The London School of Economics and Political Science

Topics in Stochastic Control with Applications to Algorithmic Trading

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I declare that my thesis consists of 122 pages.

Abstract

This thesis considers three topics in stochastic control theory. Each of these topics is motivated by an application in finance. In each of the stochastic control problems formulated, the optimal strategy is characterised using dynamic programming. Closed form solutions are derived in a number of special cases.

The first topic is about the market making problem in which a market maker manages his risk from inventory holdings of a certain asset. The magnitude of this inventory is stochastic with changes occurring due to client trading activity, and can be controlled by making small adjustments to the so-called skew, namely, the quoted price offered to the clients. After formulating the stochastic control problem, closed form solutions are derived for the special cases that arise if the asset price is modelled by a Brownian motion with drift or a geometric Brownian motion. In both cases the impact of skew is additive. The optimal controls are time dependent affine functions of the inventory size and the inventory process under the optimal skew is an Ornstein-Uhlenbeck process. As a result, the asset price is mean reverting around a reference rate.

In the second topic the same framework is expanded to include a hedging control that can be used by the market maker to manage the inventory. In particular, the market impact is assumed to be of the Almgren and Chriss type. Explicit solutions are derived in the special case where the asset price follows a Brownian motion with drift.

The third topic is about Merton's portfolio optimisation problem with the additional feature that the risky asset price is modelled in a way that exhibits support and resistance levels. In particular, the risky asset price is modelled using a skew Brownian motion. After formulating the stochastic control problem, closed form solutions are derived.

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First and foremost I would like to thank Mihail Zervos for his outstanding supervision. Not only have our discussions helped me enhance my mathematical knowledge, but he continues to act as an intellectual role model, setting a standard to which I can only aspire.

I would especially like to thank my friend and mentor John Loizides for introducing me to the world of algorithmic trading, without his patience and support I would not be where I am today. I also feel fortunate to have been able to work alongside Till, Merlin and Mark, whose support in recent years, and especially the last few months, has been greatly appreciated.

Finally I would like to thank my friends and family.

Contents

Introduction

Algorithmic trading refers to any trading system in which the decision making process is free from human intervention. The use of the phrase *algorithmic* indicates that actions are initiated by a piece of logic that listens to market data, news feeds, or any other source of information that can be expressed electronically. Although it is often confused with one of its more glamorous subsets, namely high frequency arbitrage, algorithmic trading encompasses almost every aspect of the trading of financial assets. Indeed, it is part of a much broader secular trend towards automation, driven by technological advances.

Algorithmic trading has become the main way in which assets are exchanged in financial markets. Estimates vary, but in some markets algorithmic or electronic traders are responsible for more than 90% of traded volume, see for example a 2014 report by the SEC [US 14]. The approach to trading used by algorithmic traders is usually quite different from that of manual traders. Consequently, we should consider models that take these differences into account.

In this thesis, we consider three topics in stochastic control theory. Each of these topics is motivated by an application in algorithmic trading. In each of the stochastic control problems formulated, we characterise the optimal strategy using dynamic programming. We then derive closed form solutions in a number of special cases.

The first topic is about the market making problem in which a market maker manages his risk from inventory holdings of a certain asset. The magnitude of this inventory is stochastic with changes occurring due to client trading activity, and can be controlled by making small adjustments to the so-called skew, namely, the quoted price offered to the clients. After formulating the stochastic control problem, we derive closed form solutions for the special cases that arise if the asset price is modelled by a Brownian motion with drift or a geometric Brownian motion. In both cases the impact of skew is additive. The optimal controls are time dependent affine functions of the inventory size and the inventory process under the optimal skew is an Ornstein-Uhlenbeck process. As a result, the asset price is mean reverting around a reference rate.

In the second topic the same framework is expanded to include a hedging control that can be used by the market maker to manage the inventory. In particular, the market impact is assumed to be of the Almgren and Chriss type. We then derive explicit solutions in the special case where the asset price follows a Brownian motion with drift.

The third topic is about Merton's portfolio optimisation problem with the additional feature that the risky asset price is modelled in a way that exhibits support and resistance levels. In particular, the risky asset price is modelled using a skew Brownian motion. After formulating the stochastic control problem, closed form solutions are derived.

I Market Making

In Chapter 1 we consider the problem of a market maker, that is, a market participant who provides continuously tradable prices to a set of clients. Market makers are the facilitators of market liquidity, performing a role that is needed to smooth the time inconsistencies that naturally occur in markets. These time inconsistencies appear in financial markets since it is unlikely that when one investor wishes to sell, another simultaneously wishes to buy. The purpose of market making can therefore be seen as the smoothing through time of levels of supply and demand in markets, thereby creating continuously available liquidity.

Our study casts the market maker as the main protagonist in this interaction. However, the clients with which the market maker trades also play an important role. In models of financial markets, the focus is often placed on the professional investors such as banks and hedge funds. However, the driving force and life blood of the market are those non-professional investors who come to the market because of some fundamental and exogenous economic requirement from which their desire to trade the asset stems. In the problem we study, we refer to the clients of the market maker as *noise traders*, with that name meant to convey the fact that we cannot directly discern the reason behind their trading. However, we do make some basic assumptions about their price sensitivity, that is, their reaction to being offered more or less favourable prices by the market maker.

In order to perform their role, market makers stand ready to both buy and sell assets throughout the day, thereby accumulating holdings of the asset, also known as *inventory*. They may continuously update the prices at which their clients can trade throughout the day, and the size of their inventory changes as clients buy and sell, with the inventory position increasing as clients sell and decreasing as clients buy. In this role, the market maker is said to be *making a market in the asset*.

Of course the market maker's motive is not simply the provision of this service to the market. The goal of market making is no different to any other type of trading, typically, the maximisation of revenue subject to some sort of risk constraint. Unlike traditional portfolio theory, in which only the value of the asset is changing, the market maker faces an additional randomness in the quantities held in his portfolio due to the trading activity of clients. This randomness is controlled to some degree by the actions of the market maker through adjustments to his price. Rather than simply selecting a number of units of the asset to hold, the market maker must select a desirable level for his holdings of the asset and aim to shift the inventory position toward it. The chosen intensity of this shift will depend on a number of factors, such as risk tolerance and skew impact, which correspond to implicit and explicit costs of controlling the inventory level. The cost associated with skew is due to the need to offer more favourable prices to incentivise changes in the rate of client buying to client selling, whereas the choice to change the inventory level more slowly is associated with greater inventory risk, which is itself a cost.

There is a close connection with models of market impact (with which we will enhance our model in Chapter 2). However impact models have traditionally been motivated by a very different type of market activity. The market impact literature focuses on the so called *optimal execution problem* in which an agent seeks to execute a given quantity directly into the market, whilst minimising market impact. The role of the financial institution in this exchange is usually as the provider of the algorithmic execution strategy, or sometimes even simply as the facilitator of that strategy, and in this role is said to be acting as *agent*. When a financial institution provides a market making service to its clients it is said to be acting as *principal* meaning that the financial institution is the principal risk taker of those clients' trades. This interaction between client and financial institution, which might be described as *over the counter* trading, should be distinguished from the more general notion of market making, which as well as this type of interaction would encompass trading strategies that involve leaving visible resting orders on both the bid and offer side of a limit order book market. Although the model we present focusses on the former relationship, the insights gained here are readily applied to any variant of market making, since they relate to the optimisation of expected revenue and risk generated from movements in the asset price and the inventory position. In addition most clients of over the counter market makers will aggregate the prices of several institutions all of which provide this same service, so that the dynamics become quite similar to on exchange market making.

Prior to the advent of algorithmic and electronic trading, the market maker's problem was fundamentally different. Clients would make individual requests for quotes, to which the market maker would reply with a specific bid and offer price on which the client could, if desired, execute a trade. In the event of a trade, the market maker's problem was then essentially identical to the optimal execution problem, with the aim being to rid himself of the acquired risk. The modern market maker's problem, that is to say the algorithmic or electronic market maker's problem, is quite different. Indeed, much has been made of the way in which technological advance has improved liquidity provision, see for example Hendershott et al. [HJM11] and Chaboud et al. [CCHV14]. What is certain is that these changes have dramatically increased competition between market makers, thereby compressing bid-offer spreads. This means that algorithmic market makers receive less spread and are less able to immediately clear risk from their inventory. Consequently, the modern market maker's problem has become less to do with block trade pricing and the optimal execution problem, and more to do with risk management of a portfolio, and *top of book pricing*.

We choose to omit entirely any consideration of the explicit construction of bid and offer prices. In over the counter markets this is typically done on a client by client basis, with construction of the mid price occurring beforehand. Indeed it would be simple to append a model that explicitly constructs the bid and offer price, such as Guilbaud and Pham [GP13], as an extension to our framework. Moreover, the work of Glosten and Milgrom [GM85] suggests that the formation of bid and offer prices is at least partially driven by the notion of adverse selection. In their model, an on exchange market maker forms bid and offer prices at distances from the mid price that reflect where their expectation of the fair price would be adjusted to in the event of a trade. Additionally, some unknown proportion of traders posses private information, and so would be willing to pay an additional spread. Our model does not include informed traders, and so in a Glosten and Milgrom setting the market maker would set spreads to be 0. However, we do incorporate the idea of adverse selection by including a term that captures a correlation between noise trader activity and price movements.

We model trade arrivals as a continuous process rather than a jump process. In addition to the discussion above, some further considerations have shaped our thinking in doing so. The advent of electronic trading has led to small ticket sizes and higher frequency of trading. A fairly recent survey by Menkveld [Men13] from 2010 found that in the Dutch equity market average trade sizes are around €15,000, despite fixed transaction fees of over €1, and in the super liquid foreign exchange market it is not unusual for trades of lower than $1,000 notional value to occur. The skew process then acts more like a lever changing the direction of aggregate client flow, whereas in the context of Poisson arrivals, the skew would change the probability of the direction of the next jump in inventory.

On the basis of these considerations, we focus our attention on the core component of the market maker's problem which is the management of inventory risk by skewing a mid price, expressed as an appropriate offset from a market reference price. We will model noise trader activity using a controlled diffusion process, and we feel that this is a sensible approximation.

Much of the research under the title of *market making* is focussed exclusively on one specific type, namely *on exchange market making*, which as discussed is only a subset of the general concept. Bayraktar and Ludkovski [BL14] consider the slow execution of a block order using limit orders, with a focus on controlling the intensity of trade arrivals by adjusting the placement of orders in a limit order book. Avellaneda and Stoikov [AS08] consider an on exchange market maker in a similar setting with Poisson trade arrival intensities and compute some numerical results. The focus of both of these models is the area of market market microstructure of the limit order book. Closer to the topic of this chapter is Carmona and Webster [CW12] who consider an interesting special case of the over the counter market making problem in which a high frequency market maker attempts to utilise information present in client trades to his own advantage. Gueant et al. [GLFT13] consider the problem of inventory risk for a market maker faced with Poisson arrivals, and investigate the asymptotic properties of the resulting bid and offer prices. Guilbard and Pham [GP13] present a model which focuses on the order placement of an on exchange market maker, who submits limit orders around a top of book price that evolves according to a Markov chain with finite values. They also consider the inventory risk associated with holdings and use a mean-quadratic risk criterion which is the same as the one considered in this chapter.

The model we present in Chapter 1 considers the solution to the market makers problem in continuous time with an inventory process that evolves according to a controlled diffusion process. We formulate the model in such a way as to allow for price dynamics that follow another diffusion process and allow the market maker's price to be a general function of the unaffected asset price and the skew control. We present closed form solutions in two special cases. By allowing for drift in the unaffected asset price, and correlation between the asset price and noise trader activity we allow for the study of two interesting features of the market makers problem, namely the utilisation of private information on the asset price and the adverse selection effect caused by client toxicity.

In Section 2 of the chapter, we present the formal stochastic control problem on both the finite and infinite time horizons, for which we will prove separate verification theorems in Section 3 of the chapter. Using these theorems, in Section 4 of the chapter we present two special cases for which we can find closed form expressions for the value function and the skew control. The first of these special cases models the asset price as a Bachelier process, whereas the second case uses geometric Brownian motion.

II Hedging and Market Impact

In Chapter 2, we further develop the model considered in Chaper 1. We study a market in which a market maker provides liquidity to a set of clients by setting a tradable price as an offset from a known reference price. The market maker's primary objective is to make money from the flow of client trades, or as we have been referring to them, *noise traders*. The market maker is sensitive to the amount of risk he is holding at any given time and so, as covered in Chapter 1, will try and skew his price to avoid build-ups of risk. This comes at a small cost in revenue as skewing the price to incentivise this risk reducing behaviour means lowering the price at which he will sell or raising the price at which he will buy.

We expand the setting in which the market maker operates to include an additional market, which we will refer to as the *interbank market*. If skewing is insufficient to reduce the market maker's risk he may choose to hedge this risk in the interbank market, that is, lay it off with other market makers or professional trading firms and pay a transaction fee to do so.

To this end we model a two tiered marketplace in which noise traders interact only with the market maker, whereas the market maker has access to a second pool of liquidity namely the interbank market. The interbank market is the forum in which large professional trading firms trade with each other. We might also refer to this market as the *primary market* in the sense that it represents the location where price formation occurs and where the majority of information that market makers use to set prices resides.

To justify this structure it is important to discuss the reasons why noise traders do not also possess access to the interbank market. To do so requires us to emphasise the difference between the noise traders, who represent the clients of the market maker, and professional investors such as our market maker. Typically the market making function is performed by a trading desk in an investment bank or hedge fund that specialises in electronic execution and risk management, with a client base that consists of a varied mix of smaller investors, including but not limited to the trading departments of non-financial corporations, non-execution focused hedge funds and asset managers, and smaller banks.

These investors may themselves be financial professionals, but if so they are typically not focused on execution. By this we mean that they do not have the inclination to manage the execution of their deals themselves, and do not wish to make the fixed investments in fees and technology infrastructure needed to trade in the primary market. Equally they may simply lack the scale and resources to make such an investment in execution infrastructure. Further to these considerations, some markets specify a minimum trade size which may make trading in the interbank market prohibitive for some market participants. On the other hand, trading with a bank allows for greater flexibility, including the possibility to trade in smaller sizes. Furthermore, in general the market maker is compelled by the competitive nature of markets to offer clients prices that are better than those available in the interbank market. The definition of a *better* price may be quite illusive, but roughly speaking we may assume that it means a smaller bid-offer spread. For this reason the noise traders may simply opt not to trade in the interbank market, even if such access were available to them.

Prices shown by the market maker are generally free of arbitrage in the sense that they are not *crossed* with the primary market. By this we mean that the market maker's bid price is never higher than the primary market offer price or his offer price is lower than the primary market bid. The two tiered nature of the market would mean this could not be considered pure arbitrage, but in practice such soft arbitrage will be spotted and the opportunity to profit seized upon by a client who possesses access to the liquidity provided by two market makers and can therefore trade in opposite directions with each of them.

Within our model, we wish the representation of the interbank market to capture both the random arrival of new information expressed through changes in the expected fair value of the underlying asset, as well as price changes caused by the depletion of liquidity, that is, through market impact caused by the market participants. These two notions are clearly intertwined, in some cases the act of trading is itself a signal, sending information to other market participants, who update their expectations of the fair value of the asset accordingly, see for example Glosten and Milgrom [GM85]. However for the purposes of modelling market impact, we will assume that these notions are separable and there exists a known market impact function, as well as a source of exogenous information which drives changes in the asset price in the absence of trades.

In addition, price movements in the interbank market might also be due to the actions of competing market makers, but as this information is private to those market makers, and as we do not attempt here to model the game theoretic nature of interaction in the interbank market, we assume that all impact other than that caused by our market maker is contained in a Brownian motion term. This is quite reasonable in that it allows our market maker to concentrate on his own market impact.

There is a rich literature relating to market impact models beginning with the foundational work of Bertsimas and Lo [BL98] and Almgren and Chriss [AC99, AC01, Alm03] in which they formulate the problem of how to optimally split the execution of an order into smaller pieces, with the objective being minimise some cost function over a set time horizon. This has come to be known as the *optimal execution problem* and we discuss it in more detail in Section 1.1 of Chapter 2. This model has been extended by many authors. Notably Gatheral and Schied [GS11] consider the problem with a GBM price process and a time-averaged VaR risk criteria and Forsyth [For11] considers a mean-variance type model in a continuous time setting. In contrast to much of the early research into the optimal execution problem, which focussed on *static* or *deterministic* trading strategies, Almgren and Lorenz [LA11] develop a model to produce adaptive trading strategies. They do so within the framework of the original Almgren Chriss model, as do Schied and Schöneborn [SS09] for an investor with von-Neumann-Morgenstern preferences on an infinite time horizon. Alfonsi et al. [AFS10] introduce an interesting alternative to the Almgren Chriss model in which rather than separate temporary and permanent impact they include a single temporary but persistant impact term. Further advances to the model have incorporated singular control so as to allow for block trades. For example Guo and Zervos [GZ15] develop a model of multiplicative impact and solve the optimal execution problem in this context.

To the best of our knowledge the model that we study is the first one that considers market impact from the perspective of a market maker whose revenue depends on minimising his own market impact when hedging positions. We model the fair value of the asset as a diffusion process subject to permanent market impact. The market maker's inventory process is modelled as another controlled diffusion, where both the skew and hedging controls alter the drift of the process. Our market maker will seek to to maximise a mean-quadratic performance criterion, of the same type as the one discussed in Chapter 1. We present closed form solutions in a special case where the asset price is a Bachelier process with linear permanent and temporary market impact of the Almgren and Chriss type.

In Section 2 of the chapter, we present the formal stochastic control problem on both the finite and infinite time horizons, for which we prove separate verification theorems in Section 3 of the chapter. Using these theorems, in Section 4 of the chapter we present the special case for which we can find closed form expressions for the value function and both the skew control and the hedging control.

III Support and Resistance

In Chapter 3 we consider the optimisation problem faced by a single agent who possesses wealth consisting of an initial endowment x which may be consumed or invested over an interval $[0, T]$. Consuming wealth too quickly will reduce the amount of capital available to grow via investment, whereas investment is inherently risky and is not guaranteed to result in greater wealth being available for future consumption. The agent is therefore faced with the challenge of how to set these controls in order to maximise his total utility from both consumption and investment. The study of this type of problem is known as *portfolio optimisation theory* and has a long history. At the origin of the area is the work of Merton [Mer69], who, building on earlier insights made by Markowitz [Mar52], considered stochastic dynamics for the traded assets in continuous time and power utility functions.

The model that we study in this chapter falls within the context of the Merton model. However we introduce asset prices that include singularities in their drift. The purpose of this is to represent *support and resistance levels*. Specifically of interest is how the agent should respond to optimally adjust his holdings of the asset and rate of consumption when such levels are present in the market. A standard definition of support and resistance levels is given by Murphy [Mur99]:

> *"Support is a level or area on the chart under the market where buying interest is sufficiently strong to overcome selling pressure. As a result, a decline is halted and prices turn back again . . . Resistance is the opposite of support."*

Such price points exist in markets for a variety of reasons, one key reason being the presence of clustered resting orders in the market. Resting orders are instructions left with a financial institution to execute an order for a client only when the price arrives at a pre-specified level. For a buy order, if this rate is lower than the prevailing rate then the order is a *take profit*, whereas if it is higher than the prevailing rate the order is a *stop loss*. Such orders are the standard way in which to open a new position or close an existing position at a certain price, especially for systematic strategies such as trend following strategies. In an empirical analysis of such orders in the FX market, Osler [Osl01] noted two interesting phenomena whilst investigating why trading strategies that depend on price level breakouts are persistently profitable. Firstly, there are significant differences in the clustering patterns of stop loss orders compared to take profit orders. Stop loss orders tended to spread more than take profit order, which cluster strongly. Second, the clustering locations of take profit orders were strongly linked to round numbers in the asset price, while stop loss orders tended to be clustered just above round numbers for buy stops, and just below round numbers for sell stops.

These two observations provide important motivation for our model. The empirical existence of support and resistance levels provides a general justification for the model we study. The observation that their origin is in clustered resting orders also suggest that a good model for support and resistance would involve impulses in the price dynamics, rather than an alternative such as a price dependent drift function, since the underlying source of the phenomenon is itself a cluster of single impulses, namely resting orders.

Stochastic processes that exhibit precisely this behaviour are well known. Diffusions with generalised drift, such as those considered in Lejay [Lej06], are a generalisation of the skew Brownian motion the properties of which have been studied by Walsh [Wal78], Harrison and Shepp [HS81], and Engelbert and Schmidt [ES85], among others. The skew Brownian motion behaves like a Brownian motion away from a given level, but once it hits that level the side of the level on which it makes its next excursion depends on the outcome of an independent Bernoulli random variable. It can be shown that the skew Brownian motion is the strong solution to a SDE involving its local time at the level in question.

In Section 2 of the chapter we present the formal stochastic control problem on both the finite and infinite time horizons, for which we will prove separate verification theorems in Section 3. Using these theorems, in Section 4 and 5 of the chapter we present closed form expressions for the value function and the controls.

Chapter 1

Market Making

1 Introduction

This chapter concerns the problem of a market maker, that is, a market participant who provides continuously tradable prices to a set of clients. Market makers are the facilitators of market liquidity, performing a role that is needed to smooth the time inconsistencies that naturally occur in markets. These time inconsistencies appear in financial markets since it is unlikely that when one investor wishes to sell, another simultaneously wishes to buy. The purpose of market making can therefore be seen as the smoothing through time of levels of supply and demand in markets, thereby creating continuously available liquidity.

Our study casts the market maker as the main protagonist in this interaction. However, the clients with which the market maker trades also play an important role. In models of financial markets, the focus is often placed on the professional investors such as banks and hedge funds. However, the driving force and life blood of the market are those non-professional investors who come to the market because of some fundamental and exogenous economic requirement from which their desire to trade the asset stems. In the problem we study, we refer to the clients of the market maker as *noise traders*, with that name meant to convey the fact that we cannot directly discern the reason behind their trading. However, we do make some basic assumptions about their price sensitivity, that is, their reaction to being offered more or less favourable prices by the market maker.

In order to perform their role, market makers stand ready to both buy and sell assets throughout the day, thereby accumulating holdings of the asset, also known as *inventory*. They may continuously update the prices at which their clients can trade throughout the day, and the size of their inventory changes as clients buy and sell, with the inventory position increasing as clients sell and decreasing as clients buy. In this role, the market maker is said to be *making a market in the asset*.

Of course the market maker's motive is not simply the provision of this service to the market. The goal of market making is no different to any other type of trading, typically, the maximisation of revenue subject to some sort of risk constraint. Unlike traditional portfolio theory, in which only the value of the asset is changing, the market maker faces an additional randomness in the quantities held in his portfolio due to the trading activity of clients. This randomness is controlled to some degree by the actions of the market maker through adjustments to his price. Rather than simply selecting a number of units of the asset to hold, the market maker must select a desirable level for his holdings of the asset and aim to shift the inventory position toward it. The chosen intensity of this shift will depend on a number of factors, such as risk tolerance and skew impact, which correspond to implicit and explicit costs of controlling the inventory level. The cost associated with skew is due to the need to offer more favourable prices to incentivise changes in the rate of client buying to client selling, whereas the choice to change the inventory level more slowly is associated with greater inventory risk, which is itself a cost.

There is a close connection with models of market impact (with which we will enhance our model in Chapter 2). However impact models have traditionally been motivated by a very different type of market activity. The market impact literature focuses on the so called *optimal execution problem* in which an agent seeks to execute a given quantity directly into the market, whilst minimising market impact. The role of the financial institution in this exchange is usually as the provider of the algorithmic execution strategy, or sometimes even simply as the facilitator of that strategy, and in this role is said to be acting as *agent*. When a financial institution provides a market making service to its clients it is said to be acting as *principal* meaning that the financial institution is the principal risk taker of those clients' trades. This interaction between client and financial institution, which might be described as *over the counter* trading, should be distinguished from the more general notion of market making, which as well as this type of interaction would encompass trading strategies that involve leaving visible resting orders on both the bid and offer side of a limit order book market. Although the model we present focusses on the former relationship, the insights gained here are readily applied to any variant of market making, since they relate to the optimisation of expected revenue and risk generated from movements in the asset price and the inventory position. In addition most clients of over the counter market makers will aggregate the prices of several institutions all of which provide this same service, so that the dynamics become quite similar to on exchange market making.

Prior to the advent of algorithmic and electronic trading, the market maker's problem was fundamentally different. Clients would make individual requests for quotes, to which the market maker would reply with a specific bid and offer price on which the client could, if desired, execute a trade. In the event of a trade, the market maker's problem was then essentially identical to the optimal execution problem, with the aim being to rid himself of the acquired risk. The modern market maker's problem, that is to say the algorithmic or electronic market maker's problem, is quite different. Indeed, much has been made of the way in which technological advance has improved liquidity provision, see for example Hendershott et al. [HJM11] and Chaboud et al. [CCHV14]. What is certain is that these changes have dramatically increased competition between market makers, thereby compressing bid-offer spreads. This means that algorithmic market makers receive less spread and are less able to immediately clear risk from their inventory. Consequently, the modern market maker's problem has become less to do with block trade pricing and the optimal execution problem, and more to do with risk management of a portfolio, and *top of book pricing*.

We choose to omit entirely any consideration of the explicit construction of bid and offer prices. In over the counter markets this is typically done on a client by client basis, with construction of the mid price occurring beforehand. Indeed it would be simple to append a model that explicitly constructs the bid and offer price, such as Guilbaud and Pham [GP13], as an extension to our framework. Moreover, the work of Glosten and Milgrom [GM85] suggests that the formation of bid and offer prices is at least partially driven by the notion of adverse selection. In their model, an on exchange market maker forms bid and offer prices at distances from the mid price that reflect where their expectation of the fair price would be adjusted to in the event of a trade. Additionally, some unknown proportion of traders posses private information, and so would be willing to pay an additional spread. Our model does not include informed traders, and so in a Glosten and Milgrom setting the market maker would set spreads to be 0. However, we do incorporate the idea of adverse selection by including a term that captures a correlation between noise trader activity and price movements.

We model trade arrivals as a continuous process rather than a jump process. In addition to the discussion above, some further considerations have shaped our thinking in doing so. The advent of electronic trading has led to small ticket sizes and higher frequency of trading. A fairly recent survey by Menkveld [Men13] from 2010 found that in the Dutch equity market average trade sizes are around €15,000, despite fixed transaction fees of over €1, and in the super liquid foreign exchange market it is not unusual for trades of lower than $1,000 notional value to occur. The skew process then acts more like a lever changing the direction of aggregate client flow, whereas in the context of Poisson arrivals, the skew would change the probability of the direction of the next jump in inventory.

On the basis of these considerations, we focus our attention on the core component of the market maker's problem which is the management of inventory risk by skewing a mid price, expressed as an appropriate offset from a market reference price. We will model noise trader activity using a controlled diffusion process, and we feel that this is a sensible approximation.

Much of the research under the title of *market making* is focussed exclusively on one specific type, namely *on exchange market making*, which as discussed is only a subset of the general concept. Bayraktar and Ludkovski [BL14] consider the slow execution of a block order using limit orders, with a focus on controlling the intensity of trade arrivals by adjusting the placement of orders in a limit order book. Avellaneda and Stoikov [AS08] consider an on exchange market maker in a similar setting with Poisson trade arrival intensities and compute some numerical results. The focus of both of these models is the area of market market microstructure of the limit order book. Closer to the topic of this chapter is Carmona and Webster [CW12] who consider an interesting special case of the over the counter market making problem in which a high frequency market maker attempts to utilise information present in client trades to his own advantage. Gueant et al. [GLFT13] consider the problem of inventory risk for a market maker faced with Poisson arrivals, and investigate the asymptotic properties of the resulting bid and offer prices. Guilbard and Pham [GP13] present a model which focuses on the order placement of an on exchange market maker, who submits limit orders around a top of book price that evolves according to a Markov chain with finite values. They also consider the inventory risk associated with holdings and use a mean-quadratic risk criterion which is the same as the one considered in this chapter.

The model we present in this chapter considers the solution to the market makers problem in continuous time with an inventory process that evolves according to a controlled diffusion process. We formulate the model in such a way as to allow for price dynamics that follow another diffusion process and allow the market maker's price to be a general function the unaffected asset price and the skew control. We present closed form solutions in two special cases. By allowing for drift in the unaffected asset price, and correlation between the asset price and noise trader activity we allow for the study of two interesting features of the market makers problem, namely the utilisation of private information on the asset price and the adverse selection effect caused by client toxicity.

In the remainder of this section we describe, in greater detail, some concepts that are central to the model. The purpose of this is to motivate the market maker's objective function which will take the form of a risk adjusted revenue function. Specifically, the market maker's revenue will be adjusted by a mean-quadratic risk criterion that penalises variations in the inventory position. In Section 2 we present the formal stochastic control problem on both the finite and infinite time horizons, for which we will prove separate verification theorems in Section 3. Using these theorems, in Section 4 we present two special cases for which we can find closed form expressions for the value function and the skew control. The first of these models the asset price as a Bachelier process, whereas the second case uses geometric Brownian motion.

1.1 The Market Reference Price Process

We assume that there exists an underlying reference rate for the asset, which is publicly known but not tradable. The purpose of this reference rate in our model is to allow noise traders something against which to assess the relative attractiveness of the market maker's price. Often such a reference is understood to be a consensus value given all public information, as in Glosten and Milgrom [GM85]. In practice there are many possible sources for reference rates in real markets, although the specific nature may differ somewhat in different markets. For example, in a liquid markets there is often a way for market participants to view a reference price electronically in close to real time, although truly real time access to such a rate may have an associated cost. Typically, publicly visible reference rates will be composed of aggregated prices from a wide variety of sources. Markets in which no such benchmark exists would necessarily demand a deep and one directional trust on the part of the client towards the market maker, and if they exist, such markets would not exhibit the dynamics which we wish to model here. This rate is referred to as the *unaffected price* *process* and is modelled by

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t$$

for some functions $\mu(.)$ and $\sigma(.)$, where W is a standard one-dimensional Brownian motion.

It is assumed that this reference rate process is exogenously generated in the sense that the market makers control does not factor into the reference price. The use of an unaffected price process is standard in the optimal execution literature. Bertsimas and Lo [BL98] discuss the existence of two distinct components to price moves, one of which is the dynamics when the asset is *unaffected* by trading. Almgren and Chriss [AC01] refer to it implicitly when they discuss their *equilibrium price* which moves only according to the exogenous factors of drift and volatility and the endogenous permanent impact from trading, but distinguish it from changes in the traded price due to the temporary impact from trading. Most subsequent models, for example Gatheral and Schied [GS11], Lorenz and Almgren [LS13], Schied [Sch13] and Kharroubi and Pham [KP10], refer explicitly to either a fair or an unaffected price process due to its usefulness as a way of isolating aspects of the price move due to impact from the exogenous component.

It may appear that given the interpretation of the unaffected price process as a fair price it makes sense for that price to be a martingale, but as has been mentioned in Lorenz and Schied [LS13] allowing for a drift in the unaffected asset price is a good way to allow certain interesting phenomena to be modelled. One such example is that drift may represent the expected market impact of another major market participant. Another is that the market maker may himself have some private information to exploit though his choice of skew.

1.2 The Market Maker's Price Process

Investors who wish to trade the asset are permitted to buy and sell at a price set by the market maker. Given the market maker's skew, there is an implied price at which investors can trade. We model this as a mid price, rather than an explicit bid and offer price. We therefore assume noise traders buy and sell at the same price, call it $\tilde{S}$. The market maker sets $\tilde{S}$ with reference to the unaffected price process by means of δ, which is a control process. We refer to this process as the *market maker's control* or the *market maker's skew* process.

Express the tradable asset price $\tilde{S}$ as a composition of the unaffected price process S and the skew adjustment δ which the market maker applies to the unaffected price. The general form of $\tilde{S}$ will be

$$\tilde{S}_t = k(S_t, \delta_t).$$

In general k may take any form, but we should insist that in order to maintain the intuition and intent of the model, k should at a minimum satisfy the requirement that k is a non-decreasing function of both δ_t and S_t, and that $k(S_t, 0) = S_t$.

In our special cases we will use two forms for the function k. When S is a Bachelier process we will take

$$k(S_t, \delta_t) = S_t + \delta_t$$

so that the market maker's price is determined by an offset of δ from S. When S is geometric Brownian motion we will take

$$k(S_t, \delta_t) = S_t(1 + \delta_t)$$

so that the market maker's price is proportionally greater than S by a factor of δ.

1.3 Skew impact

As δ represents the market maker's price relative to a public benchmark, we should expect the flow of new trades arriving from noise traders to be monotone increasing in δ. Two key justifications for this statement arise from the natural behaviour of noise traders. First we note that the unaffected price should be seen as representative of fair value of the asset. Prices that are quoted far above what is understood as a fair price will result in noise traders delaying their decisions to submit buy trades until the price returns closer to fair value, whilst also hurrying their decisions to submit sell trades. Secondly, as market makers price in competition with each other, noise traders are able to route their trading activity to the most attractive price available to them. Consequently when the market maker skews the price up he is implicitly making it more likely that his is the highest bid price across any given set of market makers, and less likely that his is the lowest offer price. This means that he should expect to receive more sell trades and fewer buy trades from clients and his inventory should drift upwards.

In light of the above considerations we model the market maker's inventory process by

$$dX_t = \nu(X_t, \delta_t)dt + e(X_t, \delta_t)dB_t, \quad X_0 \in \mathbb{R},$$

for some functions $\nu(.)$, $e(.)$, where B is a standard Brownian motion. We model the correlation between the noise terms involved in the asset price and the flow of trades arriving as a constant ρ so that

$$d\langle W, B\rangle_t = \rho dt.$$

Whilst the conclusions of our analysis will be valid for any choice of $\nu(.)$ and $e(.)$ satisfying certain technical assumptions, we informally require that they conform to our intuition as to how the inventory process should change as the market maker changes his price. A coherent choice for $\nu(.)$ will be both increasing in δ and $\nu(X_t, 0) = 0$. Similarly, a coherent choice for $e(.)$ will be non-negative and symmetric in δ. In both of the special cases that follow we have chosen to make $\nu(.)$ linear in δ_t and independent of X and $e(.)$ constant, e.g. equation (1.38).

1.4 Market Maker's Revenue

Imagine a sequence of trades between the market maker and any other counterparty occurring at discrete times $t_0, t_1, \ldots, t_n$, of size $x_0, x_1, \ldots, x_n$ and at prices $s_0, s_1, \ldots, s_n$. The revenue accruing to the market maker due to these trades is clearly

$$-\sum_{i=0}^{n} s_i x_i,$$

which represents the cash position of the market maker after the sequence of trades has occurred. Note that the negative sign is due to the nature of market making: the market maker always takes the opposite side of a counterparty's trade, so a trade of size x_i units of the asset means that the market maker buys x_i units whilst the counterparty sells x_i units. From the perspective of the market maker this means that he takes possession of x_i units of the asset and exchanges it for $s_i x_i$ units of cash. Therefore, after the sequence of trades is complete, the market maker also has an open position in the asset, specifically

$$\sum_{i=0}^{n} x_i$$

units of the asset.

Taking these summations to the limit, and remembering that we wish to model the market maker's net inventory holdings X which is the sum of all trades that have occurred so far, we see that the first summation becomes

$$R_t = -\int_0^t \tilde{S}_u dX_u$$

whilst the second summation is simply $X_t - X_0$.

The value of R_t identifies with the market maker's cash position, but does not equate with any notion of trading profit, since the position X_t still needs to be cleared and the rate at which this occurs will determine the profit generated. One option would be to take $R_t + \tilde{S}_t X_t$. However such a modelling choice implicitly assumes that the market maker is able and likely to clear his position at his current price. If X_t is large and the market maker's price is not attractive to sellers this is not realistic. Worse still by setting an arbitrarily high price $\tilde{S}_t$ the market maker could arbitrarily increase his revenue. Instead we follow the market convention of *marking the position to market* by using the unaffected price as a reference for the fair value of the current position and taking $R_t + S_t X_t$.

Thus, we propose the following form for the market maker's revenue: it is the sum of R_t and a mark to market term $S_t X_t$. Thus, the market maker's revenue at terminal time T is given by

$$\mathcal{R}_T(\delta) = -\int_0^T \tilde{S}_t dX_t + (S_T X_T - S_0 X_0), \tag{1.1}$$

specifically, the sum of all trades marked at the rate at which they occurred, plus a *mark to market* term.

As the market maker's price $\tilde{S}$ is a deterministic function of S, X and δ we can express revenue using the dynamics of X as

$$\mathcal{R}_T(\delta) = -\int_0^T k(S_t, \delta_t)\nu(X_t, \delta_t)dt - \int_0^T k(S_t, \delta_t)e(X_t, \delta_t)dB_t + (S_T X_T - S_0 X_0). \tag{1.2}$$

An application of Itô's product formula allows us to calculate

$$S_T X_T - S_0 X_0 = \int_0^T \left(\nu(X_t, \delta_t) S_t + \mu(S_t) X_t + \rho_t \sigma(S_t) e(X_t, \delta_t)\right) dt \\ + \int_0^T e(X_t, \delta_t) S_t dB_t + \int_0^T \sigma(S_t) X_t dW_t. \tag{1.3}$$

By substituting (1.3) back into (1.2) we can express the market maker's revenue function as

$$\mathcal{R}_T(\delta) = \int_0^T \left[\nu(X_t, \delta_t)\left(S_t - k(S_t, \delta_t)\right) + \mu(S_t) X_t + \rho\sigma(S_t) e(X_t, \delta_t)\right] dt + M_T,$$

where

$$M_T = \int_0^T e(X_t, \delta_t)\left(S_t - k(S_t, \delta_t)\right) dB_t + \int_0^T \sigma(S_t) X_t dW_t.$$

This expression provides an alternative way to understand the source of revenue for the market maker. The drift term includes $\nu(X_t, \delta_t)\left(S_t - k(S_t, \delta_t)\right)$ which should be understood as the expected cost due to the decision of the market maker to skew the price. Since we have informally stated that sensible choices for $\nu(X_t, \delta_t)$ and $k(S_t, \delta_t)$ are increasing in δ this means that this term is equal to zero only when $k(S_t, \delta_t) = S_t$. This in turn suggests that when the market maker shows a skewed price the net change in inventory should arrive with a negative inception cost.

The term $\mu(S_t)X_t$ has the obvious intuition that the market maker benefits by holding positions on the right side of the market drift, while the final term $\rho\sigma(S_t)e(X_t, \delta_t)$ has the equally obvious interpretation that if changes in inventory are positively correlated with market moves this acts in the interests of the market maker, and vice versa.

1.5 Risk and inventory costs

A risk neutral market maker would simply maximise the expected value of the revenue function $\mathcal{R}_T(\delta)$ described in the previous section. However in reality market makers are not risk neutral and so we wish to include a risk related component in the market maker's objective function. Such costs may arise due to self imposed restrictions such as a fee charged by the banks central risk desk, external sources such as the costs associated with holding open positions on a futures exchange or simply an inherent risk aversion.

It would appear desirable to include a term penalising the variance of terminal revenue, which suggests the objective function

$$\mathbb{E}\left[\mathcal{R}_T(\delta)\right] - \lambda \text{Var}\left[\mathcal{R}_T(\delta)\right]$$

where the parameter λ defines the risk aversion of the market maker. However there are inherent problems with introducing a variance term explicitly in a continuous time optimal control problem. This problem is known as *time-inconsistency* and is explored in [BM10, BMZ14, ZL00, BC10]. The source of the problem is the $\mathbb{E}\left[\mathcal{R}_T(\delta)\right]^2$ term appearing in the variance term.

We restrict ourself to time consistent measures of risk rather than the variance of terminal revenue. One such option first suggested in Brugiere [Bru96] is to use the quadratic variation of revenue. Not only is this measure time consistent but it is also intuitively appealing since it penalises large deviations in revenue throughout the entire trading period.

It is common to use a slight variant to the quadratic variation of revenue as an alternative risk criterion

$$\lambda \int_0^T \sigma(S_t) X_t dt.$$

This risk criterion, known as the time-averaged VaR process, has been widely adopted in the optimal execution literature, for example Lorenz and Schied [LS13], Schied [Sch13] and Gatheral and Schied [GS11], due to its time consistent nature, intuitive appeal and popularity with practitioners. In our setting this is not directly applicable as the inventory process of the market maker X can be negative which would result in negative risk accruing to negative inventory holdings. We would need to replace X_t, e.g., with $|X_t|$. However the resulting non-linearity would complicate our analysis.

Partially to address this issue, a common approach in the market making literature, e.g. Guilbaud and Pham [GP13], is to use a *mean-quadratic* risk criteria meaning that the expected revenue is adjusted by a term that penalises variations in the inventory position, namely $\int_0^T g(X_t^2)dt$. We therefore propose the penalty function

$$\lambda \int_0^T \sigma(S_t) X_t^2 dt. \tag{1.4}$$

We can also calculate the quadratic variation of our revenue process directly as

$$\begin{aligned}\langle \mathcal{R}_t(\delta) \rangle_t = &\int_0^T e^2(X_t, \delta_t)(S_t - k(S_t, \delta_t))^2 dt + \int_0^T \sigma^2(S_t) X_t^2 dt \\ &+ \int_0^T \rho e(X_t, \delta_t)(S_t - k(S_t, \delta_t))\sigma(S_t) X_t dt \end{aligned} \tag{1.5}$$

where the first integral represents the temporary variation in revenue caused by values of δ away from 0, the second integral represents the variation in revenue caused by the pure market risk due to price fluctuations, and the third integral represents cross correlation of the previous two effects. We then see that the penalisation term used in *mean-quadratic* risk criteria corresponds to the component of the quadratic variation of revenue that is due to market risk.

Additionally, we wish to include a terminal time cost function to represent both the cost of clearing or the cost of holding risk overnight. Market making desks can typically hold an overnight position after the end of the trading session but need to pay a fee to fund this position. Overnight risk can be seen as a single random variable, since the inventory process X_T is fixed until the next days open, and the asset price S_T will exhibit a sudden jump on the open. Consequently, market making desks are often heavily incentivised to clear their positions at the end of the trading period to avoid these risks during the market close. In the marker making literature it is common to either include such a penalty term or enforce strategies that result in $X_T = 0$, see for example Guilbaud and Pham [GP13], thereby avoiding the need for such a penalty. We choose to include a penalty for the terminal position of the same form as the continuous risk penalty

$$\lambda\sigma(S_t)X_t^2. \tag{1.6}$$

We therefore include two additional functions in the market maker's objective function which we denote by $\Phi(.)$ and $\Psi(.)$ which represent these factors. Motivated by the considerations made above we will allow these functions to take general forms, however in the examples they will take the restricted forms

$$\Psi(s,x) = kx^2 \quad \text{and} \quad \Phi(s,x) = Kx^2$$

in the Bachelier case and

$$\Psi(s,x) = ksx^2 \quad \text{and} \quad \Phi(s,x) = Ksx^2$$

in the GBM case to correspond to (1.4) and (1.6). The market maker's objective function can be written in the general form

$$\mathbb{E}\left[\mathcal{R}_T(\delta) - \int_0^T \Psi(S_t, X_t)dt - \Phi(S_T, X_T)\right].$$

Remark 1. We also note that linear-quadratic type objective functions similar in nature to those motivated in this chapter can be derived from the family of exponential utility functions

$$u(x) = -e^{-ax} \approx -1 + ax - \frac{1}{2}a^2x^2$$

for small a, where a represents risk aversion.

2 The Market Model and Control Problem

Fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supporting two standard one-dimensional $(\mathcal{F}_t)$-Brownian motions W and B. Suppose W and B are correlated with coefficient ρ, namely,

$$d\langle W, B\rangle_t = \rho dt.$$

The system we study comprises three stochastic processes S, X and δ, namely the *unaffected price process*, the market maker's *inventory process* and the market maker's *skew process*. The process S is given by

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t, \quad S_0 = s \tag{1.7}$$

and takes values in $\mathcal{S} \subseteq \mathbb{R}$, an open subset of $\mathbb{R}$. The *inventory process* is given by

$$dX_t = \nu(X_t, \delta_t)dt + e(X_t, \delta_t)dB_t, \quad X_0 = x \tag{1.8}$$

taking values in $\mathbb{R}$. This process is affected by the skew δ, which is a D-valued process where $D \subseteq \mathbb{R}$ is also open. We assume sufficient conditions to ensure the existence and uniqueness of a strong solution to these SDEs (see Assumption 1 and Definition 1 below).

Assumption 1. *The functions $\mu : \mathcal{S} \to \mathbb{R}$, $\sigma : \mathcal{S} \to \mathbb{R}$, $\nu : \mathbb{R} \times D \to \mathbb{R}$ and $e : \mathbb{R} \times D \to \mathbb{R}$ are C^1 and there exists a constant $C > 0$ such that*

$$|\mu(s)| + |\sigma(s)| \leq C(1 + |s|),$$

$$|\mu'(s)| + |\sigma'(s)| + |\nu_x(x, \delta)| + |e_x(x, \delta)| \leq C,$$

and

$$|\nu(x, \delta)| + |e(x, \delta)| \leq C(1 + |x| + |\delta|)$$

for all $(s, x, \delta) \in \mathcal{S} \times \mathbb{R} \times D$. Furthermore, $\mu(.)$ and $\sigma(.)$ are such that the solution to (1.7) *is non-explosive, namely, $S_t \in \mathcal{S}$ for all $t \geq 0$, $\mathbb{P}$-a.s..*

2.1 The Control Problem for $T < \infty$

The market maker's objective is to maximise the objective function

$$\begin{aligned} J_{T,s,x}(\delta) = \mathbb{E}\bigg[\int_0^T e^{-\Lambda_t} \Big[& (S_t - k(S_t, \delta_t))\nu(X_t, \delta_t) + \mu(S_t)X_t + \rho e(X_t, \delta_t)\sigma(S_t) \\ & - \Psi(S_t, X_t) \Big] dt - e^{-\Lambda_T} \Phi(S_T, X_T) \ \Big| \ S_0 = s, X_0 = x \bigg] \end{aligned} \tag{1.9}$$

over all admissible controls δ. Here $e^{-\Lambda}$ represents the subjective discounting of the market maker's revenue over time. We assume that Λ is of the form

$$\Lambda_t = \int_0^t \beta(S_u, X_u)du,$$

for some measurable function $\beta(.)$.

Definition 1. *Given a time horizon $T > 0$, the set of admissible controls $\mathcal{A}_T$ is all $(\mathcal{F}_t)$-progressively measurable processes δ such that δ takes values in D and*

$$\mathbb{E}\left[\int_0^T |\delta_t|^m dt \right] < \infty \tag{1.10}$$

for all $m \in \mathbb{N}$.

The conditions in Assumption 1 ensure that (1.7) and (1.8) have unique solutions S and X for each choice of δ such that $\delta \in \mathcal{A}_T$ for all $T > 0$.

We define the value function of the control problem by

$$v(T, s, x) = \sup_{\delta \in \mathcal{A}_T} J_{T,s,x}(\delta),$$

for $s \in \mathcal{S}$ and $x \in \mathbb{R}$. Using standard stochastic control theory that can be found for example in Pham [Pha09], we expect that the value function v of the stochastic control problem identifies with a function $w : [0, T] \times \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$w_t(t, s, x) + \sup_{\delta \in D} \Big[\mathcal{L}^\delta w(t, s, x) + F(s, x, \delta) \Big] = 0 \tag{1.11}$$

for $(t, s, x) \in [0, T] \times \mathcal{S} \times \mathbb{R}$, and with terminal condition

$$w(T, s, x) = -\Phi(s, x), \tag{1.12}$$

where $\mathcal{L}^\delta$ is the differential operator defined by

$$\begin{aligned} \mathcal{L}^\delta w(t, s, x) = & \frac{1}{2}\sigma^2(s) w_{ss}(t, s, x) + \rho e(x, \delta)\sigma(s) w_{sx}(t, s, x) + \frac{1}{2} e^2(x, \delta) w_{xx}(t, s, x) \\ & + \mu(s) w_s(t, s, x) + \nu(x, \delta) w_x(t, s, x) - \beta(s, x) w(t, s, x) \end{aligned} \tag{1.13}$$

for $\delta \in D$, and

$$F(s, x, \delta) = (s - k(s, \delta))\, \nu(x, \delta) + \mu(s)x + \rho e(x, \delta)\sigma(s) - \Psi(s, x) \tag{1.14}$$

for $(s, x, \delta) \in \mathcal{S} \times \mathbb{R} \times D$.

Assumption 2. *The functions $F : \mathcal{S} \times \mathbb{R} \times D \to \mathbb{R}$ and $\Phi : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ are such that*

$$|F(s, x, \delta)| + |\Phi(s, x)| \le C(1 + |s|^k + |x|^k + |\delta|^k) \tag{1.15}$$

for all $(s, x, \delta) \in \mathcal{S} \times \mathbb{R} \times D$, where $k \in \mathbb{N}$ and $C > 0$ are constants. Also the discounting rate $\beta(.)$ takes values in $\mathbb{R}_+$.

2.2 The Control Problem for $T = \infty$

Over an infinite time horizon, the market maker's objective is to maximise the performance criterion

$$\begin{aligned} J_{\infty,s,x}(\delta) = \limsup_{T \to \infty} \mathbb{E}\bigg[\int_0^T e^{-\Lambda_t} \Big[& (S_t - k(S_t, \delta_t))\nu(X_t, \delta_t) + \mu(S_t) X_t \\ & + \rho e(X_t, \delta_t)\sigma(S_t) - \Psi(S_t, X_t) \Big] dt \, \Big| S_0 = s, X_0 = x \bigg] \end{aligned} \tag{1.16}$$

over all admissible controls δ.

Assumption 3. *The discounting rate function* $\beta(.)$ *is such that*

$$\beta(s, x) > \varepsilon > 0$$

for all $s \in \mathcal{S}$ *and* $x \in \mathbb{R}$, *for some* ε.

Definition 2. *The family of all admissible controls* $\mathcal{A}_\infty$ *is the set of all processes* δ *such that* $\delta \in \mathcal{A}_T$ *for all* $T > 0$ *and*

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\Lambda_T}|\Psi(S_T, X_T)|\right] = 0, \tag{1.17}$$

where X *is the associated solution to* (1.8).

Remark 2. The condition (1.17) rules out strategies that do not sufficiently control the inventory position. Making reference to our discussion on the appropriate forms of $\Psi(.)$ in Section 1.5, any optimal strategy that fails to satisfy (1.17) would necessarily involve the build up of larger and larger positions in such a way that expected future gains offset the increasing size of the penalty term.

The value function associated with the control problem on the infinite time horizon is defined by

$$v(s, x) = \sup_{\delta\in\mathcal{A}_\infty} J_{\infty,s,x}(\delta)$$

for $s \in \mathcal{S}$ and $x \in \mathbb{R}$. We opt to repeat the usage of v to represent the value function on the infinite horizon, as the context will ensure there is no ambiguity.

Again, we expect that the value function v of the stochastic control problem on the infinite time horizon identifies with a function $w : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ that solves the HJB equation

$$\sup_{\delta\in D}\left[\mathcal{L}^\delta w(s, x) + F(s, x, \delta)\right] = 0 \tag{1.18}$$

for $(s, x) \in \mathcal{S} \times \mathbb{R}$, where $\mathcal{L}^\delta$ is the differential operator defined by (1.13) and F is defined by (1.14).

3 Verification Theorems

We now prove two verification theorems for the control problem described in Section 2, first for the finite time horizon and then for the infinite horizon.

Theorem 1 (Finite Time Horizon: $T < \infty$)**.** *Let* $w : [0, T] \times \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ *be a* $C^{1,2,2}$ *solution to the HJB equation* (1.11)–(1.14) *that satisfies the polynomial growth condition*

$$|w_s(t, s, x)| + |w_x(t, s, x)| \le C(1 + |s|^k + |x|^k) \tag{1.19}$$

for all $(t, s, x) \in [0, T] \times \mathcal{S} \times \mathbb{R}$ *for some constants* $k \in \mathbb{N}$ *and* $C > 0$. *Then*

$$w(0, s, x) \ge v(T, s, x) \tag{1.20}$$

for all $(t,s,x) \in [0,T] \times \mathcal{S} \times \mathbb{R}$*. Furthermore, suppose that there exists a measurable function* $\hat{\delta} : [0,T] \times \mathcal{S} \times \mathbb{R} \to D$ *such that*

$$\begin{aligned} &w_t(t,s,x) + \mathcal{L}^{\hat{\delta}(t,s,x)} w(t,s,x) + F(s,x,\hat{\delta}(t,s,x)) \\ &= w_t(t,s,x) + \sup_{\delta \in D} \Big[\mathcal{L}^{\delta} w(t,s,x) + F(s,x,\delta) \Big], \end{aligned} \tag{1.21}$$

for all $(t,s,x) \in [0,T] \times \mathcal{S} \times \mathbb{R}$*. Also, suppose that the controlled diffusion*

$$dX_t = \nu(X_t, \hat{\delta}(t,S_t,X_t))dt + e(X_t, \hat{\delta}(t,S_t,X_t))dB_t$$

where S *is the solution to* (1.7)*, admits a unique strong solution and*

$$\hat{\delta}_t = \hat{\delta}(t, S_t, X_t)$$

defines a process in $\mathcal{A}_T$*. Then* $\hat{\delta}$ *is an optimal skew control and*

$$w(0,s,x) = v(T,s,x) \tag{1.22}$$

for all $(t,s,x) \in [0,T] \times \mathcal{S} \times \mathbb{R}$*.*

Proof. Fix any admissible control $\delta \in \mathcal{A}_T$. Corollary 2.10 in Krylov [Kry08] implies that,

$$\sup_{u \in [0,T]} \mathbb{E}\Big[|S_u|^k\Big] < \infty \tag{1.23}$$

and

$$\sup_{u \in [0,T]} \mathbb{E}\Big[|X_u|^k\Big] < \infty. \tag{1.24}$$

for all $k \geq 1$. Using Itô's formula we obtain

$$\begin{aligned} w(T,S_T,X_T) = w(0,S_0,X_0) & \\ + \int_0^T \Big[& w_t(u,S_u,X_u) + \frac{1}{2}\sigma^2(S_u) w_{ss}(u,S_u,X_u) \\ & + \rho e(X_u,\delta_u)\sigma(S_u) w_{sx}(u,S_u,X_u) + \frac{1}{2} e^2(X_u,\delta_u) w_{xx}(u,S_u,X_u) \\ & + \mu(S_u) w_s(u,S_u,X_u) + \nu(X_u,\delta_u) w_x(u,S_u,X_u) \Big] du \\ + \int_0^T & \sigma(S_u) w_s(u,S_u,X_u) dW_u + \int_0^T e(X_u,\delta_u) w_x(u,S_u,X_u) dB_u. \end{aligned}$$

Applying the integration by parts formula we then calculate

$$\begin{aligned} e^{-\Lambda_T} w(T, S_T, X_T) = w(0, S_0, X_0) \\ + \int_0^T e^{-\Lambda_u} \Big[& w_t(u, S_u, X_u) + \frac{1}{2}\sigma^2(S_u) w_{ss}(u, S_u, X_u) \\ & + \rho e(X_u, \delta_u)\sigma(S_u) w_{sx}(u, S_u, X_u) \\ & + \frac{1}{2} e^2(X_u, \delta_u) w_{xx}(u, S_u, X_u) \\ & + \mu(S_u) w_s(u, S_u, X_u) \\ & + \nu(X_u, \delta_u) w_x(u, S_u, X_u) \\ & - \beta w(u, S_u, X_u) \Big] du + M_T \end{aligned} \tag{1.25}$$

where

$$M_T = \int_0^T e^{-\Lambda_t} \sigma(S_t) w_s(t, S_t, X_t) dW_t + \int_0^T e^{-\Lambda_t} e(X_t, \delta_t) w_x(t, S_t, X_t) dB_t.$$

Using Itô's isometry, Assumption 1, the growth condition on w_s and w_x given by (1.19), the admissibility of δ given by (1.10) and the estimates (1.23) and (1.24), we obtain

$$\begin{aligned} \mathbb{E}\left[M_T^2\right] &= \mathbb{E}\left[\int_0^T e^{-2\Lambda_u} \sigma^2(S_u) w_s^2(u, S_u, X_u) du\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{-2\Lambda_u} \rho\sigma(S_u) e(X_u, \delta_u) w_s(u, S_u, X_u) w_x(u, S_u, X_u) du\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{-2\Lambda_u} e^2(X_u, \delta_u) w_x^2(u, S_u, X_u) du\right] \\ &\le \bar{C}\, \mathbb{E}\left[\int_0^T (1 + |S_u|^{\bar{k}} + |X_u|^{\bar{k}} + |\delta_u|^{\bar{k}}) du\right] \\ &\le \bar{C} \int_0^T \left(1 + \sup_{v\in[0,T]} \mathbb{E}\left[|S_v|^{\bar{k}}\right] + \sup_{v\in[0,T]} \mathbb{E}\left[|X_v|^{\bar{k}}\right] + |\delta_u|^{\bar{k}}\right) du \\ &< \infty, \end{aligned} \tag{1.26}$$

where $\bar{k} \in \mathbb{N}$ and $\bar{C} > 0$ are appropriate constants. Therefore M is a square integrable martingale. Furthermore, Assumption 2 implies that

$$\mathbb{E}\left[e^{-\Lambda_T} |\Phi(S_T, X_T)|\right] \le C\left(1 + \mathbb{E}\left[|S_T|^k\right] + \mathbb{E}\left[|X_T|^k\right]\right) < \infty$$

and

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\Lambda_t}|F(S_t,X_t,\delta_t)|dt\right] &\leq C\left(1+\sup_{t\in[0,T]}\mathbb{E}\left[|S_t|^k\right]+\sup_{t\in[0,T]}\mathbb{E}\left[|X_t|^k\right]\right)T\\ &\quad+\mathbb{E}\left[\int_0^T|\delta_t|^k dt\right]\\ &<\infty.\end{aligned}\tag{1.27}$$

Since δ may not achieve the supremum in (1.11), we have the inequality

$$-F(S_t,X_t,\delta_t)\geq w_t(t,S_t,X_t)+\mathcal{L}^{\delta_t}w(t,S_t,X_t).\tag{1.28}$$

Consequently, by substituting (1.28) and (1.12) into (1.25) and taking expectations, we may write

$$\begin{aligned}-\mathbb{E}\left[e^{-\Lambda_T}\Phi(S_T,X_T)\right] &\leq w(0,S_0,X_0)\\ &\quad-\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\delta_u)du\right].\end{aligned}$$

Rearranging terms we derive the inequality

$$J_{T,s,x}(\delta)\equiv\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\delta_u)du-e^{-\Lambda_T}\Phi(S_T,X_T)\right]\leq w(0,S_0,X_0),$$

which implies (1.20) because $\delta\in\mathcal{A}_T$ has been arbitrary.

If we take $\hat{\delta}\in\mathcal{A}_T$ in place of δ, then (1.28) holds with equality and

$$J_{T,s,x}(\hat{\delta})\equiv\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\hat{\delta}_u)du-e^{-\Lambda_T}\Phi(S_T,X_T)\right]=w(0,S_0,X_0).$$

Together with (1.20), this identity results in (1.22) as well as the optimality of $\hat{\delta}$.

□

Theorem 2 (Infinite Time Horizon: $T=\infty$)**.** *Let $w:\mathcal{S}\times\mathbb{R}\to\mathbb{R}$ be a $C^{2,2}$ solution to the HJB equation* (1.18) *that satisfies the polynomial growth conditions*

$$|w(s,x)|\leq C(1+|\Psi(s,x)|)\tag{1.29}$$

and

$$|w_s(s,x)|+|w_x(s,x)|\leq C(1+|s|^k+|x|^k)\tag{1.30}$$

for all $(s,x)\in\mathcal{S}\times\mathbb{R}$ for some constants $k\in\mathbb{N}$ and $C>0$. Then

$$w(s,x)\geq v(s,x)\tag{1.31}$$

for all $(s,x) \in \mathcal{S} \times \mathbb{R}$*. Furthermore, suppose that there exists a measurable function* $\hat{\delta}$: $\mathcal{S} \times \mathbb{R} \to D$ *such that*

$$\bar{\mathcal{L}}^{\hat{\delta}(s,x)} w(s,x) + F(s,x,\hat{\delta}(s,x)) = \sup_{\delta \in D} \left[\bar{\mathcal{L}}^{\delta} w(s,x) + F(s,x,\delta) \right] \tag{1.32}$$

for all $(s,x) \in \mathcal{S} \times \mathbb{R}$*. Also suppose that the controlled diffusion*

$$dX_t = \nu(X_t, \hat{\delta}(S_t, X_t))dt + e(X_t, \hat{\delta}_t(S_t, X_t))dB_t$$

admits a unique strong solution,

$$\hat{\delta}_t = \hat{\delta}(S_t, X_t)$$

defines a process in $\mathcal{A}_\infty$*. Then* $\hat{\delta}$ *is an optimal skew control and*

$$w(s,x) = v(s,x) \tag{1.33}$$

for all $(s,x) \in \mathcal{S} \times \mathbb{R}$*.*

Proof. Fix any admissible control $\delta \in \mathcal{A}_\infty$. Applying Itô's formula and the integration by parts formula we obtain

$$\begin{aligned} e^{-\Lambda_T} w(S_T, X_T) = w(S_0, X_0) & \\ + \int_0^T e^{-\Lambda_u} \Big[& \frac{1}{2}\sigma^2(S_u) w_{ss}(S_u, X_u) + \rho e(X_u, \delta_u)\sigma(S_u) w_{sx}(S_u, X_u) \\ & + \frac{1}{2} e^2(X_u, \delta_u) w_{xx}(S_u, X_u) + \mu(S_u) w_s(S_u, X_u) \\ & + \nu(X_u, \delta_u) w_x(S_u, X_u) - \beta w(S_u, X_u) \Big] du + M_T, \end{aligned} \tag{1.34}$$

where

$$M_T = \int_0^T e^{-\Lambda_t} \sigma(S_t) w_s(S_t, X_t) dW_t + \int_0^T e^{-\Lambda_t} e(X_t, \delta_t) w_x(S_t, X_t) dB_t.$$

Arguing as in (1.26), we can see that M is a square integrable martingale. Furthermore, since δ may not achieve the supremum in (1.18) we have the inequality

$$-F(S_t, X_t, \delta_t) \geq \mathcal{L}^{\delta_t} w(S_t, X_t). \tag{1.35}$$

Recalling (1.27), we substitute (1.35) into (1.34) and we take expectations to obtain

$$\begin{aligned} \mathbb{E}\left[e^{-\Lambda_T} w(S_T, X_T) \right] \leq{} & w(S_0, X_0) \\ & - \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \delta_u) du \right]. \end{aligned}$$

In view of Assumption 3, (1.17) in Definition 2 and (1.29), we can pass to the limit as $T \to \infty$ through an appropriate subsequence to obtain

$$J_{\infty,s,x}(\delta) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \delta_u) du\right] \le w(S_0, X_0),$$

which implies (1.31) because $\delta \in \mathcal{A}_\infty$ has been arbitrary.

If we take $\hat{\delta} \in \mathcal{A}_\infty$ in place of δ, then (1.35) holds with equality and

$$J_{\infty,s,x}(\hat{\delta}) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \hat{\delta}_u) du\right] = w(S_0, X_0).$$

Together with (1.31), this identity results in (1.33) as well as the optimality of $\hat{\delta}$.

□

4 Two Explicitly Solvable Special Cases

We present two alternative choices for the dynamics of S and X and the form of the market maker's control and incentive structure as defined by the functions $k(.)$, $\Psi(.)$ and $\Phi(.)$ as well as the discounting rate β. Both choices allow for explicit solutions in both finite and infinite time horizon.

4.1 An Explicit Solution: Bachelier Price Dynamics

In our first special case we consider Bachelier price dynamics for the fair price process and noise trader activity that is proportional to the absolute distance of the market maker's price from the fair price. In particular we assume that $\mathcal{S} = D = \mathbb{R}$,

$$dS_t = \mu dt + \sigma dW_t \tag{1.36}$$

and that the market maker sets his price $\tilde{S}$ as a linear offset from the fair price S, so that

$$k(S_t, \delta_t) = S_t + \delta_t. \tag{1.37}$$

This is not a modelling assumption *per se*, but rather a notational convention that allows us to express our control as an intuitive quantity. In this case the difference between the market maker's price and the fair price

$$\delta_t = \tilde{S}_t - S_t.$$

Furthermore, we assume that the *skew impact* is a linear function of δ_t that does not depend on the current level of the fair price S_t. In particular, we assume that

$$dX_t = \eta \delta_t dt + \varepsilon dB_t, \tag{1.38}$$

which means that the market maker has the ability to incentivise noise trader activity at a linear cost. This makes sense in a market with arithmetic Brownian prices, where the drift and volatility of the fair price are independent of the actual price level S_t. One common criticism of such models is the potential for the occurrence of negative prices with non-zero probabilities. However in reality it is a reasonable modelling assumption to make in any market where the absolute value of the asset price is large compared to the volatility and drift, a situation that exists in almost all liquid assets. For assets for which this condition is not met it may be preferable to consider a model involving a geometric Brownian price, which we consider in Section 4.2.

As discussed in Section 1.5, the penalties for holding open inventory positions incurred by the market maker can be modelled as a quadratic function of inventory holdings. We therefore consider both time-continuous and terminal time penalties given by

$$\Psi(x) = kx^2 \quad \text{and} \quad \Phi(x) = Kx^2. \tag{1.39}$$

It is worth noting here that in our earlier discussion of the mean-quadratic risk criterion in Section 1.5, we intended for the the constants k and K to combine both the market maker's risk aversion parameter λ, and the volatility of the asset price σ. We will see later in this section that σ plays a minor role in the closed form solutions. However k and K, in which σ is implicitly present, features prominently.

The finite horizon case $T < \infty$

In the context set out at the beginning of this section, we now consider the problem of maximising the objective

$$J_{T,x}(\delta) = \mathbb{E}\left[\int_0^T \left[-\eta\delta_t^2 + \mu X_t + \rho\varepsilon\sigma - kX_t^2\right] dt - KX_T^2 \;\middle|\; X_0 = x\right]. \tag{1.40}$$

over all admissible controls $\mathcal{A}_T$, subject to the stochastic dynamics specified in (1.38).

The value function of the control problem identifies with some appropriate solution to the HJB equation (1.11)–(1.13), which under the conditions described in this section is given by the following partial differential equation

$$w_t(t,x) + \frac{1}{2}\varepsilon^2 w_{xx}(t,x) - kx^2 + \mu x + \rho\varepsilon\sigma + \sup_{\delta\in\mathbb{R}}\left[\eta\delta w_x(t,x) - \eta\delta^2\right] = 0 \tag{1.41}$$

with boundary condition

$$w(T,x) = -Kx^2. \tag{1.42}$$

The skew parameter δ that achieves the maximum in (1.41) is given by

$$\hat{\delta}(t,x) = \frac{1}{2} w_x(t,x). \tag{1.43}$$

This first order condition provides some insight into the likely form of the optimal control. The market maker's optimal choice of skew will be to adjust his price upwards by half of the change in value that would occur due to any consequent change in inventory position. This fits our intuition in the sense that the primary purpose of the skew δ is to control the value of X, which should naturally depend on changes in the value function with respect to X.

Substituting this $\hat{\delta}$ back into the HJB equation (1.41), we obtain

$$w_t(t,x) + \frac{1}{2}\varepsilon^2 w_{xx}(t,x) + \frac{\eta}{4} w_x^2(t,x) - kx^2 + \mu x + \rho\varepsilon\sigma = 0 \tag{1.44}$$

with boundary condition

$$w(T,x) = -Kx^2. \tag{1.45}$$

This non-linear Cauchy problem can be solved using a series of suitable transformations. First, we linearise the equation (1.44) using a logarithmic transformation to remove the w_x^2 term. In particular, we consider the expression

$$w(t,x) = 2\frac{\varepsilon^2}{\eta} \ln |u(\tau,x)|, \tag{1.46}$$

where the time reversal $\tau = T - t$ is intended to convert the terminal condition into an initial condition. In view of the partial derivatives

$$\begin{aligned} w_t(t,x) &= -2\frac{\varepsilon^2}{\eta}\frac{u_\tau(\tau,x)}{u(\tau,x)}, \\ w_x(t,x) &= 2\frac{\varepsilon^2}{\eta}\frac{u_x(\tau,x)}{u(\tau,x)}, \\ w_{xx}(t,x) &= 2\frac{\varepsilon^2}{\eta}\left(\frac{u_{xx}(\tau,x)}{u(\tau,x)} - \frac{u_x^2(\tau,x)}{u^2(\tau,x)}\right), \end{aligned}$$

we can see that (1.44) reduces to the Cauchy problem defined by the PDE

$$u_\tau(\tau,x) = \frac{1}{2}\varepsilon^2 u_{xx}(\tau,x) + \frac{1}{2}\frac{\eta}{\varepsilon^2}(-kx^2 + \mu x + \rho\varepsilon\sigma)u(\tau,x) \tag{1.47}$$

and the boundary condition

$$u(0,x) = e^{-\frac{K\eta}{2\varepsilon^2}x^2}. \tag{1.48}$$

Next, we introduce the transformation

$$u(\tau,x) = \bar{u}(\bar{\tau},z)\exp\left(\frac{\sqrt{k\eta}}{2\varepsilon^2}\left(x - \frac{\mu}{2k}\right)^2 + \left(\frac{1}{2}\sqrt{k\eta} + \frac{\rho\sigma\eta}{2\varepsilon} + \frac{\mu^2\eta}{8k\varepsilon^2}\right)\tau\right), \tag{1.49}$$

where

$$z = (x - \frac{\mu}{2k})e^{\sqrt{k\eta}\tau} \quad \text{and} \quad \bar{\tau} = \frac{1}{2\sqrt{k\eta}}\left(e^{2\sqrt{k\eta}\tau} - 1\right). \tag{1.50}$$

Substituting (1.49) and (1.50) back into (1.47) reduces the problem to the heat equation

$$\bar{u}_{\bar{\tau}}(\bar{\tau},z) = \frac{1}{2}\varepsilon^2 \bar{u}_{zz}(\bar{\tau},z) \tag{1.51}$$

with initial condition

$$\bar{u}\,(0, z) = \exp\left(-\frac{K\eta}{2\varepsilon^2}\left(z + \frac{\mu}{2k}\right)^2 - \frac{\sqrt{k\eta}}{2\varepsilon^2}z^2\right). \tag{1.52}$$

This problem has an initial condition but no boundary conditions on x, which can occupy the entire real line. We will therefore solve it using the Fourier transform

$$\hat{u}(\bar{\tau}, \xi) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \bar{u}(\bar{\tau}, z)e^{-iz\xi}dz.$$

The Fourier transform of $\bar{u}_{\bar{\tau}}$ is simply $\hat{u}_{\bar{\tau}}$, while the second derivative term $\bar{u}_{zz}$ has Fourier transform $(i\xi)^2\hat{u}$. Consequently, the Fourier transformed Cauchy problem (1.51) is a first order ODE in time

$$\hat{u}_{\bar{\tau}}(\bar{\tau}, \xi) = -\frac{1}{2}\varepsilon^2\xi^2\hat{u}(\bar{\tau}, \xi)$$

with initial condition

$$\hat{u}\,(0, \xi) = \hat{\Phi}(\xi),$$

where $\hat{\Phi}(.)$ denotes the Fourier transform of the initial condition (1.52).

The solution to the transformed system is given by

$$\hat{u}(\bar{\tau}, \xi) = \hat{\Phi}(\xi)e^{-\frac{1}{2}\varepsilon^2\xi^2\bar{\tau}}.$$

To recover the solution to the original system we need only invert the transformed solution

$$\bar{u}(\bar{\tau}, z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\bar{\tau}, \xi)e^{i\xi z}d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Phi}(\xi)e^{-\frac{1}{2}\varepsilon^2\xi^2\bar{\tau}}e^{i\xi z}d\xi.$$

Recalling that

$$\hat{\Phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi(z)e^{-i\xi z}dz$$

and using Fubini Theorem, we can see that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Phi}(\xi)e^{-\frac{1}{2}\varepsilon^2\xi^2\bar{\tau}}e^{i\xi z}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(y)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}\varepsilon^2\xi^2\bar{\tau}}e^{i\xi(z-y)}d\xi\right)dy. \tag{1.53}$$

The inner integral can be found using Polyanin [PM08, Supplement 3.2] from which we know that

$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx}dx = \frac{\sqrt{\pi}}{|a|}e^{\frac{b^2}{4a^2}}.$$

for any a and b. Therefore, setting

$$a = \sqrt{\frac{1}{2}\varepsilon^2\bar{\tau}} \quad \text{and} \quad b = i(z - y)$$

we get the equality

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\varepsilon^2\xi^2\bar{\tau}} e^{i\xi(z-y)} d\xi = \frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}\varepsilon^2\bar{\tau}}} e^{-\frac{1}{2}\frac{(z-y)^2}{\varepsilon^2\bar{\tau}}}.$$

Substituting this expression and (1.52) into (1.53) we obtain

$$\begin{aligned}
\bar{u}(\bar{\tau}, z) &= \frac{1}{\sqrt{2\pi\varepsilon^2\bar{\tau}}} \int_{-\infty}^{\infty} \Phi(y) e^{-\frac{1}{2}\frac{(z-y)^2}{\varepsilon^2\bar{\tau}}} dy \\
&= \frac{1}{\sqrt{2\pi\varepsilon^2\bar{\tau}}} \int_{-\infty}^{\infty} e^{-\frac{K\eta}{2\varepsilon^2}(y+\frac{\mu}{2k})^2 - \frac{\sqrt{k\eta}}{2\varepsilon^2}y^2} e^{-\frac{1}{2}\frac{(z-y)^2}{\varepsilon^2\bar{\tau}}} dy \\
&= \frac{1}{\sqrt{(K\eta+\sqrt{k\eta})\bar{\tau}+1}} e^{-\frac{K\eta(z+\frac{\mu}{2k})^2 + \sqrt{k\eta}z^2 + K\eta\sqrt{k\eta}\frac{\mu^2}{4k^2}\bar{\tau}}{2\varepsilon^2((K\eta+\sqrt{k\eta})\bar{\tau}+1)}}. \qquad (1.54)
\end{aligned}$$

We can now prove the main result of this section.

Theorem 3. *Consider the control problem with problem data described in* (1.36)–(1.40). *Given a time horizon $T \in (0, \infty)$ the value function of the control problem identifies with the function*

$$w(t, x) = \varphi(t)x^2 + \psi(t)x + \chi(t), \qquad (1.55)$$

where $\varphi(.)$, $\psi(.)$ and $\chi(.)$ are given by

$$\begin{aligned}
\varphi(t) &= \sqrt{\frac{k}{\eta}} - \frac{\left(K + \sqrt{\frac{k}{\eta}}\right) e^{2\sqrt{k\eta}(T-t)}}{\mathcal{K}(T-t)+1} \\
&= -\sqrt{\frac{k}{\eta}} \tanh\left(\sqrt{k\eta}(T-t) + \operatorname{arctanh}\left(K\sqrt{\frac{\eta}{k}}\right)\right), \qquad (1.56)
\end{aligned}$$

$$\psi(t) = \frac{\mu}{k}\left(\frac{K\left(1 - e^{-\sqrt{k\eta}(T-t)}\right) + \sqrt{\frac{k}{\eta}} e^{2\sqrt{k\eta}(T-t)}}{\mathcal{K}(T-t)+1} - \sqrt{\frac{k}{\eta}}\right) \qquad (1.57)$$

and

$$\begin{aligned}
\chi(t) &= \frac{\mu^2}{k^2}\left(\sqrt{\frac{k}{\eta}} + \frac{K\left(2e^{\sqrt{k\eta}(T-t)} - \frac{3}{2}e^{2\sqrt{k\eta}(T-t)} - \frac{1}{2}\right) - \sqrt{\frac{k}{\eta}}}{\mathcal{K}(T-t)+1}\right) \\
&\quad - 2\frac{\varepsilon^2}{\eta} \ln\sqrt{\mathcal{K}(T-t)+1} + \rho\varepsilon\sigma(T-t), \qquad (1.58)
\end{aligned}$$

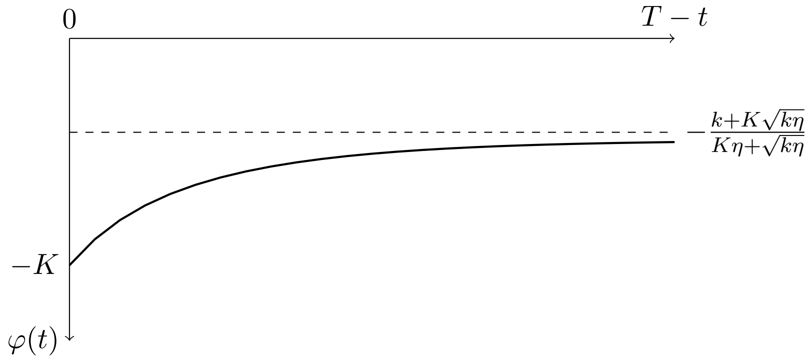

Figure 1.1: Decay function $\varphi(t)$

in which expressions,

$$\mathcal{K}(\tau) = \frac{\left(K\eta + \sqrt{k\eta}\right)\left(e^{2\sqrt{k\eta}\tau} - 1\right)}{2\sqrt{k\eta}}.$$

Furthermore, the optimal control $\hat{\delta} \in \mathcal{A}_T$ is given by

$$\hat{\delta}_t = \varphi(t)\hat{X}_t + \frac{1}{2}\psi(t) \tag{1.59}$$

where

$$d\hat{X}_t = \left[\eta\varphi(t)\hat{X}_t + \frac{1}{2}\eta\psi(t)\right]dt + \varepsilon dB_t, \tag{1.60}$$

Proof. Given explicit solution (1.54) for $\bar{u}(.)$ we can reverse the transformation (1.49) and (1.50) to obtain $u(.)$. Finally we reverse the transformation (1.46) to obtain (1.55)–(1.58) which describes a solution to the HJB equation.

It is clear from their explicit forms that (1.56), (1.57) and (1.58) are bounded, and so the function defined by (1.55) satisfies the growth condition (1.19) in the verification Theorem 1. In view of (1.43) and (1.55), the optimal control $\hat{\delta}$ is given by (1.59). The estimates in Corollary 2.10 in Krylov [Kry08] imply immediately that these controls satisfy the admissibility condition (1.10) in Definition 1. □

The function $\varphi(.)$, which determines the strength of skew due to the inventory position X, decays exponentially towards $-K$ as time approaches the terminal time. This means that, as the amount of remaining time diminishes, the market maker should become willing to *pay* to reduce it by skewing the price and increasing the speed of the drift of the inventory position towards 0. When we talk about the market maker *paying* through the skew, we mean that the market maker absorbs the instantaneous cost of the net change in the inventory position in the direction of the skew, which is $\eta\delta_t$.

As the time remaining to reduce the position increases, the market maker acts with less urgency due to the terminal time penalty. However, he remains incentivised to reduce the

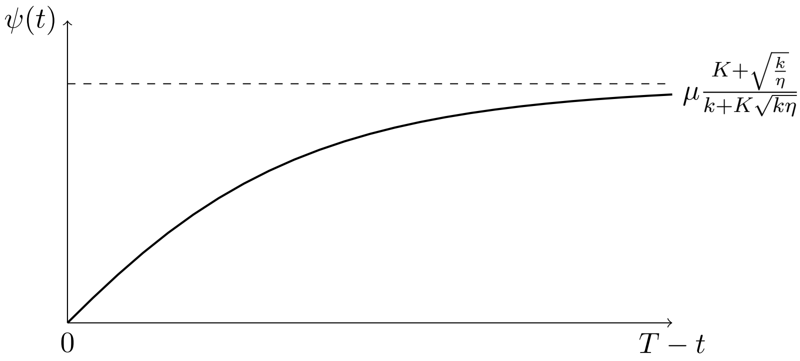

Figure 1.2: Decay function $\psi(t)$

inventory position due to the continuous penalty k. With significant time remaining until terminal time the size of skew therefore also depends on the continuous penalty term k, given by the term $-\frac{k+K\sqrt{k\eta}}{K\eta+\sqrt{k\eta}}$. Whether this skew is more or less dominant for large $T-t$ than it is for small $T-t$ depends on the relative values of k and K. It is worth noting that in the absence of any continuous penalty

$$\varphi(t) = -\frac{K}{K\eta(T-t)+1}.$$

Specifically, the size of the market maker's skew due to the inventory position goes to 0 as $T-t \to \infty$.

The function $\psi(.)$ represents the market maker's *inventory position independent* of skew. This is the skew that the market maker should set to maximise revenue due to any informational advantage about the drift of the fair price μ. However, in order to benefit from the drift, the market maker will still have to bear the risk of holding an extra position. Therefore, the size of this skew, and the implied *target position* that the market maker attempts to build up also depends on k and K.

As time remaining goes to 0 so does this component of skew, since expected revenue due to the drift also goes to 0. As time remaining goes to ∞ this skew goes to

$$\mu\frac{K+\sqrt{\frac{k}{\eta}}}{k+K\sqrt{k\eta}},$$

a risk-dampened multiple of the size of the drift in the fair price of the asset.

We can also see that the market maker's *target position* is

$$\frac{1}{2}\frac{\psi(t)}{\varphi(t)}$$

which is an interesting quantity as it is the size of inventory position at which the market maker will show no skew, so it is effectively the size of position that the market maker is aiming to hold at any given time to maturity.

Finally we note that in this context, the correlation between the the asset price S and the inventory holdings X given by ρ, appears only in $\chi(.)$, through the term $\rho\varepsilon\sigma(T-t)$ but nowhere in the optimal control $\hat{\hat{\delta}}$. The economic meaning of ρ is the *toxicity* of the noise traders, that is, how correlated their trading activity is with movements in the market. If it is positive then noise traders sell when the price is moving up, however if it is negative they buy when the price is moving up. From this we can deduce that in the case with Bachelier price dynamics, the market maker is unable to counteract trade toxicity, and simply must accept its impact on the value function.

The infinite horizon case $T = \infty$

We now consider the problem of maximising the objective

$$J_{\infty,x}(\delta) = \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\beta t} \left(-\eta\delta_t^2 + \mu X_t + \rho\sigma\varepsilon - kX_t^2\right) du \,\middle|\, X_0 = x \right], \tag{1.61}$$

where the discounting rate $\beta > 0$ is a given constant, over all admissible controls $\delta \in \mathcal{A}_\infty$ subject to the stochastic dynamics given by (1.38). In this case, the HJB equation (1.18) takes the form

$$\frac{1}{2}\varepsilon^2 w_{xx}(x) - \beta w(x) + \mu x + \rho\varepsilon\sigma - kx^2 + \sup_{\delta\in\mathbb{R}} \left[\eta\delta w_x(x) - \eta\delta^2\right] = 0. \tag{1.62}$$

The skew control $\hat{\delta}$ that achieves the maximum in (1.62) is

$$\hat{\delta}(x) = \frac{1}{2} w_x(x). \tag{1.63}$$

Substituting this into (1.62) we obtain

$$\frac{1}{2}\varepsilon^2 w_{xx}(x) + \frac{\eta}{4} w_x^2(x) - \beta w(x) + \mu x + \rho\varepsilon\sigma - kx^2 = 0 \tag{1.64}$$

which is a non-linear ordinary differential equation similar to the partial differential equation (1.44) seen in the finite horizon setting. Therefore we approach the problem by suggesting a candidate form for the solution

$$w(x) = \varphi x^2 + \psi x + \chi. \tag{1.65}$$

By substituting (1.65) into (1.64) we can see that φ, ψ, and χ should satisfy the algebraic equations

$$\eta\varphi^2 - \beta\varphi - k = 0, \tag{1.66}$$

$$(\eta\varphi - \beta)\,\psi + \mu = 0 \tag{1.67}$$

and

$$-\beta\chi + \frac{1}{4}\eta\psi^2 + \varepsilon^2\varphi + \rho\varepsilon\sigma = 0. \tag{1.68}$$

The first algebraic equation (1.66) has two distinct real roots since $\beta^2 + 4\eta k > 0$, and we note that one is positive and one is negative since $\frac{k}{\eta} > 0$. Denote the positive and negative roots of (1.66) as

$$\varphi^+,\, \varphi^- = \frac{\beta}{2\eta} \pm \frac{1}{2\eta}\sqrt{\beta^2 + 4\eta k}. \tag{1.69}$$

Lemma 1. *The control defined by* $\hat{\delta}_t = \hat{\delta}(X_t)$*, where* $\hat{\delta}$ *is defined by* (1.63)*, and* X *is the associated solution to* (1.38)*, which is given by* (1.71)*, is such that the admissibility condition* (1.17) *in Definition 2, which takes the form*

$$\lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[X_T^2\right] = 0 \tag{1.70}$$

in the current context, holds true if $\varphi = \varphi^-$ *and fails to be true if* $\varphi = \varphi^+$*.*

Proof. Under the control given by $\hat{\delta}_t = \hat{\delta}(X_t)$, the process X is an Ornstein-Uhlenbeck process with dynamics

$$dX_t = \eta\left(\varphi X_t + \frac{1}{2}\psi\right)dt + \varepsilon dB_t. \tag{1.71}$$

Using Itô's isometry

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left(\left(X_0 + \frac{\psi}{2\varphi}\right)e^{\eta\varphi T} - \frac{\psi}{2\varphi} + \varepsilon\int_0^T e^{\eta\varphi(T-s)}dB_s\right)^2\right] \\
&= \left(\left(X_0 + \frac{\psi}{2\varphi}\right)e^{\eta\varphi T} - \frac{\psi}{2\varphi}\right)^2 + \mathbb{E}\left[\varepsilon\left(\int_0^T e^{\eta\varphi(T-s)}dB_s\right)^2\right] \\
&= \left(X_0 + \frac{\psi}{2\varphi}\right)^2 e^{2\eta\varphi T} - \frac{\psi}{\varphi}\left(X_0 + \frac{\psi}{2\varphi}\right)e^{\eta\varphi T} + \frac{\psi^4}{4\varphi^2} - \frac{\varepsilon^2}{2\eta\varphi}\left(1 - e^{2\eta\varphi T}\right).
\end{aligned}$$

Since the coefficient of $e^{2\eta\varphi T}$ is strictly positive and $\eta > 0$ and $\beta > 0$,

$$\begin{aligned}
\lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[X_T^2\right] &= \lim_{T\to\infty}\left(\left(X_0 + \frac{\psi}{2\varphi}\right)^2 + \frac{\varepsilon^2}{2\eta\varphi}\right)e^{(2\eta\varphi - \beta)T} \\
&= \begin{cases} \infty & \text{if } 2\eta\varphi - \beta > 0, \\ 0 & \text{if } 2\eta\varphi - \beta < 0. \end{cases}
\end{aligned}$$

The result now follows because $2\eta\varphi^+ - \beta > 0$, whereas $2\eta\varphi^- - \beta < 0$. □

We can now prove the main result of this section.

Theorem 4. *Consider the control problem with problem data described in* (1.36)–(1.39)*. The value function of the control problem identifies with the function*

$$w(s, x) = \varphi x^2 + \psi x + \chi \tag{1.72}$$

where $\varphi = \varphi^-$,

$$\psi = -\frac{\mu}{\eta\varphi^- - \beta}$$

and

$$\chi = \frac{\frac{\eta}{4}\psi^2 + \varepsilon^2\varphi^- + \rho\varepsilon\sigma}{\beta}.$$

Furthermore, the optimal control $\hat{\delta} \in \mathcal{A}_\infty$ *is given by*

$$\hat{\delta}_t = \varphi^-\hat{X}_t + \frac{1}{2}\psi \tag{1.73}$$

where

$$d\hat{X}_t = \left[\eta\varphi^-\hat{X}_t + \frac{1}{2}\eta\psi\right] dt + \varepsilon dB_t. \tag{1.74}$$

Proof. We have already established that (1.72) satisfies the HJB equation. Plainly, this function satisfies the growth conditions (1.29) and (1.30). We can immediately see from (1.72) and (1.63) that the optimal control $\hat{\delta}$ is given by (1.73). This control $\hat{\delta}$ is admissible, namely, belongs to $\mathcal{A}_\infty$ thanks to Lemma 1.

□

4.2 An Explicit Solution: GBM Price Dynamics

We now present a model in which the price process is a geometric Brownian motion. We assume that $\mathcal{S} = (0, \infty)$ and we let changes in the unaffected price evolve according to the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1.75}$$

We consider an appropriate form for the function $k(.)$ which determines how the market maker's control changes the prices offered to clients. In Section 4.1 with Bachelier price dynamics we set $k(S_t, \delta_t) = S_t + \delta_t$ so that δ_t coincided with the difference between the market maker's price and the fair price. Here, we consider

$$k(S_t, \delta_t) = S_t(1 + \delta_t) \tag{1.76}$$

so that δ_t identifies with the classical return of a trade made at the market maker's price $\tilde{S}_t$ and revalued at the fair price S_t

$$\delta_t = \frac{\tilde{S}_t - S_t}{S_t}.$$

Remark 3. Our choice of $k(.)$ is a natural one for the GBM case as it is proportional to S. However as we will see, we do not rule out choices for the market maker's skew δ such that the market maker's price is negative. We could avoid this possibility by choosing $k(S_t, \delta_t) = S_t e^{\delta_t}$. However in doing so we would create a problem in that our objective function incentivises the market maker based on revenue, and as $\delta \to -\infty$, further decreases in δ correspond to vanishingly small sacrifices in terms of revenue.

We assume that the market maker's skew δ is real valued, namely, $D = \mathbb{R}$. We also assume that the impact of the market maker's skew is a linear function of δ. Although the definition of δ has changed, the dynamics of X remain as

$$dX_t = \eta\delta_t dt + \varepsilon dB_t. \tag{1.77}$$

These dynamics imply that noise traders are incentivised by the *returns* offered by the market maker rather than the nominal difference in price to the fair value. To put it differently, a \$1 difference in price when the asset price is \$100 is different to a \$1 difference when the price is \$1000.

As discussed in Section 1.5, penalties incurred by the market maker for holding open inventory positions can be modelled as a quadratic function of the inventory holdings, specifically

$$\Psi(s,x) = ksx^2 \quad \text{and} \quad \Phi(s,x) = Ksx^2. \tag{1.78}$$

It is worth noting here that in our earlier discussion of the mean-quadratic risk criterion in Section 1.5, we intended for the constants k and K to combine both the market maker's risk aversion parameter λ and the volatility of the asset price σ. We will see later in this section that σ plays a minor role in the closed form solutions. However k and K, in which σ is implicitly present features prominently.

We now present two explicit solutions with GBM price dynamics, one on a finite horizon and one on an infinite horizon.

The finite horizon case $T < \infty$

We now consider the problem of maximising the objective

$$J_{T,s,x}(\delta) = \mathbb{E}\left[\int_0^T e^{-\beta t} S_t \left[-\eta\delta_t^2 + \mu X_t + \rho\varepsilon\sigma - kX_t^2\right] dt - KS_T X_T^2 \,\middle|\, S_0 = s, X_0 = x\right] \tag{1.79}$$

over all admissible controls $\delta \in \mathcal{A}_T$, subject to the stochastic dynamics given by (1.75) and (1.77). The value function of the control problem should identify with an appropriate solution to the HJB equation

$$\begin{aligned} w_t(t,s,x) + \frac{1}{2}\varepsilon^2 w_{xx}(t,s,x) + \rho\varepsilon\sigma s w_{sx}(t,s,x) + \frac{1}{2}\sigma^2 s^2 w_{ss}(t,s,x) + \mu s w_s(t,s,x) \\ - ksx^2 + \mu sx + \rho\varepsilon\sigma s + \sup_{\delta\in\mathbb{R}}\left[\eta\delta w_x(t,s,x) - \eta s\delta^2\right] = 0 \end{aligned} \tag{1.80}$$

with boundary condition

$$w(T,s,x) = -Ksx^2. \tag{1.81}$$

The skew parameter δ that achieves the maximum in (1.80) is given by

$$\hat{\delta}(t,s,x) = \frac{1}{2}\frac{w_x(t,s,x)}{s}. \tag{1.82}$$

This first order condition involves w_x as it did in the case with Bachelier dynamics (1.43). However it is now scaled by the value of the risky asset s.

Substituting this $\hat{\delta}$ back into the HJB equation (1.80) we obtain

$$w_t(t,s,x) + \frac{1}{2}\varepsilon^2 w_{xx}(t,s,x) + \rho\varepsilon\sigma s w_{sx}(t,s,x) + \frac{1}{2}\sigma^2 s^2 w_{ss}(t,s,x) + \frac{\eta}{4}\frac{1}{s} w_x^2(t,s,x) + \mu s w_s(t,s,x) - ksx^2 + \mu s x + \rho\varepsilon\sigma s = 0, \tag{1.83}$$

with boundary condition

$$w(T,s,x) = -Ksx^2. \tag{1.84}$$

Notice that a function of the form $w(t,s,x) = s\hat{w}(t,x)$ will correspond to a solution to (1.83) so long as $\hat{w}$ satisfies

$$\hat{w}_t(t,x) + \frac{1}{2}\varepsilon^2 \hat{w}_{xx}(t,x) + \frac{\eta}{4}\hat{w}_x^2(t,x) + \rho\varepsilon\sigma\hat{w}_x(t,x) + \mu\hat{w}(t,x) - kx^2 + \mu x + \rho\varepsilon\sigma = 0, \tag{1.85}$$

with boundary condition

$$\hat{w}(T,x) = -Kx^2. \tag{1.86}$$

We postulate that a solution exists in the form

$$\hat{w}(t,x) = \varphi(t)x^2 + \psi(t)x + \chi(t). \tag{1.87}$$

Substituting this into (1.85), we find that the functions $\varphi(.)$, $\psi(.)$ and $\chi(.)$ satisfy

$$\Big[(\varphi'(t) + \eta\varphi^2(t) + (\mu - \beta)\,\varphi(t) - k\Big]x^2 + \Big[\psi'(t) + \eta\varphi(t)\psi(t) + (\mu-\beta)\,\psi(t) + 2\rho\varepsilon\sigma\varphi(t) + \mu\Big]x + \Big[\chi'(t) + \frac{1}{4}\eta\psi^2(t) + \varepsilon^2\varphi(t) + \rho\varepsilon\sigma\psi(t) + (\mu-\beta)\,\chi(t) + \rho\varepsilon\sigma\Big] = 0$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. This identity can be true if and only if $\varphi(.)$, $\psi(.)$ and $\chi(.)$ satisfy the system of ordinary differential equations

$$\varphi'(t) + \eta\varphi^2(t) + (\mu-\beta)\,\varphi(t) - k = 0, \tag{1.88}$$

$$\psi'(t) + (\eta\varphi(t) + \mu - \beta)\,\psi(t) + 2\rho\varepsilon\sigma\varphi(t) + \mu = 0 \tag{1.89}$$

and

$$\chi'(t) + (\mu-\beta)\,\chi(t) + \frac{1}{4}\eta\psi^2(t) + \rho\varepsilon\sigma\psi(t) + \varepsilon^2\varphi(t) + \rho\varepsilon\sigma = 0. \tag{1.90}$$

Furthermore, the function given by (1.87) will satisfy the boundary condition (1.86) if

$$\varphi(T) = -K, \quad \psi(T) = 0 \quad \text{and} \quad \chi(T) = 0.$$

The ODE (1.88) is a constant coefficient Ricatti equation which can be transformed into a constant coefficient second order ODE. Indeed, the transformation

$$\varphi(t) = \frac{z'(t)}{\eta z(t)} \tag{1.91}$$

reduces it to

$$z''(t) + (\mu - \beta)\, z'(t) - k\eta z(t) = 0. \tag{1.92}$$

The initial condition $\varphi(T) = -K$ under this transformation becomes a Robin boundary condition

$$z'(T) + K\eta z(T) = 0. \tag{1.93}$$

Since the discriminant of the equation (1.92) is positive there exists two real distinct roots to the quadratic equation $x^2 + (\mu - \beta)x - k\eta = 0$ and since $4k\eta > 0$ one of the roots is positive whist the other is negative. Therefore every solution to (1.92) is given by

$$z(t) = Ae^{nt} + Be^{mt}, \tag{1.94}$$

for some constants $A, B \in \mathbb{R}$, where

$$n, m = -\frac{1}{2}(\mu - \beta) \pm \frac{1}{2}\sqrt{(\mu - \beta)^2 + 4k\eta}. \tag{1.95}$$

We can now prove the main result of this section.

Theorem 5. *Consider the control problem with problem data described in* (1.75)–(1.78). *Then given a time horizon $T \in (0, \infty)$ the value function of the control problem identifies with the function*

$$w(t, s, x) = s(\varphi(t)x^2 + \psi(t)x + \chi(t)), \tag{1.96}$$

where

$$\varphi(t) = \frac{1}{\eta}\frac{m(n + K\eta)e^{-m(T-t)} - n(m + K\eta)e^{-n(T-t)}}{(n + \eta K)e^{-m(T-t)} - (m + K\eta)e^{-n(T-t)}}, \tag{1.97}$$

$$\begin{aligned} \psi(t) = {} & \mu \frac{\frac{1}{n+\Gamma}\left(e^{\Gamma(T-t)} - e^{-n(T-t)}\right) - \frac{\Theta}{m+\Gamma}\left(e^{\Gamma(T-t)} - e^{-m(T-t)}\right)}{e^{-n(T-t)} - \Theta e^{-m(T-t)}} \\ & + \frac{2\rho\varepsilon\sigma}{\eta}\frac{\frac{1}{n+\Gamma}\left(e^{\Gamma(T-t)} - e^{-n(T-t)}\right) - \frac{\Theta}{m+\Gamma}\left(e^{\Gamma(T-t)} - e^{-m(T-t)}\right)}{e^{-n(T-t)} - \Theta e^{-m(T-t)}} \end{aligned} \tag{1.98}$$

and

$$\chi(t) = e^{(\mu-\beta)(T-t)}\int_t^T \left(\frac{\eta}{4}\psi^2(u) + \rho\varepsilon\sigma\psi(u) + \varepsilon^2\varphi(u) + \rho\varepsilon\sigma\right) du, \tag{1.99}$$

in which expressions,

$$\Gamma = (\mu - \beta) \quad \text{and} \quad \Theta = \frac{n + K\eta}{m + K\eta}.$$

Furthermore, the optimal control $\hat{\delta} \in \mathcal{A}_T$ is given by

$$\hat{\delta}_t = \varphi(t)\hat{X}_t + \frac{1}{2}\psi(t), \tag{1.100}$$

where

$$d\hat{X}_t = \left[\eta\varphi(t)\hat{X}_t + \frac{1}{2}\eta\psi(t)\right] dt + \varepsilon dB_t. \tag{1.101}$$

Proof. As we have already established, if the function of the form (1.96) is to identify with the value function of the control problem, $\varphi(.)$, $\psi(.)$ and $\chi(.)$ must satisfy (1.88)–(1.90). To derive the solution to (1.97) we first note that the solution to (1.92) is of the form (1.94). By taking derivatives in (1.94) and solving for A and B, we can see that this solution satisfies the boundary condition (1.93) if

$$B = -A\frac{n + K\eta}{m + K\eta}e^{(n-m)T}.$$

It follows that

$$z(t) = C\left((n + \eta K)e^{-m(T-t)} - (m + K\eta)e^{-n(T-t)}\right). \tag{1.102}$$

Differentiating, we obtain

$$z'(t) = C\left(m(n + \eta K)e^{-m(T-t)} - n(m + K\eta)e^{-n(T-t)}\right).$$

Substituting these expressions into (1.91) the constant C cancels out, and we obtain (1.97).

The ordinary differential equation (1.89) which $\psi(.)$ satisfies is of the form

$$\psi'(t) + L(t)\psi(t) + 2\rho\varepsilon\sigma\varphi(t) + \mu = 0$$

with terminal condition $\psi(T) = 0$. Therefore,

$$\psi(t) = e^{\int_t^T L(u)du}\int_t^T (2\rho\varepsilon\sigma\varphi(t) + \mu)e^{-\int_v^T L(u)du}dv.$$

Since

$$L(t) = \eta\varphi(t) + (\mu - \beta) = \frac{z'(t)}{z(t)},$$

where $z(.)$ is given by (1.102), we calculate

$$\int_t^T L(u)du = \log\left\{\frac{\left(1 - \frac{n+K\eta}{m+K\eta}\right)}{\left(e^{-n(T-t)} - \frac{n+K\eta}{m+K\eta}e^{-m(T-t)}\right)}\right\} + (\mu - \beta)(T - t)$$

and

$$\int_t^T (2\rho\varepsilon\sigma\varphi(t) + \mu)e^{-\int_v^T L(u)du}dv$$

$$= \frac{\mu}{(1-\Theta)}\left(\frac{1}{n+\Gamma}\left\{1 - e^{-(n+\Gamma)(T-t)}\right\} - \frac{\Theta}{m+\Gamma}\left\{1 - e^{-(m+\Gamma)(T-t)}\right\}\right)$$
$$+ \frac{2\rho\varepsilon\sigma}{\eta(1-\Theta)}\left(\frac{n}{n+\Gamma}\left\{1 - e^{-(n+\Gamma)(T-t)}\right\} - \frac{m\Theta}{m+\Gamma}\left\{1 - e^{-(m+\Gamma)(T-t)}\right\}\right). \tag{1.103}$$

It follows that $\psi(.)$ is given by (1.98). Moreover it is immediate from (1.90) that the function $\chi(.)$ can be written as (1.99).

The functions $\varphi(.)$ and $\psi(.)$ are both bounded because

$$(n + \eta K)e^{-m(T-t)} > (m + \eta K)e^{-n(T-t)}$$

for all $t \in [0, T]$. The boundedness of these functions implies the boundedness of $\chi(.)$. It follows that the function defined by the right hand side of (1.96) satisfies the growth condition (1.19) in the verification Theorem 1. In view of (1.82) and (1.96), the optimal control $\hat{\delta}$ is given by (1.100). The estimates in Corollary 2.10 in Krylov [Kry08] imply immediately that these controls satisfy the admissibility condition (1.10) in Definition 1.

□

Corollary 1. *At all times $t \in [0, T]$, $\varphi(t) < 0$ and hence the skew $\hat{\delta}$ is an affine function with negative coefficient of x.*

Proof. The function $\varphi(.)$ is of the form (1.97), the denominator is positive since

$$(n + K\eta)e^{-m(T-t)} - (m + K\eta)e^{-n(T-t)} \geq n - m > 0$$

while the numerator is negative since

$$m(n + K\eta)e^{-m(T-t)} - n(m + K\eta)e^{-n(T-t)} \leq K\eta(m - n) < 0.$$

□

Remark 4. The explicit form for $\chi(.)$ is left in integral form. However it can be solved explicitly since the integral for $\varphi(.)$ is easy to calculate and the integral for $\psi^2(.)$ is also known in explicit form and involves the Gaussian hypergeometric function. We choose to omit the explicit form because it is rather long and $\chi(.)$ does not feature in either of the optimal controls, only as the inventory position independent time value in the value function.

The infinite horizon case $T = \infty$

We now consider the problem of maximising the objective

$$J_{\infty,s,x}(\delta) = \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\beta t} S_t \left[-\eta\delta_t^2 + \mu X_t + \rho\varepsilon\sigma - kX_t^2\right] dt \,\middle|\, S_0 = s, X_0 = x\right] \quad (1.104)$$

where the discounting rate $\beta > 0$ is a given constant, over all admissible controls $\delta \in \mathcal{A}_\infty$ subject to the stochastic dynamics given by (1.76) and (1.77). In this case, the HJB equation (1.18) takes the form

$$\begin{aligned} \frac{1}{2}\varepsilon^2 w_{xx}(s,x) + \rho\varepsilon\sigma s w_{sx}(s,x) + \frac{1}{2}\sigma^2 s^2 w_{ss}(s,x) + \mu s w_s(s,x) \\ - ksx^2 + \mu sx + \rho\varepsilon\sigma s + \sup_{\delta\in\mathbb{R}}\left[\eta\delta w_x(s,x) - \eta s\delta^2\right] = 0. \end{aligned} \quad (1.105)$$

The skew control $\hat{\delta}$ that achieves the maximum in (1.105) is

$$\hat{\delta}(s,x) = \frac{1}{2}\frac{w_x(s,x)}{s}. \quad (1.106)$$

As in the finite horizon case, its intuitive meaning is that the market maker sets δ_t such that the difference in price $\tilde{S}_t - S_t$ is equal to half the change in the value function with respect to x, scaled by the value of the risky asset. Substituting this $\hat{\delta}$ back into the HJB equation (1.105) we obtain

$$\begin{aligned} \frac{1}{2}\varepsilon^2 w_{xx}(s,x) + \rho\varepsilon\sigma s w_{sx}(s,x) + \frac{1}{2}\sigma^2 s^2 w_{ss}(s,x) + \frac{\eta}{4}\frac{1}{s} w_x^2(s,x) \\ + \mu s w_s(s,x) - \beta w(s,x) - ksx^2 + \mu sx + \rho\varepsilon\sigma s = 0. \end{aligned} \quad (1.107)$$

The non-linear PDE (1.107) can be simplified and the dependency on s removed using a substitution of the form $w(s,x) = s\hat{w}(x)$, from which we obtain the following ODE for $\hat{w}$

$$\frac{1}{2}\varepsilon^2 \hat{w}_{xx}(x) + \frac{\eta}{4}\hat{w}_x^2(x) + \rho\varepsilon\sigma\hat{w}_x(x) + (\mu-\beta)\hat{w}(x) - kx^2 + \mu x + \rho\varepsilon\sigma = 0. \quad (1.108)$$

This ODE has the same type of non-linearity that appears in the finite horizon case. Therefore, we approach it using a candidate solution of the form

$$\hat{w}(x) = \varphi x^2 + \psi x + \chi. \quad (1.109)$$

Substituting (1.109) into (1.108), we can see that φ, ψ and χ should satisfy the algebraic equations

$$\eta\varphi^2 + (\mu-\beta)\varphi - k = 0 \quad (1.110)$$

$$(\eta\varphi + (\mu-\beta))\,\psi + 2\rho\varepsilon\sigma\varphi + \mu = 0 \quad (1.111)$$

$$(\mu-\beta)\chi + \varepsilon^2\varphi + (\frac{\eta}{4}\psi + \rho\varepsilon\sigma)\psi + \rho\varepsilon\sigma = 0. \quad (1.112)$$

The first algebraic equation (1.110) has two distinct real roots since $\beta^2 + 4\eta k > 0$. We note that one is positive and one is negative since $k\eta > 0$. Denote the positive and negative roots of (1.110) as

$$\varphi^+, \varphi^- = -\frac{\mu - \beta}{2\eta} \pm \frac{1}{2\eta}\sqrt{(\mu-\beta)^2 + 4\eta k}. \tag{1.113}$$

Assumption 4. *The problem data satisfies $\mu < \beta$.*

This assumption is standard in the stochastic control literature for infinite horizon problems, and is intuitive. The objective function (1.104) includes an expectation of $e^{-\beta t}S_t$ and since S is a geometric Brownian motion it is intuitive that $\mu > \beta$ could result in $v = \infty$. Consequently we require the assumption to prove the following result.

Lemma 2. *Suppose that the problem data satisfies Assumption 4. The control defined by $\hat{\delta}_t = \hat{\delta}(S_t, X_t)$, where $\hat{\delta}$ is defined by* (1.106)*, and X is the associated solution to* (1.77)*, which is given by* (1.115)*, is such that the admissibility condition* (1.17) *in Definition 2, which takes the form*

$$\lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[S_T X_T^2\right] = 0 \tag{1.114}$$

in the current context, holds true if $\varphi = \varphi^-$ and fails to be true if $\varphi = \varphi^+$.

Proof. Under the controls given by $\delta_t = \hat{\delta}(S_t, X_t)$, the process X is an Ornstein-Uhlenbeck process with dynamics

$$dX_t = \eta\left(\varphi X_t + \frac{1}{2}\psi\right)dt + \varepsilon dB_t. \tag{1.115}$$

Using the integration by parts formula, we can calculate the dynamics of SX as

$$d(S_t X_t) = S_t\left((\eta\varphi + \mu)X_t + \frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)dt + S_t\left(\varepsilon dB_t + \sigma X_t dW_t\right)$$

This allows us to calculate the expectation using Fubini's theorem as

$$\begin{aligned}\mathbb{E}\left[S_T X_T\right] &= \int_0^T (\eta\varphi + \mu)\,\mathbb{E}\left[S_t X_t\right]dt + \int_0^T \left(\frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)\mathbb{E}\left[S_t\right]dt \\ &= \int_0^T (\eta\varphi + \mu)\,\mathbb{E}\left[S_t X_t\right]dt + \int_0^T \left(\frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)S_0 e^{\mu t}dt. \end{aligned}\tag{1.116}$$

Define a function $f(t) = \mathbb{E}\left[S_t X_t\right]$. From (1.116) we see that $f(.)$ satisfies the following differential equation

$$f'(t) = (\eta\varphi + \mu)f(t) + \left(\frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)S_0 e^{\mu t}$$

with initial condition

$$f(0) = S_0 X_0.$$

By solving this differential equation it is easy to see that

$$\mathbb{E}\left[S_T X_T\right] = S_0 e^{\mu T}\left\{\left(X_0 + \frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)e^{\eta\varphi T} - \left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)\right\}.$$

Next define $Z_t = S_t X_t$. Again using the integration by parts formula we have that

$$d(Z_t X_t) = \left\{Z_t\left(2\left(\eta\varphi X_t + \frac{1}{2}\eta\psi\rho\sigma\varepsilon\right) + \mu X_t\right) + \varepsilon^2 S_t\right\}dt + Z_t\left(2\varepsilon dB_t + \sigma X_t dW_t\right)$$

and so

$$\mathbb{E}\left[Z_T X_T\right] = \int_0^T (2\eta\varphi + \mu)\,\mathbb{E}\left[Z_t X_t\right]dt + \int_0^T 2\left(\frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)\mathbb{E}\left[Z_t\right]dt + \int_0^T \varepsilon^2\mathbb{E}\left[S_t\right]dt.$$

Define another function $g(t) = \mathbb{E}\left[Z_T S_T\right]$ and note that this satisfies

$$\begin{aligned} g'(t) &= (2\eta\varphi + \mu)\,g(t) \\ &\quad + 2S_0 e^{\mu t}\left(\frac{1}{2}\eta\psi + \rho\sigma\varepsilon\right)\left\{\left(X_0 + \frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)e^{\eta\varphi T} - \left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)\right\} + \varepsilon^2 S_0 e^{\mu t} \end{aligned}$$

as well as the initial condition

$$g(0) = Z_0 X_0.$$

By solving this differential equation, which is similar to the one solved above for $f(.)$, we see that

$$\begin{aligned} \mathbb{E}\left[Z_T X_T\right] &= S_0 e^{(2\eta\varphi+\mu)T}\left(\left(X_0 + \left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)\right)^2 + \frac{\varepsilon^2}{2\eta\varphi}\right) \\ &\quad - 2S_0 e^{(\eta\varphi+\mu)T}\left(X_0 + \frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)\left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right) \\ &\quad + S_0 e^{\mu T}\left(\left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)^2 - \frac{\varepsilon^2}{2\eta\varphi}\right). \end{aligned}$$

Since the coefficient of $e^{(2\eta\varphi+\mu)T}$ is strictly positive and $\eta > 0$ and $\beta > 0$, and in addition since $\mu < \beta$ due to Assumption 4

$$\begin{aligned} \lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[S_T X_T^2\right] &= \lim_{T\to\infty} S_0\left(\left(X_0 + \left(\frac{\psi}{2\varphi} + \frac{\rho\sigma\varepsilon}{\eta\varphi}\right)\right)^2 + \frac{\varepsilon^2}{2\eta\varphi}\right)e^{(2\eta\varphi+\mu-\beta)T} \\ &= \begin{cases} \infty & \text{if } \; 2\eta\varphi + \mu - \beta > 0 \\ 0 & \text{if } \; 2\eta\varphi + \mu - \beta < 0. \end{cases} \end{aligned}$$

The result now follows since $2\eta\varphi^+ + \mu - \beta > 0$ whereas $2\eta\varphi^- + \mu - \beta < 0$. □

We can now prove the main result of this section.

Theorem 6. *Consider the control problem with problem data described in* (1.75)–(1.77) *and suppose that the problem data are such that Assumption 4 holds. Then on the infinite time horizon $T = \infty$ the value function of the control problem identifies with the function*

$$w(s,x) = s(\varphi x^2 + \psi x + \chi) \tag{1.117}$$

where $\varphi = \varphi^-$,

$$\psi = -\frac{\mu + 2\rho\varepsilon\sigma\varphi^-}{\eta\varphi^- + (\mu - \beta)}$$

and

$$\chi = \frac{\varepsilon^2\varphi^- + \frac{\eta}{4}\psi^2 + \rho\varepsilon\sigma\psi}{\mu - \beta}.$$

Furthermore, the optimal control $\hat{\delta} \in \mathcal{A}_\infty$ is given by

$$\hat{\delta}_t = \varphi^- \hat{X}_t + \frac{1}{2}\psi \tag{1.118}$$

where

$$d\hat{X}_t = \left[\eta\varphi^- \hat{X}_t + \frac{1}{2}\eta\psi\right] dt + \varepsilon dB_t. \tag{1.119}$$

Proof. We have already established that (1.117) satisfies the HJB equation (1.105). We can also see that this function satisfies the growth conditions (1.29) and (1.30) in verification Theorem 2. We can immediately see from (1.117) and (1.106) that the optimal control $\hat{\delta}$ is given by (1.118). This control $\hat{\delta}$ is admissible, namely, $\hat{\delta} \in \mathcal{A}_\infty$ thanks to Lemma 2.

□

Chapter 2

Hedging and Market Impact

1 Introduction

As in Chapter 1, we study a market in which a market maker provides liquidity to a set of clients by setting a tradable price as an offset from a known reference price. The market maker's primary objective is to make money from the flow of client trades, or as we have been referring to them, *noise traders*. The market maker is sensitive to the amount of risk he is holding at any given time and so, as covered in Chapter 1, will try and skew his price to avoid build-ups of risk. This comes at a small cost in revenue as skewing the price to incentivise this risk reducing behaviour means lowering the price at which he will sell or raising the price at which he will buy.

We now wish to expand the setting in which the market maker operates to include an additional market, which we will refer to as the *interbank market*. If skewing is insufficient to reduce the market maker's risk he may choose to hedge this risk in the interbank market, that is, lay it off with other market makers or professional trading firms and pay a transaction fee to do so.

To this end we model a two tiered marketplace in which noise traders interact only with the market maker, whereas the market maker has access to a second pool of liquidity namely the interbank market. The interbank market is the forum in which large professional trading firms trade with each other. We might also refer to this market as the *primary market* in the sense that it represents the location where price formation occurs and where the majority of information that market makers use to set prices resides.

To justify this structure it is important to discuss the reasons why noise traders do not also possess access to the interbank market. To do so requires us to emphasise the difference between the noise traders, who represent the clients of the market maker, and professional investors such as our market maker. Typically the market making function is performed by a trading desk in an investment bank or hedge fund that specialises in electronic execution and risk management, with a client base that consists of a varied mix of smaller investors, including but not limited to the trading departments of non-financial corporations, non-execution focused hedge funds and asset managers, and smaller banks.

These investors may themselves be financial professionals, but if so they are typically not focused on execution. By this we mean that they do not have the inclination to manage the execution of their deals themselves, and do not wish to make the fixed investments in fees and technology infrastructure needed to trade in the primary market. Equally they may simply lack the scale and resources to make such an investment in execution infrastructure. Further to these considerations, some markets specify a minimum trade size which may make trading in the interbank market prohibitive for some market participants. On the other hand, trading with a bank allows for greater flexibility, including the possibility to trade in smaller sizes. Furthermore, in general the market maker is compelled by the competitive nature of markets to offer clients prices that are better than those available in the interbank market. The definition of a *better* price may be quite illusive, but roughly speaking we may assume that it means a smaller bid-offer spread. For this reason the noise traders may simply opt not to trade in the interbank market, even if such access were available to them.

Prices shown by the market maker are generally free of arbitrage in the sense that they are not *crossed* with the primary market. By this we mean that the market maker's bid price is never higher than the primary market offer price or his offer price is lower than the primary market bid. The two tiered nature of the market would mean this could not be considered pure arbitrage, but in practice such soft arbitrage will be spotted and the opportunity to profit seized upon by a client who possesses access to the liquidity provided by two market makers and can therefore trade in opposite directions with each of them.

Within our model, we wish the representation of the interbank market to capture both the random arrival of new information expressed through changes in the expected fair value of the underlying asset, as well as price changes caused by the depletion of liquidity, that is, through market impact caused by the market participants. These two notions are clearly intertwined, in some cases the act of trading is itself a signal, sending information to other market participants, who update their expectations of the fair value of the asset accordingly, see for example Glosten and Milgrom [GM85]. However for the purposes of modelling market impact, we will assume that these notions are separable and there exists a known market impact function, as well as a source of exogenous information which drives changes in the asset price in the absence of trades.

In addition, price movements in the interbank market might also be due to the actions of competing market makers, but as this information is private to those market makers, and as we do not attempt here to model the game theoretic nature of interaction in the interbank market, we assume that all impact other than that caused by our market maker is contained in a Brownian motion term. This is quite reasonable in that it allows our market maker to concentrate on his own market impact.

There is a rich literature relating to market impact models beginning with the foundational work of Bertsimas and Lo [BL98] and Almgren and Chriss [AC99, AC01, Alm03] in which they formulate the problem of how to optimally split the execution of an order into smaller pieces, with the objective being minimise some cost function over a set time horizon. This has come to be known as the *optimal execution problem* and we discuss it in more detail in Section 1.1. This model has been extended by many authors. Notably Gatheral and Schied [GS11] consider the problem with a GBM price process and a time-averaged VaR risk criteria and Forsyth [For11] considers a mean-variance type model in a continuous time setting. In contrast to much of the early research into the optimal execution problem, which focussed on *static* or *deterministic* trading strategies, Almgren and Lorenz [LA11] develop a model to produce adaptive trading strategies. They do so within the framework of the original Almgren Chriss model, as do Schied and Schöneborn [SS09] for an investor with von-Neumann-Morgenstern preferences on an infinite time horizon. Alfonsi et al. [AFS10] introduce an interesting alternative to the Almgren Chriss model in which rather than separate temporary and permanent impact they include a single temporary but persistant impact term. Further advances to the model have incorporated singular control so as to allow for block trades. For example Guo and Zervos [GZ15] develop a model of multiplicative impact and solve the optimal execution problem in this context.

To the best of our knowledge the model that we study is the first one that considers market impact from the perspective of a market maker whose revenue depends on minimising his own market impact when hedging positions. We model the fair value of the asset as a diffusion process subject to permanent market impact. The market maker's inventory process is modelled as another controlled diffusion, where both the skew and hedging controls alter the drift of the process. Our market maker will seek to to maximise a mean-quadratic performance criterion, of the same type as the one discussed in Chapter 1. We present closed form solutions in a special case where the asset price is a Bachelier process with linear permanent and temporary market impact of the Almgren and Chriss type.

In the remainder of this section we describe, in greater detail, the market impact model and motivate the market maker's objective function which takes the form of a risk adjusted revenue function. In Section 2 we present the formal stochastic control problem on both the finite and infinite time horizons, for which we prove separate verification theorems in Section 3. Using these theorems, in Section 4 we present the special case for which we can find closed form expressions for the value function and both the skew control and the hedging control.

1.1 Market Impact of the Almgren and Chriss Type

The Almgren and Chriss market impact model captures the effects on an asset's fair price due to the rate of trading ξ. The model distinguishes two types of impact. The *permanent impact* captures changes that forever remain in the fair value of the asset, and that have magnitude dependent on a function of the trading rate

$$g(\xi_t).$$

The model also considers *temporary* or *transient* market impact which is the impact that occurs as a result of instantaneous liquidity exhaustion. Such liquidity is immediately replenished with no change to the *fair* price of the asset. This impact is modelled by another function

$$h(\xi_t).$$

The temporary impact can be understood as approximation to the effect of trading in a limit order book, where a faster rate of trading will require the investor to go deeper into the order book at each point in time, meaning that it acts like a proportional transaction cost. This impact model was intended as a way of deriving solutions to the optimal execution problem, and so the agent's objective function was based on what is known as *implementation shortfall* rather than revenue. Perold [Per88] originally defined implementation shortfall as the difference between the price at inception of an execution and the average traded rate over the entire execution. It could equivalently be thought of as the revenue accrued by buying the asset at the inception price and selling it at the average rate achieved during execution.

In their original paper, Almgren and Chriss used a mean-variance type objective function where the expectation of the implementation shortfall was penalised by its variance. They were able to do this as their approach was restricted to deterministic strategies in discrete time and so does not fall foul of the time inconsistency problems associated with the mean-variance problem in continuous time. Their intention was to capture the trade off between the higher market impact created by rapid trading against the higher market risk incurred by waiting and performing a slow execution. Prior to this the simple case in which the implementation shortfall alone was minimised was solved and the rate of trading found to be constant [BL98].

We consider the continuous time version of the Almgren and Chriss model, similar to the setting considered in [GS11], meaning that the fair price of the asset evolves according to the dynamics

$$dS_t = \mu(S_t)dt + g(\xi_t)dt + \sigma(S_t)dW_t.$$

In these dynamics, the function $g(.)$ models the permanent impact of trading in the interbank market. The purpose of hedging is of course to alter the market maker's inventory position, alongside the skew control considered in Chapter 1, and so the market maker's inventory evolves according to the dynamics

$$dX_t = \nu(X_t, \delta_t)dt + \xi_t dt + e(X_t, \delta_t)dB_t$$

so that hedging feeds directly into the inventory.

1.2 Revenue with Market Impact

When the market maker hedges, that is when $\xi_t \neq 0$, there occurs a temporary impact to the price paid on the trade as well as permanent impact that moves the fair price. As its name suggests, the temporary impact disappears instantaneously after trading activity ends. This means that the market maker pays a price of

$$S_t + h(\xi_t)$$

on hedging trades rather than just S_t.

The purpose of the market maker's additional control ξ is to manage the inventory process X when it cannot be adequately managed by the normal skew process δ. The dynamics of the inventory process X show that ξ causes changes in the inventory as the market maker hedges, along with a corresponding permanent market impact in the price process S.

Now the market maker's revenue depends on whether changes to inventory are due to client trading or hedging, as hedging activity is more expensive for the market maker due to transient impact, but also because hedging activity causes permanent market impact which changes the value of the market maker's remaining inventory holdings.

We can write the revenue function, which in the absence of hedging was given by (1.1) as

$$\mathcal{R}_T(\delta,\xi) = -\int_0^T \tilde{S}_t \circ dX_t + (S_T X_T - S_0 X_0).$$

where the operator $\circ$ represents the dichotomy between changes in inventory due to client trading and due to market maker hedging. This operator is defined by

$$\tilde{S}_t \circ dX_t = \tilde{S}_t \left(\nu(X_t,\delta_t)dt + e(X_t,\delta_t)dB_t\right) + (S_t + h(\xi_t))\xi_t dt$$

This expression formalises the situation discussed above where noise traders trade at the market maker's rate $\tilde{S}_t$ while hedging trades occur as $S_t + h(\xi_t)$. Applying Itô's product formula, we obtain the expression

$$\begin{aligned} S_T X_T - S_0 X_0 = &\int_0^T \left((\nu(X_t,\delta_t) + \xi_t)S_t + (\mu(S_t) + g(\xi_t))X_t + \rho\sigma(S_t)e(X_t,\delta_t)\right) dt \\ &+ \int_0^T e(X_t,\delta_t)S_t dB_t + \int_0^T \sigma(S_t)X_t dW_t. \end{aligned}$$

Rearranging terms we derive the expression for market maker's revenue given by

$$\begin{aligned} \mathcal{R}_T(\delta,\xi) = \int_0^T &\Big[\nu(X_t,\delta_t)(S_t - k(S_t,\delta_t)) - h(\xi_t)\xi_t \\ &+ (\mu(S_t) + g(\xi_t))X_t + \rho\sigma(S_t)e(X_t,\delta_t)\Big] dt + M_T, \end{aligned}$$

where

$$M_T = \int_0^T e(X_t,\delta_t)(S_t - k(S_t,\delta_t))dB_t + \int_0^T \sigma(S_t)X_t dW_t.$$

We can rewrite this expression as

$$\mathcal{R}_T(\delta,\xi) = \int_0^T R(S_t,X_t,\delta_t,\xi_t)dt + M_T \tag{2.1}$$

where

$$\begin{aligned} R(S_t,X_t,\delta_t,\xi_t) = &\ \nu(S_t,\delta_t)(S_t - k(S_t,\delta_t)) - h(\xi_t)\xi_t \\ &+ (\mu(S_t) + g(\xi_t))X_t + \rho\sigma(S_t)e(X_t,\delta_t). \end{aligned} \tag{2.2}$$

Remark 5. Comparing (2.2) to (1.9) provides insight into the hedging control and its impact on revenue. Hedging reduces revenue directly due to transient price impact through the $-h(\xi_t)\xi_t$ term. This effect should be thought of as a transaction cost paid on hedging trades. Hedging also changes the value of the remaining inventory position through permanent price impact, namely the $g(\xi_t)X_t$ term.

As in Chapter 1, we will include two additional functions in the market maker's objective function which we will name $\Phi(.)$ and $\Psi(.)$ which represent risk factors (see also the discussion in Section 1.5. In our special case, we will restrict them to the form

$$\Psi(s,x) = kx^2 \quad \text{and} \quad \Phi(s,x) = Kx^2$$

to correspond to (1.4) and (1.6). The market maker's objective function can therefore be written in the general form

$$\mathbb{E}\left[\mathcal{R}_T(\delta) - \int_0^T \Psi(S_t, X_t)dt - \Phi(S_T, X_T)\right],$$

which in our special case will be a mean-quadratic objective function.

2 The Market Model and Control Problem

Fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supporting two standard one-dimensional $(\mathcal{F}_t)$-Brownian motions W and B. Suppose W and B are correlated with coefficient ρ, namely,

$$d\langle W, B\rangle_t = \rho dt.$$

The system we study comprises four stochastic processes S, X, δ and ξ, namely the *fair price process*, the *market maker's inventory process*, the *market maker's skew process* and the *market maker's hedging process.* The process S has dynamics given by

$$dS_t = \mu(S_t)dt + g(\xi_t)dt + \sigma(S_t)dW_t, \quad S_0 = s \tag{2.3}$$

and takes values in $\mathbb{R}$. The inventory process has dynamics given by

$$dX_t = \nu(X_t, \delta_t)dt + \xi_t dt + e(X_t, \delta_t)dB_t, \quad X_0 = x \tag{2.4}$$

taking values in $\mathbb{R}$. The inventory process is affected by the skew δ, while both the fair price process and the inventory process are affected by the hedging process ξ. The controls δ and ξ are D and U valued $(\mathcal{F}_t)$-progressively measurable process where $D \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}$ are both open. We assume sufficient conditions that ensure the existence and uniqueness of a strong solution to these SDEs (see Assumption 5 and Definition 3 below).

Remark 6. The process S, which represented the *unaffected price* of the asset in Chapter 1, now represents the *fair price* of the asset. The difference is that the fair price is the unaffected price plus permanent price impact. In the absence of hedging trades, these two concepts are identical.

Assumption 5. *The functions* $\mu : \mathbb{R} \to \mathbb{R}$, $g : U \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$, $\nu : \mathbb{R} \times D \to \mathbb{R}$ *and* $e : \mathbb{R} \times D \to \mathbb{R}$ *are* C^1 *and there exists a constant* $C > 0$ *such that*

$$|\mu(s)| + |\sigma(s)| \leq C(1 + |s|),$$

$$|\mu'(s)| + |g'(u)| + |\sigma'(s)| + |\nu_x(x, \delta)| + |e_x(x, \delta)| \leq C,$$

$$|g(u)| \leq C(1 + |u|),$$

and

$$|\nu(x, \delta)| + |e(x, \delta)| \leq C(1 + |x| + |\delta|)$$

for all $(s, x, u, \delta) \in \mathbb{R} \times \mathbb{R} \times U \times D$.

2.1 The Control Problem for $T < \infty$

The market maker's objective is to maximise the mean-quadratic revenue criterion which was derived in Section 1.2. In particular, we define the market maker's objective function by

$$J_{T,s,x}(\delta, \xi) = \mathbb{E}\Bigg[\int_0^T e^{-\Lambda_t}\left[R(S_t, X_t, \delta_t, \xi_t) - \Psi(S_t, X_t)\right] dt - e^{-\Lambda_T}\Phi(S_T, X_T)\,\Bigg|\, S_0 = s, X_0 = x\Bigg] \tag{2.5}$$

over all pairs of admissible controls (δ, ξ). Here $e^{-\Lambda_t}$ again represents the subjective discounting of the market maker's revenue over time and is of exponential form

$$\Lambda_t = \int_0^t \beta(S_u, X_u)du$$

for some measurable function $\beta(.)$ and $R(.)$ is given by (2.2).

Definition 3. *Given a time horizon* $T > 0$, *the set of admissible controls* $\mathcal{A}_T$ *is all pairs* (δ, ξ) *of* $(\mathcal{F}_t)$*-progressively measurable processes such that* δ *takes values in* D, ξ *takes values in* U,

$$\mathbb{E}\left[\int_0^T |\delta_t|^m dt\right] < \infty \tag{2.6}$$

and

$$\mathbb{E}\left[\int_0^T |\xi_t|^m dt\right] < \infty \tag{2.7}$$

for all $m \in \mathbb{N}$.

The value function of the control problem is defined by

$$v(T, s, x) = \sup_{(\delta,\xi)\in\mathcal{A}_T} J_{T,s,x}(\delta, \xi)$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}$.

Using standard stochastic control theory that can be found, e.g., in Pham [Pha09], we expect that the value function v of the stochastic control problem will identify with a function $w : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that solves the HJB equation

$$w_t(t, s, x) + \sup_{(\delta,\xi)\in D\times U} \Big[\mathcal{L}^{\delta,\xi} w(t, s, x) + F(s, x, \delta, \xi)\Big] = 0 \tag{2.8}$$

for all $(t, s, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, with terminal condition

$$w(T, s, x) = -\Phi(s, x), \tag{2.9}$$

where the differential operator $\mathcal{L}^{\delta,\xi}$ is defined by

$$\begin{aligned}\mathcal{L}^{\delta,\xi} w(t, s, x) = {}& \frac{1}{2}\sigma^2(s) w_{ss}(t, s, x) + \rho e(x, \delta)\sigma(s) w_{sx}(t, s, x) + \frac{1}{2}e^2(x, \delta) w_{xx}(t, s, x) \\ & + (\mu(s) + g(\xi)) w_s(t, s, x) + (\nu(x, \delta) + \xi) w_x(t, s, x) - \beta(s, x) w(t, s, x),\end{aligned} \tag{2.10}$$

for $\delta \in D$ and $\xi \in U$, and

$$F(s, x, u, \delta) = R(s, x, u, \delta) - \Psi(s, x) \tag{2.11}$$

for $(s, x, u, \delta) \in \mathbb{R} \times \mathbb{R} \times U \times D$.

Assumption 6. *The functions $F : \mathbb{R}\times\mathbb{R}\times U \times D \to \mathbb{R}$ and $\Phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and are such that*

$$|F(s, x, u, \delta)| + |\Phi(s, x)| \le C\left(1 + |s|^k + |x|^k + |u|^k + |\delta|^k\right)$$

for all $(s, x, u, \delta) \in \mathbb{R} \times \mathbb{R} \times U \times D$, where $k \in \mathbb{N}$ and $C > 0$ are constants. Also the discounting rate $\beta(.)$ takes values in $\mathbb{R}_+$.

2.2 The Control Problem for $T = \infty$

Over an infinite time horizon, the market maker's objective is to maximise the performance criterion

$$J_{\infty,s,x}(\delta, \xi) = \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_t}\Big[R(S_t, X_t, \delta_t, \xi_t) - \Psi(S_t, X_t)\Big] dt \,\Big|\, S_0 = s, X_0 = x\right] \tag{2.12}$$

over all pairs of admissible controls (δ, ξ).

Definition 4. *The family of all admissible controls* $\mathcal{A}_\infty$ *is the set of all pairs of processes* δ *and* ξ *such that* $(\delta, \xi) \in \mathcal{A}_T$ *for all* $T > 0$ *and*

$$\lim_{T \to \infty} \mathbb{E}\left[e^{-\Lambda_T} |\Psi(S_T, X_T)|\right] = 0 \tag{2.13}$$

where X *is the associated solution to* (2.4)*.*

Assumption 7. *The discounting rate function* $\beta(.)$ *is such that*

$$\beta(s, x) > \varepsilon > 0$$

for all $s \in \mathbb{R}$ *and* $x \in \mathbb{R}$*, for some constant* ε*.*

Remark 7. The condition (2.13) rules out strategies that do not sufficiently control the inventory position. Making reference to our discussion on the appropriate forms of $\Psi(.)$ in Chapter 1, any optimal strategy that fails to satisfy (2.13) would necessarily involve the build up of larger and larger positions in such a way that expected future gains offset the increasing size of the penalty term.

The value function associated with the control problem on the infinite time horizon is defined by

$$v(s, x) = \sup_{(\delta,\xi) \in \mathcal{A}_\infty} J_{\infty,s,x}(\delta)$$

for $s \in \mathbb{R}$ and $x \in \mathbb{R}$. We opt to repeat the usage of v to represent the value function on the infinite horizon, as the context will ensure there is no ambiguity.

Again, we expect that the value function v of the stochastic control problem on the infinite time horizon identifies with a function $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that solves the HJB equation

$$\sup_{(\delta,\xi) \in D \times U} \left[\mathcal{L}^{\delta,\xi} w(s, x) + F(S_t, X_t, \delta, \xi)\right] = 0 \tag{2.14}$$

for all $(s, x) \in \mathbb{R} \times \mathbb{R}$, where $\mathcal{L}^{\delta,\xi}$ is the differential operator (2.10) and F is given by (2.11).

3 Verification Theorems

We now prove two verification theorems for the control problem described in Section 2, first for the finite time horizon and then for the infinite horizon.

Theorem 7 (Finite Time Horizon: $T < \infty$)**.** *Let* $w : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a* $C^{1,2,2}$ *solution to the HJB equation* (2.8) *and* (2.9) *that satisfies the polynomial growth condition*

$$|w_s(t, s, x)| + |w_x(t, s, x)| \leq C(1 + |s|^k + |x|^k) \tag{2.15}$$

for all $(t, s, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ *for some constants* $k \in \mathbb{N}$ *and* $C > 0$*. Then*

$$w(0, s, x) \geq v(T, s, x) \tag{2.16}$$

for all $(t,s,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. *Furthermore, suppose that there exists a pair of measurable functions* $\hat{\delta} : [0,T] \times \mathbb{R} \times \mathbb{R} \to D$ *and* $\hat{\xi} : [0,T] \times \mathbb{R} \times \mathbb{R} \to U$ *such that*

$$\begin{aligned} &w_t(t,S_t,X_t) + \mathcal{L}^{\hat{\delta}(t,S_t,X_t),\hat{\xi}(t,S_t,X_t)} w(t,S_t,X_t) + F(S_t,X_t,\hat{\delta}(t,S_t,X_t),\hat{\xi}(t,S_t,X_t)) \\ &= w_t(t,S_t,X_t) + \sup_{(\delta,\xi)\in D\times U} \left[\mathcal{L}^{\delta,\xi} w(t,S_t,X_t) + F(S_t,X_t,\delta,\xi) \right]. \end{aligned} \tag{2.17}$$

for all $(t,s,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. *Also, suppose that the controlled diffusions*

$$dS_t = \mu(S_t)dt + g(\hat{\xi}(t,S_t,X_t))dt + \sigma(S_t)dW_t$$

and

$$dX_t = \nu(X_t,\hat{\delta}(t,S_t,X_t))dt + \hat{\xi}(t,S_t,X_t)dt + e(X_t,\hat{\delta}(t,S_t,X_t))dB_t$$

admit unique strong solutions, and

$$\hat{\delta}_t = \hat{\delta}(t,S_t,X_t)$$

and

$$\hat{\xi}_t = \hat{\xi}(t,S_t,X_t)$$

together define a pair of processes in $\mathcal{A}_T$. *Then* $(\hat{\delta},\hat{\xi})$ *is an optimal skew-hedging pair and*

$$w(0,s,x) = v(T,s,x) \tag{2.18}$$

for all $(t,s,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

Proof. Fix any admissible pair of controls $(\delta,\xi) \in \mathcal{A}_T$. Corollary 2.10 in Krylov [Kry08] implies that,

$$\sup_{u\in[0,T]} \mathbb{E}\left[|S_u|^k\right] < \infty \tag{2.19}$$

and

$$\sup_{u\in[0,T]} \mathbb{E}\left[|X_u|^k\right] < \infty. \tag{2.20}$$

for all $k \geq 1$. Using Itô's formula we obtain

$$\begin{aligned} w(T,S_T,X_T) &= w(0,S_0,X_0) \\ &+ \int_0^T \Big[w_t(u,S_u,X_u) + \frac{1}{2}\sigma^2(S_u) w_{ss}(u,S_u,X_u) \\ &\qquad + \rho e(X_u,\delta_u)\sigma(S_u) w_{sx}(u,S_u,X_u) + \frac{1}{2} e^2(X_u,\delta_u) w_{xx}(u,S_u,X_u) \\ &\qquad + (\mu(S_u) + g(\xi_u)) w_s(u,S_u,X_u) + (\nu(X_u,\delta) + \xi_u) w_x(u,S_u,X_u) \Big] du \\ &+ \int_0^T \sigma(S_u) w_s(u,S_u,X_u) dW_u + \int_0^T e(X_u,\delta_u) w_x(u,S_u,X_u) dB_u. \end{aligned}$$

Applying the integration by parts formula to $e^{-\Lambda_t}w(t,S_t,X_t)$, we calculate

$$\begin{aligned} e^{-\Lambda_T}w(T,S_T,X_T) &= w(0,S_0,X_0) \\ &\quad + \int_0^T e^{-\Lambda_u}\Big[w_t(u,S_u,X_u) + \frac{1}{2}\sigma^2(S_u)w_{ss}(u,S_u,X_u) \\ &\qquad + \rho e(X_u,\delta_u)\sigma(S_u)w_{sx}(u,S_u,X_u) \\ &\qquad + \frac{1}{2}e^2(X_u,\delta_u)w_{xx}(u,S_u,X_u) \\ &\qquad + (\mu(S_u)+g(\xi_u))w_s(u,S_u,X_u) \\ &\qquad + (\nu(X_u,\delta_u)+\xi_u)w_x(u,S_u,X_u) \\ &\qquad - \beta w(u,S_u,X_u)\Big]du + M_T \end{aligned} \tag{2.21}$$

where

$$M_T = \int_0^T e^{-\Lambda_t}\sigma(S_t)w_s(t,S_t,X_t)dW_t + \int_0^T e^{-\Lambda_t}e(X_t,\delta_t)w_x(t,S_t,X_t)dB_t. \tag{2.22}$$

Using Itô's isometry, Assumption 6, the growth condition on w_s and w_x given by (2.15), the admissibility conditions for δ and ξ given by (2.6) and (2.7) and the estimates (2.19) and (2.20), we obtain

$$\begin{aligned} \mathbb{E}\left[M_T^2\right] &= \mathbb{E}\left[\int_0^T e^{-2\Lambda_u}\sigma^2(S_u)w_s^2(u,S_u,X_u)du\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{-2\Lambda_u}\rho\sigma(S_u)e(X_u,\delta_u)w_s(u,S_u,X_u)w_x(u,S_u,X_u)du\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{-2\Lambda_u}e^2(X_u,\delta_u)w_x^2(u,S_u,X_u)du\right] \\ &\le C\,\mathbb{E}\left[\int_0^T (1+|S_u|+|X_u|+|\delta_u|)^2(1+|S_u|^k+|X_u|^k+|\delta_u|^k)^2du\right] \\ &\le \bar{C}\,\mathbb{E}\left[\int_0^T (1+|S_u|^{\bar{k}}+|X_u|^{\bar{k}}+|\delta_u|^{\bar{k}})du\right] \\ &< \infty. \end{aligned} \tag{2.23}$$

where $\bar{k} \in \mathbb{N}$ and $\bar{C} > 0$ are appropriate constants. Therefore M is a square integrable martingale. Furthermore, Assumption 6 implies that

$$\mathbb{E}\left[e^{-\Lambda_T}|\Phi(S_T,X_T)|\right] \le C\left(1+\mathbb{E}\left[|S_T|^k\right]+\mathbb{E}\left[|X_T|^k\right]\right) < \infty$$

and

$$\mathbb{E}\left[\int_0^T e^{-\Lambda_t}|F(S_t,X_t,\delta_t,\xi_t)|dt\right] \leq C\left(1+\sup_{t\in[0,T]}\mathbb{E}\left[|S_t|^k\right]+\sup_{t\in[0,T]}\mathbb{E}\left[|X_t|^k\right]\right)T$$
$$+\mathbb{E}\left[\int_0^T|\delta_t|^k dt\right]+\mathbb{E}\left[\int_0^T|\xi_t|^k dt\right]$$
$$<\infty. \tag{2.24}$$

Since the pair of controls (δ,ξ) may not achieve the supremum in (2.8) we have the following inequality

$$-F(S_t,X_t,\delta_t,\xi_t) \geq w_t(t,S_t,X_t)+\mathcal{L}^{\delta_t,\xi_t}w(t,S_t,X_t). \tag{2.25}$$

Consequently, by substituting (2.25) and (2.9) into (2.21) and taking expectations, we may write

$$-\mathbb{E}\left[e^{-\Lambda_T}\Phi(S_T,X_T)\right] \leq w(0,S_0,X_0)$$
$$-\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\delta_u,\xi_t)du\right].$$

Rearranging terms we derive the inequality

$$J_{T,s,x}(\delta,\xi)\equiv\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\delta_u,\xi_u)du-e^{-\Lambda_T}\Phi(S_T,X_T)\right]\leq w(0,S_0,X_0)$$

which implies (2.16) because $(\delta,\xi)\in\mathcal{A}_T$ has been arbitrary.

If we take $(\hat{\delta},\hat{\xi})$ in place of (δ,ξ), then (2.25) holds with equality and

$$J_{T,s,x}(\hat{\delta},\hat{\xi})\equiv\mathbb{E}\left[\int_0^T e^{-\Lambda_u}F(S_u,X_u,\hat{\delta}_u,\hat{\xi}_u)du-e^{-\Lambda_T}\Phi(S_T,X_T)\right]=w(0,S_0,X_0)$$

Together with (2.16), this identity results in (2.18) as well as the optimality of $(\hat{\delta},\hat{\xi})$.

□

Theorem 8 (Infinite Time Horizon: $T=\infty$)**.** *Let $w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a $C^{2,2}$ solution to the HJB equation* (2.14) *that satisfies the polynomial growth conditions*

$$|w(s,x)|\leq C(1+|\Psi(s,x)|) \tag{2.26}$$

and

$$|w_s(s,x)|+|w_x(s,x)|\leq C(1+|s|^k+|x|^k) \tag{2.27}$$

for all $(s,x)\in\mathbb{R}\times\mathbb{R}$ for some constants $k\in\mathbb{N}$ and $C>0$. Then

$$w(s,x)\geq v(s,x) \tag{2.28}$$

for all $(s,x) \in \mathbb{R}\times\mathbb{R}$*. Furthermore, suppose that there exists a pair of measurable functions* $\hat{\delta} : \mathbb{R}\times\mathbb{R} \to D$ *and* $\hat{\xi} : \mathbb{R}\times\mathbb{R} \to U$ *such that*

$$\begin{aligned}&\mathcal{L}^{\hat{\delta}(S_t,X_t),\hat{\xi}(S_t,X_t)}w(S_t,X_t) + F(S_t,X_t,\hat{\delta}(S_t,X_t),\hat{\xi}(S_t,X_t)) \\ &= \sup_{(\delta,\xi)\in D\times U}\Big[\mathcal{L}^{\delta,\xi}w(S_t,X_t) + F(S_t,X_t,\delta,\xi)\Big] \end{aligned} \tag{2.29}$$

for all $(s,x) \in \mathbb{R}\times\mathbb{R}$*. Also suppose that the controlled diffusions*

$$dS_t = \mu(S_t)dt + g(\hat{\xi}(S_t,X_t))dt + \sigma(S_t)dW_t$$

and

$$dX_t = \nu(X_t,\hat{\delta}(S_t,X_t))dt + \hat{\xi}(S_t,X_t)dt + e(X_t,\hat{\delta}(S_t,X_t))dB_t$$

admit unique strong solutions,

$$\hat{\delta}_t = \hat{\delta}(S_t,X_t)$$

and

$$\hat{\xi}_t = \hat{\xi}(S_t,X_t)$$

together define a pair of processes in $\mathcal{A}_\infty$*. Then* $(\hat{\delta},\hat{\xi})$ *is an optimal skew-hedging pair and*

$$w(s,x) = v(s,x) \tag{2.30}$$

for all $(s,x) \in \mathbb{R}\times\mathbb{R}$*.*

Proof. Fix any pair of admissible controls $(\delta,\xi) \in \mathcal{A}_\infty$. Applying Itô's formula and the integration by parts formula we obtain

$$\begin{aligned}e^{-\Lambda_T}w(S_T,X_T) &= w(S_0,X_0) \\ &+ \int_0^T e^{-\Lambda_u}\Big[\frac{1}{2}\sigma^2(S_u)w_{ss}(S_u,X_u) + \rho e(X_u,\delta_u)\sigma(S_u)w_{sx}(S_u,X_u) \\ &\qquad + \frac{1}{2}e^2(X_u,\delta_u)w_{xx}(S_u,X_u) + (\mu(S_u)+g(\xi_u))w_s(S_u,X_u) \\ &\qquad + (\nu(X_u,\delta_u)+\xi_u)w_x(S_u,X_u) - \beta w(S_u,X_u)\Big]du + M_T,\end{aligned} \tag{2.31}$$

where

$$M_T = \int_0^T e^{-\Lambda_t}\sigma(S_t)w_s(S_t,X_t)dW_t + \int_0^T e^{-\Lambda_t}e(X_t,\delta_t)w_x(S_t,X_t)dB_t.$$

Arguing as in (2.23), we can see that M is a square integrable martingale. Furthermore, since δ and ξ may not achieve the supremum in (2.14) we have the inequality

$$-F(S_t,X_t,\delta_t,\xi_t) \geq \mathcal{L}^{\delta_t,\xi_t}w(S_t,X_t). \tag{2.32}$$

Recalling (2.24), we substitute (2.32) into (2.31) and take expectations to obtain

$$\mathbb{E}\left[e^{-\Lambda_T} w(S_T, X_T)\right] \leq w(S_0, X_0) - \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \delta_u, \xi_u) du\right].$$

In view of Assumption 7, (2.13) in Definition 4 and (2.26), we can pass to the limit $T \to \infty$ through an appropriate subsequence to obtain

$$J_{\infty,s,x}(\delta, \xi) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \delta_u, \xi_u) du\right] \leq w(S_0, X_0),$$

which implies (2.28) because $(\delta, \xi) \in \mathcal{A}_\infty$ has been arbitrary.

If we take $(\hat{\delta}, \hat{\xi}) \in \mathcal{A}_\infty$ in place of (δ, ξ), then (2.32) holds with equality

$$J_{\infty,s,x}(\hat{\delta}, \hat{\xi}) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda_u} F(S_u, X_u, \hat{\delta}_u, \hat{\xi}_u) du\right] = w(S_0, X_0)$$

Together with (2.28), this identity results in (2.30) as well as the optimality of $(\hat{\delta}, \hat{\xi})$. □

4 A Special Case with Explicit Solution

We consider a specific choice for the dynamics of the asset price S and the inventory process X for which we are able to find explicit solutions. Our case corresponds closely to the Bachelier dynamics studied in Section 4.1, however with additional impact terms due to market maker hedging.

We assume that the market maker's skew δ as well as the rate of trading ξ are real valued, namely $D = U = \mathbb{R}$. We also assume that both permanent and transient market impact are linear functions of the rate of trading, that is

$$g(\xi) = \lambda\xi \quad \text{and} \quad h(\xi) = \gamma\xi. \tag{2.33}$$

The fair price of the asset follows the controlled arithmetic Brownian motion given by

$$dS_t = (\mu + \lambda\xi_t)dt + \sigma dW_t. \tag{2.34}$$

We also assume that the market maker's inventory process is modelled by

$$dX_t = (\eta\delta_t + \xi_t)dt + \varepsilon dB_t. \tag{2.35}$$

According to (2.1) and (2.2) the revenue evolves according to

$$\mathcal{R}_t(\delta, \xi) = \int_0^t R(S_u, X_u, \delta_u, \xi_u)du + M_t,$$

where

$$R(S_t, X_t, \delta_t, \xi_t) = -\eta\delta^2 - \gamma\xi_t^2 + (\mu + \lambda\xi_t)X_t + \rho\sigma\varepsilon. \tag{2.36}$$

We assume that the market maker's time discounting occurs at a constant rate β.

As we did in Chapter 1, we assume that the risk functions $\Phi(.)$ and $\Psi(.)$ are quadratic in x, specifically

$$\Psi(s, x) = -kx^2 \quad \text{and} \quad \Phi(s, x) = -Kx^2. \tag{2.37}$$

It is worth noting here that in our earlier discussion of the mean-quadratic risk criterion in Section 1.5, we intended for the constants k and K to combine both the market maker's risk aversion parameter λ and the volatility of the asset price σ. We will see later in this section that σ plays a minor role in the closed form solutions. However k and K, in which σ is implicitly present, features prominently.

We now present three explicit solutions in which Assumption 8 below plays an important role. In Section 4.1, we find an explicit solution on a finite time horizon in which Assumption 8 is satisfied. In Section 4.2, we do the same on an infinite time horizon. In Section 4.2 we consider the case on the finite time horizon where it does not hold, and find that given a long enough time horizon the value function may be infinite.

Assumption 8. *The problem data satisfies* $\left(\frac{\lambda}{\gamma} - \beta\right)^2 > 4\left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right)$.

4.1 An Explicit Solution: Finite Time Horizon $T < \infty$

The market maker's objective is to maximise the performance criterion

$$J_{T,x}(\delta, \xi) = \mathbb{E}\left[\int_0^T e^{-\beta t}\big[-\eta\delta^2 - \gamma\xi_t^2 + (\mu + \lambda\xi_t)X_t + \rho\sigma\varepsilon - kX_t^2\big]dt - Ke^{-\beta T}X_T^2 \,\Big|\, X_0 = x\right], \tag{2.38}$$

over all admissible controls $(\delta, \xi) \in \mathcal{A}_T$, subject to the stochastic dynamics given by (2.35). We immediate see from (2.38) that in this setting the objective function is independent of the value of S_t. We may therefore drop it from consideration and so the value function of the control problem should identify with an appropriate solution to the HJB equation

$$w_t(t, x) + \frac{1}{2}\varepsilon^2 w_{xx}(t, x) - \beta w(t, x) + \mu x + \rho\varepsilon\sigma - kx^2 + \sup_{\delta\in\mathbb{R}}\left[\eta\delta w_x(t, x) - \eta\delta^2\right] + \sup_{\xi\in\mathbb{R}}\left[\xi w_x(t, x) + \lambda\xi x - \gamma\xi^2\right] = 0 \tag{2.39}$$

with boundary condition

$$w(T, x) = -Kx^2. \tag{2.40}$$

The skew control δ and hedging control ξ that achieve the maximum in (2.39) are given by

$$\hat{\delta}(t, x) = \frac{1}{2}w_x(t, x) \tag{2.41}$$

and

$$\hat{\xi}(t,x) = \frac{1}{2\gamma}(w_x(t,x) + \lambda x). \tag{2.42}$$

These conditions can provide insight into the optimal solution. The market maker skews his price up in proportion to the marginal increase in the value function with respect to increases in inventory given by w_x. This is because doing so will help incentivise noise traders to sell to the market maker. Similarly the market maker will hedge in proportion to w_x, but also in proportion to λx, which is the increase in value of the current inventory that would be caused by the market maker's hedging, which causes permanent market impact. We also note that the market maker's rate of trading is scaled by γ, which is the temporary impact caused by trading, which reduces revenue.

Substituting (2.41) and (2.42) into the HJB equation (2.39), we see that the value function under the optimal control should satisfy

$$\begin{aligned} w_t(t,x) + \frac{1}{2}\varepsilon^2 w_{xx}(t,x) + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right) w_x^2(t,x) + \frac{\lambda}{2\gamma} x w_x(t,x) \\ - \beta w(t,x) + \left(\frac{\lambda^2}{4\gamma} - k\right) x^2 + \mu x + \rho\varepsilon\sigma = 0 \end{aligned} \tag{2.43}$$

with boundary condition

$$w(0,x) = -Kx^2. \tag{2.44}$$

We postulate that a solution exists in the form

$$w(t,x) = \varphi(t)x^2 + \psi(t)x + \chi(t). \tag{2.45}$$

Substituting this into the equation (2.43), we find that the functions $\varphi(.)$, $\psi(.)$ and $\chi(.)$ must satisfy

$$\begin{aligned} &\left(\varphi'(t) + \left(\eta + \frac{1}{\gamma}\right)\varphi^2(t) + \left(\frac{\lambda}{\gamma} - \beta\right)\varphi(t) + \left(\frac{\lambda^2}{4\gamma} - k\right)\right) x^2 \\ &+ \left(\psi'(t) + \left(\eta + \frac{1}{\gamma}\right)\varphi(t)\psi(t) + \left(\frac{\lambda}{2\gamma} - \beta\right)\psi(t) + \mu\right) x \\ &+ \left(\chi'(t) + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\psi^2(t) + \varepsilon^2\varphi(t) - \beta\chi(t) + \rho\varepsilon\sigma\right) = 0 \end{aligned}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. This identity can be true if and only if $\varphi(.)$, $\psi(.)$ and $\chi(.)$ satisfy the system of ordinary differential equations

$$\varphi'(t) + \left(\eta + \frac{1}{\gamma}\right)\varphi(t)^2 + \left(\frac{\lambda}{\gamma} - \beta\right)\varphi(t) + \left(\frac{\lambda^2}{4\gamma} - k\right) = 0, \tag{2.46}$$

$$\psi'(t) + \left(\left(\eta + \frac{1}{\gamma}\right)\varphi(t) + \frac{\lambda}{2\gamma} - \beta\right)\psi(t) + \mu = 0 \tag{2.47}$$

and

$$\chi'(t) - \beta\chi(t) + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\psi(t)^2 + \varepsilon^2\varphi(t) + \rho\varepsilon\sigma = 0. \tag{2.48}$$

Furthermore, the function given by (2.45) will satisfy the boundary condition (2.44) if

$$\varphi(T) = -K, \quad \psi(T) = 0 \quad \text{and} \quad \chi(T) = 0.$$

The ODE (2.46) is a constant coefficient Ricatti equation which can be transformed into a second order linear ODE. Indeed, the transformation

$$\varphi(t) = \frac{z'(t)}{(\eta + \frac{1}{\gamma})z(t)} \tag{2.49}$$

reduces it to

$$z''(t) + \left(\frac{\lambda}{\gamma} - \beta\right) z'(t) + \left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right) z(t) = 0. \tag{2.50}$$

The initial condition $\varphi(T) = -K$ under this transformation becomes a Robin boundary condition

$$z'(T) + K\left(\eta + \frac{1}{\gamma}\right) z(T) = 0. \tag{2.51}$$

If Assumption 8 holds, the differential equation (2.50) has positive discriminant and so every solution to it is given by

$$z(t) = Ae^{nt} + Be^{mt}, \tag{2.52}$$

for some constants $A, B \in \mathbb{R}$, where

$$n, m = -\frac{1}{2}\left(\frac{\lambda}{\gamma} - \beta\right) \pm \frac{1}{2}\sqrt{\left(\frac{\lambda}{\gamma} - \beta\right)^2 - 4\left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right)}.$$

We can now prove the main result of this section.

Theorem 9. *Consider the control problem with problem data described in* (2.35) *and suppose that Assumption 8 holds. Then given a time horizon $T \in (0, \infty)$ the value function of the control problem identifies with the function*

$$w(t, x) = \varphi(t)x^2 + \psi(t)x + \chi(t) \tag{2.53}$$

where

$$\varphi(t) = \frac{1}{\left(\eta + \frac{1}{\gamma}\right)} \frac{ne^{-n(T-t)} - m\Theta e^{-m(T-t)}}{e^{-n(T-t)} - \Theta e^{-m(T-t)}}, \tag{2.54}$$

$$\psi(t) = \mu \frac{\frac{1}{n+\Gamma}\left(e^{\Gamma(T-t)} - e^{-n(T-t)}\right) - \frac{\Theta}{m+\Gamma}\left(e^{\Gamma(T-t)} - e^{-m(T-t)}\right)}{e^{-n(T-t)} - \Theta e^{-m(T-t)}} \tag{2.55}$$

and

$$\chi(t) = e^{-\beta(T-t)} \int_t^T \left(\frac{1}{4} \left(\eta + \frac{1}{\gamma} \right) \psi^2(u) + \varepsilon^2 \varphi(u) \right) du, \tag{2.56}$$

in which expressions,

$$\Gamma = \left(\frac{\lambda}{2\gamma} - \beta \right) \quad \textit{and} \quad \Theta = \frac{n + K\left(\eta + \frac{1}{\gamma}\right)}{m + K\left(\eta + \frac{1}{\gamma}\right)}.$$

Furthermore, the optimal controls $(\hat{\delta}, \hat{\xi}) \in \mathcal{A}_T$ *are given by*

$$\hat{\delta}_t = \varphi(t)\hat{X}_t + \frac{1}{2}\psi(t) \tag{2.57}$$

and

$$\hat{\xi}_t = \left(\frac{\varphi(t)}{\gamma} + \frac{\lambda}{2\gamma} \right) \hat{X}_t + \frac{\psi(t)}{2\gamma} \tag{2.58}$$

where

$$d\hat{X}_t = \left[\left(\left(\eta + \frac{1}{\gamma} \right) \varphi(t) + \frac{\lambda}{2\gamma} \right) \hat{X}_t + \frac{1}{2} \left(\eta + \frac{1}{\gamma} \right) \psi(t) \right] dt + \varepsilon dB_t. \tag{2.59}$$

Proof. As we have already established, if the function of the form (2.53) is to identify with the value function of the control problem, $\varphi(.)$, $\psi(.)$ and $\chi(.)$ must satisfy (2.46)–(2.48). To derive the solution to (2.46), we first note that the solution to (2.50) is of the form (2.52) if Assumption 8 holds true. By taking derivatives in (2.52) and solving for A and B, we can see that this solution satisfies the boundary condition (2.51) if

$$B = -A \frac{n + K\left(\eta + \frac{1}{\gamma}\right)}{m + K\left(\eta + \frac{1}{\gamma}\right)} e^{(n-m)T}.$$

It follows that

$$z(t) = C \left(e^{-n(T-t)} - \frac{n + K\left(\eta + \frac{1}{\gamma}\right)}{m + K\left(\eta + \frac{1}{\gamma}\right)} e^{-m(T-t)} \right). \tag{2.60}$$

Differentiating, we obtain

$$z'(t) = C \left(n e^{-n(T-t)} - m \frac{n + K\left(\eta + \frac{1}{\gamma}\right)}{m + K\left(\eta + \frac{1}{\gamma}\right)} e^{-m(T-t)} \right).$$

Substituting these expressions into (2.49), the constant C cancels out, and we obtain (2.54).

The ordinary differential equation (2.47), which $\psi(.)$ satisfies, is of the form

$$\psi'(t) + L(t)\psi(t) + \mu = 0$$

with terminal time condition $\psi(T) = 0$. Therefore,

$$\psi(t) = e^{\int_t^T L(u)du} \int_t^T \mu e^{-\int_v^T L(u)du} dv.$$

Since

$$L(t) = \left(\eta + \frac{1}{\gamma}\right) \varphi(t) + \Gamma = \frac{z'(t)}{z(t)}$$

where $z(.)$ is given by (2.60), we calculate

$$\int_t^T L(u)du = \log \left\{ \frac{1 - \Theta}{e^{-n(T-t)} - \Theta e^{-m(T-t)}} \right\} + \Gamma(T - t)$$

and

$$\int_t^T \mu e^{-\int_v^T L(u)du} dv = \frac{\mu}{(1 - \Theta)} \left(\frac{1}{n + \Gamma} \left\{ 1 - e^{-(n+\Gamma)(T-t)} \right\} - \frac{\Theta}{m + \Gamma} \left\{ 1 - e^{-(m+\Gamma)(T-t)} \right\} \right).$$

It follows that $\psi(.)$ is given by (2.55). Moreover it is immediate from (2.48) that the solution $\chi(.)$ can be written as (2.56).

The functions $\varphi(.)$ and $\psi(.)$ are both bounded because either $\Theta < 0$ or $\Theta > 1$. In the former case

$$e^{-n(T-t)} - \Theta e^{-m(T-t)} > 0$$

for all $t \in [0, T]$, whereas in the latter case

$$e^{-n(T-t)} - \Theta e^{-m(T-t)} < 0$$

for all $t \in [0, T]$. The boundedness these functions implies the boundedness of $\chi(.)$. It follows that the function defined by the right hand side of (2.53) satisfies the growth condition (2.15) in the verification Theorem 7. In view of (2.41), (2.42) and (2.45), the optimal controls $\hat{\xi}$ and $\hat{\delta}$ are given by (2.57) and (2.58). The estimates in Corollary 2.10 in Krylov [Kry08] imply immediately that these controls satisfy the admissibility conditions (2.6) and (2.7) in Definition 3.

□

Remark 8. The explicit form for $\chi(.)$ is left in integral form. However it can be solved explicitly since the integral for $\varphi(.)$ is easy to calculate and the integral for $\psi^2(.)$ is also known in explicit form and involves the Gaussian hypergeometric function. We choose to omit the explicit form because it is rather long and $\chi(.)$ does not feature in either of the optimal controls, only as a function of time in the value function.

Corollary 2. *The time-reversed impact function $\bar{\varphi}(\tau) := \varphi(T - \tau)$ is increasing or decreasing depending on the sign of $\frac{n+K(\eta+\frac{1}{\gamma})}{m+K(\eta+\frac{1}{\gamma})}$.*

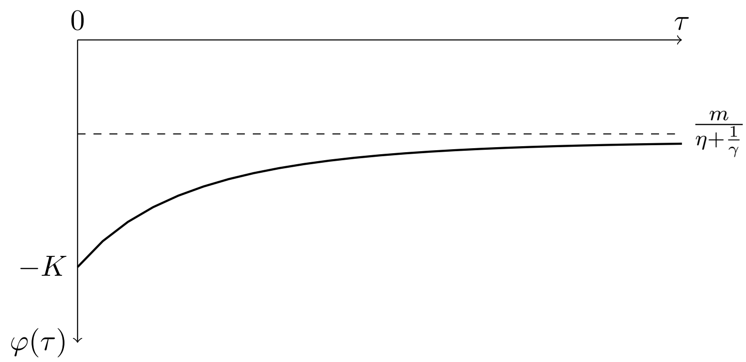

Figure 2.1: Decay function $\bar{\varphi}(\tau)$

Proof. The claim follows immediately from the calculation

$$\bar{\varphi}'(\tau) = \frac{(n-m)^2 \frac{n+K(\eta+\frac{1}{\gamma})}{m+K(\eta+\frac{1}{\gamma})} e^{-(n+m)\tau}}{\left(\eta + \frac{1}{\gamma}\right) \left(e^{-n(T-t)} - \frac{n+K\left(\eta+\frac{1}{\gamma}\right)}{m+K\left(\eta+\frac{1}{\gamma}\right)} e^{-m(T-t)}\right)^2}.$$

□

Remark 9. As the transient market impact and thereby the cost of trading increases, the magnitude the the market maker's hedging activity decreases. As $\gamma \to \infty$ we see that $\hat{\xi} \to 0$ and the optimal solution reduces to that of the market marker's problem considered in Chapter 1.

Remark 10. As $K \to \infty$ the market maker's incentive to reduce remaining inventory holdings at $T = t$ increases. Specifically, since $\Theta \to 1$ this means that $\varphi(t) \to \infty$ as $t \to T$ and the problem resembles an optimal execution problem.

Remark 11. By completing the square in the value function, which under appropriate conditions takes the form of a negative quadratic, we may rewrite the value function as

$$w(t,x) = \varphi(t) \left(x - \frac{\psi(t)}{2\varphi(t)}\right)^2 - \frac{\psi^2(t)}{4\varphi(t)} + \chi(t).$$

Using this form we can directly see the market maker's target position, namely

$$x = \frac{\psi(t)}{2\varphi(t)}.$$

4.2 An Explicit Solution: Infinite Time Horizon $T = \infty$

We now consider the problem of maximising the objective

$$J_{\infty,x}(\delta,\xi) = \limsup_{T\to\infty} \mathbb{E}\bigg[\int_0^T e^{-\beta t}\Big[-\eta\delta^2 - \gamma\xi_t^2 + (\mu+\lambda\xi_t)X_t + \rho\sigma\varepsilon - kX_t^2\Big]dt \,\Big|\, X_0 = x\bigg], \tag{2.61}$$

where the discounting rate $\beta > 0$ is a given constant, over all admissible controls $(\delta,\xi) \in \mathcal{A}_\infty$ subject to the stochastic dynamics given by (2.35). In this case, the HJB equation (2.39) takes the form

$$\frac{1}{2}\varepsilon^2 w_{xx}(x) - \beta w(x) + \mu x + \rho\varepsilon\sigma - kx^2 + \sup_{\delta\in\mathbb{R}}\Big[\eta\delta w_x(x) - \eta\delta^2\Big] + \sup_{\xi\in\mathbb{R}}\Big[\xi w_x(x) + \lambda\xi x - \gamma\xi^2\Big] = 0 \tag{2.62}$$

The skew control $\hat{\delta}$ and hedging control $\hat{\xi}$ that achieve the maximum in (2.62) are given by

$$\hat{\delta}(x) = \frac{1}{2}w_x(x) \tag{2.63}$$

and

$$\hat{\xi}(x) = \frac{1}{2\gamma}(w_x(x) + \lambda x). \tag{2.64}$$

Substituting (2.63) and (2.64) into the HJB equation (2.62), we obtain

$$\frac{1}{2}\varepsilon^2 w_{xx}(x) + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)w_x^2(x) + \frac{\lambda}{2\gamma}xw_x(x) - \beta w(x) + \left(\frac{\lambda^2}{4\gamma} - k\right)x^2 + \mu x + \rho\varepsilon\sigma = 0 \tag{2.65}$$

The non-linear ODE (2.65) is similar to one we have seen in Chapter 1 and similar to the PDE we solved for the finite horizon case. Therefore we approach it using a candidate solution of the form

$$w(x) = \varphi x^2 + \psi x + \chi. \tag{2.66}$$

Substituting (2.66) into (2.65) we can see that, φ, ψ and χ should satisfy the algebraic equations

$$\left(\eta + \frac{1}{\gamma}\right)\varphi^2 + \left(\frac{\lambda}{\gamma} - \beta\right)\varphi + \left(\frac{\lambda^2}{4\gamma} - k\right) = 0, \tag{2.67}$$

$$\left(\left(\eta + \frac{1}{\gamma}\right)\varphi + \frac{\lambda}{2\gamma} - \beta\right)\psi + \mu = 0 \tag{2.68}$$

and

$$-\beta\chi + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\psi^2 + \varepsilon^2\varphi + \rho\varepsilon\sigma = 0. \tag{2.69}$$

The quadratic equation (2.67) has two real roots provided

$$\left(\frac{\lambda}{\gamma} - \beta\right)^2 - 4\left(\eta + \frac{1}{\gamma}\right)\left(k - \frac{\lambda^2}{4\gamma}\right) \geq 0,$$

which is guaranteed by Assumption 8. Since $\eta + \frac{1}{\gamma} > 0$, these roots are both positive if $\beta > \frac{\lambda}{\gamma}$ and $k < \frac{\lambda^2}{4\gamma}$, both negative if $\beta < \frac{\lambda}{\gamma}$ and $k < \frac{\lambda^2}{4\gamma}$, and one negative one positive if $k > \frac{\lambda^2}{4\gamma}$. We denote these roots by

$$\varphi^+,\, \varphi^- = \frac{\left(\beta - \frac{\lambda}{\gamma}\right) \pm \sqrt{\left(\frac{\lambda}{\gamma} - \beta\right)^2 - 4\left(\eta + \frac{1}{\gamma}\right)\left(\frac{\lambda^2}{4\gamma} - k\right)}}{2\left(\eta + \frac{1}{\gamma}\right)}. \tag{2.70}$$

To determine which of these two roots is associated with the expression for w given by (2.66) that identifies with the problem's value function, we prove the following result.

Lemma 3. *Suppose that the problem data satisfies Assumption 8, that is*

$$\left(\frac{\lambda}{\gamma} - \beta\right)^2 - 4\left(\eta + \frac{1}{\gamma}\right)\left(k - \frac{\lambda^2}{4\gamma}\right) \geq 0. \tag{2.71}$$

The controls defined by $\hat{\delta}_t = \hat{\delta}(X_t)$ and $\hat{\xi}_t = \hat{\xi}(X_t)$, where $\hat{\delta}$ and $\hat{\xi}$ are given by (2.63) *and* (2.64)*, and X is the associated solution to* (2.35)*, which is given by* (2.73)*, are such that the admissibility condition* (2.13) *in Definition 4, which takes the form*

$$\lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[X_T^2\right] = 0 \tag{2.72}$$

in the current context, holds true if $\varphi = \varphi^-$ and fails to hold true if $\varphi = \varphi^+$.

Proof. Under the controls $\hat{\delta}$ and $\hat{\xi}$ the controlled process X is an Ornstein-Uhlenbeck process with dynamics

$$dX_t = \left[\left(\left(\eta + \frac{1}{\gamma}\right)\varphi + \frac{\lambda}{2\gamma}\right)X_t + \frac{1}{2}\left(\eta + \frac{1}{\gamma}\right)\psi\right]dt + \varepsilon dB_t. \tag{2.73}$$

We define

$$\alpha = \frac{\frac{1}{2}\left(\eta + \frac{1}{\gamma}\right)\psi}{\left(\eta + \frac{1}{\gamma}\right)\varphi + \frac{\lambda}{2\gamma}},$$

and we use Itô's isometry to calculate

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left((X_0+\alpha)\,e^{\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T} - \alpha + \varepsilon\int_0^T e^{\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)(T-s)}dB_s\right)^2\right] \\
&= \left((X_0+\alpha)\,e^{\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T} - \alpha\right)^2 + \mathbb{E}\left[\varepsilon\left(\int_0^T e^{\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)(T-s)}dB_s\right)^2\right] \\
&= (X_0+\alpha)^2\, e^{2\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T} - \alpha\,(X_0+\alpha)\,e^{\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T} + \alpha^2 \\
&\quad - \frac{\varepsilon^2}{2\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)}\left(1-e^{2\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T}\right).
\end{aligned}$$

Since the coefficient of $e^{2\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)T}$ is strictly positive, $\eta > 0$ and $\beta > 0$,

$$\begin{aligned}
\lim_{T\to\infty} e^{-\beta T}\mathbb{E}\left[X_T^2\right] &= \lim_{T\to\infty}\left((X_0+\alpha)^2 + \frac{\varepsilon^2}{2\left(\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma}\right)}\right)e^{\left(2\left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{\gamma}-\beta\right)T} \\
&= \begin{cases} \infty & \text{if } \left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma} > 0, \\ 0 & \text{if } \left(\eta+\frac{1}{\gamma}\right)\varphi+\frac{\lambda}{2\gamma} < 0. \end{cases}
\end{aligned}$$

The result now follows since $2\left(\eta+\frac{1}{\gamma}\right)\varphi^- + \frac{\lambda}{\gamma} - \beta < 0$ and $2\left(\eta+\frac{1}{\gamma}\right)\varphi^+ + \frac{\lambda}{\gamma} - \beta > 0$. □

We can now prove the main result of this section.

Theorem 10. *Consider the control problem with problem data described in* (2.35) *and* (2.61) *and suppose that the problem data are such that Assumption 8 holds. The value function of the control problem identifies with the function*

$$w(x) = \varphi x^2 + \psi x + \chi, \tag{2.74}$$

where $\varphi = \varphi^-$,

$$\psi = \frac{\mu}{\frac{\beta}{2} + \sqrt{\left(\frac{\lambda}{\gamma}-\beta\right)^2 - 4\left(\eta+\frac{1}{\gamma}\right)\left(\frac{\lambda^2}{4\gamma}-k\right)}}$$

and

$$\chi = \frac{\frac{1}{4}\left(\eta+\frac{1}{\gamma}\right)\psi^2 + \varepsilon^2\varphi + \rho\varepsilon\sigma}{\beta}.$$

Furthermore, the optimal controls $(\hat{\delta},\hat{\xi}) \in \mathcal{A}_\infty$ *are given by*

$$\hat{\delta}_t = \varphi^-\hat{X}_t + \frac{1}{2}\psi \tag{2.75}$$

and

$$\hat{\xi}_t = \left(\frac{\varphi^-}{\gamma} + \frac{\lambda}{2\gamma}\right)\hat{X}_t + \frac{\psi}{2\gamma} \tag{2.76}$$

where

$$d\hat{X}_t = \left[\left(\left(\eta + \frac{1}{\gamma}\right)\varphi^- + \frac{\lambda}{2\gamma}\right)\hat{X}_t + \frac{1}{2}\left(\eta + \frac{1}{\gamma}\right)\psi\right]dt + \varepsilon dB_t. \tag{2.77}$$

Proof. We have already established that (2.74) satisfies the HJB equation (2.62). We can also see that this function satisfies the growth conditions (2.26) and (2.27) in verification Theorem 8. We can immediately see from (2.63), (2.64) that the controls $\hat{\delta}$ and $\hat{\xi}$ are of the form (2.75) and (2.76). These controls are admissible, namely, $(\hat{\delta}, \hat{\xi}) \in \mathcal{A}_\infty$ thanks to Lemma 3.

□

Remark 12. We have characterised all solutions to the control problem on both the finite horizon $(0, T)$ and the infinite horizon under Assumption 8, that is

$$\left(\frac{\lambda}{\gamma} - \beta\right)^2 > 4\left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right).$$

Under this assumption, the value function is finite and, in general, the market maker uses both controls to attempt to reduce the outstanding inventory position whilst taking advantage of the drift μ. See Remark 13 for a caveat. This strong economic meaning matches our intuition of how a market maker should behave whilst disincentivised from holding risk. Indeed, Assumption 8 holds true when the market maker's continuous risk penalty k sufficiently outweighs the market maker's ability to influence the fair price through market impact λ. It highlights an interesting feature of market impact models that cannot be observed in the traditional market impact literature; market impact is a double edged sword which causes a negative effect when reducing a position, but a positive effect when increasing a position. Therefore Assumption 8 has the implicit meaning that situations in which the market maker is able to use the market impact caused by hedging to benefit himself by pushing up the value of the current inventory position should not be allowed in the model. We investigate in the next section what happens when these situations occur.

Remark 13. We notice that it is possible for the smaller root of (2.67) to satisfy the condition specified in Lemma 3 whilst being positive. For this to occur we need to have $\beta > \frac{\lambda}{\gamma}$ and $k < \frac{\lambda^2}{4\gamma}$ in which case both roots of (2.67) are positive. This is slightly counter intuitive, because it means that the market maker chooses to skew upwards and buy the asset when the inventory position is positive, and skew downwards and sell the asset when the inventory position is negative. This behaviour would normally be associated with an exploding value function and so would result in a failure of the transversality condition. In this case, assuming no drift in the fair price of the asset, we see that the controlled process X is given by

$$dX_t = aX_t dt + \varepsilon dB_t$$

where $0 < a < \beta$. We add that this situation is somewhat atypical, in the sense that the market maker is able to derive excess benefit by moving the market in the same direction as the current inventory position, which he does by increasing or decreasing ξ. This is possible because λ is small and k is not great enough to disincentivise the accumulation of a large position. However β is simultaneously large enough, meaning that this accumulated benefit is discounted to zero as time goes to ∞.

4.3 An Explicit Solution: Exploding Value Function $T < \infty$

We now investigate the case with a finite time horizon where Assumption 8 is false, which intuitively means that the market impact term is large relative to the cost of holding inventory. In particular, we assume that the market data are such that

$$\left(\frac{\lambda}{\gamma} - \beta\right)^2 < 4\left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right). \tag{2.78}$$

Theorem 11. *Consider the control problem with problem data described in* (2.33)–(2.35) *and* (2.37)*, and suppose that the problem data fails to satisfy Assumption 8 namely, the inequality* (2.78) *holds true. Given a time horizon $T \in (0, \infty)$ and time to maturity $\tau = T - t$ the value function of the control problem takes the form*

$$v(T, x) = \begin{cases} \varphi(t)x^2 + \psi(t)x + \chi(t), & T - t < \tau^*, \\ \infty & T - t \geq \tau^*, \end{cases}$$

where

$$\varphi(t) = \frac{\left(\zeta + \frac{1}{2}\left(\frac{\lambda}{\gamma} - \beta\right)\Theta\right)\sin\left(\zeta(T-t)\right) - K\left(\eta + \frac{1}{\gamma}\right)\cos\left(\zeta(T-t)\right)}{\left(\eta + \frac{1}{\gamma}\right)\left[\cos\left(\zeta(T-t)\right) - \Theta\sin\left(\zeta(T-t)\right)\right]}, \tag{2.79}$$

$$\psi(t) = \mu\frac{\left(\frac{\lambda}{2\gamma} - K(\eta + \frac{1}{\gamma})\right)\left[\cos(\zeta(T-t)) - e^{-\frac{1}{2}\beta(T-t)}\right] + \left(\zeta - \frac{1}{2}\beta\Theta\right)\sin(\zeta(T-t))}{(\frac{1}{4}\beta^2 + \zeta^2)\left[\cos\left(\zeta(T-t)\right) - \Theta\sin\left(\zeta(T-t)\right)\right]} \tag{2.80}$$

and

$$\chi(t) = e^{-\beta(T-t)}\int_t^T \left(\frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\psi^2(u) + \varepsilon^2\varphi(u)\right)du,$$

in which expressions,

$$\Theta = \frac{\frac{1}{2}\left(\frac{\lambda}{\gamma} - \beta\right) - K\left(\eta + \frac{1}{\gamma}\right)}{\zeta}.$$

Furthermore, τ^* is defined as the first explosion point of the time reversed functions $\bar{\varphi}(\tau) = \varphi(T - \tau)$ and $\bar{\psi}(\tau) = \psi(T - \tau)$, namely

$$\tau^* = \min\{\tau > 0 \mid \cos(\zeta\tau) = \Theta\sin(\zeta\tau)\} = \frac{1}{\zeta}\arctan\left(\frac{1}{\Theta}\right). \tag{2.81}$$

Proof. For easier reference, we recall the ODE (2.50) and its associated Robin boundary condition (2.51), that gives rise to the solution to the finite horizon problem, namely

$$z''(\tau) - \left(\frac{\lambda}{\gamma} - \beta\right) z'(\tau) + \left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right) z(\tau) = 0 \tag{2.82}$$

and

$$-z'(0) + K\left(\eta + \frac{1}{\gamma}\right) z(0) = 0. \tag{2.83}$$

If the inequality (2.78) holds true, then every solution to (2.82) takes the form

$$z(t) = e^{\frac{1}{2}\left(\frac{\lambda}{\gamma}-\beta\right)(T-t)}\Big[A\sin\left(\zeta(T-t)\right) + B\cos\left(\zeta(T-t)\right)\Big], \tag{2.84}$$

for some constants $A, B \in \mathbb{R}$, where

$$\zeta = \frac{1}{2}\sqrt{4\left(\frac{\lambda^2}{4\gamma} - k\right)\left(\eta + \frac{1}{\gamma}\right) - \left(\frac{\lambda}{\gamma} - \beta\right)^2}.$$

By taking derivatives and solving for A and B, we can see that this solution satisfies the boundary condition (2.83) if

$$A = -B\frac{\frac{1}{2}\left(\frac{\lambda}{\gamma} - \beta\right) - K\left(\eta + \frac{1}{\gamma}\right)}{\zeta}.$$

It follows that

$$z(t) = Ce^{\frac{1}{2}\left(\frac{\lambda}{\gamma}-\beta\right)(T-t)}\left[\cos\left(\zeta(T-t)\right) - \frac{\frac{1}{2}\left(\frac{\lambda}{\gamma} - \beta\right) - K\left(\eta + \frac{1}{\gamma}\right)}{\zeta}\sin\left(\zeta(T-t)\right)\right]. \tag{2.85}$$

Differentiating and substituting back into (2.49), we derive the expression (2.79).

To derive $\psi(.)$ we notice that the ordinary differential equation (2.47) which $\psi(.)$ satisfies is of the form $\psi'(t) + L(t)\psi(t) + \mu = 0$ and $\psi(T) = 0$ which means that

$$\psi(t) = e^{\int_t^T L(u)du}\int_t^T \mu e^{-\int_v^T L(u)du}dv \tag{2.86}$$

for $L(.)$ given by

$$L(t) = \left(\eta + \frac{1}{\gamma}\right)\varphi(t) + \left(\frac{\lambda}{2\gamma} - \beta\right).$$

Since $\left(\eta + \frac{1}{\gamma}\right)\varphi(.)$ identifies with $\frac{z'(.)}{z(.)}$ its integral is simply $\log z(.)$ and so

$$\int_t^T L(u)du = \log\left\{\frac{1}{z(t)}\right\} + \left(\frac{\lambda}{2\gamma} - \beta\right)(T-t).$$

Substituting this expression back into (2.86) we obtain

$$\begin{aligned}\psi(t) &= \mu\frac{e^{\left(\frac{\lambda}{2\gamma}-\beta\right)(T-t)}}{z(t)}\int_t^T e^{-\left(\frac{\lambda}{2\gamma}-\beta\right)(T-u)}z(u)du \\ &= \mu\frac{e^{\left(\frac{\lambda}{2\gamma}-\beta\right)(T-t)}}{z(t)}\int_t^T e^{\frac{1}{2}\beta(T-u)}\left[\cos(\zeta(T-t))-\Theta\sin(\zeta(T-t))\right]du\end{aligned}$$

and from this we obtain (2.80). Furthermore, we see that

$$\chi(t) = e^{-\beta(T-t)}\int_t^T \left(\frac{1}{4}\left(\eta+\frac{1}{\gamma}\right)\psi^2(u)+\varepsilon^2\varphi(u)\right)du.$$

Again, this integral can be solved explicitly.

In view of the expression

$$\begin{aligned}\bar{\varphi}'(\tau) &= \frac{\zeta K(\eta+\frac{1}{\gamma})\Big[\sin^2(\zeta\tau)+\cos^2(\zeta\tau)\Big]-\Theta\left(\zeta+\frac{1}{2}\left(\frac{\lambda}{\gamma}-\beta\right)\Theta\right)\Big[\sin^2(\zeta\tau)+\cos^2(\zeta\tau)\Big]}{(\eta+\frac{1}{\gamma})\Big[\cos(\zeta\tau)-\Theta\sin(\zeta\tau)\Big]^2} \\ &= \frac{\zeta^2(\Theta^2+1)}{(\eta+\frac{1}{\gamma})\Big[\cos(\zeta\tau)-\Theta\sin(\zeta\tau)\Big]^2},\end{aligned} \tag{2.87}$$

we can see that

$$\bar{\varphi}'(\tau) > 0 \quad \forall\, \tau < \tau^*$$

and

$$\bar{\varphi}(\tau) \to \infty \text{ as } \tau \to \tau^*.$$

All that remains to show is that the value function v goes to ∞ as $\tau \equiv T-t \to \tau^*$. To this end, we consider the control processes

$$\delta_t = \varphi(t)X_t + \frac{1}{2}\psi(t)$$

and

$$\xi_t = \left(\frac{\varphi(t)}{\gamma}+\frac{\lambda}{2\gamma}\right)X_t + \frac{\psi(t)}{2\gamma},$$

where X is the solution to the SDE

$$dX_t = \left[\left(\left(\eta+\frac{1}{\gamma}\right)\varphi(t)+\frac{\lambda}{2\gamma}\right)X_t + \frac{1}{2}\left(\eta+\frac{1}{\gamma}\right)\psi(t)\right]dt + \varepsilon dB_t. \tag{2.88}$$

X is an Ornstein-Uhlenbeck process that admits the expression

$$X_t = P(t)\left[X_0 + \int_0^t \frac{\frac{1}{2}\left(\eta+\frac{1}{\gamma}\right)\psi(s)}{P(s)}ds + \int_0^t \frac{\varepsilon}{P(s)}dB_s\right],$$

where

$$P(t) = \exp\left(\int_0^t \left(\left(\eta + \frac{1}{\gamma}\right)\varphi(s) + \frac{\lambda}{2\gamma}\right) ds\right).$$

Recalling (2.38), we can see that the performance of these controls over the interval $[t, T]$ is given by

$$\begin{aligned} J(\tau, x) = \mathbb{E}\Bigg[\int_0^\tau e^{-\beta u}\Bigg[&\left\{-\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}^2(u) + \left(\frac{\lambda^2}{4\gamma} - k\right)\right\} X_{T-u}^2 \\ &+ \left\{-\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}(u)\bar{\psi}(u) + \mu\right\} X_{T-u} \\ &+ \left\{-\frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\bar{\psi}^2(u) + \rho\varepsilon\sigma\right\}\Bigg] du - K e^{-\beta(T-t)} X_T^2 \,\Bigg|\, X_T = x\Bigg]. \end{aligned}$$

where $\tau = T - t$ as above. Applying the Feymann-Kac formula and using the dynamics of X, we obtain

$$\begin{aligned} -\frac{\partial J(\tau, x)}{\partial \tau} + \frac{1}{2}\varepsilon^2 \frac{\partial^2 J(\tau, x)}{\partial x^2} &+ \left[\left(\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}(\tau) + \frac{\lambda}{2\gamma}\right)x + \frac{1}{2}\left(\eta + \frac{1}{\gamma}\right)\bar{\psi}(\tau)\right]\frac{\partial J(\tau, x)}{\partial x} - \beta J(\tau, x) \\ &+ \left[-\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}^2(\tau) + \left(\frac{\lambda^2}{4\gamma} - k\right)\right] x^2 \\ &+ \left[-\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}(\tau)\bar{\psi}(\tau) + \mu\right] x \\ &+ \left[-\frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\bar{\psi}^2(\tau) + \rho\varepsilon\sigma\right] = 0, \end{aligned} \tag{2.89}$$

with boundary condition

$$J(0, x) = -Kx^2.$$

Substituting the choice

$$J(\tau, x) = a(\tau)x^2 + b(\tau)x + c(\tau) \tag{2.90}$$

into (2.89) we obtain the ordinary differential equations for the functions $a(.)$, $b(.)$ and $c(.)$

$$\begin{aligned} a'(\tau) + 2\left(\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}(\tau) + \frac{\lambda}{2\gamma} - \frac{\beta}{2}\right)a(\tau) - \left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}^2(\tau) + \left(\frac{\lambda^2}{4\gamma} - k\right) &= 0 \\ b'(\tau) + \left(\left(\eta + \frac{1}{\gamma}\right)\bar{\varphi}(\tau) + \frac{\lambda}{2\gamma} - \beta\right)b(\tau) + \left(\eta + \frac{1}{\gamma}\right)\bar{\psi}(\tau)\left(a(\tau) - \bar{\varphi}(\tau)\right) + \mu &= 0 \\ c'(\tau) - \beta c(\tau) + \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\bar{\psi}(\tau)c(\tau) - \frac{1}{4}\left(\eta + \frac{1}{\gamma}\right)\bar{\psi}^2(\tau) + \varepsilon^2 a(\tau) + \rho\varepsilon\sigma &= 0 \end{aligned}$$

with terminal conditions

$$a(0) = -K, \quad b(0) = 0 \quad \text{and} \quad c(0) = 0.$$

It is straightforward to check that the choices $a(.) = \bar{\varphi}(.)$, $b(.) = \bar{\psi}(.)$ and $c(.) = \bar{\chi}(.)$ where $\bar{\varphi}(.)$, $\bar{\psi}(.)$ and $\bar{\chi}(.)$ are as above satisfy this system of ODEs.

As $\tau \to \tau^*$ we know from (2.87) that $\bar{\varphi}(\tau) \to \infty$. From this we can also deduce that $\bar{\chi}(\tau) \to \infty$ as the explicit form of $\bar{\chi}(.)$, seen in (2.56), is an integral involving only positive multiples of $\psi^2(.)$ and $\varphi(.)$. We also deduce that $P(.) \to \infty$ from the observation that $P(.)$ is an exponential function of a positive multiple of $\varphi(.)$.

It is therefore easy to see that in (2.90) the first and final terms go to ∞. It follows that

$$\lim_{\tau \to \tau^*} J(\tau, x) = \infty$$

and the value function v is infinite for all $t \in [0, T]$ such that $T - t > \tau^*$. □

Remark 14. We would expect to also find that the functions $\bar{\psi}(.)$ and $\bar{\chi}(.)$ exhibit a similar monotonicity in time remaining to that found for $\bar{\varphi}(.)$ in (2.87). However since there is only marginal interest in obtaining such a result we have left this for future work.

5 Singular Hedging

The model we have studied in this chapter included a hedging control, namely ξ, which represented the rate of hedging of the market maker's inventory. The use of a rate of trading ensured that the hedging control was absolutely continuous with respect to time, a modelling assumption that was in line with the classical approach of Almgren and Chriss [AC01].

We now consider an extension of the model, specifically allowing the market maker's hedging control to be singular. The market maker would then be able to clear inventory positions in blocks rather than being restricted to a rate of trading. Unlike the previous section, we will not prove a formal verification theorem, but will instead simply derive the HJB equation for which we can find numerical solutions and consider its properties.

In this context, permanent impact on the fair price process should depend on the magnitude of trades that occur. We distinguish market maker buying Ξ^b from market maker selling Ξ^s by specifying two controls each of which is an increasing $(\mathcal{F}_t)$-progressively measurable process. We restrict our attention to the case of linear impact functions $\lambda d\Xi^b_t$ and $\lambda d\Xi^s_t$ and assume that in the absence of trade the fair price follows a Bachelier process. The fair price process will evolve according to the dynamics

$$dS_t = \mu dt - \lambda d\Xi^s_t + \lambda d\Xi^b_t + \sigma dW_t$$

for some constant $\lambda > 0$. Furthermore the dynamics of the market maker's inventory holdings are given by

$$dX_t = \eta \delta_t dt - d\Xi^s_t + d\Xi^b_t + \varepsilon dB_t$$

which is the combination of noise trader activity and the market maker's hedging trades.

Recall from Section 1.1 that transient impact in the Almgren and Chriss model is effectively a proportional transaction cost. We therefore assume that transient impact is a fixed proportion of the trade size, so that buy trades occur at a price of

$$S_t + C_b$$

and similarly, sell trades occur at a price

$$S_t - C_s$$

for some constants $C_b > 0$ and $C_s > 0$. Following the same approach as in Section 1.2, we can derive the market maker's revenue function as

$$\begin{aligned}\mathcal{R}_T(\delta, \Xi^b, \Xi^s) = &- \int_0^T \tilde{S}_t\,(\eta\delta_t dt + \varepsilon dB_t) - \int_0^T S_t^f d\Xi_t^b + \int_0^T S_t^f d\Xi_t^s \\ &- \int_0^T C_b d\Xi_t^b - \int_0^T C_s d\Xi_t^s + (S_T^f X_T - S_0^f X_0).\end{aligned}$$

An application of Itô's formula gives

$$d(S_t^f X_t) = X_t(\mu dt - \lambda d\Xi_t^s + \lambda d\Xi_t^b + \sigma dW_t) + S_t(-\eta\delta_t dt - d\Xi_t^s + d\Xi_t^b + \varepsilon dB_t) + \rho\sigma\varepsilon dt,$$

so the revenue function can be rewritten as

$$\begin{aligned}\mathcal{R}_T(\delta, \Xi^s, \Xi^b) = &\int_0^T (-\eta\delta_t^2 + \mu X_t + \rho\sigma\epsilon)dt + \int_0^T (\lambda X_t - C_b)d\Xi_t^b \\ &+ \int_0^T (-\lambda X_t - C_s)d\Xi_t^s + M_T.\end{aligned} \tag{2.91}$$

where

$$M_T = \int_0^T \sigma X_t dW_t - \int_0^T \varepsilon\delta_t dB_t$$

Consider the infinite horizon objective function, which is simply the mean-quadratic objective discussed in Chapter 1, given by

$$\begin{aligned}J_{\infty,x}(\delta, \Xi^b, \Xi^s) = \mathbb{E}\Bigg[\int_0^\infty e^{-\beta t}\Bigg[&(-\eta\delta_t^2 + \mu X_t + \rho\sigma\epsilon - kX_t^2)dt \\ &+ \int_0^T (\lambda X_t - C_b)d\Xi_t^b + \int_0^T (-\lambda X_t - C_s)d\Xi_t^s\Bigg] - e^{\beta T}\Phi(X_T)\Bigg].\end{aligned}$$

The value function of the control problem is given by

$$v(x) = \sup_{\delta, \Xi^b, \Xi^s} J_{\infty,x}(\delta, \Xi^b, \Xi^s),$$

for $x \in \mathbb{R}$. At any given time t the market maker has three options to choose from. Firstly he may set δ to a fixed value and wait for a short period Δt, and then continue optimally.

In this case, since such a course of action may or may not be optimal we can state the inequality,

$$v(X_t) \geq \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\beta s}(-\eta\delta_s^2 + \mu X_s + \rho\sigma\epsilon - kX_s^2)ds + e^{-\beta\Delta t}v(X_{t+\Delta t})\right].$$

Using Itô's formula, dividing by Δt and passing to the limit as $\Delta t \downarrow 0$, we obtain

$$-\beta v(x) + \mathcal{L}^\delta v(x) \leq 0$$

where

$$\mathcal{L}^\delta v(x) = \frac{1}{2}\varepsilon^2 v_{xx}(x) + \eta\delta v_x(x) - \eta\delta^2 + \mu x + \rho\varepsilon\sigma - kx^2.$$

The optimal choice of δ is given by

$$\hat{\delta}(x) = \frac{1}{2}v_x(x).$$

Therefore, we obtain the inequality

$$-\beta v(x) + \bar{\mathcal{L}}v(x) \leq 0$$

where

$$\bar{\mathcal{L}}v(x) = \frac{1}{2}\varepsilon^2 v_{xx}(x) + \frac{1}{4}\eta v_x^2(x) + \mu x + \rho\varepsilon\sigma - kx^2.$$

The second possibility is to buy a small amount ϵ of the asset, which we know from (2.91) will cost the market maker a transaction cost $C_b\epsilon$. There will however be a secondary effect in that the market maker will, due to the permanent component of market impact, alter the fair value of his current holdings by an amount $\lambda\epsilon X_t$. Consequently we can state the inequality

$$v(x) \geq v(x+\epsilon) + (\lambda x - C_b)\epsilon,$$

which for infinitesimally small ϵ corresponds to

$$0 \geq v_x(x) + \lambda x - C_b.$$

Similarly for selling

$$0 \geq -w_x(x) - \lambda x - C_s.$$

Therefore we expect that the value function v of the problem will identify with a smooth solution w to the variational inequality

$$\max\left\{\bar{\mathcal{L}}w(x) - \beta w(x),\ w_x(x) + \lambda x - C_b,\ -w_x(x) - \lambda x - C_s\right\} = 0. \tag{2.92}$$

We cannot hope to find a closed form solution to (2.92). However we may solve it numerically using the smoothness of w. We postulate three regions $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{S}$

$$\begin{aligned}
\mathcal{W} &= \{x \in \mathbb{R} \mid \mathcal{L}w(x) - \beta w(x) = 0\}, \\
\mathcal{B} &= \{x \in \mathbb{R} \mid w_x(x) + \lambda x - C_b = 0\}, \\
\mathcal{S} &= \{x \in \mathbb{R} \mid -w_x(x) - \lambda x - C_s = 0\},
\end{aligned}$$

and propose that the three the regions are such that

$$\mathcal{B} \cup \mathcal{W} \cup \mathcal{S} = \mathbb{R}.$$

In particular, we postulate that the regions are characterised by constants, F and G and take the form

$$\begin{aligned}\mathcal{W} &= \{x \in \mathbb{R} \mid F \leq x < G\}, \\ \mathcal{B} &= \{x \in \mathbb{R} \mid x < F\}, \\ \mathcal{S} &= \{x \in \mathbb{R} \mid x \geq G\}.\end{aligned}$$

The solution approach we take combines an analytical solution in the regions $\mathcal{B}$ and $\mathcal{S}$ with a numerical solution in the region $\mathcal{W}$. We combine these into a single smooth solution by using a variant of the so called *shooting method.* We begin by solving the problem in the regions $\mathcal{B}$ and $\mathcal{S}$, which is quite simple as in these regions we have a one dimensional ODE with solution

$$w(x) = -\frac{\lambda}{2}x^2 + C_b x + a$$

for some constant a and $x \in \mathcal{B}$,

$$w(x) = -\frac{\lambda}{2}x^2 - C_s x + b$$

for some constant b and $x \in \mathcal{S}$. The constants a and b will be used to preserve the continuity of our solution at the boundaries $x = F$ and $x = G$.

Inside the region $\mathcal{W}$, we look to find a function $\bar{w}$ that satisfies

$$\bar{\mathcal{L}}w(x) - \beta w(x) = 0 \tag{2.93}$$

as well as

$$\begin{aligned}\bar{w}'(F) &= -\lambda F + C_b, \\ \bar{w}'(G) &= -\lambda G - C_s,\end{aligned} \tag{2.94}$$

and

$$\bar{w}''(F) = \bar{w}''(G) = -\lambda. \tag{2.95}$$

so that when combined together the three solutions form a single C^2 function. Equations (2.94) and (2.95) arise from the assumed smoothness of the value function, and correspond to the so called *value matching* and *smooth pasting* properties at the optimal exercise boundary in American option pricing problems.

Denote by $\bar{w}_{F_i,G_i}(x)$ the numerical solution to the Dirichlet boundary value problem, namely a solution to (2.93) and (2.94) where the boundaries are located at F_i and G_i. Such numerical solutions, even for non-linear ODEs are simple to solve using mathematical software packages. We then proceed by choosing arbitrary initial points F_0 and G_0,

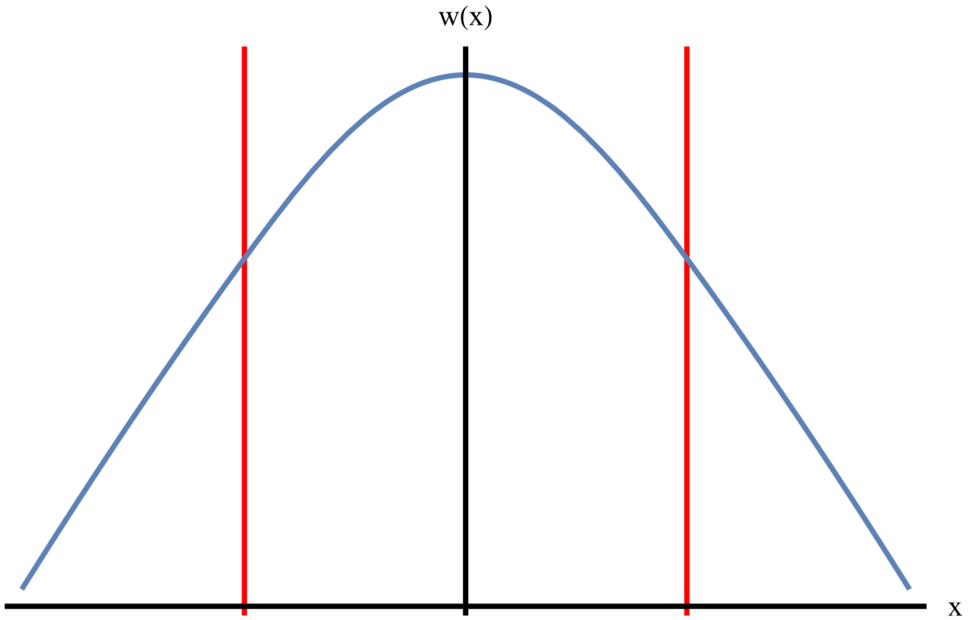

Figure 2.2: Value function $w(.)$

and calculate an initial candidate solution $\bar{w}_{F_0,G_0}(x)$. Any such solution satisfies the requirements for a solution $\bar{w}$ other than (2.95). We apply a root finding algorithm such as the secant method or the bisection method to the functions $f(F) = \bar{w}''_{F,G}(F) + \lambda$ and $g(G) = \bar{w}''_{F,G}(G) + \lambda$ together until convergence is achieved. After convergence is achieved, we are left with a numerical solution $\bar{w}$ that satisfies (2.92) in the region $\mathcal{W}$ and the mixed boundary condition (2.94) and (2.95). Furthermore, the function

$$w(x) = \begin{cases} -\frac{\lambda}{2}x^2 + C_b x + \bar{w}(F) + \frac{\lambda}{2}F^2 - C_b F, & \text{for} \quad x \in \mathcal{B}, \\ -\frac{\lambda}{2}x^2 - C_s x + \bar{w}(G) + \frac{\lambda}{2}G^2 + C_s G, & \text{for} \quad x \in \mathcal{S}, \\ \bar{w}(x), & \text{for} \quad x \in \mathcal{W}, \end{cases}$$

is a C^2 solution to (2.92).

The optimal hedging control, which is a singular control, will be to buy or sell so as to ensure that the controlled inventory process $\hat{X}$ is a reflecting diffusion inside the waiting region $\mathcal{W}$. The optimal hedging control $\hat{\xi}$ will be the difference between the local times of $\hat{X}$ at the boundaries of $\mathcal{B}$ and $\mathcal{S}$. This type of control also appears in the work of Davis and Norman [DN90] in their study of portfolio selection with transaction costs. An interesting expansion of this model would be to follow the approach of Kallsen and Muhle-Karbe [KMK15] and Muhle-Karbe, Reppen and Soner [MKRS16], who consider the case in which transaction costs are small and proportional to the asset price, allowing them to obtain explicit results for the asymptotic case as a good approximation to the full model.

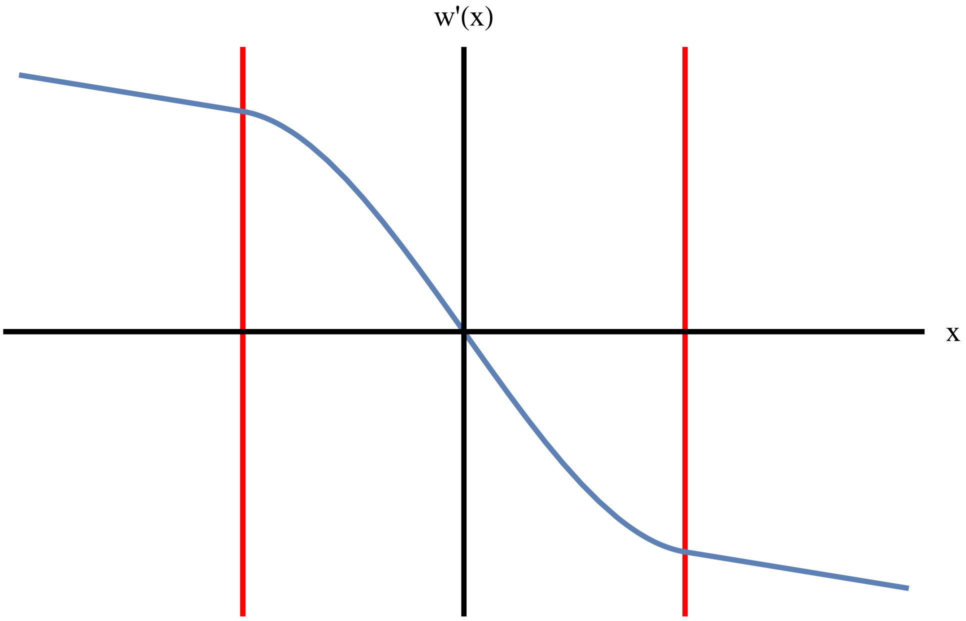

Figure 2.3: First derivative of value function $w'(.)$

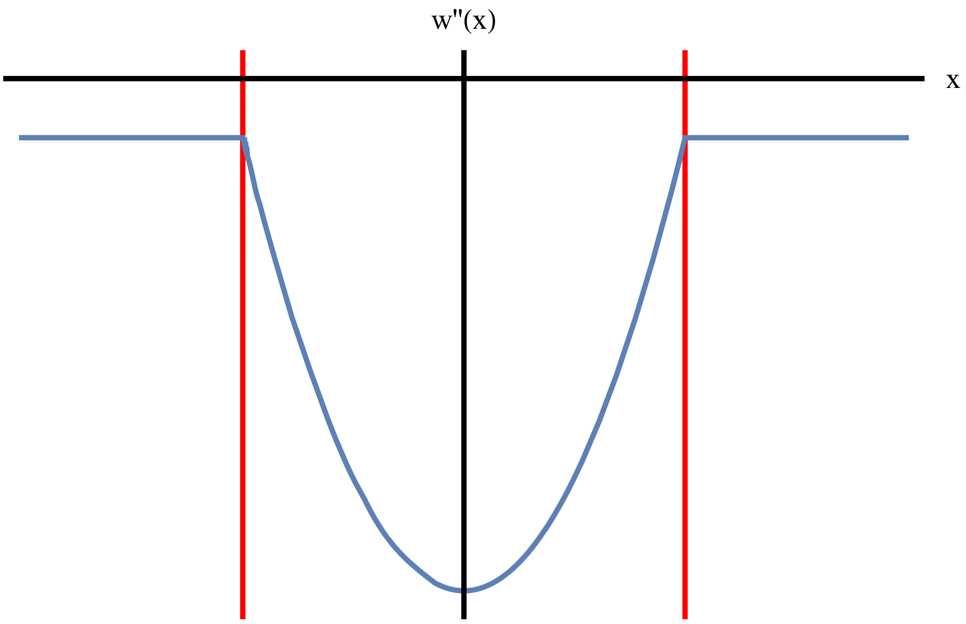

Figure 2.4: Second derivative of value function $w''(.)$

Chapter 3

Portfolio Theory in a Market with Support and Resistance Levels

1 Introduction

We consider the optimisation problem faced by a single agent who possesses wealth consisting of an initial endowment x which may be consumed or invested over an interval $[0, T]$. Consuming wealth too quickly will reduce the amount of capital available to grow via investment, whereas investment is inherently risky and is not guaranteed to result in greater wealth being available for future consumption. The agent is therefore faced with the challenge of how to set these dual controls in order to maximise his total utility from both consumption and investment. The study of this type of problem is known as *portfolio optimisation theory* and has a long history. At the origin of the area is the work of Merton [Mer69], who, building on earlier insights made by Markowitz [Mar52], considered stochastic dynamics for the traded assets in continuous time and power utility functions.

The model that we study in this chapter falls within the context of the Merton model. However we introduce asset prices that include singularities in their drift. The purpose of this is to represent *support and resistance levels*, that is, price points at which the market exhibits either an upward or downward singularity in contrast to its normal behaviour. Specifically of interest is how the agent should respond to optimally adjust his holdings of the asset and rate of consumption when such levels are present in the market.

Stochastic processes that exhibit precisely this behaviour are well known. Diffusions with generalised drift, such as those considered in Lejay [Lej06], are a generalisation of the skew Brownian motion the properties of which have been studied by Walsh [Wal78], Harrison and Shepp [HS81], and Engelbert and Schmidt [ES85], among others. The skew Brownian motion behaves like a Brownian motion away from a given level, but once it hits that level the side of the level on which it makes its next excursion depends on the outcome of an independent Bernoulli random variable. It can be shown that the skew Brownian motion is the strong solution to a SDE involving its local time at the level in question.

In the remainder of this section we provide some more details on some concepts central to this chapter. We discuss the phenomena which we study, namely, support and resistance levels and give some background on the skew Brownian motion which we use to model them. We then provide a brief summary of the results of the classic Merton model, both to provide a benchmark from which to measure the difference that including support and resistance levels creates, and also for use as a limiting case of our model away from the support and resistance levels. In Section 2 of the chapter we present the formal stochastic control problem on both the finite and infinite time horizons, for which we will prove separate verification theorems in Section 3. Using these theorems, in Section 4 and 5 of the chapter we present closed form expressions for the value function and the controls.

1.1 Modelling Support and Resistance Levels

We wish to model points of support and resistance in the asset price around which the asset price process experiences an impact in its drift. A standard definition of support and resistance levels is given by Murphy [Mur99]:

> *"Support is a level or area on the chart under the market where buying interest is sufficiently strong to overcome selling pressure. As a result, a decline is halted and prices turn back again . . . Resistance is the opposite of support."*

Such price points exist in markets for a variety of reasons, one key reason being the presence of clustered resting orders in the market. Resting orders are instructions left with a financial institution to execute an order for a client only when the price arrives at a pre-specified level. For a buy order, if this rate is lower than the prevailing rate then the order is a *take profit*, whereas if it is higher than the prevailing rate the order is a *stop loss*. Such orders are the standard way in which to open a new position or close an existing position at a certain price, especially for systematic strategies such as trend following strategies. In an empirical analysis of such orders in the FX market, Osler [Osl01] noted two interesting phenomena whilst investigating why trading strategies that depend on price level breakouts are persistently profitable. Firstly, there are significant differences in the clustering patterns of stop loss orders compared to take profit orders. Stop loss orders tended to spread more than take profit order, which cluster strongly. Second, the clustering locations of take profit orders were strongly linked to round numbers in the asset price, while stop loss orders tended to be clustered just above round numbers for buy stops, and just below round numbers for sell stops.

These two observations provide important motivation for our model. The empirical existence of support and resistance levels provides a general justification for the model we study. The observation that their origin is in clustered resting orders also suggest that a good model for support and resistance would involve impulses in the price dynamics, rather than an alternative such as a price dependent drift function, since the underlying source of the phenomenon is itself a cluster of single impulses, namely resting orders.

1.2 The Skew Diffusion Process

To model the phenomenon of support and resistance levels within an asset price process, we introduce the skew diffusion process that satisfies the SDE

$$S_t = S_0 + \int_0^t \mu(S_s)ds + \sum_{j=1}^{n} \beta_j L_t^{z_j}(S) + \int_0^t \sigma(S_s)dW_s, \tag{3.1}$$

where $\beta_1, \ldots, \beta_n \in (-1,1)$ are constants and $z_1, \ldots, z_n$ are distinct points. The SDE (3.1) contains n symmetric local times of S, each of which is defined by

$$L_t^z(S) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \sigma^2(S_s) \mathbb{1}_{(z-\varepsilon, z+\varepsilon)}(S_s)ds. \tag{3.2}$$

The local times embedded into the process dynamics mean that the process exhibits a singular deviation at each of the levels z_j, and consequently the process exhibits support or resistance at those levels depending on the sign of β_j.

Remark 15. The intuitive meaning of this property is made clear by following an argument close to that of Harrison and Shepp [HS81]. Let Y with $Y_0 = 0$ be a reflected Brownian motion and define J_n as the interval (s,t) in which Y makes its nth excursion away from zero. Next associate with each J_n an independent Bernoulli random variable e_n with $\mathbb{P}[e_n = 1] = \frac{1+\beta}{2}$ and $\mathbb{P}[e_n = -1] = \frac{1-\beta}{2}$, then the process X defined by

$$X_t = e_n Y_t, \quad \text{for } t \in J_n$$

is a skew Brownian motion.

1.3 Merton's Portfolio Problem

In his classic 1969 paper, Merton proposed a model in which an agent attempts to strike a balance between the conflicting incentives of present consumption, investment towards future consumption and a terminal time bequest. His work, which has influenced many extensions of this concept including this one, provided explicit solutions to the continuous-time problem with using power law utility functions.

It is assumed that the agent starts with an initial endowment $X_0 = x > 0$ and can transfer his holdings, continuously and without incurring transaction costs, between the risky asset S and a risk free asset B. Denote the proportion of wealth held by the agent in the risky asset by π and the rate of consumption of wealth by c. Denoting the agent's wealth by the process X, changes in this wealth will depend on the control processes c and π, specifically

$$X_t = x + \int_0^t \frac{(1-\pi_u)X_u}{B_u} dB_u + \int_0^t \frac{\pi_u X_u}{S_u} dS_u. \tag{3.3}$$

Merton assumed that

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0,$$

and

$$dB_t = rB_t dt, \quad B_0 = 1,$$

considered the family of *constant relative risk aversion* (CRRA) utility functions

$$u(x) = \frac{x^\gamma}{\gamma},$$

for some $\gamma < 1$ and looked for a solution to the control problem with objective function

$$\mathbb{E}\left[\int_0^T u(c_t X_t)dt + u(X_T)\right].$$

Merton then established that the optimal choices for the controls are

$$\hat{\pi}_t = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma}$$

and

$$\hat{c}_t = \frac{\alpha}{(\alpha - 1)e^{-\alpha(T-t)} + 1}$$

for a certain constant α. Specifically, π is a constant and c is a time dependent function that does not depend on the current risky asset price or level of wealth.

2 The Market Model and Control Problem

Fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. The system we study comprises three stochastic processes S, X and B which represent the state processes of the control problem, plus a further two stochastic processes π and c which are the agent's controls. The price process of the risky asset S is such that S_t is the prevailing market price of the risky asset at time t at which the agent can buy or sell. We model the price of the risky asset by the skew geometric Brownian motion

$$dS_t = \mu S_t dt + \beta dL_t^z(S) + \sigma S_t dW_t, \quad S_0 = s > 0, \tag{3.4}$$

where μ and $\sigma > 0$ are constants and $L^z(S)$ is the symmetric local time of S at level z. Here $\beta > 0$ corresponds to a support level whilst $\beta < 0$ corresponds to resistance. The following assumption ensures that the SDE has a unique strong solution.

Assumption 9. *The constant β in (3.4) is such that $\beta \in (-1, 1)$.*

The process B represents the price of a risk free asset, accruing interest at a constant rate r, the price of which is given by

$$dB_t = rB_t dt, \qquad B_0 = 1. \tag{3.5}$$

As we have discussed, the general dynamics of the wealth process X are given by (3.3). Substituting in the specific dynamics (3.4) and (3.5) we obtain

$$X_t = x + \int_0^t \left[(\mu - r)\,\pi_u + r - c_u\right] X_u du + \int_0^t \frac{\beta}{z} \pi_u X_u dL_u^z(S_t) + \int_0^t \sigma \pi_u X_u dW_u \tag{3.6}$$

We assume that the agent's utility from both consumption and bequest is given by the CRRA utility function defined by

$$u(x) = \frac{x^\gamma}{\gamma}, \tag{3.7}$$

for $x > 0$.

Assumption 10. $\gamma \in (0, 1)$.

We also set the agent's rate of subjective exponential discounting $\rho \geq 0$ to be a constant. This is understood to specify the agent's time preference, as opposed to r which represents the rate of return of cash not invested in the risky asset, and we shall see that these two rates play different roles in the solution.

2.1 Arbitrage Considerations

It is tempting to consider strategies in the class introduced by the following definition.

Definition 5. *The family of all admissible portfolio-consumption pairs* (c, π) *consists of all bounded* $(\mathcal{F}_t)$*-progressively measurable processes* c *and* π*.*

However, this class includes strategies that realise arbitrage as the following result shows.

Theorem 12. *A market as described in* (3.4)–(3.6) *admits arbitrage.*

Proof. Take an agent with initial wealth $x = 1$ who follows the very simple admissible strategy $\pi_t = \frac{|\beta|}{\beta} \mathbb{1}_{\{S_t = z\}}$. In other words, the agent invests X_t or $-X_t$ in the risky asset whenever $S_t = z$, where the sign of the investment is equal to the sign of β, and invests 0 in the risky asset at all other times. Without loss of generality assume that $r = 0$ so that no wealth is accrued away from the level. The corresponding wealth process is

$$X_t = e^{\frac{|\beta|}{z} L_t^z(S)},$$

because $L^z(S)$ increases on the set $\{S_t = z\}$. Since $L^z(S)$ is an increasing process, we can state the conditions for a classical arbitrage opportunity have been met, slightly altered to fit the context of portfolio management in our context. Specifically,

$$\mathbb{P}(X_t \geq 1) = 1$$

and

$$\mathbb{P}(X_t > 1) > 0.$$

□

In light of Theorem 12, we see that the skew geometric Brownian motion allows for a special type of arbitrage opportunity that involves creating an arbitrary spike in the portfolio holdings of the risky asset at the level z in order to benefit from the directional effect of the local time. It is therefore the case that the value function of the control problem is infinite.

The type of arbitrage opportunity arising at the level z is valid in the sense that indicator functions like $\mathbb{1}_{\{S_t=z\}}$ are measurable. However they are not economically realistic or of particular interest to model. Were such strategies, which require opening and closing large positions at the level z, practically feasible then they would be the exclusive domain of professional high frequency arbitrageurs. Our model seeks to understand the impact that the presence of such support and resistance levels has on overall investment and consumption across the entire problem space, rather than the specific behaviour that the singularity might provoke the instant the level is broken.

We therefore propose an additional restriction on the portfolio process π, in order to exclude these specific arbitrage strategies from the market. In particular we require a type of "smoothness" around the level z, so that the agent may plan their portfolio holdings immediately above and below the level. These two positions may be distinct, so the portfolio process may experience a jump. However when the price is exactly at the level the position must be the average of these two positions.

Definition 6. *The family $\mathcal{A}$ of all admissible portfolio-consumption pairs (c, π) consists of all $(\mathcal{F}_t)$-progressively measurable processes c and π such that*

$$0 \le c_t \quad \text{and} \quad 0 \le c_t + |\pi_t| \le C \quad \text{for all} \quad t \ge 0, \ \mathbb{P}\text{-a.s.},$$

for some constant $C > 0$, which may depend on (c, π), and

$$\pi_t = \pi(t, S_t, X_t) \quad \text{for all} \quad t \ge 0,$$

where $\pi : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ is a measurable làdlàg function that satisfies

$$\pi(t, z, x) = \frac{1}{2}(\pi(t, z-, x) + \pi(t, z+, x))$$

for all $(t, x) \in [0, T] \times \mathbb{R}_+^*$.

Remark 16. An alternative approach would be to model the phenomenon of support and resistance levels using a price process with drift that is absolutely continuous with respect to time. Doing so would ensure the existence of an equivalent martingale measure and therefore would not allow for the arbitrage we saw in Theorem 12. For example, we might consider a market consisting of a single risky asset with discontinuous drift,

$$dS_t^\varepsilon = \left[\mu(S_t^\varepsilon) + \frac{1}{2\varepsilon}\sigma^2(S_t^\varepsilon)\left(\ln(1+\beta)\mathbb{1}_{[z-\varepsilon, z]} - \ln(1-\beta)\mathbb{1}_{[z, z+\varepsilon]}\right)\right] dt + \sigma(S_t^\varepsilon) dW_t.$$

Here ε represents the intensity of the level, and smaller ε corresponds to both a smaller area of impact and a greater jump in the size of the drift. We may say that ε is related to the minimum price increment, namely, a tick, on the market in question, since markets that allow orders to be left with extremely high precision thereby allow market participants to cluster their orders extremely close to the level z. We also notice that as $\varepsilon \to 0$ the process converges to the skew diffusion process considered in Section 1.2, a fact that can be seen by comparing the speed and scale measures of both processes.

We expect that by solving the control problem with this process in place of (3.4) which should be straightforward since it is a one dimensional Itô diffusion process and the general case has been studied comprehensively in Karatzas and Shreve [KS98], we would be able to present our model as the limiting case of this model, as $\varepsilon \to 0$. However we did not complete this work in time to include it in the thesis.

2.2 The Control Problem for $T < \infty$

The agent's objective is to maximise

$$J_{T,s,x}(c,\pi) = \frac{1}{\gamma}\mathbb{E}\left[\int_0^T e^{-\rho t}(X_t c_t)^\gamma dt + e^{-\rho T}X_T^\gamma \,\middle|\, S_0 = s, X_0 = x\right] \tag{3.8}$$

over all admissible controls (c, π). The value function of the portfolio optimisation problem is defined by

$$v(T,s,x) = \sup_{(c,\pi)\in\mathcal{A}} J_{T,s,x}(c,\pi)$$

for $s, x > 0$. We expect that the value function v of the stochastic control problem identifies with a function $w : [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ satisfying the non-linear PDE

$$w_t(t,s,x) + \sup_{(c,\pi)\in\mathbb{R}_+\times\mathbb{R}}\left[\mathcal{L}^{c,\pi}w(t,s,x) + \frac{(xc)^\gamma}{\gamma}\right] = 0 \tag{3.9}$$

inside $\mathcal{C}_T \cup \mathcal{D}_T$, where

$$\mathcal{C}_T = \{(t,s,x) \in (0,T) \times \mathbb{R}_+^* \times \mathbb{R}_+^* \mid s > z\}, \tag{3.10}$$

$$\mathcal{D}_T = \{(t,s,x) \in (0,T) \times \mathbb{R}_+^* \times \mathbb{R}_+^* \mid s < z\}, \tag{3.11}$$

with terminal condition

$$w(T,s,x) = \frac{x^\gamma}{\gamma}, \tag{3.12}$$

and subject to the boundary condition

$$\frac{\beta}{z}x\pi w_x(t,z,x) + \big((1+\beta)w_s(t,z+,x) - (1-\beta)w_s(t,z-,x)\big) = 0 \tag{3.13}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+^*$. The differential operator $\mathcal{L}^{c,\pi}$ is defined by

$$\begin{aligned}\mathcal{L}^{c,\pi}w(t,s,x) = {}& \frac{1}{2}\sigma^2\pi^2x^2w_{xx}(t,s,x) + \sigma^2 sx\pi w_{sx}(t,s,x) + \frac{1}{2}\sigma^2s^2w_{ss}(t,s,x) \\ & + ((\mu-r)\pi + (r-c))xw_x(t,s,x) + \mu s w_s(t,s,x) - \rho w(t,s,x)\end{aligned} \tag{3.14}$$

for $(c,\pi) \in \mathbb{R}_+ \times \mathbb{R}$ and for $(t,s,x) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$.

2.3 The Control Problem for $T = \infty$

Over an infinite time horizon, the agent's objective is to maximise the performance criterion

$$J_{\infty,s,x}(c,\pi) = \frac{1}{\gamma}\mathbb{E}\left[\int_0^\infty e^{-\rho t}(X_t c_t)^\gamma dt \,\middle|\, S_0 = s, X_0 = x\right] \tag{3.15}$$

over all admissible controls (c,π). In this case, we restrict attention to the following class of controls.

Definition 7. *Given a discounting rate $\rho > 0$, the family of consumption-portfolio pairs $\mathcal{A}_p$ is the set of all $(c,\pi) \in \mathcal{A}$ such that the associated wealth process X satisfies the transversality condition*

$$\lim_{T\to\infty} e^{-\rho T}\mathbb{E}\left[X_T^\gamma\right] = 0 \tag{3.16}$$

The value function associated with the control problem on the infinite time horizon is defined by

$$v(s,x) = \sup_{(c,\pi)\in\mathcal{A}_p} J_{\infty,s,x}(c,\pi)$$

for $s \in \mathbb{R}_+^*$ and $x \in \mathbb{R}_+^*$. We opt to repeat the usage of v to represent the value function on the infinite horizon, as the context will ensure there is no ambiguity.

Again, we expect that the value function v of the stochastic control problem on the infinite time horizon identifies with a function $w : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$, that solves the HJB equation

$$\sup_{(c,\pi)\in\mathbb{R}_+\times\mathbb{R}} \left[\mathcal{L}^{c,\pi}w(s,x) + \frac{(xc)^\gamma}{\gamma}\right] = 0 \tag{3.17}$$

for $(s,x) \in \mathcal{C} \cup \mathcal{D}$, where

$$\mathcal{C} = \{(s,x) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid s > z\}, \tag{3.18}$$

$$\mathcal{D} = \{(s,x) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid s < z\}, \tag{3.19}$$

and subject to the boundary condition

$$\frac{\beta}{z}x\pi w_x(z,x) + \big((1+\beta)w_s(z+,x) - (1-\beta)w_s(z-,x)\big) = 0$$

for all $x \in \mathbb{R}_+^*$. Here the differential operator $\mathcal{L}^{c,\pi}$ is defined as in (3.14).

3 Verification Theorems

We now prove two verification theorems for the control problem described in Section 2, first for the finite time horizon and then for the infinite horizon.

Theorem 13 (Finite Time Horizon: $T < \infty$)**.** *Let $w : [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ be a continuous function that is $C^{1,2,2}$ in $\mathcal{C}_T \cup \mathcal{D}_T$, defined by (3.10) and (3.11), and satisfies the HJB equation (3.9) and (3.12) as well as the polynomial growth condition*

$$|w_s(t,s,x)| + |w_x(t,s,x)| \leq C(1 + |s|^k + |x|^k) \tag{3.20}$$

for all $(t,s,x) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ for some constants $k \in \mathbb{N}$ and $C > 0$. Then

$$w(0,s,x) \geq v(T,s,x) \tag{3.21}$$

for all $s, x > 0$. Furthermore, suppose that there exist measurable functions $\hat{c} : [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+$ and $\hat{\pi} : [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ such that

$$\begin{aligned} &w_t(t,s,x) + \mathcal{L}^{\hat{c}(t,s,x),\hat{\pi}(t,s,x)} w(t,s,x) + \frac{x^\gamma \hat{c}^\gamma(t,s,x)}{\gamma} \\ &= w_t(t,s,x) + \sup_{(c,\pi)\in\mathbb{R}_+\times\mathbb{R}} \left[\mathcal{L}^{c,\pi} w(t,s,x) + \frac{x^\gamma c^\gamma}{\gamma} \right] \end{aligned} \tag{3.22}$$

for all $(t,s,x) \in \mathcal{C}_T \cup \mathcal{D}_T$, and

$$\frac{\beta}{z} x \hat{\pi}(t,z,x) w_x(t,z,x) + ((1+\beta) w_s(t,z+,x) - (1-\beta) w_s(t,z-,x)) = 0 \tag{3.23}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}_+^*$. Also, suppose that the controlled diffusion

$$\begin{aligned} dX_t = &\Big[(\mu - r)\, \hat{\pi}(t,S_t,X_t) + r - \hat{c}(t,S_t,X_t) \Big] X_t dt \\ &+ \frac{\beta}{z} \hat{\pi}(t,S_t,X_t) X_t dL_t^z(S) + \sigma \hat{\pi}(t,S_t,X_t) X_t dW_t \end{aligned}$$

admits unique strong solution, and

$$\hat{c}_t = \hat{c}(t,S_t,X_t)$$

and

$$\hat{\pi}_t = \hat{\pi}(t,S_t,X_t)$$

define processes such that $(\hat{c},\hat{\pi}) \in \mathcal{A}$. Then $(\hat{c},\hat{\pi})$ is an optimal consumption-portfolio pair and

$$w(0,s,x) = v(T,s,x) \tag{3.24}$$

for all $(t,s,x) \in [0,T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$.

Proof. We first note that

$$\begin{aligned} S_t^k &= s^k \exp\left(k \left(\mu - \frac{1}{2}\sigma^2 \right) t + k \frac{\beta}{z} L_t^z(S) + k\sigma W_t \right) \\ &\leq s^k \exp\left(k \left(\mu - \frac{1}{2}\sigma^2 + \frac{|\beta|}{z} \right) t + k\sigma W_t \right) \end{aligned}$$

implies that

$$\sup_{u\in[0,T]} \mathbb{E}\left[|S_u|^k\right] < \infty \tag{3.25}$$

for all $k \geq 1$. We next fix any admissible pair of controls $(c, \pi) \in \mathcal{A}$, and we calculate

$$\begin{aligned} X_t^k &= x^k \exp\left(k \int_0^t \left((\mu - r)\,\pi_u - c_u - \frac{1}{2}\sigma^2\pi_u^2\right) du + krt + k\frac{\beta}{z}\int_0^t \pi_u dL_u^z(S) + k\int_0^t \sigma\pi_u dW_u\right) \\ &\leq x^k \exp\left(k\left(|\mu - r|C + \frac{|\beta|C}{2} + r + \frac{1}{2}\sigma^2C^2\right)t\right)\exp\left(-\frac{1}{2}k^2\sigma^2\int_0^t \pi_u^2 du + k\sigma\int_0^t \pi_u dW_u\right) \end{aligned} \tag{3.26}$$

where $C > 0$ is a constant such that $|\pi_t| \leq C$ for all $t \geq 0$ (see Definition 6). Novikov's condition and the calculation

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t k^2\sigma^2\pi_u^2 du\right)\right] \leq \exp\left(\frac{1}{2}k^2\sigma^2C^2t\right) < \infty, \quad \text{for all } t \geq 0$$

imply that the exponential local martingale on the right hand side of (3.26) is a martingale. Therefore,

$$\sup_{t\in[0,T]} \mathbb{E}\left[X_t^k\right] < \infty \quad \text{for all } k \geq 1. \tag{3.27}$$

Using the symmetric version of the Itô Tanaka formula established in Peskir [Pes07]

$$\begin{aligned} w(t, S_t, X_t) = w(0, s, x) &+ \int_0^t \frac{1}{2}\left(w_t(u, S_u+, X_u) + w_t(u, S_u-, X_u)\right) du \\ &+ \int_0^t \frac{1}{2}\left(w_s(u, S_u+, X_u) + w_s(u, S_u-, X_u)\right) dS_u \\ &+ \int_0^t \frac{1}{2}\left(w_x(u, S_u+, X_u) + w_x(u, S_u-, X_u)\right) dX_u \\ &+ \frac{1}{2}\int_0^t \frac{1}{2}\left(w_{ss}(u, S_u+, X_u) + w_{ss}(u, S_u-, X_u)\right) d\langle S, S\rangle_u \\ &+ \int_0^t \frac{1}{2}\left(w_{sx}(u, S_u+, X_u) + w_{sx}(u, S_u-, X_u)\right) d\langle S, X\rangle_u \\ &+ \frac{1}{2}\int_0^t \frac{1}{2}\left(w_{xx}(u, S_u+, X_u) + w_{xx}(u, S_u-, X_u)\right) d\langle X, X\rangle_u \\ &+ \int_0^t \frac{1}{2}\left(w_s(u, S_u+, X_u) - w_s(u, S_u-, X_u)\right) \mathbb{1}_{\{S_u=z\}} dL_t^z(S). \end{aligned} \tag{3.28}$$

Using the fact that the Lebesgue measure of $\{u \in [0, t] | S_u = z\}$ is zero, $\mathbb{P}$-a.s., we can see that, e.g.,

$$\int_0^t \frac{1}{2}\left(w_t(u, z+, X_u) + w_t(u, z-, X_u)\right) \mathbb{1}_{\{S_u=z\}} du = 0.$$

It follows that (3.28) reduces to

$$
\begin{aligned}
w(t,S_t,X_t) = w(0,s,x) &+ \int_0^t w_t(u,S_u,X_u)\mathbb{1}_{\{S_u\neq z\}}du \\
&+ \int_0^t w_s(u,S_u,X_u)\mathbb{1}_{\{S_u\neq z\}}dS_u \\
&+ \int_0^t w_x(u,S_u,X_u)\mathbb{1}_{\{S_u\neq z\}}dX_u \\
&+ \int_0^t \frac{1}{2}\left(w_s(u,z+,X_u)+w_s(u,z-,X_u)\right)\mathbb{1}_{\{S_u=z\}}dS_u \\
&+ \int_0^t \frac{1}{2}\left(w_x(u,z+,X_u)+w_x(u,z-,X_u)\right)\mathbb{1}_{\{S_u=z\}}dX_u \\
&+ \int_0^t \frac{1}{2}w_{ss}(u,S_u,X_u)\mathbb{1}_{\{S_u=z\}}d\langle S,S\rangle_u \\
&+ \int_0^t w_{sx}(u,S_u,X_u)\mathbb{1}_{\{S_u=z\}}d\langle S,X\rangle_u \\
&+ \int_0^t \frac{1}{2}w_{xx}(u,S_u,X_u)\mathbb{1}_{\{S_u=z\}}d\langle X,X\rangle_u \\
&+ \int_0^t \frac{1}{2}\left(w_s(u,z+,X_u)-w_s(u,z-,X_u)\right)\mathbb{1}_{\{S_u=z\}}dL_t^z(S).
\end{aligned}
\tag{3.29}
$$

Applying Itô's integration by parts formula and substituting in (3.4) and (3.6) we obtain

$$
\begin{aligned}
e^{-\rho T}w(T,S_T,X_T) =& w(0,S_0,X_0) \\
&+ \int_0^T e^{-\rho t}\left(w_t(t,S_t,X_t)+\mathcal{L}^{c,\pi}w(t,S_t,X_t)\right)\mathbb{1}_{\{S_t\neq z\}}du \\
&+ \int_0^T \frac{1}{2}e^{-\rho t}\Big(\frac{\beta}{z}X_t\pi_t(w_x(t,z-,X_t)+w_x(t,z+,X_t)) \\
&\quad + ((1+\beta)w_s(t,z-,X_t)-(1-\beta)w_s(t,z-,X_t))\Big)dL_t^z(S) \\
&+ M_T,
\end{aligned}
\tag{3.30}
$$

where

$$
M_t = \int_0^t e^{-\rho u}\left(\sigma\pi_u X_u w_x(u,S_u,X_u)+\sigma S_u w_s(u,S_u,X_u)\right)\mathbb{1}_{\{S_u\neq z\}}dW_u. \tag{3.31}
$$

Using Itô's isometry, the growth condition on w_s and w_x given in (3.20), the boundedness of π and the estimates (3.25) and (3.27), we obtain

$$\begin{aligned}\mathbb{E}\left[M_T^2\right] =&\mathbb{E}\Bigg[\int_0^T e^{-2\rho u}\Big(\sigma^2\pi_u^2X_u^2w_x^2(u,S_u,X_u)\\&\qquad + 2\sigma^2 S_u X_u \pi_u w_x(u,S_u,X_u)w_s(u,S_u,X_u)\\&\qquad + \sigma^2 S_u^2 w_s^2(u,S_u,X_u)\Big)\mathbb{1}_{\{S_u\neq z\}}du\Bigg]\\&\le \bar{C}\,\mathbb{E}\left[\int_0^T(1+|S_u|^{\bar{k}}+|X_u|^{\bar{k}})du\right]\\&\le \bar{C}\int_0^T\left(1+\sup_{v\in[0,t]}\mathbb{E}\left[|S_v|^{\bar{k}}\right]+\sup_{v\in[0,t]}\mathbb{E}\left[|X_v|^{\bar{k}}\right]\right)du\\&<\infty\end{aligned} \tag{3.32}$$

where $\bar{k}\in\mathbb{N}$ and $C>0$ are appropriate constants. Therefore M is a square integrable martingale. Furthermore, the estimate (3.27) and the boundedness of c imply that

$$\mathbb{E}\left[e^{-\rho T}|X_T^\gamma|\right]<\infty$$

and

$$\mathbb{E}\left[\int_0^T e^{-\rho t}|X_t^\gamma c_t^\gamma|dt\right]\le C\sup_{t\in[0,T]}\mathbb{E}\left[|X_t|^\gamma\right]T<\infty. \tag{3.33}$$

Since c and π may not achieve the supremum in (3.9), we have the inequality

$$-\frac{X_t^\gamma c_t^\gamma}{\gamma}\ge w_t(t,S_t,X_t)+\mathcal{L}^{c_t,\pi_t}w(t,S_t,X_t). \tag{3.34}$$

By substituting (3.34), the terminal condition (3.12) and the boundary condition (3.13) at $s=z$ into (3.30) and taking expectations, we obtain

$$J_{T,s,x}(c,\pi)\equiv\frac{1}{\gamma}\mathbb{E}\left[\int_0^T e^{-\rho u}X_u^\gamma c_u^\gamma du+e^{-\rho T}X_T^\gamma\right]\le w(0,S_0,X_0) \tag{3.35}$$

which establishes (3.21) because $(c,\pi)\in\mathcal{A}$ has been arbitrary.

If we take $(\hat{c},\hat{\pi})\in\mathcal{A}$ in place of (c,π), then (3.34) holds with equality and

$$J_{T,s,x}(\hat{c},\hat{\pi})\equiv\frac{1}{\gamma}\mathbb{E}\left[\int_0^T e^{-\rho u}X_u^\gamma \hat{c}_u^\gamma du+e^{-\rho T}X_T^\gamma\right]=w(0,S_0,X_0).$$

Together with (3.21), this identity results in (3.24) as well as the optimality of $(\hat{c},\hat{\pi})$. □

Theorem 14 (Infinite Time Horizon: $T = \infty$)**.** *Let $w : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ be a continuous function that is $C^{2,2}$ in $\mathcal{C} \cup \mathcal{D}$, defined by* (3.18) *and* (3.19)*, and satisfies the HJB equation* (3.17) *as well as the polynomial growth conditions*

$$|w(s,x)| \le C(1 + |x|^\gamma) \tag{3.36}$$

and

$$|w_s(s,x)| + |w_x(s,x)| \le C(1 + |s|^k + |x|^k) \tag{3.37}$$

for all $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ for some constants $k \in \mathbb{N}$ and $C > 0$. Then

$$w(s,x) \ge v(s,x) \tag{3.38}$$

for all $s, x > 0$. Furthermore, suppose that there exists measurable functions $\hat{c} : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+$ and $\hat{\pi} : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ such that

$$\begin{aligned} &\mathcal{L}^{\hat{c}(t,s,x),\hat{\pi}(t,s,x)} w(t,s,x) + \frac{x^\gamma \hat{c}^\gamma(t,s,x)}{\gamma} \\ &= \sup_{(c,\pi)\in\mathbb{R}_+\times\mathbb{R}} \left[\mathcal{L}^{c,\pi} w(t,s,x) + \frac{x^\gamma c^\gamma}{\gamma}\right] \end{aligned} \tag{3.39}$$

for all $(s,x) \in \mathcal{C} \cup \mathcal{D}$, and

$$\frac{\beta}{z} x\hat{\pi}(t,z,x) w_x(z,x) + ((1+\beta) w_s(z+,x) - (1-\beta) w_s(z-,x)) = 0 \tag{3.40}$$

for all $x > 0$. Also, suppose that the controlled diffusion

$$\begin{aligned} dX_t = &\Big[(\mu - r)\,\hat{\pi}(S_t, X_t) + r - \hat{c}(S_t, X_t)\Big] X_t dt \\ &+ \frac{\beta}{z}\hat{\pi}(S_t, X_t) X_t dL_t^z(S) + \sigma\hat{\pi}(S_t, X_t) X_t dW_t \end{aligned}$$

admits a unique strong solution, and

$$\hat{c}_t = \hat{c}(S_t, X_t)$$

and

$$\hat{\pi}_t = \hat{\pi}(S_t, X_t)$$

define processes such that $(\hat{c}, \hat{\pi}) \in \mathcal{A}_p$. Then $(\hat{c}, \hat{\pi})$ is an optimal consumption-portfolio pair and

$$w(s,x) = v(s,x) \tag{3.41}$$

for all $(s,x) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$.

Proof. Fix any admissible pair of controls $(c, \pi) \in \mathcal{A}_p$. Applying Itô's formula and the integration by parts formula we obtain

$$\begin{aligned} e^{-\rho T} w(S_T, X_T) = w(S_0, X_0) \\ &+ \int_0^T e^{-\rho t} \left(\mathcal{L}^{c,\pi} w(S_t, X_t)\right) \mathbb{1}_{\{S_t \neq z\}} du \\ &+ \int_0^T \frac{1}{2} e^{-\rho t} \Big(\frac{\beta}{z} X_t \pi_t (w_x(z-, X_t) + w_x(z+, X_t)) \\ &\qquad + ((1+\beta) w_s(z-, X_t) - (1-\beta) w_s(z-, X_t)) \Big) dL_t^z(S) \\ &+ M_T, \end{aligned} \tag{3.42}$$

where

$$M_T = \int_0^T e^{-\rho t} \left(\sigma \pi_t X_t w_x(S_t, X_t) + \sigma S_t w_s(S_t, X_t)\right) \mathbb{1}_{\{S_t \neq z\}} dW_t. \tag{3.43}$$

Arguing as in (3.32), we can see that M is a square integrable martingale. Furthermore, since c and π may not achieve the supremum in (3.39), we have the inequality

$$-\frac{X_t^\gamma c_t^\gamma}{\gamma} \geq \mathcal{L}^{c_t, \pi_t} w(t, S_t, X_t). \tag{3.44}$$

Recalling (3.27), we substitute (3.44) and the boundary condition (3.40) at $s = z$ into (3.42) and we take expectations to obtain

$$\frac{1}{\gamma} \mathbb{E} \left[\int_0^T e^{-\rho u} c_u^\gamma X_u^\gamma du \right] + \mathbb{E}\left[e^{-\rho T} w(S_T, X_T)\right] \leq w(S_0, X_0).$$

Using the monotone convergence theorem and (3.16) in Definition 7, we obtain

$$J_{\infty,s,x}(c, \pi) \equiv \frac{1}{\gamma} \mathbb{E} \left[\int_0^\infty e^{-\rho u} X_u^\gamma c_u^\gamma du \right] \leq w(0, S_0, X_0).$$

It follows that (3.38) holds because $(c, \pi) \in \mathcal{A}_p$ has been arbitrary.

It is then clear that if we take $(\hat{c}, \hat{\pi}) \in \mathcal{A}_p$ in place of (c, π), then (3.44) holds with equality and

$$J_{\infty,s,x}(\hat{c}, \hat{\pi}) \equiv \frac{1}{\gamma} \mathbb{E} \left[\int_0^\infty e^{-\rho u} X_u^\gamma \hat{c}_u^\gamma du \right] = w(0, S_0, X_0).$$

Together with (3.38), this identity results in (3.41) as well as the optimality of $(\hat{c}, \hat{\pi})$. □

4 The Solution to the Portfolio Problem over a Finite Time Horizon $T < \infty$

We now consider the problem of maximising the objective (3.8) over all admissible controls $(c, \pi) \in \mathcal{A}$ subject to the stochastic dynamics given by (3.4) and (3.5). For easier reference, recall the HJB equation (3.9), (3.12) and (3.13) for the finite horizon problem

$$w_t(t,s,x) + \frac{1}{2}\sigma^2\pi^2x^2w_{xx}(t,s,x) + \sigma^2 sx\pi w_{sx}(t,s,x) + \frac{1}{2}\sigma^2s^2w_{ss}(t,s,x)$$
$$+ \left((\mu - r)\pi + r - c\right)xw_x(t,s,x) + \mu s w_s(t,s,x) - \rho w(t,s,x) + \frac{(cx)^\gamma}{\gamma} = 0 \quad (3.45)$$

with terminal condition

$$w(T,s,x) = \frac{x^\gamma}{\gamma}, \quad (3.46)$$

and subject to the boundary condition

$$\frac{\beta}{z}x\pi(w_x(t,z-,x) + w_x(t,z+,x))$$
$$+ ((1+\beta)w_s(t,z-,x) - (1-\beta)w_s(t,z-,x)) = 0. \quad (3.47)$$

Inside the set $\mathcal{C}_T \cup \mathcal{D}_T$ the controls π and c that achieve the maximum in (3.45) are given by

$$\hat{\pi}(t,s,x) = -\frac{s}{x}\frac{w_{sx}(t,s,x)}{w_{xx}(t,s,x)} - \frac{(\mu - r)}{\sigma^2}\frac{w_x(t,s,x)}{xw_{xx}(t,s,x)}, \quad (3.48)$$

and

$$\hat{c}(t,s,x) = \frac{w_x^{\frac{-1}{1-\gamma}}(t,s,x)}{x}. \quad (3.49)$$

Substituting $\hat{\pi}$ and $\hat{c}$ back into (3.45), we obtain

$$w_t(t,s,x) - \frac{1}{2}\sigma^2s^2\frac{w_{sx}^2(t,s,x)}{w_{xx}(t,s,x)} - s(\mu - r)\frac{w_x(t,s,x)w_{sx}(t,s,x)}{w_{xx}(t,s,x)}$$
$$- \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{w_x^2(t,s,x)}{w_{xx}(t,s,x)} + \frac{1}{2}\sigma^2s^2w_{ss}(t,s,x) + \frac{1-\gamma}{\gamma}w_x^{\frac{-\gamma}{1-\gamma}}(t,s,x)$$
$$+ rxw_x(t,s,x) + \mu s w_s(t,s,x) - \rho w(t,s,x) = 0. \quad (3.50)$$

Substituting the expression

$$w(t,s,x) = \frac{x^\gamma}{\gamma}h^{1-\gamma}(t,s) \quad (3.51)$$

for w into the non-linear PDE (3.50) results in the linear PDE

$$h_t(t,s) + \frac{1}{2}\sigma^2s^2h_{ss}(t,s) + \left(\frac{\mu - r}{1-\gamma} + r\right)sh_s(t,s)$$
$$- \frac{1}{1-\gamma}\left(\rho - r\gamma - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{\gamma}{1-\gamma}\right)h(t,s) + 1 = 0 \quad (3.52)$$

with terminal condition

$$h(T, s) = 1. \tag{3.53}$$

To deal with the boundary condition (3.47), we note that if w is of the special form given by (3.51), then taking derivatives with respect to x above and below z we get

$$\begin{aligned} w_x(t, z-, x) + w_x(t, z+, x) &= x^{\gamma-1}\left(h^{1-\gamma}(t, z-) + h^{1-\gamma}(t, z+)\right) \\ &= 2x^{\gamma-1}h^{1-\gamma}(t, z) \end{aligned} \tag{3.54}$$

by the continuity of h which follows from the continuity of w. Applying the transformation (3.51) to the first order conditions for $\hat{\pi}$ and $\hat{c}$ given by (3.48) and (3.49), we see that they also admit simpler representations in terms of h, specifically

$$\hat{\pi}(t, s, x) = \frac{(\mu - r)}{\sigma^2}\frac{1}{1-\gamma} + s\frac{h_s(t, s)}{h(t, s)} \tag{3.55}$$

for $s \neq z$, and

$$\hat{c}(t, s, x) = \frac{1}{h(t, s)}. \tag{3.56}$$

In particular $\hat{\pi}$ and $\hat{c}$ are not dependent on x. Notice that the boundary condition (3.47) also contains a term involving π. As discussed in Section 2.1 the potential for arbitrage opportunities for certain choices of π at $s = z$ led to the inclusion of an additional restriction in Definition 6 which allows us to write

$$\hat{\pi}(t, z, x) = \frac{(\mu - r)}{\sigma^2}\frac{1}{1-\gamma} + \frac{1}{2}z\frac{(h_s(t, z-) + h_s(t, z+))}{h(t, z)}. \tag{3.57}$$

Substituting (3.54) and (3.57) into the original boundary condition (3.47) we arrive at a boundary condition for h at $s = z$

$$\frac{\beta}{z}\mu_\alpha h(t, z) + \frac{1}{2}\left(1 + \frac{\beta}{1-\gamma}\right)h_s(t, z+) - \frac{1}{2}\left(1 - \frac{\beta}{1-\gamma}\right)h_s(t, z-) = 0 \tag{3.58}$$

where

$$\mu_\alpha = \frac{\mu - r}{\sigma^2}\frac{\gamma}{(1-\gamma)^2}. \tag{3.59}$$

The PDE (3.52) with terminal condition (3.53) can be transformed to the heat equation using the transformation

$$h(t, s) = e^{-(\alpha + \frac{1}{2}\sigma^2\kappa^2)\tau + \kappa\zeta}u(\tau, \zeta) - \frac{1}{\alpha}\left(e^{-\alpha\tau} - 1\right), \tag{3.60}$$

where

$$\alpha = \frac{\rho - r\gamma}{1-\gamma} - \frac{1}{2}\mu_\alpha(\mu - r), \tag{3.61}$$

$$\kappa = -\frac{\frac{\mu - r}{1-\gamma} + r - \frac{1}{2}\sigma^2}{\sigma^2}, \tag{3.62}$$

$$\tau = T - t \tag{3.63}$$

and

$$\zeta = \log s. \tag{3.64}$$

Here, we reverse time and consider log prices ζ instead of absolute prices s, namely (3.63)–(3.64). In particular, we can see that u must satisfy the heat equation given by

$$u_\tau(\tau, \zeta) = \Sigma\, u_{\zeta\zeta}(\tau, \zeta), \tag{3.65}$$

where $\Sigma = \frac{1}{2}\sigma^2$, with initial condition

$$u(0, \zeta) = e^{-\kappa\zeta} \tag{3.66}$$

and boundary condition

$$\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right) u(\tau, \bar{z}) - \mu_\alpha \frac{\beta}{\alpha}\left(e^{-\alpha\tau} - 1\right) e^{\left(\alpha + \frac{1}{2}\sigma^2\kappa^2\right)\tau - \kappa\bar{z}} + \frac{1}{2}\left(1 + \frac{\beta}{1-\gamma}\right) u_\zeta(t, \bar{z}+) - \frac{1}{2}\left(1 - \frac{\beta}{1-\gamma}\right) u_\zeta(t, \bar{z}-) = 0, \tag{3.67}$$

where $\bar{z} = \log z$.

We now prove two lemmas which will be needed to solve (3.65)–(3.67). The boundary condition (3.67) prevents a simple direct solution, and so we apply a Laplace transform to the problem. In Lemma 4 we present the solution to the resulting ODE, to which we must then apply an inverse Laplace transform to recover the solution to the original problem, which we do in Lemma 5.

Our results involve the error function Erf(.) and the complementary error function Erfc(.), which are defined by

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du,$$

and

$$\mathrm{Erfc}(z) = 1 - \mathrm{Erf}(z)$$

for $z \in \mathbb{R}$.

Lemma 4. *Consider the ordinary differential equation*

$$p\,\tilde{u}(p, \zeta) - e^{-\kappa\zeta} = \Sigma\,\tilde{u}_{\zeta\zeta}(p, \zeta) \tag{3.68}$$

with boundary condition

$$\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)\tilde{u}(p,\bar{z})-\mu_\alpha\frac{\beta}{\alpha}\left(\frac{1}{p-\kappa^2\Sigma}-\frac{1}{p-(\alpha+\kappa^2\Sigma)}\right)e^{-\kappa\bar{z}}$$
$$+\frac{1}{2}\left(1+\frac{\beta}{1-\gamma}\right)\tilde{u}_\zeta(p,\bar{z}+)-\frac{1}{2}\left(1-\frac{\beta}{1-\gamma}\right)\tilde{u}_\zeta(p,\bar{z}-)=0 \quad (3.69)$$

where $p>0$ and $\Sigma>0$ are constants. There exists a unique function $\tilde{u}(p,.)$ such that if we write

$$\tilde{u}(p,\zeta)=\begin{cases}\tilde{u}^-(p,\zeta), & \zeta\le\bar{z},\\ \tilde{u}^+(p,\zeta), & \zeta>\bar{z}.\end{cases} \quad (3.70)$$

then $\tilde{u}$ is continuous, $\tilde{u}^-$ (respectively, $\tilde{u}^+$) satisfies the ordinary differential equation (3.68) *in $(-\infty,\bar{z})$ (respectively, $(\bar{z},\infty)$) and* (3.69) *holds. This function admits the expressions*

$$\tilde{u}^-(p,\zeta)=\frac{e^{-\kappa\zeta}}{p-\kappa^2\Sigma}-e^{-\kappa\bar{z}}\frac{p-\left(\alpha+\kappa^2\Sigma-1\right)}{\left(p-\kappa^2\Sigma\right)\left(p-(\alpha+\kappa^2\Sigma)\right)}\left[\frac{\beta\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\frac{\sqrt{p}}{\sqrt{\Sigma}}}\right]e^{-\frac{(\bar{z}-\zeta)}{\sqrt{\Sigma}}\sqrt{p}} \quad (3.71)$$

and

$$\tilde{u}^+(p,\zeta)=\frac{e^{-\kappa\zeta}}{p-\kappa^2\Sigma}-e^{-\kappa\bar{z}}\frac{p-\left(\alpha+\kappa^2\Sigma-1\right)}{\left(p-\kappa^2\Sigma\right)\left(p-(\alpha+\kappa^2\Sigma)\right)}\left[\frac{\beta\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\frac{\sqrt{p}}{\sqrt{\Sigma}}}\right]e^{-\frac{(\zeta-\bar{z})}{\sqrt{\Sigma}}\sqrt{p}}. \quad (3.72)$$

Proof. The general solution to the ordinary differential equation (3.68) is given by

$$\tilde{u}(p,\zeta)=\frac{e^{-\kappa\zeta}}{p-\kappa^2\Sigma}+A(p)e^{\frac{\sqrt{p}}{\sqrt{\Sigma}}\zeta}+B(p)e^{-\frac{\sqrt{p}}{\sqrt{\Sigma}}\zeta}. \quad (3.73)$$

The requirement that $\tilde{u}$ should remain bounded as $\zeta\to-\infty$ or $\zeta\to\infty$ yields the expressions

$$\tilde{u}^-(p,\zeta)=\frac{e^{-\kappa\zeta}}{p-\kappa^2\Sigma}+A(p)e^{\frac{\sqrt{p}}{\sqrt{\Sigma}}\zeta} \quad (3.74)$$

and

$$\tilde{u}^+(p,\zeta)=\frac{e^{-\kappa\zeta}}{p-\kappa^2\Sigma}+B(p)e^{-\frac{\sqrt{p}}{\sqrt{\Sigma}}\zeta}. \quad (3.75)$$

To ensure that $\tilde{u}$ is continuous at $\bar{z}$ the functions $A(.)$ and $B(.)$ must satisfy the boundary condition

$$A(p)e^{\frac{\sqrt{p}}{\sqrt{\Sigma}}\bar{z}}=B(p)e^{-\frac{\sqrt{p}}{\sqrt{\Sigma}}\bar{z}}. \quad (3.76)$$

Substituting (3.74), (3.75) and the boundary condition (3.76) into (3.73) and solving for $A(p)$ and $B(p)$ gives,

$$A(p)=-e^{-\kappa\bar{z}}\frac{p-\left(\alpha+\kappa^2\Sigma-1\right)}{\left(p-\kappa^2\Sigma\right)\left(p-(\alpha+\kappa^2\Sigma)\right)}\left[\frac{\beta\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\frac{\sqrt{p}}{\sqrt{\Sigma}}}\right]e^{-\frac{\bar{z}}{\sqrt{\Sigma}}\sqrt{p}}$$

and

$$B(p) = -e^{-\kappa\bar{z}} \frac{p - \left(\alpha + \kappa^2\Sigma - 1\right)}{\left(p - \kappa^2\Sigma\right)\left(p - (\alpha + \kappa^2\Sigma)\right)} \left[\frac{\beta\mu_\alpha}{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right) - \frac{\sqrt{p}}{\sqrt{\Sigma}}}\right] e^{\frac{\bar{z}}{\sqrt{\Sigma}}\sqrt{p}}$$

and (3.74) and (3.75) identify with (3.71) and (3.72). □

Lemma 5. *The function*

$$\tilde{f}(p,\zeta) = \frac{d(p - c_1)}{(p - c_2)(p - c_3)} \left\{\frac{1}{1 + b\sqrt{p}}\right\} e^{a\sqrt{p}} \tag{3.77}$$

where $a < 0$*,* $b \in \mathbb{R}$*,* $c_1 \in \mathbb{R}$*,* $c_2 > 0$*,* $c_3 > 0$ *and* $d \in \mathbb{R}$ *are constants, has inverse Laplace transform*

$$f(\tau,\zeta) = \frac{c_2 - c_1}{c_2 - c_3} f_{c_2}(\tau,\zeta) - \frac{c_3 - c_1}{c_2 - c_3} f_{c_3}(\tau,\zeta) \tag{3.78}$$

where

$$\begin{aligned} f_c(\tau,\zeta) = \frac{d}{1 - b^2 c} \Bigg[& e^{\frac{1}{b^2}\tau - \frac{a}{b}} \left(\mathrm{Erfc}\left(\frac{a - \frac{2}{b}\tau}{2\sqrt{\tau}}\right) - 2\right) \\ & + \frac{1}{2}\left(-1 + b\sqrt{c}\right) e^{c\tau + a\sqrt{c}} \left(\mathrm{Erf}\left(\frac{-a - 2\sqrt{c}\tau}{2\sqrt{\tau}}\right) - 1\right) \\ & + \frac{1}{2}\left(-1 - b\sqrt{c}\right) e^{c\tau - a\sqrt{c}} \left(\mathrm{Erf}\left(\frac{-a + 2\sqrt{c}\tau}{2\sqrt{\tau}}\right) - 1\right) \Bigg] \end{aligned} \tag{3.79}$$

Proof. If $g(\tau)$ is the inverse Laplace transform of $\hat{g}(\tau)$, then the inverse Laplace transform of $\tilde{g}(\sqrt{p})$ is given by

$$\int_0^\infty \frac{u}{2\sqrt{\pi\tau^3}} \exp\left(-\frac{u^2}{4\tau}\right) g(u)du.$$

Also note that the inverse Laplace transform of

$$\frac{e^{ap}}{1 + bp}$$

is known to be

$$\frac{1}{b} e^{-\frac{a+\tau}{b}} \mathcal{H}(a + \tau),$$

where $\mathcal{H}(a + \tau)$ represents the Heaviside function. Combining these two observations we get that the inverse Laplace transform of

$$g(p) = \frac{e^{a\sqrt{p}}}{1 + b\sqrt{p}}$$

is given by

$$\begin{aligned}\tilde{g}(\tau) &= \frac{1}{b}\int_0^\infty \frac{u}{2\sqrt{\pi\tau^3}}\exp\left(-\frac{u^2}{4\tau}-\frac{a+u}{b}\right)\mathcal{H}(a+u)du \\ &= \frac{1}{b}\int_{-a}^\infty \frac{u}{2\sqrt{\pi\tau^3}}\exp\left(-\frac{u^2}{4\tau}-\frac{a+u}{b}\right)du \\ &= \frac{1}{b}e^{\frac{\tau}{b^2}-\frac{a}{b}}\left[\frac{e^{-\frac{(ab-2\tau)^2}{4b^2\tau}}}{\sqrt{\pi\tau}}+\frac{1}{b}\left(\mathrm{Erfc}\left(\frac{ab-2\tau}{2b\sqrt{\tau}}\right)-2\right)\right]. \end{aligned} \tag{3.80}$$

Next we appeal to the Laplace convolution theorem which states that given the inverse transforms for two functions $\tilde{f}$ and $\tilde{g}$ calculated individually as f and g, the inverse transform of the product is given by the convolution

$$\int_0^\tau f(u)g(\tau-u)du.$$

If $g(\tau)$ is the inverse Laplace transform of $\tilde{g}(p)$, then the inverse Laplace transform of

$$\frac{p-c_1}{(p-c_2)(p-c_3)}\tilde{g}(p)$$

is given by

$$\frac{c_2-c_1}{c_2-c_3}e^{c_2\tau}\int_0^\tau e^{-c_2u}g(u)du-\frac{c_3-c_1}{c_2-c_3}e^{c_3\tau}\int_0^\tau e^{-c_3u}g(u)du.$$

Applying this observation to (3.77) and (3.80) we arrive at the inverse Laplace tranform,

$$\begin{aligned} e^{c\tau}\int_0^\tau e^{-cu}\frac{e^{a\sqrt{u}}}{1+b\sqrt{u}}du &= \frac{d}{b}e^{c\tau-\frac{a}{b}}\int_0^\tau e^{\left(\frac{1}{b^2}-c\right)u}\left[\frac{e^{-\frac{(ab-2u)^2}{4b^2u}}}{\sqrt{\pi u}}+\frac{1}{b}\left(\mathrm{Erfc}\left(\frac{ab-2u}{2b\sqrt{u}}\right)-2\right)\right]du \\ &= \frac{d}{1-b^2c}\left[e^{\frac{1}{b^2}\tau-\frac{a}{b}}\left(\mathrm{Erfc}\left(\frac{a-\frac{2}{b}\tau}{2\sqrt{\tau}}\right)-2\right)\right. \\ &\qquad +\frac{1}{2}\left(-1+b\sqrt{c}\right)e^{c\tau+a\sqrt{c}}\left(\mathrm{Erf}\left(\frac{-a-2\sqrt{c}\tau}{2\sqrt{\tau}}\right)-1\right) \\ &\qquad \left.+\frac{1}{2}\left(-1-b\sqrt{c}\right)e^{c\tau-a\sqrt{c}}\left(\mathrm{Erf}\left(\frac{-a+2\sqrt{c}\tau}{2\sqrt{\tau}}\right)-1\right)\right] \end{aligned}$$

We conclude that the inverse Laplace transform of $\tilde{f}$ identifies with (3.79). □

We can now prove the main result of this section. The functions $\Phi_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$\Phi_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $\Phi_3 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ appearing here are defined by

$$\Phi_1(T,l) = (\Upsilon_1 + \Upsilon_2) e^{\left(\beta^2\Sigma\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)^2 - \kappa^2\Sigma\right)T} l^{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)} \times \left(\operatorname{Erf}\left[\frac{\frac{\ln l}{\sqrt{\Sigma}} + \beta\sqrt{\Sigma}\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)T}{2\sqrt{T}} \right] + 1 \right),$$

$$\Phi_2(T,l) = \Upsilon_1 \left[\frac{1}{2}\left(1 + \frac{|\kappa|}{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)}\right) l^{|\kappa|} \left(\operatorname{Erf}\left(\frac{\frac{-\ln l}{\sqrt{\Sigma}} - 2|\kappa|\sqrt{\Sigma}T}{2\sqrt{T}} \right) - 1 \right) + \frac{1}{2}\left(1 - \frac{|\kappa|}{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)}\right) l^{-|\kappa|} \left(\operatorname{Erf}\left(\frac{\frac{-\ln l}{\sqrt{\Sigma}} + 2|\kappa|\sqrt{\Sigma}T}{2\sqrt{T}} \right) - 1 \right) \right],$$

$$\Phi_3(T,l) = \Upsilon_2 e^{\alpha T} \left[\frac{1}{2}\left(1 + \frac{\sqrt{\frac{\alpha}{\Sigma} + \kappa^2}}{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)}\right) l^{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}} \times \left(\operatorname{Erf}\left(\frac{\frac{-\ln l}{\sqrt{\Sigma}} - 2\sqrt{\frac{\alpha}{\Sigma} + \kappa^2}T}{2\sqrt{T}} \right) - 1 \right) + \frac{1}{2}\left(1 - \frac{\sqrt{\frac{\alpha}{\Sigma} + \kappa^2}}{\beta\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)}\right) l^{-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}} \times \left(\operatorname{Erf}\left(\frac{\frac{-\ln l}{\sqrt{\Sigma}} + 2\sqrt{\frac{\alpha}{\Sigma} + \kappa^2}\sqrt{\Sigma}T}{2\sqrt{T}} \right) - 1 \right) \right],$$

$$P(T,l) = \Phi_1(T,l) + \Phi_2(T,l) + \Phi_3(T,l)$$

and

$$Q(T,l) = \frac{\partial \Phi_1}{\partial l}(T,l) + \frac{\partial \Phi_2}{\partial l}(T,l) + \frac{\partial \Phi_3}{\partial l}(T,l).$$

in which expressions μ_α, α and κ are defined by (3.59), (3.61) and (3.62), $\Sigma = \frac{1}{2}\sigma^2$,

$$\Upsilon_1 = \frac{\frac{\alpha-1}{\alpha}}{1 - \frac{\kappa^2}{\beta^2\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)^2}} \tag{3.81}$$

and

$$\Upsilon_2 = \frac{\frac{1}{\alpha}}{1 - \frac{\alpha + \kappa^2\Sigma}{\beta^2\Sigma\left(\mu_\alpha + \frac{\kappa}{1-\gamma}\right)^2}}. \tag{3.82}$$

Theorem 15. *Consider the control problem formulated in Section 2.2. Define*

$$v^-(T,s,x) = \frac{x^\gamma}{\gamma}\Bigg\{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha T}+1\right) + \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha T}\left(\frac{s}{z}\right)^\kappa\left[\Phi_1\left(T,\frac{s}{z}\right)+\Phi_2\left(T,\frac{s}{z}\right)+\Phi_3\left(T,\frac{s}{z}\right)\right]\Bigg\}^{1-\gamma} \tag{3.83}$$

and

$$v^+(T,s,x) = \frac{x^\gamma}{\gamma}\Bigg\{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha T}+1\right) + \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha T}\left(\frac{s}{z}\right)^\kappa\left[\Phi_1\left(T,\frac{z}{s}\right)+\Phi_2\left(T,\frac{z}{s}\right)+\Phi_3\left(T,\frac{z}{s}\right)\right]\Bigg\}^{1-\gamma} \tag{3.84}$$

and suppose that the functions defined by

$$\hat{\pi}^\pm(t,s,x)(t,s,x) = -\frac{s}{x}\frac{v^\pm_{sx}(T-t,s,x)}{v^\pm_{xx}(T-t,s,x)} - \frac{(\mu-r)}{\sigma^2}\frac{v^\pm_x(T-t,s,x)}{x\,v^\pm_{xx}(T-t,s,x)} \tag{3.85}$$

and

$$\hat{c}^\pm(t,s,x) = \frac{\left(v^\pm_x(T-t,s,x)\right)^{-\frac{1}{1-\gamma}}}{x} \tag{3.86}$$

are bounded. Then the value function of the control problem takes the form

$$v(T,s,x) = \begin{cases} v^-(T,s,x), & s\le z,\\ v^+(T,s,x), & s>z,\end{cases} \tag{3.87}$$

and the optimal controls $(\hat{c},\hat{\pi})\in\mathcal{A}$ are given by

$$\begin{aligned}\hat{\pi}_t =& \frac{(\mu-r)}{\sigma^2}\frac{1}{1-\gamma}\\ &-\left[\frac{\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa\left(\kappa P\left(T-t,\frac{s}{z}\right)+\frac{s}{z}Q\left(T-t,\frac{s}{z}\right)\right)}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa P\left(T-t,\frac{s}{z}\right)}\right]\mathbb{1}_{\{S_t<z\}}\\ &-\left[\frac{\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa\left(\kappa P\left(T-t,\frac{z}{s}\right)-\frac{z}{s}Q\left(T-t,\frac{z}{s}\right)\right)}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa P\left(T-t,\frac{z}{s}\right)}\right]\mathbb{1}_{\{S_t>z\}}\\ &-\left[\frac{\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\kappa P\left(T-t,1\right)\right)}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}P\left(T-t,1\right)}\right]\mathbb{1}_{\{S_t=z\}}\end{aligned} \tag{3.88}$$

and

$$\begin{aligned}
\hat{c}_t = &\left[\frac{1}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa P\left(T-t,\frac{s}{z}\right)}\right]\mathbb{1}_{\{S_t<z\}} \\
&+\left[\frac{1}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa P\left(T-t,\frac{z}{s}\right)}\right]\mathbb{1}_{\{S_t>z\}} \\
&+\left[\frac{1}{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}P\left(T-t,1\right)}\right]\mathbb{1}_{\{S_t=z\}} \qquad (3.89)
\end{aligned}$$

Proof. We have established that using a suitable transformation, a function w that solves the HJB equation (3.9), (3.12) and (3.13) may be transformed into the heat equation with initial condition and boundary condition at the point $\bar{z} = \log z$ as in (3.65)–(3.67). Taking the Laplace transform of the differential equation (3.65)–(3.67) yields the ODE and boundary condition specified in Lemma 4. For the lower solution $\tilde{u}^-$, we can see that taking

$$\begin{aligned}
a &= \frac{(\zeta-\bar{z})}{\sqrt{\Sigma}}, \\
b &= -\frac{1}{\beta\sqrt{\Sigma}\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}, \\
c_1 &= \alpha+\Sigma\kappa^2-1, \\
c_2 &= \kappa^2\Sigma, \\
c_3 &= \alpha+\kappa^2\Sigma,
\end{aligned}$$

and

$$d = -\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\kappa\bar{z}},$$

the second term in (3.71) corresponds to the function (3.77) in Lemma 5, so its inverse Laplace transform is given by (3.79). The first term in (3.71) has inverse transform

$$e^{\kappa^2\Sigma\tau-\kappa\zeta}.$$

It is then clear that the inverse Laplace transform of $\tilde{u}^-$ can be written as the sum of these two solutions, which can be written as

$$u^-(\tau,\zeta) = e^{\kappa^2\Sigma\tau-\kappa\zeta}+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{\kappa^2\Sigma\tau-\kappa\bar{z}}\left[\phi_1(\tau,\zeta)+\phi_2(\tau,\zeta)+\phi_3(\tau,\zeta)\right],$$

where

$$\begin{aligned}
\phi_1(\tau,\zeta) = (\Upsilon_1+\Upsilon_2)&e^{\left(\beta^2\Sigma\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)^2-\kappa^2\Sigma\right)\tau+\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)(\zeta-\bar{z})} \\
&\times\left(\mathrm{Erf}\left[\frac{\frac{(\zeta-\bar{z})}{\sqrt{\Sigma}}+\beta\sqrt{\Sigma}\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)\tau}{2\sqrt{\tau}}\right]+1\right),
\end{aligned}$$

$$\phi_2(\tau,\zeta) = \Upsilon_1 \left[\frac{1}{2}\left(1+\frac{|\kappa|}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right) e^{|\kappa|(\zeta-\bar{z})}\left(\operatorname{Erf}\left(\frac{\frac{-(\zeta-\bar{z})}{\sqrt{\Sigma}}-2|\kappa|\sqrt{\Sigma}\tau}{2\sqrt{\tau}}\right)-1\right)\right.$$
$$\left. +\frac{1}{2}\left(1-\frac{|\kappa|}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right) e^{-|\kappa|(\zeta-\bar{z})}\left(\operatorname{Erf}\left(\frac{\frac{-(\zeta-\bar{z})}{\sqrt{\Sigma}}+2|\kappa|\sqrt{\Sigma}\tau}{2\sqrt{\tau}}\right)-1\right)\right]$$

and

$$\phi_3(\tau,\zeta) = \Upsilon_2 e^{\alpha\tau}\left[\frac{1}{2}\left(1+\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)e^{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}(\zeta-\bar{z})}\right.$$
$$\times\left(\operatorname{Erf}\left(\frac{\frac{-(\zeta-\bar{z})}{\sqrt{\Sigma}}-2\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}\tau}{2\sqrt{\tau}}\right)-1\right)$$
$$+\frac{1}{2}\left(1-\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)e^{-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}(\zeta-\bar{z})}$$
$$\left.\times\left(\operatorname{Erf}\left(\frac{\frac{-(\zeta-\bar{z})}{\sqrt{\Sigma}}+2\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}\sqrt{\Sigma}\tau}{2\sqrt{\tau}}\right)-1\right)\right],$$

in which expressions the constants Υ_1 and Υ_2 are defined by (3.81) and (3.82).

Next, we reverse the transformation (3.60) from h to u by substituting in our solution to u^- to retrieve h^-, which combined with the observation that $e^{n(\zeta-\bar{z})} = \left(\frac{e^\zeta}{e^{\bar{z}}}\right)^n = \left(\frac{s}{z}\right)^n$ allows us to write

$$h^-(t,s) = \frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)$$
$$+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa\left[\Phi_1\left(T-t,\frac{s}{z}\right)+\Phi_2\left(T-t,\frac{s}{z}\right)+\Phi_3\left(T-t,\frac{s}{z}\right)\right].$$

Finally by reversing the first transformation that we made in (3.51) we are able to retrieve the value function w in the form

$$w^-(t,s,x) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right)\right.$$
$$\left.+\frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}}e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa\left[\Phi_1\left(T-t,\frac{s}{z}\right)+\Phi_2\left(T-t,\frac{s}{z}\right)+\Phi_3\left(T-t,\frac{s}{z}\right)\right]\right\}^{1-\gamma}.$$

By following a similar argument we can derive the expression

$$w^{+}(t,s,x) = \frac{x^\gamma}{\gamma}\Bigg\{ \frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right) + \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}} e^{-\alpha(T-t)}\left(\frac{s}{z}\right)^\kappa \left[\Phi_1\left(T-t,\frac{z}{s}\right)+\Phi_2\left(T-t,\frac{z}{s}\right)+\Phi_3\left(T-t,\frac{z}{s}\right)\right]\Bigg\}^{1-\gamma}.$$

It is worth noting that at the boundary $s = z$ we have, the continuity of w provides the expression

$$w(t,z,x) = \frac{x^\gamma}{\gamma}\Bigg\{ \frac{1}{\alpha}\left((\alpha-1)e^{-\alpha(T-t)}+1\right) + \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}} e^{-\alpha(T-t)}\left[\Phi_1(T-t,1)+\Phi_2(T-t,1)+\Phi_3(T-t,1)\right]\Bigg\}^{1-\gamma}.$$

We notice that

$$|\Phi_1(t,l)| \le 2\,|\Upsilon_1+\Upsilon_2|\, e^{\left(\beta^2\Sigma\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)^2-\kappa^2\Sigma\right)(T-t)},$$

$$|\Phi_2(t,l)| \le \left|\Upsilon_1\left(1+\frac{|\kappa|}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\right| + \left|\Upsilon_1\left(1-\frac{|\kappa|}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\right|$$

and

$$|\Phi_3(t,l)| \le e^{\alpha(T-t)}\left|\Upsilon_2\left(1+\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\right| + e^{\alpha(T-t)}\left|\Upsilon_2\left(1-\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\right|$$

for any $l \in [0,1]$. From these inequalities and from (3.25) and (3.27), it is clear that (3.83) and (3.84) satisfy the growth condition (3.20). Furthermore, the assumption that the functions $\hat{c}$ and $\hat{\pi}$ defined by (3.86) and (3.85) are bounded implies that the processes $\hat{\pi}$ and $\hat{c}$ defined by (3.88) and (3.89) are admissible.

□

Corollary 3. *Away from the boundary z the value function converges to the value function of the classic Merton portfolio problem.*

Proof. This follows immediately from the observation that for constants a and b and any fixed $T > 0$

$$\lim_{l\to\infty} l^a \,\mathrm{Erf}\left(\frac{-\frac{\ln l}{\sqrt{\Sigma}}+bT}{2\sqrt{T}}\right) = -1$$

and

$$\lim_{l\to\infty} l^a \operatorname{Erf}\left(\frac{\frac{\ln l}{\sqrt{\Sigma}} + bT}{2\sqrt{T}}\right) = 1.$$

Therefore,

$$\lim_{s\to\infty} v^+(T,s,x) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha T}+1\right)\right\}^{1-\gamma}$$

and

$$\lim_{s\to 0} v^-(T,s,x) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha}\left((\alpha-1)e^{-\alpha T}+1\right)\right\}^{1-\gamma}.$$

□

Remark 17. Considering the contrapositive of Corollary 3, as the asset price S approaches the level z the value function, and hence the portfolio process π and consumption rate c begin to differ from that of the classic Merton problem. Recall from (3.55) that this difference depends on the ratio of h and its first derivate in s, specifically $\frac{h_s(.)}{h(.)}$. It turns out that this term, while it can be written in explicit form, is complicated and does not lead to any useful intuition about the behaviour of the risky asset holdings π in the solution. We will therefore defer any detailed analysis of the solution form until we have studied the infinite horizon case.

5 The Solution to the Portfolio Problem over an Infinite Time Horizon $T = \infty$

We now consider the problem of maximising the objective (3.15) over all admissible controls $(c,\pi) \in \mathcal{A}_p$ subject to the stochastic dynamics given by (3.4) and (3.5). For easier reference, recall that the HJB equation (3.17) takes the form

$$\begin{aligned}-\frac{1}{2}\sigma^2 s^2 \frac{w_{sx}^2(s,x)}{w_{xx}(s,x)} - s(\mu-r)\frac{w_x(s,x)w_{sx}(s,x)}{w_{xx}(s,x)} - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{w_x^2(s,x)}{w_{xx}(s,x)} \\ + \frac{1}{2}\sigma^2 s^2 w_{ss}(s,x) + \frac{1-\gamma}{\gamma} w_x^{\frac{\gamma}{\gamma-1}}(s,x) \\ + rxw_x(s,x) + \mu s w_s(s,x) - \rho w(s,x) = 0, \qquad (3.90)\end{aligned}$$

with boundary condition at the level $s = z$ given by

$$\begin{aligned}\frac{\beta}{z}x\pi_u(w_x(z-,x) + w_x(z+,x)) \\ + ((1+\beta)w_s(z-,x) - (1-\beta)w_s(z-,x)) = 0. \qquad (3.91)\end{aligned}$$

To derive the solution to this PDE, we consider functions of the form

$$w(s,x) = \frac{x^\gamma}{\gamma} h^{1-\gamma}(s) \tag{3.92}$$

which result in the linear equation

$$\begin{aligned}\frac{1}{2}\sigma^2 s^2 h_{ss}(s) + \left(\frac{\mu - r}{1-\gamma} + r\right) s h_s(s) \\ - \frac{1}{1-\gamma}\left(\rho - r\gamma - \frac{\frac{1}{2}(\mu - r)^2}{\sigma^2}\frac{\gamma}{1-\gamma}\right) h(s) + 1 = 0.\end{aligned} \tag{3.93}$$

At the boundary $s = z$ our solution must satisfy the boundary condition (3.91). Following the argument made in (3.54), (3.55) and (3.57) with regard to the form of π at the boundary, we can see that the choices of controls π and c that achieve the maximum in (3.93) are given by

$$\hat{\pi}(s,x) = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma} + s\frac{h_s(s)}{h(s)} \tag{3.94}$$

for $s \neq z$, and

$$\hat{c}(s,x) = \frac{1}{h(s)}. \tag{3.95}$$

In particular, $\hat{\pi}$ and $\hat{c}$ do not depend on x. By substituting these into the original boundary condition (3.91), we arrive at the boundary condition for h at $s = z$

$$\frac{\beta}{z}\mu_\alpha h(z) + \frac{1}{2}\left(1 + \frac{\beta}{1-\gamma}\right) h_s(z+) - \frac{1}{2}\left(1 - \frac{\beta}{1-\gamma}\right) h_s(z-) = 0, \tag{3.96}$$

where

$$\mu_\alpha = \frac{\mu - r}{\sigma^2}\frac{\gamma}{(1-\gamma)^2}. \tag{3.97}$$

This ODE has general solution given by

$$h(s) = \frac{1}{\alpha} + A s^n + B s^m,$$

for some constants $A, B \in \mathbb{R}$, where $m < 0 < n$ are defined by

$$\begin{aligned} n, m &= -\frac{\left(\frac{\mu - r}{1-\gamma} + r - \frac{1}{2}\sigma^2\right)}{\sigma^2} \pm \frac{1}{\sigma^2}\sqrt{\left(\frac{\mu - r}{1-\gamma} + r - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\alpha} \\ &:= \kappa \pm \sqrt{\kappa^2 + \frac{\alpha}{\frac{1}{2}\sigma^2}}, \end{aligned}$$

where

$$\alpha = \frac{\rho - r\gamma}{1-\gamma} - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{\gamma}{(1-\gamma)^2}.$$

Assumption 11. *The model data is such that*

$$\alpha > 0. \tag{3.98}$$

Furthermore,

$$C := \frac{-\beta\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1 + \frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1 - \frac{\beta}{1-\gamma}\right)} > -1. \tag{3.99}$$

The following result reveals that, in the absence of conditions such as the ones in this assumption, the value function may be infinite.

Lemma 6. *If* (3.98) *in Assumption 11 fails, $\mu > r$ and $\beta > 0$, the value function of the control problem*

$$v(s,x) = \infty.$$

for all $s, x > 0$.

Proof. Suppose that the agent starts with initial endowment x at time 0 and sets π and c to be constant for all $t \in [0,\infty)$. In this case the dynamics of the wealth process X are given by the SDE

$$X_t = x + \int_0^t \left((\mu - r)\pi + r - c\right) X_t dt + \int_0^t \sigma\pi X_t dW_t + \int_0^t \frac{\beta}{z}\pi X_t dL_t^x(S_t),$$

which admits the solution

$$X_t = x\exp\left(\left((\mu - r)\pi + r - c - \frac{1}{2}\sigma^2\pi^2\right)t + \frac{\beta}{z}\pi L_t^z(S) + \sigma\pi W_t\right).$$

Substituting this expression into the objective function (3.15) we see that

$$J_{\infty,s,x}(c,\pi) = x\frac{c^\gamma}{\gamma}\mathbb{E}\Bigg[\int_0^\infty \exp\Bigg(-\rho t + \gamma\left((\mu - r)\pi + r - c - \frac{1}{2}\sigma^2\pi^2\right)t + \gamma\frac{\beta}{z}\pi L_t^z(S) + \sigma\gamma\pi W_t\Bigg)dt\Bigg].$$

Setting

$$\pi = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma} \geq 0$$

it is clear that there exists $c > 0$ small enough that if Assumption 11 fails to hold

$$\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{\gamma}{1-\gamma} + r\gamma - c - \rho > 0$$

and hence,

$$J_{\infty,s,x}(c,\pi) \geq x\frac{c^\gamma}{\gamma}\int_0^\infty \exp\left(\gamma\left((\mu - r)\pi + r - c - \frac{1}{2}\sigma^2(1-\gamma)\pi^2 - \frac{\rho}{\gamma}\right)t\right)dt = \infty.$$

□

Remark 18. In Lemma 11 we restricted attention to the case $\beta > 0$ so as to simplify our demonstration that the value function may be infinite. We expect that this is also the case when $\beta < 0$ however the local time appearing in the expectation complicates the analysis.

We can now prove the main result of this section.

Theorem 16. *Consider the control problem with formulated in Section 2.3, and suppose that Assumption 11 holds. The value function of the control problem admits the expression*

$$v(x,s) = \begin{cases} v^-(x,s), & s \leq z, \\ v^+(x,s), & s > z, \end{cases} \tag{3.100}$$

where

$$v^-(x,s) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha} + \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)}\left(\frac{s}{z}\right)^n\right\}^{1-\gamma} \tag{3.101}$$

and

$$v^+(x,s) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha} + \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)}\left(\frac{s}{z}\right)^m\right\}^{1-\gamma}. \tag{3.102}$$

Furthermore, the optimal controls $(\hat{\pi}, \hat{\delta}) \in \mathcal{A}_p$ *are given by*

$$\begin{aligned}\hat{\pi}_t = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma} &+ \frac{nC\left(\frac{S_t}{z}\right)^n}{1+C\left(\frac{S_t}{z}\right)^n}\mathbb{1}_{\{S_t<z\}} + \frac{mC\left(\frac{S_t}{z}\right)^m}{1+C\left(\frac{S_t}{z}\right)^m}\mathbb{1}_{\{S_t>z\}} \\ &+ \frac{(n+m)}{2}\frac{C}{1+C}\mathbb{1}_{\{S_t=z\}}\end{aligned} \tag{3.103}$$

and

$$\hat{c}_t = \frac{\alpha}{1+C\left(\frac{S_t}{z}\right)^n}\mathbb{1}_{\{S_t<z\}} + \frac{\alpha}{1+C\left(\frac{S_t}{z}\right)^m}\mathbb{1}_{\{S_t>z\}} + \frac{\alpha}{1+C}\mathbb{1}_{\{S_t=z\}} \tag{3.104}$$

where μ_α *and* $C > -1$ *are defined by* (3.97) *and* (3.99).

Proof. We consider solutions to (3.93) of the form

$$h^-(s) = \frac{1}{\alpha} + As^n$$

and

$$h^+(s) = \frac{1}{\alpha} + Bs^m.$$

The continuity of h yields the equation

$$Az^n = Bz^m.$$

Using the boundary condition and (3.96) solving for A and B we see that

$$A = \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)} z^{-n}$$

and

$$B = \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)} z^{-m}.$$

It follows that

$$h^-(s) = \frac{1}{\alpha} + \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)} \left(\frac{s}{z}\right)^n$$

and

$$h^+(s) = \frac{1}{\alpha} + \frac{-\frac{\beta}{\alpha}\mu_\alpha}{\beta\mu_\alpha + \frac{m}{2}\left(1+\frac{\beta}{1-\gamma}\right) - \frac{n}{2}\left(1-\frac{\beta}{1-\gamma}\right)} \left(\frac{s}{z}\right)^m.$$

Reversing the transformation by substituting h^- and h^+ back into (3.92) we see that v^+ and v^- admit the expressions given by (3.101) and (3.102).

It is immediately clear from (3.101) and (3.102) that the growth conditions (3.36) and (3.37) are satisfied. Moreover, $\hat{\pi}^-$, $\hat{\pi}^+$, $\hat{c}^-$ and $\hat{c}^+$ are bounded and $(\hat{c}, \hat{\pi}) \in \mathcal{A}_p$.

□

We are now able to observe some features of the agent's optimal control around the level $s = z$. Specifically recall that (3.94) and (3.95) imply that the optimal portfolio weight as a function of the underlying asset price is characterised by the expression

$$\pi(s,x) = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma} + n\frac{C\left(\frac{s}{z}\right)^n}{1 + C\left(\frac{s}{z}\right)^n}$$

below the level $s = z$ and by the expression

$$\pi(s,x) = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma} + m\frac{C\left(\frac{s}{z}\right)^m}{1 + C\left(\frac{s}{z}\right)^m}$$

above the level $s = z$. For the case with $\beta > 0$ we can see that the portfolio process π becomes greater than the Merton solution $\frac{\mu-r}{\sigma^2}\frac{1}{1-\gamma}$ below the level, peaking at the level and then reversing, so that π is less than the Merton solution above the level.

We can interpret this as the agent adding additional holdings of the risky asset when below the level in order to take advantage of the potential upward movement in the asset price that is more likely to occur than not due to the positive β. This effect can be seen in Figure 3.1. It is an interesting feature of the solution that holdings of the risky asset will then be reduced to levels lower than that of the standard Merton solution when the

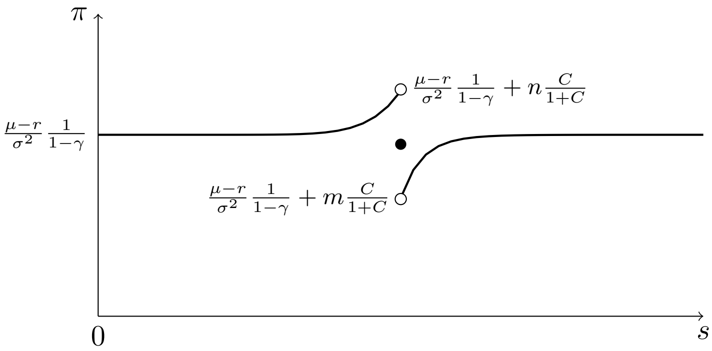

Figure 3.1: Optimal portfolio weight π as a function of the underlying asset price s when $\beta > 0$

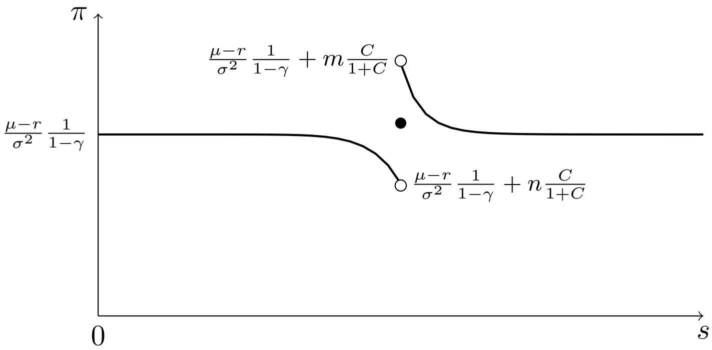

Figure 3.2: Optimal portfolio weight π as a function of the underlying asset price s when $\beta < 0$

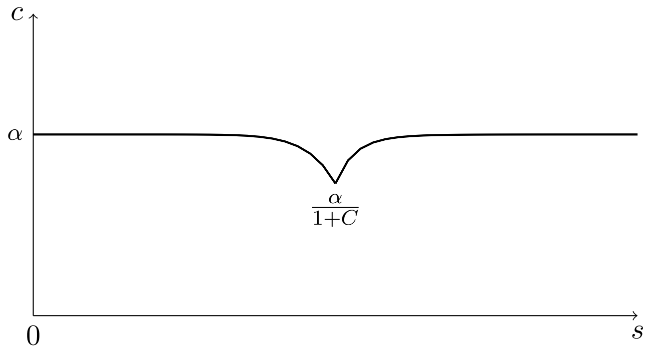

Figure 3.3: Optimal consumption rate c as a function of the underlying asset price s when $\beta > 0$

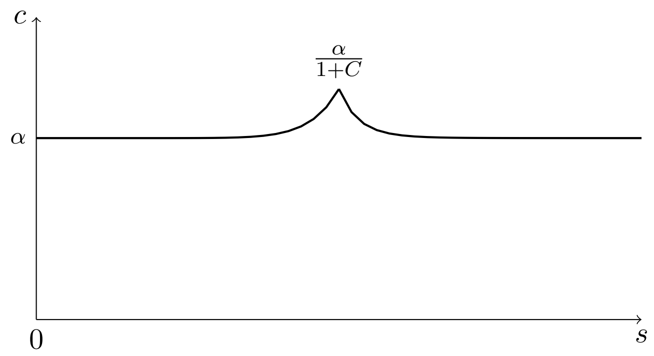

Figure 3.4: Optimal consumption rate c as a function of the underlying asset price s when $\beta < 0$

price S_t is just above the level z. We can see from (3.94) that this is due to those holding being related to the first derivative of h in s. We can also see that when $\beta < 0$ the effect is reversed, see Figure 3.2.

It is interesting to note that the size of the jump in the portfolio process π at $s = z$ can be written as

$$\frac{2\beta}{\frac{1-\gamma}{\gamma} - \frac{\beta}{\gamma}\frac{(n+m)}{(n-m)}} \frac{\mu - r}{\sigma^2} \frac{1}{1-\gamma}.$$

As the price approaches the level z from below, the portfolio holdings of the agent differ from those of the Merton problem by a multiple of

$$(1+n)\frac{2\beta}{\frac{1-\gamma}{\gamma}(n-m) - \frac{\beta}{\gamma}(n+m)},$$

and on the other hand, as the price approaches the level z from above, the portfolio holdings of the agent differ from those of the Merton problem by a multiple of

$$(1+m)\frac{2\beta}{\frac{1-\gamma}{\gamma}(n-m) - \frac{\beta}{\gamma}(n+m)}.$$

It is also easy to see that for $\beta = 0$ the jump in the portfolio process π and in the first derivate of c at z disappear and the problem reduces to the classic Merton problem as expected. Another interesting observation is that as β approaches 1, the size of the jump in the portfolio process π at z remains finite, despite the consideration made in Remark 15 that under these conditions the level at z acts like an impermeable barrier. Of course we cannot claim anything about the behaviour of the problem at $\beta = 1$ because strong solutions to the process S exist only for $|\beta| < 1$. However it is interesting to observe that

$$\lim_{\beta \to 1} n\frac{C}{1+C} = n\frac{\mu_\alpha}{\frac{\gamma}{1-\gamma}(n+m)}, \quad \lim_{\beta \to 1} n\frac{C}{1+C} = m\frac{\mu_\alpha}{\frac{\gamma}{1-\gamma}(n+m)}$$

so that the jump size is

$$\frac{(n-m)}{(n+m)}\frac{\mu-r}{\sigma^2}\frac{1}{1-\gamma}.$$

This calculation reveals that the finite jump size is related to the risk aversion of the agent, and see that as $\gamma \to 1$ the jump size goes to ∞.

Finally, Figure 3.3 and Figure 3.4 show that the consumptionrate c as a function of the price s. Around the level $s = z$ this rate of consumption, being inversely proportional to the function h, decreases if $\beta > 0$ and increases if $\beta > 0$. Intuitively this inverse relationship with the value function is expected, since when the agent is able to derive excess value from the presence of the level in the market he should consume less quickly in order to invest more in the risky asset, and vice versa.

Corollary 4. *As $T \to \infty$, the value function of the finite horizon problem converges to the value function of the infinite horizon problem*

Proof. It is straightforward to see that since

$$\lim_{T\to\infty} \frac{1}{\alpha}\left((\alpha-1)e^{\alpha(T-t)}+1\right) = \frac{1}{\alpha},$$

$$\lim_{T\to\infty} \Phi_1\left(T, \frac{s}{z}\right) = 0$$

and

$$\lim_{T\to\infty} \Phi_2\left(T, \frac{s}{z}\right) = 0.$$

Furthermore,

$$\begin{aligned}\lim_{T\to\infty} \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}} e^{-\alpha T}\left(\frac{s}{z}\right)^{\kappa} \Phi_3\left(T, \frac{s}{z}\right) &= \Upsilon_2\left(1+\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\left(\frac{s}{z}\right)^{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}} \\ &= \left(\frac{\frac{\beta}{\alpha}\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}\right)\left(\frac{s}{z}\right)^{\kappa+\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}\end{aligned}$$

and

$$\begin{aligned}\lim_{T\to\infty} \frac{\mu_\alpha}{\mu_\alpha+\frac{\kappa}{1-\gamma}} e^{-\alpha T}\left(\frac{s}{z}\right)^{\kappa} \Phi_3\left(T, \frac{z}{s}\right) &= \Upsilon_2\left(1+\frac{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)}\right)\left(\frac{s}{z}\right)^{\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}} \\ &= \left(\frac{\frac{\beta}{\alpha}\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}\right)\left(\frac{s}{z}\right)^{\kappa-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}.\end{aligned}$$

It follows that

$$\lim_{T\to\infty} v^-(T,s,x) = \frac{x^\gamma}{\gamma}\left\{\frac{1}{\alpha}+\left(\frac{\frac{\beta}{\alpha}\mu_\alpha}{\beta\left(\mu_\alpha+\frac{\kappa}{1-\gamma}\right)-\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}\right)\left(\frac{s}{z}\right)^{\kappa+\sqrt{\frac{\alpha}{\Sigma}+\kappa^2}}\right\}^{1-\gamma}$$

and

$$\lim_{T\to\infty} v^{+}(T,s,x) = \frac{x^{\gamma}}{\gamma}\left\{\frac{1}{\alpha} + \left(\frac{\frac{\beta}{\alpha}\mu_{\alpha}}{\beta\left(\mu_{\alpha}+\frac{\kappa}{1-\gamma}\right)-\sqrt{\frac{\alpha}{\Sigma}+\kappa^{2}}}\right)\left(\frac{s}{z}\right)^{\kappa-\sqrt{\frac{\alpha}{\Sigma}+\kappa^{2}}}\right\}^{1-\gamma}$$

which identify with the expressions for the value function v in the infinite horizon case. $\square$

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