On a Theoretical Background for Computing Reliable Approximations of the Barankin Bound

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Abstract. The Barankin bound is locally the greatest possible lower bound for the variance of any unbiased estimator of a deterministic parameter, under certain relatively mild conditions. Much more essential, Barankin's work determines the sufficient and necessary conditions under which an unbiased estimator with finite variance exists. Nevertheless, the computing of this bound, along with the proof of existence or non-existence of the estimator, has shown to be extremely challenging in most cases. Thereby, many approaches have been made to attain easily computable approximations of the bound, given it exists. Focusing on the rather central matter of existence, we provide a simple theoretical frame within which our approximations of the bound give a clear insight on whether an unbiased estimator does exist.

Keywords: deterministic parameter estimation, MSE lower bound, barankin bound, single-tone frequency estimation, simulation, stochastic optimization

1 Introduction

In 1949, E. W. Barankin in [1] provided an oustanding landmark in the field of deterministic parameter estimation, studying exhaustively the issue of unbiased estimators with finite variance, giving a full characterization of them therein. Among the many results he displayed, he provided the necessary and sufficient conditions for such an estimator to exist, assuming certain rather not restraining hypothesis, and he as well supplied the expression for the minimum achievable variance of them. Further, he proved that, within his assumptions, the very well known and heavily used lower bounds of Cramér-rao and Bhattacharyya can be naturally derived from his results, being both lower bounds beneath that optimum of his.

Nonwithstanding, despite the exact form of the minimum achievable variance having been provided, it is a highly complex one to compute and virtually unmanageable in most cases. In fact, it is too difficult a task to even prove the existence of an unbiased estimator of finite variance. Thus, great efforts with many different approaches have been made throughout the last forty years in an attempt to compute approximations of the bound efficiently (assuming it does exist), or at least to be able to find tight lower bounds for the Barankin bound itself, giving birth to entire families of new bounds. The literature abounds (see, for instance, [9], [10], [12], [14]), yet no approach has proved to be ultimately satisfactory.

We intend to focus our scope on the matter of the existence of unbiased estimators, by providing a theoretical background such that the computed approximations of the Barankin bound reveal some insight on whether the estimator exists or not. Therein, we provide a simple algorithm, well known in the literature, to approximate the bound. However, further we shall discuss the behaviour of these approximations also in the case that the finite variance unbiased estimator does not exist, attempting to shed some light on the subject.

Lastly, we shall apply the method to a classical, yet unsolved, estimation problem, the single-tone frequency estimation, and analyse the results in virtue of the frame exposed.

2 The Barankin Bound

2.1 Preeliminaries, Terminology and Notation

Let $(X, \mathbb{X}, \mu)$ be a measure space, and Θ any parameter set such that $\mathcal{F} = \{p_\theta, \theta \in \Theta\}$ is a parametric family of probability densities. Let $g : \Theta \longrightarrow \mathbb{R}$ be any real function of the parameter set. Our purpose is to attain an estimate of $g(\theta)$, and analyse the existence and characterization of unbiased estimators therein. Should $f : X \longrightarrow \mathbb{R}$ be an unbiased estimator of $g(\theta)$ for all $\theta \in \Theta$, then

$$\int f p_\theta \, d\mu = g(\theta) \quad , \quad \forall \theta \in \Theta, \tag{2.1}$$

which from a classical probabilistic perspective can be written as $\mathbb{E}_\theta[f] = g(\theta)$, for all $\theta \in \Theta$. Let θ_0 be the true value of the parameter from which the samples of X are obtained. We wish for our estimator to be of minimum variance at θ_0 among all other unbiased estimators. That is to say, we wish to find f_0, unbiased, such that for all unbiased f

$$\int \Big(f_0 - g(\theta_0)\Big)^2 p_{\theta_0} \, d\mu \leq \int \Big(f - g(\theta_0)\Big)^2 p_{\theta_0} \, d\mu. \tag{2.2}$$

Nonwithstanding, we will consider a more general form of (2.2). Instead of the variance, we shall work with the p-th central moment, or p-variance, with $1 < p < +\infty$ being the usual variance simply a particular case. Hence, for all unbiased f, the best unbiased estimator f_0 shall satisfy

$$\int \Big(f_0 - g(\theta_0)\Big)^p p_{\theta_0} \, d\mu \leq \int \Big(f - g(\theta_0)\Big)^p p_{\theta_0} \, d\mu. \tag{2.3}$$

Since $\int \Big(f_0 - g(\theta_0)\Big)^p p_{\theta_0} \, d\mu = \int \Big(f_0 - g(\theta_0)\Big)^p \, d\mathbb{P}_{\theta_0}$, considering the Banach space $L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ allows (2.3) to become a norm minimization issue, in light of $\int \Big(f_0 - g(\theta_0)\Big)^p d\mathbb{P}_{\theta_0} = ||f_0 - g(\theta_0)||_p^p$. Further, let $\varphi = f - g(\theta_0)$ and $h(\theta) = g(\theta) - g(\theta_0)$, it is readily seen f is an unbiased estimator of $g(\theta)$ if and only if φ is an unbiased estimator of $h(\theta)$, and (2.3) is reinterpreted as the minimizing of $||\varphi||_p^p$.

Now, let Barankin's main hypothesis come into play. Assume $\mathbb{P}_\theta << \mathbb{P}_{\theta_0}$ for all $\theta \in \Theta$, such that the Radon-Nikodym derivatives $\pi_\theta = \dfrac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}$ exist for all $\theta \in \Theta$, and further, that these π_θ lie in $L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0})$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let us have then the set

$$\mathfrak{B}_{\theta_0} = \Big\{\pi_\theta \in L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0}) : \exists \theta \in \Theta : \pi_\theta = \frac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}\Big\},$$

the unbiasedness condition in (2.1) is then redefined as

$$\int \Big(f - g(\theta_0)\Big) p_\theta \, d\mu = \int \varphi \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta) \quad , \quad \forall \theta \in \Theta.$$

Moreover, since we are interested in finding estimators with finite p-variance, these estimators should lie in $L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$. Thus, let $\mathfrak{M}_p$ be the set of all unbiased estimators with finite p-variance, i.e.

$$\mathfrak{M}_p = \Big\{\varphi \in L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0}) : \forall \theta \in \Theta : \int \varphi \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)\Big\}.$$

Therefore, we wish to find $\varphi_0 \in \mathfrak{M}_p$ such that $||\varphi_0||_p$ is minimum. However, should $\mathfrak{M}_p$ be empty, there will be no unbiased estimator with finite p-variance. Barankin's outstanding work provides the necessary and sufficient conditions for the existence of such an unbiased estimator, as well as the expression for its p-variance, given it does exist.

2.2 Barankin's Main Theorem

We shall now expose Barankin's main result, among the many displayed in his article. Onwards, we will asumme $g(\theta)$ to be a non-constant function since this case is trivial; it's treatment can be seen in [1], p. 482, Theorem 1.

Theorem 1 (Barankin).

1. *$\mathfrak{M}_p \neq \emptyset$ if and only if $\exists M \in \mathbb{R} : \forall n \in \mathbb{N} : \forall (\pi_{\theta_i})_{i=1}^n \subseteq \mathfrak{B}_{\theta_0} : \forall (a_i)_{i=1}^n \subseteq \mathbb{R}$:*

$$\left|\sum_{i=1}^n a_i h(\theta_i)\right| \leq M \cdot \left|\left|\sum_{i=1}^n a_i \pi_{\theta_i}\right|\right|_p. \tag{2.4}$$

2. *If $\mathfrak{M}_p \neq \emptyset$, then $\forall \varphi \in \mathfrak{M}_p : ||\varphi||_p \geq M_0$, such that*

$$M_0 = \inf \Bigg\{ M \in \mathbb{R} : \forall n \in \mathbb{N} : \forall (\pi_{\theta_i})_{i=1}^n \subseteq \mathfrak{B}_{\theta_0} : \\ : \forall (a_i)_{i=1}^n \subseteq \mathbb{R} : \left| \sum_{i=1}^n a_i h(\theta_i) \right| \leq M \cdot \left|\left| \sum_{i=1}^n a_i \pi_{\theta_i} \right|\right|_p \Bigg\}. \quad (2.5)$$

3. If $\mathfrak{M}_p \neq \emptyset$, then $\exists \varphi_0 \in \mathfrak{M}_p : ||\varphi_0||_p = M_0$, and moreover, for all $\varphi_1, \varphi_2 \in \mathfrak{M}_p$, $||\varphi_1||_p = ||\varphi_2||_p = M_0$ only if $\varphi_1 = \varphi_2$, $\mathbb{P}_{\theta_0} - a.e.$.

Proof.

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1. – *Sufficiency.* Let $\mathfrak{M}_p \neq \emptyset$ and $\varphi \in \mathfrak{M}_p$, thus $\varphi \in L_p(X, \mathbb{X}, \mu)$ and $\int \varphi \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)$ for all $\theta \in \Theta$. Let $n \in \mathbb{N}$, $(\pi_{\theta_i})_{i=1}^n \subseteq \mathfrak{B}_{\theta_0}$ y $(a_i)_{i=1}^n \subseteq \mathbb{R}$, then it follows

$$\sum_{i=1}^n a_i h(\theta_i) = \sum_{i=1}^n a_i \left(\int \varphi \pi_{\theta_i} \, d\mathbb{P}_{\theta_0} \right) = \int \varphi \left(\sum_{i=1}^n a_i \pi_{\theta_i} \right) d\mathbb{P}_{\theta_0}.$$

Since $\mathfrak{B}_{\theta_0} \subseteq L_q(X, \mathbb{X}, \mu)$, we have $\sum_{i=1}^n a_i \pi_{\theta_i} \in L_q(X, \mathbb{X}, \mu)$. Thus, from Hölder's inequality,

$$\left| \sum_{i=1}^n a_i h(\theta_i) \right| = \left| \int \varphi \left(\sum_{i=1}^n a_i \pi_{\theta_i} \right) d\mathbb{P}_{\theta_0} \right| \leq ||\varphi||_p \cdot \left|\left| \sum_{i=1}^n a_i \pi_{\theta_i} \right|\right|_q.$$

Therefore, setting $M = ||\varphi||_p$ proves the sufficiency.

– *Necessity.* Let $M \in \mathbb{R}$ such that $\left| \sum_{i=1}^n a_i h(\theta_i) \right| \leq M \cdot \left|\left| \sum_{i=1}^n a_i \pi_{\theta_i} \right|\right|_q$, for all $n \in \mathbb{N}$, all $(\pi_{\theta_i})_{i=1}^n \subseteq \mathfrak{B}_{\theta_0}$ and all $(a_i)_{i=1}^n \subseteq \mathbb{R}$. Now, let $T_{\mathfrak{B}_{\theta_0}} : \mathfrak{B}_{\theta_0} \longrightarrow \mathbb{R}$ be a linear functional, such that $T_{\mathfrak{B}_{\theta_0}}(\pi_\theta) = h(\theta)$. Hence, we have

$$\left| \sum_{i=1}^n a_i T_{\mathfrak{B}_{\theta_0}}(\pi_{\theta_i}) \right| = \left| \sum_{i=1}^n a_i h(\theta_i) \right| \leq M \cdot \left|\left| \sum_{i=1}^n a_i \pi_{\theta_i} \right|\right|_q. \quad (2.6)$$

The Helly-Banach theorem for the extension of linear functionals (see [4], p. 55, Theorem 4) states precisely (2.6) is the necessary and sufficient condition for the existence of a norm preserving linear extension of $T_{\mathfrak{B}_{\theta_0}}$ to all of L_q. Further, the uniqueness of this functional can be proved by means of the Taylor-Foguel theorem, given L_p is strictly convex (see [5]). Thus there exists a unique $T : L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0}) \longrightarrow \mathbb{R}$ such that T is bounded and linear and extends $T_{\mathfrak{B}_{\theta_0}}$, and moreover $||T|| = M_0$. Finally, the Riesz Representation theorem asserts the existence of $\varphi_0 \in L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ such that $T(f) = \int f\varphi_0 \, d\mathbb{P}_{\theta_0}$ for all $f \in L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ and $||\varphi_0||_p = ||T|| = M_0$. In particular, setting $f = \pi_\theta$ for any $\pi_\theta \in \mathfrak{B}_{\theta_0}$,

$$T(\pi_\theta) = \int \varphi_0 \pi_\theta \, d\mathbb{P}_{\theta_0} = T_{\mathfrak{B}_{\theta_0}}(\pi_\theta) = h(\theta). \tag{2.7}$$

Equation (2.7) simply states the fact φ_0 is unbiased for all $\theta \in \Theta$, thus $\varphi_0 \in \mathfrak{M}_p$ and the statement is proved.

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2. Let $\mathfrak{M}_p \neq \emptyset$ and $\varphi \in \mathfrak{M}_p$. In view of the above we have $\forall n \in \mathbb{N} : \forall (\pi_{\theta_i})_{i=1}^n \subseteq \mathfrak{B}_{\theta_0} : \forall (a_i)_{i=1}^n \subseteq \mathbb{R}$,

$$\left| \sum_{i=1}^n a_i h(\theta_i) \right| \leq ||\varphi||_p \cdot \left|\left| \sum_{i=1}^n a_i \pi_{\theta_i} \right|\right|_p .$$

Hence, $||\varphi||_p$ lies in the set defined in (2.5) and $||\varphi||_p \geq M_0$.

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3. Assume $\mathfrak{M}_p \neq \emptyset$. It has been already been established the existence of $\varphi_0 \in \mathfrak{M}_p$ such that $||\varphi_0||_p = M_0$. Let $\varphi_1, \varphi_2 \in \mathfrak{M}_p$ such that $||\varphi_1||_p = ||\varphi_2||_p = M_0$. We have then

$$\left|\left| \frac{\varphi_1 + \varphi_2}{2} \right|\right|_p \leq \frac{||\varphi_1||_p + ||\varphi_2||_p}{2} = \frac{M_0 + M_0}{2} = M_0.$$

In addition, it is readily seen that $\dfrac{\varphi_1 + \varphi_2}{2}$ lies in $\mathfrak{M}_p$. Let $\theta \in \Theta$,

$$\int \frac{\varphi_1 + \varphi_2}{2} \pi_\theta \, d\mathbb{P}_{\theta_0} = \frac{h(\theta) + h(\theta)}{2} = h(\theta).$$

Hence, in virtue of the above $\left|\left| \dfrac{\varphi_1 + \varphi_2}{2} \right|\right|_p \geq M_0$, and hereby $\left|\left| \dfrac{\varphi_1 + \varphi_2}{2} \right|\right|_p = M_0$. We have then $||\varphi_1 + \varphi_2||_p = 2M_0 = ||\varphi_1||_p + ||\varphi_2||_p$. However, according to Minkowski's inequality strict equality conditions, given $1 < p < +\infty$, there exists $\alpha \geq 0$ such that $\varphi_1 = \alpha\varphi_2$, $\mathbb{P}_{\theta_0} - a.e.$ Thus

$$M_0 = ||\varphi_1||_p = ||\alpha\varphi_2||_p = |\alpha| \cdot ||\varphi_2||_p = |\alpha| \cdot M_0.$$

Consider the case $M_0 = 0$, then $||\varphi_1||_p = 0$ and $\varphi_1 = 0$, $\mathbb{P}_{\theta_0} - a.e.$.As a consequence, for all $\theta \in \Theta$, $\int \varphi_1 \pi_\theta \, d\mathbb{P}_{\theta_0} = 0$. It follows $h(\theta) = 0$ for all $\theta \in \Theta$, which is absurd since we had assumed h to be a non-constant function. Therefore, $M_0 \neq 0$, which brings $|\alpha| = 1$. We have then $\alpha = 1$, and $\varphi_1 = \varphi_2$, $\mathbb{P}_{\theta_0} - a.e.$, completing the proof.

Changing the emphasis into a more statistical point of view it is we get the more popular expression for the Barankin lower bound. Let $\sigma_{p,\min} = ||\varphi_0||_p$, then (2.4) and (2.5) can be reinterpreted as

$$\sigma_{p,\min} = \sup_{\substack{\forall n\in\mathbb{N}:\forall(\theta_i)_{i=1}^n\subseteq\Theta:\forall(a_i)_{i=1}^n\subseteq\mathbb{R}:\\ ||\sum_{i=1}^n a_i\pi_{\theta_i}||_q\neq 0}} \frac{\left|\sum_{i=1}^n a_i h(\theta_i)\right|}{\left|\left|\sum_{i=1}^n a_i \pi_{\theta_i}\right|\right|_q}. \tag{2.8}$$

Thereby, $\mathfrak{M}_p \neq \emptyset$, i.e. there exists an unbiased estimator with finite p-variance, if and only if $\sigma_{p,\min}$ in (2.8) exists, in which case $\sigma_{p,\min}$ itself is the minimum p-variance achievable by any unbiased estimator. Particularly, considering the usual variance, with $p = 2$, we have

$$\sigma_{2,\min}^2 = \sup_{\substack{\forall n\in\mathbb{N}:\forall(\theta_i)_{i=1}^n\subseteq\Theta:\forall(a_i)_{i=1}^n\subseteq\mathbb{R}:\\ ||\sum_{i=1}^n a_i\pi_{\theta_i}||_2\neq 0}} \frac{\left[\sum_{i=1}^n a_i h(\theta_i)\right]^2}{\sum_{i=1}^n\sum_{j=1}^n a_i a_j \langle \pi_{\theta_i}, \pi_{\theta_j}\rangle}. \tag{2.9}$$

Given $n \in \mathbb{N}$ and $(\theta_i)_{i=1}^n \subseteq \Theta$, we shall denominate $B_n \in \mathbb{R}^{n\times n}$ such that $(B_n)_{ij} = \langle \pi_{\theta_i}, \pi_{\theta_j}\rangle$ as the Barankin matrix that results from the test-points $(\theta_i)_{i=1}^n \subseteq \Theta$.

2.3 Another of the Many Results from Barankin

We shall make use of another theorem from Barankin for our own results, a theorem which very much served as a motivation for the initiative here proposed.

Theorem 2 (Barankin). *Let $\mathfrak{M}_2 \neq \emptyset$, then φ_0 is the unique element of $\mathfrak{M}_2$ which lies in the closure of* $\operatorname{span}\mathfrak{B}_{\theta_0}$.

The proof of this theorem can be seen in [1], p. 494, Theorem 10.

3 Theoretical Background for the Computations

The approach here exposed is based upon the ideas put forward by Frederick Glave in [2], attempting to provide a formal theoretical frame.

Theorem 2 served as starting ground, as above stated. According to this theorem, if an unbiased estimator exists, the one achieving the minimum variance, φ_0, must lie in the closure of $\operatorname{span}\mathfrak{B}_{\theta_0}$. Hence, there exists a sequence in $\operatorname{span}\mathfrak{B}_{\theta_0}$ which converges in L_2 to it. Furthermore, in [1], p. 489, Theorem 7 Barankin even provides the expression of a sequence in $\operatorname{span}\mathfrak{B}_{\theta_0}$ which converges to φ_0, given a sequence of real numbers which converges to the minimum variance is known.

We wish therefore to obtain a sequence lying in span $\mathfrak{B}_{\theta_0}$ which converges to φ_0, without any prior knowledge about the minimum variance. As a matter of fact, since even proving the existence or non-existence of an unbiased estimator shows to be a humongous enterprise, we wish for our sequence to provide information in any of the two cases.

3.1 A Useful Lemma

The following lemma, though elementary, shall prove to be a fundamental stepping stone.

Lemma 1. *Let* $(\varphi_n)_{n\in\mathbb{N}}$ *be a Cauchy sequence in* $L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$*, that is to say there exists* $\varphi \in L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ *such that* $||\varphi_n - \varphi||_p \longrightarrow 0$*. Then* $\varphi \in \mathfrak{M}_p$ *if and only if for all* $\theta \in \Theta$*,* $\lim\limits_{n\to\infty} \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)$*.*

Proof.

Necessity. Let $(\varphi_n)_{n\in\mathbb{N}} \subseteq L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ such that $||\varphi_n - \varphi||_p \longrightarrow 0$, and let $\varphi \in \mathfrak{M}_p$. Then, for any $\theta \in \Theta$, the asymptotic unbiasedness follows readily from the use of Hölder's inequality, and the fact that π_θ belongs to $L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0})$,

$$\begin{aligned} \left| \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - h(\theta) \right| &= \left| \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - \int \varphi \pi_\theta \, d\mathbb{P}_{\theta_0} \right| \\ &= \left| \int (\varphi_n - \varphi) \pi_\theta \, d\mathbb{P}_{\theta_0} \right| \leq ||\pi_\theta||_q \cdot ||\varphi_n - \varphi||_p \longrightarrow 0. \end{aligned}$$

Sufficiency. Let again $(\varphi_n)_{n\in\mathbb{N}} \subseteq L_p(X, \mathbb{X}, \mu)$ such that $||\varphi_n - \varphi||_p \longrightarrow 0$, and that for all $\theta \in \Theta$, $\lim\limits_{n\to\infty} \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)$. For any given $\theta \in \Theta$, we have $\pi_\theta \in L_q(X, \mathbb{X}, \mathbb{P}_{\theta_0})$. Thus, the functional $G : L_p(X, \mathbb{X}, \mathbb{P}_{\theta_0}) \longrightarrow \mathbb{R}$, such that $G(f) = \int f \pi_\theta \, d\mathbb{P}_{\theta_0}$, is bounded and linear. Hence, G is a continuous functional, and since $(\varphi_n)_{n\in\mathbb{N}}$ converges in L_p to φ we have $G(\varphi_n) \longrightarrow G(\varphi)$; i.e.,

$$\int \varphi \pi_\theta \, d\mathbb{P}_{\theta_0} = \lim_{n\to\infty} \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta).$$

Therefore, $\varphi \in \mathfrak{M}_p$, and the sufficiency is thereby proved.

Lemma 1 simply establishes that any sequence that converges in L_2 to an unbiased estimator is asymptotically unbiased, and that any converging sequence which is asymptotically unbiased converges to an unbiased estimator.

3.2 Construction of the Sequence to Compute the Approximations

We proceed now to design the sequence that will allow us to obtain the approximations of the Barankin bound.

Definition 1. *Let $(\Theta_n)_{n\in\mathbb{N}} \subseteq 2^{\Theta}$ such that $\Theta_n = (\theta_{k,n})_{k=1}^{m_n}$ with $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$, for all $n \in \mathbb{N}$, and let B_n be the Barankin matrix that results from the test-points in Θ_n. If B_n^{-1} exists for all $n \in \mathbb{N}$, then it is well defined the sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq span\mathfrak{B}_{\theta_0}$ such that*

$$\varphi_n = \sum_{k=1}^{m_n} a_{k,n}\pi_{\theta_{k,n}} \quad , \textit{ where } \quad \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m_n,n} \end{bmatrix} = B_n^{-1} \cdot \begin{bmatrix} h(\theta_{1,n}) \\ h(\theta_{2,n}) \\ \vdots \\ h(\theta_{m_n,n})) \end{bmatrix}, \forall n \in \mathbb{N}. \tag{3.1}$$

The linear systems of equations in (3.1) are derived from the compelling of φ_n to be unbiased in all test-points of Θ_n. In other words,

$$\int \varphi_n \pi_{\theta_{j,n}}\, d\mathbb{P}_{\theta_0} = \sum_{i=1}^{m_n} a_{i,n}\langle \pi_{\theta_{i,n}}, \pi_{\theta_{j,n}}\rangle = h(\theta_{j,n}) \quad , \quad \forall j = 1, 2, \ldots, m_n.$$

Thus, the obtained φ_n is an unbiased estimator for all $\theta \in \Theta_n$, and furthermore, it can be easily seen it is the minimum variance estimator, among all estimators unbiased on Θ_n, by means of another result from Barankin's work which we have not exposed in this article. Another characterization of unbiased estimators in L_2, the Barankin integral equation; see [1], p. 495, Corollary 10-1.

3.3 On the Convergence and Reliability of the Constructed Sequence

We will make use of the following simple lemma for our main result.

Lemma 2. *Let $\langle \pi_\theta, \pi_{\theta'}\rangle : \Theta \longrightarrow \mathbb{R}$ be continuous for all $\theta \in \Theta$, for any given fixed $\theta' \in \Theta$. Let (Θ, ρ) be a metric space with distance $\rho : \Theta^2 \longrightarrow \mathbb{R}_0^+$. Then, if $\mathcal{A}$ is a dense subset of Θ, it follows $\mathcal{B} = \left\{\pi_\theta \in \mathfrak{B}_{\theta_0} : \exists \theta \in \mathcal{A} : \pi_\theta = \dfrac{d\mathbb{P}_\theta}{d\mathbb{P}_{\theta_0}}\right\}$ is a dense subset of $\mathfrak{B}_{\theta_0}$.*

Proof.

Let $\mathcal{A} = (\theta_n)_{n\in\mathbb{N}}$ be a dense subset of Θ. Let $\pi_\theta \in \mathfrak{B}_{\theta_0}$, then we have

$$\begin{aligned} ||\pi_\theta - \pi_{\theta'}||_2^2 &= \langle \pi_\theta, \pi_\theta\rangle + \langle \pi_{\theta'}, \pi_{\theta'}\rangle - 2\langle \pi_\theta, \pi_{\theta'}\rangle \\ &= \big[\langle \pi_\theta, \pi_\theta\rangle - \langle \pi_\theta, \pi_{\theta'}\rangle\big] + \big[\langle \pi_{\theta'}, \pi_{\theta'}\rangle - \langle \pi_\theta, \pi_{\theta'}\rangle\big] \\ &= \big|\langle \pi_\theta, \pi_\theta\rangle - \langle \pi_\theta, \pi_{\theta'}\rangle\big| + \big|\langle \pi_{\theta'}, \pi_{\theta'}\rangle - \langle \pi_\theta, \pi_{\theta'}\rangle\big|. \end{aligned} \tag{3.2}$$

Since $\langle \pi_\theta, \pi_{\theta'} \rangle$ is continuous, it is readily seen that for any given $\varepsilon > 0$ there exists $\theta' \in \mathcal{A}$ such that $\left| \langle \pi_\theta, \pi_\theta \rangle - \langle \pi_\theta, \pi_{\theta'} \rangle \right| < \varepsilon/2$ and $\left| \langle \pi_{\theta'}, \pi_{\theta'} \rangle - \langle \pi_\theta, \pi_{\theta'} \rangle \right| < \varepsilon/2$. Thus, from (3.2) there exists $\pi_{\theta'} \in \mathcal{B}$ such that $||\pi_\theta - \pi_{\theta'}||_2^2 < \varepsilon$, and $\mathcal{B}$ is dense in $\mathfrak{B}_{\theta_0}$.

Alas, we state the central theorem of this work.

Theorem 3. *Let $\langle \pi_\theta, \pi_{\theta'} \rangle : \Theta \longrightarrow \mathbb{R}$ be continuous for all $\theta \in \Theta$, for any given fixed $\theta' \in \Theta$ and assume $h \in \mathcal{C}(\Theta)$. Let (Θ, ρ) be a metric space with distance $\rho : \Theta^2 \longrightarrow \mathbb{R}_0^+$, and further, let $(\theta_n)_{n\in\mathbb{N}}$ be a dense subset of Θ such that the Barankin matrix that results from any finite subset of $(\theta_n)_{n\in\mathbb{N}}$ is invertible. Moreover, let $(\varphi_n)_{n\in\mathbb{N}} \subseteq \operatorname{span} \mathfrak{B}_{\theta_0}$ be the sequence defined in (3.1), such that $\bigcup_{n\in\mathbb{N}} \Theta_n = (\theta_n)_{n\in\mathbb{N}}$, and $\Theta_n \subseteq \Theta_{n+1}$ for all $n \in \mathbb{N}$. Then the following statements follow*

1. *$||\varphi_n||_2 \leq ||\varphi_{n+1}||_2$, for all $n \in \mathbb{N}$.*
2. *$(||\varphi_n||)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\mathfrak{M}_2 \neq \emptyset$, in which case $||\varphi_n||_2 \longrightarrow ||\varphi_0||_2$, and $(\varphi_n)_{n\in\mathbb{N}}$ converges in L_2 to φ_0.*

Proof.

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1. Since φ_n is unbiased for all $\theta \in \Theta_n$, we have $\int \varphi_n \pi_{\theta_{k,n}} \, d\mathbb{P}_{\theta_0} = h(\theta_{k,n})$ for all $k = 1, 2, ..., m_n$. Thus, we have the variance of φ_n can be expressed as

$$||\varphi_n||_2^2 = \langle \varphi_n, \varphi_n \rangle = \int \varphi_n \sum_{k=1}^{m_n} a_{k,n} \pi_{\theta_{k,n}} \, d\mathbb{P}_{\theta_0} = \sum_{k=1}^{m_n} a_{k,n} \int \varphi_n \pi_{\theta_{k,n}} \, d\mathbb{P}_{\theta_0} = \\ = \sum_{k=1}^{m_n} a_{k,n} h(\theta_{k,n}). \quad (3.3)$$

Now, let $m \in \mathbb{N}$ such that $m \geq n$,

$$\langle \varphi_n, \varphi_m \rangle = \int \varphi_m \sum_{k=1}^{m_n} a_{k,n} \pi_{\theta_{k,n}} \, d\mathbb{P}_{\theta_0} = \sum_{k=1}^{m_n} a_{k,n} \int \varphi_m \pi_{\theta_{k,n}} \, d\mathbb{P}_{\theta_0} = \\ = \sum_{k=1}^{m_n} a_{k,n} h(\theta_{k,n}) = \langle \varphi_n, \varphi_n \rangle. \quad (3.4)$$

It is interesting to observe (3.4) resembles the above mentioned Barankin integral equation. In virtue of (3.3) and (3.4) it is straightforward to see $||\varphi_m - \varphi_n||_2^2 = ||\varphi_m||_2^2 - ||\varphi_n||_2^2$. Since $||\varphi_m - \varphi_n||_2 \geq 0$, we have $||\varphi_n||_2 \leq ||\varphi_m||_2$ for all $m \geq n$, hence the proposition is proved.

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2. – *Sufficiency*. Let $\theta \in \Theta$ and $n \in \mathbb{N}$. Then φ_n is unbiased for all $\theta \in \Theta_n$, and therefore, for all $\theta \in \Theta_l$ such that $l \leq n$. Now, let $\theta_m \in \Theta_k$ such that $k \leq n$, we have then for the bias of φ_n at θ,

$$\begin{aligned}\left|\int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - h(\theta)\right| &= \left|\int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - \int \varphi_n \pi_{\theta_m} \, d\mathbb{P}_{\theta_0} + h(\theta_m) - h(\theta)\right| \\ &\leq \left|\int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - \int \varphi_n \pi_{\theta_m} \, d\mathbb{P}_{\theta_0}\right| + \big|h(\theta_m) - h(\theta)\big| \\ &\leq \big|\langle \varphi_n, \pi_\theta \rangle - \langle \varphi_n, \pi_{\theta_m} \rangle\big| + \big|h(\theta_m) - h(\theta)\big|. \qquad (3.5)\end{aligned}$$

In virtue of lemma 2, $(\pi_{\theta_n})_{n\in\mathbb{N}}$ is a dense subset of $\mathfrak{B}_{\theta_0}$. In light of $\langle \cdot, \cdot \rangle$ being a continuous function of both its arguments and h being continuous as well, given any $\varepsilon > 0$ there exists $\theta_N \in (\theta_n)_{n\in\mathbb{N}}$ such that both $\big|h(\theta_N) - h(\theta)\big| < \varepsilon/2$ and $\big|\langle \varphi_n, \pi_\theta \rangle - \langle \varphi_n, \pi_{\theta_N} \rangle\big| < \varepsilon/2$ for all $n \in \mathbb{N}$. Now, let $m = N$ and $n > N$ in (3.5), hereby $\left|\int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} - h(\theta)\right| < \varepsilon$ for all $n \geq N$. Thus $\lim_{n\to\infty} \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)$.

Assume $\mathfrak{M}_2 \neq \emptyset$. Then, from (3.3) and theorem 1, we have

$$||\varphi_n||_2^2 = \sum_{k=1}^{m_n} a_{k,n} h(\theta_{k,n}) \leq ||\varphi_0||_2 \cdot \left|\left| \sum_{k=1}^{m_n} a_{k,n} \pi_{\theta_{k,n}} \right|\right|_2 = ||\varphi_0||_2 \cdot ||\varphi_n||_2.$$

In consequence, $||\varphi_n||_2 \leq ||\varphi_0||_2$ for all $n \in \mathbb{N}$, and since $||\varphi_n||_2 \leq ||\varphi_{n+1}||_2$ as well, we have $(||\varphi_n||)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $||\varphi_m - \varphi_n||_2^2 = ||\varphi_m||_2^2 - ||\varphi_n||_2^2$ we also have $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L_2; i.e. there exists $\varphi \in L_2(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ such that $||\varphi_n - \varphi||_2 \longrightarrow 0$. In virtue of lemma 1, $\varphi \in \mathfrak{M}_2$. Given $(\varphi_n)_{n\in\mathbb{N}} \subseteq \operatorname{span} \mathfrak{B}_{\theta_0}$, φ lies in the closure of $\operatorname{span} \mathfrak{B}_{\theta_0}$. Then according to theorem 2, we have $\varphi = \varphi_0$. In addition, $||\varphi_n - \varphi_0||_2^2$ can be expressed as

$$||\varphi_n - \varphi_0||_2^2 = ||\varphi_0||_2^2 + ||\varphi_n||_2^2 - 2\int \varphi_n \varphi_0 \, d\mathbb{P}_{\theta_0}.$$

Since φ_0 lies in $L_2(X, \mathbb{X}, \mathbb{P}_{\theta_0})$, $G : L_2(X, \mathbb{X}, \mathbb{P}_{\theta_0}) \longrightarrow \mathbb{R}$ such that $G(f) = \int f \varphi_0 \, d\mathbb{P}_{\theta_0}$ is a bounded linear functional. Ergo, $\lim_{n\to\infty} \int \varphi_n \varphi_0 \, d\mathbb{P}_{\theta_0} = \int \varphi_0^2 \, d\mathbb{P}_{\theta_0} = ||\varphi_0||_2^2$. Since $||\varphi_n - \varphi_0||_2 \longrightarrow 0$, we have $||\varphi_n||_2 \longrightarrow ||\varphi_0||_2$, which finally proves the sufficiency.

– *Necessity*. Let us now assume $\mathfrak{M}_2 = \emptyset$. To follow with, assume $(||\varphi_n||_2)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $||\varphi_m - \varphi_n||_2^2 = ||\varphi_m||_2^2 - ||\varphi_n||_2^2$, $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence as well, and yet again there exists $\varphi \in L_2(X, \mathbb{X}, \mathbb{P}_{\theta_0})$ such that $||\varphi_n - \varphi||_2 \longrightarrow 0$. Nevertheless, since $\lim_{n\to\infty} \int \varphi_n \pi_\theta \, d\mathbb{P}_{\theta_0} = h(\theta)$ for all $\theta \in \Theta$, according to lemma 1, we have $\varphi \in \mathfrak{M}_2$, which is absurd.

What theorem 3 is asserting, is that under the stated conditions the sequence $(||\varphi_n||)_{n\in\mathbb{N}}$ defined in (3.1) will converge to the Barankin bound, given it exists, or will diverge otherwise. Thus, without having prior knowledge about the existence of an unbiased estimator, by choosing appropiate $(\Theta_n)_{n\in\mathbb{N}}$ one might verify through simulation whether the variance of the sequence diverges or not.

The monotonicity of the sequence's variance provides also certain assurance on the reliability of the computations. The Barankin matrix shows to be extremely ill-conditioned as n goes higher, and the verification of the monotonicity feature proves to be vastly useful concerning the precision needed for the numerical operations to yield accurate results.

Furthermore, in the case that the unbiased estimator with finite variance does not exist, the sequence will still be asymptotically unbiased, though its variance shall not be finite.

4 A Classical Case Study: Single-Tone Frequency Estimation

The estimation of the frequency of a single tone embedded in additive Gaussian white noise is an emblematic problem in the field of electronics and detection theory, one of multiple applications and subject to innumerable approaches. Many algorithms of all kinds have been developed to tackle this standard spectral analysis problem, (see, for instance, [3], [6], [7]) and almost every new methodology attempting to approximate the Barankin bound has had it as its test case (see [14]).

In particular, the single-tone is famous especially for being a victim of the so called threshold effect at low levels of signal to noise ratio that ML estimates often exhibit in many relevant problems (see [3]), an issue which has had attracted attention of its own though intimately related to the Barankin bound (see [11], [13], [8]).

Nevertheless, even whether the bound exists or not is a matter yet unresolved, and thus, so does the existence of an unbiased estimator with finite variance.

4.1 Problem Formulation

Consider a complex-valued sinusoidal signal buried in noise

$$X_k + jY_k = A\exp\left[j(\theta t_k + \alpha)\right] + \eta_k \quad , \quad \text{where} \quad t_k = t_0 + kT,\ k = 0, 1, ..., N-1, \tag{4.1}$$

where the amplitude A and the phase α are known. The parameter to be estimated is the frequency θ, whereas N complex samples are taken at a constant sampling rate of $1/T$ with the first one taken at $t = t_0$. The noise η_k is assumed to be zero mean complex-valued, circular and white Gaussian, with variance σ^2. The set $\Theta \subseteq \mathbb{R}$ is yet to be specified.

With this model, we have the probability density function is

$$p_\theta(x) = \frac{1}{(2\pi\sigma^2)^N} \exp\Bigg(- \frac{1}{2\sigma^2} \sum_{k=0}^{N-1} \bigg(\Big[x_k - A\cos(\theta t_k + \alpha)\Big]^2 + \\ + \Big[y_k - A\sin(\theta t_k + \alpha)\Big]^2 \bigg)\Bigg), \quad (4.2)$$

where $t_k = t_0 + kT$, with $k = 0, 1, ..., N - 1$.

4.2 Analysis of the Barankin Actors

Let θ_0 be the true value of the unknown parameter out of which the samples are obtained, i.e. the value at which the estimator variance is desired to be minimum. The Radon-Nikodym derivatives are in this case simply $\pi_\theta(x) = p_\theta(x)/p_{\theta_0}(x)$, thus, from (4.2) we have

$$\pi_\theta(x) = \exp\Bigg(- \frac{1}{2\sigma^2} \sum_{k=0}^{N-1} \bigg(\Big[x_k - A\cos(\theta t_k + \alpha)\Big]^2 + \Big[y_k - A\sin(\theta t_k + \alpha)\Big]^2 - \\ - \Big[x_k - A\cos(\theta_0 t_k + \alpha)\Big]^2 - \Big[y_k - A\sin(\theta_0 t_k + \alpha)\Big]^2 \bigg)\Bigg).$$

After some tedious algebra, the elements of the Barankin matrix, $(B_n)_{ij}$, yield

$$\langle \pi_{\theta_i}, \pi_{\theta_j} \rangle = \exp\Bigg(\frac{A^2}{\sigma^2} \bigg[N + \frac{\sin\left((\theta_i - \theta_j)\frac{NT}{2}\right)}{\sin\left((\theta_i - \theta_j)\frac{T}{2}\right)} \cos\left((\theta_i - \theta_j)\Big[t_0 + (N-1)\frac{T}{2}\Big]\right) - \\ - \frac{\sin\left((\theta_i - \theta_0)\frac{NT}{2}\right)}{\sin\left((\theta_i - \theta_0)\frac{T}{2}\right)} \cos\left((\theta_i - \theta_0)\Big[t_0 + (N-1)\frac{T}{2}\Big]\right) - \\ - \frac{\sin\left((\theta_j - \theta_0)\frac{NT}{2}\right)}{\sin\left((\theta_j - \theta_0)\frac{T}{2}\right)} \cos\left((\theta_j - \theta_0)\Big[t_0 + (N-1)\frac{T}{2}\Big]\right)\bigg]\Bigg). \quad (4.3)$$

In view of (4.3), using the usual definition for the sinc function, we have $\langle \pi_\theta, \pi_{\theta'} \rangle$ is continuous for all $\theta \in \Theta$, for any fixed $\theta' \in \Theta$. Since $||\pi_\theta||_2^2 = \langle \pi_\theta, \pi_\theta \rangle$ it is clear π_θ lies in $L_2(X, \mathbb{X}, \mathbb{P}_{\theta_0})$, no matter what set $\Theta \subseteq \mathbb{R}$ is. Furthermore, it can be shown that given Θ is properly restricted and $\theta_i \neq \theta_j$ for all $i \neq j$, the Barankin matrix is invertible for all $n \in \mathbb{N}$. Thus, in this case, the conditions of theorem 3 are fullfilled, and we are able to freely construct the sequence $(\varphi_n)_{n\in\mathbb{N}}$ by choosing appropiately the sets $(\Theta_n)_{n\in\mathbb{N}} \subseteq 2^\Theta$.

4.3 Simulation Results

The simulations were carried out with the following arbitrary choice of parameters, $\Theta = [0, 1]$, $\theta_0 = 0.3$, $t_0 = 0.2$, $T = 0.07$, $N = 20$, considering that with the given Θ the Barankin matrix is invertible under the stated conditions. The sets chosen for the succesive iterations were $\Theta_n = \left(k/n\right)_{k=0}^{n}$ with $n = 10, 50, 100, 200, 300$, such that $\Theta_n \subseteq \Theta_{n+1}$. Therefore, the propositions proven in theorem 3 hold.

We also show in our results the very well known Cramér-Rao lower bound for the variance (see [3], p. 592.),

$$\sigma^2_{\min} \geq \frac{6\sigma^2}{A^2\Big[6t_0^2N + 6t_0TN(N-1) + T^2N(N-1)(2N-1)\Big]}.$$

In spite of the mentioned threshold effect, it is known ML estimates's variances converge to the Cramér-Rao bound at high signal to noise ratios, suggesting that in this situations the Barankin and the Cramér-Rao bounds coincide. Thus, this should be verified for the variance of our sequence.

Our results are displayed in Fig. 4.1, showing the variance of φ_n as a function of the signal to noise ratio, $SNR = A^2/\sigma^2$, for the different number of test points n stated above, as well as the Cramér-Rao bound. It is readily seen that at high levels of SNR, given n is sufficiently large, our curves achieve the Cramér-Rao bound as expected, and moreover, the variance of φ_n seem to converge for all levels of signal to noise ratio, strongly suggesting the Barankin bound does exist, and thus that there exists an unbiased estimator for the frequency of the complex single-tone.

5 Conclusion

In this paper, we have discussed Barankin's main theorem and with it a landmark in deterministic parametrical estimation theory. We have provided a theoretical frame in order to attain reliable numerical approximations of the bound, given it does exists, and to be able to discern if it does not. It was shown the analysed sequence's variance converges if and only if the unbiased estimator exists. Moreover, the sequence is in all circumstances asymptotically unbiased and monotonically increasing. This monotonicity proves to be extremely effective as to assuring the numerical precision utilized is high enough to yield accurate, or rather, reliable results. On these grounds, given the finite variance unbiased estimator does not exist, the variance of the sequence will monotonically diverge. Therein, through numerical simulation we are able to develop some insight concerning the so far elusive matter of the existence. Lastly, we applied the exposed methodology to an iconic, and yet unresolved, problem of spectral analysis, the single-tone frequency estimation.

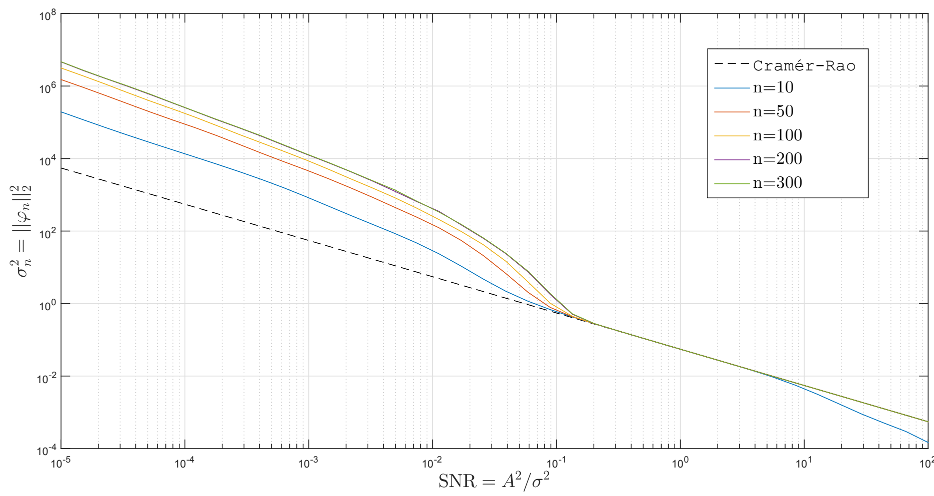

Fig. 4.1. Variances $||\varphi_n||_2$ for $n = 10, 50, 100, 200, 300$, parameters $\Theta = [0, 1]$, $\theta_0 = 0.3$, $t_0 = 0.2$, $T = 0.07$, $N = 20$, and sets $\Theta_n = \left(k/n\right)_{k=0}^{n}$, as functions of the signal to noise ratio, $SNR = A^2/\sigma^2$; as well as the Cramér-Rao bound.

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