F}_{gf}"] & & \\
Fa \arrow[urr, "Ff"] \arrow[rrrr, "F(gf)"] \arrow[dd, "\beta_{aa}"'] \arrow[drr, "\beta_{ba}"] & & {} & & Fc \arrow[dd, "\beta_{cc}"] \\
& \Downarrow\beta^{-1}_{(gf)a} & G(c,c,a) \arrow[dd, "{G(1,gf,1)}"] \arrow[drr, "{G(1,1,gf)}"] & \Downarrow\beta_{c(gf)} & \\
G(a,a,a) \arrow[dd, "\gamma_{aa}"'] \arrow[drr, "{G(gf,1,1)}"] & & & & G(c,c,c) \arrow[dd, "\gamma_{cc}"] \\
& & G(c,a,a) \arrow[drr, "\gamma_{ca}"] & \Downarrow\gamma^{-1}_{c(gf)} & \\
Hc. \arrow[rrrr, "H(gf)"] \arrow[drr, "Gf"'] & \Downarrow\gamma_{(gf)a} & {} \arrow[d, phantom, "\Downarrow(\phi^{G}_{gf})^{-1}"] & & Hc \\
& & Hb \arrow[urr, "Gg"'] & &
\end{tikzcd}$$

The final diagram is given below.

$$\begin{tikzcd}
Fa \arrow[rr, "Ff"] \arrow[dr, "\beta_{ba}"] \arrow[dd, "\beta_{aa}"'] & & Fb \arrow[rr, "Fg"] \arrow[dr, "\beta_{cb}"] \arrow[dd, "\beta_{bb}"] & & Fc \arrow[dd, "\beta_{cc}"] \\
& G(b,b,a) \arrow[dr, "{G(1,1,f)}"] \arrow[dd, "{G(1,f,1)}"] \arrow[ur, phantom, "\Downarrow\beta_{bf}"] & & G(c,c,b) \arrow[dr, "{G(1,1,g)}"] \arrow[dd, "{G(1,g,1)}"] \arrow[ur, phantom, "\Downarrow\beta_{cg}"] & \\
G(a,a,a) \arrow[dr, "{G(f,1,1)}"] \arrow[dd, "\gamma_{aa}"'] \arrow[ur, phantom, "\Downarrow\beta_{fa}^{-1}"] & & G(b,b,b) \arrow[dr, "{G(g,1,1)}"] \arrow[dd, "\gamma_{bb}"] \arrow[ur, phantom, "\Downarrow\beta_{gb}^{-1}"] & & G(c,c,c) \arrow[dd, "\gamma_{cc}"] \\
& G(b,a,a) \arrow[dr, "\gamma_{ba}"] \arrow[ur, phantom, "\Downarrow\gamma_{bf}^{-1}"] & & G(c,b,b) \arrow[dr, "\gamma_{cb}"] \arrow[ur, phantom, "\Downarrow\gamma_{cg}^{-1}"] & \\
Ha \arrow[rr, "Gf"'] \arrow[ur, phantom, "\Downarrow\gamma_{fa}"] & & Hb \arrow[rr, "Gg"'] \arrow[ur, phantom, "\Downarrow\gamma_{gb}"] & & Hc
\end{tikzcd}$$

The first step is to use the composition axioms for the pseudonatural components of β and γ, replacing the components at gf with those for g and f, whilst also introducing the composition coherence cells for $G(-,a,a)$ and $G(c,c,-)$. Now we can apply axioms EP2 and EP1, respectively, for the extrapseudonatural components of β and γ. This gives a diagram with extrapseudonatural components for β and γ at g and f, along with a large number of coherence cells in the middle of the diagram. At this point many of the coherence cells cancel out and we are then able to apply axioms EP4 and EP3, respectively, for the mixed components of β and γ. The coherence cells introduced by the use of these axioms then cancel in the middle of the diagram, yielding the second diagram pictured above.

The component δ_{id_a} is plainly seen to the be the identity. Axiom EP5 ensures the extrapseudonatural components at identity morphisms are themselves identities and what remains is simply a composition of pseudonatural components at identities. The axioms EP6 and EP7 are a simple chase of 2-cells through the component pasting diagram defined at the start of the proof. □

2.3 Bicodescent Objects

Descent objects, the dual notion to codescent objects, first appeared in [Str76] before being formally defined by Street in [Str87]. We will base our definition of bicodescent object upon that given by Lack in [Lac02], where codescent objects are used to study coherence for the algebras of 2-monads. Each of these treatments of codescent objects goes on to define them as weighted colimits, whereas our weaker notion of bicodescent object, being a bicolimit, has no description using weights. However, one could investigate the connection between weighted bicolimits, following [Str80], [Str87], and bicoends. We will go on to recast the definition of bicodescent object as a universal object amongst extrapseudonatural transformations (sometimes referred to as *bicoends*), allowing us to obtain a Fubini theorem for bicodescent objects.

Definition 2.3.1. Let $\mathcal{B}$ be a bicategory. Coherence data consists of a diagram

$$\begin{tikzcd}
X_1 \arrow[r, "v" description] & X_2 \arrow[l, shift right=2, "u"'] \arrow[l, shift left=2, "w"] & X_3. \arrow[l, shift right=2, "p"'] \arrow[l, "q" description] \arrow[l, shift left=2, "r"]
\end{tikzcd}$$

in $\mathcal{B}$ along with invertible 2-cells

$$\delta\colon uv \Rightarrow id_{X_1}, \gamma\colon id_{X_1} \Rightarrow wv, \kappa\colon up \Rightarrow uq,$$
$$\lambda\colon wr \Rightarrow wq, \rho\colon ur \Rightarrow wp.$$

The *bicodescent object* of this coherence data consists of a 0-cell X, a 1-cell $x\colon X_1 \to X$, and an invertible 2-cell $\chi\colon xu \Rightarrow xw$ in $\mathcal{B}$ satisfying the following axioms.

BC1 The following pasting diagrams are equal.

$$\begin{tikzcd}
 & X_3 \arrow[dl, "q"'] \arrow[dr, "r"] \arrow[rrr, "p"] \arrow[rrr, phantom, shift right=5, "\Downarrow\rho"] & & & X_2 \arrow[dl, "w"'] \arrow[dr, "u"] & \\
X_2 \arrow[dr, "w"'] & \Downarrow\lambda & X_2 \arrow[r, "u" description] \arrow[dl, "w"] & X_1 \arrow[dr, "x"] & \Downarrow\chi & X_1 \arrow[dl, "x"] \\
 & X_1 \arrow[rrr, "x"'] \arrow[rrr, phantom, shift left=5, "\Downarrow\chi"] & & & X; &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & X_3 \arrow[dl, "q"'] \arrow[rrr, "p"] & & & X_2 \arrow[dr, "u"] & \\
X_2 \arrow[rrrrr, "u" description] \arrow[dr, "w"'] & & & \Downarrow\kappa & & X_1 \arrow[dl, "x"] \\
 & X_1 \arrow[rrr, "x"'] & & \Downarrow\chi & X &
\end{tikzcd}$$

BC2 The following pasting diagrams are equal.

$$\begin{tikzcd}
 & & X_1 \arrow[dr, "x"] & \\
X_1 \arrow[r, "v"] & X_2 \arrow[ur, "u"] \arrow[dr, "w"'] \arrow[rr, phantom, "\Downarrow\chi"] & & X \\
 & & X_1 \arrow[ur, "x"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & X_2 \arrow[dr, "u"] \arrow[d, phantom, "\Downarrow\delta"] & & \\
X_1 \arrow[ur, "v"] \arrow[dr, "v"'] \arrow[rr, "id_{X_1}"] & {} \arrow[d, phantom, "\Downarrow\gamma"] & X_1 \arrow[r, "x"] & X \\
 & X_2 \arrow[ur, "w"'] & &
\end{tikzcd}$$

BC3 Given any other 0-cell Y, 1-cell $y\colon X_1 \to Y$, and 2-cell $\upsilon\colon yu \Rightarrow yw$ which satisfy the previous two axioms, there exists a 1-cell $h\colon X \to Y$ and an isomorphism $\zeta\colon hx \Rightarrow y$ such that the following pasting diagrams are equal.

$$\begin{tikzcd}
X_2 \arrow[rr, "u"] \arrow[dr, "w"'] & & X_1 \arrow[rr, "x"] \arrow[dr, "y"] \arrow[dl, phantom, "\Downarrow\upsilon"] & & X \arrow[dl, "h"] \arrow[dll, phantom, "\Downarrow\zeta"] \\
 & X_1 \arrow[rr, "y"'] & & Y &
\end{tikzcd}
=
\begin{tikzcd}
X_2 \arrow[r, "u"] \arrow[d, "w"'] \arrow[dr, phantom, "\Downarrow\chi"] & X_1 \arrow[d, "x"] \\
X_1 \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[d, "h"] \arrow[dl, phantom, "\Downarrow\zeta"] \\
 & Y
\end{tikzcd}$$

BC4 Given a 0-cell Y, 1-cells $h, k\colon X \Rightarrow Y$, and a 2-cell $\beta\colon h \Rightarrow kx$ satisfying

$$\begin{tikzcd}
 & & X \arrow[dr, "h"] & \\
 & X_1 \arrow[ur, "x"] \arrow[dr, "x"] \arrow[rr, phantom, "\Downarrow\beta"] & & Y \\
X_2 \arrow[ur, "u"] \arrow[dr, "w"'] \arrow[rr, phantom, "\Downarrow\chi"] & & X \arrow[ur, "k"'] & \\
 & X_1 \arrow[ur, "x"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & X_1 \arrow[dr, "x"] & & \\
X_2 \arrow[ur, "u"] \arrow[dr, "w"'] \arrow[rr, phantom, "\Downarrow\chi"] & & X \arrow[dr, "h"] & \\
 & X_1 \arrow[ur, "x"] \arrow[dr, "x"'] \arrow[rr, phantom, "\Downarrow\beta"] & & Y \\
 & & X \arrow[ur, "k"'] &
\end{tikzcd}$$

there exists a unique 2-cell $\beta'\colon h \Rightarrow k$ such that $\beta' * 1_x = \beta$.

Now we have defined bicodescent objects we will liken them to coends. (See Remark 3.2.1.) Coends can be described as a colimit for functors of the form $F\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{C}$, being given as the coequalizer

$$\int^a F(a,a) \xleftarrow[i]{} \coprod_a F(a,a) \underset{\rho}{\overset{\lambda}{\leftleftarrows}} \coprod_{a \xrightarrow{f} b} F(b,a)$$

where λ and ρ act in a similar manner to u and w below. In our case, a bicodescent object will be a bicolimit for pseudofunctors of the form $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$. The previous definition only has two axioms but requires setting up a lot of data, whereas using extrapseudonatural transformations requires little in the specification of data with the trade-off of checking a few more axioms.

Given a pseudofunctor $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ we describe its coherence data as follows.

$$\begin{tikzcd}
\coprod\limits_{a \in ob\mathcal{B}} P(a,a) \arrow[r, "v" description] & \coprod\limits_{f} P(b,a) \arrow[l, shift right=2, "u"'] \arrow[l, shift left=2, "w"'] & \coprod\limits_{\theta} P(c,a) \arrow[l, shift right=2, "p"'] \arrow[l, "q" description] \arrow[l, shift left=2, "r"']
\end{tikzcd}$$

The middle coproduct is indexed over 1-cells $f\colon a \to b$ while the last coproduct is indexed over 2-cells $\theta\colon gf \to h$ for 1-cells $f\colon a \to b$, $g\colon b \to c$, and $h\colon a \to c$.

In the following, the 1-cells I_a and J_f are coproduct inclusions. The 1-cell u is determined by the 1-cells

$$P(b,a) \xrightarrow{P(f,1)} P(a,a) \xrightarrow{I_a} \coprod_{a} P(a,a),$$

the 1-cell w is determined by the 1-cells

$$P(b,a) \xrightarrow{P_{bf}} P(b,b) \xrightarrow{I_b} \coprod_{a} P(a,a),$$

and the 1-cell v is determined by the inclusion on identities. The 1-cell p is characterised by the 1-cells

$$P(c,a) \xrightarrow{P_{ga}} P(b,a) \xrightarrow{J_f} \coprod_{f\colon a \to b} P(b,a),$$

the 1-cell q is determined by the 1-cells

$$P(c,a) \xrightarrow{J_h} \coprod_{f\colon a \to b} P(b,a),$$

and the 1-cell r is characterised by the 1-cells

$$P(c,a) \xrightarrow{P_{cf}} P(c,b) \xrightarrow{J_g} \coprod_{f\colon a \to b} P(b,a).$$

When we consider a pseudofunctor, say $F\colon \mathcal{A} \to \mathcal{B}$, we will write its coherence cells as follows. For composition, on 1-cells $f\colon a \to b$, $g\colon b \to c$ in $\mathcal{A}$, we write

$$\phi^F_{gf}\colon F(g) \cdot F(f) \Rightarrow F(g \cdot f)$$

with

$$\phi^F_{aa} = \phi^F_{id_a, id_a}$$

while for identities we write

$$\phi^F_a \colon F(id_a) \Rightarrow id_{Fa}$$

where $a \in \mathcal{A}$. We now use the corresponding cells for P in order to determine the coherence data for the bicodescent object of P. We will omit certain indices on the pseudofunctor coherence cells for P when it is obvious. The 2-cell $\delta\colon uv \Rightarrow id$ is determined by the 2-cells

$$\begin{tikzcd}
P(a,a) \arrow[r, bend left=40, "P(id_a{,}id_a)"] \arrow[r, bend right=40, "id_{P(a,a)}"'] \arrow[r, phantom, "\Downarrow\phi^P_{aa}"] & P(a,a)
\end{tikzcd}$$

Similarly the 2-cell $\gamma\colon id \Rightarrow wv$ is characterised by the inverses of those that give δ. The 2-cell $\kappa\colon up \Rightarrow uq$ is characterised by the 2-cells

$$\begin{tikzcd}
& P(b,a) \arrow[dr, "P(f{,}1)"] \arrow[d, phantom, "\Downarrow\phi^P"] & \\
P(c,a) \arrow[ur, "P(g{,}1)"] \arrow[rr, bend left=15, "P(gf{,}11)"] \arrow[rr, bend right=15, "P(h{,}1)"'] \arrow[rr, phantom, "\Downarrow P(\theta{,}r_1)"] & {} & P(a,a)
\end{tikzcd}$$

and the 2-cell $\lambda\colon wr \Rightarrow wq$ is characterised by the 2-cells

$$\begin{tikzcd}
& P(c,b) \arrow[dr, "P(1{,}g)"] \arrow[d, phantom, "(\phi^P)\Downarrow"] & \\
P(c,a) \arrow[ur, "P(1{,}f)"] \arrow[rr, bend left=15, "P(11{,}gf)"] \arrow[rr, bend right=15, "P(1{,}h)"'] \arrow[rr, phantom, "P(l_1{,}\theta)\Downarrow"] & {} & P(c,c).
\end{tikzcd}$$

The remaining 2-cell, $\rho\colon ur \Rightarrow wp$, is characterised by the 2-cells

$$\begin{tikzcd}[row sep=large]
& P(c,b) \arrow[ddr, "P(g{,}1)"] \arrow[d, phantom, "\Downarrow\phi^P"] & \\
& {} & \\
P(c,a) \arrow[uur, bend left, "P(1{,}f)"] \arrow[ddr, bend right, "P(g{,}1)"'] \arrow[rr, bend left=30, "P(1g{,}1f)"] \arrow[rr, "P(g{,}f)"] \arrow[rr, bend right=30, "P(g1{,}f1)"'] & \Downarrow P(r{,}l) \quad P(l{,}r)^{-1}\Downarrow & P(b,b). \\
& {} & \\
& P(b,a) \arrow[uur, "P(1{,}f)"'] \arrow[u, phantom, "(\phi^P)^{-1}\Downarrow"] &
\end{tikzcd}$$

The following results will characterise bicodescent objects as objects which are universal amongst extrapseudonatural transformations. By this we mean that the morphisms

$$i_a \colon P(a,a) \to \mathrm{Cod}P$$

are part of the data for an extrapseudonatural transformation, satisfying universal properties as in Definition 2.3.3.

Lemma 2.3.2. *Let (Y, y, υ) be data as in 2.3.1 satisfying only the axioms $BC1$ and $BC2$ for the coherence data of a pseudofunctor P. Then it is necessary and sufficient for the corresponding (Y, y_a, υ_f) to constitute an extrapseudonatural transformation.*

Proof. Collectively the (Y, y_a, υ_f) give a triple (Y, y, υ). The axiom BC1 then follows from the axioms EP1, to change an υ_{gf} into a composite of υ_g and υ_f, and EP6 for the modification-like property. For BC2 we see that one side of the pasting diagram corresponds to $\upsilon_{id_a} = id_{y_a \cdot P_{aa}}$ by EP5, whilst the other side is a composite of δ and $\gamma = \delta^{-1}$, giving the identity required.

Now we will show that the 1-cells

$$y_b \colon P(b,b) \to Y$$

along with 2-cells, to be described, constitute an extrapseudonatural transformation. As P is a pseudofunctor out of $\mathcal{B}^{op} \times \mathcal{B}$ then we require pseudonatural transformations between the pseudofunctor

$$\begin{aligned} \overline{\Delta}_{P(b,b)} \colon \mathbf{1} &\longrightarrow \mathcal{C} \\ \cdot &\longmapsto P(b,b) \\ 1. &\longmapsto P(1_b, 1_b) \\ 1_{1.} &\longmapsto id_{P(1_b,1_b)} \end{aligned}$$

and the constant pseudofunctor Δ_Y. The 2-cell components at the identity are given by

$$\begin{tikzcd}
P(b,b) \arrow[r, "id"'] \arrow[r, bend left=60, "P(1_b,1_b)"] \arrow[r, phantom, bend left=30, "\Downarrow \phi_{bb}"] \arrow[d, "y_b"'] \arrow[dr, "y_b" description] \arrow[dr, phantom, shift left=4ex, "\Swarrow r_{y_b}"] \arrow[dr, phantom, shift right=4ex, "\Swarrow l_{y_b}^{-1}"] & P(b,b) \arrow[d, "y_b"] \\
Y \arrow[r, "id"'] & Y
\end{tikzcd}$$

which we will denote by $\bar{j}_b$. Using the fact that in any bicategory the left and right unitors at the identity are equal, along with the naturality of many of the coherence cells we can see that this constitutes a pseudonatural transformation. The first collection of extrapseudonatural 2-cells are given by

$$\begin{tikzcd}
P(b,a) \arrow[r, "{P(f,1)}"] \arrow[d, "{P(1,f)}"'] \arrow[dr, phantom, "\Downarrow \upsilon_f"] & P(a,a) \arrow[d, "y_a"] \\
P(b,b) \arrow[r, "y_b"'] & Y
\end{tikzcd}$$

where υ is the 2-cell given in the triple. The second collection of extrapseudonatural 2-cells is given by the identity on the 1-cell $id_Y \cdot y_b$.

The axiom EP1 holds by an instance of BC1 using the identity 2-cell $id\colon g \cdot f \Rightarrow gf$, whilst EP2 holds trivially. The third axiom, EP3, requires an equality of the following two pasting diagrams.

$$\begin{tikzcd}
 & P(b,a) \arrow[dl, "{P(1,f)}"'] \arrow[dr, "{P(1,1)}"] \arrow[ddd, "{P(f,1)}"] & \\
P(b,b) \arrow[dddd, "j_b"'] & & P(b,a) \arrow[dddd, "{P(f,1)}"] \\
 & & \\
 & & \\
 & P(a,a) \arrow[dl, "j_a"'] \arrow[dr, "{P(1,1)}"] & \\
Y \arrow[dr, "id_Y"'] & & P(a,a) \arrow[dl, "j_a"] \\
 & Y &
\arrow[from=2-1, to=5-2, phantom, "\Swarrow j_f"]
\arrow[from=2-3, to=5-2, phantom, "\cong"]
\arrow[from=6-1, to=6-3, phantom, "\Swarrow \bar{j}_a"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P(b,a) \arrow[dl, "{P(1,f)}"'] \arrow[dr, "{P(1,1)}"] & \\
P(b,b) \arrow[dr, "{P(1,1)}"] \arrow[dddd, "j_b"'] & \cong & P(b,a) \arrow[dl, "{P(1,f)}"'] \arrow[dddd, "{P(f,1)}"] \\
 & P(b,b) \arrow[dddd, "j_b"] & \\
 & & \\
 & & \\
Y \arrow[dr, "id_Y"'] & & P(a,a) \arrow[dl, "j_a"] \\
 & Y &
\arrow[from=4-1, to=4-2, phantom, "\Swarrow \bar{j}_b"]
\arrow[from=4-2, to=4-3, phantom, "\Swarrow j_f"]
\end{tikzcd}$$

Written out as a commutative diagram of 2-cells, including all coherence cells, this can plainly be seen to hold as a result of naturality of various coherence 2-cells, unit axioms for P, and triangle identities in $\mathcal{C}$. The fourth axiom, EP4, follows by a similar, though simpler, argument, whilst EP5 holds immediately due to δ and γ being inverse to each other, giving

$$\chi_{id_a} = id_{j_a \cdot P(1_a, 1_a)}.$$

Clearly axiom EP7 holds since we are considering a constant pseudofunctor Δ_Y. It remains then to check EP6 which requires, for each $\theta\colon g \Rightarrow g'$ between $g, g'\colon a \to b$ in

$\mathcal{B}$, an equality of pasting diagrams as follows.

$$\begin{tikzcd}
P(b,a) \arrow[r, bend left=50, "{P(g,1)}"] \arrow[r, "{P(g',1)}"'] \arrow[r, phantom, shift left=3, "\Downarrow P(\theta{,}1)"] \arrow[d, "{P(1,g)}"'] \arrow[dr, phantom, "\Downarrow \upsilon_{g'}"] & P(a,a) \arrow[d, "y_a"] \\
P(b,b) \arrow[r, "y_b"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
P(a,b) \arrow[r, "{P(g,1)}"] \arrow[d, bend right=50, "{P(1,g')}"'] \arrow[d, "{P(1,g)}"] \arrow[d, phantom, shift right=3, "\overset{P(1{,}\theta)}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow \upsilon_g"] & P(a,a) \arrow[d, "y_a"] \\
P(b,b) \arrow[r, "y_b"'] & Y
\end{tikzcd}$$

This is slightly tricky to prove but really relies on making a suitable choice of 2-cell in $\mathcal{B}$ when considering BC1. First we can check that EP6 holds for the 2-cell $r_g\colon g \cdot id_a \Rightarrow g$, this relies on the fact that many of the pseudofunctor coherence cells for P, and the image of some unitors, can be cancelled in the resulting diagrams. If we then have a 2-cell $\theta\colon g \Rightarrow g'$ then choosing the 2-cell $\gamma \cdot r_g$

$$\begin{tikzcd}
 & a \arrow[dr, bend left, "g"] \arrow[d, phantom, "\Downarrow r_g"] & \\
a \arrow[ur, bend left, "id_a"] \arrow[rr, "g" description] \arrow[rr, bend right=40, "g'"'] \arrow[rr, phantom, shift right=3, "\Downarrow \theta"] & {} & b
\end{tikzcd}$$

in an instance of BC1 proves that EP6 holds in all cases. □

We now describe the bicoend of P as the universal extrapseudonatural transformation out of P, which we will later use as our definition of bicodescent object.

Definition 2.3.3. Let $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor. The bicoend of P is given by $i\colon P \ddot{\Rightarrow} \int^b P(b,b)$ satisfying the following universal properties:

- EB1 Given another object X with an extrapseudonatural transformation $j\colon P \ddot{\Rightarrow} X$, there is a 1-cell $\tilde{j}\colon \int^b P(b,b) \to X$ and isomorphisms $J_a\colon \tilde{j} \cdot i_a \cong j_a$ such that the following equality of pasting diagrams holds.

$$\begin{tikzcd}
P_{ba} \arrow[r, "P_{f1}"] \arrow[dr, "P_{1f}"'] & P_{aa} \arrow[r, "i_a"] \arrow[dr, "j_a" description] \arrow[d, phantom, "\Downarrow j_f"] & \int^b P(b,b) \arrow[d, "\tilde{j}"] \arrow[dl, phantom, "\Downarrow J_a"] \\
 & P_{bb} \arrow[r, "j_b"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
P_{ba} \arrow[r, "P_{f1}"] \arrow[d, "P_{1f}"'] \arrow[dr, phantom, "\Downarrow i_f"] & P_{aa} \arrow[d, "i_a"] \\
P_{bb} \arrow[r, "i_b" description] \arrow[dr, "j_b"'] & \int^b P(b,b) \arrow[d, "\tilde{j}"] \arrow[dl, phantom, "\Downarrow J_b"] \\
 & Y
\end{tikzcd}$$

- EB2 Given two 1-cells h, $k\colon \int^b P(b,b) \to Y$ and 2-cells $\Gamma_a\colon h \cdot i_a \Rightarrow k \cdot i_a$ satisfying

$$\begin{tikzcd}
 & & \int^b P(b,b) \arrow[dr, "h"] & & & & P_{aa} \arrow[dr, "i_a"] & & \\
 & P_{aa} \arrow[ur, "i_a"] \arrow[dr, "i_a"] & \Downarrow\Gamma_a & Y & & P_{ba} \arrow[ur, "P_{f1}"] \arrow[dr, "P_{1f}"'] & \Downarrow i_f & \int^b P(b,b) \arrow[dr, "h"] & \\
P_{ba} \arrow[ur, "P_{f1}"] \arrow[dr, "P_{1f}"'] & \Downarrow i_f & \int^b P(b,b) \arrow[ur, "k"'] & & = & & P_{bb} \arrow[ur, "i_b"] \arrow[dr, "i_b"'] & \Downarrow\Gamma_b & Y \\
 & P_{bb} \arrow[ur, "i_b"'] & & & & & & \int^b P(b,b) \arrow[ur, "k"'] &
\end{tikzcd}$$

  there is a unique 2-cell $\gamma\colon h \Rightarrow k$ such that $\Gamma_a = \gamma * 1_{i_a}$ for all $a \in \mathcal{A}$.

Lemma 2.3.4. *Let $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor and suppose that $i\colon P \overset{..}{\Rightarrow} \int^b P(b,b)$ exists. Let $j\colon P \overset{..}{\Rightarrow} X$ be another extrapseudonatural transformation which also satisfies the axioms EB1 and EB2. Then there is an adjoint equivalence between $\int^b P(b,b)$ and X.*

In the sense of the above lemma we can consider bicodescent objects to be essentially unique.

Proposition 2.3.5. *Let $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor. The bicodescent object corresponding to the coherence data for P is equivalent to the bicoend of P.*

Proof. By Lemma 2.3.2 the triple of a bicodescent object for P, $(\mathrm{Cod}P, x, \chi)$, provides an extrapseudonatural transformation $x\colon P \overset{..}{\Rightarrow} \mathrm{Cod}P$. It is clear that $\mathrm{Cod}P$ satisfies the same universal properties as $\int^b P(b,b)$. Similarly the extrapseudonatural transformation i from P to $\Delta_{\int^b P(b,b)}$ satisfies all of the axioms of a bicodescent object for P, including the universal properties. We will describe how to show that these objects are equivalent.

The bicoend has invertible 2-cells $i_f\colon i_a \cdot P_{fa} \Rightarrow i_b \cdot P_{bf}$, where $f\colon a \to b$ is a 1-cell in $\mathcal{B}$. Collectively this family of 2-cells corresponds to a 2-cell i.

$$\begin{tikzcd}
\coprod_f \arrow[r, "u"] \arrow[d, "w"'] & \coprod_a \arrow[d, "j"] \\
\coprod_a \arrow[r, "j"'] \arrow[ur, phantom, "\Downarrow i"] & \int^b P_{aa}
\end{tikzcd}$$

Each of the 1-cells in the above diagram are induced using the universal properties of the displayed coproducts. The conditions in BC3 are met which induces a 2-cell as below.

$$\begin{tikzcd}
\mathrm{Cod}P \arrow[d, "s"'] \arrow[dr, phantom, "\cong"] & \coprod_a P_{aa} \arrow[l, "j"'] \arrow[dl, "j"] \\
\int^a P_{aa} &
\end{tikzcd}$$

Similarly the bicodescent object $\mathrm{Cod}P$ is an extrapseudonatural transformation, as previously described, satisfying the appropriate axioms. Since $\int^a P_{aa}$ is the universal such transformation there is an induced 1-cell $t\colon \int^a P_{aa} \to \mathrm{Cod}P$ satisfying the properties described in the previous definition. Our claim now is that s and t form an equivalence in $\mathcal{B}$.

It is simple to check this claim. In analogous 1-dimensional cases this would be immediate following from the uniqueness inherent in the 1-dimensional universal property. Since the 1-dimensional properties now no longer contain a uniqueness statement, we do not find that s and t are inverses but that we instead obtain isomorphisms $1 \cong st$ and $ts \cong 1$, as in the diagrams below.

$$\begin{tikzcd}
 & \mathrm{Cod}P \arrow[d, "s"] \arrow[dd, bend right=70, "id"'] & \\
 \cong & \int^a P_{aa} \arrow[d, "t"] & \coprod_{a\in\mathcal{B}} P_{aa} \arrow[l, "j"] \arrow[ul, "x"'] \arrow[dl, "x"] \arrow[ul, phantom, bend left, "\cong"] \arrow[dl, phantom, bend right, "\cong"] \\
 & \mathrm{Cod}P &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \int^a P_{aa} \arrow[d, "t"] \arrow[dd, bend right=70, "id"'] & \\
 \cong & \mathrm{Cod}P \arrow[d, "s"] & \coprod_{a\in\mathcal{B}} P_{aa} \arrow[l, "x_a"] \arrow[ul, "i_a"'] \arrow[dl, "i_a"] \arrow[ul, phantom, bend left, "\cong"] \arrow[dl, phantom, bend right, "\cong"] \\
 & \int^a P_{aa} &
\end{tikzcd}$$

These isomorphisms can then be used to show that the two objects $\mathrm{Cod}P$ and $\int^a P_{aa}$

are equivalent in $\mathcal{B}$. ☐

We will use the notation $i\colon P \ddot{\Rightarrow} \int^b P(b,b)$ to refer to the bicodescent object corresponding to a pseudofunctor P.

Lemma 2.3.6. *Let $P\colon \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor. Assume that, for each $a \in \mathcal{A}$, the bicodescent object*

$$j^a\colon P_{a--} \ddot{\Rightarrow} \int^b P(a,b,b)$$

exists in $\mathcal{C}$. Then

$$a \longmapsto \int^b P(a,b,b)$$

is the object part of a pseudofunctor

$$\int^b P(-,b,b)\colon \mathcal{A} \to \mathcal{C}.$$

Proof. Each $f\colon a \to a'$ in $\mathcal{A}$ gives a pseudonatural transformation

$$P_{f--}\colon P_{a--} \Rightarrow P_{a'--}.$$

By Lemma 2.2.1 there is then an extrapseudonatural transformation

$$j^{a'} \cdot P_{f--}\colon P_{a--} \ddot{\Rightarrow} \int^b P_{abb},$$

inducing the following invertible 2-cells.

$$\begin{tikzcd}
P_{abb} \arrow[r, "P_{fbb}"] \arrow[d, "j^a_b"'] \arrow[dr, phantom, "\Downarrow j^f_b"] & P_{a'bb} \arrow[d, "j^{a'}_b"] \\
\int^b P_{abb} \arrow[r, "\int^b P_{fbb}"'] & \int^b P_{a'bb}
\end{tikzcd}$$

The coherence cells of P along with these j^f_b induce coherence cells for these new 1-cells. This can be seen more clearly in the remarks following the proof. The uniqueness in the 2-dimensional universal property of each $\int^b P_{abb}$ shows that each of the axioms for a pseudofunctor are satisfied. Furthermore, the above 2-cells constitute pseudonatural transformations $j_b\colon P_{-bb} \Rightarrow \int^b P_{-bb}$. ☐

For reference, we will describe the properties of the coherence cells for the pseudofunctors $\int^b P_{-bb} \colon \mathcal{A} \to \mathcal{C}$. The inverse of the invertible 2-cell

$$\begin{tikzcd}
& \int^b P_{a'bb} \arrow[dr, bend left, "\int^b P_{f'bb}"] \arrow[d, phantom, "\Downarrow \phi_{f',f}"] & \\
\int^b P_{abb} \arrow[ur, bend left, "\int^b P_{fbb}"] \arrow[rr, "\int^b P_{(f'f)bb}"'] & {} & \int^b P_{a''bb}
\end{tikzcd}$$

induced by the 2-dimensional universal property of $\int^b P_{abb}$, upon being whiskered by j_b^a, yields the invertible pasting diagram below.

$$\begin{tikzcd}
P_{abb} \arrow[rrrr, "j_b^a"] \arrow[dr, "P_{fbb}"] \arrow[drrr, "P_{(f'f)bb}"] \arrow[dd, "j_b^a"'] & & & & \int^b P_{abb} \arrow[dd, "\int^b_{(f'f)bb}"] \\
& P_{a'bb} \arrow[rr, "P_{f'bb}"] \arrow[ur, phantom, "\cong"] \arrow[dl, phantom, "\Downarrow j_b^f"] \arrow[dr, "j_b^{a'}"] & & P_{a''bb} \arrow[dr, "j_b^{a''}"] \arrow[ur, phantom, "\Downarrow j_b^{f'f}"] \arrow[dl, phantom, "\Downarrow j_b^{f'}"] & \\
\int^b P_{abb} \arrow[rr, "\int^b P_{fbb}"'] & & \int^b P_{a'bb} \arrow[rr, "\int^b P_{f'bb}"'] & & \int^b P_{a''bb}
\end{tikzcd}$$

The unlabeled isomorphism is the composite coherence cell for P_{-bb} consisting of $P(1_{f'f}, l_1, l_1)$ and $\phi^P_{f'bb,fbb}$. Similarly, the inverse of the invertible 2-cell

$$\begin{tikzcd}
\int^b P_{abb} \arrow[r, bend left=50, "\int^b P_{abb}"] \arrow[r, bend right=50, "id"'] \arrow[r, phantom, "\Downarrow \phi_a^{\int^b P_{-bb}}"] & \int^b P_{abb}
\end{tikzcd}$$

when whiskered by j_b^a, gives the invertible pasting diagram

$$\begin{tikzcd}
P_{abb} \arrow[rr, "j_b^a"] \arrow[dd, "j_b^a"'] \arrow[dr, "P_{abb}" description] & \arrow[d, phantom, "\cong"] & \int^b P_{abb} \arrow[dd, "id"] \\
\arrow[r, phantom, "\Downarrow j_b^{id_a}"] & P_{abb} \arrow[dr, "j_b^a" description] & \\
\int^b P_{abb} \arrow[rr, "\int^b P_{abb}"'] & & \int^b P_{abb}
\end{tikzcd}$$

where the unlabeled isomorphism is the composite coherence cell consisting of $l_{{j^a}_b}$, $(r_{j_b^a})^{-1}$, and $\phi_a^{P_{-bb}}$.

Lemma 2.3.7. *Let $\mathcal{A}$ be a bicategory. There is a pseudofunctor*

$$I = \int^a -(a) \times \mathcal{A}(-,a) \colon \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat}) \to \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat}).$$

Proof. Since **Cat** is bicocomplete the bicodescent object

$$I(F) = \int^a F(a) \times \mathcal{A}(-,a)$$

exists for each pseudofunctor $F\colon \mathcal{A}^{op} \to \mathbf{Cat}$. Given a pseudonatural transformation $\gamma\colon F \Rightarrow G$, we can define another pseudonatural transformation

$$\gamma \times 1_{\mathcal{A}(-,-)} \colon F \times \mathcal{A}(-,-) \Rightarrow G \times \mathcal{A}(-,-).$$

Since $I(F)$ and $I(G)$ are bicodescent object then we also have extrapseudonatural transformations $i^F\colon F \times \mathcal{A}(-,-) \overset{..}{\Rightarrow} I(F)$, $i^G\colon G \times \mathcal{A}(-,-) \overset{..}{\Rightarrow} I(G)$, and so the composite of i^G and $\gamma \times 1_{\mathcal{A}(-,-)}$, in the manner of Lemma 2.2.1, induces a pseudonatural transformation $I(\gamma)\colon I(F) \Rightarrow I(G)$ via the universal property of i^F. This also means there are invertible modifications

$$\begin{tikzcd}
Fa \times \mathcal{A}(-,a) \arrow[r, Rightarrow, "i_a^F"] \arrow[d, Rightarrow, "\gamma_a \times 1_{\mathcal{A}(-,)}"'] \arrow[dr, phantom, "\Downarrow \Gamma_a"] & I(F) \arrow[d, Rightarrow, "I(\gamma)"] \\
Ga \times \mathcal{A}(-,a) \arrow[r, Rightarrow, "i_a^G"'] & I(G)
\end{tikzcd}$$

satisfying the pasting axiom EB1 of Definition 2.3.3.

The action of I on 2-cells is described as follows. If $\Sigma\colon \gamma \Rrightarrow \delta$ is a modification then for each $a \in \mathcal{A}$ there is a natural transformation $\Sigma_a \colon \gamma_a \Rightarrow \delta_a$, giving rise to a modification $\Sigma_a \times 1 \colon \gamma_a \times 1 \Rightarrow \delta_a \times 1$. (Note that in the following diagram we switch the style of arrow.) The composite modification

$$\begin{tikzcd}
Fa \times \mathcal{A}(-,a) \arrow[rr, bend left=10, "i_a^F"] \arrow[dr, bend left, "\gamma_a \times 1_{\mathcal{A}(-,a)}"] \arrow[dr, bend right, "\delta_a \times 1_{\mathcal{A}(-,a)}"'] \arrow[dr, phantom, "\Swarrow \Sigma_a \times 1"] \arrow[dd, bend right, "i_a^F"'] & \arrow[d, phantom, "\Downarrow \Gamma_a"] & I(F) \arrow[dd, "I(\gamma)"] \\
& Ga \times \mathcal{A}(-,a) \arrow[dr, "i_a^G"] \arrow[d, phantom, "\Downarrow \Delta_a^{-1}"] & \\
I(F) \arrow[rr, "I(\delta)"'] & & I(G)
\end{tikzcd}$$

satisfies the requirements of axiom EB2, yielding a unique 2-cell $I(\Delta) \colon I(\gamma) \Rrightarrow I(\delta)$. The action of I on 2-cells preserves the strict composition of modifications due to the uniqueness property inherent in the universal property. It remains to describe the data for the pseudofunctor on 1-cell composition and check the appropriate axioms, however this clearly follows from similar arguments to the above. $\square$

2.4 Fubini for codescent objects

This section makes use of the previous definitions and technical lemmas in order to prove a bicategorical analogue of the Fubini theorem for coends. Similar results have been established via a different approach [Nun16].

Proposition 2.4.1. *Let $P \colon \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor and assume that the bicodescent objects*

$$j^{a'a} \colon P(a', -, a, -) \ddot{\Rightarrow} \int^b P(a', b, a, b)$$

and

$$i \colon P \ddot{\Rightarrow} \int^{a,b} P(a, b, a, b)$$

exist, where $(a', a) \in \mathcal{A}^{op} \times \mathcal{A}$. Then there is a 1-cell

$$\sigma \colon \int^a \int^b P(a, b, a, b) \to \int^{a,b} P(a, b, a, b)$$

if the left side exists.

Proof. Suppose that the bicodescent object $k\colon \int^b P_{-b-b} \ddot{\Rrightarrow} \int^a \int^b P_{abab}$ exists. By Lemma 2.1.4, fixing $a \in \mathcal{A}$ results in an extrapseudonatural transformation $i_{a-}\colon P_{a-a-} \ddot{\Rrightarrow} \int^{a,b} P_{abab}$ yielding a family of 1-cells $\phi^a\colon \int^b P_{abab} \to \int^{a,b} P_{abab}$ along with corresponding families of invertible 2-cells

$$\begin{tikzcd}
 & \int^b P_{abab} \arrow[dd, "\phi^a"] \\
P_{abab} \arrow[ur, "j_b^{aa}"] \arrow[dr, "i_{ab}"'] \arrow[r, phantom, "\Downarrow \Phi_b^a"] & {} \\
 & \int^{a,b} P_{abab}
\end{tikzcd}$$

in $\mathcal{C}$, satisfying the usual axioms. To induce σ as in the statement of the theorem we now need find invertible 2-cells

$$\begin{tikzcd}
\int^b P_{a'bab} \arrow[r, "\int^b P_{fbab}"] \arrow[d, "\int^b P_{a'bfb}"'] \arrow[dr, phantom, "\Downarrow \phi^f"] & \int^b P_{abab} \arrow[d, "\phi^a"] \\
\int^b P_{a'ba'b} \arrow[r, "\phi^{a'}"'] & \int^{a,b} P_{abab}
\end{tikzcd}$$

and show that there is an extrapseudonatural transformation $\phi^-\colon \int^b P_{-b-b} \ddot{\Rrightarrow} \int^{a,b} P_{abab}$.

To find the ϕ^f we will use the 2-dimensional universal properties of the bicodescent objects $\int^b P_{a'bab}$. For each $b \in \mathcal{B}$ we have an invertible 2-cell

$$\begin{tikzcd}
 & \int^b P_{a'bab} \arrow[rr, "\int^b P_{fbab}"] & & \int^b P_{abab} \arrow[ddr, "\phi^a"] & \\
 & \Downarrow \left(j_b^{fa}\right)^{-1} & P_{abab} \arrow[ur, "j_b^{aa}"] \arrow[drr, "i_{ab}"] & \Downarrow \Phi_b^a & \\
P_{a'bab} \arrow[uur, "j_b^{a'a}"] \arrow[urr, "P_{fbab}"] \arrow[drr, "P_{a'bfb}"'] \arrow[ddr, "j_b^{a'a}"'] & & \Downarrow i_{fb} & & \int^{a,b} P_{abab} \\
 & \Downarrow j_b^{a'f} & P_{a'ba'b} \arrow[urr, "i_{a'b}"'] \arrow[dr, "j_b^{a'a'}"] & \Downarrow \left(\Phi_b^{a'}\right)^{-1} & \\
 & \int^b P_{a'bab} \arrow[rr, "\int^b P_{a'bfb}"'] & & \int^b P_{a'ba'b} \arrow[uur, "\phi^{a'}"'] &
\end{tikzcd}$$

which satisfies the pasting conditions of axiom EB2. This is seen by pasting these 2-cells with $j_g^{a'a}$ for some $g\colon b \to b'$ in $\mathcal{B}$ and using properties of the j_b^{fa}, properties of the Φ_b^a, axiom EP1, and axiom EP6, before using similar applications of these in the reverse order. Hence there is a unique 2-cell ϕ^f as required, which satisfies appropriate pasting conditions, namely that the whiskering of ϕ^f by $j_b^{a'a}$ yields the composite 2-cell displayed above.

We must now check that these ϕ^f satisfy the axioms of an extrapseudonatural transformation. As the codomain of the 1-cells is an object of $\mathcal{C}$ then some of the axioms again become redundant, namely EP2-4, and EP7. For EP1 we whisker each of the diagrams by $j_b^{a''a}$. On one side we get an instance of the above 2-cell for $f'f$, while on the other we have to use the properties of the coherence cells of $\int^b P_{-b-b}$, in the manner described following Lemma 2.1.4. To equate the two pasting diagrams is a case of using the pseudonaturality of j_b, extrapseudonaturality of i, and instances of the above composite 2-cell for both f and f'. The uniqueness in the 2-dimensional universal property of $\int^b P_{a''bab}$ is used to show that EP1 then holds. For EP5, most of the 2-cells in $\phi^{id_a} * 1_{j_b^{aa}}$ are identities, leaving Φ_b^a to cancel with itself, before again using axiom EB2 to show the equality. Axiom EP6 is simple to check.

Since $\phi\colon \int^b P_{-b-b} \overset{\cdot\cdot}{\Rightarrow} \int^{a,b} P_{abab}$ is then an extrapseudonatural transformation there exists an invertible 2-cell

$$\begin{tikzcd}
& \int^a \int^b P_{abab} \arrow[dd, "\sigma"] \\
\int^b P_{abab} \arrow[ur, "k_a"] \arrow[dr, "\phi^a"'] \arrow[r, phantom, "\Downarrow \Sigma_a"] & \\
& \int^{a,b} P_{abab}
\end{tikzcd}$$

for each $a \in \mathcal{A}$. □

Lemma 2.4.2. *Let $P\colon \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a pseudofunctor and assume that for each fixed pair $(a', a) \in \mathcal{A}^{op} \times \mathcal{A}$ the bicodescent object*

$$j^{a'a}\colon P(a', -, a, -) \overset{\cdot\cdot}{\Rightarrow} \int^b P(a', b, a, b)$$

exists. Similarly suppose that the bicodescent object

$$\theta\colon \int^{a,b} P(a, b, a, b) \to \int^a \int^b P(a, b, a, b)$$

exists. For each fixed $b \in \mathcal{B}$, considering j_b as a pseudonatural transformation $j\colon P_{-b-b} \Rightarrow \int^b P_{-b-b}$, the composite of k and j, as in Lemma 2.2.1 satisfies the compatibility condition of Lemma 2.1.3.

Proof. The proof relies on the equality of certain pasting diagrams, as in Lemma 2.1.3.

The easiest way to prove this equality is to show that one of the pasting diagrams acts as an inverse for the other. The steps required depend on how the j_b interact with the $j^{a'a}$, as specified by Lemma 2.3.6. □

Proposition 2.4.3. *Let* $P\colon \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *be a pseudofunctor and assume that the bicodescent objects*

$$j^{a'a}\colon P(a', -, a, -) \mathrel{\ddot{\Rightarrow}} \int^b P(a', b, a, b)$$

and

$$k\colon \int^b P(-, b, -, b) \mathrel{\ddot{\Rightarrow}} \int^a \int^b P(a, b, a, b)$$

exist, where $(a', a) \in \mathcal{A}^{op} \times \mathcal{A}$*. Then there is a* 1*-cell*

$$\theta\colon \int^{a,b} P(a, b, a, b) \to \int^a \int^b P(a, b, a, b)$$

if the left side exists.

Proof. Suppose that the bicodescent object $i\colon P \mathrel{\ddot{\Rightarrow}} \int^{a,b} P_{abab}$ exists. By Lemma 2.2.1, the composite of j and k is extrapseudonatural in a. This composite is also extrapseudonatural in b, following from the extrapseudonaturality of j, simply by whiskering diagrams with the 1-cells of k. By the previous lemma, the composite of j and k is then extrapseudonatural in (a, b), so there exists an invertible 2-cell

$$\begin{tikzcd}
P_{abab} \arrow[r, "i_{ab}"] \arrow[d, "j_b^{aa}"'] \arrow[dr, phantom, "\Downarrow \Theta_{ab}"] & \int^{a,b} P_{abab} \arrow[d, "\theta"] \\
\int^b P_{abab} \arrow[r, "k_a"'] & \int^a \int^b P_{abab}
\end{tikzcd}$$

for each $(a, b) \in \mathcal{A} \times \mathcal{B}$. □

Theorem 2.4.4. *Under the conditions of Proposition 2.4.1 and Proposition 2.4.3 there is an adjoint equivalence*

$$\int^{a,b} P(a, b, a, b) \simeq \int^a \int^b P(a, b, a, b).$$

Proof. The equivalence is provided by the 1-cells and invertible 2-cells induced in the previous theorems. We require isomorphisms

$$\sigma \cdot \theta \cong id, \theta \cdot \sigma \cong id$$

before showing that they satisfy appropriate axioms. For the first isomorphism, we can show that the invertible 2-cells

$$\begin{tikzcd}
 & \int^{a,b} P_{abab} \arrow[rr, "\theta"] \arrow[d, phantom, "\Downarrow \Theta_{ab}"] & & \int^{a}\int^{b} P_{abab} \arrow[dd, bend left, "\sigma"] \arrow[d, phantom, "\Downarrow \Sigma_a"] \\
P_{abab} \arrow[ru, bend left, "i_{ab}"] \arrow[rr, "j_b^{aa}"] \arrow[rrrd, "i_{ab}"] \arrow[rd, bend right, "i_{ab}"'] & \Downarrow l^{-1}_{i_{ab}} & \int^{b} P_{abab} \arrow[ru, "k_a"] \arrow[rd, "\phi^a"] \arrow[d, phantom, "\Downarrow \Phi_b^a"] & \\
 & \int^{a,b} P_{abab} \arrow[rr, "id"'] & & \int^{a,b} P_{abab}
\end{tikzcd}$$

satisfy the requirements of axiom EB2, giving a unique invertible 2-cell $\kappa \colon \sigma \cdot \theta \Rightarrow id$ such that $\kappa * 1_{i_{ab}}$ is the pasting diagram above.

The second isomorphism requires two steps. The first uses invertible 2-cells

$$\begin{tikzcd}
 & \int^{b} P_{abab} \arrow[r, "\phi^a"] \arrow[d, phantom, "\Downarrow \Phi_b^a"] & \int^{a,b} P_{abab} \arrow[rd, "\theta"] & \\
P_{abab} \arrow[ru, "j_b^{aa}"] \arrow[rru, bend right, "i_{ab}"] \arrow[rd, "j_b^{aa}"'] & \Downarrow \Theta_{ab} & & \int^{a,b} P_{abab} \\
 & \int^{b} P_{abab} \arrow[r, "k_a"'] \arrow[rru, bend left, "k_a"] & \int^{a,b} P_{abab} \arrow[ru, "id"'] \arrow[u, phantom, "\Downarrow l^{-1}_{i_{k_a}}"] &
\end{tikzcd}$$

satisfying the requirements of axiom EB2 to give unique invertible 2-cells $\Omega_a \colon \theta \cdot \phi^a \Rightarrow id \cdot k_a$ such that $\Omega_a * 1_{j_b^{aa}}$ is the pasting diagram above. The second step uses invertible

2-cells

$$\begin{tikzcd}
& \int^a\int^b P_{abab} \arrow[rr, "\sigma"] & \Downarrow\Sigma_a & \int^{a,b} P_{abab} \arrow[dd, bend left, "\theta"] \\
\int^b P_{abab} \arrow[ur, "k_a"] \arrow[urrr, bend right=15, "\phi^a" description] \arrow[dr, "k_a"'] & & \Downarrow\Omega_a & \\
& \int^a\int^b P_{abab} \arrow[rr, "id"'] & & \int^a\int^b P_{abab}
\end{tikzcd}$$

which again satisfy the requirements of axiom EB2, in order to give unique invertible 2-cells $\lambda\colon \theta\cdot\sigma \Rightarrow id$ such that $\lambda * 1_{k_a}$ is the pasting diagram above. To apply EB2 in this instance requires that, for some $f\colon a \to a'$, the pasting of the two instances of the previous diagram, for a and a', with k_f are equal. We show that they are equal by using the universal property of $\int^b P_{abab}$, whiskering the diagrams with $j_b^{a'a}$ gives an equality of pasting diagrams and by uniqueness the original diagrams are equal.

Checking that this is then an adjoint equivalence relies again on the axiom EB2. The check here is somewhat simpler than the previous calculations. For each pair $(a,b) \in \mathcal{A}\times\mathcal{B}$ we have invertible 2-cells

$$\begin{tikzcd}
& & \Downarrow l_\theta^{-1} & \\
& & \int^a\int^b P_{abab} \arrow[dd, "\sigma"] \arrow[dr, "id"] & \\
P_{abab} \arrow[r, "i_{ab}"] & \int^{a,b} P_{abab} \arrow[ur, "\theta"] \arrow[dr, "id"'] \arrow[rr, bend left=90, "\theta"] \arrow[rr, bend right=90, "\theta"'] \arrow[r, phantom, "\Downarrow\kappa"] & & \int^a\int^b P_{abab} \arrow[l, phantom, "\Downarrow\lambda^{-1}"'] \\
& & \int^{a,b} P_{abab} \arrow[ur, "\theta"'] & \\
& & \Downarrow r_\theta &
\end{tikzcd}$$

which plainly satisfy the requirements of EB2. We also note that this whiskered pasting diagram is equal to the identity on $\theta\cdot i_{ab}$ and so by uniqueness we find that the composite 2-cell, when not whiskered by i_{ab}, is the identity on θ. A similar argument shows that the other triangle identity also holds, hence the equivalence is in fact an adjoint equivalence. □

Corollary 2.4.5. *Let* $P\colon \mathcal{A}^{op}\times\mathcal{B}^{op}\times\mathcal{A}\times\mathcal{B} \to \mathcal{C}$ *be a pseudofunctor and assume that* *the bicodescent objects*

$$j^{a'a}\colon P(a',-,a,-) \stackrel{..}{\Rightarrow} \int^b P(a',b,a,b)$$

and

$$l^{b'b}\colon P(-,b',-,b) \stackrel{..}{\Rightarrow} \int^a P(a,b',a,b)$$

exist, where $(a',a) \in \mathcal{A}^{op} \times \mathcal{A}$ *and* $(b',b) \in \mathcal{B}^{op} \times \mathcal{B}$*. Then there is an adjoint equivalence*

$$\int^a \int^b P(a,b,a,b) \simeq \int^b \int^a P(a,b,a,b).$$

2.5 Example

Whilst we will work with bicodescent objects for the most part, our main example is actually a bidescent object. The definition of a bidescent object is similar to that of a bicodescent object. We now consider the universal extranatural transformation for a pseudofunctor $P\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$

$$p\colon \int_b P(b,b) \stackrel{..}{\Rightarrow} P$$

satisfying similar axioms to those in Definition 2.3.3.

Lemma 2.5.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be bicategories and let* F*,* $G\colon \mathcal{A} \to \mathcal{B}$ *be pseudofunctors. The category of pseudonatural transformations*

$$\mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G)$$

is given by the bidescent object for the pseudofunctor $\mathcal{B}(F,G)\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Cat}$*.*

Proof. We will describe an extrapseudonatural transformation

$$i\colon \mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G) \stackrel{..}{\Rightarrow} \mathcal{B}(F,G)$$

and show that it is universal. We need to give functors

$$i_a\colon \mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G) \to \mathcal{B}(Fa,Ga)$$

along with, for each $f\colon a \to a'$ in $\mathcal{A}$, a natural transformation

$$\begin{tikzcd}
\mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G) \arrow[r, "i_{a'}"] \arrow[d, "i_a"'] \arrow[dr, phantom, "\Downarrow i_f"] & \mathcal{B}(Fa',Ga') \arrow[d, "\mathcal{B}(Ff{,}Gb)"] \\
\mathcal{B}(Fa,Ga) \arrow[r, "\mathcal{B}(Fa{,}Gf)"'] & \mathcal{B}(Fa,Ga').
\end{tikzcd}$$

For a pseudonatural transformation $\alpha\colon F \Rightarrow G$ we define $i_a(\alpha) = \alpha_a$ whilst for a modification $\Gamma\colon \alpha \Rightarrow \beta$ we define $i_a(\Gamma) = \Gamma_a$, clearly defining a functor. For the natural transformations we define the components as $(i_f)_\alpha = \alpha_f$. Naturality follows from the modification axioms.

The axioms required for this to be an extrapseudonatural transformation are all instances of axioms for pseudonatural transformations between F and G. It remains to show that this is the universal such extrapseudonatural transformation, or bicoend. Let $j\colon X \overset{\cdot\cdot}{\Rightarrow} \mathcal{B}(F,G)$ be another extrapseudonatural transformation, where X is a category. We will define a functor $h\colon X \to \mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G)$ along with natural isomorphisms $H_a\colon i_a \cdot h \Rightarrow j_a$. Notice that for each $x \in X$ there is a family of 1-cells

$$\{j_a(x)\colon Fa \to Ga\}_{a\in\mathcal{A}}$$

and a family of invertible 2-cells

$$\{(j_f)_x : j_{a'}(x) \cdot Ff \Rightarrow Gf \cdot j_a(x)\}_{f\colon a\to a'}.$$

The extrapseudonaturality axioms for j show that there is a pseudonatural transformation $j_-(x)$ for each $x \in X$, defining the on-objects part of the functor h. For a morphism $k\colon x \to y$ in X we have a family of 2-cells

$$\{j_a(k)\colon j_a(x) \Rightarrow j_a(y)\}_{a\in\mathcal{A}}.$$

This data provides a modification, with the axioms holding since each j_f is a natural isomorphism, defining the mapping of morphisms for the functor h. Now we must provide natural isomorphisms $H_a\colon j_a \Rightarrow i_a \cdot h$. Note that $ih(x) = i_a(j_-(x)) = j_a(x)$, so we pick H_a to be the identity, immediately satisfying naturality.

Suppose now that there are two functors $s, t\colon X \to \mathbf{Bicat}(\mathcal{A},\mathcal{B})(F,G)$, along with natural transformations $\psi_a\colon i_a \cdot s \Rightarrow i_a \cdot t$ for each $a \in \mathcal{A}$, satisfying the usual pasting diagrams. We want to provide a natural transformation $\bar{\psi}\colon s \Rightarrow t$ such that $1_{i_a} * \bar{\psi} = \psi_a$. For each $x \in X$ there are pseudonatural transformations $s(x)$ and $t(x)$, along with families of 2-cells

$$\{(\psi_a)_x \colon s(x)_a \Rightarrow t(x)_a\}_{a \in \mathcal{A}}$$

in $\mathcal{B}$. These 2-cells define a modification $(\psi_{-})_x$ for each $x \in X$, following from the initial pasting conditions assumed of the ψ_a. Defining $\bar{\psi}_x = (\psi_{-})_x$ completes the proof. □

It will be useful to express every category $\mathcal{C}$ as a bicolimit in **Cat**. It is well known, though explicitly described in [Bou10], that every small category is the codescent object of the diagram

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "id" description] & \mathcal{C}_1 \arrow[l, shift right=2ex, "s"'] \arrow[l, shift left=2ex, "t"] & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1. \arrow[l, shift right=2ex, "p"'] \arrow[l, "q" description] \arrow[l, shift left=2ex, "r"]
\end{tikzcd}$$

where $\mathcal{C}_0$ is the object set of $\mathcal{C}$, $\mathcal{C}_1$ is the set of morphisms in $\mathcal{C}$, and $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ is the set of composable morphisms in $\mathcal{C}$. This diagram is the coherence data for the 2-functor

$$\Delta_{\mathbf{1}} \colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Cat},$$

the constant 2-functor at the terminal category $\mathbf{1}$. Each of the sets in the diagram can be seen as a coproduct of terminal categories indexed over objects, morphisms, and pairs of composable morphisms, respectively. It is easy to see that $\mathcal{C}_0$ and $\mathcal{C}_1$ are indexed in the same way as we originally described coherence data. For $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ note that the coproduct should be indexed over 2-cells $\alpha\colon gf \Rightarrow h$ in $\mathcal{C}$. Since $\mathcal{C}$ is a category it has no non-trivial 2-cells, so the only such triangles are those declaring the identity $id\colon g \cdot f \Rightarrow gf$. Hence the last object in the diagram is indexed as required.

It is simple to see that an object satisfying the axioms of a codescent object [Lac02], [Str87] also satisfies those of a bicodescent object. A bicodescent object only requires existence of an induced 1-cell in the 1-dimensional property whereas a codescent object requires this to also be unique.

Chapter 3

Free Cocompletions for Bicategories

3.1 Copowers in bicategories

When we previously considered free cocompletions in the $\mathcal{V}$-enriched case we made use of copowers. Given a $\mathcal{V}$-category $\mathcal{C}$, an object $v \in \mathcal{V}$, and an object $c \in \mathcal{C}$, the copower $v \cdot c \in \mathcal{C}$ is the weighted colimit characterised by the isomorphism

$$\mathcal{C}(v \cdot c, d) \cong \mathcal{V}(v, \mathcal{C}(c, d))$$

in $\mathcal{V}$. Copowers are used when defining the induced functor $\widehat{F} \colon \widehat{\mathcal{C}} \to \mathcal{D}$ in the characterisation of the free cocompletion of $\mathcal{C}$. We will need a similar bicolimit to exist in order to define the free bicocompletion of a bicategory.

Definition 3.1.1. Let $\mathcal{B}$ be a bicategory. Given an object $b \in \mathcal{B}$ and a category $\mathcal{C}$, the *bicopower* $\mathcal{C} \cdot b$ is defined to be the object of $\mathcal{B}$ characterised by the following equivalence of categories

$$\mathcal{B}(\mathcal{C} \cdot b, d) \simeq \mathbf{Cat}(\mathcal{C}, \mathcal{B}(b, d)),$$

for all $d \in \mathcal{B}$.

We can describe bicopowers explicitly as bicolimits in $\mathcal{B}$.

Lemma 3.1.2. *Let $\mathcal{B}$ be a bicategory. If $\mathcal{C}$ is a category and $b \in \mathcal{B}$ then the bicopower $\mathcal{C} \cdot b$, if it exists, is the bicolimit of the pseudofunctor*

$$\Delta_b \colon \mathcal{C} \to \mathcal{B}$$

where $\mathcal{C}$ is considered as a locally discrete 2-category.

Proof. The bicolimit of the pseudofunctor Δ_b consists of an object $\mathrm{bic}\Delta_b$, 1-cells $\beta_c \colon b \to \mathrm{bic}\Delta_b$ for each $c \in \mathcal{C}$, and invertible 2-cells

$$\begin{tikzcd} b \arrow[r, bend left, "\beta_c"{name=U}] \arrow[r, bend right, "\beta_{c'}"'{name=D}] & \mathrm{bic}\Delta_b \arrow[from=U, to=D, phantom, "\Downarrow \beta_f"] \end{tikzcd}$$

for each $f \colon c \to c'$ in $\mathcal{C}$, subject to usual axioms of a bicolimit. We can then use the axioms of the bicolimit to show that there exists a functor

$$P \colon \mathbf{Cat}(\mathcal{C}, \mathcal{B}(b, d)) \to \mathcal{B}(\mathrm{bic}\Delta_b, d)$$

for each $d \in \mathcal{B}$ which is furthermore an equivalence. Since we have equivalences

$$\mathcal{B}(\mathcal{C} \cdot b, d) \simeq \mathbf{Cat}(\mathcal{C}, \mathcal{B}(b, d)) \simeq \mathcal{B}(\mathrm{bic}\Delta_b, d)$$

then $\mathrm{bic}\Delta_b \simeq \mathcal{C} \cdot b$. □

In many cases these bicopowers will be taken in **Cat** or $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ (where bicolimits can be computed pointwise) which reduces to the usual product formula, giving

$$\mathcal{C} \cdot G \simeq \mathcal{C} \times G$$

where $\mathcal{C} \in \mathbf{Cat}$ and $G \colon \mathcal{A}^{op} \to \mathbf{Cat}$ is a pseudofunctor.

3.2 Bicategorical cocompletion

In the first chapter we recalled the free cocompletion of a strict 2-category under strict **Cat**-weighted colimits. In this brief chapter we will describe the free bicocompletion of a bicategory, by which we mean the free cocompletion of a bicategory under bicolimits. The proofs in this chapter are much the same as the standard results for cocompletion. Showing that each pseudofunctor $\mathcal{A}^{op} \to \mathbf{Cat}$ is a bicolimit of representables takes a little more effort than before but the rest of the section is standard.

Remark 3.2.1. The colimit of any functor $F \colon \mathcal{C} \to \mathcal{D}$ between categories can be described as a coend. While more needs to be checked, in much the same way it can be shown that the bicolimit of any pseudofunctor $P\colon \mathcal{A} \to \mathcal{B}$ can be described as a bicodescent object:

$$\int^a P(a) \simeq \mathrm{bic}P.$$

Lemma 3.2.2. *Let $\mathcal{B}$ be a bicategory. The pseudofunctor $\mathcal{B}(-,b)\colon \mathcal{B}^{op} \to \mathbf{Cat}$ preserves bicolimits as bilimits, up to equivalence.*

Proof. Let $\mathcal{K}$ be a bicategory and $P\colon \mathcal{K} \to \mathcal{B}$ a pseudofunctor for which the bicolimit $\mathrm{bic}P$ exists. We wish to show that $\mathcal{B}(\mathrm{bic}P, b) \simeq \mathrm{bilim}\mathcal{B}(P-, b)$. First we note that the bilimit of a pseudofunctor $F\colon \mathcal{A} \to \mathcal{B}$ is equivalently given by the bidescent object $\int_a F(a)$, similarly to the previous remark. This then allows us to have the following series of equivalences:

$$\begin{aligned}
\mathrm{bilim}\mathcal{B}(P-, b) &\simeq \int_k \mathcal{B}(P(k), b) \\
&= \int_k \mathcal{B}(P(k), \Delta_b(k)) \\
&\simeq \mathbf{Bicat}(\mathcal{K}, \mathcal{B})(P, \Delta_b) \\
&\simeq \mathcal{B}(\mathrm{bic}P, b).
\end{aligned}$$

The first equivalence is the one just described, the second equality is clear, the third equivalence is the use of Lemma 2.5.1, and the final equivalence is the defining property of the bicolimit of P. ☐

By Lemma 1.5.8[Str80] we compute bicolimits of pseudofunctors $P\colon \mathcal{K} \to \widehat{\mathcal{A}}$ locally, as **Bicat** is pseudo-closed [HP02].

Lemma 3.2.3. *Each pseudofunctor $F\colon \mathcal{B}^{op} \to \mathbf{Cat}$ is a bicolimit of representable pseudofunctors given by an equivalence*

$$F \simeq \int^b Fb \times \mathcal{B}(-, b).$$

Proof. We will show that F can be described as the bicodescent object of the pseudofunctor

$$F \times \mathcal{B}(-,-)\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathbf{Bicat}(\mathcal{B}^{op}, \mathbf{Cat}).$$

Note that $(F \times \mathcal{B}(-,-))(a,b) = Fa \times \mathcal{B}(-,b)$. We will use the first definition of bicodescent object, using specific coherence data as described in Section 2.3. The coherence data for this pseudofunctor is spelled out below. As shorthand we will write Y_1^a, Y_2^a, and

Y_3^a for the objects in the diagram for the coherence data corresponding to $F \times \mathcal{B}(a.-)$, where Y_1^a is the coproduct indexed over objects in $\mathcal{B}$, Y_2^a over morphisms, and Y_3^a over 2-cells.

Working pointwise by fixing an object $a \in \mathcal{B}$ gives a pseudofunctor $F \times \mathcal{B}(a,-)\colon \mathcal{B}^{op} \times \mathcal{B} \to \mathbf{Cat}$. We describe each of the functors where in each case the subscript denotes which piece of the coproduct the object lives in, $h\colon m \to m'$ is a 1-cell in $\mathcal{B}$, and $\beta\colon g \Rightarrow g'$ is a 2-cell in $\mathcal{B}$.

$$\begin{aligned}
u\colon (m,g)_f &\longmapsto (Ff(m),g)_x\\
(h,\beta)_f &\longmapsto (Ff(h),\beta)_x\\
v\colon (m,g)_x &\longmapsto (m,g)_{id_x}\\
(h,\beta)_x &\longmapsto (h,\beta)_{id_x}\\
w\colon (m,g)_f &\longmapsto (m,f\cdot g)_y\\
(h,\beta)_f &\longmapsto (h,f*\beta)_y\\
p\colon (m,g)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (Ff_2(m),g)_{f_1}\\
(h,\beta)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (Ff_2(h),\beta)_{f_1}\\
q\colon (m,g)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (m,g)_{f_3}\\
(h,\beta)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (h,\beta)_{f_3}\\
r\colon (m,g)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (m,f_1\cdot g)_{f_2}\\
(h,\beta)_{\theta\colon f_2f_1\Rightarrow f_3} &\longmapsto (h,f_1*\beta)_{f_2}
\end{aligned}$$

The coherence 2-cells are given as below.

$$\begin{aligned}
\delta_{(m,g)_x} &= ((\phi_x^F)_m, id_g)_x\\
\gamma_{(m,g)_x} &= (id_m, l_g)_x\\
\kappa_{(m,g)_\theta} &= (F(\theta)_m\cdot(\phi_{f_2f_1}^F)_m, id_g)_x\\
\lambda_{(m,g)_\theta} &= (id_m, (\theta*1_g)\cdot\alpha_{f_2f_1g})_x\\
\rho_{(m,g)_\theta} &= (id_{Ff_2(m)}, id_{f_1\cdot g})
\end{aligned}$$

For each object $a \in \mathcal{B}$ there is a functor

$$i_a\colon \coprod_{x\in ob\mathcal{B}} Fx \times \mathcal{B}(a,x) \to Fa$$

sending objects $(m,g)_x$ to $Fg(m)$ and sending morphisms $(h,\beta)_x$ to the composite

$(F\beta)_{m'} \cdot Fg(h)$ as in the following naturality square for $F\beta$.

$$\begin{tikzcd}
Fg(m) \arrow[r, "Fg(h)"] \arrow[d, "(F\beta)_m"'] & Fg(m') \arrow[d, "(F\beta)_{m'}"] \\
Fg'(m) \arrow[r, "Fg'(h)"'] & Fg'(m')
\end{tikzcd}$$

There is a natural isomorphism

$$\begin{tikzcd}
Y_2^a \arrow[r, "u"] \arrow[d, "w"'] \arrow[dr, phantom, "\Downarrow\sigma_a"] & Y_1^a \arrow[d, "i"] \\
Y_1^a \arrow[r, "i"'] & Fa
\end{tikzcd}$$

with components $(\sigma_a)_{(m,g)_f} = (\phi^F_{fg})_m$ which satisfies the usual pasting axioms of a bicodescent object, the first corresponding to the composition axiom for the pseudofunctor F, and the second to the unit axiom for F.

We will show that each (Fa, i, σ) is the bicodescent object of the coherence data we previously described. Suppose that $G\colon \mathcal{B}^{op} \to \mathbf{Cat}$ is a pseudofunctor,

$$j\colon \coprod_{x\in ob\mathcal{B}} Fx \times \mathcal{B}(a,x) \Rightarrow G$$

is a pseudonatural transformation, and that there is an invertible modification Γ between ju and jw which satisfies the pasting axioms of a bicodescent object. We will now work locally, finding functors $t\colon Fa \to Ga$ which together constitute a pseudonatural transformation $\tau\colon F \Rightarrow G$. (We will abuse notation slightly here and often write t for t_a, and similarly for i, to avoid having too many subscripts.)

We want to describe a pseudonatural transformation $\tau\colon F \Rightarrow G$ such that $\tau \cdot i \cong j$, compatible with Γ and σ. For $m \in Fa$ put $t(m) = j(m, id_a)$. Given

$$(m,g) \in \coprod_{x\in ob\mathcal{B}} Fx \times \mathcal{B}(a,x)$$

there is an isomorphism

$$ti(m,g) = t(Fg(m)) = j(Fg(m), id_a) \cong j(m, g\cdot id_a) \cong j(m,g),$$

natural in (m,g), where the first isomorphism is given by $(\Gamma_a)_{(m,g)}$ and the second by

$j_a(id_m, r_g)$. This isomorphism is compatible with σ and Γ in the sense that we have an equality of pasting diagrams as follows.

$$\begin{tikzcd} Fa \arrow[d, "t"'] \arrow[dr, phantom, "\Downarrow\tau"] & Y_1^a \arrow[l, "i"'] \arrow[dl, "j"] \arrow[dr, phantom, "\Downarrow\Gamma"] & Y_2^a \arrow[l, "u"'] \arrow[dl, "w"] \\ Ga & Y_1^a \arrow[l, "j"] & \end{tikzcd} = \begin{tikzcd} Y_1^a \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\sigma"] & Y_2^a \arrow[l, "u"'] \arrow[d, "w"] \\ Fa \arrow[d, "t"'] \arrow[r, phantom, shift right=6, "\Downarrow\tau"] & Y_1^a \arrow[l, "i"'] \arrow[dl, "j"] \\ Ga & \end{tikzcd}$$

The equality follows from a combination of the bicodescent conditions that Γ satisfies, the right unit axiom in $\mathcal{B}$, and the naturality of Γ.

The final thing to check is that Fa satisfies the 2-dimensional aspect of the universal property. This means that if there are two functors $h, k\colon Fa \to Ga$ and a natural isomorphism $\Theta\colon h \cdot i \Rightarrow k \cdot i$ satisfying

$$\begin{tikzcd} & & Fa \arrow[dr, "h"] & \\ & Y_1^a \arrow[ur, "i"] \arrow[dr, "i"] \arrow[rr, phantom, "\Downarrow\Theta_a"] & & Ga \\ Y_2^a \arrow[ur, "u"] \arrow[dr, "w"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & Fa \arrow[ur, "k"'] & \\ & Y_1^a \arrow[ur, "i"'] & & \end{tikzcd} = \begin{tikzcd} & Y_1^a \arrow[dr, "i"] & & \\ Y_2^a \arrow[ur, "u"] \arrow[dr, "w"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & Fa \arrow[dr, "h"] & \\ & Y_1^a \arrow[ur, "i"] \arrow[dr, "i"'] \arrow[rr, phantom, "\Downarrow\Theta_b"] & & Ga \\ & & Fa \arrow[ur, "k"'] & \end{tikzcd}$$

then there is a unique natural isomorphism $\xi\colon h \Rightarrow k$ such that $\xi * i = \Sigma$. The component ξ_m is given by the composite

$$h(m) \xrightarrow{h(\phi_a^F)_m^{-1}} hFid_a(m) \xrightarrow{\Sigma_{(m,id_a)}} kFid_a(m) \xrightarrow{k(\phi_a^F)_m} k(m).$$

We then have the required equality of natural transformations following from unit axioms for F and the naturality for Σ. $\square$

The bicategory $\widehat{\mathcal{A}} = \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ has the following property. If $\mathcal{B}$ is a bicategory and $F\colon \mathcal{A} \to \mathcal{B}$ a pseudofunctor then there exists a bicolimit preserving pseudofunctor $\widehat{F}\colon \widehat{\mathcal{A}} \to \mathcal{B}$ which is a left adjoint to the pseudofunctor $\mathcal{B}(G-,-)\colon \mathcal{B} \to \widehat{\mathcal{A}}$, giving an equivalence of categories

$$\mathcal{B}(\widehat{F}(G), b) \simeq \widehat{\mathcal{A}}(G, \mathcal{B}(F-, b)).$$

Proposition 3.2.4. *Let $\mathcal{A}$ be a bicategory. The bicategory of pseudofunctors* $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ *satisfies the statement given above.*

Proof. We know that $\widehat{\mathcal{A}}$ has all bicolimits from the earlier discussion. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a pseudofunctor, where $\mathcal{B}$ is a bicategory with all bicolimits. Define

$$\widehat{F}\colon \widehat{\mathcal{A}} \to \mathcal{B}$$

as follows. We require $\widehat{F}$ to preserve bicolimits so we define

$$\widehat{F}(G) = \int^{a} Ga \cdot Fa$$

where $Ga \cdot Fa$ is the bicopower of Fa by the category Ga in $\mathcal{B}$. We must now show that $\widehat{F}$ preserves bicolimits in $\widehat{\mathcal{A}}$. Suppose that $P\colon \mathcal{K} \to \widehat{\mathcal{A}}$ is a pseudofunctor. We want to show that

$$\mathrm{bic}(\widehat{F}P) \simeq \widehat{F}(\mathrm{bic}P)$$

in $\mathcal{B}$. We have described how bicolimits can be expressed as bicodescent objects and so this becomes the requirement of an equivalence between the bicodescent objects

$$\int^{k}\int^{a} (P(k)(a) \cdot Fa) \simeq \int^{a} \left(\int^{k} P(k)\right)(a) \cdot Fa.$$

The first step is to use Fubini on the left hand side to swap the variables a and k. Since bicopowers and bicodescent objects are both bicolimits then they commute with each other, so we are done.

The final thing that we will prove here is the adjunction described above, namely that $\widehat{F}$ is a left adjoint to the pseudofunctor $\mathcal{B}(F-,-)\colon \mathcal{B} \to \widehat{\mathcal{A}}$. We appeal to the following sequence of equivalences, though it can be shown directly:

$$\begin{aligned}
\mathcal{B}(\widehat{F}(G), b) &\simeq \mathcal{B}\left(\int^{a} Ga \cdot Fa, b\right) \\
&\simeq \int_{a} \mathcal{B}(Ga \cdot Fa, b) \\
&\simeq \int_{a} \mathbf{Cat}(Ga, \mathcal{B}(Fa, b)) \\
&\simeq \widehat{\mathcal{A}}(G, \mathcal{B}(F-, b)).
\end{aligned}$$

The first equivalence is simply the definition of $\widehat{F}$, the second follows from Lemma 3.2.2, the third equivalence uses the fact that $Ga \cdot Fa$ is a bicopower, and the final equivalence is simply an instance of Lemma 2.5.1. □

It would be prudent to go on to show that the bicategory $\widehat{\mathcal{A}} = \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ is in fact the free bicocompletion of $\mathcal{A}$ by which we mean that there is a pseudonatural biequivalence

$$\mathbf{Bicocom}(\widehat{\mathcal{A}}, \mathcal{B}) \simeq \mathbf{Bicat}(\mathcal{A}, \mathcal{B}),$$

where $\mathbf{Bicocom}(\widehat{\mathcal{A}}, \mathcal{B})$ is the bicategory of bicolimit-preserving pseudofunctors $\widehat{\mathcal{A}} \to \mathcal{B}$.

Chapter 4

Promonoidal Bicategories

This chapter will introduce promonoidal bicategories, a categorification of Day's promonoidal categories, before beginning to investigate the connection with monoidal bicategories.

4.1 Promonoidal Bicategories

Promonoidal categories generalise the notion of monoidal categories, encompassing a number of familiar structures such as monoidal categories and closed categories (the latter actually arising from a copromonoidal structure). Many of the bicodescent objects we consider in the following chapters are over multiple variables. A previous lemma, Lemma 2.4.2, ensures that in all situations that follow, the Fubini theorem for bicodescent objects will hold.

Definition 4.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories. A *pseudoprofunctor* $P\colon \mathcal{A} \nrightarrow \mathcal{B}$ is a pseudofunctor $P\colon \mathcal{B}^{op} \times \mathcal{A} \to \mathbf{Cat}$.

Composition of pseudoprofunctors is given by a bicodescent object, exactly as profunctors are composed using coends. Given $P\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $Q\colon \mathcal{B} \to \mathcal{C}$ we define

$$(Q \cdot P)(c, a) = \int^{b} P(b, a) \times Q(c, b).$$

There is a tricategory, 2-**Prof**, with objects bicategories, 1-cells pseudoprofunctors, pseudonatural transformations as 2-cells, and modifications as 3-cells. This structure is described in [Chi15], where pseudoprofunctors are therein called biprofunctors. It would be useful to be able to express promonoidal bicategories as a pseudomonoid structure in 2-**Prof**. The definition given here is motivated on this basis: the axioms of a promonoidal bicategory are themselves essentially those of a tricategory. However, make no claim that our definition is in fact a pseudomonoid in 2-**Prof**, only that it would be desirable.

In the following definition we will be considering pseudofunctors $P\colon \mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \to \mathbf{Cat}$, $J\colon \mathcal{A}^{op} \to \mathbf{Cat}$, and $\mathcal{A}(-,-)\colon \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Cat}$. Often we will write $P(a,b,c)$ as abc, Ja as a, and $\mathcal{A}(a,b)$ as ab, where no confusion will arise. In a similar way, the components of the modifications in the following definition have objects in their source and target which are the result of profunctor composition and we denote

$$\int^x P(x,b,c) \times P(a,x,d)$$

by $xbc \cdot axd$.

Another notational convention that we will use is as follows. As we are omitting associations by deferring to the coherence theorem for bicategories, we will often label 1-cells slightly differently to usual to make explicit the identities involved in large cartesian products. For example, there are 1-cells such as $\mathbf{l}$ and $\mathbf{r}$, which we display as

$$PJPP \xrightarrow{1\mathbf{l}1} P\mathcal{A}P.$$

The above is then a shorthand for a 1-cell

$$\int^x P_{abc} \times Jx \times P_{dxe} \times P_{fgh} \xrightarrow{1\times\mathbf{l}\times 1} P_{abc} \times \mathcal{A}(d,e) \times P_{fgh}.$$

Similarly for $\mathbf{a}$: $\int^x P_{xbc} \times P_{axc} \to \int^x P_{xcd} \times P_{abx}$. We display this simply as

$$PP \xrightarrow{\mathbf{a}} PP.$$

The axioms later consist of up to four copies of P, involving such composites as

$$PPPP \xrightarrow{\mathbf{a}2} PPPP \xrightarrow{1\mathbf{a}1} PPPP \xrightarrow{\mathbf{a}2} PPPP,$$

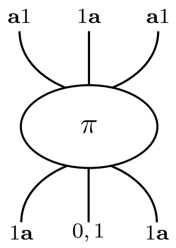
where $\mathbf{a}2$ means that $\mathbf{a}$ is being applied to the first two copies of P in the sequence, with an identity for the others. Similarly, $1\mathbf{a}1$ is an identity on the first P, an instance of $\mathbf{a}$, and an identity on the final P. A braiding such as $PJP \to JPP$ where the first P is interchanged with the J will be written as

$$PJP \xrightarrow{0,1} JPP,$$

meaning to braid the first two objects and leave an identity on the remaining P. In the axioms we will similarly depict coherence 2-cells, e.g., $\pi 1$ or $2\lambda 1$. A final notional issue is the use of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ when applying the Yoneda equivalences to specific inputs of P, with the index of each $\mathbf{y}_i$ corresponding in the obvious way.

Definition 4.1.2. A *promonoidal bicategory* $\mathcal{A}$ consists of the following data.

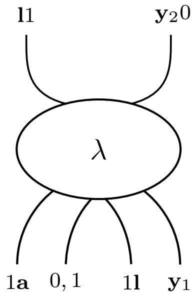
- A bicategory $\mathcal{A}$.
- A pseudofunctor $P\colon \mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \to \mathbf{Cat}$.
- A pseudofunctor $J\colon \mathcal{A}^{op} \to \mathbf{Cat}$.
- An adjoint equivalence $\mathbf{a}$ in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$, with components
$$\mathbf{a}_{abcd}\colon \int^x P(x,b,c) \times P(a,x,d) \to \int^x P(x,c,d) \times P(a,b,x).$$
- Adjoint equivalences $\mathbf{l}$ and $\mathbf{r}$ in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A}, \mathbf{Cat})$, with components
$$\mathbf{l}_{ab}\colon \int^x Jx \times P(a,x,b) \to \mathcal{A}(a,b),$$
$$\mathbf{r}_{ab}\colon \int^x Jx \times P(a,b,x) \to \mathcal{A}(a,b).$$
- An invertible modification π in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$ with components
$$\begin{tikzcd}
xbc \cdot yxd \cdot aye \arrow[r, "\mathbf{a}1"] \arrow[d, "1\mathbf{a}"'] & xcd \cdot ybx \cdot aye \arrow[dr, "1\mathbf{a}"] & \\
xbc \cdot yde \cdot axy \arrow[d, "0{,}1"'] & & xcd \cdot yxe \cdot aby \arrow[dd, "\mathbf{a}1"] \\
yde \cdot xbc \cdot axy \arrow[dr, "1\mathbf{a}"'] & \Downarrow \pi_{abcde} & \\
& yde \cdot xcy \cdot abx \arrow[r, equal, "id"'] & xde \cdot ycx \cdot aby
\end{tikzcd}$$
in $\mathbf{Cat}$, which we depict as follows.



- An invertible modification λ in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$ with components

$$\begin{tikzcd}
y \cdot xyb \cdot axc \arrow[rr, "\mathbf{1l}"] \arrow[d, "\mathbf{1a}"'] & & xb \cdot axc \arrow[dr, "\mathbf{y}_2"] & \\
 & \Downarrow \lambda_{abc} & & abc \\
y \cdot xbc \cdot ayx \arrow[dr, "{0,1}"'] & & & \\
 & xbc \cdot y \cdot ayx \arrow[r, "\mathbf{1l}"'] & xbc \cdot ax \arrow[uur, "\mathbf{y}_1"'] &
\end{tikzcd}$$

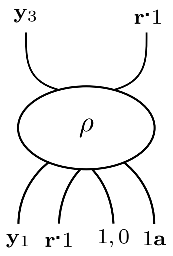
in **Cat**, which we depict as follows.



- An invertible modification in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$ with components

$$\begin{tikzcd}
abc \arrow[rr, "\mathbf{y}_3"] \arrow[d, "\mathbf{y}_1"'] & & xc \cdot abx \arrow[dr, "\mathbf{r}^{\cdot}1"] & \\
 & \Downarrow \rho_{abc} & & y \cdot xcy \cdot abx \\
ax \cdot xbc \arrow[dr, "\mathbf{r}^{\cdot}1"'] & & & \\
 & y \cdot axy \cdot xbc \arrow[r, "{1,0}"'] & y \cdot xbc \cdot axy \arrow[uur, "\mathbf{1a}"'] &
\end{tikzcd}$$

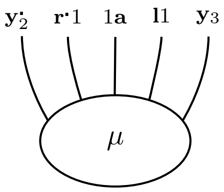
in **Cat**, which we depict as follows.



- An invertible modification μ in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$ with components

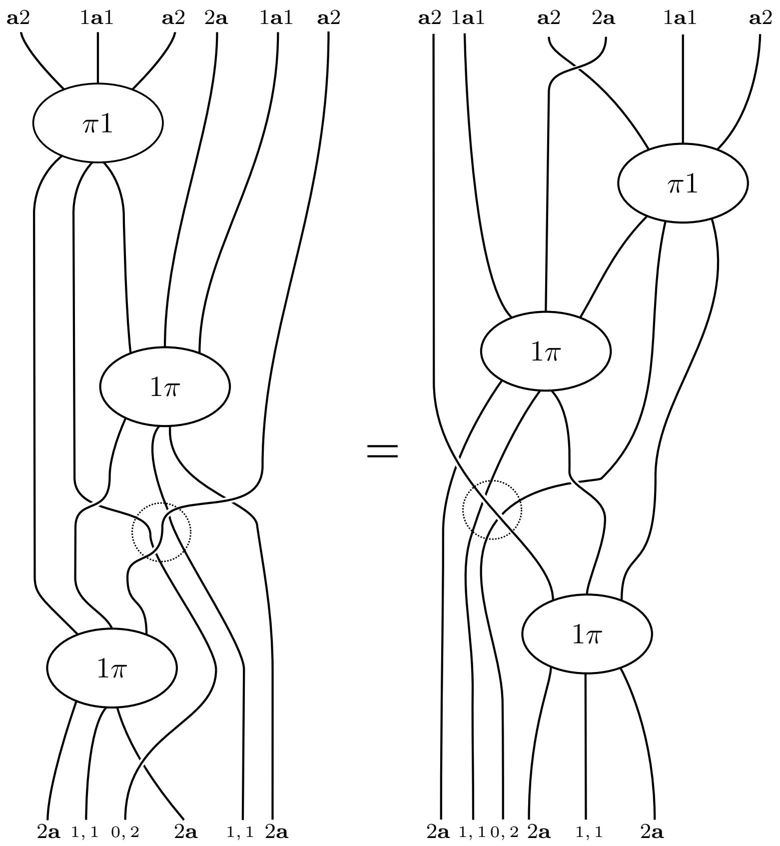
$$\begin{tikzcd}
 & & y \cdot xby \cdot axc \arrow[rr, "1\mathbf{a}"] & & y \cdot xyc \cdot abx \arrow[dr, "\mathbf{l}1"] & & \\
 & xb \cdot axc \arrow[ur, "\mathbf{r}^{\cdot}1"] & & \Downarrow \mu_{abc} & & xc \cdot abx \arrow[dr, "\mathbf{y}_3"] & \\
abc \arrow[ur, "\mathbf{y}^{\cdot}_2"] \arrow[rrrrrr, "id"'] & & & & & & abc
\end{tikzcd}$$

in **Cat**, which we depict as follows.

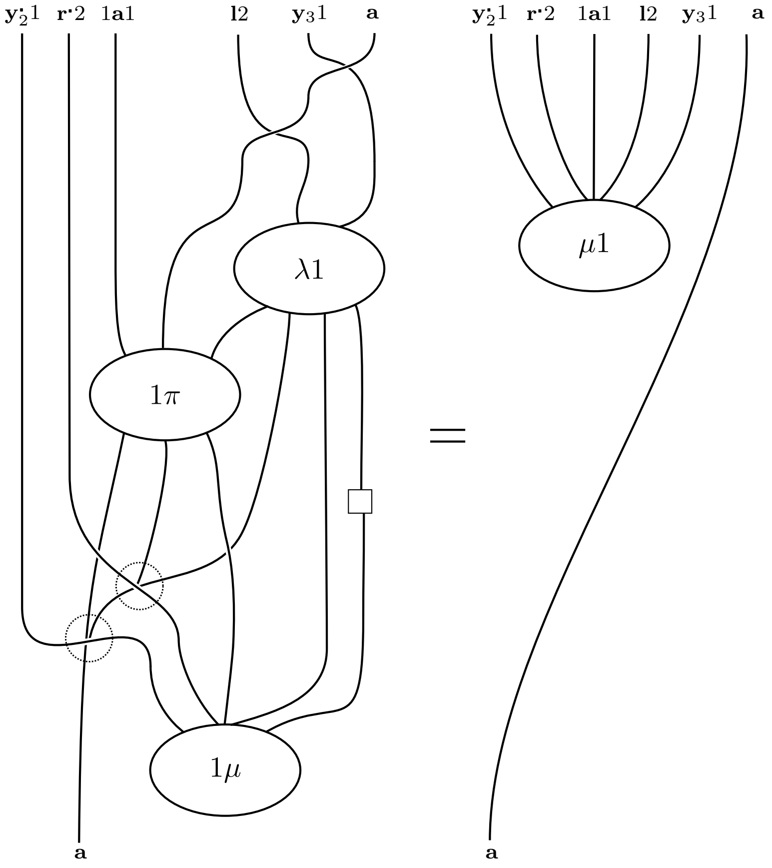


These data are subject to the following axioms.

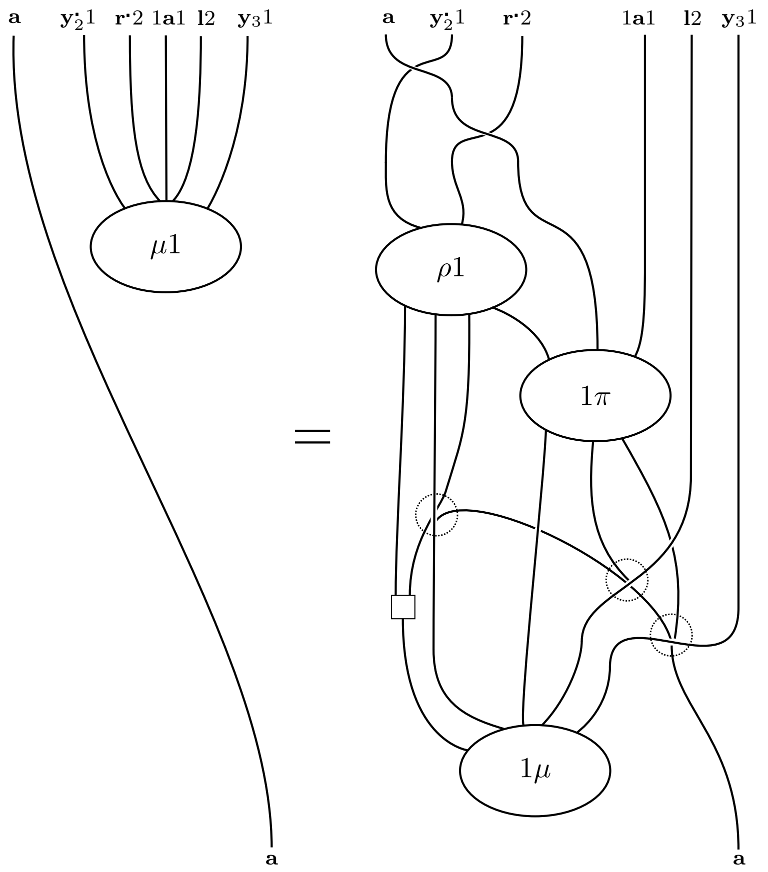
PB1 The following string diagram equation holds in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A}\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$.



PB2 The following string diagram equation holds in **Bicat**($\mathcal{A}^{op}\mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat}$).


y˙₂1
r˙2
1a1
l2
y₃1
a
λ1
1π
1μ
a
=
y˙₂1
r˙2
1a1
l2
y₃1
a
μ1
a

PB3 The following string diagram equation holds in $\mathbf{Bicat}(\mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}, \mathbf{Cat})$.



In the definition above the axioms involving λ and ρ appear to be very similar - one is almost the mirror image of the other, as we might expect. This slight mismatch seems to arise from the specification of λ and ρ and the placement of the Yoneda equivalences therein.

4.2 Monoidal bicategories as promonoidal bicategories

Let $(\mathcal{A}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r}, \pi, \lambda, \rho, \mu)$ be a monoidal bicategory. We can give $\mathcal{A}$ the structure of a promonoidal bicategory by defining $P(a, b, c) = \mathcal{A}(a, b \otimes c)$ and $Ja = \mathcal{A}(a, I)$. The 1-cells for the promonoidal structure are defined in the same way as the 1-dimensional case. Corresponding to **a** we have the composite

$$\begin{aligned}
\int^x P(x,b,c)\times P(a,x,d) &= \int^x \mathcal{A}(x,b\otimes c)\times\mathcal{A}(a,x\otimes d)\\
&\simeq \int^x \mathcal{A}(a,(b\otimes c)\otimes d)\\
&\xrightarrow{\int^x \mathcal{A}(1,\mathbf{a})} \int^x \mathcal{A}(a,b\otimes(c\otimes d))\\
&\simeq \int^x \mathcal{A}(x,c\otimes d)\times\mathcal{A}(a,b\otimes x).
\end{aligned}$$

For **l** we use the composite

$$\begin{aligned}
\int^x Jx\times P(a,x,b) &= \int^x \mathcal{A}(x,I)\times\mathcal{A}(a,x\otimes b)\\
&\simeq \mathcal{A}(a,I\otimes b)\\
&\xrightarrow{\mathcal{A}(1,\mathbf{l})} \mathcal{A}(a,b),
\end{aligned}$$

while for **r** we use

$$\begin{aligned}
\int^x Jx\times P(a,b,x) &= \int^x \mathcal{A}(x,I)\times\mathcal{A}(a,b\otimes x)\\
&\simeq \mathcal{A}(a,b\otimes I)\\
&\xrightarrow{\mathcal{A}(a,\mathbf{r})} \mathcal{A}(a,b).
\end{aligned}$$

Since each of **a**, **l**, and **r** are adjoint equivalences in $\mathcal{A}$, their images under the Yoneda mapping are also adjoint equivalences. They are then composed with Yoneda equivalences which are adjoint equivalences themselves and so the above composites are all adjoint equivalences.

We now have to define the coherence cells for the rest of the promonoidal structure on $\mathcal{A}$. In the following diagrams we write (a,b) in place of $\mathcal{A}(a,b)$, in order to save space.

The coherence cell corresponding to π is drawn below.

$$\begin{tikzcd}
&&&& (x,cd)(a,(bx)e) \arrow[rr, "1(1,\mathbf{a})"] && (x,cd)(a,b(xe)) \arrow[dr] &&&& \\
&&& (x,cd)(y,bx)(a,ye) \arrow[ur] && \cong && (x,cd)(y,xe)(a,by) \arrow[dr] &&& \\
&& (y,b(cd))(a,ye) \arrow[ur] \arrow[drr] && \cong && \cong && (y,(cd)e)(a,by) \arrow[dr, "(1,\mathbf{a})1"] && \\
& (y,(bc)d)(a,ye) \arrow[ur, "(1,\mathbf{a})1"] \arrow[drrr] & \cong && (a,(b(cd))e) \arrow[uuu, bend right] \arrow[rr, "(1,a)"] && (a,b((cd)e)) \arrow[uuu, bend left] \arrow[urr] \arrow[d, "(1,1\mathbf{a})"] & \cong && (y,c(de))(a,by) \arrow[dr] & \\
(x,bc)(y,xd)(a,ye) \arrow[ur] \arrow[dr] && \cong && (a,((bc)d)e) \arrow[u, "(1,\mathbf{a}1)"] \arrow[dr, "(1,\mathbf{a})"] & \Downarrow\mathcal{A}(1,\pi) & (a,b(c(de))) \arrow[urrr] \arrow[drrr] && \cong && (x,de)(y,cx)(a,by) \\
& (x,bc)(a,(xd)e) \arrow[urrr] \arrow[dr, "1(1,\mathbf{a})"] && \cong && (a,(bc)(de)) \arrow[ur, "(1,\mathbf{a})"] \arrow[drrr] && \cong && (y,de)(a,b(cy)) \arrow[ur] & \\
&& (x,bc)(a,x(de)) \arrow[urrr] \arrow[dr] &&& \cong &&& (y,de)(a,(bc)y) \arrow[ur, "1,(1,\mathbf{a})"] && \\
&&& (x,bc)(y,de)(a,xy) \arrow[rrrr] &&&& (y,de)(x,bc)(a,xy) \arrow[ur] &&&
\end{tikzcd}$$

The unlabelled 1-cells in the diagram above, as well as those that follow, are all Yoneda equivalences or their adjoints, apart from the few 1-cells which are clearly braidings. We depict the cell corresponding to λ below.

$$\begin{tikzcd}
&&& (x,Ib)(a,xc) \arrow[rr, "(1,\mathbf{l})1"] \arrow[dd] && (x,b)(a,xc) \arrow[ddrrr] &&& \\
&&&&& \cong &&& \\
(y,I)(x,yb)(a,xc) \arrow[uurrr] \arrow[dd] & \cong && (a,(Ib)c) \arrow[rrrrr, "(1,\mathbf{l}1)"] \arrow[ddr, "(1,\mathbf{a})"'] &&&&& (a,bc) \\
&&&&& \Downarrow\mathcal{A}(1,\lambda) &&& \\
(y,I)(a,(yb)c) \arrow[uurrr] \arrow[dr, "1(1,\mathbf{a})"'] && \cong && (a,I(bc)) \arrow[uurrrr, "(1,\mathbf{l})"'] \arrow[drrr] && \cong && (a,x)(x,bc) \arrow[uu] \\
& (y,I)(a,y(bc)) \arrow[urrr] \arrow[dr] &&& \cong &&& (a,Ix)(x,bc) \arrow[ur, "(1,\mathbf{l})1"'] & \\
&& (y,I)(x,bc)(a,yx) \arrow[rrrr] &&&& (y,I)(a,yx)(x,bc) \arrow[ur] &&
\end{tikzcd}$$

Finally, the coherence cell associated with ρ is as follows.

$$\begin{tikzcd}
 & & (x,c)(a,bx) \arrow[rr, "(1{,}\mathbf{r}^\cdot)1"] \arrow[d, phantom, "\cong"] & & (x,cI)(a,bx) \arrow[drr] & & \\
(a,bc) \arrow[urr] \arrow[rrrr, "(1{,}1\mathbf{r}^\cdot)"] \arrow[drrr, "(1{,}\mathbf{r}^\cdot)"'] \arrow[d] & & {} \arrow[d, phantom, "\Downarrow\mathcal{A}(1{,}\rho)"] & & (a,b(cI)) \arrow[u] \arrow[drr] & \cong & (y,I)(x,cy)(a,bx) \\
(a,x)(x,bc) \arrow[dr, "(1{,}\mathbf{r}^\cdot)1"'] & \cong & {} & (a,(bc)I) \arrow[ur, "(1{,}\mathbf{a})"'] \arrow[drr] & & \cong & (y,I)(a,b(cy)) \arrow[u] \\
 & (a,xI)(x,bc) \arrow[urr] \arrow[dr] & & \cong & & (y,I)(a,(bc)y) \arrow[ur, "1(1{,}\mathbf{a})"'] & \\
 & & (y,I)(a,xy)(x,bc) \arrow[rr] & & (y,I)(x,bc)(a,xy) \arrow[ur] & &
\end{tikzcd}$$

Theorem 4.2.1. *Let $\mathcal{A}$ be a monoidal bicategory. Then $\mathcal{A}$ posesses a promonoidal structure with data described as in Section 4.2.*

Proof. The size of the pasting diagrams involved in the axioms gets too large for us to include here. Proving this is a long but simple exercise in using modifications and naturality to shuffle the appropriate 2-cells around until they match up as per the monoidal bicategory axioms. ☐

Chapter 5

Day Convolution for Monoidal Bicategories

5.1 Day Convolution

In this section we will describe how a promonoidal structure on a bicategory $\mathcal{A}$ extends to a monoidal structure on the bicategory $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$. We begin by describing the monoidal product of two pseudofunctors $R, S \colon \mathcal{A}^{op} \to \mathbf{Cat}$ using the familiar convolution product [Day70]

$$R \star S = \int^{x} Rx \times \int^{y} Sy \times P_{-xy},$$

using bicodescent objects now rather than coends. We will often write such expressions as

$$Rx \underline{\times} Sy \underline{\times} P_{-xy},$$

ignoring associations. By Lemma 2.3.6, this is a pseudofunctor $\mathcal{A}^{op} \to \mathbf{Cat}$ and using the composition lemmas of Section 2.1 we can also see that the assignment $R \star S$ constitutes a pseudofunctor.

Lemma 5.1.1. *Let $\mathcal{A}$ be a promonoidal bicategory. Then*

$$\star \colon \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat}) \times \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat}) \to \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat}).$$

is a pseudofunctor.

Proof. The proof is similar to Lemma 2.3.7. In the following we will write $R \times S \times P \colon \mathcal{A}^{op} \times \mathcal{A}^{op} \times \mathcal{A} \times \mathcal{A} \to \mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ which takes (a', b', a, b) to $Ra' \times Sb' \times P_{-ab}$.

Given pseudofunctors $R, S, T, U \colon \mathcal{A}^{op} \to \mathbf{Cat}$, the bicodescent objects

$$i \colon R \times S \times P \overset{..}{\Rightarrow} \int^{a,b} Ra \times Sb \times P_{-ab}$$

and

$$j \colon T \times U \times P \overset{..}{\Rightarrow} \int^{a,b} Ta \times Ub \times P_{-ab}$$

exist in $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$. Given pseudonatural transformations $\delta \colon R \Rightarrow T$, $\gamma \colon S \Rightarrow U$, the action of $\star$ is induced by the universal property of the first bicodescent object above, using the composition lemmas, as in the diagram below.

$$\begin{tikzcd}
Ra \times Sb \times P_{-ab} \arrow[r, Rightarrow, "i_{ab}"] \arrow[d, Rightarrow, "\delta_a \times \gamma_b \times 1"'] \arrow[dr, phantom, "\Downarrow \Sigma_{ab}"] & \int^{a,b} Ra \times Sb \times P_{-ab} \arrow[d, Rightarrow, "\int^{a,b} \delta_a \times \gamma_b \times 1"] \\
Ta \times Ub \times P_{-ab} \arrow[r, Rightarrow, "j_{ab}"'] & \int^{a,b} Ta \times Ub \times P_{-ab}
\end{tikzcd}$$

□

We describe the associator and unitors in the same way as Day. Given three pseudofunctors $R, S, T \colon \mathcal{A}^{op} \to \mathbf{Cat}$, the associator for the monoidal structure is given by the sequence of adjoint equivalences

$$\begin{aligned}
(R \star S) \star T &\simeq Rx \underline{\times} Sy \underline{\times} P_{wxy} \underline{\times} Tz \underline{\times} P_{-wz} \\
&\simeq Rx \underline{\times} Sy \underline{\times} Tz \underline{\times} P_{wxy} \underline{\times} P_{-wz} \\
&\xrightarrow{1 \underline{\times} 1 \underline{\times} 1 \underline{\times} \mathbf{a}} Rx \underline{\times} Sy \underline{\times} Tz \underline{\times} P_{wyz} \underline{\times} P_{-xw} \\
&\simeq R \star (S \star T).
\end{aligned}$$

The left unitor is given by

$$\begin{aligned}
J \star R &\simeq Jx \underline{\times} Ry \underline{\times} P_{-xy} \\
&\simeq Ry \underline{\times} Jx \underline{\times} P_{-xy} \\
&\xrightarrow{1 \underline{\times} \mathbf{l}} Ry \underline{\times} \mathcal{A}(-, y) \\
&\xrightarrow{\mathbf{y}} R,
\end{aligned}$$

whilst the right unitor is similarly given by

$$\begin{aligned} R \star J &\simeq Rx \underline{\times} Jy \underline{\times} P_{-xy} \\ &\xrightarrow{1 \times \mathbf{r}} Rx \underline{\times} \mathcal{A}(-,x) \\ &\xrightarrow{\mathbf{y}} R, \end{aligned}$$

where in each case $\mathbf{y}$ denotes the Yoneda equivalence.

Day, on proving the convolution theorem in the case of promonoidal categories [Day70], comments:

> The proof of PC2 ⇒ MC3 requires a diagram that is too large for the space available . . .

The diagram that Day refers to is commutative, which is not the case for our promonoidal bicategories, which further does not bode well since we have to check coherence conditions on these non-commutative diagrams which involves sticking numerous instances of them together. To say that the diagrams that we require are too large for the space available may now be a gross understatement.

As such, we will depict the rest of the data in string diagram form. In our string diagrams we will label the source composite 1-cell, including an indication to the braiding. We will use a similar labelling in each diagram as previously done. For instance, the associator is a composite of a braiding and an instance of $\mathbf{a}$ from the promonoidal structure. We will label this composite by

$$RSPTP \xrightarrow{2,1} RSTPP \xrightarrow{3\mathbf{a}} RSTPP.$$

Here $2,1$ describes two identities whiskered with a braiding of T and P, followed by a whiskering with an identity, whilst $3\mathbf{a}$ describes three identities whiskered with $\mathbf{a}$.

Some of the following diagrams have small boxes in them. These boxes represent an isomorphism borne out of the extrapseudonaturality of the bicodescent objects being used. Often this means we are moving between a Yoneda equivalence on a contravariant pseudofunctor S, say, to a Yoneda equivalence on one of the covariant variable of P. Sometimes this also involves a braiding, resulting in the addition or removal of strings in the input or output of a box in the diagram.

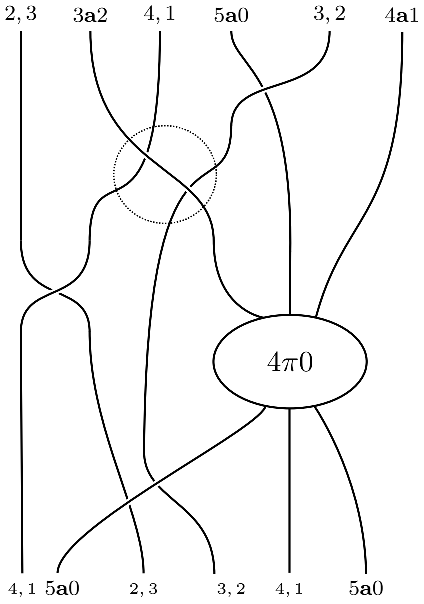
Certain 1-cells specified in the data, such as $\mathbf{y}$, $\mathbf{l}$, and $\mathbf{r}$, have domain bicodescent object featuring two objects of the image of P and J but only the hom-category in the codomain. There are then isomorphisms in the string diagrams which feature such

1-cells. For example, there is an invertible 2-cell in the following diagram that introduces extra braidings due to the number of objects increasing from the use of $\mathbf{r}^{\bullet}$. Such isomorphisms are highlighted in the string diagrams with dashed circles. An example of one of these isomorphisms can be seen as a 2-cell below.

$$\begin{tikzcd}
APP \arrow[r, "\mathbf{r}^{\bullet}2"] \arrow[dd, "0{,}1"'] & JPPP \arrow[dr, "1{,}1"] & \\
 & \cong & JPPP \\
PAP \arrow[r, "1\mathbf{r}^{\bullet}1"'] & PJPP \arrow[ur, "0{,}2"'] &
\end{tikzcd}$$

Before we specify the rest of the data for the convolution monoidal structure, a final remark is warranted. The string diagrams that we are using are largely for presentational purposes. Certain manipulations of the string diagrams correspond precisely to changes in the pasting diagram associated to them, certain braidings for example corresponding to interchange of 2-cell composition. Each pasting diagram can be reconstructed from a given string diagram in order to check the axioms required; the author, in checking these axioms, found it useful to write out each step as both a string diagram and a pasting diagram, especially for steps involving pseudonaturality of modifications as these are not easily spotted as manipulations in the string diagrams. We emphasise that the string diagrams themselves were not used to infer any of the following results. A good example of this sort of string notation can be found in the thesis of Buhné [Buh15].

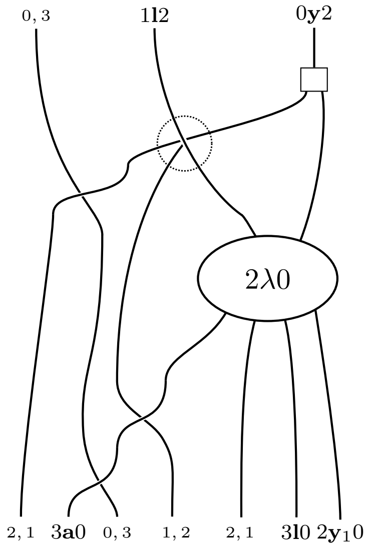
The first coherence cell, corresponding to π, is given by the following diagram.



As a pasting diagram it is as follows.

$$\begin{tikzcd}
& & & & RSTPPUP \arrow[rrr, "4{,}1"] & & & RSTPUPP \arrow[rrr, "5\mathbf{a}"] \arrow[drr, "3{,}2"'] & & \cong & RSTPUPP \arrow[drr, "3{,}2"] & & & & \\
& & RSTPPUP \arrow[urr, "3\mathbf{a}2"] \arrow[drr, "4{,}1"] & & & \cong & & & & RSTUPPP \arrow[rrr, "5\mathbf{a}"'] & & & RSTUPPP \arrow[drr, "4\mathbf{a}1"] & & \\
RSPTPUP \arrow[urr, "2{,}3"] \arrow[drr, "4{,}1"'] & & \cong & & RSTPUPP \arrow[rrr, "3{,}2"] \arrow[drr, "5\mathbf{a}"'] & & & RSTUPPP \arrow[urr, "4\mathbf{a}1"] \arrow[drr, "5\mathbf{a}"] & & & \Downarrow 4\pi & & & & RSTUPPP \\
& & RSPTUPP \arrow[urr, "2{,}3"] \arrow[drr, "5\mathbf{a}"'] & & \cong & & RSTPUPP \arrow[rrr, "3{,}2"'] & \cong & & RSTUPPP \arrow[rrr, "4{,}1"'] & & & RSTUPPP \arrow[urr, "5\mathbf{a}"'] & & \\
& & & & RSPTUPP \arrow[urr, "2{,}3"'] & & & & & & & & & &
\end{tikzcd}$$

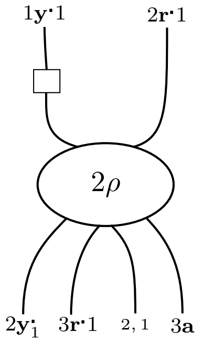
The coherence cell corresponding to λ is given by the following string diagram.



Displayed as a pasting diagram this gives the following.

$$\begin{tikzcd}
JRPSP \arrow[r, "{0,3}"] \arrow[dd, "{2,1}"'] \arrow[ddr, phantom, "\cong"] & RJPSP \arrow[rr, "1\mathbf{l}2"] \arrow[dd, "{2,1}"'] \arrow[ddrr, phantom, "\cong"] & & RASP \arrow[d, "{1,1}"'] \arrow[dr, "\mathbf{y}2"] \arrow[dr, phantom, "\cong", near start, bend right] & \\
& & & RSAP \arrow[r, "2\mathbf{y}_2 1"'] \arrow[dd, phantom, "\Downarrow 2\lambda"] & RSP \\
JRSPP \arrow[r, "{0,3}"'] \arrow[dd, "3\mathbf{a}"'] \arrow[ddr, phantom, "\cong"] & RJSPP \arrow[r, "{1,2}"] \arrow[dd, "3\mathbf{a}"] \arrow[ddr, phantom, "\cong"] & RSJPP \arrow[ur, "2\mathbf{l}1"] \arrow[dd, "3\mathbf{a}"'] & & \\
& & & & RSPA \arrow[uu, "2\mathbf{y}_1"'] \\
JRSPP \arrow[r, "{0,3}"'] & RSJPP \arrow[r, "{1,2}"'] & RSJPP \arrow[r, "{2,1}"'] & RSPJP \arrow[ur, "3\mathbf{l}"'] &
\end{tikzcd}$$

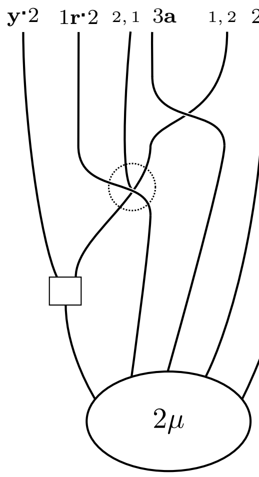
For the coherence cell corresponding to ρ we use the following string diagram.



This can also be interpreted as the following pasting diagram.

$$\begin{tikzcd}
RSP \arrow[rr, bend left, "1\mathbf{y}^\bullet 1"] \arrow[rr, bend right, "2\mathbf{y}^\bullet_3"'] \arrow[rr, phantom, "\cong"] \arrow[d, "2\mathbf{y}^\bullet_1"'] & & RSAP \arrow[dr, "2\mathbf{r}^\bullet"] & \\
RSPA \arrow[dr, "3\mathbf{r}^\bullet 1"'] & \Downarrow 2\rho & & RSJPP \\
& RSPJP \arrow[r, "{2,1}"'] & RSJPP \arrow[ur, "3\mathbf{a}"'] &
\end{tikzcd}$$

The final coherence cell, corresponding to μ, is given by the following string diagram.



We depict this final coherence cell as the pasting diagram below.

$$\begin{tikzcd}
 & & & RJSPP \arrow[dr, "1{,}2"] & & & \\
 & & RJSPP \arrow[ur, "3\mathbf{a}"] \arrow[dr, "1{,}2"] & \cong & RSJPP \arrow[ddrr, bend left, "2\mathbf{l}1"] \arrow[ddrr, bend right, "2\mathbf{y}_3"'] & & \\
 & RJPSP \arrow[ur, "2{,}1"] & \cong & RSJPP \arrow[ur, "3\mathbf{a}"] & & \cong & \\
RASP \arrow[ur, "1\mathbf{r}^{\bullet}2"] \arrow[drr, "1{,}1"] & & & \Downarrow 2\mu & & & RSAP \arrow[dd, "1\mathbf{y}1"] \\
 & \cong & RSAP \arrow[uur, "2\mathbf{r}^{\bullet}1"] & & & & \\
RSP \arrow[uu, "\mathbf{y}^{\bullet}2"] \arrow[urr, "2\mathbf{y}^{\bullet}_2"'] \arrow[rrrrrr, "id"'] & & & & & & RSP
\end{tikzcd}$$

The following proof refers to figures in the appendix.

Theorem 5.1.2. *If $\mathcal{A}$ is a promonoidal bicategory then* $\mathbf{Bicat}(\mathcal{A}^{op}, \mathbf{Cat})$ *is a monoidal bicategory as described above.*

Proof. We begin by showing that the first monoidal bicategory axiom, MB1, is satisfied via a sequence of equalities between string diagrams. There are four diagrams in the sequence, with the first and last being the two required to be equal for the axiom MB1 to hold. Careful manipulation of 2-cells using pseudonaturality and coherence cells shows that the left hand diagram in Figure 5.1 is equal to the diagram on the right. The diagram on the right hand side of Figure 5.1 is then equal to the diagram on the left hand side of Figure 5.2, using the promonoidal bicategory axiom PB1. Similar manipulation of the diagram on the left hand side of Figure 5.2 then results in the final diagram on the right hand side of Figure 5.2, showing that MB1 holds for the monoidal structure described.

The axiom MB2 requires the diagram on the left hand side of Figure 5.3 to be equal to the diagram in Figure 5.4. This is the most intricate to prove of the monoidal bicategory axioms, simply due to how complex the coherence cell for λ we specified is. The first step is to take the diagram in Figure 5.3 involving $2\mu 2$ and move this past the braiding $2, 1$. This introduces many instances of the highlighted isomorphisms, as seen in the diagram on the right hand side of Figure 5.3. However, once we have established this then we can use the promonoidal bicategory axiom PB2 to change the resulting $3\mu 1$ into the bottom of the right hand diagram of Figure 5.3.

The proof of axiom MB3 is similar to the way in which we proved MB2 but ends up being far simpler to check. We can immediately use axiom PB3 to obtain the equality in Figure 5.5. The tricky part of this proof is shuffling the resulting $3\rho 1$ upwards in the diagram to become a $2\rho 2$ as in Figure 5.6.

□

5.2 Monoidal bicocompletion

In the standard setting for Day convolution it is true that if $\mathcal{C}$ is a monoidal category then $[\mathcal{C}^{op}, \mathbf{Set}]$ is the free monoidal cocompletion of $\mathcal{C}$. This means that $\widehat{\mathcal{C}}$ is the free cocompletion and each of the endofunctors $- \star G, F \star -\colon \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ preserve colimits. More precisely there is an adjunction given by an equivalence of categories

$$\mathbf{MonCocomp}(\widehat{\mathcal{C}}, \mathcal{D}) \simeq \mathbf{MonCat}(\mathcal{C}, \mathcal{D})$$

where, for a monoidally cocomplete category $\mathcal{D}$, $\mathbf{Cocomp}(\widehat{\mathcal{C}}, \mathcal{D})$ is the category of monoidal colimit-preserving functors and monoidal natural transformations [IK86]. We have a similar setting for monoidal bicategories in that if $\mathcal{A}$ is a monoidal bicategory then $\widehat{\mathcal{A}}$ is monoidally bicocomplete, so each of the endopseudofunctors $- \star S, R \star -\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ preserve bicolimits.

Proposition 5.2.1. *Let $\mathcal{A}$ be a monoidal bicategory. The pseudofunctors*

$$- \star S, R \star -\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$$

preserve bicolimits, where $R, S\colon \mathcal{A}^{op} \to \mathbf{Cat}$ are pseudofunctors.

Proof. This is a simple exercise in commutativity of bicolimits and we will briefly describe one of the cases above. Let $P\colon \mathcal{K} \to \widehat{\mathcal{A}}$ be a pseudofunctor. There is then a pseudofunctor

$$\begin{aligned}(\mathrm{bic}P) \star S &= \int^{a,b} (\mathrm{bic}P)(a) \times Sb \times \mathcal{A}(-, a \otimes b) \\ &\simeq \int^{a,b} \left(\int^{k} P(k) \right) (a) \times Sb \times \mathcal{A}(-, a \otimes b)\end{aligned}$$

whereas the other bicolimit is given by the pseudofunctor

$$\begin{aligned}\mathrm{bic}((- \star S) \cdot P) &= \int^{k} P(k) \star S \\ &\simeq \int^{k} \int^{a,b} P(k)(a) \times Sb \times \mathcal{A}(-, a \otimes b).\end{aligned}$$

The other case is similar. □

We would like to further prove that $\widehat{\mathcal{A}}$ is the free monoidal bicocompletion of $\mathcal{A}$. This would mean that given any other monoidally bicocomplete bicategory $\mathcal{B}$ and a monoidal pseudofunctor $F \colon \mathcal{A} \to \mathcal{B}$ there exists a monoidal bicolimit-preserving pseudofunctor $\widehat{F} \colon \widehat{\mathcal{A}} \to \mathcal{B}$. Furthermore, given two monoidal bicolimit-preserving functors $H, K \colon \widehat{\mathcal{A}} \to \mathcal{B}$, where $\mathcal{B}$ is monoidally bicocomplete, along with a monoidal pseudonatural transformation $\beta \colon Hy \Rightarrow Ky$, there is a unique monoidal pseudonatural transformation $\beta' \colon H \Rightarrow K$ such that $\beta' * 1_y = \beta$. The above requirements would then constitute a pseudonatural biequivalence

$$\mathbf{MonBicocom}(\widehat{\mathcal{A}}, \mathcal{B}) \simeq \mathbf{MonBicat}(\mathcal{A}, \mathcal{B}),$$

where $\mathbf{MonBicocom}(\widehat{\mathcal{A}}, \mathcal{B})$ is the bicategory of monoidal bicolimit-preserving pseudofunctors $\widehat{\mathcal{A}} \to \mathcal{B}$, monoidal pseudonatural transformations, and modifications. The monoidal pseudofunctors we speak of are those defined by Day and Street [DS97], being a distinguished kind of functor between tricategories [GPS95]. However, we consider the algebraic variant [Gur13b] in line with our definitions of monoidal and promonoidal bicategories.

We begin to set up the adjunction above, however we do not present all of the details required to prove the biequivalence. First we should check that each $\widehat{F} \colon \widehat{\mathcal{A}} \to \mathcal{B}$ is in fact monoidal, the first step in which we establish an adjoint equivalence between

$$\widehat{F}(R) \otimes \widehat{F}(S) = \left(\int^{a} Ra \cdot Fa\right) \otimes \left(\int^{b} Sb \cdot Fb\right)$$

and

$$\widehat{F}(R \star S) = \int^{c} \left(\int^{a,b} Ra \times Sb \times \mathcal{A}(c, a \otimes b)\right) \cdot Fc.$$

This makes use of the fact that $\mathcal{B}$ is monoidally bicocomplete and so its tensor product preserves bicolimits in each variable, as well various properties of bicopowers and bicodescent objects, along with some of the coherence cells which make F a monoidal pseudofunctor. There is an obvious pseudofunctor

$$\mathbf{MonBicocom}(\widehat{\mathcal{A}}, \mathcal{B}) \to \mathbf{MonBicat}(\mathcal{A}, \mathcal{B})$$

given by precocomposition with $y \colon \mathcal{A} \to \widehat{\mathcal{A}}$ which is the basis of this biequivalence.

5.3 Biclosed structure on presheaves

We give a remark on using Day convolution to define a biclosed structure on $\widehat{\mathcal{A}}$. The ideas involved are the essentially the same as those in [Day70], again using bicodescent and bidescent objects rather than coends and ends. Closed bicategories were defined originally by Bénabou [Bén73]. We use the definition from [Sta16], though omit the requirement of symmetry and so consider a biclosed structure rather than a closed structure on a symmetric monoidal bicategory.

Definition 5.3.1. A monoidal category $\mathcal{A}$ is *biclosed* if each of the pseudofunctors

$$- \otimes b \colon \mathcal{A} \to \mathcal{A}, a \otimes - \colon \mathcal{A} \to \mathcal{A}$$

has a right pseudoadjoint, where $a, b \in \mathcal{A}$.

We begin to define the biclosed monoidal structure on the bicategory $\widehat{\mathcal{A}}$. By Theorem 5.1.2 $\widehat{\mathcal{A}}$ is a monoidal bicategory and we define right pseudoadjoints to the pseudofunctors

$$- \star S, R \star - \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$$

as follows. For $- \star S$, define

$$T/S = \int_c \mathbf{Cat}\left(\int^b Sb \times P_{c-b}, Tc\right).$$

Similarly, for $R \star -$, define

$$R\backslash T = \int_c \mathbf{Cat}\left(\int^a Ra \times P_{ca-}, Tc\right).$$

5.4 Further Remarks

The axioms in the definition of a promonoidal bicategory were motivated by the desire for these to be pseudomonoid objects in a monoidal tricategory 2-**Prof** of bicategories, pseudoprofunctors, pseudonatural transformations, and modifications. It would seem that such a result could be shown directly though the work involved in the proof would surely be cumbersome. The definition of promonoidal bicategories could be recast in the setting of **Gray**-monoids in order to reduce the complexity of the diagrams involved.

The setting of our results is rather specific. In the 1-dimensional setting a convolution product is defined on $[\mathcal{C}^{op}, \mathcal{V}]$ whenever $\mathcal{C}$ is a promonoidal $\mathcal{V}$-category, for a suitable enriching category $\mathcal{V}$. However our 2-dimensional result only applies when $\mathcal{A}$ is a promonoidal bicategory, with no notion of enrichment. One suspects the results could be developed further in the setting where $\mathcal{A}$ is a $\mathcal{V}$-enriched bicategory, where $\mathcal{V}$ is a suitable monoidal bicategory [GS16], in order to define a convolution product on $\mathbf{Bicat}(\mathcal{A}^{op}, \mathcal{V})$. Previous comments addressing **Gray**-monoids would also be suitable to consider here.

Many of the cocompletion statements lack a full resolution. It would be desirable to give a more full treatment to these ideas, including discussions of size. Such a treatment would surely use the notions of relative pseudomonads [FGHW16].

Appendix: String Diagrams

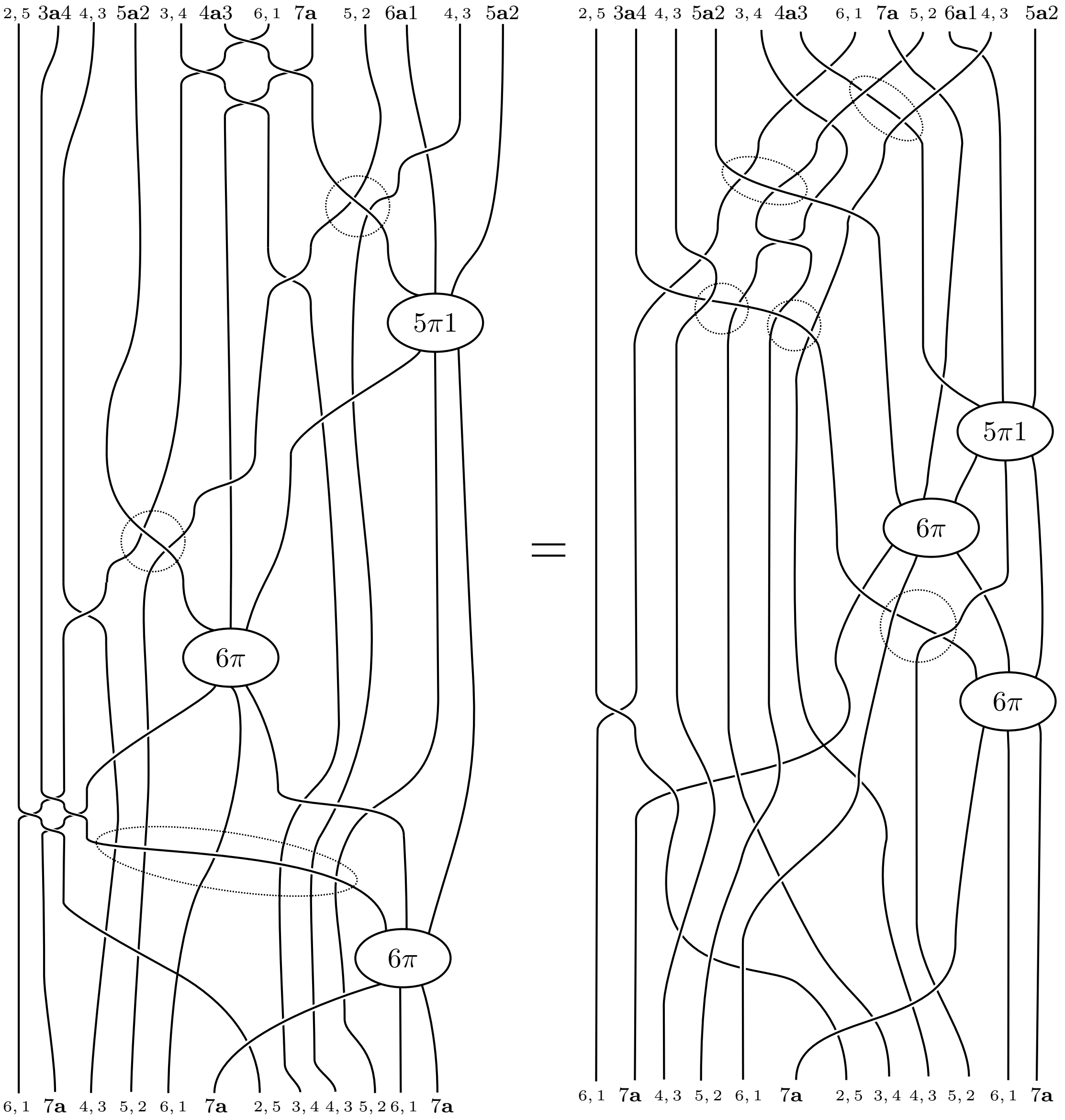


Figure 5.1: First equation to prove MB1.

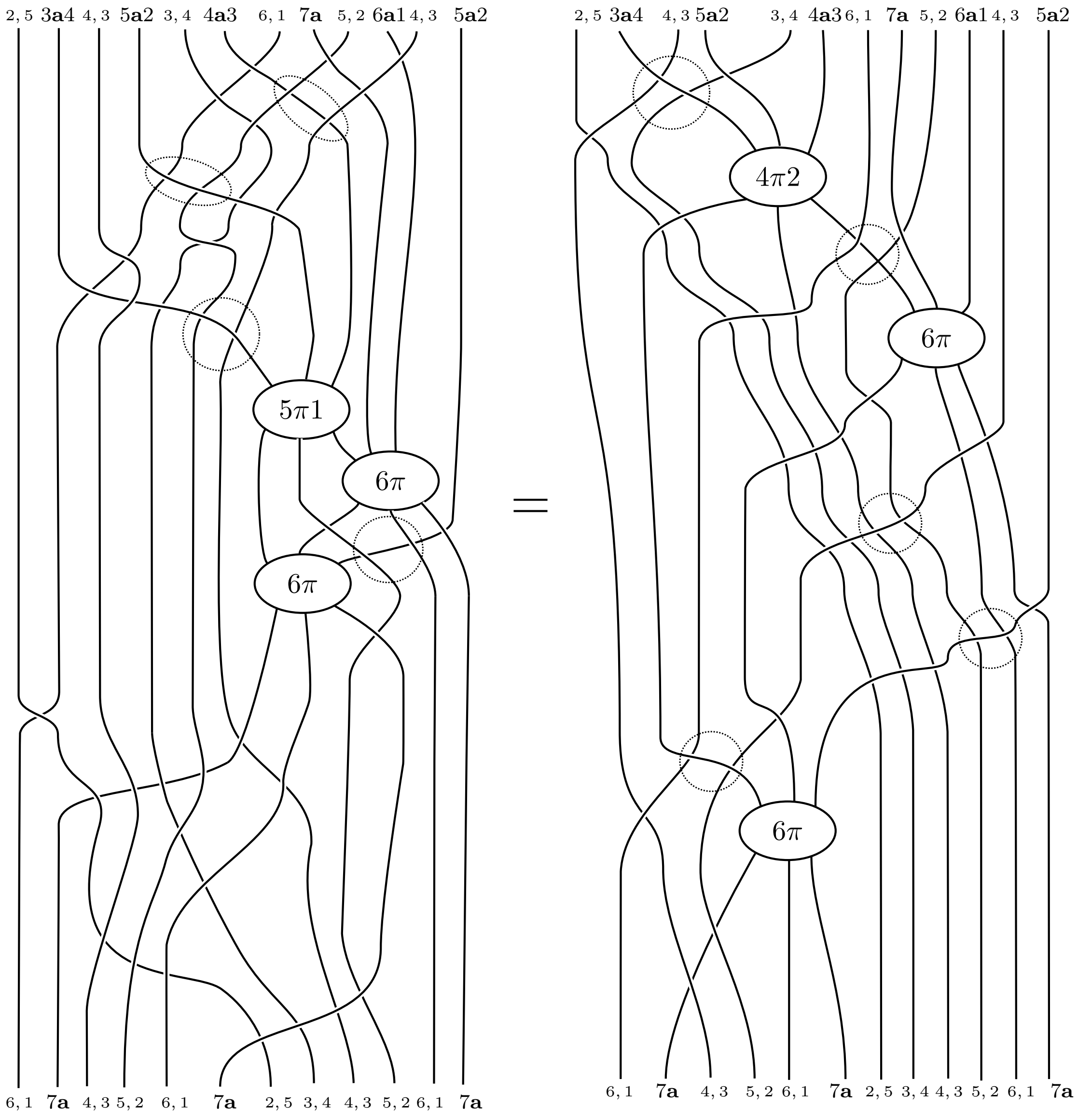


Figure 5.2: Third equation to prove MB1.

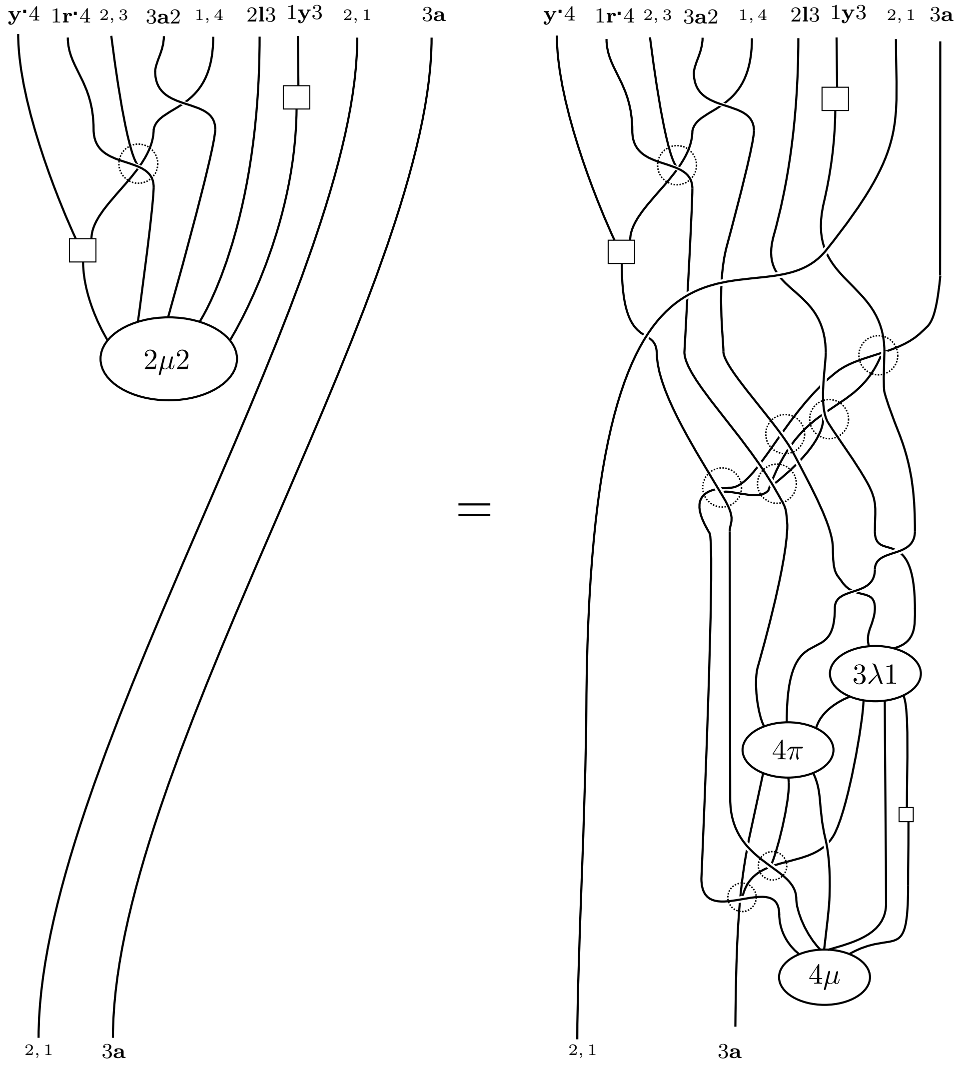


Figure 5.3: First equation to prove MB2, using PB2.

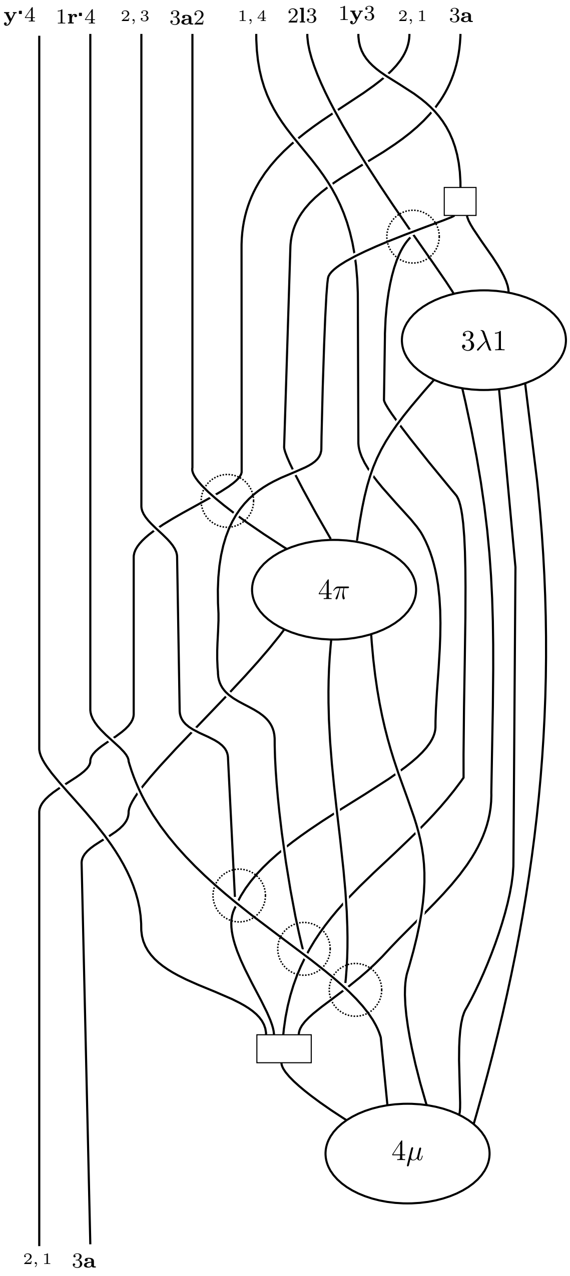


Figure 5.4: Final step in proving MB2.

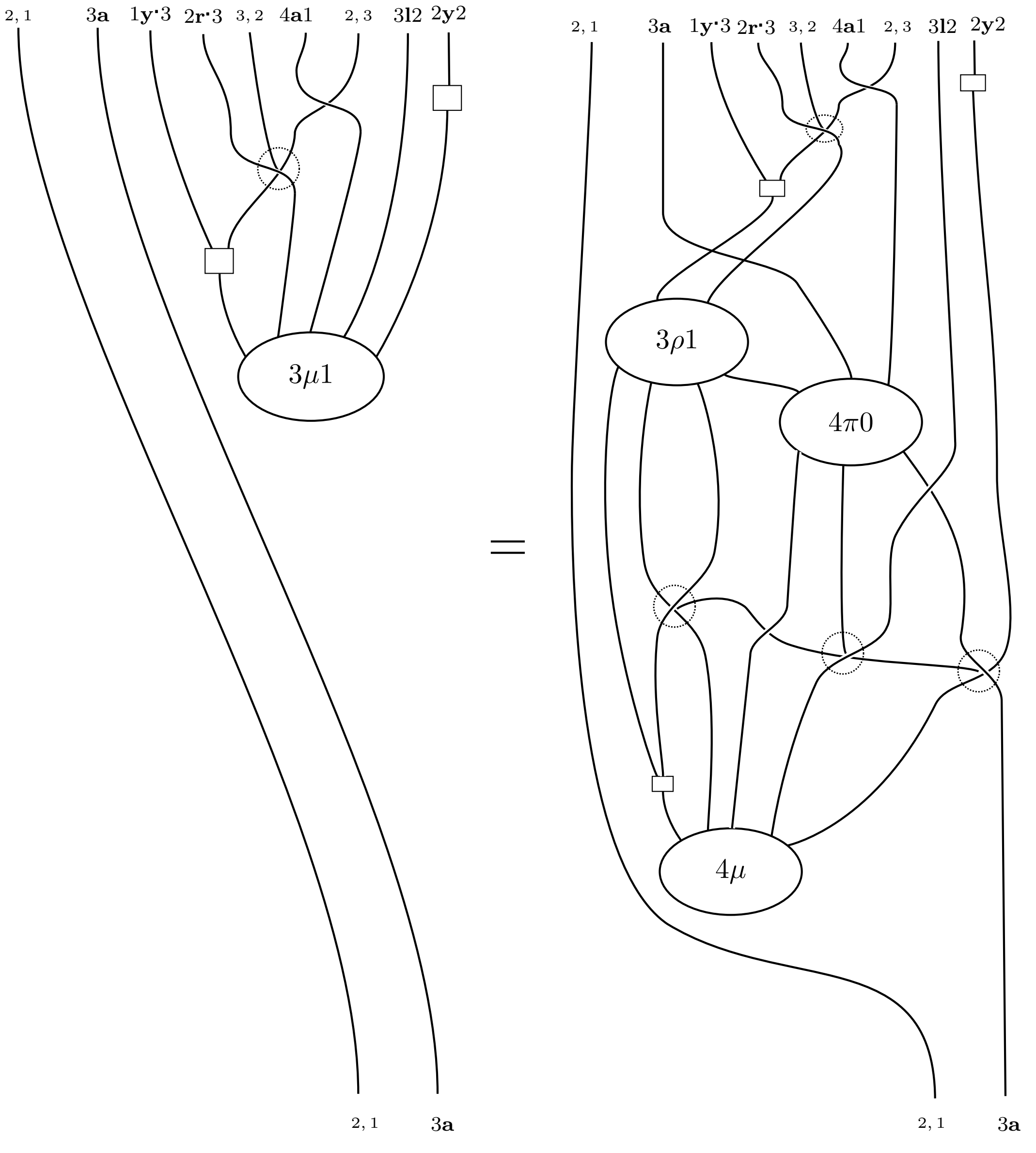


Figure 5.5: Use of axiom PB3 to prove MB3.

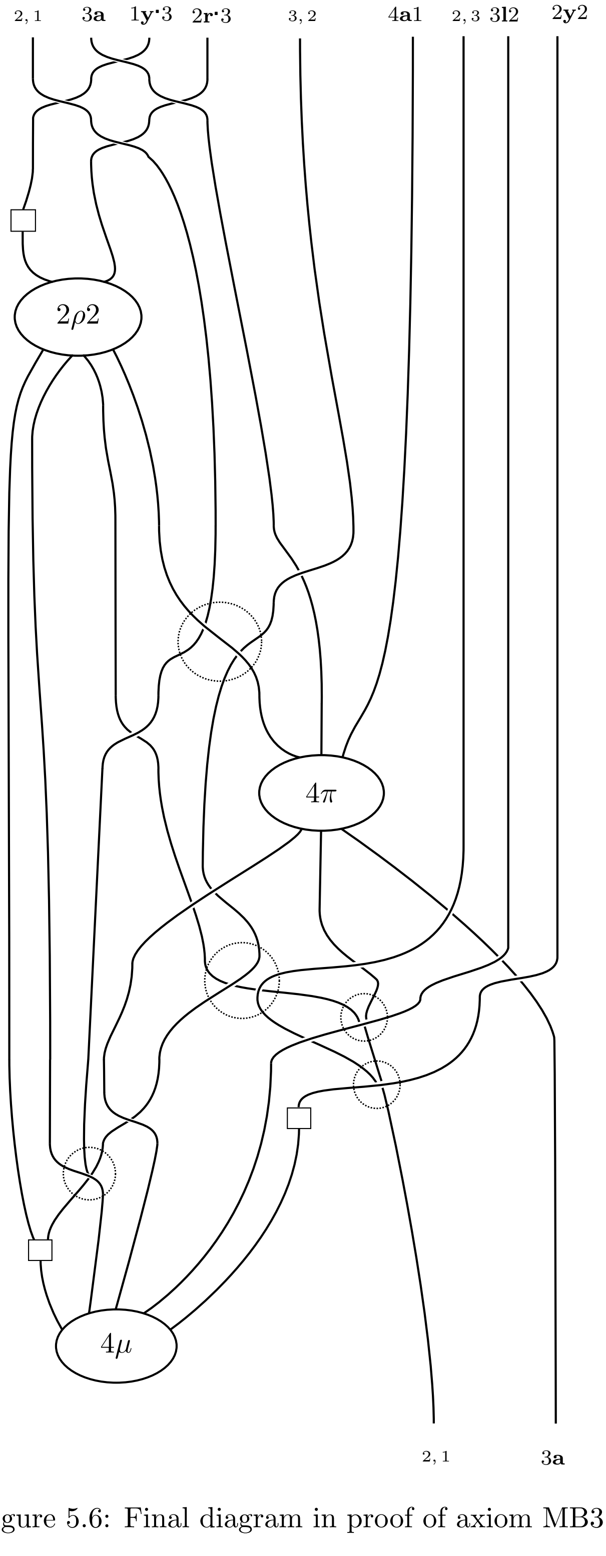


Figure 5.6: Final diagram in proof of axiom MB3.

References

[ACU10] T. Altenkirch, J. Chapman, and T. Uustalu. *Monads Need Not Be Endofunctors*, pages 297–311. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010. 3

[ACU14] T. Altenkirch, J. Chapman, and T. Uustalu. Relative monads formalised. *Journal of Formalized Reasoning*, 7(1):1–43, 2014. 3

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. 3, 11

[Bén67] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, volume 47 of *Lecture Notes in Mathematics*, pages 1–77. Springer Berlin Heidelberg, 1967. 3, 4, 16, 17

[Bén73] J. Bénabou. Les distributeurs. *Univ. Catholique de Louvain, Séminaires de Math. Pure*, Rapport No. 33, 1973. 86

[BG88] R. Betti and M. Grandis. Complete theories in 2-categories. *Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques*, 29(1):9–57, 1988. 20

[Bor94] F. Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures. 7, 12

[Bou10] J. Bourke. Codescent objects in 2-dimensional universal algebra. *PhD Thesis, University of Sydney*, 2010. 56

[Buh15] L. Buhné. Topics in three-dimensional descent theory. *PhD Thesis, Universität Hamburg*, 2015. 80

[CF07] G. L. Cattani and M. P. Fiore. The bicategory-theoretic solution of recursive domain equations. *Electronic Notes in Theoretical Computer Science*, 172:203 – 222, 2007. 19

[CG11] E. Cheng and N. Gurski. The periodic table of n-categories for low-dimensions ii: degenerate tricategories. *Cahiers Topologie Géom. Différentielle Catég. 52 (2011), no 2., 82125*, 2011. 4

[Chi15] D. Chikhladze. Lax formal theory of monads, monoidal approach to bicategorical structures and generalized operads. *Theory Appl. Categ.*, 30:No. 10, 332–386, 2015. 5, 65

[CHP13] E. Cheng, M. Hyland, and J. Power. Pseudo-distributive laws. *Electron. Notes Theor. Comput. Sci.*, 83:227–245, January 2013. 3

[Cot13] T. Cottrell. Comparing algebraic and non-algebraic foundations of n-category theory. *PhD Thesis, University of Sheffield; arXiv:1405.7263*, 2013. 4

[Day70] B. Day. *Reports of the Midwest Category Seminar IV*, chapter On closed categories of functors, pages 1–38. Springer Berlin Heidelberg, Berlin, Heidelberg, 1970. i, 2, 77, 79, 86

[Day71] B. Day. Construction of biclosed categories. *PhD Thesis, University of New South Wales*, 1971. 2, 14

[DS97] B. Day and R. Street. Monoidal bicategories and hopf algebroids. *Advances in Mathematics*, 129(1):99 – 157, 1997. 12, 85

[EK66a] S. Eilenberg and G. M. Kelly. Closed categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562. Springer, New York, 1966. 25

[EK66b] S. Eilenberg and G. M. Kelly. A generalization of the functorial calculus. *J. Algebra*, 3:366–375, 1966. i, 2, 8, 25, 30

[FGHW16] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. Relative pseudomonads, kleisli bicategories, and substitution monoidal structures. *Preprint, arXiv:1612.03678*, 2016. 3, 11, 87

[GG09] R. Garner and N. Gurski. The low-dimensional structures formed by tricategories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146(3):551589, May 2009. 20

[GPS95] R. Gordon, A. J. Power, and R. Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995. 4, 85

[Gra74] J.W. Gray. *Formal Category Theory : Adjointness for 2-Categories*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1974. 4

[Gra76] J.W. Gray. *Algebra, Topology, and Category Theory: A Collection of Papers in Honor of Samuel Eilenberg*. Academic Press 63-76, 1976. 4

[GS16] R. Garner and M. Shulman. Enriched categories as a free cocompletion. *Advances in Mathematics*, 289:1–94, 2016. 87

[Gur12] N. Gurski. Biequivalences in tricategories. *Theory Appl. Categ.*, 26:349–384, 2012. 19

[Gur13a] N. Gurski. *Coherence in Three-Dimensional Category Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 2013. 4, 19, 20

[Gur13b] N. Gurski. The monoidal structure of strictification. *Theory and applications of Categories*, 28(1):1–23, 2013. 85

[HP02] M. Hyland and J. Power. Pseudo-commutative monads and pseudo-closed 2-categories. *Journal of Pure and Applied Algebra*, 175(13):141 – 185, 2002. Special Volume celebrating the 70th birthday of Professor Max Kelly. 59

[IK86] G. B. Im and G.M. Kelly. A universal property of the convolution monoidal structure. *Journal of Pure and Applied Algebra*, 43(1):75 – 88, 1986. 3, 16, 84

[JS93] A. Joyal and R. Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993. 2

[Kel64] G. M. Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra*, 1:397–402, 1964. 2

[Kel74] G. M. Kelly. *Coherence theorems for lax algebras and for distributive laws*, pages 281–375. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974. 3

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. i, 2, 7, 10, 12

[Lac02] S. Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175(1-3):223–241, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly. 35, 56

[Law15] F. Lawler. Fibrations of predicates and bicategories of relations. *PhD Thesis, Trinity College, Dublin; arXiv:1502.08017*, 2015. 25

[Lei02] T. Leinster. A survey of definitions of n-category. *Theory and applications of Categories*, 10(1):1–70, 2002. 4

[Mar99] F. Marmolejo. Distributive laws for pseudomonads. *Theory and Applications of Categories*, 5:91–147, 1999. 3

[ML63] S. Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies*, 49(4):28–46, 1963. 1

[ML98] S. Mac Lane. *Categories for the working mathematician.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1998. 2, 7

[MM94] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Universitext. Springer New York, 1994. 3, 10

[MP85] S. Maclane and R. Paré. Coherence for bicategories and indexed categories. *Journal of Pure and Applied Algebra*, 37:59 – 80, 1985. 4

[MW08] F. Marmolejo and R.J. Wood. Coherence for pseudodistributive laws revisited. *Theory and Applications of Categories*, 20:74–84, 2008. 3

[MW12] F. Marmolejo and R.J. Wood. Kan extensions and lax idempotent pseudomonads. *Theory and Applications of Categories*, 26:1–29, 2012. 3

[Nun16] F. L. Nunes. Pseudo-kan extensions and descent theory. *arXiv:1606.04999v2*, 2016. 48

[Sta16] M. Stay. Compact closed bicategories. *Theory and applications of Categories*, 31(26):755–798, 2016. 86

[Str76] R. Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181, 1976. 4, 35

[Str80] R. Street. Fibrations in bicategories. *Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques*, 21(2):111–160, 1980. 4, 20, 36, 59

[Str87] R. Street. Correction to fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 28(1):53–56, 1987. 4, 35, 36, 56

[TP06] M. Tanaka and J. Power. Pseudo-distributive laws and axiomatics for variable binding. *Higher-Order and Symbolic Computation*, 19(2):305–337, 2006. 3, 11