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## A bifurcated circular waveguide problem

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A rigorous and exact solution is obtained for the problem of the radiation of sound from a semi-infinite rigid duct inserted axially into a larger acoustically lined tube of infinite length. The solution to this problem is obtained by the Wiener–Hopf technique. The transmission and reflection coefficients, when the fundamental mode propagates in the semi-infinite tube, are obtained. The present results could be of use for exhaust design, and as a possible instrument for impedance measurement.

### 1. Introduction

In our present industrial environment, situations often arise where noise generated by a particular source propagates through ducts to produce unwanted noise in locations removed from the source. It is necessary to try to eliminate this unwanted noise in, for example, architectural acoustics (duct noise produced in heating and ventilation systems), experimental aerodynamics (noise propagation wind tunnels), aircraft transport (noise from aircraft jets and turbo fan engines), and, of major importance in recent years, road transportation (exhaust noise from internal combustion engines). One method of reducing noise is to introduce expansion chambers to muffle the noise as it travels along the duct. The introduction of acoustically absorbent lining into the duct is another method that has proved useful in reducing unwanted noise (see Rawlins, 1978). The insertion of expansion chambers with acoustically lined walls is yet another effective method of reducing sound radiated from duct terminations of motor cars and lorries. We are interested in the effect of lining an exhaust chamber to reduce noise exiting from an exhaust. To try to solve this problem theoretically ‘head-on’ would be extremely complicated. A typical exhaust chamber is depicted in Fig. 1.

The theoretical or numerical analysis of the system shown in Fig. 1 would be extremely complicated because of the large number of design parameters. An approach for analysing the system shown in Fig. 1 is the so called ‘building block method’ (see Nilsson & Branders, 1980a,b). This assumes that the longitudinal lengths between the various discontinuities is such that the canonical problem for each discontinuity can be considered in isolation. When the various discontinuities which make up the exhaust system have been separately analysed, these separate field calculations can be combined by an effective matching procedure to produce a composite result. Such an approach has been successfully applied by Nilsson & Branders (1980a,b, 1981a,b) and Taylor *et al.* (1993).

In this work we shall obtain a solution to the discontinuity problem shown in Fig. 2. An incident mode propagates in the smaller tube of radius  $a$  and this

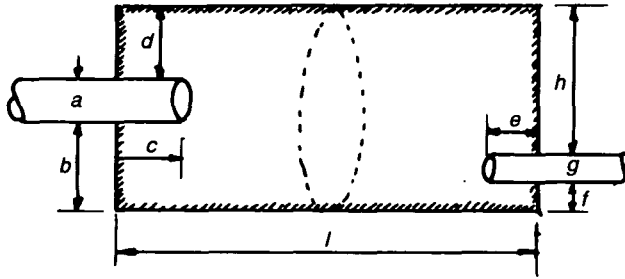


FIG. 1. An exhaust muffler.

enters an infinite duct of radius  $b$  which is lined with acoustically absorbent material. This particular problem may also be of some interest as a mathematical model of an instrument for the measurement of the acoustic lining properties from observation of the acoustic field reflected into the semi-infinite tube. Problems of a similar nature have been considered by Bailin (1951) and Vainshtein (1948), who assume all the surfaces are either rigid or soft. The latter boundary condition has applications in electromagnetism. The present work would have applications in electromagnetism when dealing with waveguides with lossy walls.

The present problem is solved by means of a relatively standard Wiener-Hopf approach and an exact expression obtained for the reflection coefficient. In Section 2 we shall formulate the mathematical boundary value problem. In Section 3 the problem formulated in Section 2 will be solved by means of the Wiener-Hopf technique. The solution will be in the form of complex contour integrals. These integrals will be evaluated in Section 4 by an application of Cauchy's residue theorem. This produces a representation for the acoustic field in the various regions as an infinite series of modes. In Section 5 the fields in the various regions are obtained when only the fundamental mode propagates in the semi-infinite duct. This will result in expressions for the reflection and transmission coefficient of the dominant propagating mode. The effect of the absorbent lining on the reflection coefficient is then analysed for the low-frequency situation. Finally, various appendices are supplied. These contain analytical details required in the main body of the paper.

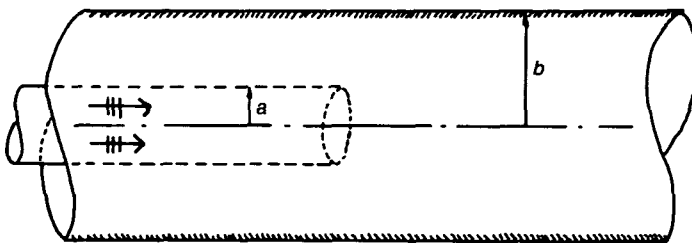


FIG. 2. Geometry of the diffraction problem.

**2. Formulation of the boundary value problem**

We shall consider the acoustic diffraction of a plane wave mode propagating out of the open end of a semi-infinite rigid cylindrical tube. This semi-infinite tube is surrounded by an infinite cylindrical casing which is lined with an acoustically absorbing material. The cylinder casing and its lining are located at  $\{r = b, -\infty < z < \infty\}$ , and the semi-infinite rigid cylinder, assumed infinitely thin, is located at  $\{r = a < b, -\infty < z < 0\}$ , in cylindrical polar coordinates  $(r, \theta, z)$ , as shown in Fig. 2. The sound source field, which is located at  $z = z_0$  ( $z_0 < 0$ ), propagates cylindrically symmetric modes along the rigid tube. Therefore the source field may be represented as plane wave modes independent of  $\theta$ . Such a situation arises, for example, when the source field is a point source located at  $(0, 0, z_0)$ . From the symmetry of the geometry of the problem and of the incident field, the acoustic field everywhere will be independent of  $\theta$ . We shall therefore introduce a scalar complex potential of the form  $\psi(r, z; t)$  which defines the acoustic pressure and velocity by  $p = -\rho_0 \partial\psi/\partial t$  and  $\mathbf{u} = \text{grad } \psi$  respectively, where  $\rho_0$  is the density of the undisturbed medium. The acoustic impedance  $Z$  of the lining is defined by the ratio  $Z = p/(\mathbf{u} \cdot \hat{\mathbf{n}})$  (see Morse & Ingard, 1961), where the normal  $\hat{\mathbf{n}}$  is directed into the lining. Thus, in terms of the velocity potential function  $\psi$ , the boundary condition on the absorbent surface is given by  $(\hat{\mathbf{n}} \cdot \text{grad} + \Xi c^{-1} \partial/\partial t)\psi = 0$ , where  $\Xi (= \rho_0 c/Z)$  is the specific admittance and  $c$  is the velocity of sound.

The incident sound field which can propagate down the tube is taken to be

$$\psi_0(r, z; t) = e^{-i\omega t} \phi_0(r, z) = A_0 J_0(\alpha_n r) e^{i\chi_n z - i\omega t} \quad (r < a, -\infty < z \leq 0), \tag{2.1}$$

where  $A_0 = e^{-i\chi_n z_0}$ , and  $\alpha_n$  are the real roots of the equation

$$J_1(a\alpha_n) = 0, \tag{2.2}$$

with  $\chi_n = (k^2 - \alpha_n^2)^{1/2}$ ,  $k = \omega/c$ ,  $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \dots$ . Having specified the incident field, we now set up the boundary value problem for the potential field everywhere:  $\psi(r, z; t) = e^{-i\omega t} \phi(r, z)$ . The time factor  $e^{-i\omega t}$  will be dropped in future calculations.

Thus, for such a primary field, if  $\phi(r, z) = \phi_0(r, z) + \hat{\phi}(r, z)$  is the total field in  $r < a$  and  $\phi(r, z)$  the total field in  $a < r < b$ , then  $\phi(r, z)$  must satisfy the following:

$$\begin{aligned} \text{(i)} \quad & \left. \begin{aligned} (\nabla^2 + k^2)\phi(r, z) &= 0 \quad (0 \leq r \leq a) \\ (\nabla^2 + k^2)\phi(r, z) &= 0 \quad (a \leq r \leq b) \end{aligned} \right\} \quad (|z| < \infty), \\ \text{(ii)} \quad & \frac{\partial \phi}{\partial r}(a^-, z) = \frac{\partial \phi}{\partial r}(a^+, z) = 0 \quad (z < 0), \\ \text{(iii)} \quad & \left. \begin{aligned} \frac{\partial \phi}{\partial r}(a^+, z) &= \frac{\partial}{\partial r} \phi(a^-, z) \\ \left( \frac{\partial}{\partial r} - ik\Xi \right) \phi(b^-, z) &= 0 \quad (\text{Re } \Xi > 0) \end{aligned} \right\} \quad (-\infty < z < \infty), \end{aligned}$$

$$(iv) \quad \phi(a^-, z) = \phi(a^+, z) \quad (z > 0).$$

To the above conditions we add those radiation conditions at infinity which are relevant to the nature of the propagating modes that the various duct regions can sustain. In Appendix A it is shown that:

(v) For  $z \rightarrow -\infty$ ,  $0 \leq r \leq a$ ,

$$\phi(r, z) - \phi_0(r, z) = R_1(r)e^{-ikx} + O(e^{-i\chi_1 z}),$$

where  $\chi_1[k^2 - (3.832/a)^2]^{\frac{1}{2}}$ ,  $\chi_n = (k^2 - \alpha_n^2)^{\frac{1}{2}}$ , with  $J_1(\alpha_n a) = 0$  ( $n = 0, 1, 2, \dots$ ).

For  $z \rightarrow -\infty$ ,  $a \leq r \leq b$ ,

$$\phi(r, z) = R_2(r)e^{-i\eta_1 z} + O(e^{-i\eta_2 z}).$$

For  $z \rightarrow \infty$ ,  $0 \leq r \leq b$ ,

$$\phi(r, z) = T(r)e^{i\xi_1 z} + O(e^{i\xi_2 z}),$$

where precise details about  $\chi_n$ ,  $\eta_n$ , and  $\xi_n$  ( $n = 1, 2, \dots$ ) are given in the appendix.

Finally, we require the edge field behaviour at  $r = a$ ,  $x \rightarrow 0$ . Namely

$$(vi) \quad \phi(a, z) = O(1), \quad \left| \frac{\partial}{\partial r} \phi(a, z) \right| = O(z^{-\frac{1}{2}}) \quad \text{as } z \rightarrow 0.$$

The satisfaction of the conditions (i)–(vi) will result in a unique solution to the boundary value problem formulated.

### 3. Solution of the boundary value problem

For analytic convenience we shall assume that  $k = k_r + ik_i$  ( $k_r > k_i \geq 0$ ). A suitable representation for the total field  $\phi(r, z)$  in all space  $\{-\infty < z < \infty, r \leq b\}$  which satisfies (i) is given by

$$\phi(r, z) = \phi_0(r, z) + \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} A(v) J_0(\kappa r) dv \quad (r < a), \quad (3.1)$$

$$\phi(r, z) = \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} [B(v) J_0(\kappa r) + C(v) H_0^{(1)}(\kappa r)] dv \quad (a < r < b), \quad (3.2)$$

where  $\kappa = (k^2 - v^2)^{\frac{1}{2}}$  and the branch cuts are from  $k$  to  $i\infty$  and from  $-k$  to  $-i\infty$ . The cut sheet on which we shall work is defined by  $0 \leq \arg \kappa \leq \pi$ . The quantities  $A(v)$ ,  $B(v)$ , and  $C(v)$  are as yet unknown; however, the edge condition (vi) requires that, as  $|v| \rightarrow \infty$ ,

$$A(v) = O(|v|^{-1} e^{-a|v|}), \quad \frac{e^{a|v|}}{\sqrt{\pi}} B(v) + \sqrt{\pi} e^{-a|v|} C(v) = O(|v|^{-1}). \quad (3.3)$$

We shall see later that the integrands of (3.1) and (3.2) have poles that produce

exponential wave modes propagating down the ducts. These wave modes, in the various duct regions, must be of the form given in the radiation condition (v). This requires that the integrands of (3.1) and (3.2) have pole singularities at  $v = -k$  and  $v = -\eta_1$ ,  $v = \xi_1$  respectively. Thus the real parameter  $\tau$  in (3.1) and (3.2) is restricted by requiring that the asymptotic behaviour (v) is achieved. This necessitates that the contour of integration lies in a strip such that

$$\max \{-\text{Im } k, -\text{Im } \eta_1\} < \tau < \text{Im } \xi_1.$$

It is shown in Appendix B that  $-\text{Im } k < 0$ ,  $-\text{Im } \eta_1 < 0$ , and  $\text{Im } \xi_1 > 0$ , so such a strip does exist. Thus we have from Appendix B that the above field representations are such that the integrands have no singularities in the strip  $-\varepsilon < \text{Im } v < \varepsilon$  ( $\varepsilon > 0$ ), so that the field representations (3.1) and (3.2) will exist for  $-\varepsilon < \tau < \varepsilon$  ( $\varepsilon > 0$ ).

To determine  $A(v)$ ,  $B(v)$ , and  $C(v)$ , we substitute (3.1) and (3.2) into the remaining boundary conditions (ii)–(iv), giving:

$$\int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} A(v) \kappa J'_0(\kappa a) dv = \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} \kappa [B(v) J'_0(\kappa a) + C(v) H_0^{(1)'}(\kappa a)] dv = 0 \quad (z < 0), \quad (3.4)$$

$$\left. \begin{aligned} & \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} \kappa [B(v) J'_0(\kappa a) + C(v) H_0^{(1)'}(\kappa a)] dv \\ & = \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} A(v) \kappa J'_0(\kappa a) dv \\ & \int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} [B(v) J(b, v) + C(v) H(b, v)] dv = 0 \end{aligned} \right\} \quad (-\infty < z < \infty), \quad (3.5)$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} e^{ivz} \left( \frac{A_0 J_0(\alpha_n a)}{2\pi i (v - \chi_n)} + A(v) J_0(\kappa a) - B(v) J_0(\kappa a) - C(v) H_0^{(1)}(\kappa a) \right) dv = 0 \quad (z > 0), \quad (3.6)$$

where

$$\left. \begin{aligned} J(b, v) &= \kappa J'_0(\kappa b) - ik \Xi J_0(\kappa b), \\ H(b, v) &= \kappa H_0^{(1)'}(\kappa b) - ik \Xi H_0^{(1)}(\kappa b). \end{aligned} \right\} \quad (3.7)$$

A solution to the above system of equations can be written as

$$A(v) \kappa J'_0(\kappa a) = \kappa [B(v) J'_0(\kappa a) + C(v) H_0^{(1)'}(\kappa a)] = \Phi^-(v), \quad (3.8)$$

$$B(v) J(b, v) + C(v) H(b, v) = 0, \quad (3.9)$$

$$\frac{A_0 J_0(\alpha_n a)}{2\pi i (v - \chi_n)} + A(v) J_0(\kappa a) - B(v) J_0(\kappa a) - C(v) H_0^{(1)}(\kappa a) = \Phi^+(v), \quad (3.10)$$

where  $\Phi^\pm(v)$  are regular and analytic in  $\text{Im } v > -\varepsilon$  and  $\text{Im } v < \varepsilon$  respectively. By

eliminating  $A(v)$ ,  $B(v)$ , and  $C(v)$  from the equations (3.8)–(3.10) we get the standard Wiener–Hopf functional equation:

$$\frac{A_0 J_0(\alpha_n \alpha)}{2\pi i(v - \chi_n)} - \frac{2iJ(b, \mu)\phi^-(v)}{\pi \kappa^2 a D(a, b, v) J'_0(\kappa a)} = \Phi^+(v), \quad (3.11)$$

where

$$D(a, b, v) = J'_0(\kappa a)H(b, v) - H_0^{(1)'}(\kappa a)J(b, v). \quad (3.12)$$

Before we can go any further with the analysis of equation (3.11), we shall require the following asymptotic growth estimates as  $v \rightarrow \pm\infty$ :

$$\begin{aligned} \kappa &\sim i|v|, \\ J_0(\kappa a) &\sim \text{const. } J'_0(\kappa a) = O(|v|^{-\frac{1}{2}}e^{a|v|}), \\ H_0^{(1)}(\kappa a) &\sim \text{const. } H_0^{(1)'}(\kappa a) = O(|v|^{-\frac{1}{2}}e^{-a|v|}), \\ J(b, v) &= O(|v|^{\frac{1}{2}}e^{b|v|}), \quad H(b, v) = O(|v|^{\frac{1}{2}}e^{-b|v|}), \\ D(a, b, v) &= O(e^{(b-a)|v|}) \quad (b > a). \end{aligned}$$

These asymptotic estimates, together with (3.3) and (3.8)–(3.10), give, as  $|v| \rightarrow \infty$ ,

$$\left. \begin{aligned} \Phi^-(v) &= O(|v|^{-\frac{1}{2}}) \quad \text{for } \text{Im } v < \varepsilon, \\ \Phi^+(v) &= O(|v|^{-1}) \quad \text{for } \text{Im } v > -\varepsilon. \end{aligned} \right\} \quad (3.13)$$

By letting

$$K(v) = K_+(v)K_-(v) = \frac{-2iJ(b, v)}{\pi \kappa^2 a D(a, b, v) J'_0(\kappa a)}, \quad (3.14)$$

we can rewrite equation (3.11) in the form

$$\begin{aligned} \frac{A_0 J_0(\alpha_n a)}{2\pi i(v - \chi_n) K_+(\chi_n)} + K_-(v) \Phi^-(v) \\ = \frac{\Phi^+(v)}{K_+(v)} - \frac{A_0 J_0(\alpha_n a)}{2\pi i(v - \chi_n)} \left( \frac{1}{K_+(v)} - \frac{1}{K_+(\chi_n)} \right). \end{aligned} \quad (3.15)$$

The detailed factorization of  $K(v)$  defined by (3.14) is carried out in detail in Appendix C. In particular it is shown there that

$$K_{\pm}(v) = O(|v|^{-\frac{1}{2}}) \quad \text{as } |v| \rightarrow \infty \quad (3.16)$$

in their respective domain of analyticity.

By using the asymptotic estimates (3.13) and (3.16) it can be shown that the left-hand side of the equation (3.15) is regular, analytic, and asymptotic to  $O(|v|^{-1})$  as  $|v| \rightarrow \infty$  in  $\text{Im } v < \varepsilon$ . Similarly, the right-hand side is regular, analytic, and asymptotic to  $O(|v|^{-\frac{1}{2}})$  as  $|v| \rightarrow \infty$  in  $\text{Im } v > -\varepsilon$ . Hence, by Liouville's

theorem, the function which is the analytic continuation of both sides of (15) in the entire complex  $\nu$ -plane is the constant zero. Hence from (15) we have

$$\Phi^-(\nu) = \frac{-A_0 J_0(\alpha_n a)}{2\pi i(\nu - \chi_n) K_+(\chi_n) K_-(\nu)}, \tag{3.17}$$

which on substituting into (3.8) and (3.9) gives

$$A(\nu) = \frac{\Phi^-(\nu)}{\kappa J'_0(\kappa a)} = \frac{-A_0 J_0(\alpha_n a)}{2\pi i \kappa J'_0(\kappa a)(\nu - \chi_n) K_+(\chi_n) K_-(\nu)}, \tag{3.18}$$

$$B(\nu) = \frac{H(b, \nu)}{\kappa D(a, b, \nu)} \Phi^-(\nu) = \frac{-A_0 H(b, \nu) J_0(\alpha_n a)}{2\pi i(\nu - \chi_n) \kappa D(a, b, \nu) K_+(\chi_n) K_-(\nu)}, \tag{3.19}$$

$$C(\nu) = \frac{-J(b, \nu)}{\kappa D(a, b, \nu)} \Phi^-(\nu) = \frac{A_0 J(b, \nu) J_0(\alpha_n a)}{2\pi i(\nu - \chi_n) \kappa D(a, b, \nu) K_+(\chi_n) K_-(\nu)}. \tag{3.20}$$

Thus, the acoustic field everywhere is now known and given by substituting (3.18)–(3.20) into (3.1) and (3.2), giving

$$\phi(r, z) = A_0 J_0(\alpha_n r) e^{i\chi_n z} - \frac{A_0 J_0(\alpha_n)}{2\pi i K_+(\chi_n)} \int_{-\infty+i\tau}^{\infty+i\tau} e^{i\nu z} \frac{J_0(\kappa r) d\nu}{\kappa J'_0(\kappa a) K_-(\nu)(\nu - \chi_n)} \quad (r < a), \tag{3.21}$$

$$\phi(r, z) = \frac{-A_0 J_0(\alpha_n a)}{2\pi i K_+(\chi_n)} \int_{-\infty+i\tau}^{\infty+i\tau} e^{i\nu z} \frac{[H(b, \nu) J_0(\kappa r) - J(b, \nu) H_0^{(1)}(\kappa r)] d\nu}{\kappa D(a, b, \nu) K_-(\nu)(\nu - \chi_n)} \quad (a < r < b). \tag{3.22}$$

**4. Model field representation**

To get a physical realization of the acoustic field in the various regions of the ducts, we can convert the expressions (3.21) and (3.22) into series of propagating wave modes. This is achieved by closing the path of integration by a suitable contour and applying Cauchy’s residue theorem. In order to close the path of integration in (3.21) and (3.22) by an infinite semicircle in either  $\text{Im } \nu \geq \tau - \epsilon$  or  $\text{Im } \nu \leq \tau + \epsilon$ , we take cognizance of the fact that if

$$F(\nu) = \frac{J_0(\kappa r)}{\kappa J'_0(\kappa a) K_-(\nu)(\nu - \chi_n)} \tag{4.1}$$

and

$$G(\nu) = \frac{H(b, \nu) J_0(\kappa r) - J(b, \nu) H_0^{(1)}(\kappa r)}{\kappa D(a, b, \nu) K_-(\nu)(\nu - \chi_n)}, \tag{4.2}$$

then

$$F(\nu) = O(|\nu|^{-\frac{1}{2}}) \quad (0 \leq r \leq a) \tag{4.3}$$

and

$$G(v) = O(|v|^{-\frac{3}{2}}) \quad (a \leq r \leq b) \quad (4.4)$$

as  $|v| \rightarrow \infty$ . We also note (see Appendix C) that  $F(v)$  and  $G(v)$  have no branch point singularities in the entire  $v$ -plane. Thus an application of Jordan's lemma enables us to close the contour of integration in (3.21) and (3.22) by an infinite semicircle in either  $\text{Im } v \geq \tau > -\varepsilon$  or  $\text{Im } v \leq \tau < \varepsilon$  (depending on the sign of  $z$ ) without affecting the value of the integral. The value of the appropriate integral can then be determined by summing the residue contributions from the poles enclosed by the contour.

*Field in  $r < a$ ,  $z < 0$*

Thus, by enclosing the contour of integration in (3.21) by an infinite semicircle in  $\text{Im } v \leq \tau < \varepsilon$  and by summing residues from the only simple poles of  $F(v)$  enclosed, i.e.  $v = -\chi_n$  ( $n = 0, 1, \dots$ ) (see Appendix B), we obtain

$$\phi(r, z) = A_0 J_0(\alpha_n r) e^{-i\chi_n z} + \frac{A_0 J_0(\alpha_n a)}{a K_+(\chi_n)} \sum_{m=0}^{\infty} \frac{J_0(\alpha_m r) e^{-i\chi_m z}}{\chi_m (\chi_m + \chi_n) J_0(\alpha_n a) K_+(\chi_m)} \quad (0 \leq r \leq a, -\infty < z < 0). \quad (4.5)$$

*Field in  $r < a$ ,  $z > 0$*

If we close the contour of integration in (3.21) by an infinite semicircle in  $\text{Im } v \geq \tau \geq -\varepsilon$ , with  $z > 0$ , by using Jordan's lemma, and rewrite the integrand by means of (3.14), we have the equivalent representation for equation (3.21):

$$\phi(r, z) = A_0 J_0(\alpha_n r) e^{i\chi_n z} - \frac{A_0 J_0(\alpha_n a)}{2\pi i K_+(\chi_n)} \left( \frac{-\pi a}{2i} \right) \oint_{C_+} \frac{e^{ivz} \kappa J_0(\kappa r) D(a, b, v) K_+(v) dv}{(v - \chi_n) J(b, v)}, \quad (4.6)$$

where  $C_+$  is the infinite semicircle  $\{z = x + i\tau, -\infty < x < \infty\} \cup \{z = x + iy, |z| = R, y \geq \tau, R \rightarrow \infty\}$ . The only poles enclosed by  $C_+$  are  $v = \chi_n$  and the roots of  $J(b, v) = 0$ , i.e.  $v = \xi_m$  ( $m = 1, 2, \dots$ ) (see Appendix B). The residue contribution from the pole  $v = \chi_n$  exactly cancels the incident wave mode, and the contribution from the remaining poles gives the field

$$\phi(r, z) = -\frac{1}{2} i \pi \left( \frac{a}{b} \right) \frac{A_0 J_0(\alpha_n a)}{K_+(\chi_n)} \sum_{m=1}^{\infty} e^{i\xi_m z} \frac{\beta_m^3 J_0(\beta_m r) J_0'(\beta_m a) H(b, \xi_m) K_+(\xi_m)}{(\xi_m - \chi_n) \xi_m (\beta_m^2 - k^2 \Xi^2) J_0(\beta_m b)} \quad (0 \leq r \leq a, 0 < z < \infty), \quad (4.7)$$

where  $v = \xi_m$  are the roots of  $J(b, v) \equiv \kappa J_0'(\kappa b) - ik \Xi J_0(\kappa b) = 0$ , and  $\beta_m = (k^2 - \xi_m^2)^{\frac{1}{2}}$  ( $m = 1, 2, \dots$ ). The location of these roots is analysed in Appendix B.



Field in  $a < r < b, z < 0$

In the expression (3.22) in the region  $\text{Im } v \leq \tau < \varepsilon, K_-(v)$  and  $v - \chi_n$  do not vanish, and so the only singularities in this region are the poles corresponding to the zeros of  $\kappa D(a, b, v) = 0$ . If we close the contour of integration in (3.22) by an infinite semicircle in  $\text{Im } v \leq \tau < \varepsilon, \text{ with } z < 0$ , we get, on summing the residue contributions,

$$\phi(r, z) = -\frac{1}{2\pi i} \frac{A_0 J_0(\alpha_n a)}{K_+(\chi_n)} \sum_{m=1}^{\infty} \frac{e^{-i\eta_m z} \delta_m [H_0^{(1)'}(\delta_m a) J_0(\delta_m r) - H_0^{(1)'}(\delta_m r) J_0'(\delta_m a)]}{K_+(\eta_m) \eta_m (\chi_n + \eta_m) \{1 + (k^2 \Xi^2 - \delta_m^2) [J_0'(\delta_m a) / J(b, -\eta_m)]^2\}} \quad (a < r < b, z < 0), \quad (4.8)$$

where  $v = -\eta_m$  ( $m = 1, 2, \dots$ ) are the zeros of  $D(a, b, v) \equiv J_0'(\kappa a) H(b, v) - H_0^{(1)'}(\kappa a) J(b, v) = 0$  that lie in  $\text{Im } v \leq -\varepsilon, \delta_m = (k^2 - \eta_m^2)^{1/2}$ , and  $J(b, -\eta_m) = \delta_m J_0'(\delta_m b) - ik \Xi J_0(\delta_m b)$ . The disposition and nature of the poles  $\eta_m$  and  $\delta_m$  are analysed in Appendix B.

Field in  $a < r < b, z > 0$

If we close the contour of integration in (3.22) by an infinite semicircle in  $\text{Im } v \geq \tau > -\varepsilon, \text{ with } z > 0$ , by applying Jordan's lemma, and rewrite the integrand by means of (3.14), we have the equivalent representation for  $\phi(r, z), (a < r < b, z > 0)$ :

$$\phi(r, z) = \frac{A_0 J_0(\alpha_n a)}{2\pi i K_+(\chi_n)} \left( \frac{i\pi a}{2} \right) \oint_{C_+} \frac{e^{ivz} K_+(v) [H(b, v) J_0(\kappa r) - J(b, v) H_0^{(1)}(\kappa r)] \kappa J_0'(\kappa a) dv}{J(b, v)(v - \chi_n)},$$

where  $C_+$  is the closed semicircular contour  $\{z = x + i\tau, \tau > -\varepsilon, -\infty < x < \infty\} \cup \{z = x + iy, |z| = R, y \geq \tau > -\varepsilon, R \rightarrow \infty\}$ . There is no residue contribution from the apparent pole  $v = \chi_n$  because this pole is cancelled by the zero  $v = \chi_n$  of  $J_0'(\kappa a) = 0$ . Thus the only residue contributions arise from the zeros of  $J(b, v) = 0$ , i.e.  $v = \xi_m$  (see Appendix B.) Thus

$$\phi(r, z) = -\frac{A_0 J_0(\alpha_n a)}{2\pi i K_+(\chi_n)} \left( \frac{i\pi a}{2b} \right) \sum_{m=1}^{\infty} \frac{e^{i\xi_m z} K_+(\xi_m) H(b, \xi_m) J_0(\beta_m r) \beta_m^3 J_0'(\beta_m a)}{\xi_m (\xi_m - \chi_n) (\beta_m^2 - k^2 \Xi^2) J_0(\beta_m b)} \quad (a \leq r \leq b, 0 < z < \infty). \quad (4.9)$$

It is refreshing to notice that this last expression (4.9) is identical in form to that of (4.7), as we should expect it to be.

The physical interpretation of (4.5), (4.7), (4.8), and (4.9) can now be made.

### 5. Propagation of only the fundamental mode

If we restrict the dimensions of the semi-infinite tube such that  $0 < ka < 3.832\dots$ , then only the fundamental mode propagates along the semi-infinite cylinder. In this case, with  $\alpha_n = 0$ ,  $\chi_n = k$ , and  $A_0 = 1$ , the incident wave is given by  $\phi_0(r, z) = e^{ikz}$ ; then the total field  $\phi(r, z)$  in the various regions is given from (4.5), (4.7), (4.8), and (4.9) by

$$\begin{aligned} \phi(r, z) &= e^{ikz} + \frac{1}{aK_+(k)} \sum_{m=0}^{\infty} \frac{J_0(\alpha_m r) e^{-i\chi_m z}}{\chi_m(\chi_m + k)J_0(\alpha_m a)K_+(\chi_m)} \quad (0 < r \leq a, -\infty < z < 0), \\ \phi(r, z) &= -\left(\frac{a}{4b}\right) \frac{1}{K_+(k)} \sum_{m=1}^{\infty} \frac{K_+(\xi_m)H(b, \xi_m)J_0(\beta_m r)\beta_m^3 J_0'(\beta_m a)e^{i\xi_m z}}{\xi_m(\xi_m - k)(\beta_m^2 - k^2 \Xi^2)J_0(\beta_m b)} \\ &\quad (0 \leq r \leq b, 0 < z < \infty), \\ \phi(r, z) &= -\left(\frac{\pi i}{2}\right) \frac{1}{K_+(k)} \sum_{m=1}^{\infty} \frac{\delta_m [H_0^{(1)'}(\delta_m a)J_0(\delta_m r) - H_0^{(1)'}(\delta_m r)J_0(\delta_m a)]e^{-i\eta_m z}}{K_+(\eta_m)\eta_m(k + \eta_m)\{1 + (k^2 \Xi^2 - \delta_m^2)[J_0'(\delta_m a)/J(b, -\eta_m)]^2\}} \\ &\quad (a < r < b, -\infty < z < 0). \end{aligned}$$

The dominant behaviour of the field in the various regions is given by

$$\begin{aligned} \phi(r, z) &= e^{ikz} + \left(\frac{1}{2a[K_+(k)]^2 k^2}\right) e^{-ikz} + O(e^{-i\chi_1 z}) \quad (0 < r \leq a, -\infty < z < 0), \\ \phi(r, z) &= -\left(\frac{a}{4b}\right) \frac{K_+(\xi_1)H(b, \xi_1)J_0(\beta_1 r)\beta_1^3 J_0'(\beta_1 a)e^{i\xi_1 z}}{\xi_1(\xi_1 - k)(\beta_1^2 - k^2 \Xi^2)J_0(\beta_1 b)} + O(e^{i\xi_2 z}) \\ &\quad (0 \leq r \leq b, 0 < z < \infty), \\ \phi(r, z) &= -\left(\frac{i\pi}{2}\right) \frac{\delta_1 [H_0^{(1)'}(\delta_1 a)J_0(\delta_1 r) - H_0^{(1)'}(\delta_1 r)J_0(\delta_1 a)]e^{-i\eta_1 z}}{K_+(k)K_+(\eta_1)\eta_1(\eta_1 + k)\{1 + (k^2 \Xi^2 - \delta_1^2)[J_0'(\delta_1 a)/J(b, -\eta_1)]^2\}} \\ &\quad + O(e^{-i\eta_2 z}) \quad (a \leq r \leq b, -\infty < z < 0). \end{aligned}$$

From these last results it is easy to obtain the reflection and transmission coefficients for the dominant mode propagation in the various regions. In particular the reflection coefficient  $R_1$  back into the duct  $\{0 \leq r \leq a, -\infty < z < 0\}$  is given by the coefficient of  $e^{-ikz}$ , i.e.

$$R_1 = \frac{1}{2ak^2[K_+(k)]^2} = -|R_1| e^{i2kl},$$

*Low-frequency results for the reflection coefficient,  $\Xi \rightarrow 0$ ,  $ka \ll 1$ ,  $kb \ll 1$*

The value of this reflection coefficient will depend on  $\Xi$  and its evaluation could offer a means of the measurement of  $\Xi$ . Also the value of  $R_1$  will decide the amount of sound that is reflected back into the duct of radius  $a$ . We shall now carry out some low-frequency asymptotics to give an expression for the ratio of

the reflection coefficient with and without an impedance lining on the duct wall at  $r = b$ . Thus, from the last expression for  $R_1$ , we have

$$\frac{R_1|_{\Xi \neq 0}}{R_1|_{\Xi = 0}} = \frac{[K_+(k)]^2|_{\Xi \neq 0}}{[K_+(k)]^2|_{\Xi = 0}} = \left[ \frac{K_{1+}(k)}{K_{3+}(k)} \right]_{\Xi \neq 0} \bigg/ \left[ \frac{K_{1+}(k)}{K_{3+}(k)} \right]_{\Xi = 0},$$

since, from Appendix C,  $K_{2+}(k)$  is independent of  $\Xi$ . We now use the asymptotic approximation of Appendix D which gives

$$\begin{aligned} \frac{R_1|_{\Xi \neq 0}}{R_1|_{\Xi = 0}} &= \left( 1 + \frac{i\Xi}{2kb} + O(\Xi^2) \right) \bigg/ \left( 1 + \frac{i\Xi a}{2k(b^2 - a^2)} + O(\Xi^2) \right) \\ &= 1 + \frac{i\Xi}{kb} \left( 1 - \frac{E}{R} \right) + O(\Xi), \end{aligned}$$

where  $E = \pi ab$  is the area of an ellipse of minor axis  $a$  and major axis  $b$ , and  $R = \pi(b^2 - a^2)$  is the area of annulus of inner radius  $a$  and outer radius  $b$ . It is interesting to note that if the geometry of the ducts is chosen so that  $E = A$  then the effect of the lining, to first order, vanishes. This corresponds to the situation where the ratio  $b/a = \frac{1}{2}(1 + \sqrt{5}) = 1.6180\dots$ , the *golden ratio*! This would seem to show that, by choosing the duct dimensions, the effect of the lining can be reduced, at least at low frequency. It would be of some moment if this phenomenon was replicated for higher frequency ranges.

**6. Conclusions**

We have obtained an exact solution to a cylindrical bifurcation problem. This solution will contribute to the analysis of exhaust systems. It can be considered as a first approximation to a finite absorbent cowel at the exit of a duct (see Fig. 3).

In carrying out some low-frequency asymptotic approximations, it was revealed that the effect of the acoustic lining can be reduced by choosing suitable dimensions for the cylindrical ducts. It would be of some significance if this phenomenon was still valid for a greater frequency range.

We note that by letting  $a \rightarrow b$  we also obtain the solution to another problem of an infinite cylindrical waveguide of radius  $a$  with an impedance mismatch at  $z = 0$ . We could include the effect of exhaust gas flows without substantially changing

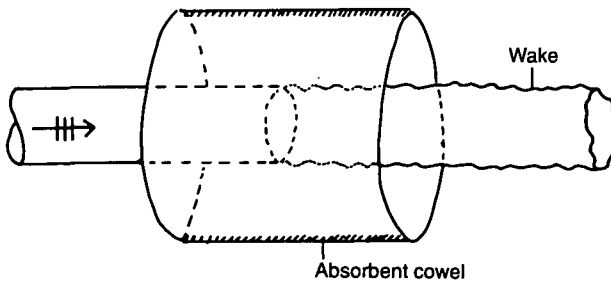


FIG. 3. A shielded jet exhaust.

the mathematical method to obtain an exact solution. This would mean the medium  $0 \leq r \leq a$  moves with a velocity greater than that in  $a \leq r \leq b$ . One would need to incorporate a wake along  $\{r = a, x > 0\}$ . We hope to consider this problem in the future.

## Appendix A

In this appendix we shall derive the permissible normal wave modes that can propagate in the various duct regions. For this purpose we need only consider the infinite region  $-\infty < z < \infty$ . We shall also assume here that  $k$  is real and positive.

### Normal modes in $0 \leq r \leq a$

Here we have to solve the boundary value problem

$$\begin{aligned} (\nabla^2 + k^2)\psi &= 0 \quad (0 \leq r \leq a, \quad -\infty < z < \infty), \\ \frac{\partial \psi}{\partial r} &= 0 \quad (r = a, \quad -\infty < z < \infty). \end{aligned}$$

By separation of variables, it is not difficult to show the only permissible modes are given by

$$\psi(r, z) = e^{\pm i\chi_n z} J_0((k^2 - \chi_n^2)^{1/2} r) \quad (n = 0, 1, \dots), \quad (\text{A.1})$$

where  $J_1(\alpha_n a) = 0$  ( $n = 0, 1, \dots$ ),  $\alpha_0 = 0$ ,  $\alpha_1 = 3.832/a$ ,  $\alpha_2 = 7.016/a$ , etc.,  $\chi_0 = k$ ,  $\chi_1 = [k^2 - (3.832/a)^2]^{1/2}$ ,  $\chi_2 = [k^2 - (7.016/a)^2]^{1/2}$  etc. In expression (A.1), when  $\chi_n$  is real and positive, the upper sign represents an outgoing wave at  $z = \infty$ , whereas the lower sign represents an outgoing wave at  $z = -\infty$ . From the way the square root has been defined,  $\chi_n$  can only be real positive or purely imaginary positive. In the situation where  $\chi_n$  is purely imaginary and positive, the upper (lower) sign in (A.1) represents bounded evanescent waves at  $z = \infty(-\infty)$ .

### Normal modes in $0 \leq r \leq b$

Here we have to solve the boundary value problem

$$\begin{aligned} (\nabla^2 + k^2)\psi &= 0 \quad (0 \leq r < b, \quad -\infty < z < \infty), \\ \frac{\partial \psi}{\partial r} - ik\Xi\psi &= 0 \quad (r = b, \quad -\infty < z < \infty, \quad \text{Re } \Xi > 0). \end{aligned}$$

By applying the usual method of the separation of variables, we obtain

$$\psi(r, z) = e^{\pm i\xi_n z} J_0(\beta_n r) \quad (n = 1, 2, \dots), \quad (\text{A.2})$$

where  $\beta_n = (k^2 - \xi_n^2)^{1/2}$  and  $\beta_n$  are the roots of the equation

$$\beta_n J_1(\beta_n b) + ik\Xi J_0(\beta_n b) = 0 \quad (n = 1, 2, \dots). \quad (\text{A.3})$$

If  $\beta_n$  is a solution of (A.3), then  $\bar{\beta}_n$  is a solution of

$$\bar{\beta}_n J_1(\bar{\beta}_n b) - ik\Xi J_0(\bar{\beta}_n b) = 0 \quad (n = 1, 2, \dots). \tag{A.4}$$

If we now substitute (A.3) and (A.4) into the right-hand side of the well-known expression (Watson, 1944: p. 134(8) or p. 482)

$$\int_0^1 J_0(zt)J_0(\bar{z}t) dt = \frac{\bar{z}J_0(z)J_1(\bar{z}) - zJ_0(\bar{z})J_1(z)}{\bar{z}^2 - z^2}.$$

We get

$$\int_0^1 t |J_0(\beta_n bt)|^2 dt = -\frac{8k \operatorname{Re} \Xi |J_0(\beta_n b)|^2}{b |\bar{\beta}_n^2 - \beta_n^2|^2} \operatorname{Re} \beta_n \operatorname{Im} \beta_n.$$

Now, for consistency, the roots of (A.3) must satisfy

$$\operatorname{Re} \beta_n \operatorname{Im} \beta_n < 0. \tag{A.5}$$

From the way the square root has been defined, i.e.  $\arg \xi_n \in [0, \pi]$ ,

$$\xi_n = (k^2 - \beta_n^2)^{\frac{1}{2}} = [k^2 - (\operatorname{Re} \beta_n)^2 + (\operatorname{Im} \beta_n)^2 - 2i \operatorname{Re} \beta_n \operatorname{Im} \beta_n]^{\frac{1}{2}},$$

and hence from (A.5)

$$\operatorname{Re} \xi_n > 0 \quad \text{and} \quad \operatorname{Im} \xi_n > 0. \tag{A.6}$$

Thus the upper (lower) sign of (A.2) corresponds to outgoing bounded waves at  $z = \infty$  ( $-\infty$ ).

*Normal modes in  $a \leq r \leq b$*

Here we have to solve the boundary value problem

$$\left. \begin{aligned} (\nabla^2 + k^2)\psi &= 0 & (a \leq r \leq b) \\ \frac{\partial \psi}{\partial r} &= 0 & (r = a) \\ \frac{\partial \psi}{\partial r} - ik\Xi\psi &= 0 & (r = b) \end{aligned} \right\} \quad (-\infty < z < \infty).$$

By an application of separation of variables, the possible modes are given by

$$\psi(r, z) = e^{\pm i\eta_n z} [J_0(\delta_n r)H_0^{(1)'}(\delta_n a) - H_0^{(1)'}(\delta_n r)J_0'(\delta_n a)] \quad (n = 1, 2, \dots), \tag{A.7}$$

where  $\delta_n = (k^2 - \eta_n^2)^{\frac{1}{2}}$  and  $\eta_n$  are the roots of the equation

$$D(a, b, \eta_n) = J_0'(\delta_n a)H(b, \eta_n) - H_0^{(1)'}(\delta_n a)J(b, \eta_n) = 0, \tag{A.8}$$

where  $H(b, \eta_n)$  and  $J(b, \eta_n)$  are defined by (3.7). Thus we can write (A.8) out fully as

$$\delta_n [J_0'(\delta_n a)H_0^{(1)'}(\delta_n b) - H_0^{(1)'}(\delta_n a)J_0'(\delta_n b)] + ik\Xi [J_0(\delta_n b)H_0^{(1)'}(\delta_n a) - J_0'(\delta_n a)H_0^{(1)'}(\delta_n b)] = 0. \tag{A.9}$$

In order to locate the disposition of the roots of (A.9), we make the more convenient change of variables  $\delta_n a = z_n$  and  $b/a = t > 0$ , giving

$$z_n [J_1(z_n)H_1^{(1)}(z_n t) - H_1^{(1)}(z_n)J_1(z_n t)] + ika\Xi [J_0(z_n t)H_0^{(1)'}(z_n) - J_0'(z_n)H_0^{(1)}(z_n t)] = 0. \quad (\text{A.10})$$

Now let  $C_0(z_n t) = J_0(z_n t)H_0^{(1)'}(z_n) - J_0'(z_n)H_0^{(1)}(z_n t)$ . Then  $C_0(u)$  is a cylinder function of  $u$ . Hence, from Watson (1944: p. 134(8)),

$$\int_1^t \tau C_0(z_n \tau) C_0(\bar{z}_n \tau) d\tau = \frac{\bar{z}_n C_0(z_n t) C_1(\bar{z}_n t) - z_n C_0(\bar{z}_n t) C_1(z_n t)}{\bar{z}_n^2 - z_n^2}, \quad (\text{A.11})$$

where

$$\begin{aligned} \frac{\partial C_0}{\partial t}(z_n t) &= z_n J_0'(z_n t) H_0^{(1)'}(z_n) - z_n J_0'(z_n) H_0^{(1)'}(z_n t) \\ &= -z_n (J_1(z_n t) H_1^{(1)}(z_n) - J_1(z_n) H_1^{(1)}(z_n t)) = -z_n C_1(z_n t). \end{aligned}$$

Thus we can write (A.10) as

$$z_n C_1(z_n t) + ika\Xi C_0(z_n t) = 0. \quad (\text{A.12})$$

If  $z_n$  is a root of (A.12), then  $\bar{z}_n$  is a root of

$$\bar{z}_n C_1(\bar{z}_n t) - ika\Xi C_0(\bar{z}_n t) = 0. \quad (\text{A.13})$$

If we now eliminate  $C_1(\bar{z}_n t)$  and  $C_1(z_n t)$  from the right-hand side of (A.11) by using (A.12) and (A.13), we get

$$\int_1^t \tau |C_0(z_n t)|^2 d\tau = -\frac{8ka \operatorname{Re} \Xi |C_0(z_n t)|^2}{|z_n^2 - \bar{z}_n^2|^2} \operatorname{Re} z_n \operatorname{Im} z_n.$$

For consistency the roots of the equation (A.10) must satisfy  $\operatorname{Re} z_n \operatorname{Im} z_n < 0$ , and hence the roots of equation (A.9) must satisfy

$$\operatorname{Re} \delta_n \operatorname{Im} \delta_n < 0 \quad (n = 1, 2, \dots). \quad (\text{A.14})$$

From the way the square root has been defined, i.e.  $\arg \eta_n \in [0, \pi]$ ,

$$\eta_n = (k^2 - \delta_n^2)^{\frac{1}{2}} = [k^2 - (\operatorname{Re} \delta_n)^2 + (\operatorname{Im} \delta_n)^2 - 2i \operatorname{Re} \delta_n \operatorname{Im} \delta_n]^{\frac{1}{2}},$$

and hence from (A.14)

$$\operatorname{Re} \eta_n > 0 \quad \text{and} \quad \operatorname{Im} \eta_n > 0. \quad (\text{A.15})$$

Thus the upper (lower) sign of (A.7) corresponds to outgoing bounded waves at  $z = \infty$  ( $-\infty$ ).

**Appendix B**

Here we shall determine the strip of regularity. In Appendix A we have shown, for  $\text{Im } k = 0$ , that  $\text{Re } \chi_n \text{Im } \chi_n > 0$ ,  $\text{Re } \xi_n \text{Im } \xi_n > 0$ , and  $\text{Re } \eta_n \text{Im } \eta_n > 0$ . If we introduce a small imaginary part to  $k$ , so that  $\text{Re } k > \text{Im } k \geq 0$ , then we shall show that the disposition of  $\chi_n$ ,  $\xi_n$ , and  $\eta_n$  does not substantially change.

$\chi_n$  for  $\text{Im } k \geq 0$

Since  $\chi_n = (k^2 - \alpha_n^2)^{1/2}$ , where  $\alpha_n$  are the real roots of  $J_1(a\alpha_n) = 0$ , we can write

$$\chi_n = [(\text{Re } k)^2 - (\text{Im } k)^2 - \alpha_n^2 + 2i \text{Re } k \text{Im } k]^{1/2}.$$

From the way we have defined the cut sheet, we have

$$\text{Re } \chi_n > 0 \quad \text{and} \quad \text{Im } \chi_n > 0,$$

since in the cut sheet  $\text{Re } (\cdot)^{1/2} > 0$  and  $\text{sgn } \text{Im } (\cdot)^{1/2} = \text{sgn } (\text{Re } k \text{Im } k) > 0$ .

We now make use of the following result.

If  $(k^2 - \sigma^2)^{1/2} = p + iq$ , then  $|q| \geq \text{Im } k$  and  $|p| < \text{Re } k$ , where  $k = \text{Re } k + i \text{Im } k$ , with  $\text{Re } k > 0$ ,  $\text{Im } k > 0$ , and  $\sigma$  real. Also, for  $\text{Re } k > 0$  and  $\text{Im } k > 0$ , we have  $pq > 0$ .

The proof is as follows.

$$(k^2 - \sigma^2) = p^2 + 2ipq - q^2$$

$$(\text{Re } k)^2 - \sigma^2 - (\text{Im } k)^2 + 2i \text{Re } k \text{Im } k = p^2 + 2ipq - q^2,$$

so that

$$(\text{Re } k)^2 - (\text{Im } k)^2 - \sigma^2 = p^2 - q^2, \tag{B.1}$$

$$\text{Re } k \text{Im } k = pq. \tag{B.2}$$

If  $|q| < \text{Im } k$ , then, from (B.2),  $\text{Re } k \text{Im } k = |pq| \leq |p| \text{Im } k \Rightarrow \text{Re } k \leq |p|$  or  $p^2 \geq (\text{Re } k)^2$ , and hence, from (B.1),  $[(\text{Re } k)^2 - p^2] - (\text{Im } k)^2 - \sigma^2 + q^2 = 0$ , or  $q^2 - (\text{Im } k)^2 - \sigma^2 \geq q^2 - (\text{Im } k)^2 \geq 0$ , or  $q^2 \geq (\text{Im } k)^2$ , or  $|q| > \text{Im } k$ , contrary to our assumption. Thus  $|q| \geq \text{Im } k$ . Since  $|q| \geq \text{Im } k$ , equation (B.2) gives  $\text{Re } k \text{Im } k \geq \text{Im } k |p|$ , or  $\text{Re } k \geq |p|$ .

The result  $p > 0$  follows directly from (B.2). This means that  $(k^2 - \sigma^2)^{1/2}$  can only lie in the first and third quadrants.

Thus we have the result for all  $n = 0, 1, \dots$ ,  $\text{Im } \chi_n \geq \text{Im } k$ , and

$$\text{Re } \chi_n > 0 \quad \text{and} \quad \text{Im } \chi_n > 0. \tag{B.3}$$

$\xi_n$  for  $\text{Im } k \geq 0$

The roots of the equation

$$\beta_n J_1(\beta_n b) + ik \Xi J_0(\beta_n b) = 0$$

satisfy  $\text{Re } \beta_n \text{Im } \beta_n < 0$  for  $\text{Im } k = 0$  and  $\text{Re } \Xi > 0$ . If we introduce a small imaginary part to  $k$  such that  $\text{Re } k \Xi > 0$ , then the proof of Appendix A follows through to give again

$$\text{Re } \beta_n \text{Im } \beta_n < 0.$$

Now the corresponding value of the propagation constant

$$\begin{aligned} \xi_n &= (k^2 - \beta_n^2)^{\frac{1}{2}} \\ &= [(\text{Re } k)^2 - (\text{Im } k)^2 + (\text{Im } \beta_n)^2 - (\text{Re } \beta_n)^2 + 2i(\text{Re } k \text{Im } k - \text{Re } \beta_n \text{Im } \beta_n)]^{\frac{1}{2}}, \end{aligned}$$

and since  $\text{Re } k \text{Im } k - \text{Re } \beta_n \text{Im } \beta_n > 0$  we have

$$\text{Re } \xi_n > 0 \quad \text{and} \quad \text{Im } \xi_n > 0. \quad (\text{B.4})$$

$\eta_n$  for  $\text{Im } k \geq 0$

The roots of the equation

$$\begin{aligned} \delta_n [J'_0(\delta_n a) H_0^{(1)}(\delta_n b) - H_0^{(1)}(\delta_n a) J'_0(\delta_n b)] \\ + ik \Xi [J_0(\delta_n b) H_0^{(1)}(\delta_n a) - J'_0(\delta_n a) H_0^{(1)}(\delta_n b)] = 0 \end{aligned}$$

satisfy  $\text{Re } \delta_n \text{Im } \delta_n < 0$  for  $\text{Im } k = 0$  and  $\text{Re } \Xi > 0$ . If we introduce a small imaginary part to  $k$  such that  $\text{Re } k \Xi > 0$ , then the proof of Appendix A follows through to give

$$\text{Re } \delta_n \text{Im } \delta_n < 0.$$

Now the corresponding value of the propagation constant

$$\begin{aligned} \eta_n &= (k^2 - \delta_n^2)^{\frac{1}{2}} \\ &= [(\text{Re } k)^2 - (\text{Im } k)^2 + (\text{Im } \delta_n)^2 - (\text{Re } \delta_n)^2 + 2i(\text{Re } k \text{Im } k - \text{Re } \delta_n \text{Im } \delta_n)]^{\frac{1}{2}}, \end{aligned}$$

and since  $\text{Re } k \text{Im } k - \text{Re } \delta_n \text{Im } \delta_n > 0$  we have

$$\text{Re } \eta_n > 0 \quad \text{and} \quad \text{Im } \eta_n > 0. \quad (\text{B.5})$$

From (B.3), (B.4), and (B.5), it can be seen that a real positive quantity  $\varepsilon$  can be found such that  $\min \{\text{Im } \chi_0, \text{Im } \xi_0, \text{Im } \eta_0\} = \varepsilon > 0$ .



**Appendix C**

In this appendix we shall carry out the factorization  $K(\nu) = K_+(\nu)K_-(\nu) = K_-(\nu)K_+(\nu)$ , where, from (3.14), (3.12), and (3.7),

$$K(\nu) = \frac{K_1(\nu)}{K_2(\nu)K_3(\nu)}, \tag{C.1}$$

where

$$K_1(\nu) = 1 + \left(\frac{ik\xi}{\kappa}\right) \frac{J_0(\kappa b)}{J_1(\kappa b)}, \tag{C.2}$$

$$K_2(\nu) = \left(\frac{i\pi a \kappa^2}{2}\right) \frac{J_1(\kappa a)}{J_1(\kappa b)} [J_1(\kappa a)H_1^{(1)}(\kappa b) - H_1^{(1)}(\kappa a)J_1(\kappa b)], \tag{C.3}$$

$$K_3(\nu) = 1 + \left(\frac{ik\xi}{\kappa}\right) \frac{J_1(\kappa a)H_0^{(1)}(\kappa b) - J_0(\kappa b)H_1^{(1)}(\kappa a)}{J_1(\kappa a)H_1^{(1)}(\kappa b) - H_1^{(1)}(\kappa a)J_1(\kappa b)}. \tag{C.4}$$

It is not difficult to see that  $K_1(\nu)$ ,  $K_2(\nu)$ , and  $K_3(\nu)$  are even functions of  $\nu$ . Thus we have  $K(\nu) = K(-\nu)$ . It can also be shown that  $K_1(\nu)$ ,  $K_2(\nu)$ , and  $K_3(\nu)$  are even functions of  $\kappa$ . Thus we have the result that  $K_1(\nu)$ ,  $K_2(\nu)$ , and  $K_3(\nu)$  are free of branch points. Consequently  $K(\nu)$  is also free of branch point singularities. The only singularities of  $K_1(\nu)$ ,  $K_2(\nu)$ , and  $K_3(\nu)$ , and consequently of  $K(\nu)$ , are zeros and poles. The function  $K_2(\nu)$  has only simple zeros and poles, whereas  $K_1(\nu)$  and  $K_3(\nu)$  could have multiple zeros and simple poles depending upon the parameter  $\xi$ . For analytic convenience we shall assume  $\xi$  is such that only simple zeros occur for  $K_1(\nu)$  and  $K_3(\nu)$ . We could, with an increase in complexity of formula, deal with the multiple zero situation without difficulty by the present method. The functions  $K_1(\nu)$  and  $K_3(\nu)$  are analytic functions of  $\xi$ , so that the zeros will also be analytic functions of  $\xi$  and vary continuously with  $\xi$ . The way (C.1) has been written is particularly useful for the situation where  $k\xi = 0$ , i.e. a nearly rigid duct wall, for then  $k(\nu)$  reduces to  $[K_2(\nu)]^{-1}[1 + O(k\xi)]$ , where  $K_2(\nu)$  has been explicitly factorized for the rigid duct situation (Bailin, 1951).

We could write (C.1) in a form suitable for when  $k\xi \rightarrow \infty$ , i.e. a nearly soft duct wall, as

$$K(\nu) = \frac{L_1(\nu)}{L_2(\nu)L_3(\nu)}, \tag{C.5}$$

where

$$L_1(\nu) = 1 + \frac{\kappa J_1(\kappa b)}{ik\xi J_0(\kappa b)}, \tag{C.6}$$

$$L_2(\nu) = \left(\frac{i\pi a}{2}\right) \kappa^2 \frac{J_1(\kappa a)}{J_0(\kappa b)} [J_1(\kappa a)H_0^{(1)}(\kappa b) - J_0(\kappa b)H_1^{(1)}(\kappa a)], \tag{C.7}$$

$$L_3(\nu) = 1 + \frac{\kappa [J_1(\kappa a)H_1^{(1)}(\kappa b) - H_1^{(1)}(\kappa a)J_1(\kappa b)]}{ik\xi [J_1(\kappa a)H_0^{(1)}(\kappa b) - J_0(\kappa b)H_1^{(1)}(\kappa a)]}. \tag{C.8}$$

Again it is not difficult to show that  $L_1(\nu)$ ,  $L_2(\nu)$ , and  $L_3(\nu)$  are even functions of  $\kappa$  and  $\alpha$ , so that they only have poles and zeros.

The factorization of  $K(\nu)$  thus depends on the factorization of  $K_1(\nu)$ ,  $K_2(\nu)$ , and  $K_3(\nu)$  for  $k\Xi \approx 0$ , and  $L_1(\nu)$ ,  $L_2(\nu)$ , and  $L_3(\nu)$  for  $k\Xi \approx \infty$ . This procedure is now fairly standard (see Noble, 1958; Morse & Feshbach, 1953), so we quote the results for the situation  $k\Xi \approx 0$ . For expression (C.1), we have

$$K_+(\nu) = K_-(-\nu) = K_{1+}(\nu)/[K_{2+}(\nu)K_{3+}(\nu)], \tag{C.9}$$

with

$$K_1(\nu) = K_{1+}(\nu)K_{1-}(\nu), \quad K_{1+}(\nu) = K_{1-}(-\nu), \tag{C.10}$$

$$K_{1+}(\nu) = \left( \frac{J_1(kb) + i\Xi J_0(kb)}{J_1(kb)} \right)^{\frac{1}{2}} e^{\nu T} \prod_{n=1}^{\infty} \frac{(1 + \nu/\xi_n) e^{-\nu/\xi_n}}{(1 + \nu/\xi_n^0) e^{-\nu/\xi_n^0}}, \tag{C.11}$$

where  $\xi_n$  are the roots of  $(k^2 - \xi_n^2)^{\frac{1}{2}} J_1((k^2 - \xi_n^2)^{\frac{1}{2}} b) + ik\Xi J_0((k^2 - \xi_n^2)^{\frac{1}{2}} b) = 0$  ( $n = 1, 2, \dots$ ) and  $\xi_n^0$  are the roots of the same equation with  $\Xi = 0$ . The exponential factors  $e^{-\nu/\xi_n}$  and  $e^{-\nu/\xi_n^0}$  are inserted to ensure the absolute convergence of the product, and  $e^{\nu T}$  is included to ensure the factor has algebraic growth at infinity. The parameter  $T$  is determined explicitly by using the following result.

Given

$$P(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{z_n} \right) e^{-z/z_n}, \tag{C.12}$$

with  $z_n = na + b + c/n + O(n^{-1})$ , where  $a$ ,  $b$ , and  $c$  are complex constants, we have

$$P(z) = A e^{z(1-\gamma)/a} \left( \frac{z}{a} \right)^{-(z+b)/a - \frac{1}{2}} \left[ 1 + O\left( \frac{\ln z}{z} \right) \right] \quad \text{as } |z| \rightarrow \infty, \tag{C.13}$$

where  $\gamma = 0.577215\dots$  and  $A$  is a constant (see Nilsson and Brander, 1981a). Thus application of (C.12) and (C.13) to (C.11), together with the asymptotic estimates

$$\left. \begin{aligned} \xi_n &= \frac{in\pi}{b} + \frac{i\pi}{4b} + O(n^{-1}) \\ \xi_n^0 &= \frac{in\pi}{b} + \frac{i\pi}{4b} + O(n^{-1}) \end{aligned} \right\} \text{ as } n \rightarrow \infty, \tag{C.14}$$

gives  $T = 0$ . Thus

$$K_{1+}(\nu) = \left( \frac{J_1(kb) + i\Xi J_0(kb)}{J_1(kb)} \right)^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{1 + \nu/\xi_n}{1 + \nu/\xi_n^0}, \tag{C.15}$$

and also

$$K_{1+}(\nu) = K_{1-}(\nu) = O(1) \quad \text{as } |\nu| \rightarrow \infty. \tag{C.16}$$

Similarly, it is not difficult to show that

$$K_2(v) = K_{2+}(v)K_{2-}(v), \quad K_{2+}(v) = K_{2-}(-v)$$

$$K_{2+}(v) = (k + v) \left[ \left( \frac{i\pi a}{2} \right) \frac{J_1(ka)}{J_1(kb)} [J_1(ka)H_1^{(1)}(kb) - H_1^{(1)}(ka)J_1(kb)] \right]^{\frac{1}{2}}$$

$$\times e^{vT} \prod_{n=1}^{\infty} \frac{(1 + v/\xi_n^a)e^{-v/\xi_n^a}(1 + v/\gamma_n)e^{-v/\gamma_n}}{(1 + v/\chi_n^b)e^{-v/\chi_n^b}},$$

where

$$(k^2 - \chi_n^a)^{\frac{1}{2}}J_1(a(k^2 - \chi_n^a)^{\frac{1}{2}}) = 0, \quad (k^2 - \chi_n^b)^{\frac{1}{2}}J_1(b(k^2 - \chi_n^b)^{\frac{1}{2}}) = 0 \quad (n = 1, 2, \dots)$$

and  $\gamma_n$  are the roots of

$$J_1(a(k^2 - \gamma_n^2)^{\frac{1}{2}})H_1^{(1)}(b(k^2 - \gamma_n^2)^{\frac{1}{2}}) - H_1^{(1)}(a(k^2 - \gamma_n^2)^{\frac{1}{2}})J_1(b(k^2 - \gamma_n^2)^{\frac{1}{2}}) = 0$$

$$(n = 1, 2, \dots).$$

It is not difficult to show that

$$\chi_n^a = \frac{i\pi}{a} + \frac{i\pi}{4a} + O(n^{-1}), \quad \chi_n^b = \frac{i\pi}{b} + \frac{i\pi}{4b} + O(n^{-1}), \quad \gamma_n = \frac{i\pi}{b-a} + O(n^{-1})$$

as  $n \rightarrow \infty$ , (C.17)

and consequently

$$T = \frac{1}{\pi} [b \ln b - a \ln a - (b - a) \ln (b - a)]. \tag{C.18}$$

Thus

$$K_{2+}(v) = (k + v) \left[ \left( \frac{i\pi a}{2} \right) \frac{J_1(ka)}{J_1(kb)} [J_1(ka)H_1^{(1)}(kb) - H_1^{(1)}(ka)J_1(kb)] \right]^{\frac{1}{2}}$$

$$\times e^{vT} \prod_{n=1}^{\infty} \frac{(1 + v/\chi_n^a)(1 + v/\gamma_n)}{1 + v/\chi_n^b} \tag{C.19}$$

and

$$K_{2+}(v) = O(v^{\frac{1}{2}}) \quad \text{as } |v| \rightarrow \infty. \tag{C.20}$$

Finally

$$K_3(v) = K_{3+}(v)K_{3-}(v), \quad K_{3+}(v) = K_{3-}(-v)$$

$$K_{3+}(v) = \left( \frac{[J_1(ka)H_1^{(1)}(kb) - J_1(kb)H_1^{(1)}(ka)] + i\Xi[J_0(ka)H_0^{(1)}(kb) - J_0(kb)H_0^{(1)}(ka)]}{J_1(ka)H_1^{(1)}(kb) - J_1(kb)H_1^{(1)}(ka)} \right)^{\frac{1}{2}}$$

$$\times e^{vT} \prod_{n=1}^{\infty} \frac{(1 + v/\eta_n)e^{-v/\eta_n}}{(1 + v/\eta_n^0)e^{-v/\eta_n^0}},$$

where  $\eta_n$  are the roots of  $\kappa_n[J_1(\kappa_n a)H_1^{(1)}(\kappa_n b) - J_1(\kappa_n b)H_1^{(1)}(\kappa_n a)] + ik\Xi[J_0(\kappa_n a)H_0^{(1)}(\kappa_n b) - J_0(\kappa_n a)H_0^{(1)}(\kappa_n a)] = 0$ , with  $\kappa_n = (k^2 - \eta_n^2)^{1/2}$  ( $n = 1, 2, \dots$ ), and  $\eta_n^0 = \gamma_n$  are the roots of the same equation with  $\Xi = 0$ . Using the fact that

$$\eta_n \sim \eta_n^0 \sim \frac{i n \pi}{b - a} + O(n^{-1}) \quad \text{as } n \rightarrow \infty,$$

together with (C.12) and (C.13), gives  $T = 0$ . Thus

$$K_{3+}(v) = \left( \frac{[J_1(ka)H_1^{(1)}(kb) - J_1(kb)H_1^{(1)}(ka)] + i\Xi[J_0(ka)H_0^{(1)}(kb) - J_0(kb)H_0^{(1)}(ka)]}{J_1(ka)H_1^{(1)}(kb) - J_1(kb)H_1^{(1)}(ka)} \right)^{1/2} \\ \times \prod_{n=1}^{\infty} \frac{(1 + v/\eta_n)}{(1 + v/\eta_n^0)} \quad (\text{C.21})$$

and

$$K_{3+}(v) = O(1) \quad \text{as } |v| \rightarrow \infty. \quad (\text{C.22})$$

Combining the results (C.15), (C.19), and (C.21) into (C.9) we have carried out the explicit factorization of (C.1). We also have from (C.9), (C.16), (C.20), and (C.22)

$$K_+(v) = K_-(-v) = O(v^{-1/2}) \quad \text{as } |v| \rightarrow \infty. \quad (\text{C.23})$$

## Appendix D

In this appendix we shall obtain some approximations for the split function  $K_+(v)$  as  $k\Xi \rightarrow 0$ . In particular for the reflection coefficient of the low-frequency fundamental mode reflected back into the duct  $\{0 \leq r \leq a, -\infty < z < 0\}$ , we require  $K_+(k)$ , and this quantity will be asymptotically approximated for  $k\Xi \rightarrow 0$  and  $ka \ll 1$ . The expression for  $K_+(k)$ , with  $k\Xi$  unrestricted, is given in Appendix C, but it is a complicated product expression. Rather than try to derive an asymptotic expression from the infinite product representation for  $k\Xi \approx 0$ , we shall use a simpler more direct method. The expression for  $K_+(v)$  is given by (C.1)–(C.4) as

$$K_+(v) = K_{1+}(v)/K_{2+}(v)K_{3+}(v).$$

The lining parameter  $\Xi$  only occurs in  $K_{1+}(v)$  and  $K_{3+}(v)$ . Thus we shall derive approximate expressions for  $K_{1+}(k)$  and  $K_{3+}(v)$ .

*Approximation for  $K_{1+}(\kappa)$ ,  $k\Xi \rightarrow 0$ ,  $kb \rightarrow 0$*

$$K_{1+}(v) = 1 + ikb\Xi \frac{J_0(\kappa b)}{\kappa b J_1(\kappa b)}.$$

By taking logarithms, and provided that  $v$  is not near  $\pm k$ , we can expand for  $kb\Xi \rightarrow 0$ , since  $|J_0(\kappa b)/\kappa b J_1(\kappa b)| < \infty$ . Thus

$$\ln K_1(v) = ikb\Xi \frac{J_0(\kappa b)}{\kappa b J_1(\kappa b)} + O(kb\Xi)^2 = \ln K_{1+}(v) + \ln K_{1-}(v). \tag{D.1}$$

By considering the integral

$$\frac{1}{2\pi i} \oint_C \frac{J_0(z) dz}{(z-t)zJ_0'(z)},$$

where  $C$  is a sufficiently large square contour, and applying Cauchy's residue theorem, and then letting the contour expand to infinity, it is possible to express  $J_0(z)/zJ_1(z)$  in its partial fraction form. Thus one obtains

$$\frac{J_0(\kappa b)}{\kappa b J_1(\kappa b)} = \frac{2}{b^2(k^2 - v^2)} - 2 \sum_{n=1}^{\infty} \frac{1}{(k\kappa)^2 - \gamma_n^2},$$

where  $\gamma_n$  are the roots of  $J_1(\gamma_n) = 0$  ( $n = 1, 2, \dots$ ). We can now write the last expression as

$$\begin{aligned} \frac{J_0(\kappa b)}{\kappa b J_1(\kappa b)} &= \frac{1}{kb^2(k+v)} - \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{1}{[k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} \{ [k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} + v \}} \\ &+ \frac{1}{kb^2(k-v)} - \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{1}{[k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} \{ [k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} - v \}}. \end{aligned} \tag{D.2}$$

Comparing (D.1) and (D.2) we have

$$K_{1+}(v) = \exp \left[ ikb\Xi \left( \frac{1}{kb^2(k+v)} - \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{1}{[k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} \{ [k^2 - (\gamma_n/b)^2]^{\frac{1}{2}} + v \}} \right) \right],$$

$$K_{1+}(v) = K_{1+}(-v).$$

Although we have carried out the additive factorization, it is not unique. We could add a polynomial to  $\ln K_{1+}(v)$  and subtract the same polynomial from  $\ln K_{1-}(v)$ . However, the above choice is the unique factorization that ensures  $\ln K_{1+}(v) \rightarrow 0$  as  $v \rightarrow \infty$ .

For  $v = k$ ,  $kb \rightarrow 0$ ,  $\Xi \rightarrow 0$ , we have

$$K_{1+}(k) = \exp \left[ ikb\Xi \left( \frac{1}{2(kb)^2} - \frac{1}{(kb)^2} \sum_{n=1}^{\infty} \frac{1}{[1 - (\gamma_n/kb)^2]^{\frac{1}{2}} \{ [1 - (\gamma_n/kb)^2]^{\frac{1}{2}} + 1 \}} \right) \right], \tag{D.3}$$

$$K_{1+}(k) = 1 + \frac{i\Xi}{2kb} + O(kb\Xi).$$

A more direct and even simpler method of obtaining the result (D.3) is as follows. Since

$$K_1(\nu) = K_{1+}(\nu)K_{1-}(\nu) = 1 + \left(\frac{ik\xi}{\kappa}\right) \frac{J_0(\kappa b)}{J_1(\kappa b)},$$

with  $\kappa = (k^2 - \nu^2)^{\frac{1}{2}}$ , it follows that, for  $\nu \rightarrow k$ , we have  $\kappa \rightarrow 0$  and

$$J_0(\kappa b) = 1 - \frac{1}{4}(\kappa b)^2 + O((\kappa b)^4), \quad J_1(\kappa b) = \frac{1}{2}\kappa b - \frac{1}{16}(\kappa b)^3 + O((\kappa b)^5).$$

Thus

$$K_1(\nu) = 1 + \frac{2ik\xi}{b\kappa^2} - \frac{ikb\xi}{4} + O((\kappa b)^2).$$

Now, although  $k\xi \rightarrow 0$ ,  $1/\kappa^2 \rightarrow \infty$ , so we cannot ignore the second term in comparison to 1. Thus

$$K_1(\nu) \sim \frac{\kappa^2 + 2ik\xi/b - \frac{1}{4}ikb\xi\kappa^2}{\kappa^2} = \frac{\nu^2 - k^2 - 2ik\xi/b}{\nu^2 - k^2} + O(kb\xi),$$

where we have ignored the term  $\kappa^2(-\frac{1}{4}ikb\xi) \rightarrow 0$  as  $\xi \rightarrow 0$ ,  $\nu \rightarrow k$ ,

$$K_1(\nu) \simeq \frac{\nu - k(1 + 2i\xi/kb)^{\frac{1}{2}}}{\nu - k} \frac{\nu + k(1 + 2i\xi/kb)^{\frac{1}{2}}}{\nu + k} + O(kb\xi).$$

The factorization is now obvious, giving

$$K_{1+}(\nu) = \frac{\nu + k(1 + 2i\xi/kb)^{\frac{1}{2}}}{\nu + k} + O(kb\xi) = K_{1-}(-\nu), \quad \nu \rightarrow k, \quad \xi \rightarrow 0.$$

Now, for  $k\xi \rightarrow 0$ , we have

$$\left(1 + \frac{2ik\xi}{k^2b}\right)^{\frac{1}{2}} = 1 + \frac{i\xi}{kb} + O(\xi^2).$$

Thus

$$\lim_{\nu \rightarrow k} K_{1+}(\nu) = \lim_{\nu \rightarrow k} \frac{\nu + k + i\xi/b}{\nu + k} + O(kb\xi) = 1 + \frac{i\xi}{2kb} + o(\xi),$$

which agrees with (A.3).

We shall use this simplified approach to factorize  $K_3(\nu)$ .

*Approximate factorization for  $K_{3+}(k)$ ,  $\xi \rightarrow 0$ ,  $kb \rightarrow 0$ ,  $ka \rightarrow 0$*

Here

$$K_3(\nu) = 1 + \frac{ik\xi}{\kappa} \left( \frac{J_1(\kappa a)H_0^{(1)}(\kappa b) - J_0(\kappa b)H_1^{(1)}(\kappa a)}{J_1(\kappa a)H_1^{(1)}(\kappa b) - H_1^{(1)}(\kappa a)J_1(\kappa b)} \right) = K_{3+}(\nu)K_{3-}(\nu).$$

Now, as  $\nu \rightarrow k$ , we have  $\kappa \rightarrow 0$  and

$$H_1^{(1)}(\kappa b) = -\frac{i}{\pi} \left( \frac{2}{\kappa b} \right) - \frac{i(\kappa b)}{\pi} \ln \left( \frac{2}{\kappa b} \right) + O(\kappa b), \quad H_0^{(1)}(\kappa b) = -\frac{2i}{\pi} \ln \left( \frac{2}{\kappa b} \right) + O(1).$$

Thus we get

$$K_3(\nu) = \frac{\nu - k[1 + 2i\Xi a/k(b^2 - a^2)]^{\frac{1}{2}}}{\nu - k} \frac{\nu + k[1 + 2i\Xi a/k(b^2 - a^2)]^{\frac{1}{2}}}{\nu + k} + o(\Xi),$$

$$\lim_{\nu \rightarrow k} K_{3+}(\nu) = \lim_{\nu \rightarrow k} \frac{\nu + k + i\Xi a/(b^2 - a^2)}{\nu + k} + O(kab\Xi) = 1 + \frac{i\Xi a}{2k(b^2 - a^2)} + o(\Xi),$$

so that

$$K_{3+}(k) = 1 + \frac{i\Xi a}{2k(b^2 - a^2)} + o(\Xi).$$

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