SUBTENDED ANGLES

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Abstract. Suppose that $d \geqslant 2$ and m are fixed. For which n is it the case that any n angles can be realised by placing m points in $\mathbb{R}^d$?

A simple degrees of freedom argument shows that m points in $\mathbb{R}^2$ cannot realise more than $2m-4$ general angles. We give a construction to show that this bound is sharp when $m \geqslant 5$.

In d dimensions the degrees of freedom argument gives an upper bound of $dm - \binom{d+1}{2} - 1$ general angles. However, the above result does not generalise to this case; surprisingly, the bound of $2m-4$ from two dimensions cannot be improved at all. Indeed, our main result is that there are sets of $2m-3$ of angles that cannot be realised by m points in any dimension.

1. Introduction

We consider the following question. Suppose that $d \geqslant 2$ and m are fixed. What is the largest n such that, given any n distinct angles $0 < \theta_1, \theta_2, \ldots, \theta_n < \pi$, we can realise all these angles by placing m points in $\mathbb{R}^d$? (We say an angle θ is *realised* if there exist points A, B and C such that $A\widehat{B}C = \theta$.)

There is a natural 'degrees of freedom' argument. There are d degrees of freedom for each of the m points, so dm in total, except we have to consider all similarities of $\mathbb{R}^d$. There are d degrees for translation, one for scaling, and then $\binom{d}{2}$ for the orthogonal group of isometries. Thus in total there are $dm - \binom{d+1}{2} - 1$ degrees of freedom, and we cannot hope to realise more than this many angles in general. (We give a rigorous proof of the relevant cases in Section 3.)

However, it is far from clear that we can realise this many, although one might guess that they can be realised, at least for large n. Indeed, this is the case in two dimensions.

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Theorem 1. *Suppose that $m \geqslant 5$ and $n \leqslant 2m-4$. Then, given any n distinct angles, there is an arrangement of m points in the plane realising all n of these angles. Moreover, these points may be chosen in convex position.*

It is trivial to see that, even without the convex condition, the theorem does not hold for $m = 3$: the configuration must be a triangle so two angles can be realised if and only if their sum is less than π. Thus, if the angles are chosen independently from the uniform distribution on $(0, \pi)$, the probability of realising two angles is $1/2$. Füredi and Szigeti [7] showed that the probability of realising four independent uniform $(0, \pi)$ angles by four points is $79/84$. This is less than 1, so Theorem 1 is also false for $m = 4$. One can also check that it is impossible for four points to represent four distinct angles $\theta_1 > \theta_2 > \theta_3 > \theta_4$ when $\theta_4 > (2/3)\pi$ and $\theta_2 + \theta_3 > \pi + \theta_4$, even in three dimensions.

Having seen that degrees of freedom gives the correct answer in two dimensions, it is natural to guess that the same is true in higher dimensions. Somewhat surprisingly this is not the case: it is not possible to guarantee to realise *any* more angles than in two dimensions. Indeed, there are sets of $2m-3$ angles that cannot be realised by m points in any dimension.

Theorem 2. *Suppose that $m \geqslant 2$ and that $n = 2m-3$. Then there exists a set of n (distinct) angles such that no arrangement of m points in any dimension realises all angles in the set.*

This theorem says that we could not guarantee to achieve more than $2m-4$ general angles with m points even in arbitrarily large dimension. Our example of an unachievable set contains angles very close to either 0 or π. Thus, it is natural to ask if we can do better if the angles are bounded away from 0 and π. Of course, the degrees of freedom bound will still hold so there will be no change in two dimensions but, in higher dimensions, there might be. Our final result shows that constraining the angles away from 0 and π makes a huge difference: in this case we can nearly obtain the degrees of freedom bound in any dimension.

Theorem 3. *Suppose that d and ε are fixed. Then there exists a constant c such that any $n = dm-c$ angles, all lying between ε and $\pi-\varepsilon$, can be realised using m points.*

The layout of the paper is as follows. In Section 2 we prove the lower bound in two dimensions (Theorem 1), and in Section 3 the upper bound in two dimensions. Then in Section 4 we prove the higher dimensional upper bound (Theorem 2). Finally, in Section 5 we prove Theorem 3. We conclude with some open questions.

We end this section with some background. These problems were posed by Füredi and Szigeti [7]; however there is a long history of related problems, both for realising angles and for realising distances.

Erdős [4, 5] asked how many points can be placed in $\mathbb{R}^d$ such that no angle greater than $\pi/2$ is realised; the hypercube provides an obvious lower bound of 2^d and Erdős asked whether this was extremal. This was answered positively by Danzer and Grünbaum in [3], who also conjectured that if all the angles must be acute (rather than just non-obtuse) then there could not be a superlinear number of points. Erdős and Füredi [6] disproved this in a very strong sense: they showed that there can be exponentially many points in $\mathbb{R}^d$ with all realised angles acute. More recently, Harangi [10] significantly improved the exponent in this lower bound. Also, in [2], Conway, Croft, Erdős and Guy initiated the study of the more general questions of how many (or few) angles can have size at least (at most) α for arbitrary angles α.

Of course, our question is somewhat different. These latter questions are asking for a set all of whose angles have some property (e.g., are non-obtuse) whereas here we are asking for some of the angles to take specific values.

There is a vast literature on the many closely related questions asking about distances rather than angles. Indeed, there are far too many references for us to do justice to here, and we just mention a few; see, for example, Brass, Moser and Pach [1] for a more complete survey. These questions date back to a question of Hopf and Pannwitz [11], who asked how many times an n point set in $\mathbb{R}^2$ with diameter 1 can realise the distance 1, with many people submitting solutions to the journal. This led Erdős [4] to ask at least how many different distances must be realised by n points in $\mathbb{R}^2$, and he gave a simple argument giving a lower bound of order $\sqrt{n}$. Moser [12] gave the first of many improvements to the exponent in the lower bound. This work has culminated with the very recent breakthrough of Guth and Katz [9] who showed that at least $\Omega(n/\log n)$ distances must be realised.

2. The lower bound in two dimensions

We start by showing that if we can realise a subset of the angles that includes the maximum angle 'optimally' then we can realise all the angles optimally.

Lemma 4. *Suppose that $m \in \mathbb{N}$, that $\theta_1, \theta_2, \dots, \theta_{2m-4}$ are $2m-4$ distinct angles, and that P is an arrangement of $m' < m$ points realising $2m'-4$ of the angles including the maximum angle. Then there is an arrangement of m points realising all $2m-4$ angles.*

Proof. Suppose θ_1 is the maximum angle. We show that we can add a single point realising any two of the remaining angles. By doing this repeatedly we obtain the result.

Suppose the two angles we want to add are θ_2 and θ_3. Let A, B and C be three points of the configuration such that $A\widehat{B}C$ realises the angle θ_1; see Figure 1. We add a point D on the ray (i.e., half-line) BD such that $A\widehat{B}D = \theta_2$ and $A\widehat{D}C = \theta_3$. Since θ_1 is the maximum angle, the ray BD lies between the side BA and BC. By varying the point D on that ray between

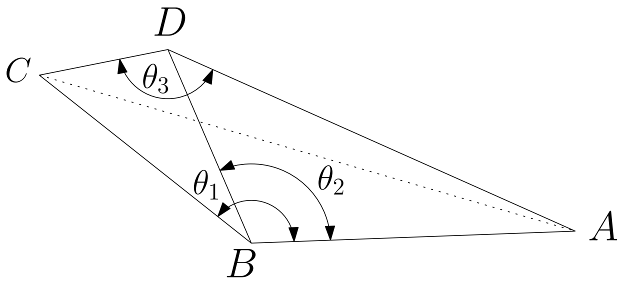

FIGURE 1. Adding a point to an existing configuration.

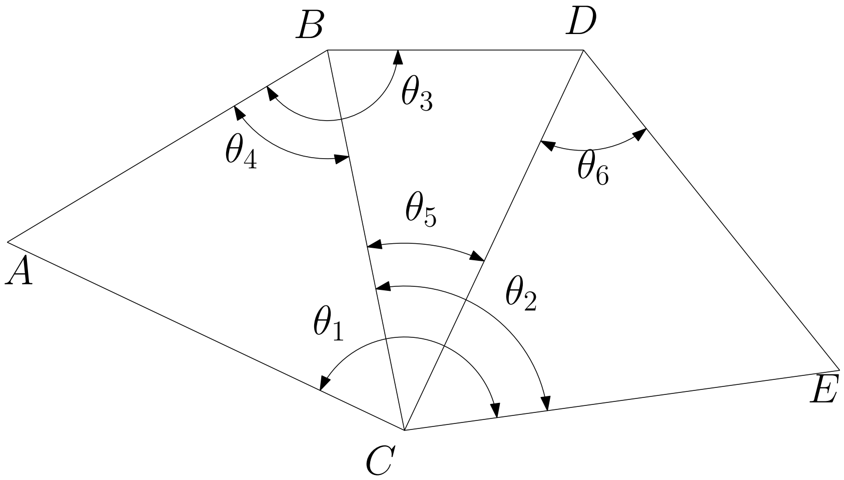

FIGURE 2. Five points realising six angles.

the point of intersection of the ray and AC, and the point at infinity we can make $A\widehat{D}C$ any angle between π and 0; in particular we can obtain θ_3. $\square$

We remark that we could realise the angle θ_3 as $E\widehat{D}F$ for any pair of points E and F lying on opposite sides of the ray BD in the above construction. We shall use this extra freedom when proving that the points may be chosen in convex position in Theorem 1.

Using this lemma we can extend an optimally realised subset to the full set. Thus, the key step in proving Theorem 1 (except for the convex condition) is showing that we can realise any six angles with five points.

Lemma 5. *Let $\theta_1, \theta_2, \ldots, \theta_6$ be six distinct angles. Then there is an arrangement of five points in the plane realising all six of the angles.*

Proof. We suppose that the angles θ_i are given in decreasing order. Our aim is to show that these angles can be realised by the arrangement shown in Figure 2 in which $A\widehat{C}E = \theta_1$, $B\widehat{C}E = \theta_2$, $A\widehat{B}D = \theta_3$, $A\widehat{B}C = \theta_4$, $B\widehat{C}D = \theta_5$ and $C\widehat{D}E = \theta_6$. To prove that this realisation is possible we start by placing CE. We then define the ray CD using $\theta_2 - \theta_5$, and then the point D using θ_6; the ray CB using θ_2 then the point B using $\theta_3 - \theta_4$; the ray CA using θ_1 then the point A using θ_4. By the choice of the ordering of the angles the rays CD, CB, CA are as in Figure 2.

To complete the proof all we have to check is that CDE, BCD, and ABC are all valid triangles. We do this by checking that the angles we have defined sum to less than π. In CDE we have $C\widehat{D}E + E\widehat{C}D = \theta_2 - \theta_5 + \theta_6 < \pi$; in BCD we have $B\widehat{C}D + C\widehat{B}D = \theta_5 + \theta_3 - \theta_4 < \pi$; and in ABC we have $A\widehat{B}C + A\widehat{C}B = \theta_1 - \theta_2 + \theta_4 < \pi$. □

Proof of Theorem 1. Lemmas 4 and 5 prove Theorem 1 except for showing that we may insist that the points lie in convex position. To prove this we modify our construction slightly.

Start with the configuration given by Lemma 5 where θ_1 is the largest angle and $\theta_2, \theta_3, \ldots, \theta_6$ are the five smallest angles, and suppose that the remaining angles are $\phi_1, \phi_2, \ldots, \phi_k$ in increasing order. We add these angles on rays between CA and CB. Place a point F such that $F\widehat{C}E = \phi_1$ and $A\widehat{F}B = \phi_2$. We just need to check that this point F is convex position. Let F' be the intersection of the (extended) lines CF and BD. Then $A\widehat{F'}B = A\widehat{F'}D < A\widehat{B}D < \phi_2$. Thus F must lie closer to C than F' and hence is in convex position. Repeating this construction replacing B with F and D with B for the remaining angles gives the result. □

Finally, we consider the case where some angles may be repeated; i.e., we have a tuple of angles, rather than a set. We view an angle as being realised multiple times if it is realised by multiple distinct pairs of rays. Note that this is more restrictive than just requiring distinct triples; in particular, we view collinear points giving rise to the same ray as being the same realisation of an angle. This result follows easily from our proofs so far, but is a little technical. We only need one more geometric idea: the following folklore lemma showing that we can efficiently realise one angle several times.

Lemma 6. *Suppose that θ is any angle. Then we can realise θ with multiplicity $t(t-1)$ using $2t$ points.*

Proof. Place t pairs of points $x_1, y_1, x_2, y_2, \ldots, x_t, y_t$ on a circle such that all the x's occur followed by all the y's and make the distance between each pair x_i, y_i such that the angle subtended by any point between x_i and y_i on the circle is θ. Then any chord and any point between the endpoints of the chords subtend angle θ. Thus, the angle θ is realised $t(t-1)$ times. □

Since this is quadratic in the number of points (as opposed to the linear bound given by Theorem 1) we expect the case of repeated angles to be easier, at least for large numbers of points or angles. As we shall show, this is indeed the case. However, a little thought shows that for small numbers of points the reverse may be true; for example, five points can only realise an angle of $\pi - \varepsilon$ four times.

Lemma 7. *Suppose that $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_{n'})$ are two tuples of angles realised by m and m' points respectively. Then the union of the two tuples can be realised with $m + m' - 2$ points.*

Proof. This is essentially trivial: we reuse two of the points realising θ when realising ϕ. We just need to be a little careful to avoid coincident rays.

Fix two points x, y of the configuration realising θ such that xy is a line segment of the convex hull of the configuration. By a similarity transformation we may assume that $x = (0, 0)$, $y = (1, 0)$ and that all the other points lie in the lower half plane.

Similarly, by a similarity, we may assume that all the points realising ϕ lie in the upper half plane and that $(0, 0)$ and $(1, 0)$ are two of these points.

Then, the union of these points sets has size $m+m'-2$ and since we have no other coincident points all rays occurring in any of the realised angles, except rays contained in the x-axis, are distinct. Thus, this a realisation of the union of the tuples θ and ϕ. □

Next we prove a weak version of our result for tuples. Somewhat surprisingly it is easy to deduce a tight bound from this weak result. Note we have made no effort to optimise the constant in this lemma or the subsequent proposition.

Lemma 8. *Suppose that n and m satisfy $m \geqslant n/2 + 30$, and $\theta_1, \theta_2, \ldots, \theta_n$ are any n (not necessarily distinct) angles. Then, we can realise all the angles with m points.*

Proof. This is a simple induction. Suppose $m \geqslant n/2 + 30$. If there are at least six distinct angles then, by the inductive hypothesis, we can realise the remaining $n - 6$ angles with $m - 3$ points and, by Lemma 5, we can realise these six distinct angles with five points. Thus, by Lemma 7, we can realise all the original n angles with $m - 3 + 5 - 2 = m$ points

On the other hand suppose any angle θ occurs with multiplicity at least twelve. By the inductive hypothesis, we can realise the $n - 12$ tuple of the angles with this angle's multiplicity reduced by twelve using $m - 6$ points, and by Lemma 6 with $t = 4$, we can realise θ twelve times with eight points. Thus, by Lemma 7, we can realise all the original n angles with $m - 6 + 8 - 2 = m$ points.

Thus, the only remaining case is when no angle occurs more than eleven times and there are no more than five distinct angles; so, in particular, $n \leqslant 55$. Trivially, we can realise these n angles with at most $n+2 < n/2+30 = m$ points (three points for the first angle and one extra point for each extra angle). Thus, in this case the results holds and the proof of the lemma is complete. □

Proposition 9. *There exists m_0 such that, for any $m \geqslant m_0$, any $n \leqslant 2m - 4$, and any (not necessarily distinct) angles $\theta_1, \theta_2, \ldots, \theta_n$, there is an arrangement of m points in the plane realising all n angles.*

Proof. Suppose that some angle occurs at least 110 times. Then we can realise this angle 110 times using 22 points ($t = 11$ in Lemma 6). Since $n \leqslant 2m - 4$, by Lemma 8, we can realise the $n - 110$ tuple of all the remaining angles using at most $m - 2 - 55 + 30$ points. In total this uses at most $m - 3$ points and the proof is complete in this case.

Thus, the only remaining case is where no angle occurs more than 110 times. Provided m_0, and so n, is large enough we can partition the remaining angles into sets of angles $A_1, A_2, \dots, A_k$ where each A_i has size at least six, each A_i consists of distinct angles, and at most one A_i has odd size. Using Theorem 1 we can realise each set A_i of angles with $\lceil |A_i|/2 \rceil + 2$ points. Thus, by Lemma 7 repeatedly, we can realise all n angles with

$$2 + \sum_{i=1}^{k} \lceil |A_i|/2 \rceil = 2 + \left\lceil \sum_{i=1}^{k} |A_i|/2 \right\rceil = 2 + \lceil n/2 \rceil \leqslant m$$

points (where the first equality used the fact that at most one $|A_i|$ is odd). □

3. The upper bound in two dimensions

In the next section we prove a result (Corollary 12) which formalises the degrees of freedom intuition we gave in the introduction and shows that we can almost never realise a set of $2m-3$ of angles with m points in the plane. In fact, we prove a stronger result as we shall need this in the next section where we deal with the higher dimensional case.

We start with a simple algebraic observation (see, for example, Chapter 18 of [8]).

Theorem 10. *Suppose that $\phi_i(x_1, .., x_{d-1})$, $i = 1, ..., d$, are rational functions. Then there exists a non-zero polynomial g in $\mathbb{R}[z_1, \dots, z_d]$ such that $g(\phi_1, ..., \phi_d) = 0$.*

Proof. The rational functions ϕ_i represent d elements in the field $\mathbb{R}(x_1, ..., x_{d-1})$. However, this field has transcendence degree $d-1$ over $\mathbb{R}$, so there must be a non-trivial algebraic relation between them; i.e., there exists $g \in \mathbb{R}[z_1, ..., z_d]$ with $g(\phi_1, ..., \phi_d) = 0$. □

Lemma 11. *Suppose that $m \in \mathbb{N}$; $n = 2m-3$. Then there exists a non-zero polynomial g in n variables $z_1, ..., z_n$ with the following property. For any configuration of m points in the plane realising angles $\theta_1, \theta_2, \dots, \theta_n$ we have*

$$g(s(\theta_1), s(\theta_2), \dots, s(\theta_n)) = 0$$

where s is the function defined by $s(\theta) = \sin^2 \theta$.

Proof. Suppose that $v_1, v_2, \dots, v_m$ are the m points of the configuration and we may assume that v_1 and v_2 are fixed with $v_1 = (0, 0)$ and $v_2 = (1, 0)$.

Let $\phi_1, \phi_2, \dots, \phi_r$ be all the $r = 3\binom{m}{3}$ angles realised by these m points. Each of these angles is a function of the positions of the three points forming that angle and, for each $1 \leqslant i \leqslant r$, we let $f_i\colon \mathbb{R}^{2m-4} \to \mathbb{R}$ be the map that sends the $2m-4$ coordinates of the $m-2$ (non-fixed) points $v_3, v_4, \ldots, v_m$ to $s(\phi_i)$. Since

$$f_i(v_3, v_4, \ldots, v_m) = s(\phi_i) = 1 - \cos^2 \phi_i = 1 - \frac{((v_j - v_k).(v_l - v_k))^2}{\|v_j - v_k\|^2 \|v_l - v_k\|^2},$$

where v_j, v_k, v_l are the triple giving angle ϕ_i, we see that each f_i is a rational function of the coordinates of the points.

For each set $I \subset [r]$ with $|I| = 2m-3$, Theorem 10 applied to the functions f_i, $i \in I$, implies that there exists a polynomial g_I in $\mathbb{R}[x_1, x_2, \ldots, x_{2m-3}]$ such that g_I applied to f_i, $i \in I$, is identically zero. In particular, if the angles $\theta_1, \theta_2, \ldots, \theta_n$ chosen are exactly ϕ_i, $i \in I$, then $g_I(s(\theta_1), s(\theta_2), \ldots, s(\theta_n)) = 0$. Let $g = \prod_{I \subset [r];\ |I| = 2m-3} g_I$ be the product of all the possible g_I. Obviously g is zero for any configuration and any choice of $n = 2m-3$ angles $\theta_1, \theta_2, \ldots, \theta_n$ from among $\phi_1, \phi_2, \ldots, \phi_r$. □

Corollary 12. *Suppose $m \in \mathbb{N}$ and $n = 2m-3$. Let $S \subset [0, \pi]^n$ be the set of n-tuples of angles that can be realised by m points in the plane. Then the set S has measure zero. Moreover S has Hausdorff dimension $n-1$.*

Proof. That S has measure zero follows immediately from Lemma 11, as does the fact that the Hausdorff dimension at most $n-1$. To complete the proof note that we can realise any n angles with $\theta_1 > \theta_2 > \cdots > \theta_n$ and $\theta_n = \theta_1 - \theta_2$ with m points. Indeed, the construction (Lemma 5 together with Lemma 4) shows realises the angles $\theta_1, \theta_2, \ldots, \theta_{n-1}$ and we see that the angle $\theta_n = \theta_1 - \theta_2$ also occurs in the construction. Hence this co-dimension one subset is contained in S and the result follows. □

4. Higher dimensions

We start this section by examining the effect that a random projection onto a two dimensional subspace has on an angle: in particular on very small angles. First we consider the length of a vector after a random one-dimensional projection. Note, that both of the following two lemmas are far from tight but suffice for our needs.

Lemma 13. *Suppose that v is a unit vector in $\mathbb{R}^d$, and that p is a projection onto a uniformly chosen one-dimensional subspace. Then, for all $\varepsilon > 0$*

$$\mathbb{P}(\|p(v)\| < \varepsilon) \leqslant \sqrt{\varepsilon} + d\varepsilon.$$

Proof. Instead of the setup in the lemma we consider a fixed projection p on to the first coordinate and a random unit vector. Now one way of generating a random unit vector is to generate d independent variables X_i all with $N(0,1)$ distribution and then let $v = X/\|X\|$.

If $\|p(v)\| < \varepsilon$ then either $|X_1| < \sqrt{\varepsilon}$ or $\|X\| \geqslant 1/\sqrt{\varepsilon}$. Since the density function for the normal distribution is always less than $1/2$ we see that

$\mathbb{P}(|X_1| < \sqrt{\varepsilon}) < \sqrt{\varepsilon}$. Also $\|X\|^2$ is a random variable with mean d. Hence, by Markov's inequality,

$$\mathbb{P}(\|X\| \geqslant 1/\sqrt{\varepsilon}) = \mathbb{P}(\|X\|^2 \geqslant 1/\varepsilon) \leqslant d\varepsilon$$

Thus

$$\mathbb{P}(v_1 < \varepsilon) \leqslant \sqrt{\varepsilon} + d\varepsilon. \qquad \square$$

Next we show that a random projection does not change angles 'too much'. [In the following lemma we refer to a 'uniformly chosen two-dimensional subspace'. Formally, this refers to the probability measure on the space of all two-dimensional subspaces that is induced by the action of $SO(d)$ equipped with its standard Haar measure.]

Lemma 14. *Suppose that v_1, v_2 are two vectors with angle $\theta < \pi/3$ between them, and that p is a projection onto a uniformly chosen two-dimensional subspace. Let ϕ be the angle between $p(v_1)$ and $p(v_2)$. Then*

$$\mathbb{P}(\phi \notin [\theta\varepsilon, \theta/\varepsilon]) < 4\sqrt{\varepsilon} + 6d\varepsilon.$$

Proof. Similarly to the previous proof, we will fix the projection p, this time onto the first two coordinates, and choose random unit vectors v_1 and v_2 subject to the constraint that the angle between them is θ. We do this by picking a random unit vector v and a random unit vector u orthogonal to v, and setting $v_1 = v$ and $v_2 = \alpha v + \beta u$ where $\alpha = \cos\theta$ and $\beta = \sin\theta$. Since $\theta < \pi/3$ we have that $\alpha > 1/2$, and $\theta < \tan\theta = \beta/\alpha < 2\theta$.

We need to bound the angle ϕ between $p(v_1)$ and $p(v_2) = \alpha p(v_1) + \beta p(u)$. First we bound the probability that this angle is big. The angle ϕ satisfies

$$\phi \leqslant \sin^{-1}\left(\frac{\beta\|p(u)\|}{\alpha\|p(v_1)\|}\right) \leqslant \frac{2\beta}{\alpha\|p(v_1)\|} \leqslant \frac{4\theta}{\|p(v_1)\|}.$$

Thus, by Lemma 13 for any $\varepsilon > 0$,

$$\mathbb{P}(\phi > \theta/\varepsilon) \leqslant \mathbb{P}(\|p(v_1)\| < 4\varepsilon) \leqslant 2\sqrt{\varepsilon} + 4d\varepsilon.$$

Next we bound the chance that ϕ is small. The vector u is chosen orthogonal to v so lies on a $d-1$ dimensional sphere. Take a basis of this sphere such that the first coordinate direction is in the plane spanned by the first two coordinate directions and perpendicular to $p(v_1)$. Now the component of $p(u)$ perpendicular to $p(v)$ is exactly the first component u_1 of u in this basis. Since the component of $p(v_2)$ in the $p(v_1)$ direction is at most 1 we have

$$\phi \geqslant \tan^{-1}(\beta|u_1|) \geqslant \tan^{-1}\left(\frac{\beta|u_1|}{2\alpha}\right) \geqslant \tfrac{1}{2}|u_1|\theta.$$

Applying Lemma 13 again we see that, for any $\varepsilon > 0$,

$$\mathbb{P}(\phi < \theta\varepsilon) \leqslant \mathbb{P}(|u_1| < 2\varepsilon) \leqslant \sqrt{2}\sqrt{\varepsilon} + 2(d-1)\varepsilon.$$

Combining these two bounds gives the result. $\square$

This result shows that projecting onto a random plane typically only changes the angles by at most a constant factor. We want to use the upper bound of $2m-4$ for the number of angles that can be realised by m points in the plane to deduce the same bound in higher dimensions. Since we do not have control over exactly what happens to the angles when we project we need a bound for the two dimensional case that can cope with the uncertainty introduced by the projection. We use Lemma 11 to prove such a result.

Theorem 15. *Suppose that h is a strictly positive real valued function, $m \in \mathbb{N}$, $n = 2m-3$, and that $\varepsilon > 0$ is given. Then there exists $\varepsilon_i \leqslant \varepsilon$ for $1 \leqslant i \leqslant n$ such that for any sequence of angles θ_i for $1 \leqslant i \leqslant n$ with $\theta_i \in [h(\varepsilon_i), \varepsilon_i]$ there is no arrangement of m points in the plane achieving the angles θ_i, $i = 1, \dots, n$.*

Proof. Let g be the polynomial given by Lemma 11. Consider the terms of g and order them reverse lexicographically so that the leading term has the lowest power of z_n, and among those, is the one with lowest power of z_{n-1}, etc.

As above define $s\colon (0,\pi) \to (0,1]$ by $s(x) = \sin^2(x)$. Then s is a bijection on $(0,\pi/2]$, and whenever we write s^{-1} we mean the preimage in $(0,\pi/2]$. Let $\tilde{h} = s \circ h \circ s^{-1}$ and note that $\tilde{h}$ is strictly positive.

Assume, without loss of generality, that the leading term of g is $z_1^{a_1} \dots z_n^{a_n}$ (with coefficient 1). Let $C \geqslant 1$ be larger than the sum of the absolute values of all the coefficients of the other terms. Set $\delta = s(\varepsilon)$ and $\delta_1 = \min(1/C, \delta)$. Inductively define

$$\delta_i = \min\left(\delta,\ \frac{\prod_{j<i} \tilde{h}(\delta_j)^{a_j}}{C}\right).$$

Then for any $z = (z_1, z_2, \dots, z_n)$ with $z_i \in [\tilde{h}(\delta_i), \delta_i]$ the leading term is larger than the sum of all the other terms. Indeed, by our ordering of terms, for any other term $z_1^{b_1} \dots z_n^{b_n}$ there exists k such that $b_k > a_k$ and $b_i = a_i$ for $i > k$. Thus, by the definition of δ_i and that $1/C < 1$,

$$z_1^{b_1} \dots z_n^{b_n} \leqslant z_k^{b_k} \dots z_n^{b_n} \leqslant \frac{z_k}{z_1^{a_1} \dots z_{k-1}^{a_{k-1}}} z_1^{a_1} \dots z_n^{a_n} \leqslant \frac{1}{C} z_1^{a_1} \dots z_n^{a_n}.$$

In particular this shows that $g(z) \neq 0$.

Define $\varepsilon_i = s^{-1}(\delta_i)$. By definition $\delta_i \leqslant \delta$ so, since s^{-1} is increasing, $\varepsilon_i \leqslant \varepsilon$. Moreover, any configuration with angles $\theta_i \in [h(\varepsilon_i), \varepsilon_i]$ would have $s(\theta_i) \in [s \circ h(\varepsilon_i), s(\varepsilon_i)] = [\tilde{h}(\delta_i), \delta_i]$ and thus

$$g(s(\theta_1), s(\theta_2), \dots, s(\theta_n)) \neq 0$$

which would contradict Lemma 11. Therefore there cannot be a configuration realising all the angles θ_i, $1 \leqslant i \leqslant n$. □

We are now in a position to prove Theorem 2; we just need to combine Lemma 14 and Theorem 15.

Proof of Theorem 2. Recall that $m \geqslant 2$ and $n = 2m-3$. Trivially, for $m = 2$ there is nothing to prove so we may assume $m \geqslant 3$. We start with a trivial observation: however we place m points in Euclidean space they actually lie in an $(m-1)$-dimensional (affine) subspace. Thus, it is sufficient to prove that the angles cannot be realised in $\mathbb{R}^{m-1}$ or, as we shall prove, in $\mathbb{R}^m$.

Let δ be such that $4\sqrt{\delta} + 6m\delta < \frac{1}{2n}$ (for example $\delta = \frac{1}{100n^2}$ will do). Fix $\varepsilon < \pi/3$ and define $h(x) = \delta^2 x$. Now apply Theorem 15 to get a sequence ε_i for $1 \leqslant i \leqslant n$ such that for any sequence θ_i with $h(\varepsilon_i) < \theta_i < \varepsilon_i$ there is no arrangement of m points in the plane realising these angles.

Let $\phi_i = \varepsilon_i \delta$. We claim that the set of angles ϕ_i is not realisable by m points in $\mathbb{R}^m$. Indeed, suppose that there is such an arrangement of m points realising these angles. Consider a random two-dimensional projection p of these points. With a slight abuse of notation for an angle ϕ we use $p(\phi)$ to denote the angle between the lines after the projection. By our choice of h and Lemma 14 we see that

$$\mathbb{P}(p(\phi_i) \notin [h(\varepsilon_i), \varepsilon_i]) = \mathbb{P}(p(\phi_i) \notin [\phi_i \delta, \phi_i/\delta]) < 4\sqrt{\delta} + 6m\delta < \tfrac{1}{2n}.$$

Thus, with positive probability we have $p(\phi_i) \in [h(\varepsilon_i), \varepsilon_i]$ for all i. Fix such a projection p. The m points $p(z_i)$ in the plane realise the $2m-3$ angles $p(\phi_i)$ which contradicts Theorem 15. □

Finally, we remark that a slight adjustment to the proof above shows not only that there is a set of n angles that is not realisable, but also that the set of n-tuples that are not realisable has non-empty interior.

5. Angles bounded away from 0 and π

In this section we prove Theorem 3. We remark that the constraint on θ_i in Theorem 15 was, in fact, a constraint on $\sin \theta_i$. Thus, our proof of Theorem 2 shows not only that we cannot realise sets containing very small angles but, also, that we cannot realise sets containing very large angles (i.e. close to π). Hence it is natural to bound the angles away from 0 and π as we do in Theorem 3.

First we show a local result: that for any $\theta \in (0, \pi)$ we can realise angles near θ efficiently.

Lemma 16. *Given d and θ there exists a finite collection of points A in $\mathbb{R}^d$ and an open set $U \subset \mathbb{R}$ containing θ such that, for any k, we can realise any dk distinct angles all in U with k points in $\mathbb{R}^d$ together with A.*

The idea is to choose our set A such that it subtends d angles of size θ at the origin. Then if we moved the point at the origin a small amount we could modify these d angles slightly and use the inverse function theorem to show that these perturbations must include a small open set around θ.

Proof. We start by defining the set A. Fix λ very large and let $a = \lambda \cos \theta$ and $b = \frac{1}{\sqrt{d-1}} \lambda \sin \theta$. For $1 \leqslant j \leqslant d$ define $f_j = \sum_i b e_i + (a-b) e_j$ where

$e_1, e_2, \dots, e_d$ denote the standard basis vectors of $\mathbb{R}^d$. Let A be the set

$$\{e_1, e_2, \dots, e_d\} \cup \{f_1, f_2, \dots, f_d\}.$$

Since

$$\cos(e_i \widehat{0} f_i) = \frac{e_i \cdot f_i}{\|e_i\|\|f_i\|} = \frac{a}{\sqrt{a^2 + (d-1)b^2}} = \frac{\lambda \cos\theta}{\lambda} = \cos\theta$$

we see that, for each $1 \leqslant i \leqslant d$ we have $e_i \widehat{0} f_i = \theta$.

Let $x = (x_1, x_2, \dots, x_d)$ be a point near 0 and let F be the function mapping x to the d-tuple $(\cos(e_1 \widehat{x} f_1), \cos(e_2 \widehat{x} f_2), \dots, \cos(e_d \widehat{x} f_d))$. Obviously the function f is continuously differentiable near zero. Thus, to complete the proof we just need to show that the derivative of f is non-singular at 0.

Writing θ_i for the angle $e_i \widehat{x} f_i$ we have

$$\begin{aligned}
\frac{\partial}{\partial x_j} \cos\theta_i \bigg|_{x=0} &= \frac{\partial}{\partial x_j} \left(\frac{f_i - x}{\|f_i - x\|} \cdot \frac{e_i - x}{\|e_i - x\|} \right) \bigg|_{x=0} \\
&= \frac{\partial}{\partial x_j} \frac{f_i - x}{\|f_i - x\|} \bigg|_{x=0} \cdot \frac{e_i - x}{\|e_i - x\|} + \frac{f_i - x}{\|f_i - x\|} \cdot \frac{\partial}{\partial x_j} \frac{e_i - x}{\|e_i - x\|} \bigg|_{x=0} \\
&= \frac{\partial}{\partial x_j} \frac{f_i - x}{\|f_i - x\|} \bigg|_{x=0} \cdot e_i + \frac{f_i}{\|f_i\|} \cdot \frac{\partial}{\partial x_j} \frac{e_i - x}{\|e_i - x\|} \bigg|_{x=0}.
\end{aligned}$$

Now

$$\left\| \frac{\partial}{\partial x_j} \frac{f_i - x}{\|f_i - x\|} \right\| = O(1/\lambda)$$

and

$$\frac{\partial}{\partial x_j} \frac{e_i - x}{\|e_i - x\|} = \begin{cases} 0 & \text{if } i = j \\ -e_j & \text{otherwise} \end{cases}$$

Thus

$$\frac{\partial}{\partial x_j} \cos\theta_i = \begin{cases} O(1/\lambda) & \text{if } i = j \\ -\frac{b}{\lambda} + O(1/\lambda) & \text{otherwise.} \end{cases}$$

Hence the derivative matrix of F is

$$-\frac{b}{\lambda}(J - I) + O(1/\lambda) = -\frac{\sin\theta}{\sqrt{d-1}}(J - I) + O(1/\lambda)$$

(where J is the all one matrix), and is thus invertible provided λ is sufficiently large. The result follows. □

Finally we prove Theorem 3.

Proof of Theorem 3. First we consider the case where all angles are distinct.

For each $\phi \in [\varepsilon, \pi - \varepsilon]$ let U_ϕ and A_ϕ be the open neighbourhoods and sets of points given by the previous lemma. The U_ϕ form an open cover of $[\varepsilon, \pi - \varepsilon]$. Let $U_{\phi_1}, U_{\phi_2}, \dots, U_{\phi_k}$ be a finite subcover and let A be the union of the corresponding A_{ϕ_i}. Note, in particular, that A is a finite set.

We claim that we can realise any collection of n angles $\theta_1, \theta_2, \dots, \theta_n$ with $n/d + k + |A|$ points. First we partition the angles $\theta_1, \theta_2, \dots, \theta_n$ into sets Φ_i such that all angles in Φ_i are in U_{ϕ_i}. Now, we place all the points in A and then, for each i, use the previous lemma to realise all the angles in Φ_i using

$\lceil |\Phi_i|/d \rceil$ more points. In total this uses (at most) $|A|+k+n/d$ points. This rearranges to the claimed bound.

Now, suppose that there are some repeated angles. If any angle occurs more than $2d(2d+1)$ times then, by Lemma 6, we can realise it using $2(2d+1)$ points. Repeating this argument reduces to the case where no angle occurs more than $2d(2d+1)$ times. This final case is easy: we take $2d(2d+1)$ copies of the above construction. Thus, we get a bound of $2d(2d+1)(|A|+k)+n/d$ as required. □

6. Open Problems

It is clear that some sets of more than $2m-4$ angles are realisable by m points in $\mathbb{R}^d$. Our main question asks how common this is.

To formalise this suppose that $d \geqslant 2$, $m \geqslant 2$ and $n \leqslant dm - \binom{d+1}{2} - 1$, and that $\theta_1, \theta_2, \ldots, \theta_n$ are n angles chosen uniformly from $(0,\pi)$. Let $P(d,m,n)$ be the probability that the angles $\theta_1, \theta_2, \ldots, \theta_n$ are realisable with m points in $\mathbb{R}^d$. Then our question becomes 'how does P behave?'

We start with the case where d is fixed.

Question 1. *Suppose d is fixed and $n = dm - \binom{d+1}{2} - 1$. Does $P(d,m,n) \to 1$ as $m \to \infty$? More weakly, is $\liminf P(d,m,n) > 0$?*

We believe that this probability does tend to 1. The situation is less clear when d is allowed to vary. We ask about the case when $m = d+1$ (i.e., the minimal number of points to actually use d dimensions).

Question 2. *Suppose $m = d+1$ and $n = dm - \binom{d+1}{2} - 1$. Does $P(d,m,n) \to 1$ as $d \to \infty$?*

In the previous section we considered the case of angles bounded away from 0 and π. Our bound there was that any n angles can be realised with m points provided that $n \leqslant dm - c$. The degrees of freedom upper bound (i.e. the higher dimensional analogue of Corollary 12) shows that we cannot hope to realise more than $n = dm - \binom{d+1}{2} - 1$ in general. These bounds have the same order but differ by a constant.

Question 3. *Suppose that ε and d are fixed. Is it possible to realise an arbitrary set of $n = dm - \binom{d+1}{2} - 1$ angles all between ε and $\pi - \varepsilon$ with m points for all sufficiently large m?*

Of course, if the answer to this is negative then one would like to determine the correct value of the constant.

Our final question asks about the number of different ways that a set of angles can be realised. We consider two similar point sets (i.e, two point sets related by a translation, rotation, reflection or scaling) as the same.

Question 4. *Suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are n angles and suppose that m is such that $n = 2m-4$. How many different m-point sets in the plane are there realising these angles?*

By Theorem 1 we know that there is at least one m-point configuration realising these angles. By the remark following Lemma 4 we see that there are at least $\Omega(k)$ chords we could use to realise the k^{th} angle. This gives a lower bound of roughly the order of $m!$ on the number of configurations.

In the other direction, Lemma 11 shows that, for almost all tuples $(\theta_1, \theta_2, \ldots, \theta_n)$, there are only finitely many other angles that can occur in configurations realising these angles. This shows that the number of such configurations is almost surely finite. By considering the possible arrangements of the points one can check that m^{10m} is an upper bound. But we do not know the correct order.

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