Proof Mining for Nonlinear Operator Theory

Four Case Studies on Accretive Operators, the Cauchy Problem and Nonexpansive Semigroups

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> "Die Mathematik ist eine Methode der Logik.[...] Die Erforschung der Logik bedeutet die Erforschung aller *Gesetzmäßigkeit*. Und außerhalb der Logik ist alles Zufall."

Ludwig Wittgenstein, *Tractatus Logico-Philosophicus (Logisch-Philosophische Abhandlung)*, 1921, §6.234- 6.30 ([86]).

Preface

The logical form of a mathematical statement can, within a specific formal framework, a priori guarantee by certain logical metatheorems the extraction of additional information via a study of the underlying logical structure of its proof. This new information is usually of quantitative nature, in the form of effective (computable) bounds -even if the original proof is prima facie ineffective- and highly uniform. Logically analyzing proofs of mathematical statements (of a certain logical form) and making their quantitative content explicit to derive new, constructive information is described as *proof mining*, and constitutes an ongoing research program in applied proof theory introduced by Ulrich Kohlenbach about 15 years ago though it finds its origins in the ideas of Georg Kreisel from the 1950's who initiated it under the name *unwinding of proofs*.

This thesis is a contribution of proof mining to nonlinear analysis. More specifically, this thesis contains the first applications of proof mining to the theory of partial differential equations and moreover to the theory of accretive, set-valued operators on Banach spaces, as well as applications of proof mining to nonexpansive semigroups and their fixed point theory. Essentially all the results presented in this thesis can be classified under operator theory in general and they all involve the study of nonlinear one-parameter semigroups of nonexpansive mappings on a subset of a Banach space (nonexpansive semigroups for short). We finally include a short comment on an idea for future work involving a proof-theoretic application in analysis of a different nature, in particular a study of a proof for the existence of a weak solution of the Navier-Stokes equations in the light of *reverse mathematics*.

This thesis contains a number of new results that are currently unpublished as well as results that have been published in two papers coauthored with Ulrich Kohlenbach (which had been formulated by the author of this thesis except from their introductions and comments not reproduced in this thesis):

- Kohlenbach, U. and K.-A., A. : *Effective asymptotic regularity for one-parameter nonexpansive semigroups* , J. Math. Anal. Appl. 433, 1883-1903 (2016),

- Kohlenbach, U. and K.-A., A. : *Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators*, J. Math. Anal. Appl. 423, 1089-1112 (2015).

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Abstract

We present the first applications of proof mining to the theory of partial differential equations as well as to set-valued operators in Banach spaces, in particular to abstract Cauchy problems generated by set-valued nonlinear operators that fulfill certain accretivity conditions. In relation to (various versions of) uniform accretivity we introduce a new notion of modulus of accretivity. A central result is an extraction of effective bounds on the convergence of the solution of the Cauchy problem to the zero of the operator that generates it. We also provide an example of an application for a specific partial differential equation.

For such operators as well as for operators fulfilling the so-called ϕ-expansivity property, again in general real Banach spaces, we give computable rates of convergence of their resolvents to their zeros.

We give two applications of proof mining to nonlinear nonexpansive semigroups, analysing two completely different proofs of essentially the same statement and obtaining completely different bounds. More specifically we obtain effective bounds for the computation of the approximate common fixed points of one-parameter nonexpansive semigroups on a subset of a Banach space and (for a convex subset) we give corollaries on their asymptotic regularity with respect to Krasnoselskii's and Kuhfittig's iteration schemata.

The bounds obtained in all the above works are all not only effective, but also highly uniform and of low complexity.

We finally include a short comment on a different perspective of a (potential) proof-theoretic application to partial differential equations, namely a reverse mathematical study of a proof for the existence of a weak solution of the Navier-Stokes equations motivating future work.

Zusammenfassung

In dieser Arbeit werden die ersten Proof-Mining-Anwendungen auf die Theorie Partieller Differentialgleichungen sowie auf mehrwertige Operatoren in Banachräumen präsentiert. Vor allem werden abstrakte Cauchy-Probleme, die durch mehrwertige nichtlineare Operatoren generiert werden, welche bestimmte Akkretivitätsbedingungen erfüllen, betrachtet. Ein neuer Begriff von Akkretivitätsmodul wird eingeführt, der sich auf verschiedene Varianten von uniformer Akkretivität bezieht. Das zentrale Ergebnis dieser Arbeit ist die Extraktion von effektiven Schranken für die Konvergenz der Lösung des Cauchy-Problems gegen die Nullstelle des erzeugenden Operators der es generiert. Ein Anwendungsbeispiel auf eine spezifische partielle Differentialgleichung wird ebenfalls präsentiert.

Für solche Operatoren in allgemeinen reellen Banachräumen sowie für Operatoren, welche die sogenannte ϕ-Expansivitätseigenschaft haben, werden berechenbare Konvergenzraten ihrer Resolventen gegen ihre entsprechenden Nullstellen angegeben. Unter der Annahme, dass der Raum darüber hinaus gleichmäßig konvex ist, wird eine Konvergenzrate von geringer Komplexität bewiesen.

Zwei Proof-Mining-Anwendungen auf nichtlineare, nichtexpansive Halbgruppen werden durch die Analyse von zwei komplett unterschiedlichen Beweisen derselben Aussage und durch das Erreichen von unterschiedlichen Schranken vorgestellt. Genauer gesagt werden effektive Schranken für die Berechnung von approximativen gemeinsamen Fixpunkten von einparameter-nichtexpansiven Halbgruppen auf einer Teilmenge eines Banachraums gewonnen. Desweiteren präsentieren wir für eine konvexe Teilmenge Korollare über die asymptotische Regularität der Iterationsschemata von Krasnoselskii und Kuhfittig.

Alle erreichten Schranken sind nicht nur effektiv, sondern auch in hohem Maße uniform und haben eine geringe Komplexität.

Diese Dissertation endet mit einem kurzen Kommentar zu einer Idee aus dem Bereich der Reverse Mathematics zu partiellen Differentialgleichungen, die eine andere Perspektive auf potentielle beweistheoretische Anwendungen aufzeigt, um zukünftige Arbeit zu motivieren.

Contents

Chapter 1

Introduction

1.1 Historical Notes

At the Second International Congress of Mathematicians in Paris on August 8, 1900, David Hilbert presented ten (from a list of a total of twenty-three) problems that he considered of fundamental significance. His second problem was entitled: *"Die Widerspruchslosigkeit der arithmetischen Axiome"* which was translated as: *"The compatibility of the arithmetical axioms"*([30]). In modern terminology, this is often interpreted as: *"the consistency of Peano arithmetic"* (PA), though *"the consistency of second order arithmetic"* would be more accurate, as the latter axiomatic system formalises not only natural numbers-as is the case with first order PA- but also their subsets, thus also real numbers, and most of ordinary mathematics is expressible in the language of second order arithmetic, which is also known as *"Analysis"*.

Namely, what Hilbert envisioned was essentially the axiomatization of mathematics - axiomatization meaning here a logical definition by using the method of formal systems, that is, by finitistic proofs starting from an agreed-upon set of axioms.

In Hilbert's own words ([30]):*"When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps.*[...]*But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results.*[...]*A direct method is* *needed for the proof of the compatibility of the arithmetical axioms.*[...]"

The solvability of Hilbert's Second Problem received a major blow in 1931, when Kurt Gödel published his notorious *Second Incompleteness Theorem* ([26] according to which any formal system (strong enough to formulate its own consistency) can prove its own consistency if and only if it is inconsistent. In that sense it is therefore impossible to prove the consistency of the axioms of PA within PA.

Nagel and Newman comment in their book ([64]) on the impact of Gödel's Second Incompleteness Theorem: (Gödel)*"presented mathematicians with the astounding and melancholy conclusion that the axiomatic method has certain inherent limitations, which rule out the possibility that even the ordinary arithmetic of the integers can ever be fully axiomatized. What is more, he proved that it is impossible to establish the internal logical consistency of a very large class of deductive systems -elementary arithmetic, for example- unless one adopts principles of reasoning so complex that their internal consistency is as open to doubt as that of the systems themselves. In the light of these conclusions, no final systematization of many important areas of mathematics is attainable, and no absolutely impeccable guarantee can be given that many significant branches of mathematical thought are entirely free from internal contradiction."*

In the years that followed, *relative consistency proofs* were developed: let T_1 and T_2 be formal theories with languages $\mathcal{L}(T_1)$, $\mathcal{L}(T_2)$. If it can be proved that the consistency of T_2 follows from the consistency of T_1 then we say that T_2 is consistent relative to T_1. (The fact that the consistency of T_2 follows from the consistency of T_1 must of course be provable in a system not stronger than the one in which the consistency of T_1 is provable). In this spirit, via the Gödel-Gentzen negative translation already in 1933 the consistency of PA was reduced to the consistency of the intuitionistic Heyting Arithmetic (HA) ([27]). In 1958, by Gödel's functional "Dialectica" Interpretation the consistency of PA was reduced to a quantifier-free calculus of primitive recursive functionals of finite type ([28]). Obtaining relative consistency proofs has been a motivation for the development of various other such proof interpretations in addition to negative translation and Dialectica. But except from their aforementioned contribution to foundational questions, it turned out that such proof interpretations could serve another significant purpose, which was brought to light by the ideas of Georg Kreisel.

In the early 1950's Kreisel proposed a shift of focus of proof theory towards applying proof-theoretic methods for applications in mathematics -rather than looking exclusively at foundational questions- introducing the program of *Unwinding of Proofs* ([52], [53]). Kreisel's motivation and the scope of his program can be summarized in his question:

"*What more do we know if we have proved a theorem by restricted means than if we merely know that it is true ?*"

In Kreisel's program proof interpretations are used as tools to extract constructive (i.e. computational) information from given proofs by recursion on the proofs. Such information in the original proof is implicit -hidden behind the use of quantifiers. Kreisel drew attention to the study of proofs of existential statements in particular (in contrast to Hilbert's original quest which referred to universal statements) aiming at extracting realizers for the existential quantifiers as functions of parameters from the proof.

For several decades that followed, applications of proof unwinding were scarce and far-between, the two most known works being in algebra and number theory: by Charles Delzell on Hilbert's 17th problem ([18]) continuing work of Kreisel and by Horst Luckhardt ([58], [59]) on Roth's theorem. Starting in [36], Ulrich Kohlenbach re-initiated Kreisel's program focusing in particular to applications in analysis. The program was later renamed, as suggested by Dana Scott, as *Proof Mining*.

Early work by Kohlenbach was focused on uniqueness proofs in best approximation theory and made use of Heine-Borel compactness in the form of the noneffective binary weak König's lemma (WKL) and of a proof-theoretic approach to eliminate WKL from the proofs (see, for example, [37], [38], [39]). In 2000-2003, the focus was shifted to strong convergence results for iterative procedures of nonexpansive and other classes of mappings in general normed spaces and hyperbolic spaces (see, for example, [42], [43], [49]). The results were the extraction of explicit, computable and highly uniform bounds. By "highly uniform" it is meant that the dependence of the bounds is only limited to general bounding information (majorants) and input data from the spaces involved, while they are independent from the mapping used in the iteration and from the starting point.

The above findings would soon be described as instances of proof-theoretic phenomena by general logical metatheorems that were discovered in 2003-2005 ([44], [24]). The metatheorems were based on modifications and extensions of the aforementioned functional Dialectica interpretation. In particular, the *monotone* functional interpretation is applied ([40]) together with Gödel's negative translation ([27]) [1]. The latter provides an embedding of the classical reasoning into an intuitionistic system so that the resulting interpretation can be applied also to classical proofs, not just to constructive/ intuitionistic ones.

[1] To be more specific, in [45] Kuroda's negative translation ([56]) is actually applied.

It is therefore possible to extract constructive (i.e. computable) information even from proofs that are prima facie nonconstructive. The passage to the resulting interpretation is survived by mathematical statements of the logical form $\forall x\ \exists y\ A_\exists(x, y)$ (where $A_\exists$ denotes an existential formula). The metatheorem thus guarantees the possibility to extract an explicit, computable bound on y from the proof of the mathematical statement of the above mentioned logical form (within some formal framework). The general idea is that T_1 is transformed into T_2 by transforming every theorem $\phi \in \mathcal{L}(T_1)$ into $\phi^I \in \mathcal{L}(T_2)$ via the proof interpretation I such that the implication $T_1 \vdash \phi \Rightarrow T_2 \vdash \phi^I$ holds. Then a given proof p of ϕ in T_1 is transformed into a proof p^I of ϕ^I in T_2 by a simple recursion over ϕ in T_1 as proof interpretations respect the logical deduction rules. This gives new quantitative information. Moreover, we may obtain qualitative information as well, in the sense that it may occur that p^I uses a restricted version of the assumptions of ϕ, thus ϕ^I can turn out to be a generalisation of ϕ. Note that, as we will see in the next section in more detail, the benefit of transforming proofs into functional programs, as is the case with Dialectica and its variations, is that it is possible to make use of the mathematical properties of the functionals (for example majorizability, continuity etc.) so as to capture and make explicit the quantitative and computational content of a proof. As a functional interpretation can be seen as a reduction of a infinitary theory to a theory of functionals, proof interpretations can be seen as a generalization of the "finitary" standpoint of analysing mathematical systems that Hilbert favored. Thus we can see proof mining as a practice within the generalized Hilbert's program. For an interesting discussion we refer the reader to Section 4.2. in [89].

Proof mining has since been applied by Kohlenbach and his collaborators to works in analysis in general and more specifically to approximation theory, ergodic theory, fixed point theory, optimization theory and (recently by Kohlenbach and the author in [47] for the first time) to the theory of partial differential equations, and has produced a vast number of results. A review of the results until 2008 can be found in the book by Kohlenbach ([45]) and after 2008 in the report [46]. In this thesis we will present our contribution in the first applications of proof mining to the theory of partial differential equations and to works on nonexpansive semigroups and their fixed point theory. Essentially all the works presented in this thesis can also be seen as applications to operator theory in general as they all involve the study of nonlinear one-parameter semigroups of nonexpansive mappings on a subset of a Banach space.

1.2 Proof Mining : An Elementary Introduction

In this section we will present a very basic introduction to the most essential proof-theoretic background, as well as the notions and metatheorems that will be employed throughout this thesis. As we will restrict to a very limited presentation, for more details we refer the reader to [45].

1.2.1 On Herbrand's Theorem, Kreisel's No-Counterexample Interpretation and Tao's Metastability

Throughout Section 1.2 the set of natural numbers is defined as $\mathbb{N} := \{0, 1, ...\}$.

Consider a statement of the form

$$A \equiv \forall k\ \exists n\ \forall m\ A_0(k, n, m)\ (+)$$

where A_0 denotes a quantifier-free formula. It is in general not possible to extract a realizer i.e. a computable function f so that

$$\forall k\ \forall m\ A_0(k, f(k), m)$$

or even a computable bound on n i.e. a computable function f so that

$$\forall k\ \exists n \leq f(k)\ \forall m\ A_0(k, n, m).$$

To see this, consider the following example (given in [45]). In (+) let us set

$$A_0 :\equiv T(k, k, n) \vee \neg T(k, k, m)$$

where T is the primitive recursive Kleene-T-predicate i.e. $T(x_1, x_2, x_3)$ means : "the Turing machine with Gödel number x_1 applied to the input x_2 terminates with a computation whose Gödel number is x_3". Supposing there existed a computable bound f so that

$$\forall k\ \exists n \leq f(k)\ \forall m\ (T(k, k, n) \vee \neg T(k, k, m)),$$

then, the special halting problem

$$\{k \in \mathbb{N} : \exists n \in \mathbb{N}\ T(k, k, n)\}$$

would be decidable by the computable function

$$\tilde{f}(n) := \begin{cases} 0, & \text{if } \exists n \leq f(k)\ T(k, k, n) \\ 1, & \text{otherwise.} \end{cases}$$

Even though in general it is not possible to extract computable witnesses/ realizers/ bounds for A, it is possible to compute a bound on n for A^H, the *Herbrand normal form* of A which is a weakened form of A :

$$A^H :\equiv \forall k\ \exists n\ A_0(k, n, g(n))$$

where g is called the Herbrand index function (in theories allowing function variables and function quantifiers we would write $A^H :\equiv \forall g, k\, \exists n\, A_0(k, n, g(n)))$. The general definition is as follows:

Definition 1. *(see [45]) Let*

$$A :\equiv (\forall y_0)\ \exists x_1\ \forall y_1 ...\ \exists x_n\ \forall y_n\ A_0(y_0, x_1, y_1, ..., x_n, y_n).$$

The Herbrand normal form of A is defined as

$$A^H :\equiv (\forall y_0)\ \exists x_1, ..., x_n\ A_0(y_0, x_1, f_1(x_1), ..., x_n, f_n(x_1, ..., x_n))$$

where $f_1, ..., f_n$ are new function symbols called Herbrand index functions. In theories that permit function variables and function quantifiers we write

$$A^H :\equiv \forall (y_0), f_1, ..., f_n\ \exists x_1, ..., x_n\ A_0(y_0, x_1, f_1(x_1), ..., x_n, f_n(x_1, ..., x_n)).$$

The dual normal form of the Herbrand normal form, where instead of the universally quantified variables being replaced by new function symbols depending on the existentially quantified variables, it is the existentially quantified variables being replaced by new function symbols depending on the universally quantified variables, is the Skolem normal formal defined as follows:

Definition 2. *(see [45]) Let*

$$A :\equiv (\exists y_0)\ \forall x_1\ \exists y_1 ...\ \forall x_n\ \exists y_n\ A_0(y_0, x_1, y_1, ..., x_n, y_n).$$

The Skolem normal form of A is defined as

$$A^S :\equiv (\exists y_0)\ \forall x_1, ..., x_n\ A_0(y_0, x_1, f_1(x_1), ..., x_n, f_n(x_1, ..., x_n)).$$

The new function symbols $f_1, ..., f_n$ are called Skolem functions.

Note that Herbrandization preserves logical validity and Skolemization preserves logical satisfiability (however none of them preserves logical equivalence, except in the presence of the axiom of choice).

Considering now again a situation :

$$A :\equiv \forall k\ \exists n\ \forall m (P(k, n) \vee \neg P(k, m))$$

where P is some predicate symbol, for the Herbrand normal form A^H of A written as:

$$A^H \equiv \forall k\ \exists n (P(k, n) \vee \neg P(k, g(n))$$

we can have a list of candidates for $\exists n$ in particular $c, g(c)$ for any constant c, since the disjunction

$$A^{H,D} :\equiv (P(k, c) \vee \neg P(k, g(c)) \vee (P(k, g(c)) \vee \neg P(k, g(g(c))))$$

is a tautology.

Now let us see how the extraction of such lists of candidates that can serve as witnesses/ bounds for the Herbrand normal form of any sentence is in general formally guaranteed. In the following let PL denote the first order predicate logic with equality. We will say that a formula in the language of PL is a quasi-tautology if it is a tautological consequence of instances of equality axioms.

Theorem 1. *(Herbrand's Theorem, see [45]) Let*

$$A \equiv \forall y_0\ \exists x_1\ \forall y_1 ...\ \exists x_n\ \forall y_n\ A_0(y_0, x_1, y_1, ..., x_n, y_n).$$

Then PL $\vdash A$ *if and only if there are terms* $t_{1,1}, ..., t_{1,k_1}, ..., t_{n,1}, ..., t_{n,k_n}$ *that are built up out of the constants, free variables and function symbols of* A *and the index functions used for the formation of* A^H *such that the Herbrand disjunction*

$$A^{H,D} :\equiv \bigvee_{j_1=1}^{k_1} ... \bigvee_{j_n=1}^{k_n} A_0(y_0, t_{1,j_1}, f_1(t_{1,j_1}), ..., t_{n,j_n}, f_n(t_{1,j_1}, ..., t_{n,j_n}))$$

is a quasi-tautology.

The terms $t_{i,j}$ *can be extracted constructively from a given* PL*-proof of* A *and conversely, one can construct a* PL*-proof for* A *out of a given quasi-tautology* $A^{H,D}$.

(Note that the above also holds for PL without equality if "quasi-tautology" is replaced by "tautology".) We therefore see that for A^H via Herbrand's theorem we obtain a list of candidates that serve as bounds. Note that replacing all occurences of terms with variables in $A^{H,D}$, we obtain a formula A^D from which we can re-derive A.

It is possible to extend Herbrand's theorem for so called "open" theories i.e. first-order theories whose non-logical axioms are all purely universal, however it is not possible to extend it straightforwardly for PA which is not open. Kreisel extended the idea behind Herbrand's theorem for PA by introducing a new proof interpretation : the so-called *no-counterexample interpretation* ([52], [53]).

Definition 3. *(see [45]) Let*

$$A :\equiv \forall y_0\ \exists x_1\ \forall y_1 ... \exists x_n\ \forall y_n\ A_0(y_0, x_1, y_1, ..., x_n, y_n).$$

We say that a tuple of functionals $\Phi_1, ..., \Phi_n (=: \underline{\Phi})$ *satisfies the no-counterexample interpretation of* A *if it realizes the Herbrand normal form* A^H *of* A*, that is, if*

$$\forall \underline{f}\ A_0(\Phi_1(y_0, \underline{f}), f_1(\Phi_1(y_0, \underline{f})), ..., \Phi_n(y_0, \underline{f}), f_n(\Phi_1(y_0, \underline{f}), ..., \Phi_n(y_0, \underline{f})))$$

where $\underline{f}$ *denotes the tuple* $f_1, ..., f_n$.

Let us now see how the above ideas are applied in analysis.

Consider the following formulation of the Cauchy property of a sequence of reals $\{a_n\} \subseteq [0, M]$:

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall i, j \in [n; n+m]\ (|a_i - a_j| <_{\mathbb{R}} 2^{-k})\ (i)$$

where $[n; n+g(n)] := \{i \in \mathbb{N}\ :\ n \leq i \leq n+g(n)\}$. As shown by Specker in [77], it is in principle not possible to extract a computable bound on n. (In particular Specker showed that it is impossible even for primitive recursive sequences of rational numbers where each a_n can be coded as a number theoretic function $\mathbb{N} \to (\mathbb{N} \times \mathbb{N})$). However, for the Herbrand normal form of (i) which is the following reformulation of (i):

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ \forall i, j \in [n; n+g(n)](|a_i - a_j| <_{\mathbb{R}} 2^{-k})\ (ii)$$

it is possible to find a computable bound $\Phi(k, g, M)$ so that $n \leq \Phi(k, g, M)$. This bound solves the no-counterexample interpretation of (i). In particular:

Proposition 1. *(Kohlenbach, see Propositions 2.26, 2.27 and Remark 2.29 in [45]) Let $\{a_n\}$ be any nonincreasing sequence in $[0, M]$. Then, for (ii) as above, there is a primitive recursive bound*

$$\Phi(g, k, M) := \tilde{g}^{(M2^k)}(0)$$

with

$$\tilde{g}(n) := n + g(n)$$

where the function iterations are defined as:

$$g^{(0)}(k) := k,\ \ g^{(i+1)}(k) := g(g^{(i)}(k))$$

so that $n \leq \Phi(k, g, M)$.

In this thesis we will see an instance of the above result in the proof of Theorem 9 in Section 2.3 as well as in the proof of Lemma 12 in Section 3.3.

The above bound is called a *rate of metastability* and we usually refer to the statement (ii) as the *metastable* version of (i). This terminology was introduced by T. Tao (see [83, 84]), who rediscovered this phenomenon in analysis without any use of proof theory. We stress that in general there is no constructive way to obtain a bound on n in (i) using the rate of metastability of (ii).

We can extend this phenomenon for convergence statements, namely, considering a statement of the form

$$\lim_{t \to \infty} P(t) = 0$$

which can be formally written as

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \geq n\ (|P(t)| <_{\mathbb{R}} 2^{-k})\ (a).$$

In principle it is not possible to extract a computable rate of convergence, i.e. a computable bound on $n \in \mathbb{N}$ (in general there exist cases where this is impossible). However, it is always possible to extract a computable rate of metastability so that $n \leq \Phi(k, g, \cdot)$ for its metastable version that coincides with its Herbrand normal form:

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \in [n, n + g(n)]\ (|P(t)| <_{\mathbb{R}} 2^{-k})\ (b).$$

Note that $\Phi(k, g, \cdot)$ will moreover be highly uniform and will depend on general uniform bounds on the input data of the proof. Although obviously (b) follows from (a) constructively just by restricting $[n, \infty)$ to $[n, n + g(n)]$, because for the other direction we have to argue by contradiction, there is no way to pass from a bound on (b) to a bound on (a) in a computational way. It is moreover noteworthy that (b), unlike (a) is a finitary statement as we now restrict things to the interval $[n, n + g(n)]$.

In Theorem 7 of Chapter 2.3, we will see that we are able to extract a rate of convergence indeed as the proof happens to be constructive, however as we explained this is not expected in general for a classical proof. (In Chapter 2.4 it happens that we extract full rates of convergence instead of metastability because of monotonicity that reduces a statement of a form $\forall\exists\forall$ to a statement of the form $\forall\exists$).

For statements of the form $\forall\exists\forall$ that we have so far seen in our examples, the no-counterexample interpretation is a special case of the Gödel functional interpretation (combined with negative translation) which we will discuss in the next subsection.

When attempting to extract realizing functionals in a modular way, that is, by recursion over the proof-tree keeping the basic structure of the proof unchanged, the no-counterexample interpretation admits only functionals of type 2 and manifests a problem in relation to modus ponens. This problem stems from the fact that for formulas of complexity $\forall\exists\forall\exists$ and higher, the passage of the Herbrand normal form back to the original formula requires the axiom of choice (AC) for universal formulas which are undecidable. (see [45] and for more details [41]). This problem is resolved by using Gödel's functional interpretation (and its variations) that is modular thus well-behaved with respect to modus ponens and, unlike the no-counterexample interpretation, also works for formulas with a higher complexity than $\forall\exists\forall$ as it admits counterfunctions of arbitrary (finite) types and only uses the quantifier-free axiom of choice.

1.2.2 On Gödel's Functional "Dialectica" Interpretation

The set $\mathbf{T}$ of all finite types over $\mathbb{N}$ is generated inductively by the clauses:

$$(i)\ 0 \in \mathbf{T},\ (ii) \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

The type 0 is the type of natural numbers, while objects of type $\tau(\rho)$ are functions that map objects of type ρ to objects of type τ. For example, for number theoretic functions instead of writing that they are of type 1 we can also write that they are of type 0(0) i.e. of type $0 \to 0$. Likewise, the type of functionals mapping number theoretic functions to naturals is $(0 \to 0) \to 0$ for which we can alternatively write 0(00).

Let WE-HA$^\omega$ and WE-PA$^\omega$ denote the weakly extensional intuitionistic Heyting arithmetic in all finite types and weakly extensional classical Peano arithmetic in all finite types respectively. The latter follows from the former by adding the law of excluded middle schema $A \vee \neg A$. For the full definition including the axioms and rules we refer the reader to Chapter 3.3 in [45]. What we want to stress here is that by *weak* extensionality it is meant that the full extensionality axioms :

$$E_\rho\ :\ \forall z^\rho, x_1^{\rho_1}, y_1^{\rho_1}, ..., x_k^{\rho_k}, y_k^{\rho_k}\ (\bigwedge_{i=1}^{k} (x_i =_{\rho_i} y_i) \to z\underline{x} =_0 z\underline{y}),$$

where $\rho = 0(\rho_k)...(\rho_1)$ which are assumed in the corresponding extensional theories E-HA$^\omega$ and E-PA$^\omega$, are weakened to a quantifier-free rule of extensionality

$$\frac{A_0 \to s =_\rho t}{A_0 \to r[s/x^\rho] =_\tau r[t/x^\rho]}$$

where A_0 is quantifier-free and s^ρ, t^ρ, r^τ are terms of WE-HA$^\omega$. The reason for this weakening is that, as explained in [45] (see Chapter 8 there) any interpretation that satisfies the Markov principle

$$\neg\neg\exists\underline{x} A_0(\underline{x}) \to \exists\underline{x} A_0(\underline{x})$$

(where $\underline{x}$ denotes tuples of arbitrary finite type) and extracts computational witnesses from proofs cannot admit full extensionality.

We now present the complete definition of Gödel's functional Dialectica interpretation ([28], also [45]) combined with Krivine's negative translation ([54]). The role of negative translation is to implement classical reasoning within an intuitionistic system so that the resulting interpretation can be applied also to classical proofs. For this reason as we have earlier mentioned proof mining can provide the extraction of effective bounds even from nonconstructive proofs. In [78] it was shown that this combination of Gödel's functional Dialectica interpretation with Krivine's negative translation coincides with the so-called Shoenfield's functional interpretation ([72]) $A \mapsto A^S$. This fulfills the following (where $\underline{x}, \underline{y}$ denote tuples of functionals of finite type over the base type of the system):

- To every formula A in $\mathcal{L}(\text{WE-HA}^\omega)$ we assign a a translation $A^S \equiv \forall \underline{x}\ \exists \underline{y}\ A_S(\underline{x}, \underline{y})$ where A_S is quantifier-free.
- For $A \equiv \forall \underline{x}\ \exists \underline{y}\ A_0(\underline{x}, \underline{y})$ we have $A^S \equiv A$.
- By classical logic and the quantifier-free axiom of choice QF-AC defined by
$$\forall \underline{x}\ \exists \underline{y}\ F_0(\underline{x}, \underline{y}) \to \exists \underline{B}\ \forall \underline{x}\ F_0(\underline{x}, \underline{B}(\underline{x}))$$
we have $A^S \leftrightarrow A$.

(In the following, for simplicity we omit the tuple notation): Let $A^S \equiv \forall u\ \exists x\ A_S(u, x)$ and $B^S \equiv \forall v\ \exists y\ B_S(v, y)$. We have:

- $P^S \equiv P \equiv P_S$ for atomic P.
- $(\neg A)^S \equiv \forall f\ \exists u\ \neg A_s(u, f(u))$
- $(A \vee B)^S \equiv \forall u, v\ \exists x, y\ (A_S(u, x) \vee B_S(v, y))$
- $(\forall z A)^S \equiv \forall z, u\ \exists x\ A_S(z, u, x)$
- $(A \to B)^S \equiv \forall f, v\ \exists u, y\ (A_S(u, f(u)) \to B_s(v, y))$
- $(\exists z\ A_S)^S \equiv \forall U\ \exists z, f\ A_S(z, U(z, f), f(U(z, f)))$
- $(A \wedge B)^S \equiv \forall n, u, v\ \exists x, y\ ((n = 0 \to A_S(u, x)) \wedge (n \neq 0 \to B_S(v, y)))$
$\leftrightarrow \forall u, v\ \exists x, y\ (A_S(u, x) \wedge B_S(v, y))$.

The idea is that that S extracts from a given proof p:

$$p \vdash \forall x\ \exists y\ A(x, y)$$

an explicit effective functional that realizes A^S, that is:

$$\forall x\ A_S(x, \Phi(x)).$$

A variation of the above is the *monotone* functional interpretation introduced by Kohlenbach in [40] (again combined with negative translation) which extracts a Φ^* such that:

$$\exists Y\ (\Phi^* \gtrsim Y \wedge \forall x\ A_S(x, Y(x))).$$

The relation $\gtrsim$ corresponds to a notion of being a "bound" that applies also to higher order function spaces, originally introduced by W.A. Howard ([32]). Such a bound is called a *majorant* and the corresponding relation is called a *majorizability* relation. If an element has a majorant, we say that it is *majorizable*. The relation is defined as follows:

$$x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x,$$

$$x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y\ (y^* \gtrsim_\rho y \to x^*(y^*) \gtrsim_\tau x(y)).$$

1.2.3 Proof Mining Metatheorems

In this section we will present two main metatheorems of proof mining (extensions of) which will manifest themselves in the course of this thesis, in the sense that in the concrete cases that will be studied, the metatheorems will guarantee the extractability of computable and highly uniform bounds. This will be the motivation behind our study of the proofs at hand, and after our analysis such bounds will indeed be extracted. The proofs of the following metatheorems involve running the Dialectica interpretation as introduced in the previous section thus ensuring the extractability of the bounds.

Consider the system of E-PA$^\omega$ enriched with WKL i.e. the axiom that every infinite subtree of the tree of all finite sequences of 0's and 1's has an infinite path (which is a non-effective axiom) and the quantifier-free versions of the axiom of choice QF-AC^{1,0} and QF-AC^{0,1} i.e.

$$\text{QF-AC}\ \forall x^\rho\ \exists y^\tau\ A_0(x,y) \to \exists Y^{\rho\to\tau}\ \forall x^\rho\ A_0(x, Y(x))$$

where $(\rho,\tau) = (1,0)$ and $(\rho,\tau) = (0,1)$ respectively.

Metatheorem 1. *(Kohlenbach, Theorem 15.1 in [45]) Let X be a Polish space, K a compact metric space and $A_\exists(n^0, x^1, y^1, m^0)$ a purely existential formula of $\mathcal{L}$(E-PA$^\omega$) where the types of the existential quantifiers are of degree at most* 1 *and n, x, y, m are the only free variables. Assume that*

$$\textit{E-PA}^\omega + \textit{QF-AC}^{1,0} + \textit{QF-AC}^{0,1} + \textit{WKL}$$

proves

$$\forall n^0, m^0, x_1^1, x_2^1, y_1^1, y_2^1\ (x_1 =_X x_2 \wedge y_1 =_X y_2 \wedge A_\exists(n, x_1, y_1, m) \to A_\exists(n, x_2, y_2, m)).$$

If a sentence

$$\forall n \in \mathbb{N}\ \forall x \in X\ \forall y \in K\ \exists m \in \mathbb{N}\ A_\exists(n, x, y, m)$$

is proved in the above system, then one can extract a uniform bound $\Phi(n, x)$ that is primitive recursive in the sense of Gödel's T for $\exists m$, that is,

$$\textit{WE-HA}^\omega \vdash \forall n \in \mathbb{N}\ \forall x \in X\ \forall y \in K\ \exists m \leq \Phi(n, x)\ A_\exists(n, x, y, m)$$

such that $\Phi(n, x)$ *depends on* $x \in X$ *via a representation* $f_x \in \mathbb{N} \to \mathbb{N}$ *of* $x \in X$*. If* $\exists m\ A_\exists$ *is monotone in* m*, then* $\Phi(n, x)$ *provides a uniform realizer for* $\exists m$*.*

It is important to stress that $\Phi(n, x)$ does not depend on $y \in K$.

We moreover stress that the noneffective WKL can be *eliminated* from the system in which the bound is extracted. An instance of this phenomenon will be observed in Lemma 7, see Remark 10 in connection to that.

Compactness means constructively completeness and total boundedness and we note that both the total boundedness and the completeness of the compact space K are necessary for the above result to hold in general (see [45]). We saw that Metatheorem 1 guarantees the possibility to extract computable uniform bounds that are independent from parameters in compact metric spaces K but only depend on representatives of elements in Polish spaces X. However while working with general classes of abstract metric spaces it is possible in certain contexts to obtain bounds that are independent from noncompact but only metrically bounded spaces, provided that no separability assumptions on the spaces are used. To this end, we will consider the system $\mathcal{A}^\omega$, defined as:

$$\mathcal{A}^\omega :\equiv \text{WE-PA}^\omega\text{+QF-AC+WKL}$$

(Note that WKL is a weak form of the axiom schema of dependent choice DC:

$$\text{DC}: \ \forall x^0, y^\rho\ \exists z^\rho\ A(x, y, z) \to \exists f^{\rho \to 0}\ \forall x^0\ A(x, f(x), f(S(x)))$$

where $\rho := \rho_1 \to (\rho_2 \to ...(\rho_{n-1} \to \rho_n))$ and S is a function symbol of type $0 \to 0$ for the successor function).

We extend the system $\mathcal{A}^\omega$ to $\mathcal{A}^\omega[X, d]$. This is done by extending the previously defined type system $\mathbf{T}$ to $\mathbf{T}^X$ over both ground types $\mathbb{N}, X$ by introducing the clauses

$$(i) 0 \in \mathbf{T}^X, X \in \mathbf{T}^X \ \ (ii) \rho, \tau \in \mathbf{T}^X \Rightarrow \tau(\rho) \in \mathbf{T}^X.$$

Moreover we add constants $0_X, 1_X$ of type X as well as b_X of type 0 and d_X of type $1(X)(X)$. (Note that in $\mathcal{A}^\omega$ real numbers are represented by sequences of rationals thus of objects of type 1). All the axioms and rules of $\mathcal{A}^\omega$ are extended to the new set of types $\mathbf{T}^X$ (see [45]) and the new constants fulfill a number of new (universal) axioms, see [45], Chapter 17. Without the constant b_X and the axiom that corresponds to it we obtain the system $\mathcal{A}^\omega[X, d]_{-b}$.

For $\rho \in \mathbf{T}^X$ we define $\hat{\rho} \in \mathbf{T}$ inductively by

$$\hat{0} := 0,\ \hat{X} := 0,\ \widehat{\tau(\rho)} := \hat{\tau}(\hat{\rho}).$$

Majorants of functionals of type $\rho \in \mathbf{T}^X$ are functionals of type $\hat{\rho} \in \mathbf{T}$. Kohlenbach introduced a new notion of strong majorizability relation for all types $\rho \in \mathbf{T}^X$ between objects x, y, α of type $\hat{\rho}, \rho, X$ respectively, that is denoted by $\gtrsim_\rho^\alpha$ as follows (see for example Def. 17.50 in [45]):

- $x^0 \gtrsim_0^\alpha y^0 :\equiv x \geq_0 y$,
- $x^0 \gtrsim_X^\alpha y^X :\equiv (x)_\mathbb{R} \geq_\mathbb{R} d_X(y, \alpha)$,
- $x \gtrsim_{\tau(\rho)}^\alpha y :\equiv \forall z', z\ (z' \gtrsim_\rho^\alpha z \to xz' \gtrsim_\tau^\alpha yz) \wedge \forall z', z\ (z' \gtrsim_{\hat{\rho}}^\alpha z \to xz' \gtrsim_{\hat{\tau}}^\alpha xz)$.

For any nonempty set X, the full set-theoretic type structure $\mathcal{J}^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$ over $\mathbb{N}$ and X is defined by

$$S_0 := \mathbb{N},\ S_X := X,\ S_{\rho(\tau)} := S_\rho^{S_\tau}$$

where $S_\rho^{S_\tau}$ is the set of all set-theoretic functions $S_\tau \to S_\rho$. $\mathcal{J}^{\omega,X}$ is a model of $\mathcal{A}^\omega[X, d]$ that satisfies the quantifier-free rule of extensionality since in a metric space we have $d(x, y) = 0 \leftrightarrow x = y$.

The following fundamental theorem of proof mining refers to any nonempty metric space (X, d).

Metatheorem 2. *(Gerhardy and Kohlenbach, Theorem 17.52 in [45], also see [24]) Let ρ be any finite type. Let $B_\forall(x, u)$ $C_\exists(x, v)$ containing only x, u free respectively x, v free. Assume that 0_X does not occur in $B_\forall$, $C_\exists$. From a proof of*

$$\forall x^\rho\ (\forall u^0\ B_\forall(x, u) \to \exists v^0\ C_\exists(x, v))$$

in $\mathcal{A}^\omega[X, d]_{-b}$ one can extract a total functional Φ, that is primitive recursive in the sense of Gödel's T, from the strongly majorizable elements $S_{\hat{\rho}}$ to $\mathbb{N}$ so that for all nonempty metric spaces (X, d) and for all $x \in S_\rho$, $x^* \in S_{\hat{\rho}}$, if there exists an $\alpha \in X$ so that $x^* \gtrsim^\alpha x$ then

$$\forall u \leq \Phi(x^*)\ B_\forall(x, u) \to \exists v \leq \Phi(x^*)\ C_\exists(x, v)$$

holds in the sense of Definition 17.29 in in [45].

Note that because here we have defined $\mathcal{A}^\omega :\equiv$ WE-PA$^\omega$+QF-AC+WKL i.e. with WKL instead of DC, we can allow for arbitrary finite types, compare with Theorem 17.52 in [45] and see Remark 4.11 in [24] as well as Remark 17.37 in [45]. Because all constructions are realized on the level of the majorants, the bounds are primitive recursive in the sense of Gödel's T. As we extract functionals over $\mathbb{N}$, these are computable without the need to impose any computability structure on the underlying space.

The above metatheorem is adapted accordingly in various concrete cases. For abstract normed spaces we extend the system $\mathcal{A}^\omega$ to $\mathcal{A}^\omega[X, \|\cdot\|]$ again by extending to $\mathbf{T}^X$ over both ground types $\mathbb{N}, X$, adding constants $0_X, 1_X$ of type X and instead of b_X and d_X as for $\mathcal{A}^\omega[X, d]$ we add $+_X$ of type $X(X)(X)$, $-_X$ of type $X(X)$, $\cdot_X$ of type $X(X)(1)$, $\|\cdot\|_X$ of type $1(X)$ for vector addition, substraction, scalar multiplication and the norm respectively. For the latter a number of universal axioms are introduced (see [45], Chapter 17.3). Note that for normed spaces we take always $\alpha := 0_X$.

We mention in particular how Kohlenbach's strong majorizability relation applies for the concrete cases that will be studied in this thesis : If $x \in \mathbb{N}$, then also $x^* \in \mathbb{N}$ and it is $x^* \geq x$. If $x \in \mathbb{N} \to \mathbb{N}$, then also $x^* \in \mathbb{N} \to \mathbb{N}$ and x^* is a nondecreasing upper bound on x. If $x \in X$ where X is a normed space, then $x^* \in \mathbb{N}$ and $x^* \gtrsim x := x^* \geq \|x\|$. If $x \in X \to \mathbb{N}$ then $x^* \in \mathbb{N} \to \mathbb{N}$ and it is nondecreasing with $x^*(n) \geq \|x(m)\|$ whenever $n \geq m$. Finally, If $x \in X \to X$ then $x^* \in \mathbb{N} \to \mathbb{N}$ and it is nondecreasing with $x^*(n) \geq \|x(y)\|$ whenever $n \geq \|y\|$. Note that all elements of $\mathbb{N}, X, \mathbb{N} \to \mathbb{N}, X \to \mathbb{N}$ are majorizable but this is not always the case for $X \to X$, it is however true in particular for the class of nonexpansive mappings in $X \to X$.

Without going into any more detail here, we only mention that it is possible to enrich such metatheorems also for extended theories by adding new axioms, provided that these are universal, and new constants (corresponding to the new axioms), provided that these have majorants. This has already been done for many abstract types of metric structures (see [45], [46]). In Section 3.1.3 of this thesis we will see such adaptations formalized for concrete mathematical settings. Another example (that is also relevant for this thesis) is that of the theory $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ for uniformly convex spaces which follows from $\mathcal{A}^\omega[X, \|\cdot\|]$ by adding a constant η of type 1 together with the following universal axiom:

$$\forall x^X, y^X\ \forall k^0\ (\|x\|, \|y\| <_{\mathbb{R}} 1_{\mathbb{R}} \wedge \|\frac{x +_X y}{2}\| >_{\mathbb{R}} 1 - 2^{-\eta(k)} \to \|x -_X y\| \leq 2^{-k}).$$

Essentially, as we will see in Section 1.3 where some preliminary mathematical notions are introduced, the above amounts to η being a modulus of uniform convexity for the space.

1.2.4 Remarks on the Possibilities and Limits of Proof Mining

We have so far seen a very minimal introduction on the logical background and tools as well as the main metatheorems of proof mining that will be used in the course of this thesis. In relation to the above and regarding the possibilities and limits of proof mining in actual mathematical practice, we can make the following comments.

The information *a priori given* by the proof mining metatheorems (in contexts where the latter are applicable) regardless of whether the given proof of a certain statement is constructive or not, is the following :

- That it is possible to extract a bound from the given proof.
- That the bound will be computable (moreover in the setting introduced above it will be primitive recursive in the sense of Gödel's T).
- That the bound will depend on general bounding information on the input data (majorants).
- That the bound will not depend on anything else, in other words, the bound will be 'uniform'.

If it happens to turn out that the obtained bound depends on less input data than expected, then we would have also achieved to obtain a strengthening of the original statement by eliminating one (or more) of the assumptions.

The information that *we cannot know a priori* by the proof mining metatheorems is the following:

- How easy or difficult it will eventually be to extract the bound from the proof.
- The complexity of the bound. (An estimation about the complexity of the bound, however, can be made by inspecting the original proof itself. More specifically, for a sufficiently weak system, for a single use of sequential compactness we can obtain a bound that is at most primitive recursive in the sense of Kleene, for instance see Proposition 13.27 in [45]. If Heine-Borel compactness is used, again for a sufficiently weak system, we will obtain bounds which are polynomial in the input data, for instance see Corollary 12.37 in [45]. Also see Remark 17.37 in [45].)
- The precise method of extracting the bound. Typically, this is done in three stages: (i) The statements involved must be written in a formal version using quantifiers. (ii) The mathematical objects involved must have the correct uniformity. While ensuring this, the quantitative content of their properties is made explicit (i.e. modulus of continuity for uniform continuity, modulus of accretivity for uniform accretivity, modulus of convexity for uniform convexity, effective irrationality measure for irrationality etc). In that way we obtain quantitative versions of the statements/ lemmas involved. (iii) Finally, we put the latter together in a deduction schema just like the one of the original proof, i.e. the structure of the original proof is typically preserved.

However, the above process and the steps thereof are *not automated* and (even though they are not ad hoc) they are open to the manipulations of the mathematician(s) performing proof mining on a given proof. Moreover, after having obtained the bound, it is in principle not possible to show that it is the optimal one for a specific proof.

In the case that we have two or more different proofs of the same statement, it is expected that we will obtain different bounds from each one. In general, the bounds may differ not only by numerical factors, but also with respect to their complexity. An instance of this phenomenon appears in Section 3.

1.3 Basic Mathematical Notions, Notation and Conventions

From now on by $\mathbb{N}$ we denote the set of natural numbers $\{1,2,...\}$. By $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ the sets of integers, rational and real numbers respectively and by $\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$ the sets of nonnegative integers, rationals and reals respectively. For $x \in \mathbb{R}$, by $[x]$ we denote the floor function (the largest integer not exceeding x) and by $\lceil x \rceil$ the ceiling function (the smallest integer exceeding or equal to x).

Let X be a Banach space. We recall the following basic definitions.

Definition 4. *(Clarkson ([14]), also see [43]) A Banach space X is called uniformly convex if*

$$\forall \epsilon \in (0,2]\ \exists \delta \in (0,1]\ \forall x,y \in X$$

$$(\|x\|,\|y\| \leq 1\ \wedge \|x-y\| \geq \epsilon \to \|\frac{1}{2}(x+y)\| \leq 1-\delta).$$

A mapping $\eta : (0,2] \to (0,1]$ giving such a $\delta := \eta(\epsilon)$ is called a modulus of uniform convexity.

For example, as a modulus of uniform convexity one may consider Clarkson's modulus of convexity ([14]) defined for any Banach space X as the function $\eta_X : (0,2] \to (0,1]$ given by

$$\eta_X(\epsilon) := \inf\{1 - \|\frac{x+y}{2}\|\ :\ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}.$$

Remark 1. *For the Banach spaces $L_p, p \geq 2$, it is useful to consider the "asymptotically optimal" modulus of convexity $\eta(\epsilon) := \frac{\epsilon^p}{p 2^p}$ (see, for example, [29], also [43] and [42]). The motivation is that it has the advantage that it can be written in the form $\eta(\epsilon) = \epsilon\tilde{\eta}(\epsilon)$ where $\tilde{\eta}$ is increasing with ϵ, in which case it can often be shown that then in the bound we can replace η with $\tilde{\eta}$. There will be several instances of applications of this remark throughout this thesis.*

Definition 5. *Given a Banach space X and a subset $C \subseteq X$, a mapping T on C is called nonexpansive if*

$$\forall x, y \in C \ \|Tx - Ty\| \leq \|x - y\|.$$

Definition 6. *(Bruck [10], also see [66]) Let C be a convex, nonempty subset of a real Banach space X. For $t \in (0,1)$ a mapping $T : C \to X$ is called t-firmly nonexpansive if T is nonexpansive and*

$$\forall x, y \in C \ \|Tx - Ty\| \leq \|(1-t)x + tTx - ((1-t)y + tTy)\|.$$

If T is t-firmly nonexpansive for every $t \in (0,1)$, then we say that T is firmly nonexpansive.

Definition 7. *(Krasnoselskii ([51]), also see [45]) Let C be a convex subset of a Banach space X and let $T : C \to C$ nonexpansive. The sequence*

$$x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Tx_n$$

is called the Krasnoselskii iteration of T starting at x_0.

If

$$\|x_n - Tx_n\| \stackrel{n\to\infty}{\to} 0$$

for all $x_0 \in C$, where $\{x_n\}$ is a given iteration starting at x_0, then T, or more precisely

$$T_{1/2} := \frac{1}{2}(I + T),$$

is called asymptotically regular. A rate of convergence for the above convergence is called a rate of asymptotic regularity for T.

Although the above notion was originally introduced by Browder and Petryshyn in [9] in particular with respect to the Picard iteration $x_{n+1} := Tx_n$, in the following we will refer to convergence results of the above form also for different iterations $\{x_n\}$ as asymptotic regularity results. The following classical results will be of use in the course of this thesis :

Theorem 2. *(Ishikawa ([33]) Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ convex and $T : C \to C$ nonexpansive. If the Krasnoselskii iteration $\{x_n\}_{n\in\mathbb{N}}$ of T is bounded, then*

$$\|x_n - Tx_n\| \stackrel{n\to\infty}{\to} 0,$$

that is, T, or more precisely

$$T_{1/2} := \frac{1}{2}I + \frac{1}{2}T,$$

is asymptotically regular.

Theorem 3. *(Baillon and Bruck ([3]) Let X be a Banach space, $C \subseteq X$ convex and $T : C \to C$ nonexpansive. Then for the Krasnoselskii iteration x_n of T we have*

$$\forall \epsilon > 0 \ \forall n \geq \theta(\epsilon, d) \ (\|x_n - Tx_n\| < \epsilon)$$

with a rate of asymptotic regularity

$$\theta(\epsilon, d) := \frac{4d^2}{\pi \epsilon^2}$$

where $d > 0$ is such that $d \geq \|x_0 - Tx_n\|$ for all $n \in \mathbb{N}$.

It is well-known that firmly nonexpansive selfmappings $T : C \to C$ on a convex subset $C \subseteq X$ have the interesting property that if they have a fixed point, then they are asymptotically regular.

A central mathematical object that will be studied throughout this thesis (and can be considered as the common ground between the different applications of proof mining that we will carry out, both in PDE theory and fixed point theory) is a one-parameter semigroup of nonexpansive mappings (or nonexpansive semigroup for short) defined as follows.

Definition 8. *(See, for instance, [4], [20]) Let X be a Banach space with a subset $T \subseteq X$. Let $\mathcal{F} = \{T(t) : \ C \to C, \ t \geq 0\}$ be a family of self-mappings of $C \subseteq X$. $\mathcal{F}$ is said to be a nonexpansive semigroup acting on C if*

1. *$T(0) = I$, where I is the identity mapping on C,*
2. *$T(s + t)x = T(s) \circ T(t)x$ for all $s, t \in [0, \infty)$ and $x \in C$,*
3. *$\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in [0, \infty)$,*
4. *$t \mapsto T(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$.*

Property 4 in Definition 8 refers here to continuity with respect to the strong operator topology (hence in the literature the above is often referred to as "strongly continuous"). It is important to point out that here we deal with *nonlinear* nonexpansive semigroups. Such nonlinear nonexpansive semigroups play a central role in the study of Cauchy problems (some classical references are [4], [7], [15], [68])

In this thesis such semigroups are studied indirectly in Chapter 2 as they arise within the Crandall-Liggett formula which gives the solution of a homogeneous Cauchy problem generated by an accretive operator and moreover within the definition of an almost-orbit.

In Chapter 3 nonexpansive semigroups are studied directly in terms of their (approximate) common fixed points. (However note that in the definition used throughout Chapter 3, as is the case in the fixed point literature in general, Property 1 of Definition 8 is formally not included, see for instance [80], [79].)

Chapter 2

Proof Mining for Classes of Accretive Operators and PDE Theory

2.1 Preliminaries

Let X be a real Banach space. By $B[x,r]$ we denote the closed ball of X with radius r and center $x \in X$. Let $\tilde{X}$ be the dual space of X. The normalized duality mapping is defined by

$$\mathcal{J}(x) := \{j \in \tilde{X} : \langle x, j\rangle = \|x\|^2, \ \|j\| = \|x\|\}.$$

Let

$$\langle y, x\rangle_+ := \max\{\langle y, j\rangle : \ j \in \mathcal{J}(x)\}.$$

Note that for all $x, y \in X$, $\langle y, x\rangle_+ \leq \|x\|\|y\|$ (for example see (1.4) in [4]).

A mapping $A : X \to 2^X$ will be called an operator on X. The domain and range of A will be denoted by $D(A)$ and $R(A)$ respectively. Here $x \in D(A) :\equiv Ax \neq \emptyset$.

Note that for such a set-valued operator A, "$\forall (x,u) \in A$ " is equivalent to writing "$\forall x \in D(A)\ \forall u \in Ax$". In the following, both alternative notations will be used.

We say that A satisfies the range condition if $\overline{D(A)} \subseteq R(I + \lambda A)$ for all $\lambda > 0$ where $\overline{D(A)}$ denotes the closure of $D(A)$.

Definition 9. *An operator A is said to be accretive if for all $\lambda \geq 0$, $u \in Ax$, $v \in Ay$,*

$$\|x - y + \lambda(u - v)\| \geq \|x - y\|.$$

This notion was originally introduced in 1967 independently by F.E. Browder ([8]), T. Kato ([34]) and Y. Komura ([50]). Note that A is accretive if and only if $-A$ is dissipative.

Some standard references on accretive and dissipative operators and applications thereof, especially in differential equations, are [4], [13], [17], [7], [15], [68].

Definition 10. *An accretive operator A is said to be m-accretive if for all $\lambda > 0$, $R(I + \lambda A) = X$.*

For an accretive A, for each $\lambda > 0$ we may define a single-valued mapping $J_\lambda^A : R(I+\lambda A) \to D(A)$ by $J_\lambda^A := (I+\lambda A)^{-1}$ where I is the identity operator. J_λ^A is called the *resolvent* of A. If it is clear from the context what A is, for simplicity we will write J_λ instead of J_λ^A. We will make use of the following well-known facts about accretive operators:

Proposition 2. *(Proposition 2.1. in [21] for a proof see, for example, [4])*
(i) An operator A on X is accretive if and only if for all $(x, u), (y, v) \in A$

$$\langle u - v, x - y\rangle_+ \geq 0.$$

(ii) An operator $A : X \to 2^X$ is accretive if and only if for each $\lambda > 0$ the resolvent J_λ^A is a single-valued nonexpansive mapping.
(iii) If A is accretive, for all $x \in R(I + \lambda A)$ with $\lambda > 0$, $\frac{I-J_\lambda^A}{\lambda} x \in AJ_\lambda^A x$.

If assuming $x \neq y$ and $u \neq v$ only the strict inequality in (i) above holds, we will say that A is strictly accretive ([1], [17]).

Consider the following initial value problem :

Problem 1. *(Non-Homogeneous Abstract Cauchy Problem)*

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$

$$u(0) = x$$

where $A : D(A) \to 2^X$ is an accretive operator with the range condition and $f \in L^1(0, \infty, X)$.

Definition 11. *A continuous function $u : [0, \infty) \to X$ is an integral solution of Problem 1 if $u(0) = x$ and for $s \in [0, t]$ and $(w, y) \in A$*

$$\|u(t) - w\|^2 - \|u(s) - w\|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - w\rangle_+ d\tau.$$

It is known (for instance see [4], Chapter III.2) that for each $x \in \overline{D(A)}$ Problem 1 has a unique integral solution u so that $u(t) \in \overline{D(A)}$ for all t. Moreover, it is known ([15]) that for $x_0 \in \overline{D(A)}$ the following initial value problem :

Problem 2. *(Homogeneous Abstract Cauchy Problem)*

$$u'(t) + A(u(t)) \ni 0, t \in [0, \infty)$$

$$u(0) = x_0,$$

where $A : D(A) \to 2^X$ *is an accretive operator with the range condition,*

has a unique integral solution given by the Crandall-Liggett formula :

$$u(t) := S(t)(x_0) = \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}(x_0),$$

see for example Chapter III 1.2 in [4], [15]. We say that the operator $-A$ generates the nonexpansive semigroup $\mathcal{F} : \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$. (Note that the Crandall-Liggett formula for an operator A that is accretive instead of dissipative it is written with $\lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}$).

Definition 12. *A continuous function* $u : [0, \infty) \to X$ *is said to be a strong solution of Problem 2 if it is Lipschitz on every bounded subinterval of* $[0, \infty)$*, almost everywhere differentiable on* $[0, \infty)$*,* $u(t) \in D(A)$ *almost everywhere,* $u(0) = x_0$ *and* $u'(t) + A(u(t)) \ni 0$ *for almost every* $t \in [0, \infty)$*.*

A classical reference where the theory of such Cauchy problems generated by linear, single-valued operators A, is treated, is [69]. For the theory with nonlinear and moreover set-valued and dissipative A (equivalently accretive $-A$) that we will study in this thesis see [4], [7], [15], [68].

2.2 Special Notions of Accretivity and Introduction of the Modulus

This section covers certain special notions of accretivity introduced by García-Falset (see [20], [23], [21]) that we will work on, as well as the new notion of the modulus of accretivity that was introduced by Kohlenbach and the author in [47] which is definable for uniform versions of such accretivity notions and corresponds to quantitative versions for the latter.

As discussed in Chapter 1, our approach is based on a logical metatheorem by Kohlenbach which uses the proof-theoretic extraction algorithm of monotone functional "Dialectica" interpretation combined with negative translation. to obtain uniform effective bounds. Because this algorithm keeps track of uniform bounding information by recursion over the given proof, starting from the assumptions used and proceeding to the conclusion of the proof, the assumptions have to have the right uniformity. Imposing sufficient uniformity for the assumptions amounts to precisely guaranteeing the existence of a modulus of accretivity, which is well-defined for corresponding uniform accretivity notions. Such a modulus of accretivity was first introduced by Kohlenbach and the author in [47] and can be understood in an analogy with the concept of a modulus of continuity for uniform continuity, or a modulus of convexity for uniform convexity. We we will see in praxis how imposing the appropriate uniformity for accretivity provides a modulus in the course of this section.

Analogously, for the notion of ψ-expansivity we will see that we can introduce a similar notion of a modulus of ψ-expansivity. The said moduli will be employed in the proof mining analysis in the next sections, as it will be essential to capture and make explicit the quantitative content of the accretivity, respectively expansivity notion. This quantitative content will appear in the resulting bounds that will be derived.

Definition 13. *(See e.g. [63], [23]) Let X be a real Banach space and let $\psi : [0,\infty) \to [0,\infty)$ be a continuous function such that $\psi(0) = 0$ and $\psi(r) > 0$ for $r > 0$. A mapping $A : D(A) \to 2^X$ is said to be ψ-expansive if for every $x, y \in D(A)$ and every $u \in Ax$ and $v \in Ay$*

$$\|u - v\| \geq \psi(\|x - y\|).$$

The above notion is already uniform, therefore we can introduce here a modulus of ψ-expansivity (in analogy to the notions that had been introduced by Kohlenbach and the author in [47]) as follows:

Definition 14. *Given a real Banach space X and a function $\Delta_{(\cdot)}(\cdot) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ we say that a ψ-expansive operator $A : D(A) \to 2^X$ has a modulus of ψ-expansivity Δ if*

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall (x,u),(y,v) \in A$$
$$(\|x - y\| \in [2^{-k}, K] \to \|u - v\| \geq 2^{-\Delta_K(k)}).$$

Proposition 3. *Let X be a real Banach space. Every ψ-expansive operator $A : D(A) \to 2^X$ has a modulus of ψ-expansivity Δ.*

Proof. For $k \in \mathbb{N}, K \in \mathbb{N}$ define

$$\Delta_K(k) := \min n\, (2^{-n} \leq \inf\{\psi(y) : y \in [2^{-k}, K]\})$$

The above is well-defined since $\psi : [0,\infty) \to [0,\infty)$ is continuous with $\psi(x) > 0$ for $x > 0$.
Then,

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall (x,u),(y,v) \in A$$

$$(\ \|x-y\| \in [2^{-k}, K] \to \|u-v\| \geq \psi(\|x-y\|) \geq 2^{-\Delta_K(k)} \ ).$$

□

Definition 15. *(See e.g. [23], [20]) Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous function such that $\psi(0) = 0$ and $\psi(x) > 0$ for $x \neq 0$. A mapping $A : D(A) \to 2^X$ on a real Banach space X is ψ-strongly accretive if*

$$\forall (x,u),(y,v) \in A \ \langle u-v, x-y\rangle_+ \geq \psi(\|x-y\|)\|x-y\|).$$

Operators that are accretive and ψ-expansive are a larger family than the family of operators that are ψ-strongly accretive. This is easy to see as follows: if A is ψ-strongly accretive i.e. if

$$\forall (x,u),(y,v) \in A \ \langle u-v, x-y\rangle_+ \geq \psi(\|x-y\|)\|x-y\|$$

then

$$\forall (x,u),(y,v) \in A \ \|u-v\|\|x-y\| \geq \psi(\|x-y\|)\|x-y\|$$

thus

$$\forall (x,u),(y,v) \in A \ \|u-v\| \geq \psi(\|x-y\|)$$

so A is ψ-expansive. However, the converse does not hold. For a counterexample see for instance Example 3.6. in [21].

Notice that if A is ψ-strongly accretive, and if A has a zero $z \in D(A)$ then z is unique. To prove this, let $z, z' \in D(A)$ so that $Az \ni 0$, $Az' \ni 0$ and $z \neq z'$. Then

$$\langle 0, z-z'\rangle_+ \geq \psi(\|z-z'\|)\|z-z'\|$$

but by definition

$$\psi(\|z-z'\|) \leq 0 \to \|z-z'\| = 0,$$

hence $z = z'$.

We introduce a quantitative form of the above already uniform notion that we call *modulus of accretivity* Θ *for* ψ*-strong accretivity*:

Definition 16. *(Kohlenbach and K.-A. ([47])) Given a real Banach space X and a function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we say that a ψ-strongly accretive operator $A : D(A) \to 2^X$ has a modulus of accretivity Θ if*

$$\forall k \in \mathbb{N} \ \forall K \in \mathbb{N} \ \forall (x,u),(y,v) \in A$$

$$(\|x-y\| \in [2^{-k}, K] \to \langle u-v, x-y\rangle_+ \geq 2^{-\Theta_K(k)}\|x-y\|).$$

Proposition 4. *(Kohlenbach and K.-A. ([47])) Let X be a real Banach space. Every ψ-strongly accretive operator $A : D(A) \to 2^X$ has a modulus of accretivity Θ.*

Proof. For $k \in \mathbb{N}, K \in \mathbb{N}$ define

$$\Theta_K(k) := \min n \, (2^{-n} \leq \inf\{\psi(y) : y \in [2^{-k}, K]\})$$

The above is well-defined since $\psi : [0, \infty) \to [0, \infty)$ is continuous with $\psi(x) > 0$ for $x > 0$.
Then,

$$\forall k \in \mathbb{N} \; \forall K \in \mathbb{N} \; \forall (x, u), (y, v) \in A \; (\; \|x - y\| \in [2^{-k}, K] \to$$

$$\langle u - v, x - y \rangle_+ \geq \psi(\|x - y\|)\|x - y\| \geq 2^{-\Theta_K(k)}\|x - y\| \;).$$

□

In [20] the notion of *ϕ-accretivity at zero* for an operator $A : D(A) \to 2^X$ is introduced:

Definition 17. *(García-Falset, Definition 2 in [20]) Let X be a real Banach space, let $\phi : X \to [0, \infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ so that for every sequence (x_n) in X such that $(\|x_n\|)$ is nonincreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$. An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is said to be ϕ-accretive at zero if*

$$\forall (x, u) \in A \; (\langle u, x - z \rangle_+ \geq \phi(x - z)) \; (\sharp).$$

Proposition 5. *(García-Falset, Proposition 4 in [20]) Let $A : D(A) \to 2^X$ be an m-accretive operator on X such that there exists $z \in X$ satisfying expression $(\sharp)$. Then A is ϕ-accretive at zero (i.e. it is moreover $0 \in Az$).*

Theorem 4. *(García-Falset and Morales, Theorem 8 in [23]) Let X be a real Banach space. Let $A : D(A) \to 2^X$ be an m-accretive and ψ-expansive mapping on $D(A)$. Then A is surjective.*

By the above theorem, if $A : D(A) \to 2^X$ is m-accretive and ψ-strongly accretive A (as we saw that every ψ-strongly accretive A is also ψ-expansive so the above applies) then A is also ϕ-accretive at zero with $\phi(\cdot) := \psi(\|\cdot\|)\|\cdot\|$ (as surjectivity guarantees the existence of a $z \in D(A)$ so that $0 \in Az$).

So, in short, the situation is summarized in the following diagram:

ψ-strongly accretive

$\Downarrow$ $\qquad\qquad$ $\Downarrow$ if moreover m-accretive

ψ-expansive $\quad \neq \quad$ ϕ-accretive at zero

Now notice that for a ϕ-accretive at zero operator A as in Definition 17, for $z \in D(A)$ so that $Az \ni 0$ we can again show uniqueness of z, as assuming that there exists a $D(A) \ni z' \neq z$ so that $Az' \ni 0$, for $(z', 0) \in A$ we obtain

$$\langle 0, z' - z \rangle_+ \geq \phi(z' - z)$$

therefore

$$\phi(z' - z) = 0,$$

and because $z' - z \neq 0 \to \phi(z' - z) > 0$,

$$z = z'.$$

However, here there exists no uniform notion of a modulus of accretivity as the distance that $\phi(x)$ has from 0 not only depends on the distance that $\|x\| > 0$ has from 0 but on $x \neq 0$ itself.

In our proof-theoretic analysis of the proofs we will have to consider operators $A : D(A) \to 2^X$ that have a well-defined modulus of accretivity, in order to capture and make explicit the quantitative content of the accretivity notion. This is the case, for instance, when A has the -more restrictive (in the sense that we impose additionally uniformity)- accretivity property that we introduce below.

Definition 18. *(Kohlenbach and K.-A. ([47])) We say that a ϕ-accretive at zero operator $A : D(A) \to 2^X$, where X is a real Banach space, is uniformly ϕ-accretive at zero if $\phi : X \to [0, \infty)$ is in particular of the form*

$$\phi(x) = g(\|x\|)$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous function such that $g(0) = 0$ and $g(\alpha) > 0$ for $\alpha \neq 0$.

The motivation for this choice is the possibility to define again a uniform notion of modulus of accretivity Θ for uniformly ϕ-accretive at zero operators in the following sense:

Definition 19. *(Kohlenbach and K.-A. ([47]))Given a real Banach space X and a function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we say that a uniformly ϕ-accretive at zero operator $A : D(A) \to 2^X$ with $Az \ni 0$ has a modulus of accretivity Θ if*

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall (x, u) \in A\ (\|x - z\| \in [2^{-k}, K] \to \langle u, x - z \rangle_+ \geq 2^{-\Theta_K(k)}).$$

Proposition 6. *(Kohlenbach and K.-A. ([47]))Let X be a real Banach space. Every uniformly ϕ-accretive at zero operator $A : D(A) \to 2^X$ with $Az \ni 0$ has a modulus of accretivity Θ.*

Proof. By assumption

$$\forall (x,u) \in A \ (\langle u, x-z\rangle_+ \geq \phi(x-z) = g(\|x-z\|)).$$

In a similar spirit as in the previous proof for the case of a ψ-strongly accretive operator, we have

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall x \in D(A)\ (\|x-z\| \in [2^{-k}, K]\ \to g(\|x-z\|) \geq 2^{-\Theta_K(k)})$$

where we have defined

$$\Theta_K(k) := \min\ n(2^{-n} \leq \inf\{g(\alpha) : \alpha \in [2^{-k}, K]\}).$$

□

Remark 2. *For a uniformly ϕ-accretive at zero operator, in the case where the function g is nondecreasing, the modulus of accretivity Θ does not depend on K, as in this case, clearly,*

$$\inf\{g(\alpha)\ :\ \alpha \in [2^{-k}, K]\} = g(2^{-k}).$$

This is usually the case in many applications and, in particular, in the application that we discuss in Section 2.3.3. Clearly, the analogous conclusion holds for ψ-strongly accretive operators.

Note that given a ψ-strongly accretive operator A, if A is also uniformly ϕ-accretive at zero with $0 \in Az$, given a modulus of ψ-strong accretivity $\Theta^{\psi}_K(k)$, we easily obtain a modulus of uniform ϕ-accretivity at zero $\Theta^{\phi}_K(k)$ by noticing that

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall (x,u) \in A$$

$$(\|x-z\| \in [2^{-k}, K] \to (\langle u, x-z\rangle_+ \geq 2^{-\Theta^{\psi}_K(k)}\|x-z\| \geq 2^{-\Theta^{\psi}_K(k)} \cdot 2^{-k}))$$

which gives

$$\Theta^{\phi}_K(k) := \Theta^{\psi}_K(k) + k$$

as a modulus of uniform ϕ-accretivity at zero.

In [21] García-Falset introduces a definition of ϕ-accretivity at zero that is different to his aforementioned definition (Definition 17 i.e. Definition 2 in [20]):

Definition 20. *(García-Falset, Definition 3.3. in [21]) Let X be a real Banach space and let $\phi : [0,\infty) \to [0,\infty)$ be a continuous function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. We say that an accretive operator $A : D(A) \to 2^X$ is ϕ-accretive at zero whenever there exists $z \in X$ such that for all $(x,u) \in A$*

$$\langle u, x-z\rangle_+ \geq \phi(\|x-z\|).$$

The above Definition 20 (i.e. Definition 3.3. in [21]) differs from the aforementioned Definition 17 (i.e. Definition 2 in [20]) in two ways:

(1) In Definition 17 the existence of the zero of the operator A, i.e. the fact that z is actually such that $0 \in Az$ is included, though it is not included here. However, note that for an m-accretive operator A, by Proposition 5, Definition 20 is consistent with Definition 17, i.e. m-accretivity together with ϕ-accretivity at zero as in [21] give that $0 \in Az$ Another way to see that $0 \in Az$ (again for an m-accretive operator A) is by the following theorem:

Theorem 5. *(García-Falset, Llorens-Fuster and Prus, Theorem 6 in [22] or Theorem 4.1 (a) in [21]) Let X be a Banach space, let $A : D(A) \subseteq X \to 2^X$ and $z \in X$ be as in Definition 20. Then, assuming $z \in R(I + A)$, we have $0 \in Az$.*

The above is applicable as $z \in R(I + A)$ clearly follows from m-accretivity. m-accretivity for A is indeed always assumed throughout both [20] and [21] as well as in this work.

(2) "Uniformity" due to the norm : Definition 20 involving the norm is already uniform as it is considered $\phi : [0, \infty) \to [0, \infty)$, however Definition 17 was not as it was considered $\phi : X \to [0, \infty)$ thus the distance of $\phi(\cdot)$ from 0 would not depend only on the norm $\|x\|$ for an input $x \in X$ but on $x \in X$ itself. So we will refer to this notion as in Definition 20 as *"uniform"* ϕ-accretivity at zero as it now coincides with Definition 18.

At this point we introduce in higher generality (in the sense of omitting the information of the function ϕ) the property of *uniform accretivity at zero* for an operator $A : D(A) \to 2^X$ with $0 \in Az$ as follows:

Definition 21. *(Kohlenbach and K.-A. ([47])) Let X be a real Banach space. An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is called uniformly accretive at zero if*

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall (x, u) \in A$$

$$(\|x - z\| \in [2^{-k}, K] \to \langle u, x - z\rangle_+ \geq 2^{-m})\ (*).$$

Any function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is called a modulus of accretivity at zero for A if $m := \Theta_K(k)$ satisfies $(*)$.

In the following sections we will see that for all the results by García-Falset that we will analyse in this thesis as well as for all the quantitative versions thereof that we will show, instead of uniform ϕ-accretivity at zero, it would actually be sufficient to assume just uniform accretivity at zero as above (thus eliminating the information of the function ϕ).

We are now in the position to make the observation that a modulus of accretivity at zero can follow as a consequence just of the weaker notion of strict accretivity at zero (defined below) by imposing the correct uniformity as expected from the discussion in Chapter 1 and in the beginning of this chapter.

Definition 22. *For a real Banach space X, an operator $A : X \to 2^X$ with $0 \in Az$ is called strictly accretive at zero if*

$$\forall (x,u) \in A\ (x - z \neq 0 \to \langle u, x - z\rangle_+ > 0)(**).$$

Notice that Definition 22 also guarantees the uniqueness of the zero z.

Clearly if an operator is strictly accretive (i.e. if $\langle u - v, x - y\rangle_+ > 0$ for all $(x,u),(y,v) \in A$ where $x \neq y$ and $u \neq v$), it is also in particular strictly accretive at zero, but the converse does not hold. Therefore one should be careful not to restrict to this weaker notion of Definition 22 in general as in praxis for the operator A we need also full accretivity for all $(x,u),(y,v) \in A$ and *not just at zero*.

The important point that we want to make here is that $(*)$ of Definition 21 can be seen only as a consequence of $(**)$ of Definition 22 by 'uniformizing' the latter as follows: The statement $(**)$ can be written as

$$\forall (x,u) \in A\ (\exists k \in \mathbb{N}\ \|x - z\| \geq 2^{-k} \to \exists m \in \mathbb{N} \langle u, x - z\rangle_+ > 2^{-m})$$

which by prenexation gives

$$\forall (x,u) \in A\ \forall k \in \mathbb{N}\ \exists m \in \mathbb{N}(\|x - z\| \geq 2^{-k} \to \langle u, x - z\rangle_+ > 2^{-m})$$

or equivalently

$$\forall K, k \in \mathbb{N}\ \forall x \in D_K\ \forall u \in Ax\ \exists m \in \mathbb{N}(\|x - z\| \geq 2^{-k} \to \langle u, x - z\rangle_+ > 2^{-m})$$

where $D_K := \{x \in D(A)\ :\ \|x - z\| \leq K\}$. Now the above statement is made uniform by moving the quantifier $\exists m \in \mathbb{N}$ forward and we write the above as:

$$\forall K, k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall (x,u) \in A$$

$$(\|x - z\| \leq K \wedge \|x - z\| \geq 2^{-k} \to \langle u, x - z\rangle_+ > 2^{-m}).$$

A computable bound $\chi(k, K) \geq m$ can be seen as a modulus Θ as in $(*)$.

Remark 3. *It is interesting to investigate how a modulus of accretivity/ expansivity for an operator A is associated with a modulus of uniqueness (see [38, 45]) for the zero z so that $0 \in Az$ (if the zero exists). We distinguish the following four cases.*

- *For a ψ-expansive operator A, any modulus of ψ-expansivity Δ yields a 'modulus of uniqueness for the zero z of A' (if the zero z exists) in the following sense: Let $Az \ni 0$ and suppose $\exists v \in Az'$ with $\|v\| \leq 2^{-\delta}$, where $\delta \in \mathbb{N}$. Then, for all $k \in \mathbb{N},\ K \in \mathbb{N}$ by making the choice $(0, z), (v, z') \in A$,*
$$\|z - z'\| \in [2^{-k}, K] \to \|v - 0\| \geq 2^{-\Delta_K(k)}$$
and so
$$\|z - z'\| \in [2^{-k}, K] \to 2^{-\delta} \geq 2^{-\Delta_K(k)}.$$
Let us take
$$\delta = \delta_K(k) := \Delta_K(k) + 1 > \Delta_K(k).$$
Then
$$2^{-\delta_K(k)} < 2^{-\Delta_K(k)}$$
and therefore
$$\|z - z'\| \leq K \wedge \exists v \in Az'(\|v\| \leq 2^{-\delta_K(k)} \to \|z - z'\| < 2^{-k}).$$
In the special case where $\psi(\cdot)$ is nondecreasing, the K dependence for the modulus of uniqueness disappears and the condition $\|z - z'\| \leq K$ is not needed (see Remark 2).

- *For a ψ-strongly accretive operator A, any modulus of accretivity Θ yields a 'modulus of uniqueness for the zero z of A' (if the zero z exists) in the following sense: Let $Az \ni 0$ and suppose $\exists v \in Az'$ with $\|v\| \leq 2^{-\delta}$, where $\delta \in \mathbb{N}$. Then, for all $k \in \mathbb{N},\ K \in \mathbb{N}$*
$$\|z - z'\| \in [2^{-k}, K] \to \|v\|\|z - z'\| \geq \langle v, z - z'\rangle_+ \geq 2^{-\Theta_K(k)}\|z - z'\|$$
and so
$$\|z - z'\| \in [2^{-k}, K] \to 2^{-\delta} \geq 2^{-\Theta_K(k)}.$$
Let us take
$$\delta = \delta_K(k) := \Theta_K(k) + 1 > \Theta_K(k).$$
Then
$$2^{-\delta_K(k)} < 2^{-\Theta_K(k)}$$
and therefore
$$\|z - z'\| \leq K \wedge \exists v \in Az'(\|v\| \leq 2^{-\delta_K(k)} \to \|z - z'\| < 2^{-k}).$$
As above, in the special case where $\psi(\cdot)$ is nondecreasing, the K dependence for the modulus of uniqueness disappears and the condition $\|z - z'\| \leq K$ is not needed (see Remark 2).

- *For a ϕ-accretive at zero operator A, as it has been already stressed, there exists no well-defined modulus of accretivity, thus we cannot associate a modulus of uniqueness for the zero z of A with a modulus of accretivity for A.*

- *For a uniformly accretive at zero operator A, any modulus of accretivity Θ also yields a modulus of uniqueness for the zero z of A as follows: Let $Az \ni 0$ and suppose $\exists v \in Az'$ with $\|v\| \leq 2^{-\delta}$, where $\delta \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $K \in \mathbb{N}$*

$$\|z-z'\| \in [2^{-k}, K] \to 2^{-\delta} \cdot K \geq \|v\|\|z-z'\| \geq \langle v, z-z' \rangle_+ \geq 2^{-\Theta_K(k)}.$$

Let us take

$$\delta = \delta_K(k) := \log_2 K + \Theta_K(k) + 1 > \log_2 K + \Theta_K(k).$$

Then

$$2^{-\delta_K(k)} < \frac{1}{K \cdot 2^{\Theta_K(k)}}$$

and so

$$\|z - z'\| \leq K \wedge \exists v \in Az'(\|v\| \leq 2^{-\delta_K(k)} \to \|z - z'\| < 2^{-k}).$$

Here, even in the special case where A is uniformly ϕ-accretive at zero with $g(\cdot)$ nondecreasing thus making the K dependence for the modulus of accretivity disappear (Remark 2), the K dependence for the modulus of uniqueness does not disappear as we still have the term $\log_2 K$.

2.3 Effective Information on the Solution of Cauchy Problems Generated by Accretive Operators

2.3.1 Overview

In this section we present some results by García-Falset in [20] of which we will obtain quantitative versions by logically analysing their proofs in Section 2.3.2. We start with a basic definition:

Definition 23. *Given a nonexpansive semigroup $\mathcal{F} = \{S(t) : C \to C, \ t \geq 0\}$ on $C \subseteq X$, a continuous function $u : [0, \infty) \to C \subseteq X$ is said to be an almost-orbit of $\mathcal{F}$ if*

$$\lim_{s\to\infty} (\sup_{t\in[0,\infty)} \|u(t + s) - S(t)u(s)\|) = 0.$$

The following theorem shown in [20] is of fundamental importance.

Theorem 6. *(García-Falset, Theorem 8 in [20]) Let X be a real Banach space. If A is an operator on X with the range condition that is ϕ-accretive at zero and such that Problem 2 has a strong solution for each $x_0 \in D(A)$ and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ is the nonexpansive semigroup generated by $-A$ via the Crandall-Liggett formula, then every almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$ is strongly convergent to the zero z of A.*

What we can view as the central result is the following corollary of the above theorem :

Corollary 1. *(García-Falset, Corollary 9 in [20]) Let X be a real Banach space. Suppose that $A : D(A) \to 2^X$ is an m-ψ-strongly accretive operator on X. Suppose that Problem 2 has a strong solution for each $x_0 \in D(A)$. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to the zero z of A as $t \to \infty$.*

Note that although in [20] the above corollary is stated with the assumption of m-ψ-strong accretivity, ϕ-accretivity at zero and the range condition are sufficient conditions for the proof. (Note that clearly m-accretivity implies the range condition and as already mentioned in Section 2.2, every m-ψ-strongly accretive operator is ϕ-accretive at zero).

The proof of the above Corollary 1 follows from the following key lemma :

Lemma 1. *(Miyadera and Kobayasi, Proposition 7.1 in [62] or Lemma 1(b) in [20]) Let X be a Banach space. Let A be an accretive operator $A : D(A)(\subseteq X) \to 2^X$ with the range condition. Then the integral solution of the initial value problem*

$$u'(t) + Au(t) \ni f(t),\ t \geq 0,\ u(0) = x \in \overline{D(A)},$$

with $f(.) \in L^1(0, \infty, X)$ is an almost-orbit of the nonexpansive semigroup generated by $-A$.

So as the above lemma states that under the conditions of the corollary the integral solution $u(\cdot)$ of Problem 1 *is* an almost-orbit of the nonexpansive semigroup

$$\mathcal{F} := \{S(t)\ :\ \overline{D(A)} \to \overline{D(A)}\ :\ t \geq 0\}$$

generated by $-A$ via the Crandall-Liggett formula, it enables the application of Theorem 6 (Theorem 8 in [20]) giving directly Corollary 1.

We also state the following:

Corollary 2. *(García-Falset, Corollary 10 in [20]) Let X be a Banach space with the Radon-Nikodym property. Suppose that $A : D(A)(\subseteq X) \to 2^X$ is an m-accretive operator satisfying condition $(\sharp)$ of Definition 17 for some $z \in D(A)$. Then for each $x \in \overline{D(A)}$, the integral solution $u(\cdot)$ of Problem 1 converges strongly to z as $t \to \infty$.*

2.3.2 Proof-theoretic Analysis and Results

The work presented in this section is included in [47].

This is the first case study within the program of proof mining where abstract Cauchy problems given by general set-valued accretive operators $A : X \to 2^X$ are treated. More specifically, this is both the first time set-valued accretive operators are treated within proof mining, and the first application of proof mining to partial differential equations.

We will show here the extraction of computable and highly uniform rates of convergence and metastability in the sense of Tao (recall the discussion in Chapter 1) for the convergence results regarding abstract Cauchy problems generated by ϕ-accretive at zero operators $A : D(A)(\subseteq X) \to 2^X$ for a real Banach space X that were proved by García-Falset in [20] and presented in the previous section. That is, we will establish explicit quantitative versions of the asymptotic behavior of solutions of a certain very general class of abstract Cauchy problems. In particular, recall from the previous section that the operator A is assumed to satisfy the range condition, to have a zero $z \in D(A)$ i.e. $0 \in Az$, and to satisfy the condition of being ϕ-accretive at zero that implies that z is unique.

The bound extractions will be achieved by proof-theoretic analysis of the proofs in [20] and having assumed *uniform* accretivity at zero, yielding the new notion of modulus of accretivity as firstly introduced by Kohlenbach and the author in [47] and presented in Section 2.2. A central result is the computation of the rate of metastability for the convergence of the solution of the abstract Cauchy problem generated by a uniformly accretive at zero operator to the unique zero of A.

More specifically, recall from the previous section (Corollary 1) that in [20] it is shown that the integral solution of the problem

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$

$$u(0) = x_0,$$

where $f \in L^1(0, \infty, X)$ and A is as described above, converges to z as $t \to \infty$.

In fact, this followed as a corollary of Theorem 6, a general theorem in [20] about this convergence for almost-orbits $v(t)$ of the nonexpansive semigroup $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ generated by $-A$. Analyzing the proof of Theorem 6, we will extract an explicit computation which eventually translates any given rate of convergence for an almost-orbit into a rate of convergence of the integral solution of the Cauchy problem towards the unique zero z of A.

In the case of $f \in L^1(0,\infty,X)$ such a rate of convergence for an almost-orbit amounts to knowing a rate of convergence of $(*)$ $\int_s^\infty \|f(\xi)\|d\xi \stackrel{s\to\infty}{\longrightarrow} 0$. Such a rate of convergence, however, would in general not be possible to compute in just f and an upper bound $M \geq \int_0^\infty \|f(\xi)\|d\xi$, and even when it is computable it will strongly depend on the particulars of f.

In the case at hand it is however possible to compute a rate of *metastability* for $(*)$ which *only* depends on M but not on f itself, by applying Proposition 1. To this end, also the conclusion of Theorem 6 has to be rephrased in metastable terms. Therefore, after obtaining a quantitative version of Theorem 6 (extracting a rate of convergence which here happens to be possible although this is in general not the case) we will also obtain a *metastable* quantitative version of Theorem 6 (i.e. extracting a rate of metastability). Even though the latter is weaker, it is essential as we will insert in it the information related to $(*)$ so as to eventually obtain the main result of this paper which the explicit construction for the metastable version of the convergence of the solution of our Cauchy problem towards the unique zero z of A i.e. a quantitative and metastable version of Corollary 1.

The uniformity of the bounds extracted is witnessed by the fact that f only enters via a bound on its L^1-norm and also by the fact that they do not depend on the (uniformly accretive at zero) operator A itself but only on its modulus of accretivity at zero as introduced in Section 2.2.

After the above very general outline, we now present and prove our results.

Theorem 7. *(Kohlenbach and K.-A., ([47]), Quantitative version of Theorem 6) Let X be a real Banach space. Let A be an operator on X with the range condition that is uniformly accretive at zero with a modulus of accretivity Θ, and such that Problem 2 has a strong solution for each $x_0 \in D(A)$ and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\ \}$ is the nonexpansive semigroup generated by $-A$ via the Crandall-Liggett formula. Then every $u : [0,\infty) \to \overline{D(A)}$ that fulfills the condition*[1]*:*

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \exists s \in [0,\Phi(k)]\ (\sup_{t\geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k}),$$

is strongly convergent to the zero z of A, i.e.

$$\forall k \in \mathbb{N}\ \forall x \geq \Psi(k,B,\Phi,\Theta)\ (\|u(x) - z\| < 2^{-k})$$

[1] clearly this is a weakening of the assumption of $u : [0,\infty) \to \overline{D(A)}$ being an almost-orbit of $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ i.e.

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \forall s \geq \Phi(k)\ (\sup_{t\geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k}).$$

with rate of convergence

$$\Psi(k,B,\Phi,\Theta) = (B(\Phi(k+1))+2)\cdot 2^{\Theta_{K(\Phi(k+1))}(k+2)+1} + \Phi(k+1)$$

where

$$K(s) := \lceil \sqrt{2(B(s)+1)} \rceil$$

and $B(s)$ *is any nondecreasing upper bound on* $\frac{1}{2}\|u(s)-z\|^2$.

In the entire section we will assume that $\Theta_K(k)$ is nondecreasing in K. Note that this assumption is possible without loss of generality, as for any $\Theta_K(k)$ we may define a nondecreasing

$$\Theta_K^{\mathcal{M}}(k) := \max\{\Theta_i(k) \ : \ i \le K\}.$$

Proof. The proof is based on performing proof mining on the proof of Theorem 6 (i.e. Theorem 8 in [20]).

Let $u : [0,\infty) \to \overline{D(A)}$ be as in the assumption of the theorem. Let $s \ge 0$ be fixed.

Case 1. Assume that $u(s) \in D(A)$.

Consider the following initial value problem, which is of the form of Problem 2:

Problem 3.

$$w_s'(t) + A(w_s(t)) \ni 0,$$
$$w_s(0) = u(s).$$

Problem 3 has a unique solution

$$w_s(t) = S(t)u(s)$$

which is a strong solution by assumption. Thus the derivative $w_s'(t)$ is defined almost everywhere and $-w_s'(t) \in Aw_s(t)$ almost everywhere, i.e.

$$\exists \mathcal{S} \subset [0,\infty)(\mu(\mathcal{S}) = 0 \ \wedge \ \forall t \in [0,\infty) \setminus \mathcal{S} \ w_s'(t) \downarrow),$$

$$\exists \mathcal{S}' \subset [0,\infty)(\mu(\mathcal{S}') = 0 \ \wedge \ \forall t \in [0,\infty) \setminus \mathcal{S}' \ -w_s'(t) \in Aw_s(t)),$$

where $\mu(\cdot)$ denotes the Lebesgue measure. There exists $j(t) \in \mathcal{J}(w_s(t)-z)$, where $\mathcal{J}(\cdot)$ is the normalized duality mapping as defined in Section 2, so that, for all $t \in [0,\infty) \setminus \mathcal{S}$:

$$\langle -w_s'(t), w_s(t)-z\rangle_+ = \langle -w_s'(t), j(t)\rangle = \langle -\frac{1}{h}(w_s(t)-w_s(t-h)) + \xi(t,h), j(t)\rangle$$

$$= \frac{1}{h}\langle w_s(t-h)-w_s(t), j(t)\rangle + \langle \xi(t,h), j(t)\rangle \quad (1)$$

where $\lim_{h\to 0}\xi(t,h) = 0$.

(Notice: $\langle w_s(t) - z, j(t)\rangle = \|w_s(t) - z\|^2 = \|j(t)\|^2$.) Now :

$$\langle w_s(t-h) - w_s(t), j(t)\rangle = \langle w_s(t-h) - w_s(t) + z - z, j(t)\rangle$$
$$= \langle w_s(t-h) - z, j(t)\rangle + \langle z - w_s(t), j(t)\rangle$$
$$= \langle w_s(t-h) - z, j(t)\rangle - \langle w_s(t) - z, j(t)\rangle$$
$$= \langle w_s(t-h) - z, j(t)\rangle - \|w_s(t) - z\|^2 \quad (2).$$

Notice that by the properties of the duality mapping (see [4] page 12 (1.4))

$$\langle w_s(t-h) - z, j(t)\rangle$$
$$\leq \|w_s(t-h) - z\|\|j(t)\|$$
$$\leq \frac{1}{2}\|w_s(t-h) - z\|^2 + \frac{1}{2}\|j(t)\|^2$$
$$= \frac{1}{2}\|w_s(t-h) - z\|^2 + \frac{1}{2}\|w_s(t) - z\|^2 (3)$$

By (2) and (3)

$$\langle w_s(t-h) - w_s(t), j(t)\rangle \leq \frac{1}{2}\|w_s(t-h) - z\|^2 - \frac{1}{2}\|w_s(t) - z\|^2 \quad (4).$$

Define $q_s(t) := \langle -w_s'(t), j(t)\rangle = \langle -w_s'(t), w_s(t) - z\rangle_+$. Note that $q_s(t)$ is defined almost everywhere, as $w_s'(t)$ is defined almost everywhere. Now $-w_s'(t) \in Aw_s(t)$ almost everywhere. By the accretivity of A, the condition $q_s(t) \geq 0$ holds for all $t \in [0,\infty) \setminus (\mathcal{S} \cup \mathcal{S}')$. By (4), for all $t \in [0,\infty) \setminus (\mathcal{S} \cup \mathcal{S}')$, (1) gives :

$$0 \leq q_s(t) = \langle -w_s'(t), j(t)\rangle \leq \frac{1}{h}\frac{1}{2}(\|w_s(t-h) - z\|^2 - \|w_s(t) - z\|^2) + \langle \xi(t,h), j(t)\rangle$$

thus

$$0 \leq q_s(t) \leq -\frac{1}{2}\frac{d}{dt}\|w_s(t) - z\|^2 \text{ a.e.} \quad (5)$$

as the derivative of $\|w_s(t) - z\|$ is defined almost everywhere i.e.

$$\exists \mathcal{S}'' \subset [0,\infty)(\mu(\mathcal{S}'') = 0 \ \wedge \ \forall t \in [0,\infty) \setminus \mathcal{S}'' \ \frac{d}{dt}\|w_s(t) - z\| \downarrow)$$

since $t \to \|w_s(t) - z\|$ is Lipschitzian, because by assumption $w_s(t)$ is Lipschitzian, as we can see that assuming

$$\|w_s(t_1) - w_s(t_2)\| \leq \lambda |t_1 - t_2|$$

for some $\lambda \in (0,1)$, we have

$$|\|w_s(t_1) - z\| - \|w_s(t_2) - z\|| \leq \|w_s(t_1) - z - w_s(t_2) + z\|$$

$$= \|w_s(t_1) - w_s(t_2)\| \leq \lambda |t_1 - t_2|.$$

By (5) we deduce that $\frac{d}{dt}\|w_s(t) - z\|^2 \leq 0$ almost everywhere. Therefore $\|w_s(t) - z\|^2$ is nonincreasing in t (see [11], p.120 and note that $\|w_s(t) - z\|^2$ is Lipschitz on bounded intervals thus absolutely continuous). For all $t \in [0, \infty)$ by (5) we have

$$0 \leq \int_0^t q_s(t)dt \leq -\frac{1}{2}\int_0^t \frac{d}{dt}\|w_s(t) - z\|^2 dt$$

$$= -\frac{1}{2}\|w_s(t) - z\|^2 + \frac{1}{2}\|w_s(0) - z\|^2 \leq \frac{1}{2}\|w_s(0) - z\|^2.$$

Thus $q_s(t)$ is Lebesgue integrable on $[0, \infty)$. Therefore

$$\lim_{t \to \infty} \inf q_s(t) = 0$$

and

$$\forall k \in \mathbb{N}\ \exists t \in [0, \infty) \setminus (\mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}'')\ (q_s(t) \leq 2^{-k}).$$

We now construct an upper bound $T(k, s)$ on t as follows. Let $B(s)$ be a nondecreasing upper bound on

$$\frac{1}{2}\|w_s(0) - z\|^2.$$

For instance let

$$B(s) = \frac{1}{2}\max\{\|w_r(0) - z\|^2\ :\ r \leq s\}.$$

We set

$$T(k, s) := (B(s) + 1) \cdot 2^k.$$

We claim that

$$\forall k \in \mathbb{N}\ \exists t \in [0, T(k, s)] \setminus (\mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}'')\ (q_s(t) \leq 2^{-k}).$$

Assume the contrary, i.e. assume that

$$\exists k \in \mathbb{N}\ \forall t \in [0, T(k, s)] \setminus (\mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}'')\ (q_s(t) > 2^{-k});$$

then, by the monotonicity property of the Lebesgue integral

$$\int_0^{T(k,s)} 2^{-k} dt \leq \int_0^{T(k,s)} q_s(t)dt$$

therefore

$$T(k, s) \cdot 2^{-k} = B(s) + 1 \leq \int_0^{T(k,s)} q_s(t)dt$$

which is a contradiction.

Hence we have for each k a $t_k \le T(k,s)$ with $t_k \notin (\mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}'')$ and $q_s(t_k) \le 2^{-k}$. By the definition of the modulus of accretivity Θ we have

$$\forall n, k \in \mathbb{N}\ \forall K \in \mathbb{N}\ (\|w_s(t_k) - z\| \in [2^{-n}, K] \to 2^{-\Theta_K(n)} \le q_s(t_k))$$

therefore

$$\forall n, k \in \mathbb{N}\ \forall K \in \mathbb{N}\ (\|w_s(t_k) - z\| \in [2^{-n}, K] \to 2^{-\Theta_K(n)} \le 2^{-k}).$$

We set $k = \Theta_K(n) + 1$ and thus obtain :

$$\forall n \in \mathbb{N}\ \forall K \in \mathbb{N}\ (\|w_s(t_{\Theta_K(n)+1}) - z\| \in [2^{-n}, K] \to 2^{-\Theta_K(n)} \le 2^{-\Theta_K(n)-1})$$

whose conclusion is obviously false. Thus the premise is false. Since for all $t \ge 0$ (recall that $\|w_s(t) - z\|$ is nonincreasing in t)

$$K = K_0(s) := \lceil \sqrt{2B(s)} \rceil \ge \|w_s(0) - z\| \ge \|w_s(t) - z\|$$

we, therefore, have

$$\forall n \in \mathbb{N}\ (\|w_s(t_{\Theta_{K_0(s)}(n)+1}) - z\| < 2^{-n})$$

where

$$t_{\Theta_{K_0(s)}(n)+1} \le (B(s)+1) \cdot 2^{\Theta_{K_0(s)}(n)+1}$$

and so, using again the fact that $\|w_s(t) - z\|$ is nonincreasing in t,

$$\forall n \in \mathbb{N}\ \forall t \ge (B(s)+1) \cdot 2^{\Theta_{K_0(s)}(n)+1}\ (\|w_s(t) - z\| < 2^{-n})\ (6).$$

Case 2. Now assume that $u(s) \in \overline{D(A)}$. Then there exists a sequence $(x_k(s)) \subseteq D(A)$ such that $x_k(s) \to u(s)$. Let

$$\tilde{w}_{k,s}(t) := S(t)x_k(s) \subseteq D(A).$$

We want to show that

$$\lim_{t \to \infty} \|w_s(t) - z\| = 0$$

where

$$w_s(t) := S(t)u(s).$$

By the triangle inequality:

$$|\|\tilde{w}_{k,s}(t) - z\| - \|w_s(t) - z\|| \le \|\tilde{w}_{k,s}(t) - w_s(t)\|.$$

Notice that

$$\|\tilde{w}_{k,s}(t) - w_s(t)\| = \|S(t)x_k(s) - S(t)u(s)\| \le \|x_k(s) - u(s)\|,$$

the above inequality following from Definition 8(3) . Thus

$$| \|\tilde{w}_{k,s}(t) - z\| - \|w_s(t) - z\|| \leq \|x_k(s) - u(s)\|.$$

We assume, without loss of generality, that we can make an appropriate choice of $x_{\tilde{k}}(s) \in D(A)$ so that

$$\|x_{\tilde{k}}(s) - u(s)\| \leq 2^{-\tilde{k}}$$

thus

$$\Big| \|\tilde{w}_{\tilde{k},s}(t) - z\| - \|w_s(t) - z\| \Big| \leq 2^{-\tilde{k}}.$$

In particular this gives

$$\frac{1}{2}\|\tilde{w}_{\tilde{k},s}(t) - z\|^2 \leq \frac{1}{2}(\|w_s(t) - z\|^2 + 2 \cdot 2^{-\tilde{k}}\|w_s(t) - z\| + (2^{-\tilde{k}})^2)$$

$$\leq \frac{1}{2}\|w_s(0) - z\|^2 + 2^{-\tilde{k}}\|w_s(t) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2$$

$$\leq B(s) + 2^{-\tilde{k}}\|w_s(t) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2.$$

Now for $k \in \mathbb{N}$, take $\mathbb{N} \ni \tilde{k} > k$ such that:

$$2^{-\tilde{k}}\|w_s(0) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2 \leq 1.$$

Then (by evaluating the previous estimate at $t = 0$) an upper bound on $\frac{1}{2}\|\tilde{w}_{\tilde{k},s}(0) - z\|^2$, denoted by $\tilde{B}(s, \tilde{k})$, can be taken as $\tilde{B}(s, \tilde{k}) := B(s) + 1$, where $B(s)$ is a nondecreasing upper bound on $\frac{1}{2}\|w_s(0) - z\|^2$.

By (6) of Case 1 applied to $\|\tilde{w}_{\tilde{k},s}(t) - z\|$ here:

$$\forall t \geq (\tilde{B}(s, \tilde{k}) + 1) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|\tilde{w}_{\tilde{k},s}(t) - z\| < 2^{-k})$$

i.e.

$$\forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|\tilde{w}_{\tilde{k},s}(t) - z\| < 2^{-k})$$

and thus

$$\forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|w_s(t) - z\| < 2^{-\tilde{k}} + 2^{-k} < 2 \cdot 2^{-k}).$$

Since $k \in \mathbb{N}$ and $s \geq 0$ were arbitrary we thus have

$$\forall k \in \mathbb{N} \ \forall s \geq 0 \ \forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k+1)+1} \ (\|w_s(t) - z\| < 2^{-k}) \ (7).$$

Note that here we have taken

$$K(s) := \lceil \sqrt{2(B(s)+1)} \rceil,$$

as

$$\tilde{B}(s,\tilde{k}) = B(s) + 1 \geq \frac{1}{2}\|\tilde{w}_{\tilde{k},s}(0) - z\|^2.$$

Now, by assumption, $u : [0,\infty) \to \overline{D(A)}$ fulfills the condition:

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ (\varphi(s) \leq 2^{-k})\ (8)$$

where we have set

$$\varphi(s) := \sup_{t \geq 0} \|u(s+t) - S(t)u(s)\|.$$

The triangle inequality gives, for all $t \geq 0$,

$$\begin{aligned} \|u(t+s) - z\| &\leq \|u(t+s) - S(t)u(s)\| + \|S(t)u(s) - z\| \\ &= \|u(t+s) - S(t)u(s)\| + \|w_s(t) - z\| \\ &\leq \varphi(s) + \|w_s(t) - z\|. \end{aligned}$$

By (7)

$$\forall k \in \mathbb{N}\ \forall s \geq 0\ \forall t \geq (B(s)+2) \cdot 2^{\Theta_{K(s)}(k+1)+1}\ (\|u(t+s) - z\| < \varphi(s) + 2^{-k}).$$

For Φ as in (8) the above gives

$$\forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ \forall t \geq (B(s)+2) \cdot 2^{\Theta_{K(s)}(k+1)+1}\ (\|u(t+s) - z\| < 2 \cdot 2^{-k})$$

which, using that $(B(\cdot)+2) \cdot 2^{\Theta_{K(\cdot)}(k+1)+1}$ is nondecreasing, implies

$$\forall k \in \mathbb{N}\ \forall x \geq (B(\Phi(k))+2) \cdot 2^{\Theta_{K(\Phi(k))}(k+1)+1} + \Phi(k)\ (\|u(x) - z\| < 2 \cdot 2^{-k}),$$

that is,

$$\forall k \in \mathbb{N}\ \forall x \geq (B(\Phi(k+1))+2) \cdot 2^{\Theta_{K(\Phi(k+1))}(k+2)+1} + \Phi(k+1)\ (\|u(x) - z\| < 2^{-k}).$$

□

Remark 4. *Our logical analysis shows that Theorem 1 (Theorem 8 in [20]) is true not only under the assumption that the continuous function* $u(\cdot) : [0,\infty) \to \overline{D(A)}$ *is an almost-orbit, i.e.*

$$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall s \geq s_0\ (\sup\{\|u(t+s) - S(t)u(s)\| \ :\ t \in [0,\infty)\} \leq 2^{-k}),$$

i.e., equivalently,

$$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall s \geq s_0\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+s) - S(t)u(s)\| \ :\ t \leq m\} \leq 2^{-k})\ (*),$$

but also under the weaker assumption

$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+s_0) - S(t)u(s_0)\|\ :\ t \leq m\} \leq 2^{-k})\ (**)$.

Notice, moreover, that $(*)$ *implies*

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+n) - S(t)u(n)\|\ :\ t \leq m\} \leq 2^{-k}),$$

whose noneffectively equivalent metastable form is

$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ (\sup\{\|u(t+n) - S(t)u(n)\|\ :\ t \leq g(n)\} \leq 2^{-k})\ (+)$,

while $(**)$ *implies*

$$\forall k \in \mathbb{N}\ \exists q \in \mathbb{Q}^+\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+q) - S(t)u(q)\|\ :\ t \leq m\} \leq 2^{-k}),$$

whose noneffectively equivalent metastable form is

$\forall k \in \mathbb{N}\ \forall g : \mathbb{Q}^+ \to \mathbb{N}\ \exists q \in \mathbb{Q}^+\ (\sup\{\|u(t+q) - S(t)u(q)\|\ :\ t \leq g(q)\} \leq 2^{-k})\ (++)$.

Moreover, note that instead of using metastability in the form of $(+)$ *one could also work with the still weaker form* $(++)$ *which, however, makes things more complicated without any apparent benefit.*

Remark 5. *The reader will notice that in the above theorem we have obtained a full rate of convergence instead of a rate of metastability. This is due to the fact that the proof is constructive. It should be stressed that even for (semi)-constructive proofs the extraction of highly uniform bounds is guaranteed by general logical metatheorems ([25]). The rate of convergence given by the above theorem, however, contains the black box information* Φ *which is the unknown rate of convergence of an almost-orbit (note that the definition of an almost orbit involves in itself a convergence statement).*

In the following theorem we show a metastable (in the sense of Tao) version of Theorem 7 above, namely a version where the statement referring to the (weakening of the condition of the) almost-orbit is replaced by a metastable statement in the form $(+)$, thus giving a metastable version for the convergence of the result. That is, the rate of convergence $\Phi(k)$ that appeared in Theorem 7 so that

$$\forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ (\sup_{t \geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k})$$

is substituted with a rate of metastability $\Phi : \mathbb{N} \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ in Theorem 8 below so that

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)]$$
$$(\|u(t+n) - S(t)u(n)\| \leq 2^{-k}).$$

Clearly, the conclusion of Theorem 8, being metastable, is weaker than that of Theorem 7, however it will serve a useful purpose; it will illustrate the pattern of metastability, which will be realized in Theorem 9 later on. We will later see that Theorem 9 can be regarded as a corollary of Theorem 8 below, where in particular the quantity $\Phi(k, g, \ldots)$ corresponding to the metastability information relating to the almost-orbit in Theorem 8 (i.e. $\Phi(k, g)$), can be computed by applying Proposition 1, thus providing, as we will see, a computable rate of metastability for the strong convergence of the solution of the abstract Cauchy problem generated by a uniformly accretive at zero operator A to the zero z of A, which can be considered as the central result of this chapter.

Theorem 8. *(Kohlenbach and K.-A., ([47]), Quantitative and metastable version of Theorem 6) Let X be a real Banach space. Let A be an operator on X with the range condition that is uniformly accretive at zero with modulus of accretivity Θ and such that Problem 2 has a strong solution for each $x_0 \in D(A)$ and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0 \}$ is the nonexpansive semigroup generated by $-A$ via the Crandall-Liggett formula. Then every almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$ is strongly convergent to the zero z of A with rate of metastability $\Psi(k, \bar{g}, B, \Phi, \Theta)$ so that*

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, B, \Phi, \Theta)\ \forall x \in [\bar{n}, \bar{n}+\bar{g}(\bar{n})]\ (\|u(x)-z\| < 2^{-k}),$$

where

$$\Psi(k, \bar{g}, B, \Phi, \Theta) = \Phi(k+1, g) + h(\Phi(k+1, g))$$

with

$$g(n) := \bar{g}(n + h(n)) + h(n),$$

$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$

$$K(n) := \lceil \sqrt{2(B(n) + 1)} \rceil.$$

Here $\Phi : \mathbb{N} \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is a rate of metastability (in the sense that we discussed above) corresponding to a given almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$, i.e.

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)]\ (\|u(t+n) - S(t)u(n)\| \leq 2^{-k})$$

and $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(n) - z\|^2$.

Proof. Let $u(\cdot) \in \overline{D(A)}$.

We consider the metastable (in the sense of Tao [83, 84]) version of an almost-orbit discussed in the form (+) as in Remark 4:

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)]\ (\varphi_t(n) \leq 2^{-k})\ (9)$$

where we have defined, for $n \in \mathbb{N}$,

$$\varphi_t(n) := \|u(t+n) - S(t)u(n)\|.$$

Notice that by the triangle inequality :

$$\|u(t+n) - z\| \leq \|u(t+n) - S(t)u(n)\| + \|w_n(t) - z\| = \varphi_t(n) + \|w_n(t) - z\|,$$

where $w_n(t)$ is as in Problem 3. Note that here $n \in \mathbb{N}$. We claim that

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}$$
$$\forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ \ (\|u(x) - z\| < 2^{-k})$$

where

$$\bar{n} := n + (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$g(m) := \bar{g}(m + (B(m) + 2) \cdot 2^{\Theta_{K(m)}(k+2)+1}) + (B(m) + 2) \cdot 2^{\Theta_{K(m)}(k+2)+1}$$

so that

$$g(n) + n = \bar{g}(\bar{n}) + \bar{n},$$

where $B(m) : \mathbb{N} \to \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(m) - z\|^2$ and $K(m) := \lceil \sqrt{2(B(m) + 1}\rceil$. Note that here $n \leq \Phi(k+1, g)$ is chosen for $k+1$ and g as defined above according to (9). By the monotonicity of $B(m)$ and $K(m)$ it follows that

$$\bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}.$$

To show the above claim, let $x = t + n \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]$ with

$$t \in [(B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}, \bar{g}(\bar{n}) + (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}] \subseteq [0, g(n)].$$

By (7) in the proof of Theorem 7 with t chosen in the above interval we have

$$\|w_n(t) - z\| < 2^{-k-1}$$

and, therefore,

$$\|u(x) - z\| = \|u(t+n) - z\| < \|u(t+n) - S(t)u(n)\| + 2^{-k-1} = \varphi_t(n) + 2^{-k-1}.$$

Thus, by our choice of n to be $n \leq \Phi(k+1, g)$ based on (9), we obtain :

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}$$
$$\forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ \ (\|u(x) - z\| < 2 \cdot 2^{-k-1} = 2^{-k}).$$

□

As previously mentioned, we will now show a result where the information on the almost-orbit is not any more "black box information" but explicitly given and the rate of metastability $\Phi(k, g, ...)$- thus also the rate of metastability for the final result- can be computed using Proposition 1.

In particular, we will show the following theorem, which can be seen as a corollary of Theorem 8, in an analogy to Corollary 1 (Corollary 9 in [20]) being a corollary of Theorem 6 (Theorem 8 in [20]):

Theorem 9. *(Kohlenbach and K.-A., ([47]), Quantitative version of Corollary 1) Let X be a real Banach space. Suppose that $A : D(A) \to 2^X$ is a uniformly accretive at zero operator on X with the range condition that has a modulus of accretivity Θ. Suppose that Problem 2 has a strong solution for each $x_0 \in D(A)$. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to the zero z of A as $t \to \infty$ and one has*

$$\forall k \in \mathbb{N}\, \forall \overline{g} : \mathbb{N} \to \mathbb{N}\, \exists \bar{n} \le \Psi(k, \overline{g}, M, B, \Theta)\, \forall x \in [\bar{n}, \bar{n}+\overline{g}(\bar{n})]\, (\|u(x)-z\| < 2^{-k})$$

with rate of metastability

$$\Psi(k, \overline{g}, M, B, \Theta) = \tilde{g}^{(M\cdot 2^{k+1})}(0) + h(\tilde{g}^{(M\cdot 2^{k+1})}(0)),$$

where

$$\tilde{g}(n) := g(n) + n$$

with

$$g(n) := \overline{g}(n + h(n)) + h(n),$$
$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$K(n) := \lceil \sqrt{2(B(n) + 1)} \rceil.$$

Here $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(n) - z\|^2$, $M \in \mathbb{N}$ is any upper bound on the integral $I := \int_0^\infty \|f(\xi)\| d\xi$, and in general the function iterations are defined recursively in the following way:

$$g^{(0)}(k) := k$$

$$g^{(i+1)}(k) := g(g^{(i)}(k)).$$

Proof. Let $s \ge 0$ be arbitrarily fixed. Set $u_s(t) = u(t + s)$, $f_s(t) = f(t + s)$, $v(t) = S(t)u(s)$ for $t \ge 0$. Then u_s is an integral solution of

$$(d/dt)u_s \in Au_s + f_s, u_s(0) = u(s).$$

and $(d/dt)v \in Av, v(0) = u(s)$ respectively. By a result in [4] (see (2.4) in p.124),

$$\|u_s(t) - v(t)\|^2 \le 2 \int_0^t \|f_s(\xi)\| \|u_s(\xi) - v(\xi)\| d\xi.$$

Therefore [2]

$$\|u(t+s) - S(t)u(s)\| \le \int_0^t \|f(s+\xi)\| d\xi.$$

(Note: the above argumentation was taken from the proof of Lemma 1 which is given in [62].)

We now set

$$\varphi_t(s) := \|u(t+s) - S(t)u(s)\|$$

so we have

$$\varphi_t(s) \le \int_0^t \|f(s+\xi)\| d\xi = \int_s^{s+t} \|f(\xi)\| d\xi.$$

Let $I := \int_0^\infty \|f(\xi)\| d\xi$ and let $M \in \mathbb{N}$ be any upper bound of I. We define a nonincreasing path on $[0, M]$ by

$$\bar{\varphi}(s) := M - \int_0^s \|f(\xi)\| d\xi$$

so that

$$\varphi_t(s) \le \int_s^{s+t} \|f(\xi)\| d\xi = |\bar{\varphi}(s+t) - \bar{\varphi}(s)|.$$

We now claim that we can give a bound $\Phi(k, g, M)$ on the metastable version of the Cauchy property of this path, i.e.

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \le \Phi(k, g, M)\, \forall s, t \in [n, n+g(n)]\, (|\bar{\varphi}(s) - \bar{\varphi}(t)| < 2^{-k})\ (*)$$

[2] We verify this conclusion using Bihari's inequality [6] as follows: Assuming

$$\|u_s(t) - v(t)\|^2 \le 0 + \int_0^t 2\|f_s(\xi)\| \|u_s(\xi) - v(\xi)\| d\xi$$

we have

$$G(\|u_s(t) - v(t)\|^2) \le G(0) + \int_0^t 2\|f_s(\xi)\| d\xi$$

where

$$G(x) := \int_{\chi_0}^x \frac{1}{\sqrt{t}} dt = 2\sqrt{x} - \sqrt{\chi_0}.$$

(with $\chi_0 > 0$, $x \ge 0$), so

$$G(\|u_s(t) - v(t)\|^2) = \int_{\chi_0}^{\|u_s(t)-v(t)\|^2} \frac{1}{\sqrt{t}} dt$$
$$= 2\|u_s(t) - v(t)\| - 2\chi_0$$

and thus

$$2\|u_s(t) - v(t)\| - 2\chi_0 \le -2\chi_0 + \int_0^t 2\|f_s(\xi)\| d\xi$$

so

$$\|u_s(t) - v(t)\| \le \int_0^t \|f_s(\xi)\| d\xi.$$

which by the monotonicity of $\bar{\varphi}(\cdot)$ can be equivalently written as

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k, g, M)\ (|\bar{\varphi}(n + g(n)) - \bar{\varphi}(n)| < 2^{-k})\ (**)$$

which is the no-counterexample interpretation of

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall m \in \mathbb{N}\ (|\bar{\varphi}(n + m) - \bar{\varphi}(n)| < 2^{-k}).$$

$(*)$ (and hence $(**)$) yields

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k, g, M)\ \forall t \in [0, g(n)]\ (|\bar{\varphi}(n + t) - \bar{\varphi}(n)| < 2^{-k})$$

so that in turn

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k, g, M)\ \forall t \in [0, g(n)]\ (\varphi_t(n) < 2^{-k}).$$

To show the claim that we can give a bound $\Phi(k, g, M)$ for $(**)$, we refer to Proposition 1 (shown as Proposition 2.26/ 2.27 in [45], also see Remark 2.29 there). Here we follow the proof of the aforementioned Propositions in [45] adapted to the case at hand. For $g : \mathbb{N} \to \mathbb{N}$ define $\tilde{g} : \mathbb{N} \to \mathbb{N}$ by

$$\tilde{g}(n) := n + g(n)$$

and notice that

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists i \leq 2^k \cdot M\ (\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i+1)}(0)) < 2^{-k}),$$

where in general the function iterations are defined recursively in the following way:

$$g^{(0)}(k) := k$$
$$g^{(i+1)}(k) := g(g^{(i)}(k)),$$

for assuming on the contrary that for some $k \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$

$$\forall i \leq 2^k \cdot M(\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i+1)}(0)) \geq 2^{-k})$$

by $\tilde{g}^{(0)}(0) = 0$ we obtain

$$\bar{\varphi}(0) - \bar{\varphi}(\tilde{g}^{(2^k \cdot M + 1)}(0)) \geq (2^k \cdot M + 1) \cdot 2^{-k} > M$$

which is a contradiction. Now, because $\bar{\varphi}(\cdot)$ is nonincreasing, we obtain

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists i \leq 2^k \cdot M\ (|\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i)}(0) + g(\tilde{g}^{(i)}(0)))| < 2^{-k}).$$

Hence we may take

$$\Phi(k, g, M) := \tilde{g}^{(M \cdot 2^k)}(0) (= \max\{\tilde{g}^{(i)}(0)\ :\ i \leq M \cdot 2^k\}).$$

From this point on the same pattern as in the proof of Theorem 8 is followed.

Assume that $u(\cdot) \in \overline{D(A)}$.

Recall that

$$K(n) := \lceil \sqrt{2B(n)+1} \rceil$$

where $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|w_n(0)-z\|^2$ and that

$$w_s'(t) + A(w_s(t)) \ni 0, w_s(0) =: u(s).$$

We claim that we obtain

$$\forall k \in \mathbb{N}\, \forall \bar{g} : \mathbb{N} \to \mathbb{N}\, \exists \bar{n} \le \Psi(k, \bar{g}, M, \Theta, B)\, \forall x \in [\bar{n}, \bar{n}+\bar{g}(\bar{n})]\ (\|u(x)-z\| < 2^{-k})$$

with a rate of metastability

$$\Psi(k, \overline{g}, M, B, \Theta) = \Phi(k+1, g, M) + h(\Phi(k+1, g, M))$$
$$= \tilde{g}^{(M\cdot 2^{k+1})}(0) + h(\tilde{g}^{(M\cdot 2^{k+1})}(0))$$

where

$$h(n) := (B(n)+2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$

$M \in \mathbb{N}$ is any upper bound on the integral $I := \int_0^\infty \|f(\xi)\| d\xi$,

$$\tilde{g}(n) := g(n) + n,$$

and we have defined

$$g(n) := \overline{g}(n + h(n)) + h(n).$$

To prove the above claim, let

$$x \in [n + h(n), n + h(n) + \overline{g}(n + h(n))],$$

where $x := n + t$ for some $t \in [h(n), h(n) + \overline{g}(n + h(n))]$.

By (7) of Theorem 7 we have

$$\forall k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall t \ge (B(n)+2) \cdot 2^{\Theta_{K(n)}(k+1)+1}\ (\|w_n(t) - z\| < 2^{-k})$$

Therefore, as we assumed that $t \ge h(n)$, the condition

$$\|w_n(t) - z\| < 2^{-k-1}$$

is satisfied.

Now, by the triangle inequality, we obtain

$$\|u(n+t) - z\| \le \|u(n+t) - S(t)u(n)\| + \|S(t)u(n) - z\|$$

$$
\begin{aligned}
&= \|u(n+t) - S(t)u(n)\| + \|w_n(t) - z\| \\
&= \varphi_t(n) + \|w_n(t) - z\| \\
&< \varphi_t(n) + 2^{-k-1} \\
&\leq \int_n^{n+t} \|f(\xi)\| d\xi + 2^{-k-1} \\
&\leq \int_n^{n+h(n)+\bar{g}(n+h(n))} \|f(\xi)\| d\xi + 2^{-k-1} \\
&= |\bar{\varphi}(n+g(n)) - \bar{\varphi}(n)| + 2^{-k-1}.
\end{aligned}
$$

By the (metastable) Cauchy property above we have

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k{+}1, g, M)\ (\varphi_{g(n)}(n) \leq |\bar{\varphi}(n{+}g(n)){-}\bar{\varphi}(n)| < 2^{-k-1})$$

where we recall that $g(n) = h(n) + \bar{g}(n + h(n))$ and $\Phi(k, g, M)$ is as before. Hence we obtain

$$
\begin{aligned}
&\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists n \leq \tilde{g}^{(M \cdot 2^{k+1})}(0) \\
&\qquad \forall t \in [h(n), h(n) + \bar{g}(n + h(n))]\ (\|u(n+t) - z\| < 2 \cdot 2^{-k-1} = 2^{-k}).
\end{aligned}
$$

Thus, for n chosen as above (taking $\bar{n} := n + h(n)$ and using the monotonicity of $h(\cdot)$) we get

$$
\begin{aligned}
&\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \tilde{g}^{(M \cdot 2^{k+1})}(0) + h(\tilde{g}^{(M \cdot 2^{k+1})}(0)) \\
&\qquad\qquad \forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2^{-k}).
\end{aligned}
$$

□

Corollary 3. *(Kohlenbach and K.-A., ([47])), Quantitative form of Corollary 2) Let X be a real Banach space with the Radon-Nikodym property. Let $A : D(A) \to 2^X$ be an m-accretive and uniformly accretive at zero operator with $0 \in Az$ and modulus of accretivity Θ. Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to z as $t \to \infty$ with a rate of metastability as in Theorem 9.*

Proof. As shown in [5] Chapter 7, because X has the Radon-Nikodym property, the integral solution of Problem 2 is a strong solution. Because A is m-accretive, A satisfies the range condition. Thus the result follows directly by Theorem 9. □

Note that it is well-known that every reflexive Banach space has the Radon-Nikodym property. (Also note that in [4](Theorem 2.2, page 131) it is shown that assuming that X is a reflexive Banach space, the integral solution of Problem 2 is a strong solution).

2.3.3 An Application

To demonstrate an application of our result, we present a case studied in [20] where both Corollary 2 is applicable and A fulfills at the same time the uniform accretivity condition, so a modulus of accretivity exists and our result can be directly applied.

In [20] the following nonlinear boundary value problem is studied :

Problem 4.

$$u_t - div\ (|Du|^{p-2}Du) + \varphi(x,u) = f, \ \ on\ (0,\infty)\times\Omega,$$

$$-\frac{\partial u}{\partial \eta} \in \beta(u) \ \ on\ [0,\infty)\times\partial\Omega,$$

$$u(0,x) = u_0(x) \in L^q(\Omega) \ \ in\ \Omega,$$

where Ω *is a bounded open domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\Omega$, $f \in L^1((0,\infty), L^q(\Omega))$, $1 \leq p, q < \infty$, $\frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta\rangle$, η *the unit outward normal on* $\partial\Omega$, Du *the gradient of* u, β *a maximal monotone graph in* $\mathbb{R}\times\mathbb{R}$ *with* $0 \in \beta(0)$ *and* $\varphi : \Omega\times\mathbb{R}\to\mathbb{R}$ *satisfying the following conditions:*

1. *for almost all* $x \in \Omega$, $r \to \varphi(x,r)$ *is continuous and nondecreasing,*
2. *for every* $r \in \mathbb{R}$, $x \to \varphi(x,r)$ *is in* $L^1(\Omega)$,
3. $\varphi(x,0) = 0$, $\varphi(x,r) \neq 0$ *whenever* $r \neq 0$ *and there exist* $\lambda > 0$, $\alpha \geq 2$ *such that* $\varphi(x,r)r \geq \lambda|r|^\alpha$.

In [20] it is shown that the above problem can be written in the form :

$$u'(t) + \mathcal{B}u(t) = f(t), \quad 0 < t < \infty$$

$$u(0) = u_0$$

where for any $q \geq 1$, $u_0 \in L^q(\Omega)$, $f \in L^1((0,\infty), L^q(\Omega))$ and $\mathcal{B}$ is shown to be an m-ϕ-accretive at zero operator in $L^q(\Omega)$ with

$$\phi(x) := C_{\alpha,\Omega,\lambda}\|x\|_q^\alpha$$

for zero $z = 0$ and some constant $C_{\alpha,\Omega,\lambda}$ which can be explicitly computed in α, λ and any upper bound $\mu_\Omega \geq 1$ on the measure of Ω. In particular, using Hölder's inequality we compute that we can set

$$C_{\alpha,\Omega,\lambda} := \frac{\lambda}{\mu_\Omega^{\alpha-2}}.$$

It is shown in [20] that $u(t)$ converges strongly in $L^q(\Omega)$ to $z = 0$ as $t \to \infty$ as it is shown that Problem 2 has a strong solution. Then, because $\mathcal{B}$ is, moreover, uniformly ϕ-accretive at zero (thus has a well-defined modulus of accretivity), the rate of metastability is given by Theorem 9. In particular, we have

$$\forall k \in \mathbb{N}\ \forall \bar{g} \in \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, M, B, \alpha, C_{\alpha,\Omega,\lambda})\ \forall t \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]$$

$$(\|u(t)\| < 2^{-k})$$

with a rate of metastability

$$\Psi(k, \bar{g}, M, B, \alpha, C_{\alpha,\Omega,\lambda}) = \tilde{g}^{(M\cdot 2^{k+1})}(0) + h(\tilde{g}^{(M\cdot 2^{k+1})}(0))$$

where $h(n) := (B(n) + 2) \cdot 2^{\Theta(k+2)+1}$,

$$\tilde{g}(n) := g(n) + n,$$

$$g(n) := \bar{g}(n + h(n)) + h(n),$$

$$\bar{n} := n + h(n),$$

$B(n)$ is a nondecreasing upper bound on

$$\frac{1}{2}\|u(n)\|^2,$$

$M \in \mathbb{N}$ is any upper bound on the integral $I = \int_0^\infty \|f(t)\|dt$, and $\Theta(k)$ may be estimated (in terms of $C_{\alpha,\Omega,\lambda}$ and α) as follows:

For any $q \geq 1$ and assuming $\|x\|_q \geq 2^{-k}$, let $\Theta(k)$ be such that

$$\Theta(k) \geq \min n\{C_{\alpha,\Omega,\lambda}\|x\|_q^\alpha \geq C_{\alpha,\Omega,\lambda} \cdot (2^{-k})^\alpha \geq 2^{-n}\}.$$

We have

$$\log_2(C_{\alpha,\Omega,\lambda} \cdot (2^{-k})^\alpha) \geq \log_2 2^{-n}$$

therefore for n fulfilling

$$n \geq k \cdot \alpha - \log_2 C_{\alpha,\Omega,\lambda}$$

we may take

$$\Theta(k) \geq k \cdot \alpha - \log_2 C_{\alpha,\Omega,\lambda}.$$

Notice that because $g(r) := C_{\alpha,\Omega,\lambda} \cdot r^\alpha$ is nondecreasing, the modulus of accretivity Θ for $\mathcal{B}$ depends only on $k \in \mathbb{N}$ but not on K (see Remark 2).

2.4 Effective Information on the Behaviour of Resolvents of Accretive Operators

2.4.1 Overview

In [21] García-Falset showed the convergence of the resolvents of set-valued, m-accretive and either uniformly ϕ-accretive at zero or ϕ-expansive operators on general real Banach spaces, to the zero of each operator. His work had been inspired by a classical result by Reich in [70], later improved by Takahashi and Ueda in [81]. In [21] there are not any additional assumptions on the Banach space (as was the case in [70] and [81]), instead, the special notion of uniform ϕ-accretivity at zero is assumed- in addition to m-accretivity. In an earlier, related work by García-Falset and Morales ([23]) the removal of additional assumptions on the Banach space had already been achieved only for operators that are m-accretive and ϕ-expansive.

Theorem 10. *(García-Falset, Theorem 4.4 in [21]) Let X be a Banach space. Let $A : D(A) \subseteq X \to 2^X$ be an m-accretive and either ϕ-expansive or uniformly ϕ-accretive at zero* [3] *operator. Then there exists a unique $z \in D(A)$ such that $0 \in Az$. In addition,*

$$(i) \lim_{\lambda\to\infty} J_\lambda x = z \text{ for each } x \in X$$

$$(ii) \lim_{n\to\infty} J^n_\lambda x = z \text{ for each } \lambda > 0 \text{ and } x \in X.$$

The following two recent results by Nicolae and Ariza-Ruiz, Leuştean, López-Acedo respectively that we will present refer to the asymptotic regularity of t-firmly nonexpansive mappings and will be applied in our analysis in Section 2.4.2. It is worthwhile to mention that both results involve computable bounds that were also extracted by proof mining. As they will be applied for firmly nonexpansive mappings, we will make in particular the choice

$$t = \frac{1}{2}$$

for our application, nevertheless here we present them in full generality for any $t \in (0, 1)$.

Theorem 11. *(Nicolae, Corollary 4.6 in [66]) Let X be a normed space and let $C \subseteq X$ be a nonempty and bounded set with diameter $d_C \leq b$. Suppose $T : C \to C$ is t-firmly nonexpansive. Then, for all $x \in C$,*

$$\forall k \in \mathbb{N}\ \forall n \geq \tilde{\Phi}(k, b, t)\ \|T^n x - T^{n+1}x\| \leq 2^{-k}$$

[3] in Theorem 4.4 in [21] Definition 20 is used for ϕ-accretivity at zero, so here this amounts to *uniform* ϕ-accretivity at zero in the sense of Definition 18 and thus A has a modulus of ϕ- accretivity at zero (recall the discussion in Section 2.2.).

where

$$\tilde{\Phi}(k,b,t) := M\lceil 2b(1+e^{KM})2^k\rceil,$$

with

$$K = \lceil \frac{1}{t} \rceil \text{ and } M = \lceil b2^{k+2}\rceil.$$

The following result in [2] had been formulated in the more general case of uniformly convex W- (UCW) hyperbolic spaces [4], however we state it below for the more special case of uniformly convex Banach spaces.

Theorem 12. *(Ariza-Ruiz, Leuştean and López-Acedo, Corollary 7.3 in [2]) Let X be a uniformly convex Banach space with a modulus of convexity $\eta : (0,2] \to (0,1]$. Let $b > 0$, $\lambda \in (0,1)$. For any bounded subset $C \subseteq X$ with diameter $d_C \leq b$, for all t-firmly nonexpansive mappings $T : C \to C$ and all $x \in C$,*

$$\forall k \in \mathbb{N}\ \forall n \geq \tilde{\Phi}(k,\eta,b,t)\ \|T^n x - T^{n+1}x\| \leq 2^{-k}$$

with

$$\tilde{\Phi}(k,\eta,b,t) = \begin{cases} \lceil \frac{2^k(b+1)}{t(1-t)\eta(\frac{2^{-k}}{b+1})}\rceil, & \text{for } 2^{-k} < 2b \\ 0, & \text{otherwise.} \end{cases}$$

Remark 6. *As it is also the case for Theorem 7.1. in [2], thus also in the above result, if the modulus of convexity η can be written as $\eta(\epsilon) = \epsilon\tilde{\eta}(\epsilon)$ where $\tilde{\eta}$ is increasing with ϵ, then in the bound we can replace η with $\tilde{\eta}$ (see Remark 7.2. in [2] and recall Remark 1).*

2.4.2 Proof-theoretic Analysis and Results

We extract explicit, effective and uniform rates for the convergence of the resolvents of set-valued, m-accretive and uniformly accretive at zero operators on general real Banach spaces to the zero of each operator. This is achieved via proof mining on the proof of Theorem 10. For the derivation of the bound here we need to make explicit the quantitative content of uniform ϕ-accretivity at zero assumed by García-Falset in [21] and for that we make use of the notion of a modulus of accretivity at zero. (We also show here that even for Theorem 10, rather than assuming uniform ϕ-accretivity at zero for the operator A it would be sufficient to just assume uniform accretivity at zero thus eliminating the introduction of the function ϕ.) Analogously, for the case of ϕ-expansive operators, we will make use of the corresponding notion of a modulus of ϕ-expansivity.

We show the following result which is a quantitative version of Theorem 10.

[4] W-hyperbolic spaces were originally introduced by Kohlenbach in [44] and uniformly convex W-hyperbolic spaces by Leuştean in [57].

Theorem 13. *(Quantitative version of Theorem 10) Let X be a real Banach space, let $A : D(A) \subseteq X \to 2^X$ be an m-accretive and either uniformly accretive at zero operator with a modulus of accretivity at zero Θ or ϕ-expansive operator with a modulus of ϕ-expansivity Δ. Then there exists a unique $z \in D(A)$ such that $0 \in Az$. Moreover, for all $x \in X$, for the case of a uniformly accretive at zero A :*

$$(i)\ \forall k \in \mathbb{N}\ \forall \lambda > 2^{\Theta_R(k)+1}R^2\ \ \|J_\lambda x - z\| < 2^{-k}$$

$$(ii)\ \forall \lambda > 0\ \forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1+e^{2M})2^{\lceil \log_2 \Lambda R\rceil + \Theta_R(k)+1}\rceil + 1$$
$$\|J^n_\lambda x - z\| < 2^{-k}$$

with

$$M = \lceil b2^{\lceil \log_2 \Lambda R\rceil + \Theta_R(k)+2}\rceil$$

and for the case of a ϕ-expansive A :

$$(i)'\ \forall k \in \mathbb{N}\ \forall \lambda > 2^{\Delta_R(k)+1}R\ \ \|J_\lambda x - z\| < 2^{-k}$$

$$(ii)'\ \forall \lambda > 0\ \forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1+e^{2M})2^{\lceil \log_2 \Lambda\rceil + \Delta_R(k)+1}\rceil + 1$$
$$\|J^n_\lambda x - z\| < 2^{-k}$$

with

$$M = \lceil b2^{\lceil \log_2 \Lambda\rceil + \Delta_R(k)+2}\rceil$$

where $\Lambda \in \mathbb{N}$ is such that $\frac{1}{\Lambda} \leq \lambda$, $R \in \mathbb{N}$ is such that $\|x - z\| < R$ and $b > 0$ is an upper bound on the diameter of $\overline{co}(D(A)) \cap B[z, R]$ (for example we can take $b := 2R$).

Remark 7. *For a uniformly accretive at zero A, about the existence of the zero z so that $0 \in Az$, see Definition 21, compare with Definition 20 and see the first point of the discussion below Definition 20. (Note that even if Definition 21 had been formulated assuming $(*)$ only for some $z \in D(A)$ and without incorporating the assumption $0 \in Az$ for z, it would be possible to show $0 \in Az$ by Theorem 5. The latter would be applicable here as A is assumed to be m-accretive, and by making the observation that it holds also for a uniformly accretive at zero A, as in its proof (given in [21]) just the fact $0 < \langle u, x - z\rangle_+$, which is indeed covered by the modulus as in $(*)$ of Definition 21 for any $z \in D(A)$, is sufficient).*

Proof. For the case of a uniformly accretive at zero operator A, considering the existence of z see the above remark. The uniqueness of z follows directly by the fact that $0 \in Az$ and $(*)$ in Definition 21, as assuming we had $z' \neq z$ so that $0 \in Az'$, for $(0, z') \in A$ we would have $\langle 0, z' - z\rangle_+ = 0$ therefore $(*)$ of Definition 21 could not hold for any $m \in \mathbb{N}$ which gives a contradiction.

For the case of a ϕ-expansive operator A, by Theorem 8 in [23], A is surjective therefore 0 is in the range of A i.e. for some $z \in D(A)$ we have $0 \in Az$. To show the uniqueness of such a z, assuming that for some $z' \in D(A)$ so that $z' \neq z$ we had $0 \in Az'$, then $\phi(\|z - z'\|) = 0$ gives a contradiction.

(Regarding uniqueness in both cases also recall Remark 3).

We will firstly show (i).

Let $x \in X$ and let $R \in \mathbb{N}$ be such that $\|x - z\| < R$. As A is m-accretive, for all $\lambda > 0$ the range of $I + \lambda A$ is X, so we have $J_\lambda x \in D(A)$ i.e. for all $\lambda > 0$

$$x \in J_\lambda x + \lambda A J_\lambda x.$$

If $0 \in Az$ clearly $z \in (I + \lambda A)^{-1} z$ for all $\lambda > 0$ and because for an accretive A the resolvent $J_\lambda := (I + \lambda A)^{-1}$ is single-valued (by Proposition 2 (ii)),

$$J_\lambda z = z \ (I).$$

Again by Proposition 2 (ii), J_λ is nonexpansive, therefore by (I) we obtain:

$$\|J_\lambda x - z\| = \|J_\lambda x - J_\lambda z\| \leq \|x - z\| \ (II).$$

Therefore, for any given $\lambda > 0$ the mapping J_λ is a self-mapping of

$$\overline{co}(D(A)) \cap B[z, R].$$

By Proposition 2 (iii) we have $A_\lambda x \in A J_\lambda x$ (where $A_\lambda := \frac{I - J_\lambda}{\lambda}$). Therefore

$$\begin{aligned}
\langle A_\lambda x, J_\lambda x - z\rangle_+ &\leq \|\frac{J_\lambda x - x}{\lambda}\|\|J_\lambda x - z\| \\
(\text{by}(II)) &\leq \frac{1}{\lambda}\|J_\lambda x - x\|\|x - z\| \\
&= \frac{1}{\lambda}\|J_\lambda x - x + J_\lambda z - J_\lambda z\|\|x - z\| \\
&\leq \frac{1}{\lambda}(\|J_\lambda x - J_\lambda z\| + \|J_\lambda z - x\|)\|x - z\| \\
&\leq \frac{1}{\lambda}(\|x - z\| + \|J_\lambda z - x\|)\|x - z\|
\end{aligned}$$

(because J_λ is nonexpansive by Proposition 2 (ii))

$$\begin{aligned}
(\text{and by}(I)) &= \frac{1}{\lambda}(\|x - z\| + \|z - x\|)\|x - z\| \\
&= \frac{2}{\lambda}\|x - z\|^2 < \frac{2R^2}{\lambda}.
\end{aligned}$$

Now clearly for $\lambda \to \infty$, $\frac{2R^2}{\lambda} \to 0$ i.e.

$$\forall k \in \mathbb{N}\ \exists k_0\ \forall \lambda > k_0\ (\frac{2R^2}{\lambda} < 2^{-k})$$

and by the decreasingness of $\frac{2R^2}{\lambda}$ in λ it is enough to write

$$\forall k \in \mathbb{N}\ \exists k_0\ (\frac{2R^2}{k_0} < 2^{-k})$$

[5] so we directly take

$$k_0 := 2^{k+1}R^2.$$

Thus we also have

$$\forall k \in \mathbb{N}\ \forall \lambda > 2^{k+1}R^2\ (\langle A_\lambda x, J_\lambda x - z\rangle_+ < 2^{-k}).$$

By assumption

$$\forall k, K \in \mathbb{N}\ (\|J_\lambda x - z\| \in [2^{-k}, K] \to \langle A_\lambda x, J_\lambda x - z\rangle_+ \geq 2^{-\Theta_K(k)}).$$

Considering $k, K \in \mathbb{N}$ fixed, and taking the equivalent contrapositive of the above

$$\langle A_\lambda x, J_\lambda x - z\rangle_+ < 2^{-\Theta_K(k)} \to \|J_\lambda x - z\| < 2^{-k} \vee \|J_\lambda x - z\| > K\ (III).$$

Considering that moreover

$$\forall m \in \mathbb{N}\ \forall \lambda > 2^{m+1}R^2\ (\langle A_\lambda x, J_\lambda x - z\rangle_+ < 2^{-m})$$

as previously determined and setting

$$m := \Theta_K(k)$$

for our fixed $k, K \in \mathbb{N}$ to fulfill the premise of (III) we obtain

$$\forall \lambda > 2^{\Theta_K(k)+1}R^2\ (\langle A_\lambda x, J_\lambda x - z\rangle_+ < 2^{-\Theta_K(k)})$$

and thus

$$\forall \lambda > 2^{\Theta_K(k)+1}R^2\ (\|J_\lambda x - z\| < 2^{-k} \vee \|J_\lambda x - z\| > K)$$

and because $k, K \in \mathbb{N}$ had been arbitrary,

$$\forall k, K \in \mathbb{N}\ \forall \lambda > 2^{\Theta_K(k)+1}R^2$$
$$(\|J_\lambda x - z\| < 2^{-k} \vee \|J_\lambda x - z\| > K.)$$

[5] Thanks to the decreasingness of the function, the statement : $\forall\exists\forall$ became $\forall\exists$ for which it is possible to find a bound. Indeed, we see that this is the case here.

Choosing in the above $K := R$ we have

$$\forall k \in \mathbb{N}\ \forall \lambda > 2^{\Theta_R(k)+1} R^2$$

$$(\|J_\lambda x - z\| < 2^{-k} \vee \|J_\lambda x - z\| > R)$$

however recall that by (II)

$$\|J_\lambda x - z\| \leq \|x - z\| < R.$$

We can therefore conclude that

$$\forall k \in \mathbb{N}\ \forall \lambda > 2^{\Theta_R(k)+1} R^2\ (\|J_\lambda x - z\| < 2^{-k}).$$

We will now show (ii).

Again let $x \in X$ and let $R \in \mathbb{N}$ be such that $\|x - z\| < R$. Let $\lambda > 0$ be fixed. Again recall that by Proposition 2 (iii) we have $A_\lambda J_\lambda^n x \in A J_\lambda^{n+1} x$ where A_λ is defined as previously. We have

$$\langle A_\lambda J_\lambda^n x, J_\lambda^{n+1} x - z\rangle_+ \leq \|A_\lambda J_\lambda^n x\| \|J_\lambda^{n+1} x - z\|$$

$$= \frac{1}{\lambda} \|J_\lambda^n x - J_\lambda^{n+1} x\| \|J_\lambda^{n+1} x - z\|$$

(by (I) :)

$$= \frac{1}{\lambda} \|J_\lambda^n x - J_\lambda^{n+1} x\| \|J_\lambda^{n+1} x - J_\lambda^{n+1} z\|$$

(since by Proposition 2 (ii) J_λ and thus J_λ^n is nonexpansive :)

$$\leq \frac{1}{\lambda} \|J_\lambda^n x - J_\lambda^{n+1} x\| \|x - z\|$$

$$< \frac{R}{\lambda} \|J_\lambda^n x - J_\lambda^{n+1} x\|\ (IV).$$

Note that J_λ is moreover firmly nonexpansive, and thus in particular t-firmly nonexpansive (a mapping is firmly nonexpansive if and only if it is the resolvent for some accretive operator, see [71]). As a firmly nonexpansive mapping with a fixed point z, by Corollary 1 of [71], J_λ is asymptotically regular, which means that for $n \to \infty$, we have that

$$\|J_\lambda^n x - J_\lambda^{n+1} x\| \to 0.$$

A computable and highly uniform rate of asymptotic regularity is given by Theorem 11 (recall that J_λ is a self-mapping on $\overline{co}(D(A)) \cap B[z, R]$ so we can indeed apply Theorem 11 .) In particular we apply Theorem 11 with the choice $t := \frac{1}{2}$ and we obtain that (for fixed $\lambda > 0$)

$$\forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1 + e^{2M}) 2^{k+1} \rceil\ \|J_\lambda^n x - J_\lambda^{n+1} x\| \leq 2^{-k}$$

where $b > 0$ is any upper bound on the diameter of $\overline{co}(D(A)) \cap B[z, R]$ (e.g we may take $b := 2R$) and

$$M = \lceil b 2^{k+2} \rceil.$$

This gives that

$$\forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1 + e^{2M})2^{k+1}\rceil$$
$$\left(\frac{R}{\lambda}\|J_\lambda^n x - J_\lambda^{n+1} x\| \leq \frac{R}{\lambda} 2^{-k}\right)$$

and letting

$$k := \lceil -\log_2 \frac{\lambda}{R} \rceil + m$$

for some $m \in \mathbb{N}$ we have (by the above shift and then renaming m to k and by (IV) :)

$$\forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1 + e^{2M})2^{\lceil -\log_2 \frac{\lambda}{R}\rceil + k + 1}\rceil$$
$$(\langle A_\lambda J_\lambda^n x, J_\lambda^{n+1} x - z\rangle_+ < \frac{R}{\lambda}\|J_\lambda^n x - J_\lambda^{n+1} x\| \leq 2^{-k})$$

and shifting n by 1

$$\forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1 + e^{2M})2^{\lceil -\log_2 \frac{\lambda}{R}\rceil + k + 1}\rceil + 1$$
$$(\langle A_\lambda J_\lambda^{n-1} x, J_\lambda^n x - z\rangle_+ < 2^{-k})$$

with

$$M = \lceil b 2^{\lceil -\log_2 \frac{\lambda}{R}\rceil + k + 2}\rceil.$$

We finally associate the rate of convergence of $\langle A_\lambda J_\lambda^{n-1} x, J_\lambda^n x - z\rangle_+$ to 0 with the rate of convergence of $\|J_\lambda^n x - z\|$ to 0. This is achieved by using the analogous argument as in the end of part (i) considering (III) with $\|J_\lambda^n x - z\|$ instead of $\|J_\lambda x - z\|$ i.e. setting in (III) $J_\lambda^{n-1} x$ instead of x and we obtain (as $\lambda > 0$ was arbitrary)

$$\forall \lambda > 0\ \forall k \in \mathbb{N}\ \forall n \geq M\lceil b(1 + e^{2M})2^{\lceil -\log_2 \frac{\lambda}{R}\rceil + \Theta_R(k) + 1}\rceil + 1$$
$$(\|J_\lambda^n x - z\| < 2^{-k})$$

with

$$M = \lceil b 2^{\lceil -\log_2 \frac{\lambda}{R}\rceil + \Theta_R(k) + 2}\rceil.$$

Finally, in order to ensure that the resulting bound will be computable for any real Banach space X, we can write the above obtained bound substituting the input $\lambda > 0$ with some $\frac{1}{\Lambda}$ so that $\lambda \geq \frac{1}{\Lambda}$ where $\Lambda \in \mathbb{N}$. That is, the computable $\Lambda \in \mathbb{N}$ functions as a Skolem constant witnessing the positivity of $\lambda > 0$ (recall Definition 2).

We will show (i').

Notice that we have by assumption

$$\|A_\lambda x - 0\| \leq \frac{1}{\lambda}\|x - J_\lambda x\|$$

(recall that by Proposition 2 (iii) we have $A_\lambda x \in AJ_\lambda x$ (where $A_\lambda := \frac{I - J_\lambda}{\lambda}$). giving

$$\|A_\lambda x\| \leq \frac{2}{\lambda}\|x - z\| < \frac{2R}{\lambda}.$$

Similarly as in (i), this gives

$$\forall k \in \mathbb{N}\ \forall \lambda > 2^{k+1}R\ \|A_\lambda x\| < 2^{-k}$$

and by assumption

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall x \in D(A)$$
$$(\|J_\lambda x - z\| \in [2^{-k}, K] \to \|A_\lambda x\| \geq 2^{-\Delta_K(k)}).$$

The above gives, arguing exactly as in (i), that

$$\forall k \in \mathbb{N}\ \forall \lambda > 2^{\Delta_R(k)+1}R\ \|J_\lambda x - z\| < 2^{-k}$$

so we get a bound that differs by a factor of R.

To show (ii'), it is enough to notice that

$$\|A_\lambda J_\lambda^n x\| = \frac{1}{\lambda}\|J_\lambda^n x - J_\lambda^{n+1} x\|$$

and proceed in exactly the same way as in (ii).

□

Under the supplementary assumption that the real Banach space X is moreover uniformly convex, instead of Theorem 13 (ii), we may obtain a bound that is polynomial if the modulus of convexity $\eta(\epsilon)$ is polynomial in ϵ (for example this is the case for the Banach spaces L_p, that, for $\infty > p \geq 2$, have an asymptotically optimal modulus of convexity $\frac{\epsilon^p}{p2^p}$).

We show the following:

Theorem 14. *(Quantitative version of Theorem 10 (ii)) Let X be a real Banach space that is uniformly convex with a modulus $\eta : (0,2] \to (0,1]$, let $A : D(A) \subseteq X \to 2^X$ be an m-accretive and either uniformly accretive at zero with a modulus Θ or ϕ-expansive with a modulus Δ operator. Then there exists a unique $z \in D(A)$ such that $0 \in Az$. Moreover, for all $x \in X$*

$$\forall \lambda > 0\ \forall k \in \mathbb{N}\ \forall n \geq \Phi\ \|J_\lambda^n x - z\| < 2^{-k}$$

with

$$\Phi(k,\eta,b,R,\Lambda,\Theta)=\begin{cases}[\frac{2^{\lceil\log_2\Lambda R\rceil+\Theta_R(k)+2}(b+1)}{\eta(\frac{2^{-(\lceil\log_2\Lambda R\rceil+\Theta_R(k))}}{b+1})}]+1, & \text{for } 2^{-\Theta_R(k)}<2b\\ 1, & \text{otherwise.}\end{cases}$$

for a uniformly accretive at zero A, or

$$\Phi(k,\eta,b,R,\Lambda,\Delta)=\begin{cases}[\frac{2^{\lceil\log_2\Lambda\rceil+\Delta_R(k)+2}(b+1)}{\eta(\frac{2^{-(\lceil\log_2\Lambda\rceil+\Delta_R(k))}}{b+1})}]+1, & \text{for } 2^{-\Delta_R(k)}<2b\\ 1, & \text{otherwise.}\end{cases}$$

for a ϕ-expansive A,

where $\Lambda\in\mathbb{N}$ is such that $\frac{1}{\Lambda}\leq\lambda$, $R\in\mathbb{N}$ is such that $\|x-z\|<R$ and $b>0$ is an upper bound on the diameter of $\overline{co}(D(A))\cap B[z,R]$ (for example $b:=2R$).

Proof. The proof is exactly the same as the proof of Theorem 13(ii) above, with the difference that instead of Theorem 11 we apply Theorem 12 (again with the choice $t:=\frac{1}{2}$ so that $t(1-t)=\frac{1}{4}$) by which we obtain that, for $J_\lambda:\overline{co}(D(A))\cap B[z,R]\to\overline{co}(D(A))\cap B[z,R]$ (where $\lambda>0$ is fixed), for all $x\in\overline{co}(D(A))\cap B[z,R]$, letting $b>0$ be an upper bound on the diameter of $\overline{co}(D(A))\cap B[z,R]$

$$\forall k\in\mathbb{N}\ \forall n\geq\tilde{\Phi}(k,\eta,b)\ \|J_\lambda^n x-J_\lambda^{n+1}x\|\leq 2^{-k}$$

with

$$\tilde{\Phi}(k,\eta,b)=\begin{cases}[\frac{2^{k+2}(b+1)}{\eta(\frac{2^{-k}}{b+1})}], & \text{for } 2^{-k}<2b\\ 0, & \text{otherwise.}\end{cases}$$

To complete the proof we proceed with the exact same argumentation as in Theorem 13 (ii) and (ii') in the respective cases.

□

If the modulus of convexity η can be written as $\eta(\epsilon)=\epsilon\tilde{\eta}(\epsilon)$ where $\tilde{\eta}$ is increasing with ϵ, then in the bound we can replace η with $\tilde{\eta}$. (See Remark 6 and Remark 1).

Chapter 3

Proof Mining for the Fixed Point Theory of Nonexpansive Semigroups

3.1 Preliminaries and Overview

Considering a one-parameter nonexpansive semigroup $\{T(t) : t \geq 0\}$ as defined in Definition 8 (possibly without Property (1)), for a given $t \geq 0$, we denote the set of the fixed points of $T(t)$, i.e. all the points $q \in C$ for which $T(t)q = q$, by $F(T(t))$. It would be of interest to ask the question :

Question 1. *How could one determine the set of the common fixed points of the semigroup, i.e.* $\bigcap_{t\geq 0} F(T(t))$*, that is all the points* $q \in C$ *such that*

$$\forall t \in [0, \infty)\ T(t)q = q\ ?$$

Two different answers to the above question were given by T. Suzuki via two different results that we present below.

Theorem 15. *(Suzuki, Theorem 1 in [80]) Let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on a subset* $C \subseteq X$ *for some Banach space* X*. Let* $\alpha, \beta \in \mathbb{R}^+$ *satisfying* $\alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$*. Then for all* $\lambda \in (0, 1)$ *:*

$$\bigcap_{t\geq 0} F(T(t)) = F(\lambda T(\alpha) + (1 - \lambda)T(\beta)),$$

where

$$\lambda T(\alpha) + (1 - \lambda)T(\beta)$$

is a mapping from C *into* X *defined by*

$$(\lambda T(\alpha) + (1 - \lambda)T(\beta))x = \lambda T(\alpha)x + (1 - \lambda)T(\beta)x$$

for $x \in C$*.*

Theorem 16. *(Suzuki, Proposition 2 in [79]) Let X be a Banach space and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a subset C of X. Let α, β be positive real numbers so that $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then we have*

$$\bigcap_{t \geq 0} F(T(t)) = F(T(\alpha)) \cap F(T(\beta))$$

Clearly if $q \in F(T(\alpha)) \cap F(T(\beta))$, then $q \in F(\lambda T(\alpha) + (1-\lambda)T(\beta))$ for any $\lambda \in (0,1)$ as assuming that for some $q \in C$ we have

$$T(\alpha)q = q \wedge T(\beta)q = q$$

we obtain

$$(\lambda T(\alpha) + (1-\lambda)T(\beta))q = \lambda T(\alpha)q + (1-\lambda)T(\beta)q = \lambda q + (1-\lambda)q = q.$$

However, the converse does not hold. In that sense, Theorem 15 is a generalisation of Theorem 16.

It is worthwhile to mention that Theorem 15 has the following important consequence:

Theorem 17. *(Suzuki, Theorem 4 in [80]) Let X be a Banach space and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a subset C of X. Assume that every nonexpansive mapping on C has a fixed point. Then $\{T(t) : t \geq 0\}$ has a common fixed point.*

Proof. (Suzuki, ([80])) Having assumed that every nonexpansive mapping on C has a fixed point, then in particular the nonexpansive mapping $\frac{1}{2}T(1) + \frac{1}{2}T(\sqrt{2})$ on C has a fixed point, i.e. $F(\frac{1}{2}T(1) + \frac{1}{2}T(\sqrt{2})) \neq \emptyset$. By Theorem 15, $\bigcap_{t \geq 0} F(T(t)) = F(\frac{1}{2}T(1) + \frac{1}{2}T(\sqrt{2})) \neq \emptyset$. □

Observe that the above could *not* follow from Theorem 16, as the assumption that $T(t_1) \neq \emptyset \wedge T(t_2) \neq \emptyset$ does *not* give $T(t_1) \cap T(t_2) \neq \emptyset$.

We say that Theorem 15 and Theorem 16 shown in [80] and in [79] respectively are “different” even though the two statements giving $\bigcap_{t \geq 0} F(T(t))$ are similar- to be more specific as we saw the statement in [80] is a generalisation of the statement in [79]. The point is, that the methodology, structure and content of their proofs is completely different and this reflects to the different bounds that we will eventually obtain after performing proof mining on both said proofs in view of answering the question :

Question 2. *What quantitative, computable information can one obtain regarding the computation of the set of the approximate common fixed points of $\{T(t) : t \geq 0\}$?*

In Sections 3.2 and 3.3 we will show in detail our extraction of the bounds by giving quantitative versions for the results of the two approaches by Suzuki respectively, thus providing two different answers to the above question.

To this end, in Sections 3.1.1 and 3.1.2 we will explain certain notions that will be later used in our quantitative analysis and bound extraction. After that, in Section 3.1.3 we will present two corollaries of Metatheorem 2 for the general case of a normed space $(X, \|\cdot\|)$ with a subset $C \subseteq X$ adapted in two different concrete settings involving nonexpansive semigroups so as to illustrate how the extractability of the bounds is a priori guaranteed proof-theoretically in the specific cases that we will study.

3.1.1 Effective Irrationality Measure

Because both results by Suzuki make an irrationality assumption on $\gamma := \alpha/\beta$, we will need a quantitative version of this assumption in both of our quantitative analyses of his proofs. For that we will make use of the following notion from number theory.

Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Then

$$\forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ \exists z \in \mathbb{N}\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{z}).$$

The Skolem normal form of the above is

$$\exists f_\gamma : \mathbb{N} \times \mathbb{Z}^+ \to \mathbb{N}\ \forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p, p')})\ (i)$$

and f_γ is the corresponding Skolem function (recall Definition 2).

Notice that , if $\gamma \in (0, 1)$, assuming that we have

$$|\gamma - \frac{p'}{p}| \geq \frac{1}{\tilde{f}(p, p')},$$

in the case where $p' \geq p + 1$, because $\frac{p'}{p} \geq 1 + \frac{1}{p}$ we have $|\gamma - \frac{p'}{p}| \geq \frac{1}{p}$ so we can take $f(p) := p$. For the other finitely many cases where $p' < p + 1$, we can take $f(p) := \max\{\tilde{f}(p, p') : p' \leq p\}$. So in any case we may take

$$f(p) := \max\{p, \max\{\tilde{f}(p, p) : p' \leq p\}\}.$$

Therefore without loss of generality in the case that $\gamma \in (0, 1)$, the following is instead used :

$$\exists f_\gamma : \mathbb{N} \to \mathbb{N}\ \forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)})\ (I)$$

Definition 24. *Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. The function f_γ as in (I) (or f_γ as in (i)) above is called an effective irrationality measure for γ.*

Remark 8. *Since $\gamma > 0$, both (I) and (i) can easily be seen to imply the claim also for $p' \in \mathbb{Z}$.*

Definition 25. *Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. The function f_γ as in (I) is called an effective irrationality measure for γ.*

Remark 9. *Note that, in the context of this work (as in both works by Suzuki one make any choice of $\alpha, \beta \in \mathbb{R}^+$ as long they have an irrational ratio), one may choose without loss of generality $\alpha, \beta \in \mathbb{R}^+$ with $0 < \alpha < \beta$ so that $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with an effective irrationality measure f_γ that is known or easy to compute. This may be computed by proof-mining on (i.e. an effective version of) the proof of the irrationality of γ.*

Example 1. *For example, let us make the choice $\alpha := \sqrt{2}, \beta := 2$. Then, $f_{\frac{\sqrt{2}}{2}}$ is specified by writing down a quantitative version of the classical proof of the irrationality of $\frac{\sqrt{2}}{2}$ as follows: Assuming that $\frac{\sqrt{2}}{2} \in \mathbb{Q}$, there will exist $p, p' \in \mathbb{Z}$ so that $\frac{\sqrt{2}}{2} = \frac{p'}{p}$. Without loss of generality we may assume that $\frac{p'}{p}$ is in maximally simplified form. Then $p^2 = 2p'^2$ and so p^2 is even. Hence p is even, i.e. there exists $k \in \mathbb{Z}$ so that $p = 2k$. Thus $4k^2 = 2p'^2$, so p'^2 is even and thus p' is even contradicting the assumption that $\frac{p'}{p}$ was maximally simplified. Now, by the above proof (assuming without loss of generality that $\frac{p'}{p}$ is in maximally simplified form, thus also $\frac{p'^2}{p^2}$ is in maximally simplified form), we cannot have $2p'^2 - p^2 = 0$. Thus,*

$$|2p'^2 - p^2| \geq 1$$

and therefore for all $p, p' \in \mathbb{Z}$

$$|\frac{\sqrt{2}}{2} - \frac{p'}{p}| = \frac{|\frac{2}{4} - \frac{p'^2}{p^2}|}{|\frac{\sqrt{2}}{2} + \frac{p'}{p}|} = \frac{|p^2 - 2p'^2|}{2p^2|\frac{\sqrt{2}}{2} + \frac{p'}{p}|} \geq \frac{1}{2p^2|\frac{\sqrt{2}}{2} + \frac{p'}{p}|} \geq \frac{1}{4p^2}$$

having assumed that

$$\frac{p'}{p} \leq 2 - \frac{\sqrt{2}}{2}.$$

If we had assumed $\frac{p'}{p} > 2 - \frac{\sqrt{2}}{2}$, then $|\frac{\sqrt{2}}{2} - \frac{p'}{p}| > 2 - \sqrt{2} > \frac{1}{4p^2}$ for all $p \in \mathbb{N}$ therefore in any case we may set $f_{\frac{\sqrt{2}}{2}}(p) := 4p^2$.

3.1.2 Equicontinuity

We introduce the following concepts of *uniform equicontinuity* for a nonexpansive semigroup and *modulus of uniform equicontinuity*:

Definition 26. *(Kohlenbach and K.-A. ([48]))We say that a nonexpansive semigroup $\{T(t) : t \geq 0\}$ on a subset C of a Banach space X is uniformly equicontinuous if the mapping $t \mapsto T(t)q$ is uniformly continuous on each compact interval $[0, K]$ for all $K \in \mathbb{N}$ and given a $b \in \mathbb{N}$ it has a common modulus of continuity for all $q \in C_b$ where $C_b := \{q \in C : \|q\| \leq b\}$. Namely if there exists a function $\omega : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that*

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall m \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall t, t' \in [0, K]$$

$$(|t - t'| < 2^{-\omega_{K,b}(m)} \to \|T(t)q - T(t')q\| < 2^{-m}).$$

We call ω a modulus of uniform equicontinuity for the nonexpansive semigroup $\{T(t) : t \geq 0\}$.

In the following we will assume uniform equicontinuity as defined above for the nonexpansive semigroup $\{T(t) : t \geq 0\}$. The motivation from introducing equicontinuity i.e. the property of having a *common* modulus of continuity for all q that are norm-bounded by a specific $b \in \mathbb{N}$, and assuming this requirement for our semigroup, comes from the need to fit the framework of the logical metatheorems that will guarantee the extractability of the bounds (see Section 3.1.3.). We will see that to achieve the majorizability of the semigroup equicontinuity is required. In praxis one may a posteriori remove equicontinuity but then the bound would be less uniform as it would depend on each point which would not be desirable. Moreover it would then not be possible to obtain the results on asymptotic regularity.

In the literature one may find several examples where uniform equicontinuity is fulfilled. For instance, any nonexpansive semigroup generated from a bounded accretive operator via the Crandall-Liggett formula (recall Chapter 2.1. in this thesis) fulfills the property of uniform equicontinuity, as can been seen in [15] (see in particular (1.11) there). Moreover, we mention the following example:

Example 2. *In [67] the following mapping is studied (referring to [65] where it is attributed to G.F. Webb) :*

Let $X = C = C_{[0,1]}$ and for $f \in C_{[0,1]}$ and $x \in [0, 1]$ define:

$$(T(t)f)(x) := \begin{cases} t + f(x) \ \text{if}\ f(x) \geq 0, \\ t + \frac{1}{2}f(x) \ \text{if}\ f(x) < 0 \ \text{and}\ t + \frac{1}{2}f(x) \geq 0, \\ 2t + f(x) \ \text{if}\ t + \frac{1}{2}f(x) < 0. \end{cases}$$

It is easy to see that the above semigroup is nonexpansive: we distinguish the following cases; In the case that for both $f(x)$ and $f(y)$ the first case holds nonexpansivity is trivial. Similarly in the cases where for both $f(x)$ and $f(y)$ the second case holds and where for both $f(x)$ and $f(y)$ the third case holds. Now, in the cases where the first and second cases are combined, assume, without loss of generality, that we have $f(x) \geq 0$ and $f(y) < 0 \wedge t + \frac{1}{2}f(y) \geq 0$ we have

$$|(T(t)f)(x) - (T(t)f)(y)| = |t + f(x) - (t + \frac{1}{2}f(y))| = |f(x) - \frac{1}{2}f(y)| \leq |f(x) - f(y)|$$

because both $f(x)$ and $-f(y)$ are positive quantities. In the cases where the first and third cases are combined, assume, without loss of generality, that we have $f(x) \geq 0$ and $t + \frac{1}{2}f(y) < 0$ we have

$$|(T(t)f)(x) - (T(t)f)(y)| = |t + f(x) - (2t + f(y))| = |f(x) - f(y) - t| \leq |f(x) - f(y)|$$

again because both $f(x)$ and $-f(y)$ are positive quantities. Finally, in the cases where the second and third cases coincide, assume, without loss of generality, that we have $f(x) < 0 \wedge t + \frac{1}{2}f(x) \geq 0$ and $t + \frac{1}{2}f(y) < 0$ notice that we have

$$\frac{1}{2}f(y) < -t \leq \frac{1}{2}f(x) < 0$$

thus

$$\frac{1}{2}f(y) + \frac{1}{2}f(x) < -t + \frac{1}{2}f(x) \leq f(x) < \frac{1}{2}f(x).$$

Now by

$$\frac{1}{2}f(y) < \frac{1}{2}f(x)$$

it is

$$f(y) < \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

thus

$$f(y) < \frac{1}{2}f(x) + \frac{1}{2}f(y) < -t + \frac{1}{2}f(x) \leq f(x)$$

so

$$|(T(t)f)(x) - (T(t)f)(y)| = |t + \frac{1}{2}f(x) - (2t + f(y))| = |\frac{1}{2}f(x) - t - f(y)|$$

$$= |f(y) - (-t + \frac{1}{2}f(x))| \leq |f(x) - f(y)|.$$

Now, considering the modulus, in the first case where $f(x) \geq 0$ we have

$$|(T(t)f)(x) - (T(t')f)(x)| = |t - t'|$$

thus we may take $\omega(m) := m$ as we can see that

$$(|t - t'| < 2^{-m} \to |(T(t)f)(x) - (T(t')f)(x)| = |t - t'| < 2^{-m}).$$

The same modulus can be taken in the case where $f(x) < 0$ and both $t+\frac{1}{2}f(x) \geq 0$ and $t' + \frac{1}{2}f(x) \geq 0$. In the case where $f(x) < 0$ and both $t + \frac{1}{2}f(x) < 0$ and $t' + \frac{1}{2}f(x) < 0$ we have

$$|(T(t)f)(x) - (T(t')f)(x)| = 2|t - t'|$$

thus we may take $\omega(m) := m + 1$ as we can see that

$$(|t-t'| < 2^{-(m+1)} \rightarrow |(T(t)f)(x)-(T(t')f))x| = 2|t-t'| < 2\cdot 2^{-(m+1)} = 2^{-m}).$$

Finally, in the cases where $f(x) < 0$ and $t + \frac{1}{2}f(x) < 0 \wedge t' + \frac{1}{2}f(x) \geq 0$ or $t + \frac{1}{2}f(x) \geq 0 \wedge t' + \frac{1}{2}f(x) < 0$ (let's take without loss of generality the second one:) we have

$$|(T(t)f)(x)-(T(t')f)(x)| = |t+\frac{1}{2}f(x)-(2t'+f(x))| \leq |-t'-\frac{1}{2}f(x))|+|-t'+t|$$

$$\leq 2|-t'+t|$$

(that is because $|-t' - \frac{1}{2}f(x))| \leq |-t' + t|$ as here $t' < -\frac{1}{2}f(x)$) thus also in this case we may take $\omega(m) := m+1$. In conclusion as a common modulus of continuity we may take $\omega(m) := m + 1$ to cover all cases.

3.1.3 Corollaries of Metatheorem 2 adapted for Nonexpansive Semigroups

Consider a nonexpansive semigroup $\{T(t) : C \rightarrow C, t \geq 0\}$ as defined in Definition 8 (without Property 1) on $C \subseteq X$ for a Banach space X. As mentioned in Section 3.1, we will present two corollaries of Metatheorem 2 adapted to specific settings for the theory of nonexpansive semigroups. We will shortly see that these settings correspond to the statements of Theorems 19 and 15 by Suzuki and therefore the extractability of the bounds is a priori guaranteed proof-theoretically. (Since in this subsection we are referring again to the logical background, the natural numbers are here defined as $\mathbb{N} := \{0, 1, ...\}$ as was done throughout Section 1.2.)

As we briefly explained in the end of Section 1.2.3. the system $\mathcal{A}^\omega$ is extended to $\mathcal{A}^\omega[X, \|\cdot\|]$ (for the details again we refer to [45]). In turn the latter must now be extended to $\mathcal{A}^\omega[X, \|\cdot\|, C]$ as in this theory we introduce a subset $C \subseteq X$. To this end new constants b_X, c_X, χ_C of type $0, X$ and $0(X)$ respectively are added, together with the axioms:

- $\forall x^X\ (\chi_C(x) =_0 0 \rightarrow \|x\|_X \leq_{\mathbb{R}} (b_X)_{\mathbb{R}})$,
- $\chi_C(c_X) =_0 0$,
- $\forall x^X\ (\chi_C(x) \leq_0 1)$

In the above χ_C is to be interpreted as the characteristic function of $C \subseteq X$. The latter fulfills a weak form of extensionality:

$$\frac{A_0 \to s =_X t}{A_0 \to \chi_C(s) =_X \chi_C(t)}$$

where A_0 is a quantifier-free formula.

(Note that here C is not assumed to be convex, therefore we omit the convexity axiom (axiom (13) of the construction as given in page 414 of [45]).

Without the constant b_X and the axiom that corresponds to it from the system $\mathcal{A}^\omega[X, \|\cdot\|, C]$ we obtain the system $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$.

Notice that the definition of a nonexpansive semigroup on a subset C of a Banach space X involves only universal axioms, which are permitted to be added for the extention of the theory. Considering that a nonexpansive semigroup is an object of type $1 \to (X \to X)$ and having $\alpha := 0_X$ as we are now in a normed space setting, by the definition of the strong majorizability relation (see Section 1.2.3) we have

$$T^* \gtrsim^{0_X}_{1\to(X\to X)} T :\equiv$$

$$\forall t^*, t(t^* \gtrsim^{0_X}_1 t \to T^*(t^*, \cdot) \gtrsim^{0_X}_{X\to X} T(t, \cdot))$$
$$\wedge \forall t^*, t(t^* \gtrsim^{0_X}_1 t \to T^*(t^*, \cdot) \gtrsim^{0_X}_1 T^*(t, \cdot)).$$

In turn

$$T^*(t^*, \cdot) \gtrsim^{0_X}_{X\to X} T(t, \cdot) :\equiv$$
$$\forall x^*, x(x^* \gtrsim^{0_X}_X x \to T^*(t^*, x^*) \gtrsim^{0_X}_X T(t, x))$$
$$\wedge \forall x^*, x(x^* \gtrsim^{0_X}_0 x \to T^*(t^*, x^*) \gtrsim^{0_X}_0 T^*(t^*, x))$$

and

$$T^*(t^*, \cdot) \gtrsim^{0_X}_1 T^*(t, \cdot) :\equiv$$
$$\forall x^*, x(x^* \gtrsim^{0_X}_0 x \to T^*(t^*, x^*) \gtrsim^{0_X}_0 T^*(t, x))$$
$$\wedge \forall x^*, x(x^* \gtrsim^{0_X}_0 x \to T^*(t^*, x^*) \gtrsim^{0_X}_0 T^*(t^*, x))$$

In total :

$$\forall x^*, x, t^*, t\Big(t^* \gtrsim^{0_X}_1 t \wedge x^* \gtrsim^{0_X}_X x$$
$$\to T^*(t^*, x^*) \gtrsim^{0_X}_X T(t, x) \wedge T^*(t^*, x^*) \gtrsim^{0_X}_0 T^*(t, x) \wedge T^*(t^*, x^*) \gtrsim^{0_X}_0 T^*(t^*, x)\Big).$$

The above can easily be reduced to (also see Lemma 17.80 in [45]):

$$\forall x^*, x, t^*, t\Big(t^* \gtrsim^{0_X}_1 t \wedge x^* \gtrsim^{0_X}_X x$$

$$\rightarrow T^*(t^*, x^*) \gtrsim_X^{0_X} T(t,x) \wedge T^*(t^*, x^*) \gtrsim_0^{0_X} T^*(t^*, x)\Big).$$

We will find such a T^* as above so that

$$\forall \tilde{t}, t, x^*, x\ (x^* \geq \|x\| \wedge \tilde{t} \geq t \rightarrow T^*(\tilde{t}, x^*) \geq \|T(t)x\| \wedge T^*(\tilde{t}, x^*) \geq T^*(\tilde{t}, x))$$

assuming that for some $z \in X$ with $\|z\| \leq K \in \mathbb{N}$ and that for some $t' \in \mathbb{R}^+$ with $t' \leq n \in \mathbb{N}$ we have $\|T(t')z - z\| \leq K$.

Now notice that $x^* \in \mathbb{N}$ can be taken as a majorant of $x \in X$ as it has the correct type (type 0 in this case). However from the theory we would not expect $\mathbb{N} \ni \tilde{t} \geq t$ to be considered as a majorant of $t \in \mathbb{R}^+$ as here the theory says that the majorant should have type 1 and not type 0 (recall that the reals have type 1 so they should be majorized by a number-theoretic function). However let us see how a natural number $\tilde{t}$ playing the role of a majorant of a real could be "produced" : Each real number t is represented by a Cauchy sequence of rational numbers that has a fixed rate of convergence while the rationals are represented by naturals via a monotone coding function so that $t(m) \in \mathbb{N}$ is an upper bound for the absolute value of tha rational number that it encodes (see Chapter 4 in [45]). In Chapter 4 in [45] a 2^{-k} rational approximation $\hat{t}(k)$ of t is constructed so that

$$\forall k (|t -_{\mathbb{R}} \lambda n.\hat{t}(k)|_{\mathbb{R}} <_{\mathbb{R}} \langle 2^{-k} \rangle)$$

where $n \mapsto \langle n \rangle$ denotes the embedding of $\mathbb{N}$ into $\mathbb{Q}$ defined in Chapter 4 in [45]. So $|t -_{\mathbb{R}} \lambda n.\hat{t}(0)|_{\mathbb{R}} <_{\mathbb{R}} 1$ meaning that the natural $t(0) + 1$ is an upper bound for the real number represented by $|t|_{\mathbb{R}}$. If $t^* \gtrsim_1 t$ we have $t^*(0) + 1 \geq_0 t(0) + 1$ therefore we see that we have indeed "produced" a natural number $t^*(0)+1 := \tilde{t}$ that can be seen as a majorant for the real t.

We now consider the estimates:

$$\|T(t)x\| = \|T(t)x - T(t)z + T(t)z + T(t')z - T(t')z + z - z\|$$

$$\leq \|T(t)x - T(t)z\| + \|T(t)z - T(t')z\| + \|T(t')z - z\| + \|z\|$$

$$\leq \|x - z\| + \|T(t)z - T(t')z\| + \|T(t')z - z\| + \|z\|$$

$$\leq \|x\| + 3K + \|T(t)z - T(t')z\| (!).$$

Now to bound the term $\|T(t)z - T(t')z\|$ we consider that:

$$\forall k \in \mathbb{N}\ \forall t, t' \in [0, \max\{n, t^*(0) + 1\}]$$

$$(|t - t'| < 2^{-\omega_{K, \max\{n, t^*(0)+1\}}(k)} \rightarrow \|T(t)z - T(t')z\| \leq 2^{-k}).$$

Setting $k = 0$

$$\forall t, t' \in [0, \max\{n, t^*(0) + 1\}]$$

$$(|t-t'| < 2^{-\omega_{K,\max\{n,t^*(0)+1\}}(0)} \to \|T(t)z - T(t')z\| \leq 1).$$

Now construct $2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n, t^*(0)+1\}$ many points so that

$$|t-t_1|, |t_1-t_2|, ...,$$

$$|t_{2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n,t^*(0)+1\}} - t'| < 2^{-\omega_{K,\max\{n,t^*(0)+1\}}(0)}$$

which give that

$$\|T(t)z - T(t_1)z\|, \|T(t_1)z - T(t_2)z\|, ...,$$

$$\|T(t_{2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n,t^*(0)+1\}}z - T(t')z\| \leq 1$$

and by applying the triangle inequality repeatedly the above give

$$\|T(t)z - T(t')z\| \leq 2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n, t^*(0)+1\} + 1$$

and substituting this in (!) we obtain

$$\|T(t)x\| \leq 3K + \|x\| + 2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n, t^*(0)+1\} + 1.$$

So,

$$\forall x^*, x, t^*, t$$

$$(\|x\| \leq x^* \wedge t \leq t^*(0) + 1 \to \|T(t)x\| \leq T^*(t^*, x^*) \wedge T^*(t^*, x) \leq T^*(t^*, x^*))$$

with

$$T^*(t^*, x^*) := 3K + x^* + 2^{\omega_{K,\max\{n,t^*(0)+1\}}(0)}\max\{n, t^*(0)+1\}) + 1.$$

As we have just seen that a nonexpansive semigroup on a subset C of a Banach space X is majorizable, we can state the following two versions of Metatheorem 2 but for the theory $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$ (we emphasize again that throughout this thesis $\mathcal{A}^\omega$ is defined as in Chapter 1 with WKL and not with DC as in [45]) and are formulated in an analogy to Corollary 17.71 in [45] that refers to a single nonexpansive mapping, but adapted here to the case of an equicontinuous nonexpansive semigroup $\{T(t) : t \geq 0\}$ with a modulus ω. (Also see Corollary 17.55 in [45] where the majorants are treated as in Corollary 17.54 in [45].) In the following, the real numbers are assumed to be extensional with respect to equality (as explained in page 398 in [45]). Also note that an object of type C in this context is to be understood as described by the axioms on page 415 in [45]. Firstly we state a corollary of the metatheorem explicitly for the setting as in Theorem 15 :

Metatheorem 3. *(Relevant to Metatheorem 2 and Corollary 17.71 in [45]) Assume that we have a proof of a sentence in* $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$

$$\forall \alpha, \beta, t \in \mathbb{R}^+ \; \forall N \in \mathbb{N} \; \forall \lambda \in (0,1) \; \forall z \in C \; \forall T \in C \times \mathbb{R}^+ \to C$$

$$\forall \omega \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{N}\ \forall f_{\frac{\alpha}{\beta}} \in \mathbb{N}\to\mathbb{N}\ \forall m\in\mathbb{N}\ \exists k\in\mathbb{N}$$

$$\Big((\forall t\in\mathbb{R}^+\ \forall x,y\in C\ \|T(t)x-T(t)y\| \leq_{\mathbb{R}} \|x-y\|)$$

$$\wedge(\forall x\in C\ \forall t,s\in\mathbb{R}^+\ T(s)\circ T(t)(x) =_X T(s+t)(x)) \wedge (\frac{1}{N} \leq_{\mathbb{R}} \beta)$$

$$\wedge(\forall p\in\mathbb{N}\forall p'\in\mathbb{Z}^+\ (|\frac{\alpha}{\beta}-\frac{p'}{p}| \geq_{\mathbb{R}} \frac{1}{f_{\frac{\alpha}{\beta}}(p)}))$$

$$\wedge(\forall b\in\mathbb{N}\ \forall q\in C\ \forall m\in\mathbb{N}\ \forall K\in\mathbb{N}\ \forall t,t'\in[0,K]$$

$$(\|q\| <_{\mathbb{R}} b \wedge |t-t'| <_{\mathbb{R}} 2^{-\omega_{K,b}(m)} \to \|T(t)q-T(t')q\| \leq_{\mathbb{R}} 2^{-m}))$$

$$\wedge(\|(\lambda T(\alpha)+(1-\lambda)T(\beta))z-z\| \leq_{\mathbb{R}} 2^{-k}) \to \|T(t)z-z\| <_{\mathbb{R}} 2^{-m}\Big).$$

Then one can extract from the proof a primitive recursive in the sense of Gödel's T functional Φ *so that*

$$\forall D\in\mathbb{N}\ \forall\alpha,\beta\in[0,D]\ \forall N\in\mathbb{N}\ \forall M\in\mathbb{N}\ \forall t\in[0,M]\ \forall\lambda\in(0,1)\ \forall\Lambda\in\mathbb{N}$$

$$\forall B,B'\in\mathbb{N}\ \forall z\in C_B\ \forall T\in C\times\mathbb{R}^+\to C\ \forall\omega\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{N}$$

$$\forall f_{\frac{\alpha}{\beta}}\in\mathbb{N}\to\mathbb{N}\ \forall m\in\mathbb{N}\ \exists k\leq\Phi(B,D,M,\Lambda,N,m,f'_{\frac{\alpha}{\beta}},\omega')$$

$$\Big((\forall t\in\mathbb{R}^+\ \forall x,y\in C\ \|T(t)x-T(t)y\| \leq_{\mathbb{R}} \|x-y\|)$$

$$\wedge(\forall x\in C\ \forall t,s\in\mathbb{R}^+\ T(s)\circ T(t)(x) =_X T(s+t)(x)) \wedge (\frac{1}{N} \leq_{\mathbb{R}} \beta)$$

$$\wedge(\frac{1}{\Lambda} \leq_{\mathbb{R}} \lambda) \wedge (\frac{1}{\Lambda} \leq_{\mathbb{R}} 1-\lambda)$$

$$\wedge(\forall p\in\mathbb{N}\ \forall p'\in\mathbb{Z}^+\ (|\frac{\alpha}{\beta}-\frac{p'}{p}| \geq_{\mathbb{R}} \frac{1}{f_{\frac{\alpha}{\beta}}(p)}))$$

$$\wedge(\forall b\in\mathbb{N}\ \forall q\in C\ \forall m\in\mathbb{N}\ \forall K\in\mathbb{N}\ \forall t,t'\in[0,K]$$

$$(\|q\| <_{\mathbb{R}} b \wedge |t-t'| <_{\mathbb{R}} 2^{-\omega_{K,b}(m)} \to \|T(t)q-T(t')q\| \leq_{\mathbb{R}} 2^{-m}))$$

$$\wedge\|z\| \leq_{\mathbb{R}} B \wedge \|T(\tau)z-z\| \leq_{\mathbb{R}} B'$$

$$\wedge\tau \leq_{\mathbb{R}} M \wedge (\|(\lambda T(\alpha)+(1-\lambda)T(\beta))z-z\| \leq_{\mathbb{R}} 2^{-k}) \to \|T(t)z-z\| \leq_{\mathbb{R}} 2^{-m}\Big)$$

holds (in the sense of Definition 17.68 in [45]) for any nontrivial normed space X *with a nonempty* $C \subseteq X$.

The bound Φ will not depend on B' as the latter can eventually be reduced to a function of other input data [1]. It suffices to replace the number theoretic functions f, ω with the nondecreasing functions f', ω' so that the latter are their own majorants (Note that for any function $g : \mathbb{N} \to \mathbb{N}$ one can define a nondecreasing $g'(n) := \max\{g(i) : i \leq n\}$ and we can replace a ternary function -as is here the case of the function ω - with one that is nondecreasing in all three arguments by simply taking $g'(n, k, m) := \max\{g(i, j, l) : i \leq n \wedge j \leq k \wedge l \leq m\}$).

It is important to stress that we have ensured that all the axioms above are formulated as universal statements. This can be easily verified by prenexation, by using that $\forall \to \exists \equiv \neg\forall \vee \exists \equiv \exists \vee \exists$ and $\exists \to \forall \equiv \neg\exists \vee \forall \equiv \forall \vee \forall$, and by the fact that $=_{\mathbb{R}}$ thus also $=_X$ are universal while $<_{\mathbb{R}}$ are existential statements (see [45]).

We now also state another corollary explicitly fitting the setting of Theorem 19 :

Metatheorem 4. *(Relevant to Metatheorem 2 and Corollary 17.71 in [45]) Assume that we have a proof of a sentence in $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$*

$$\forall t \in \mathbb{R}^+ \ \forall z \in C \ \forall T \in C \times \mathbb{R}^+ \to C \ \forall \{\alpha_n\} \subseteq \mathbb{R}^+ \ \forall \alpha_\infty \in \mathbb{R}^+$$

$$\forall \omega \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\forall \Phi, \Psi \in \mathbb{N} \to \mathbb{N} \ \forall m \in \mathbb{N} \ \exists k \in \mathbb{N} \ \exists n \in \mathbb{N}$$

$$\Big((\forall t \in \mathbb{R}^+ \ \forall x, y \in C \ \|T(t)x - T(t)y\| \leq_{\mathbb{R}} \|x - y\|)$$

$$\wedge (\forall x \in C \ \forall t, s \in \mathbb{R}^+ \ T(s) \circ T(t)(x) =_X T(s + t)(x))$$

$$\wedge (\forall b \in \mathbb{N} \ \forall q \in C \ \forall m \in \mathbb{N} \ \forall K \in \mathbb{N} \ \forall t, t' \in [0, K]$$

$$(\|q\| <_{\mathbb{R}} b \wedge |t - t'| <_{\mathbb{R}} 2^{-\omega_{K,b}(m)} \to \|T(t)q - T(t')q\| \leq_{\mathbb{R}} 2^{-m}))$$

$$\wedge (\forall n \in \mathbb{N} \ |\alpha_n - \alpha_\infty| \geq_{\mathbb{R}} 2^{-\Psi(n)})$$

$$\wedge (\forall k \in \mathbb{N} \ \forall n \geq \Phi(k) \ |\alpha_n - \alpha_\infty| \leq_{\mathbb{R}} 2^{-k})$$

$$\wedge (\|T(\alpha_n)z - z\| \leq_{\mathbb{R}} 2^{-k}) \to \|T(t)z - z\| <_{\mathbb{R}} 2^{-m} \Big).$$

[1](In particular: we take $\tau := \alpha$ and we will see in the proof in page 92 that we will make use of the estimate $\|T(\alpha)z - z\| \leq \|\lambda T(\alpha)z + (1-\lambda)T(\beta)z - z\| + (1-\lambda)\|T(\alpha)z - T(\beta)z\|$ $\leq 1 + (1 - \frac{1}{\Lambda})\|T(\alpha)z - T(\beta)z\|$. To estimate the quantity $\|T(\alpha)z - T(\beta)z\|$, as $\alpha, \beta \in [0, D]$, construct $D \cdot 2^{\omega_{b,D}(0)}$ many points such that $|\alpha - t_1|, |t_1 - t_2|, ..., |t_{D \cdot 2^{\omega_{b,D}(0)}} - \beta| < 2^{-\omega_{b,D}(0)}$. Then by equicontinuity we will accordingly have $\|T(\alpha)z - T(t_1)z\|, \|T(t_1)z - T(t_2)z\|, ..., \|T(t_{D \cdot 2^{\omega_{b,D}(0)}})z - T(\beta)z\| < 2^{-0} = 1$.) By repeatedly applying the triangle inequality $\|T(\alpha)z - T(\beta)z\| \leq D \cdot 2^{\omega_{b,D}(0)} + 1$ so overall $\|T(\alpha)z - z\| \leq 1 + (1 - \frac{1}{\Lambda})(D \cdot 2^{\omega_{b,D}(0)} + 1)$.

Then one can extract from the proof primitive recursive in the sense of Gödel's T functionals $W, \tilde{W}$ *so that*

$$\forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall L \in \mathbb{N}\ \forall \{\alpha_n\} \in [0, L]^{\mathbb{N}}\ \forall \alpha_\infty \in [0, L]\ \forall B \in \mathbb{N}\ \forall z \in C_B$$

$$\forall T \in C \times \mathbb{R}^+ \to C\ \forall \omega \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}\ \forall \Phi, \Psi \in \mathbb{N} \to \mathbb{N}$$

$$\forall m \in \mathbb{N}\ \exists k \leq W(B, M, L, \Psi', \Phi', m, \omega')\ \exists n \leq \tilde{W}(B, M, L, \Psi', \Phi', m, \omega')$$

$$\Big(((\forall t \in \mathbb{R}^+\ \forall x, y \in C\ \|T(t)x - T(t)y\| \leq_{\mathbb{R}} \|x - y\|)$$

$$\wedge(\forall x \in C\ \forall t, s \in \mathbb{R}^+\ T(s) \circ T(t)(x) =_X T(s + t)(x))$$

$$\wedge(\forall b \in \mathbb{N}\ \forall q \in C\ \forall m \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall t, t' \in [0, K]$$

$$(\|q\| <_{\mathbb{R}} b \wedge |t - t'| <_{\mathbb{R}} 2^{-\omega_{K,b}(m)} \to \|T(t)q - T(t')q\| \leq_{\mathbb{R}} 2^{-m}))$$

$$\wedge(\forall n \in \mathbb{N}\ |\alpha_n - \alpha_\infty| \geq_{\mathbb{R}} 2^{-\Psi(n)})$$

$$\wedge(\forall k \in \mathbb{N}\ \forall n \geq \Phi(k)\ |\alpha_n - \alpha_\infty| \leq_{\mathbb{R}} 2^{-k})$$

$$\wedge(\|T(\alpha_n)z - z\| \leq_{\mathbb{R}} 2^{-k}) \to \|T(t)z - z\| <_{\mathbb{R}} 2^{-m}\Big)$$

holds (in the sense of Definition 17.68 in [45]) for any nontrivial normed space X *with a nonempty* $C \subseteq X$*.*

Here the displacement assumption for some arbitrary element of the sequence is trivially covered because of the premise $\|T(\alpha_n)z - z\| \leq_{\mathbb{R}} 2^{-k}$. As before, we can replace the input number theoretic functions Φ, Ψ, ω with the nondecreasing functions Φ', Ψ', ω' so that the latter are their own majorants, and, as before, we have ensured that all the axioms introduced are formulated as universal statements. Notice that the bound will not depend on representatives of the sequence of reals $\{\alpha_n\} \subseteq \mathbb{R}^+$ nor its limit $\alpha_\infty \in \mathbb{R}^+$ but on $L \in \mathbb{N}$ as we can instead write $\forall L \in \mathbb{N}\ \forall \alpha_n \in [0, L]^{\mathbb{N}}\ \forall \alpha_\infty \in [0, L]$ and they can therefore be seen as elements not of the Polish space $\mathbb{R}^+$ but of the compact spaces $[0, L]^{\mathbb{N}}$ and $[0, L]$ respectively. Therefore the bound will depend only on such a parameter $L \in \mathbb{N}$. This approach has similarly been followed for $t \in \mathbb{R}^+$, as instead of $\forall t \in \mathbb{R}^+$ we write $\forall M \in \mathbb{N}\ \forall t \in [0, M]$ so that t will be considered an element of the compact space $[0, M]$ and the bound will only depend on the parameter $M \in \mathbb{N}$.

3.2 First Approach: Proof Theoretic Analysis and Results

The work presented in this section is included in [48].

The main result of this section will be a quantitative version of Theorem 15.

In particular we will give an explicit, uniform and effective bound for the computation of approximate common fixed points of nonexpansive semigroups on a subset C of a general Banach space by applying proof mining on the proof of Theorem 15 by Suzuki. Moreover, for a convex $C \subseteq X$ we provide the first explicit and highly uniform rate of convergence for an iterative procedure to compute such points, in particular the main result will be applied to extract rates of asymptotic regularity for $\{T(t) : t \geq 0\}$ with respect to the Krasnoselskii iteration.

The inclusion

$$\bigcap_{t\geq 0} F(T(t)) \subseteq F(\lambda T(\alpha) + (1-\lambda)T(\beta))$$

is trivial to show as assuming $q \in \bigcap_{t\geq 0} F(T(t))$, then

$$(\lambda T(\alpha) + (1-\lambda)T(\beta))q = \lambda T(\alpha)q + (1-\lambda)T(\beta)q = \lambda q + (1-\lambda)q = q,$$

thus $q \in F(\lambda T(\alpha) + (1-\lambda)T(\beta))$.

We will extract a bound from (the proof of) the nontrivial inclusion

$$\bigcap_{t\geq 0} F(T(t)) \supseteq F(\lambda T(\alpha) + (1-\lambda)T(\beta))$$

in the following sense: Notice that the above inclusion gives

$$\forall q \in C \ ((\lambda T(\alpha) + (1-\lambda)T(\beta))q = q \rightarrow \forall t \geq 0 \ T(t)q = q)$$

which can be written as:

$$\forall q \in C \ \forall m \in \mathbb{N} \ \forall t \geq 0 \ \exists k \in \mathbb{N}$$

$$(\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq 2^{-k} \rightarrow \|T(t)q - q\| < 2^{-m}).$$

The above statement is a $\forall\exists(\forall \rightarrow \exists)$ equivalently a $\forall\exists$ statement. Therefore as guaranteed by Metatheorem 3, also see [24], [44]), it is possible to extract a computable bound $\Psi > 0$ depending on bounds on the input data so that

$$\forall \lambda \in (0,1) \ \forall b \in \mathbb{N} \ \forall q \in C_b \ \forall M \in \mathbb{N} \ \forall t \in [0,M] \ \forall m \in \mathbb{N}$$

$$(\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq \Psi(M,m,b,...) \rightarrow \|T(t)q - q\| < 2^{-m}),$$

where, given $b \in \mathbb{N}$, $C_b := \{q \in C : \|q\| \leq b\}$. Note that here we have replaced $\forall t \geq 0$ with the equivalent $\forall M \in \mathbb{N} \ \forall t \in [0,M]$ so as to have a natural instead of a real number as input data (serving as a majorant).

As mentioned, we will achieve the above bound extraction by proof mining on the proof of Theorem 15 given in [80]. To this end, we will first obtain quantitative versions of a number of preliminary lemmas by Suzuki in [80] which will then be combined in a deduction schema following the pattern of the original proof of Theorem 15 to obtain a quantitative version of the latter, i.e. our Theorem 18.

Notice that because, for any $t \in \mathbb{R}, z \in \mathbb{Z}$ we have

$$t + z - [t + z] = t - [t],$$

for all $t \in \mathbb{R}, z \in \mathbb{Z}$ we have

$$[t + z] = [t] + z. \; (II)$$

Let $\gamma \in (0, 1)$ and $\theta \in [0, 1]$. Define a sequence $\{\tilde{A}_n\}$ of subsets of $[0, 1]$ by $\tilde{A}_1 = \{\theta\}$ and

$$\tilde{A}_{n+1} = \bigcup_{t \in \tilde{A}_n} \{|1 - t|, |\gamma - t|\}$$

for $n \in \mathbb{N}$ and set

$$\tilde{A}(\theta) := \bigcup_{n=1}^{\infty} \tilde{A}_n.$$

Moreover define a sequence $\{A_n\}$ of subsets of $[0, \beta]$ by

$$A_1 = \{\theta\beta\},$$

$$A_{n+1} = \bigcup_{t \in A_n} \{|\alpha - t|, |\beta - t|\}$$

for $n \in \mathbb{N}$. Set

$$A(\theta) := \bigcup_{n=1}^{\infty} A_n.$$

The following lemma in [80] will be used:

Lemma 2. *(Suzuki, Lemma 1 in [80]) Let $\gamma \in (0, 1)$ and $t \in \mathbb{R}$. Then the following statements hold.*

$$(i) \; t - [t] \in [0, 1).$$

$$(ii) \; t - [t] = 0 \to t \in \mathbb{Z}.$$

$$(iii) \; 0 < t - [t] < 1 \to -[-t] = [t] + 1.$$

$$(iv) \; \gamma \leq t - [t] \to [t - \gamma] = [t]).$$

$$(v) \; t - [t] < \gamma \to [t - \gamma] = [t] - 1 \wedge [\gamma - t] = -[t].$$

In [80] the proof was omitted, but for completeness we give a proof here:

Proof. (i) By the definition we have $[t] \le t < [t]+1$ so trivially $0 \le t-[t] < 1$.

(ii) Trivial, as then $t = [t] \in \mathbb{Z}$.

(iii) Notice that, because

$$[-t] < -t < -[t]$$

we have $(-[t]-(-t))+(-t-[-t]) = 1$ i.e. $-[t]-[-t] = 1$ i.e. $-[-t] = [t]+1$.

(iv) By $\gamma \le t-[t]$ we have $t-\gamma \ge [t]$ thus $[t-\gamma] \ge [[t]] = [t]$. But, moreover, it is $t-\gamma \le t$ thus $[t-\gamma] \le [t]$. By combining the above $[t-\gamma] = [t]$.

(v) By assumption we have $t-[t] < \gamma < 1$ i.e. $-[t] < \gamma - t < 1-t$ i.e. $[-[t]] \le [\gamma - t] \le [1-t]$ i.e. by (II) $[-[t]] \le [\gamma - t] \le [1-t] = 1+[-t]$.

In the case where $0 < t-[t]$, by (iii) we get

$$[-t] = -[t]-1$$

therefore

$$[-[t]] \le [\gamma - t] \le 1-[t]-1$$

i.e.

$$-[t] \le [\gamma - t] \le -[t].$$

So we showed that

$$[\gamma - t] = -[t].$$

The above, by applying (iii) to $\gamma - t$ (this is possible because $0 < t-[t] < \gamma < 1$ gives here $0 < \gamma - t - [\gamma - t] < 1$) gives:

$$-[-(\gamma - t)] = -[t]+1$$

i.e. $-[t-\gamma] = -[t]+1$ i.e. $[t-\gamma] = [t]-1$.

In the other case where $0 = t-[t]$, by (ii) we have $t \in \mathbb{Z}$ and thus (II) gives

$$[t-\gamma] = t+[-\gamma] = t-1$$

and similarly, again by $t \in \mathbb{Z}$ here we have

$$[\gamma - t] = [\gamma] - t = 0 - t = -[t].$$

□

In the following, let $\alpha, \beta, \gamma, \theta \in \mathbb{R}$ with $0 < \alpha < \beta, 0 \le \theta \le 1$ and let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with a modulus of irrationality f_γ.

Lemma 3. *(Suzuki, Lemma 2 in [80]) Let* $\{\tilde{A}_n\}$ *be the sequence of subsets of* $[0,1]$ *and* $\tilde{A}(\theta) := \bigcup_{n=1}^{\infty} \tilde{A}_n$ *as previously defined. Then*

$$\tilde{A}(\theta) \setminus \{1\} = \{e\theta + l\gamma - [e\theta + l\gamma] : e \in \{+1,-1\}, l \in \mathbb{Z}\}.$$

Moreover, if $h \in \tilde{A}(\theta)$*, then* $\tilde{A}(h) = \tilde{A}(\theta)$.

Lemma 4. *(Suzuki, Lemma 3 in [80]) Let* $\{A_n\}$ *be the sequence of subsets of* $[0,\beta]$ *and* $A(\theta) := \bigcup_{n=1}^{\infty} A_n$ *as previously defined. Then*

$$A(\theta) \setminus \{\beta\} = \{(e\theta + l\gamma - [e\theta + l\gamma])\beta : e \in \{+1,-1\}, l \in \mathbb{Z}\}.$$

Moreover, if $h \in A(\theta)$*, then* $A(h) = A(\theta)$.

(Note that the notation we choose to use here slightly differs from the one in [80]).

The following quantitative version of (relevant parts of) Lemmata 3 and 4 is extracted below by inspecting step-by-step the proof of Lemma 3 in [80] and then extending it to also show a quantitative version of Lemma 4 (the proof of the latter is not given in [80]).

Lemma 5. *(Kohlenbach and K.-A. ([48]), Quantitative version of relevant parts of Lemmas 2 and 3 in [80])*
Let $\{A_n\}$ *be the sequence of subsets of* $[0,\beta]$ *and let* $\{\tilde{A}_n\}$ *be the sequence of subsets of* $[0,1]$ *as previously defined. Then*
(i)For $t \in \mathbb{R}$*, if* $t - [t] \in \tilde{A}_n$ *then* $-t - [-t] \in \tilde{A}_{n+1}$ *in the case* $0 < t - [t] < 1$ *and* $-t - [-t] \in \tilde{A}_n$ *in the case* $t - [t] = 0$.
(ii) For $t \in \mathbb{R}$*, if* $t - [t] \in \tilde{A}(\theta)$ *then* $t - \gamma - [t - \gamma] \in \tilde{A}_{n+1}$ *in the case where* $\gamma \le t - [t]$ *and* $t - \gamma - [t - \gamma] \in \tilde{A}_{n+2}$ *in the case* $t - [t] < \gamma$.
(iii) Define for $l \in \mathbb{N}$

$$B_l := \{(e\theta + i\gamma - [e\theta + i\gamma])\beta : e \in \{+1,-1\}, i \in \mathbb{Z}, |i| \le l\}$$

and

$$A_n^* := \bigcup_{i \le n} A_i.$$

Then for each $l \in \mathbb{N}$ *we have*

$$B_l \subseteq A_{2l+8}^*$$

i.e. for each $x \in B_l$ *there exists an* $n \le 2l + 8$ *with* $x \in A_n$.

Proof. First of all, recall by the definition of $\{\tilde{A}_n\}$ that if $t \in \tilde{A}_n$, then $|1-t|, |\gamma - t| \in \tilde{A}_{n+1}$.

We will show step (*i*). Let $t \in \mathbb{R}$ with $t - [t] \in \tilde{A}_n$. If $t - [t] = 0$, (i) follows immediately by Lemma 2(*ii*) as then $t \in \mathbb{Z}$ thus

$$-t - [-t] = 0 = t - [t].$$

If $0 < t - [t] < 1$, Lemma 2(*iii*) gives $-[-t] = [t] + 1$ therefore

$$-t - [-t] = -t + [t] + 1 = |1 - (t - [t])|$$

but recall that assuming $t - [t] \in \tilde{A}_n$, then $|1 - (t - [t])| \in \tilde{A}_{n+1}$, thus $-t - [-t] \in \tilde{A}_{n+1}$.

We will now show step (*ii*). Let $t \in \mathbb{R}$ with $t - [t] \in \tilde{A}_n$. We distinguish cases: In the case where $\gamma \leq t - [t]$, by Lemma 2 (*iv*) we have $[t - \gamma] = [t]$. Thus:

$$t - \gamma - [t - \gamma] = t - [t] - \gamma = |\gamma - (t - [t])|$$

and by $|\gamma - (t - [t])| \in \tilde{A}_{n+1}$ we have $t - \gamma - [t - \gamma] \in \tilde{A}_{n+1}$. Now consider the other case where $t - [t] < \gamma$. Then Lemma 2(*v*) gives:

$$[t - \gamma] + 1 = [t].$$

Therefore

$$\gamma - t + [t - \gamma] + 1 = \gamma - t + [t] = |\gamma - (t - [t])|$$

but recall that $|\gamma - (t - [t])| \in \tilde{A}_{n+1}$ thus also $\gamma - t + [t - \gamma] + 1 \in \tilde{A}_{n+1}$. Now, considering the quantity $\gamma - t + [t - \gamma] + 1$, we will therefore have
$|1 - (\gamma - t + [t - \gamma] + 1)| \in \tilde{A}_{n+2}$ i.e.

$$t - \gamma - [t - \gamma] \in \tilde{A}_{n+2}.$$

We will now show step (*iii*). First of all, notice that for all $n \in \mathbb{N}$,

$$\forall t(t \in \tilde{A}_n \leftrightarrow \beta t \in A_n).$$

The above claim is shown by induction on n; For $n = 1$, we have

$$t \in \tilde{A}_1 = \{\theta\} \text{ if and only if } t\beta \in A_1 = \{\theta\beta\}$$

as clearly $t = \theta$ if and only if $t\beta = \theta\beta$.

Now, assuming

$$\forall t(t \in \tilde{A}_{n_0} \leftrightarrow \beta t \in A_{n_0})$$

for some $n_0 \in \mathbb{N}$, we have,

$$t \in \tilde{A}_{n_0+1} \leftrightarrow$$

$$t\beta \in \beta\tilde{A}_{n_0+1} = \beta \bigcup_{s \in \tilde{A}_{n_0}} \{|1 - s|, |\gamma - s|\}$$

$$= \bigcup_{s \in \tilde{A}_{n_0}} \{|\beta - \beta s|, |\alpha - \beta s|\}$$

(because, by the induction hypothesis $s \in \tilde{A}_{n_0} \leftrightarrow \beta s \in A_{n_0}$)

$$= \bigcup_{\beta s \in A_{n_0}} \{|\beta - \beta s|, |\alpha - \beta s|\}$$

$$= A_{n_0+1}$$

so the claim is proved. Therefore, to show (iii) it suffices to show the following: For $l \in \mathbb{N}$, defining

$$\tilde{B}_l := \{(e\theta + i\gamma - [e\theta + i\gamma]), e \in \{+1, -1\}, i \in \mathbb{Z}, |i| \leq l\}$$

and

$$\tilde{A}_n^* := \bigcup_{i \leq n} \tilde{A}_i$$

we have, for each $l \in \mathbb{N}$,

$$\tilde{B}_l \subseteq \tilde{A}^*_{2l+8},$$

i.e. for each $x \in \tilde{B}_l$ there exists an $n \leq 2l + 8$ with $x \in \tilde{A}_n$.

Case I : $\theta \neq 1$. We have $[\theta] = 0$, therefore $\theta - [\theta] = \theta \in \tilde{A}_1$. Now we apply step (ii) l times which results in an increase by at most 2 in each step. Hence we obtain

$$\theta - l\gamma - [\theta - l\gamma] \in \tilde{A}^*_{1+2l}.$$

Now we apply step (i) which increases the level at most by 1, hence we have

$$-\theta + l\gamma - [-\theta + l\gamma] \in \tilde{A}^*_{2+2l}.$$

This in particular holds for $l = 1$ and so

$$-\theta + \gamma - [-\theta + \gamma] \in \tilde{A}^*_4.$$

We now apply again step (ii) $l + 1$ times resulting in at most

$$-\theta - l\gamma - [-\theta - l\gamma] \in \tilde{A}^*_{4+2(l+1)} = \tilde{A}^*_{6+2l}.$$

At this point we have obtained the result for the $e = -1$ case, having now covered the $e = -1$ case for both positive and negative $l \in \mathbb{Z}$. We now apply step (i) which gives a shift by at most 1 and therefore we obtain

$$\theta + l\gamma - [\theta + l\gamma] \in \tilde{A}^*_{7+2l}.$$

So we have now also covered the $e = +1$ case for both positive and negative $l \in \mathbb{Z}$.

Case II : $\theta = 1$. Here $\theta - [\theta] = 0 \in \tilde{A}_2$, therefore there is a shift by 1 on all the above.

Combining Cases I and II, we obtain at most

$$\theta + l\gamma - [\theta + l\gamma] \in \tilde{A}^*_{8+2l}$$

where $l \in \mathbb{Z}$.

□

The proof of the following lemma was omitted in [80] because it originates from well-known classical results. However, we give a proof here because we will later make use of it so as to extract our quantitative version of this lemma that will be needed for the proof of Theorem 18.

Lemma 6. *(Suzuki, Lemma 4 in [80]) It is* $\overline{A(\theta)} = [0, \beta]$ *where by* $\overline{A(\theta)}$ *we denote the closure of* $A(\theta)$.

Proof. By Lemma 3 in [80]

$$A(\theta) \setminus \{\beta\} \supseteq \{(e\theta + l\gamma - [e\theta + l\gamma])\beta : e \in \{+1, -1\}, l \in \mathbb{Z}\}.$$

Notice that it is always true, by the definition of the floor function $[\cdot]$, that

$$\forall l \in \mathbb{Z}\ (e\theta + l\gamma - [e\theta + l\gamma] \in [0, 1)).$$

We will show that $A(\theta)$ is dense in $[0, \beta]$. It is enough to show that $A(\theta) \setminus \{\beta\}$ is dense in $[0, \beta]$. For that we will first show that the set $\{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$ is dense in $[0, 1]$ [2]. We argue as follows. Fix $k \in \mathbb{N}$. Cut $[0, 1]$ into pieces of length $\frac{1}{k+1}$ each. Then by the pigeonhole principle there must exist $i, j \in \mathbb{Z}$ with $i \neq j$ so that

$$0 \leq j, i \leq k + 1$$

such that $i\gamma - [i\gamma]$ and $j\gamma - [j\gamma]$ belong to the same piece so that

$$|i\gamma - [i\gamma] - (j\gamma - [j\gamma])| \leq 1/(k + 1) < 1/k.$$

Notice that because $\gamma \notin \mathbb{Q}$ and since $i \neq j$ we have

$$i\gamma - [i\gamma] \neq j\gamma - [j\gamma],$$

for $i\gamma - [i\gamma] = j\gamma - [j\gamma]$ would give

$$\gamma = \frac{[i\gamma] - [j\gamma]}{i - j} \in \mathbb{Q}$$

[2] This is a classical fact. We provide a proof here inspired by a proof given in [61] but we replaced the use of Bolzano-Weierstrass by the finitary pigeonhole principle.

which is a contradiction. Without loss of generality assume that

$$i\gamma - [i\gamma] - (j\gamma - [j\gamma]) > 0.$$

We now define

$$X := \max\{x \in \mathbb{Z}^+ \; : \; x(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1\}.$$

Now notice that for all $p \in \mathbb{N}$ we have

$$|p(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - (p+1)(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))|$$

$$= |i\gamma - [i\gamma] - (j\gamma - [j\gamma]))| < 1/k.$$

Therefore, for any $m \in [0, 1, ..., k-1]$ we can find a $\tilde{m} \in [1, ..., X]$ so that

$$\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) \in [\frac{m}{k}, \frac{m+1}{k}].$$

Moreover notice that, because of

$$0 < \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1$$

we have

$$[\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))] = 0$$

therefore

$$\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))$$

$$= \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - [\tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma]))]$$

(by (II))

$$= \tilde{m}(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) - \tilde{m}(-[i\gamma] + [j\gamma]) - [\tilde{m}(i-j)\gamma]$$

$$= \tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma].$$

Therefore

$$\tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma] \in [\frac{m}{k}, \frac{m+1}{k}] \cap \{l\gamma - [l\gamma] \; : \; l \in \mathbb{Z}\}$$

and, because $k \in \mathbb{N}$ was arbitrary, we conclude that $\{l\gamma - [l\gamma] \; : \; l \in \mathbb{Z}\}$ is dense in $[0, 1]$. Therefore, by (II) for $\theta \in \{0, 1\}$ the set

$$\{e\theta + l\gamma - [e\theta + l\gamma] \; : \; l \in \mathbb{Z}, e \in \{+1, -1\}\}$$

is dense in $[0, 1]$. To show the density of the set

$$\{e\theta + l\gamma - [e\theta + l\gamma] \; : \; l \in \mathbb{Z}, e \in \{+1, -1\}\}$$

in $[0,1]$ where $\theta \in (0,1)$ it is enough to show the density of

$$\{\theta + l\gamma - [\theta + l\gamma] \ : \ l \in \mathbb{Z}\} \subset \{e\theta + l\gamma - [e\theta + l\gamma] \ : \ l \in \mathbb{Z}, e \in \{+1, -1\}\}$$

in $[0,1]$.

Fix $k \in \mathbb{N}$.

Case A : Let $x \in [\theta, 1 - \frac{1}{k}]$. Then $x' := x - \theta \in [0,1]$. Hence there exists an $i \in \mathbb{Z}$ so that

$$|x' - (i\gamma - [i\gamma])| < \frac{1}{k} \ (!).$$

Then

$$i\gamma - [i\gamma] + \theta < x' + \theta + \frac{1}{k} = x + \frac{1}{k} \leq 1.$$

Notice that $i\gamma - [i\gamma] + \theta < 1$, by Lemma 2 (v), gives us $[i\gamma - (1-\theta)] = [i\gamma] - 1$ and by (II) we have

$$[i\gamma - (1-\theta)] = [i\gamma + \theta] - 1.$$

Therefore

$$\theta + i\gamma - [\theta + i\gamma] = \theta + i\gamma - [i\gamma].$$

By (!) we have (since $x = x' + \theta$)

$$|x - (i\gamma - [i\gamma] + \theta)| < \frac{1}{k}$$

and so

$$|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k}.$$

Case B : Let $x \in [\frac{1}{k}, \theta)$. Then $x' := x - \theta + 1 \in [0,1)$. Again there exists an $i \in \mathbb{Z}$ so that

$$|x' - (i\gamma - [i\gamma])| < \frac{1}{k} \ (!).$$

Then

$$i\gamma - [i\gamma] + \theta \geq x' - \frac{1}{k} + \theta$$

$$= x + 1 - \frac{1}{k} \geq 1.$$

Therefore, by Lemma 2 (iv), we have $[i\gamma - (1-\theta)] = [i\gamma]$. Moreover, by (II),

$$[i\gamma - (1-\theta)] = [i\gamma + \theta] - 1.$$

Therefore

$$\theta + i\gamma - [\theta + i\gamma] = \theta + i\gamma - ([i\gamma] + 1) = i\gamma - [i\gamma] + \theta - 1.$$

By (!) (since $x = x' + \theta - 1$)

$$|x - (i\gamma - [i\gamma] + \theta - 1)| < \frac{1}{k}$$

and so

$$|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k}.$$

By combining Cases A and B together, we have that

$$\forall x \in [1/k, 1 - 1/k]\ \exists i \in \mathbb{Z}\ (|x - (\theta + i\gamma - [\theta + i\gamma])| < \frac{1}{k})$$

and hence

$$\forall x' \in [0,1]\ \exists i \in \mathbb{Z}\ \exists x \in [1/k, 1 - 1/k]$$

$$(|x' - (\theta + i\gamma - [\theta + i\gamma])| \leq |x - (\theta + i\gamma - [\theta + i\gamma])| + |x - x'| < \frac{2}{k}).$$

Therefore

$$\forall \tilde{x} \in [0,\beta]\ \exists x' := \tilde{x}/\beta \in [0,1]\ \exists i \in \mathbb{Z}$$

$$(|\tilde{x} - (\theta + i\gamma - [\theta + i\gamma])\beta| = |x'\beta - (\theta + i\gamma - [\theta + i\gamma])\beta| = \beta|x' - (\theta + i\gamma - [\theta + i\gamma])| < \frac{2\beta}{k}).$$

Hence for $\mathbb{N} \ni D \geq \beta$ we have

$$\forall \tilde{x} \in [0,\beta]\ \exists i \in \mathbb{Z}\ (|\tilde{x} - (\theta + i\gamma - [\theta + i\gamma])\beta| < \frac{2D}{k}).$$

Since $k \in \mathbb{N}$ was arbitrary, the claim follows.

□

We will show a quantitative version of the above lemma.

Lemma 7. *(Kohlenbach and K.-A. ([48]), Quantitative version of Lemma 4 in [80]) Let $\alpha, \beta, \theta \in \mathbb{R}$ be as before and define a sequence $\{A_n\}$ of subsets of $[0,\beta]$ and the set $A(\theta)$ as before. Let $\mathbb{N} \ni D \geq \beta$. Let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Then*

$$\forall k \in \mathbb{N}\ \forall f_\gamma : \mathbb{N} \to \mathbb{N}\ \exists n_k \in \mathbb{N}\ \forall s \in [0,\beta]\ \exists p \in \mathbb{N}\ \exists p' \in \mathbb{Z}^+\ \exists s' \in \bigcup_{r \leq n_k} A_r$$

$$(|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \to |s - s'| < \frac{2D}{k})(*)$$

and we can extract a computable bound $\phi(k,f) \geq n_k$ with

$$\phi(k,f) := \max\{2f(i-j) + 6\ :\ 0 \leq j < i \leq k+1\}.$$

Proof. In the previous lemma we essentially showed that

$$\forall k \in \mathbb{N}\ \forall s \in [0,\beta]\ \exists s' \in A(\theta) = \bigcup_{n=1}^{\infty} A_n$$

$$(\exists f_\gamma\ \forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ |\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \rightarrow |s - s'| < \frac{2D}{k})$$

i.e.

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists p \in \mathbb{N}\ \exists p' \in \mathbb{Z}^+\ \exists s' \in A(\theta) = \bigcup_{n=1}^{\infty} A_n$$

$$(|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \rightarrow |s - s'| < \frac{2D}{k}).$$

The above essentially means that

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists p \in \mathbb{N}\ \exists p' \in \mathbb{Z}^+\ \exists s' \in \bigcup_{r \leq n} A_r$$

$$(|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \rightarrow |s - s'| < \frac{2D}{k}).\ (III)$$

Notice that (III) is a $\forall\exists(\forall \rightarrow \exists)$ i.e. a $\forall\exists$ statement. Therefore we can apply Metatheorem 1, which ensures that we can find a computable bound ϕ on n . This will be done by analyzing the proof of the previous lemma as follows. Recall that in the previous lemma the conclusion that gave the density of the set $\{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$ in $[0,1]$, thus also the density of the set $\{(e\theta + l\gamma - [e\theta + l\gamma])\beta\ :\ e \in \{+1,-1\},\ l \in \mathbb{Z}\}$ in $[0,\beta]$ for a fixed $\theta \in [0,1]$, was

$$\tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma] \in [\frac{m}{k}, \frac{m+1}{k}] \cap \{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}\}$$

for arbitrary $k \in \mathbb{N}$ (where $\tilde{m}, X$ are as in the previous lemma). Moreover recall that

$$X(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1$$

where $i \neq j \in \mathbb{Z}$ with $0 \leq j, i \leq k+1$ and we had assumed, without loss of generality,

$$i\gamma - [i\gamma] - (j\gamma - j\gamma]) > 0.$$

To bound the quantity $\tilde{m}|i-j|$ it will therefore be enough to bound the quantity $X|i-j|$. Now, notice that by the proof of the previous lemma, as γ is by assumption irrational with an effective irrationality measure f_γ,

$$\exists f_\gamma : \mathbb{N} \rightarrow \mathbb{N}(|\gamma - \frac{[i\gamma] - [j\gamma]}{i-j}|) \geq \frac{1}{f_\gamma(|i-j)|}.$$

Therefore, taking in (III)

$$p' := [i\gamma] - [j\gamma]$$

and

$$p := i - j$$

if $i > j$ and $-p', -p$ otherwise using Remark 8, we have

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in \bigcup_{r \leq n} A_r\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$

$$(|\gamma - \frac{[i\gamma] - [j\gamma]}{i-j}| \geq \frac{1}{f_\gamma(|i-j|)} \rightarrow |s - s'| < \frac{2D}{k})$$

i.e.

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in \bigcup_{r \leq n} A_r\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$

$$(|\frac{\gamma(i-j) - ([i\gamma] - [j\gamma])}{i-j}| \geq \frac{1}{f_\gamma(|i-j|)} \rightarrow |s - s'| < \frac{2D}{k})$$

thus

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in \bigcup_{r \leq n} A_r\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$

$$(|\gamma(i-j) - ([i\gamma] - [j\gamma])| \geq \frac{|i-j|}{f_\gamma(|i-j|)} \rightarrow |s - s'| < \frac{2D}{k})$$

therefore

$$\forall k \in \mathbb{N}\ \forall f_\gamma\ \forall s \in [0,\beta]\ \exists n \in \mathbb{N}\ \exists s' \in \bigcup_{r \leq n} A_r\ \exists j \neq i \in \mathbb{Z} : 0 \leq j, i \leq k+1$$

$$(|\gamma(i-j) - ([i\gamma] - [j\gamma])| \geq \frac{|i-j|}{f_\gamma(|i-j|)} \wedge X(i\gamma - [i\gamma] - (j\gamma - [j\gamma])) < 1$$

$$\rightarrow |s - s'| < \frac{2D}{k} \wedge X < \frac{f_\gamma(|i-j|)}{|i-j|}).$$

Having bounded X means having bounded $\tilde{m}$ (where $X, \tilde{m}$ are as in the previous lemma) and recall that in the previous lemma our conclusion that gave the density of the set $\{l\gamma - [l\gamma] \in [0,1]$, thus (replacing k by $2D/k$) also the density of the set $\{(e\theta + l\gamma - [e\theta + l\gamma])\beta\ :\ e \in \{+1,-1\}, l \in \mathbb{Z}\} \in [0,\beta]$ for a fixed $\theta \in [0,1]$, was

$$\tilde{m}(i-j)\gamma - [\tilde{m}(i-j)\gamma] \in [\frac{m}{k}, \frac{m+1}{k}] \cap \{l\gamma - [l\gamma]\ :\ l \in \mathbb{Z}$$

for arbitrary $k \in \mathbb{N}$. Note that the proof of the previous lemma shows that in order to construct an $l \in \mathbb{Z}$ such that for a given $x \in [0,\beta]$ one has

$$|x - (\theta + l\gamma - [\theta + l\gamma])\beta| < \frac{2D}{k},$$

it suffices to construct for a suitable $x' \in [0,1]$ an $l \in \mathbb{Z}$ such that

$$|x' - (l\gamma - [l\gamma])| < \frac{1}{k}.$$

Hence a bound on $|l|$ for the latter problem gives also a bound on $|l|$ for the former problem.

We have

$$|\tilde{m}(i-j)| = \tilde{m}|i-j| \leq X|i-j| < \frac{f_\gamma(|i-j|)}{|i-j|}|i-j| = f_\gamma(|i-j|)$$

and so

$$|\tilde{m}(i-j)| \leq f_\gamma(|i-j|) - 1.$$

thus

$$2\tilde{m}|i-j| + 8 \leq 2f_\gamma(|i-j|) + 6.$$

Recall now that by Lemma 5(iii)

$$\bigcup_{i \leq 2\tilde{m}(|i-j|)+8} A_i \supseteq \{(e\theta + \tilde{m}(i-j)\gamma - [e\theta + \tilde{m}(i-j)\gamma])\beta \ : \ e \in \{+1,-1\}\}.$$

so we may set

$$\phi(k,f) := \max\{2f(i-j) + 6 \ : \ 0 \leq j < i \leq k+1\}.$$

Finally, notice that the bound extracted for n in (III) is also a witness because (III) is clearly monotone in n , and does not depend on s. Therefore the same bound serves as a bound for n_k in $(*)$ (i.e. the reversal of the quantifiers $\forall s\ \exists n$ in (III) to $\exists n_k\ \forall s$ in $(*)$ plays no role here) and our proof is complete.

□

Lemma 8. *(Suzuki, Lemma 6 in [80]) Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset C of a Banach space X. Assume that there exist $q \in C$, $\lambda \in (0,1)$ such that*

$$\lambda T(\alpha)q + (1-\lambda)T(\beta)q = q$$

and assume that $\tau \in A(\theta)$ where $A(\theta)$ is as defined previously so that

$$\|T(\tau)q - q\| = \max\{\|T(t)q - q\| \ : \ t \in A(\theta)\}.$$

Define a sequence $\{H_n\}$ of subsets of $[0,\beta]$ by $H_1 = \{\tau\}$ and

$$H_{n+1} = \bigcup_{t \in H_n} \{|\alpha - t|, |\beta - t|\}.$$

Then

$$\forall n \in \mathbb{N}\ \forall t \in H_n\ \|T(\tau)q - q\| = \|T(t)q - q\|.$$

We show the following:

Lemma 9. *(Kohlenbach and K.-A. ([48]), Quantitative version of Lemma 8) Let* $\{T(t) : t \geq 0\}$ *be a strongly continuous semigroup of nonexpansive mappings on a subset* C *of a Banach space* X*. Let* $\Lambda \in \mathbb{N}$ *be such that* $1/\Lambda \leq \lambda, 1-\lambda$*. Let* $\delta > 0$ *and* $q \in C$ *be such that*

$$\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq \delta.$$

Let $\tau \in A(\theta)$ *where* $A(\theta)$ *is defined as previously so that*

$$\|T(\tau)q - q\| = \max\{\|T(t)q - q\| \ : \ t \in A(\theta)\}.$$

Define a sequence $\{H_n\}$ *of subsets of* $[0, \beta]$ *as in the lemma above. Then*

$$\forall n \in \mathbb{N}\ \forall t \in H_n\ \|T(\tau)q - q\| \leq \|T(t)q - q\| + \delta \sum_{i=1}^{n-1} \Lambda^i) \quad (**)$$

Proof. Note that by Lemma 3 in [80] our assumption that $\tau \in A(\theta)$ gives us that

$$A(\theta) = \bigcup_{n=1}^{\infty} H_n.$$

and by the definition of the sets

$$\bigcup_{n=1}^{\infty} H_n = A(\tau).$$

The proof is by induction. Let $n = 1$. Then by definition $H_1 = \{\tau\}$. Notice that it is true that

$$\|T(\tau)q - q\| \leq \|T(\tau)q - q\| + \delta \sum_{i=1}^{0} \Lambda^i = \|T(\tau)q - q\|$$

therefore we see that for $n = 1$, $(**)$ holds. Assume that $(**)$ holds for some fixed n. Let $t \in H_n$. Then

$$|\alpha - t|, |\beta - t| \in H_{n+1}.$$

We therefore have:

$$\|T(\tau)q - q\| \leq \|T(t)q - q\| + \delta \sum_{i=1}^{n-1} \Lambda^i$$

$$= \|T(t)q - q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q\| + \delta \sum_{i=1}^{n-1} \Lambda^i$$

$$\leq \|T(t)q-(\lambda T(\alpha)q+(1-\lambda)T(\beta)q)\|+\|\lambda T(\alpha)q+(1-\lambda)T(\beta)q-q\|+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$\leq \|T(t)q-(\lambda T(\alpha)q+(1-\lambda)T(\beta)q)\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \|T(t)q-\lambda T(\alpha)q-(1-\lambda)T(\beta)q+\lambda T(t)q-\lambda T(t)q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$\leq \lambda\|T(t)q-T(\alpha)q\|+(1-\lambda)\|T(t)q-T(\beta)q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \lambda\|T(t)q-T(\alpha)q\|+(1-\lambda)\|T(t)q-T(\beta-t+t)q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$= \lambda\|T(t)q-T(\alpha)q\|+(1-\lambda)\|T(t)q-T(t)T(\beta-t)q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

$$\leq \lambda\|T(t)q-T(\alpha)q\|+(1-\lambda)\|q-T(\beta-t)q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

and similarly, by replacing, in $\|T(t)q-T(\alpha)q\|$, t with $t-\alpha+\alpha$ in the case where $t>\alpha$ or α by $\alpha-t+t$ in the case where $t\leq\alpha$, (notice that we always have $t\leq\beta$) again by the definition of a nonexpansive semigroup the above gives

$$\leq \lambda\|T(|t-\alpha|)q-q\|+(1-\lambda)\|T(|t-\beta|)q-q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i.$$

Therefore (since by $\bigcup_{n=1}^{\infty}H_n=A(\theta)$ we have $\|T(|t-\alpha|)q-q\|\leq\|T(\tau)q-q\|$)

$$\|T(\tau)q-q\|\leq\lambda\|T(\tau)q-q\|+(1-\lambda)\|T(|t-\beta|)q-q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

i.e.

$$(1-\lambda)\|T(\tau)q-q\|\leq(1-\lambda)\|T(|t-\beta|)q-q\|+\delta+\delta\sum_{i=1}^{n-1}\Lambda^i$$

i.e.

$$\|T(\tau)q-q\|\leq\|T(|t-\beta|)q-q\|+\frac{1}{1-\lambda}(\delta+\delta\sum_{i=1}^{n-1}\Lambda^i)$$

$$\leq\|T(|t-\beta|)q-q\|+\Lambda(\delta+\delta\sum_{i=1}^{n-1}\Lambda^i)$$

$$= \|T(|t-\beta|)q - q\| + \Lambda\delta + \Lambda\delta(\Lambda + \Lambda^2 + ... + \Lambda^{n-1})$$
$$= \|T(|t-\beta|)q - q\| + \delta \sum_{i=1}^{n} \Lambda^i,$$

and similarly: (since by $\bigcup_{n=1}^{\infty} H_n = A(\theta)$ we have $\|T(|t-\beta|)q-q\| \leq \|T(\tau)q-q\|$)

$$\|T(\tau)q - q\| \leq \lambda\|T(|t-\alpha|)q - q\| + (1-\lambda)\|T(\tau)q - q\| + \delta + \delta\sum_{i=1}^{n-1} \Lambda^i$$

i.e.

$$\lambda\|T(\tau)q - q\| \leq \lambda\|T(|t-\alpha|)q - q\| + \delta + \delta\sum_{i=1}^{n-1} \Lambda^i$$

i.e.

$$\|T(\tau)q - q\| \leq \|T(|t-\alpha|)q - q\| + \frac{1}{\lambda}(\delta + \delta\sum_{i=1}^{n-1} \Lambda^i)$$
$$\leq \|T(|t-\alpha|)q - q\| + \Lambda(\delta + \delta\sum_{i=1}^{n-1} \Lambda^i)$$
$$= \|T(|t-\alpha|)q - q\| + \Lambda\delta + \Lambda\delta(\Lambda + \Lambda^2 + ... + \Lambda^{n-1})$$
$$= \|T(|t-\alpha|)q - q\| + \delta \sum_{i=1}^{n} \Lambda^i.$$

We have thus shown that for all $s \in H_{n+1}$

$$\|T(\tau)q - q\| \leq \|T(s)q - q\| + \delta \sum_{i=1}^{n} \Lambda^i$$

so we have shown that $(**)$ holds for $n+1$ which concludes the inductive proof of $(**)$ for all n.

□

We can now proceed to show our main theorem Theorem 18.

Theorem 18. *(Kohlenbach and K.-A. ([48]), Quantitative version of Theorem 15) Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on $C \subseteq X$ for some Banach space X. Let $\alpha, \beta \in \mathbb{R}^+$ with $0 < \alpha < \beta$. Let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with an effective irrationality measure f_γ. Let*

$$\lambda T(\alpha) + (1-\lambda)T(\beta)$$

be a mapping of C into X defined by

$$(\lambda T(\alpha) + (1-\lambda)T(\beta))x = \lambda T(\alpha)x + (1-\lambda)T(\beta)x$$

for $x \in C$ with $\lambda \in (0,1)$. Let $\Lambda \in \mathbb{N}$ be such that $1/\Lambda \leq \lambda, 1-\lambda$ and $N \in \mathbb{N}$ so that $\beta \geq 1/N$, $\mathbb{N} \ni D \geq \beta$, Moreover assume that $\{T(t) : t \geq 0\}$ is uniformly equicontinuous with a modulus of uniform equicontinuity ω. Then

$$\forall b \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall q \in C_b\ \forall m \in \mathbb{N}$$

$$(\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq \Psi \rightarrow \forall t \in [0,M] \|T(t)q - q\| < 2^{-m})$$

with

$$\Psi = \Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) = \frac{2^{-m}}{4(\sum_{i=1}^{\phi(k,f_\gamma)-1} \Lambda^i + 1)(1 + MN)}$$

where $k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1} \in \mathbb{N}$ and

$$\phi(k, f) := \max\{2f(i-j) + 6 \ : \ 0 \leq j < i \leq k+1\} \in \mathbb{N}.$$

Proof. As already mentioned, we will obtain a quantitative version of

$$\bigcap_{t \geq 0} F(T(t)) \supseteq F(\lambda T(\alpha) + (1-\lambda)T(\beta))$$

by proof mining on the proof of Theorem 15 given in [80].

We will follow the same pattern as in the proof of Theorem 15 shown in [80] but use our quantitative versions of the corresponding lemmata in [80] that we have obtained.

Recall that in general by assumption we have

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall K \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall s, s' \in [0, K]$$

$$(|s - s'| < 2^{-\omega_{K,b}(m)} \rightarrow \|T(s)q - T(s')q\| < 2^{-m}).$$

Assume that given $b \in \mathbb{N}$, for an arbitrary $q \in C_b$, for any given $\lambda \in (0,1)$ and for an unknown $\delta > 0$

$$\|(\lambda T(\alpha) + (1-\lambda)T(\beta))q - q\| \leq \delta.$$

The map $t \mapsto T(t)q$ is by assumption continuous, hence the map $h(t) := ||T(t)q - q||$ is continuous. Because $[0, \beta]$ is compact, h attains its maximum on $[0, \beta]$ at a point $\tau \in [0, \beta]$, i.e.

$$\exists \tau \in [0, \beta]\ \forall t \in [0, \beta](\|T(\tau)q - q\| \geq \|T(t)q - q\|).$$

Note that the statement that a general continuous function $h : [a, b] \rightarrow \mathbb{R}$ attains its maximum is equivalent to WKL over RCA_0 (see Theorem $I.10.3(6)$ in [76]).

Let $\gamma := \alpha/\beta \in (0,1)$ and $\theta := \tau/\beta \in [0,1]$, let $A(\theta)$ be as in the previous lemmata.

Then, by definition, $\tau = \theta\beta \in \{\theta\beta\} = A_1 \subseteq A(\theta) \subseteq [0,\beta]$. So

$$\|T(\tau)q - q\| = \max\{\|T(t)q - q\| \ : \ t \in A(\theta)\}.$$

Without loss of generality we may set

$$K := D$$

where $\mathbb{N} \ni D \geq \beta$ i.e. here we have

$$\forall m \in \mathbb{N}\ \forall s, s' \in [0, D]$$

$$(|s - s'| < 2^{-\omega_{D,b}(m)} \rightarrow \|T(s)q - T(s')q\| < 2^{-m})\ (***).$$

From now on recall the assumption that γ is irrational with an effective irrationality measure f_γ.

Now recall $(*)$ shown in Lemma 7:

$$\forall k \in \mathbb{N}\, \forall f_\gamma : \mathbb{N} \rightarrow \mathbb{N}\, \exists n_k \leq \phi(k, f_\gamma)\ \forall s \in [0,\beta]\ \exists p \in \mathbb{N}\ \exists p' \in \mathbb{Z}\ \exists s' \in \bigcup_{r \leq n_k \leq \phi(k,f_\gamma)} A_r$$

$$(|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \rightarrow |s - s'| < \frac{2D}{k})(*)$$

and notice that the premise of $(***)$ is fulfilled for

$$\frac{2D}{k} \leq 2^{-\omega_{D,b}(m)}$$

i.e. for

$$k \geq D2^{\omega_{D,b}(m)+1}.$$

We therefore set $k := D2^{\omega_{D,b}(m)+1}$ in $(*)$ of Lemma 7 and we get

$$\forall m \in \mathbb{N}\, \forall f_\gamma : \mathbb{N} \rightarrow \mathbb{N}\, \forall s \in [0,\beta]\ \exists p \in \mathbb{N}\ \exists p' \in \mathbb{Z}\ \exists s' \in \bigcup_{r \leq \phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)} A_r \subseteq [0,\beta]$$

$$(|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)} \rightarrow |s - s'| < 2^{-\omega_{D,b}(m)}).$$

By $(***)$ the above gives

$$\forall m \in \mathbb{N}\ \forall s \in [0,\beta] \subseteq [0,D]\ \exists s' \in \bigcup_{r \leq \phi(D2^{\omega_{D,b}(m)+1}, f_\gamma)} A_r$$

$$(\|T(s)q - T(s')q\| < 2^{-m}).$$

By the triangle inequality:

$$\|T(s')q - q\| \leq \|T(s)q - T(s')q\| + \|T(s)q - q\|,$$

therefore

$$\forall m \in \mathbb{N}\, \forall s \in [0,\beta]\, \exists s' \in \bigcup_{r \leq \phi(D2^{\omega_D, b(m)+1}, f_\gamma)} A_r \; (\|T(s')q - q\| < \|T(s)q - q\| + 2^{-m}).$$

Thus by (∗∗) shown in Lemma 9 and using that $H_n = \bigcup_{i \leq n} A_i$ (since $\tau = \theta\beta$)

$$\forall m \in \mathbb{N}\ \forall s \in [0,\beta]\ \exists s' \in \bigcup_{r \leq \phi(D2^{\omega_D, b(m)+1}, f_\gamma)} A_r$$

$$(\|T(\tau)q - q\| \leq$$

$$\|T(s')q - q\| + \delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i < \|T(s)q - q\| + 2^{-m} + \delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i).$$

Now, applying the above to both $s = \alpha, \beta \in [0,\beta]$, we have, for all $m \in \mathbb{N}$

$$2\|T(\tau)q - q\| < \|T(\alpha)q - q\| + \|T(\beta)q - q\| + 2\delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 2 \cdot 2^{-m}$$

$$= \|T(\alpha)q - q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q\|$$
$$+ \|T(\beta)q - q + \lambda T(\alpha)q + (1-\lambda)T(\beta)q - \lambda T(\alpha)q - (1-\lambda)T(\beta)q\|$$
$$+ 2\delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 2^{-m+1}$$

$$\leq \|\lambda T(\alpha)q + (1-\lambda)T(\beta)q - q\| + (1-\lambda)\|T(\alpha)q - T(\beta)q\|$$
$$+ \|\lambda T(\alpha)q + (1-\lambda)T(\beta)q - q\| + \lambda\|T(\beta)q - T(\alpha)q\|$$
$$+ 2\delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 2^{-m+1}$$

$$\leq 2\delta + \|T(\alpha)q - T(\beta)q\| + 2\delta \sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 2^{-m+1}$$

$$= \|T(\alpha)q - T(\alpha)T(\beta - \alpha)q\| + 2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1}$$

$$\leq \|q - T(\beta-\alpha)q\| + 2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1}$$

$$\leq \|T(\tau)q - q\| + 2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1}$$

Therefore

$$\forall m \in \mathbb{N}\ (\|T(\tau)q - q\| < 2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1})$$

and, because for all $t \in [0, \beta]$, by the definition of $\tau \in [0, \beta]$ we have

$$\|T(t)q - q\| \leq \|T(\tau)q - q\|,$$

it is :

$$\forall m \in \mathbb{N}\ \forall t \in [0, \beta](\|T(t)q - q\| < 2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1}).$$

Now for all $\mathbb{R}^+ \ni t > \beta$ there exist $r \in \mathbb{N}$, $s \in [0, \beta]$ such that

$$t = r\beta + s.$$

Therefore

$$\begin{aligned}
\|T(t)q - q\| = \|T(r\beta + s)q - q\| &= \|T(r\beta)T(s)q - q\| = \|T^r(\beta)T(s)q - q\| \\
&= \|T^r(\beta)T(s)q - q + T^r(\beta)q - T^r(\beta)q\| \\
&\leq \|T^r(\beta)T(s)q - T^r(\beta)q\| + \|T^r(\beta)q - q\| \\
&\leq \|T(s)q - q\| + \|T^r(\beta)q - q\| \\
&= \|T(s)q - q\| + \|T^r(\beta)q - q + T(\beta)q - T(\beta)q\| \\
&\leq \|T(s)q - q\| + \|T(\beta)T^{r-1}(\beta)q - T(\beta)q\| + \|T(\beta)q - q\| \\
&\leq \|T(s)q - q\| + \|T^{r-1}(\beta)q - q\| + \|T(\beta)q - q\| \\
&= \|T(s)q - q\| + \|T^{r-1}(\beta)q - q + T(\beta)q - T(\beta)q\| + \|T(\beta)q - q\| \\
&\leq \|T(s)q - q\| + \|T(\beta)T^{r-2}(\beta)q - T(\beta)q\| + 2\|T(\beta)q - q\| \\
&\leq \|T(s)q - q\| + \|T^{r-2}(\beta)q - q\| + 2\|T(\beta)q - q\| \\
&\leq .... \\
&\leq \|T(s)q - q\| + r\|T(\beta)q - q\| \\
&\leq \|T(\tau)q - q\|(1 + r)
\end{aligned}$$

$$< (2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1})(1+r).$$

Let $M \in \mathbb{N}$ so that $t \leq M$ and $N \in \mathbb{N}$ so that $\beta \geq 1/N$. We may then write:

$$M \geq t = r\beta + s \geq r/N \ + s \geq r/N$$

thus we have

$$r \leq MN.$$

Therefore

$$\forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall m \in \mathbb{N}$$

$$(\|T(t)q - q\| < (2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1})(1+MN)).$$

For a yet to be determined $m \in \mathbb{N}$, we set $\delta > 0$ to be so small so that

$$2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) \leq 2^{-m+1}$$

i.e.

$$(2\delta(\sum_{i=1}^{\phi(D2^{\omega_D, b(m)+1}, f_\gamma)-1} \Lambda^i + 1) + 2^{-m+1})(1+MN) \leq 2 \cdot 2^{-m+1}(1+MN)$$

$$= 4 \cdot 2^{-m}(1+MN) \leq 2^{-\tilde{m}}$$

for some $\tilde{m} \in \mathbb{N}$. We have

$$4 \cdot 2^{-m}(1+MN) \leq 2^{-\tilde{m}}$$

thus

$$\log_2(4 \cdot 2^{-m}(1+MN)) \leq \log_2(2^{-\tilde{m}})$$

thus

$$2 - m + \log_2(1+MN) \leq -\tilde{m}$$

thus

$$m \geq 2 + \log_2(1+MN) + \tilde{m}$$

we may thus choose

$$m := 3 + [\log_2(1+MN)] + \tilde{m}$$

i.e. we have

$$(2\delta(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+\tilde{m})+1},f_\gamma)-1}\Lambda^i+1)+2^{-(2+[\log_2(1+MN)]+\tilde{m})})(1+MN)\leq 2^{-\tilde{m}}.$$

Renaming

$$(2\delta(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1},f_\gamma)-1}\Lambda^i+1)+2^{-(2+[\log_2(1+MN)]+m)})(1+MN)\leq 2^{-m}$$

and solving for δ we obtain

$$\delta\leq\frac{2^{-m}-2^{-(2+[\log_2(1+MN)]+m)}(1+MN)}{2(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1},f_\gamma)-1}\Lambda^i+1)(1+MN)}.$$

Now notice that we can find a lower bound on the right hand side as follows. Notice that

$$2^{-(2+[\log_2(1+MN)]+m)}(1+MN)$$
$$=2^{-2}2^{-[\log_2(1+MN)]}2^{-m}(1+MN)$$

(by using that for $x>0$ we have $-[x]\leq -x+1$)

$$\leq\frac{1}{4}2^{-\log_2(1+MN)}2^{-m+1}(1+MN)$$
$$=\frac{1}{4}\frac{1}{1+MN}2^{-m+1}(1+MN)$$
$$=\frac{1}{4}2^{-m+1}=2^{-m-1}$$

Therefore

$$2^{-m}-2^{-(2+[\log_2(1+MN)]+m)}(1+MN)\geq 2^{-m}-2^{-m-1}=2^{-m-1}$$

i.e. we have shown that

$$\forall b\in\mathbb{N}\;\forall q\in C_b\;\forall M\in\mathbb{N}\;\forall t\in[0,M]\;\forall m\in\mathbb{N}$$
$$(\|(\lambda T(\alpha)+(1-\lambda)T(\beta))q-q\|$$
$$\leq\frac{2^{-m}}{4(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1},f_\gamma)-1}\Lambda^i+1)(1+MN)}$$
$$\to\|T(t)q-q\|<2^{-m}).$$

□

Corollary to the proof. If the semigroup $\{T(t) : t \geq 0\}$ is just a strongly continuous semigroup of nonexpansive mappings without the equicontinuity condition, then the bound holds with $\omega_{D,b}$ being replaced by a modulus $\omega_{D,q}$ of uniform continuity for $[0, D] \ni t \mapsto T(t)q$. Then, however, the bound would no longer be independent of q.

Remark 10. *The statement that a general continuous function $h : [0, \beta] \to \mathbb{R}$ attains its maximum at some $\tau \in [0, \beta]$ used in the proof above is noneffective (even when, as in our case, h is given with a modulus of uniform continuity) as even for computable h such a point τ will in general not be computable (see Theorem I.10.3(6) in [76] where this principle is shown to be equivalent to WKL, which is not computable). The reason why this noneffectiveness does not cause a problem in the quantitative analysis is that τ is only used via θ and that the bound obtained in Lemma 7 is independent of θ where the latter is obtained by a majorization argument applied to $\theta \in [0, 1]$. Therefore it is indeed possible to eliminate WKL while carrying out a verification of the obtained bound, see the remark below Metatheorem 1 (Lemma 7 is in fact an instance of Metatheorem 1) and [45] for a general logical discussion of this point.*

We will now give a result following by the above theorem on the asymptotic regularity of a nonexpansive semigroup $\{T(t) : t \geq 0\}$ (imposing the extra assumption that is defined on a convex $C \subseteq X$) with respect to the Krasnoselskii iteration. Considering the classical result by Ishikawa (Theorem 2) we will apply our main result Theorem 18 to obtain Corollary 4.

Corollary 4. *(Kohlenbach and K.-A., [48]) Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a convex subset $C \subseteq X$ for some Banach space X. Let $\alpha, \beta \in \mathbb{R}^+$ with $0 < \alpha < \beta$ and let $\gamma := \alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with an effective irrationality measure f_γ. Let $S : C \to C$ be defined by*

$$S := \lambda T(\alpha) + (1 - \lambda)T(\beta)$$

with $\lambda \in (0, 1)$. Let $\Lambda \in \mathbb{N}$ such that $1/\Lambda \leq \lambda, 1 - \lambda$ and $N \in \mathbb{N}$ so that $\beta \geq 1/N$, $\mathbb{N} \ni D \geq \beta$. Moreover assume that $\{T(t) : t \geq 0\}$ is uniformly equicontinuous with a modulus of uniform equicontinuity ω. Then for the Krasnoselskii iteration $\{x_n\}_{n\in\mathbb{N}}$ of the mapping $\lambda T(\alpha) + (1 - \lambda)T(\beta)$, starting at x_0, if $\{x_n\}_{n\in\mathbb{N}}$ is bounded by $b \in \mathbb{N}$, we have

$$\forall b \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall n \geq \Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d)$$
$$(\forall t \in [0, M]\ \|T(t)x_n - x_n\| < 2^{-m})$$

with a rate of asymptotic regularity

$$\Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d) = \frac{2^{2m+6}d^2}{\pi}((\sum_{i=1}^{\phi(k,f_\gamma)-1} \Lambda^i + 1)(1 + MN))^2,$$

where $d > 0$ is such that

$$d \geq \|x_0 - Sx_n\|$$

for all $n \in \mathbb{N}$,

$$k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1} \in \mathbb{N}$$

and

$$\phi(k, f) := \max\{2f(i-j)+6 \ : \ 0 \leq j < i \leq k+1\} \in \mathbb{N}.$$

Proof. The mapping $Sx := (\lambda T(\alpha) + (1-\lambda)T(\beta))x$ is nonexpansive as for all $x, y \in C$ we have

$$\|(\lambda T(\alpha) + (1-\lambda)T(\beta))x - (\lambda T(\alpha) + (1-\lambda)T(\beta))y\|$$

$$\leq \lambda\|T(\alpha)x - T(\alpha)y\| + (1-\lambda)\|T(\beta)x - T(\beta)y\| \leq$$
$$\leq \lambda\|x-y\| + (1-\lambda)\|x-y\| = \|x-y\|.$$

By Theorem 3, for the nonexpansive $S : C \to C$ and for its Krasnoselskii iteration x_n we have

$$\forall \epsilon > 0 \ \forall n \geq \theta(\epsilon, d) \ (\|x_n - Sx_n\| < \epsilon)$$

with a rate of asymptotic regularity (using that $\|x_n - Sx_n\| = 2\|x_{n+1} - x_n\|$)

$$\theta(\epsilon, d) := \frac{4d^2}{\pi\epsilon^2}$$

where $d > 0$ is such that

$$d \geq \|x_0 - Sx_n\|$$

for all $n \in \mathbb{N}$. In Theorem 18 we showed that

$$\forall b \in \mathbb{N} \ \forall q \in C_b \ \forall M \in \mathbb{N} \ \forall m \in \mathbb{N}$$
$$(\|Sq - q\| \leq \Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) \to \ \forall t \in [0, M] \ \|T(t)q - q\| < 2^{-m})$$

with

$$\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) = \frac{2^{-m}}{4(\sum_{i=1}^{\phi(D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1}, f_\gamma)-1} \Lambda^i + 1)(1+MN)}.$$

Thus, by the above it directly follows that (having substituted ϵ with Ψ)

$$\forall b \in \mathbb{N} \ \forall M \in \mathbb{N} \ \forall t \in [0, M] \ \forall m \in \mathbb{N} \ \forall n \geq \Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d)$$
$$(\|T(t)x_n - x_n\| < 2^{-m})$$

with a rate of asymptotic regularity

$$\Phi(m, M, N, \Lambda, D, b, f_\gamma, \omega, d) := \theta(\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega), d)$$
$$= \frac{4d^2}{\pi(\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega))^2}.$$

□

We emphasize that the rate above is highly uniform, as it depends on the semigroup only via the modulus ω.

Remark 11. *Instead of the Krasnoselskii iteration of* $S := \lambda T(\alpha)+(1-\lambda)T(\beta)$ *as above one may also have the more general iteration from Theorem 6 in [80] defined by*

$$x_1 \in C, x_{n+1} = \kappa T(\alpha)x_n + \lambda T(\beta)x_n + (1 - \kappa - \lambda)x_n$$

for $n \in \mathbb{N}$ *where* $\kappa, \lambda > 0$ *are fixed and* $\kappa + \lambda < 1$*, which is the Krasnoselskii–Mann iteration with* $\kappa + \lambda$ *instead of* $\frac{1}{2}$*) of* $S := \frac{\kappa}{\kappa+\lambda}T(\alpha) + \frac{\lambda}{\kappa+\lambda}T(\beta))$*. Then one uses the bound from Theorem 3 with* $\kappa + \lambda$ *and applies our Theorem 18 with* $\frac{\kappa}{\kappa+\lambda}$ *instead of* λ*.*

3.3 Second Approach : Proof Theoretic Analysis and Results

We give an -alternative to the one obtained in the previous section- explicit, computable and uniform bound for the computation of approximate common fixed points of one-parameter nonexpansive semigroups on a subset C of a Banach space, by proof mining on the proof of Theorem 16 by Suzuki. The bound obtained here is completely different to the bound obtained in the previous section. For uniformly convex C, as a corollary to our result we will afterwards give a computable rate of asymptotic regularity with respect to Kuhfittig's [55] classical iteration schema, by applying a theorem by Khan and Kohlenbach ([35]) which had been derived via proof mining on a the proof of a result by Kuhfittig ([55]).

Our main result will be a quantitative version of Theorem 16. Exactly as was the case for the inclusion

$$\bigcap_{t\geq 0} F(T(t)) \subseteq F(\lambda T(\alpha) + (1 - \lambda)T(\beta))$$

in the previous section, also the inclusion

$$\bigcap_{t\geq 0} F(T(t)) \subseteq F(T(\alpha)) \cap F(T(\beta))$$

is trivial, so in the proof of Theorem 16 the nontrivial inclusion

$$\bigcap_{t\geq 0} F(T(t)) \supseteq F(T(\alpha)) \cap F(T(\beta))$$

is proved. Our main result (Theorem 21) which we will show here, constitutes in particular a quantitative version of the latter statement. As this can be written as

$$\forall z \in C \ (\forall m \in \mathbb{N} \ \|T(\alpha)z - z\| \leq 2^{-m} \wedge \|T(\beta)z - z\| \leq 2^{-m} \rightarrow$$
$$\forall k \in \mathbb{N} \ \forall M \in \mathbb{N} \ \forall t \in [0, M] \ \|T(t)z - z\| < 2^{-k})$$

i.e. (by prenexation)

$$\forall z \in C \ \forall k \in \mathbb{N} \ \forall M \in \mathbb{N} \ \forall t \in [0, M] \ \exists m \in \mathbb{N}$$
$$(\|T(\alpha)z - z\| \leq 2^{-m} \wedge \|T(\beta)z - z\| \leq 2^{-m} \rightarrow \|T(t)z - z\| < 2^{-k}),$$

we have a a $\forall\exists(\forall \rightarrow \exists)$ i.e. $\forall\exists$ statement, so it is possible to extract a computable bound on $m \in \mathbb{N}$ as guaranteed by (an obvious variant of) Metatheorem 3. This will be done by proof mining on the proof of Theorem 16.

To derive our quantitative versions of Suzuki's results, we will need to make use of the concepts of uniform equicontinuity for the nonexpansive semigroup $\{T(t) : t \geq 0\}$ and modulus of equicontinuity as already presented. Without the equicontinuity assumption for $\{T(t) : t \geq 0\}$, the bound that we will obtain would be less uniform as it would depend on the point $z \in C$ instead of the input $b \in \mathbb{N}$ so that $C_b := \{z \in C : \|z\| \leq b\}$.

We will obtain quantitative versions of Theorem 19 and Lemma 11 which will be used to obtain our main result Theorem 21 (which is a quantitative version of Theorem 16). Lemma 10 will be auxiliary.

Lemma 10. *(Suzuki, Lemma 2 in [79]) Let $t \in \mathbb{R}^+$ and let $\{\beta_n\}$ be a sequence in $(0, \infty)$ converging to 0. Define sequences $\{\delta_n\} \in [0, \infty)$ and $\{k_n\} \in \mathbb{N} \cup \{0\}$ as follows:*

- $\delta_1 = t$,
- $k_n = [\delta_n/\beta_n]$ *for* $n \in \mathbb{N}$,
- $\delta_{n+1} = \delta_n - k_n\beta_n$ *for* $n \in \mathbb{N}$.

Then the following hold:

1. $0 \leq \delta_{n+1} < \beta_n$ *for all* $n \in \mathbb{N}$,
2. $k_n \in \mathbb{N} \cup \{0\}$ *for all* $n \in \mathbb{N}$,

3. $\{\delta_n\}$ *converges to* 0,

4. $\sum_{j=1}^{n} k_j\beta_j + \delta_{n+1} = t$ *for all* $n \in \mathbb{N}$,

5. $\sum_{j=1}^{\infty} k_j\beta_j = t$.

Lemma 11. *(Suzuki, Lemma 3 in [79]) Let* $\alpha, \beta \in \mathbb{R}^+$ *satisfying* $\alpha/\beta \notin \mathbb{Q}$. *Define sequences* $\{\alpha_n\} \in (0, \infty)$ *and* $\{k_n\} \in \mathbb{N}$ *as follows:*

- $\alpha_1 = \max\{\alpha, \beta\}$,
- $\alpha_2 = \min\{\alpha, \beta\}$,
- $k_n = [\alpha_n/\alpha_{n+1}]$ *for all* $n \in \mathbb{N}$,
- $\alpha_{n+2} = \alpha_n - k_n\alpha_{n+1}$ *for all* $n \in \mathbb{N}$.

Then the following hold:

- $0 < \alpha_{n+1} < \alpha_n$ *for all* $n \in \mathbb{N}$,
- $k_n \in \mathbb{N}$ *for all* $n \in \mathbb{N}$,
- $\alpha_n/\alpha_{n+1} \notin \mathbb{Q}$ *for all* $n \in \mathbb{N}$,
- $\{\alpha_n\}$ *converges to* 0.

Theorem 19. *(Suzuki, Proposition 1 in [79]) Let* X *be a Banach space and let* $\{T(t) : t \geq 0\}$ *be a one-parameter nonexpansive semigroup on* $C \subseteq X$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0, \infty)$ *converging to* $\alpha_\infty \in [0, \infty)$, *and satisfying* $\alpha_n \neq \alpha_\infty$ *for all* $n \in \mathbb{N}$. *Suppose that* $z \in C$ *satisfies* $T(\alpha_n)z = z$ *for all* $n \in \mathbb{N}$. *Then* z *is a common fixed point of* $\{T(t) : t \geq 0\}$.

Theorem 20. *(Quantitative version of Theorem 19) Let* X *be a Banach space and let* $\{T(t) : t \geq 0\}$ *be a one-parameter uniformly equicontinuous semigroup of nonexpansive mappings on a subset* C *of* X, *with a modulus of uniform equicontinuity* ω. *Let* $\{\alpha_n\}$ *be a sequence of reals in* $[0, \infty)$ *converging to* $\alpha_\infty \in [0, \infty)$ *with a rate of convergence* $\Phi : \mathbb{N} \to \mathbb{N}$, *and so that* $\forall n \in \mathbb{N}(|\alpha_n - \alpha_\infty| > 2^{-\Psi(n)})$ *where* $\Psi : \mathbb{N} \to \mathbb{N}$. *Let* $L \in \mathbb{N}$ *be such that for all* $n \in \mathbb{N}$ $\{\alpha_n\}, \alpha_\infty \in [0, L]$. *Then*

$$\forall k \in \mathbb{N}\ \forall b \in \mathbb{N}\ \forall z \in C_b\ \forall M \in \mathbb{N}\ \forall L \in \mathbb{N}$$

$$(\forall n \leq \tilde{W}\ \|T(\alpha_n)z - z\| \leq W \to \forall t \in [0, M]\ \|T(t)z - z\| < 2^{-k})\ (*)$$

with

$$\tilde{W} = \tilde{W}(k, b, M, L, \Phi, \Psi, \omega) =$$

$$\max\{\Phi(\omega_{b,M+1}(k+1)), \Phi(\omega_{b,L}(k+1+\lceil\log_2(3M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)})\rceil))\}$$

and

$$W = W(k, b, M, \Phi, \Psi, \omega) = \frac{2^{-(k+1)}}{3M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}.$$

Remark 12. *By Theorem 19, if $z \in C$ is a fixed point of $T(\alpha_n)$ for all $n \in \mathbb{N}$, then $z \in C$ is a fixed point of $T(t)$ for all $t \in [0, \infty)$. A formalized version of the above statement is the following:*

$$\forall z \in C(\forall \delta > 0\, \forall n \in \mathbb{N}\, \|T(\alpha_n)z - z\| \le \delta \to \forall k \in \mathbb{N}\, \forall t \in [0, \infty)\|T(t)z - z\| < 2^{-k}).$$

By prenexing the above we have

$$\forall z \in C\, \forall k \in \mathbb{N}\, \forall t \in [0, \infty)\, \exists \delta > 0\, \exists n \in \mathbb{N}(\|T(\alpha_n)z - z\| \le \delta \to \|T(t)z - z\| < 2^{-k})$$

i.e. (setting $C_b := \{z \in C \ : \ \|z\| \le b\}$)

$$\forall b \in \mathbb{N}\, \forall z \in C_b\, \forall k \in \mathbb{N}\, \forall M \in \mathbb{N}\, \forall t \in [0, M]\, \exists \delta > 0\, \exists n \in \mathbb{N}$$
$$(\|T(\alpha_n)z - z\| \le \delta \to \|T(t)z - z\| < 2^{-k}).$$

For the above statement which is of the logical form $\forall\exists$, Metatheorem 4 guarantees the extraction of computable bounds on δ and n. We will extract such bounds by proof mining on Theorem 19 and therefore obtain $(*)$.

Proof. As in Theorem 19, we define

$$\beta_n := |\alpha_n - \alpha_\infty|.$$

By this definition, clearly the rate of convergence of $\{\beta_n\}$ to 0 is the same as the rate of convergence Φ of $\{\alpha_n\}$ to α_∞ and, moreover, by the assumption we have

$$\forall n \in \mathbb{N}\, \beta_n > 2^{-\Psi(n)}.$$

Now, in both cases $\alpha_n = \alpha_\infty + \beta_n$ and $\alpha_\infty = \alpha_n + \beta_n$ we claim that we have, for all $n \in \mathbb{N}$,

$$\|T(\beta_n)z - z\| \le \|T(\alpha_\infty)z - z\| + \|T(\alpha_n)z - z\|.$$

The above claim follows directly by the semigroup properties and the triangle inequality i.e. if $\alpha_n = \alpha_\infty + \beta_n$ we have

$$\|T(\beta_n)z - z\| = \|T(\beta_n)z - z + T(\alpha_n)z - T(\alpha_n)z\|$$
$$\le \|T(\beta_n)z - T(\alpha_n)z\| + \|T(\alpha_n)z - z\|$$
$$= \|T(\beta_n)z - T(\alpha_\infty + \beta_n)z\| + \|T(\alpha_n)z - z\|$$

$$= \|T(\beta_n)z - T(\beta_n)T(\alpha_\infty)z\| + \|T(\alpha_n)z - z\|$$
$$\leq \|z - T(\alpha_\infty)z\| + \|T(\alpha_n)z - z\|$$

and analogously if $\alpha_\infty = \alpha_n + \beta_n$ we have

$$\|T(\beta_n)z - z\| = \|T(\beta_n)z - z + T(\alpha_\infty)z - T(\alpha_\infty)z\|$$
$$= \|T(\beta_n)z - z + T(\alpha_n + \beta_n)z - T(\alpha_n + \beta_n)z\|$$
$$\leq \|T(\beta_n)z - T(\beta_n)T(\alpha_n)z\| + \|T(\alpha_n + \beta_n)z - z\|$$
$$\leq \|z - T(\alpha_n)z\| + \|T(\alpha_\infty)z - z\|.$$

Now let $t \in [0, M]$ for some $M \in \mathbb{N}$. By Lemma 10, there exists a sequence $\{k_n\} \in \mathbb{N} \cup \{0\}$ (as defined in Lemma 10) such that

$$\forall m \in \mathbb{N}\ \forall n \geq \Phi(m)(|\sum_{i=1}^{n} k_i\beta_i - t| < 2^{-m}).$$

(Note that as by Lemma 10 we have that for all $n \in \mathbb{N}$

$$0 \leq t - \sum_{i=1}^{n} k_i\beta_i < \beta_n,$$

as a rate of convergence of $\{\sum_{i=1}^{n} k_i\beta_i\}$ to t we can take the rate of convergence Φ of $\{\beta_n\}$ to 0 (i.e. the rate of convergence Φ of $\{\alpha_n\}$ to α_∞).
Now the triangle inequality gives

$$\|T(t)z - z\| = \|T(t)z - z + T(\sum_{i=1}^{n} k_i\beta_i)z - T(\sum_{i=1}^{n} k_i\beta_i)z\|$$

$$\leq \|T(\sum_{i=1}^{n} k_i\beta_i)z - z\| + \|T(t)z - T(\sum_{i=1}^{n} k_i\beta_i)z\|.$$

We moreover have

$$\|T(\sum_{i=1}^{n} k_i\beta_i)z - z\|$$
$$= \|T(k_1\beta_1)T(k_2\beta_2)...T(k_n\beta_n)z - z\|$$
$$= \|T(k_1\beta_1)T(k_2\beta_2)...T(k_n\beta_n)z - z + T(k_1\beta_1)z - T(k_1\beta_1)z\|$$
$$\leq \|T(k_1\beta_1)T(k_2\beta_2)...T(k_n\beta_n)z - T(k_1\beta_1)z\| + \|T(k_1\beta_1)z - z\|$$
$$\leq \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + \|T(k_1\beta_1)z - z\|$$
$$= \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + \|T^{k_1}(\beta_1)z - z\|$$
$$= \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + \|T^{k_1}(\beta_1)z - z + T(\beta_1)z - T(\beta_1)z\|$$

$$\leq \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + \|T^{k_1}(\beta_1)z - T(\beta_1)z\| + \| - z + T(\beta_1)z\|$$
$$\leq \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + \|T^{k_1-1}(\beta_1)z - z\| + \| - z + T(\beta_1)z\|$$
$$\leq \|T(k_2\beta_2)...T(k_n\beta_n)z - z\| + k_1\|T(\beta_1)z - z\|$$
$$\leq k_1\|T(\beta_1)z - z\| + k_2\|T(\beta_2)z - z\| + ... + k_n\|T(\beta_n)z - z\|$$
$$\leq k_1(\|T(\alpha_1)z - z\| + \|T(\alpha_\infty)z - z\|) + ... + k_n(\|T(\alpha_n)z - z\| + \|T(\alpha_\infty)z - z\|)$$
$$= (k_1\|T(\alpha_\infty)z - z\| + ... + k_n\|T(\alpha_\infty)z - z\|) + (k_1\|T(\alpha_1)z - z\|$$
$$+... + k_n\|T(\alpha_n)z - z\|)$$
$$= \|T(\alpha_\infty)z - z\| \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} k_i\|T(\alpha_i)z - z\|$$

and by the triangle inequality

$$\|T(\alpha_\infty)z - z\| \leq \|T(\alpha_m)z - T(\alpha_\infty)z\| + \|z - T(\alpha_m)z\|$$

for any arbitrary $m \in \mathbb{N}$, so, overall, we have calculated that

$$\|T(t)z - z\| \leq$$
$$(\|T(\alpha_m)z - T(\alpha_\infty)z\| + \|z - T(\alpha_m)z\|) \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} k_i\|T(\alpha_i)z - z\|$$
$$+\|T(t)z - T(\sum_{i=1}^{n} k_i\beta_i)z\|.$$

By the construction of Lemma 10, it is

$$k_n = [\frac{\delta_n}{\beta_n}], n \in \mathbb{N}$$

where $\{\delta_n\}$ is a sequence in $[0, \infty)$ defined by

$$\delta_1 = t;$$
$$\delta_{n+1} = \delta_n - k_n\beta_n.$$

So, as for all $n \in \mathbb{N}$

$$\delta_n - \delta_{n+1} = k_n\beta_n \geq 0,$$

$\{\delta_n\}$ is decreasing. Therefore, for all $n \in \mathbb{N}$,

$$k_n = [\frac{\delta_n}{\beta_n}] \leq \frac{\delta_n}{\beta_n} \leq \frac{\delta_1}{\beta_n} = \frac{t}{\beta_n} < t\ 2^{\Psi(n)} \leq M\ 2^{\Psi(n)}.$$

Therefore we may write the above calculated estimate as

$$\|T(t)z - z\| \leq$$

$$(\|T(\alpha_m)z - T(\alpha_\infty)z\| + \|z - T(\alpha_m)z\|)M\sum_{i=1}^{n} 2^{\Psi(i)} + M\sum_{i=1}^{n} 2^{\Psi(i)}\|T(\alpha_i)z - z\|$$

$$+\|T(t)z - T(\sum_{i=1}^{n} k_i\beta_i)z\| \ (**).$$

Now consider, together with the uniform equicontinuity assumption for the semigroup (as $m \in \mathbb{N}$ was arbitrary):

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall k \in \mathbb{N}\ \forall L \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall \alpha_\infty, \alpha_m \in [0, L]$$
$$(|\alpha_\infty - \alpha_m| < 2^{-\omega_{b,L}(k)} \to \|T(\alpha_\infty)z - T(\alpha_m)z\| < 2^{-k}),$$

the convergence assumption :

$$\forall k \in \mathbb{N}\ \forall m \geq \Phi(k)(|\alpha_m - \alpha_\infty| < 2^{-k})$$

that combined give:

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall k \in \mathbb{N}\ \forall L \in \mathbb{N}\ \forall \alpha_\infty, \alpha_m \in [0, L]\ \forall m \geq \Phi(\omega_{b,L}(k))$$
$$\|T(\alpha_m)z - T(\alpha_\infty)z\| < 2^{-k}.$$

Now, the convergence statement (as already mentioned by Lemma 10 here we have again the same rate of convergence Φ) :

$$\forall m \in \mathbb{N}\ \forall n \geq \Phi(m)(|\sum_{i=1}^{n} k_i\beta_i - t| < 2^{-m})$$

combined with the uniform equicontinuity assumption for the semigroup (notice that for $t \in [0, M]$ if $n \geq \Phi(m)$ by the above we have $|\sum_{i=1}^{n} k_i\beta_i - t| \in [0, M+1]$) gives

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall k \in \mathbb{N}\ \forall t \in [0, M]\ \forall n \geq \Phi(\omega_{b,M+1}(k))$$
$$\|T(\sum_{i=1}^{n} k_i\beta_i)z - T(t)z\| < 2^{-k}.$$

Substituting in $(**)$, for a given $j \in \mathbb{N}$ which satisfies

$$\forall n \in \mathbb{N}\ \|T(\alpha_n)z - z\| < 2^{-j}$$

, we obtain:

$$\forall k \in \mathbb{N}\, \forall t \in [0, M]\ \|T(t)z - z\| \leq (2^{-j} + \|z - T(\alpha_{\Phi(\omega_{b,L}(j))}z)\|)M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)}$$

$$+M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)} \|T(\alpha_i)z - z\| + 2^{-k}.$$

Because

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}(\|T(\alpha_n)z - z\| < 2^{-j} \to$$

$$\|z - T(\alpha_{\Phi(\omega_{b,L}(j))}z)\| < 2^{-j} \wedge \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)} \|T(\alpha_i)z - z\| < 2^{-j} \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)})$$

we have

$$n \leq \max\{\Phi(\omega_{b,M+1}(k)), \Phi(\omega_{b,L}(j))\}.$$

In total, as $M \in \mathbb{N}$ was arbitrary:

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall L \in \mathbb{N}\ \forall \alpha_n, \alpha_\infty \in [0, L]$$

$$\exists n \leq \max\{\Phi(\omega_{b,M+1}(k)), \Phi(\omega_{b,L}(j))\}$$

$$(\|T(\alpha_n)z - z\| \leq 2^{-j} \to$$

$$\|T(t)z - z\| < (2^{-j} + 2^{-j})M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)} + 2^{-j}M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)} + 2^{-k}$$

$$= 3 \cdot 2^{-j}M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)} + 2^{-k})$$

Now let the above arbitrary $j \in \mathbb{N}$ be such that for a yet to be determined $k \in \mathbb{N}$,

$$2^{-j} \leq \frac{2^{-k}}{3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)}}.$$

Choosing

$$j := k + \lceil \log_2(3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)}) \rceil$$

we thus have

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall t \in [0, M]\ \forall L \in \mathbb{N}$$

$$\exists n \leq \max\{\Phi(\omega_{b,M+1}(k)), \Phi(\omega_{b,L}(k + \lceil \log_2(3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)}) \rceil))\}$$

$$(\|T(\alpha_n)z - z\| \leq \frac{2^{-k}}{3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k))} 2^{\Psi(i)}} \to \|T(t)z - z\| < 2^{-k} + 2^{-k}).$$

i.e. we have extracted the bounds:

$$\tilde{W} := \max\{\Phi(\omega_{b,M+1}(k+1)), \Phi(\omega_{b,L}(k+1+\lceil\log_2(3M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)})\rceil))\},$$

$$W := \frac{2^{-(k+1)}}{3M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}.$$

□

Lemma 12. *(Quantitative version of Lemma 11) Let* $2^{-G} < \alpha, \beta \in \mathbb{R}^+$ *for some* $G \in \mathbb{N}$ *and satisfying* $\alpha < \beta$ *and* $\beta/\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *with an effective irrationality measure (with domain restricted to* $\mathbb{N} \times \mathbb{N}$*)* $f_{\frac{\beta}{\alpha}}$*. Define a sequence* $\{\alpha_n\} \in (0, \infty)$ *as*

$$\alpha_1 := \beta, \alpha_2 := \alpha, \alpha_{n+2} := \alpha_n - [\frac{\alpha_n}{\alpha_{n+1}}]\alpha_{n+1}.$$

Then
(A) $\forall n \in \mathbb{N}\ \alpha_n > \alpha_{n+1} > 0$ *and* $\forall n \in \mathbb{N}\ \frac{\alpha_n}{\alpha_{n+1}} \in \mathbb{R} \setminus \mathbb{Q}$.
(B) $\forall n \in \mathbb{N}\ \alpha_n > 2^{-\Psi(n)}$ *where* $\Psi(n)$ *is defined simultaneously with* $f_{\frac{\alpha_n}{\alpha_{n+1}}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *as follows :*

$$f_{\frac{\alpha_1}{\alpha_2}}(p,q) := f_{\frac{\beta}{\alpha}}(p,q)$$

$$f_{\frac{\alpha_{n+1}}{\alpha_{n+2}}}(p,q) := \max_{k \leq \lceil\beta\rceil 2^{\Psi(n+1)}} \{f_{\frac{\alpha_n}{\alpha_{n+1}}}(kp+q,p)\}|\lceil\frac{q}{p}\rceil|,$$

and

$$\Psi(1) := G, \Psi(2) := G$$

and for $n > 2$*:*

$$\Psi(n) := \sum_{i=2}^{n-2} \lceil\log_2(\max_{l \leq \lceil\beta\rceil 2^{\Psi(i+1)}} \{f_{\frac{\alpha_i}{\alpha_{i+1}}}(l,1)\})\rceil + G.$$

(C) $\forall k \in \mathbb{N}\ \forall n \geq \Phi(k)\ \alpha_n < 2^{-k}$ *with*

$$\Phi(k) := \lceil\beta\rceil 2^k + 2.$$

Proof. The proof of (A) is carried out by induction, it is in fact the proof of Lemma 11 that is given in [79] and we present it here as for the next step we will write down a quantitative version of it. By definition $\alpha_1/\alpha_2 = \beta/\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $\alpha_1 = \beta > \alpha_2 = \alpha > 0$ so $A(n=1)$ holds. Consider the induction hypothesis :

$$\text{for some } j \in \mathbb{N}\ 0 < \alpha_{j+1} < \alpha_j \text{ and } \frac{\alpha_j}{\alpha_{j+1}} \in \mathbb{R} \setminus \mathbb{Q}\ (A(j))$$

(Note that then in particular $[\alpha_j/\alpha_{j+1}] \geq 1$). By the definition of the sequence we have

$$\alpha_{j+2} = (\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}])\alpha_{j+1}$$

and by the definition of the floor function $[\cdot]$ for any $x \in \mathbb{R}$ we have $x - [x] \in [0, 1)$ while here, by $(A(j))$,

$$\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}] > 0 \text{ and } \alpha_{j+1} > 0$$

therefore

$$0 < \alpha_{j+2} < \alpha_{j+1}.$$

As by definition

$$\frac{\alpha_{j+2}}{\alpha_{j+1}} = \frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}],$$

it is $\frac{\alpha_{j+2}}{\alpha_{j+1}} \in \mathbb{R} \setminus \mathbb{Q}$ and thus $\frac{\alpha_{j+1}}{\alpha_{j+2}} \in \mathbb{R} \setminus \mathbb{Q}$. So we have shown $(A(j + 1))$ and thus by induction we have shown $(A(n))$ i.e. $0 < \alpha_{n+1} < \alpha_n$ and $\frac{\alpha_{n+1}}{\alpha_{n+2}} \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$.

Showing (B) amounts to writing down a quantitative version of (A). Since (A) was shown by induction and the statements on the irrationality of $\frac{\alpha_n}{\alpha_{n+1}}$ for all $n \in \mathbb{N}$ and the fact that for all $n \in \mathbb{N}$ $\alpha_n > 0$ were shown simultaneously, $f_{\frac{\alpha_n}{\alpha_{n+1}}}$ is defined recursively and simultaneously with $\Psi(n)$ (the latter is in fact the quantitative information that is of interest here) thus the proof will be carried out again by induction. For $n = 1$ by the definition of $\{\alpha_n\}$ we have

$$f_{\frac{\alpha_1}{\alpha_2}}(p, q) = f_{\frac{\beta}{\alpha}}(p, q)$$

where $f_{\frac{\beta}{\alpha}}$ is a function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that

$$\forall p, q \in \mathbb{N} \, |\frac{\beta}{\alpha} - \frac{p}{q}| \geq \frac{1}{f_{\frac{\beta}{\alpha}}(p, q)}$$

i.e. the effective irrationality measure of $\frac{\beta}{\alpha}$ with the domain restricted to exclude zero (compare with Definition 24) and moreover $\alpha_1 = \beta > 2^{-\Psi(1)}$ respectively $\alpha_2 = \alpha > 2^{-\Psi(2)}$ are clearly fulfilled when

$$\Psi(1) := G, \; \Psi(2) := G$$

as by assumption

$$\beta > \alpha > 2^{-G}.$$

Consider the following induction hypothesis $B(j)$: Let us assume, that for some $j \in \mathbb{N}$ we have

$$\forall p, q \in \mathbb{N} \, |\frac{\alpha_j}{\alpha_{j+1}} - \frac{p}{q}| \geq \frac{1}{f_{\frac{\alpha_j}{\alpha_{j+1}}}(p, q)} \text{ and } \alpha_{j+1} > 2^{-\Psi(j+1)} \; (B(j)).$$

Using $(B(j))$ we will show $(B(j+1))$.

Notice that for all $p, q \in \mathbb{N}$,

$$|\frac{\alpha_{j+2}}{\alpha_{j+1}} - \frac{p}{q}| = |\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}] - \frac{p}{q}| = |\frac{\alpha_j}{\alpha_{j+1}} - (\frac{[\frac{\alpha_j}{\alpha_{j+1}}]q + p}{q})|$$

$$\geq \frac{1}{f_{\frac{\alpha_j}{\alpha_{j+1}}}([\frac{\alpha_j}{\alpha_{j+1}}]q + p, q)}$$

$$\geq \frac{1}{\max_{k \leq \lceil\beta\rceil 2^{\Psi(j+1)}}\{f_{\frac{\alpha_j}{\alpha_{j+1}}}(kq + p, q)\}}$$

(because

$$[\frac{\alpha_j}{\alpha_{j+1}}] \leq [\frac{\beta}{\alpha_{j+1}}] \leq [\beta 2^{\Psi(j+1)}] \leq \beta 2^{\Psi(j+1)} \leq \lceil\beta\rceil 2^{\Psi(j+1)}).$$

Therefore

$$\forall p, q \in \mathbb{N}\ |\frac{\alpha_{j+1}}{\alpha_{j+2}} - \frac{p}{q}| = |\frac{\alpha_{j+1}}{\alpha_{j+2}}\frac{p}{q}|\ |\frac{q}{p} - \frac{\alpha_{j+2}}{\alpha_{j+1}}|$$

$$\geq |\frac{\alpha_{j+1}}{\alpha_{j+2}}\frac{p}{q}| \frac{1}{\max_{k \leq \lceil\beta\rceil 2^{\Psi(j+1)}}\{f_{\frac{\alpha_j}{\alpha_{j+1}}}(kp + q, p)\}}$$

$$\geq |\frac{p}{q}| \frac{1}{\max_{k \leq \lceil\beta\rceil 2^{\Psi(j+1)}}\{f_{\frac{\alpha_j}{\alpha_{j+1}}}(kp + q, p)\}}$$

(as by (A) $\alpha_{j+1} > \alpha_{j+2} > 0$) so

$$f_{\frac{\alpha_{j+1}}{\alpha_{j+2}}}(p, q) = \max_{k \leq \lceil\beta\rceil 2^{\Psi(j+1)}}\{f_{\frac{\alpha_j}{\alpha_{j+1}}}(kp + q, p)\}|\lceil\frac{q}{p}\rceil|$$

indeed gives

$$\forall p, q \in \mathbb{N}\ |\frac{\alpha_{j+1}}{\alpha_{j+2}} - \frac{p}{q}| \geq \frac{1}{f_{\frac{\alpha_{j+1}}{\alpha_{j+2}}}(p, q)}.$$

We will now show that $\alpha_{j+2} > 2^{-\Psi(j+2)}$. To this end, in $(B(j))$ we make the choice (recall that $[\alpha_j/\alpha_{j+1}] \geq 1$)

$$p = [\frac{\alpha_j}{\alpha_{j+1}}],\ q = 1$$

and thus obtain (as by the definition of the floor function $[\cdot]$ for any $x \in \mathbb{R}$ we have $[x] \leq x$ and moreover $[\frac{\alpha_j}{\alpha_{j+1}}] \leq \lceil\beta\rceil 2^{\Psi(j+1)}$):

$$\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}] = |\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}]| \geq \frac{1}{f_{\frac{\alpha_j}{\alpha_{j+1}}}([\frac{\alpha_j}{\alpha_{j+1}}], 1)} \geq \frac{1}{\max_{l \leq \lceil\beta\rceil 2^{\Psi(j+1)}}\{f_{\frac{\alpha_j}{\alpha_{j+1}}}(l, 1)\}}.$$

By recalling that by the definition of $\{\alpha_n\}$ we have

$$\alpha_{j+2} = (\frac{\alpha_j}{\alpha_{j+1}} - [\frac{\alpha_j}{\alpha_{j+1}}])\alpha_{j+1}$$

and by $B(j)$ we obtain

$$\alpha_{j+2} \geq \frac{1}{\max_{l \leq \lceil \beta \rceil 2^{\Psi(j+1)}} \{f_{\frac{\alpha_j}{\alpha_{j+1}}}(l,1)\}} \alpha_{j+1}$$

$$> \frac{1}{\max_{l \leq \lceil \beta \rceil 2^{\Psi(j+1)}} \{f_{\frac{\alpha_j}{\alpha_{j+1}}}(l,1)\}} 2^{-\Psi(j+1)}.$$

Therefore, as by having set

$$\Psi(n) := \sum_{i=1}^{n-2} \lceil \log_2 (\max_{l \leq \lceil \beta \rceil 2^{\Psi(i+1)}} \{f_{\frac{\alpha_i}{\alpha_{i+1}}}(l,1)\}) \rceil + \Psi(2)$$

we have

$$\Psi(j+2) - \Psi(j+1) = \lceil \log_2 (\max_{l \leq \lceil \beta \rceil 2^{\Psi(j+1)}} \{f_{\frac{\alpha_j}{\alpha_{j+1}}}(l,1)\}) \rceil$$

we have shown by the above that

$$\alpha_{j+2} > 2^{-\Psi(j+2)}$$

so the proof of $(B(j+1))$ is complete and by induction

$$\forall n \in \mathbb{N} \ \alpha_n > 2^{-\Psi(n)}.$$

We will now show (C). By (A) $\{\alpha_n\}$ is convergent, and thus Cauchy. We will show that the limit of $\{\alpha_n\}$ is zero, i.e. that

$$\forall m \in \mathbb{N} \ \exists n \ \forall i,j \geq n \ |\alpha_i - \alpha_j| < 2^{-m} \to \forall k \in \mathbb{N} \ \exists l \in \mathbb{N} \ \forall n \geq l \ \alpha_n < 2^{-k}$$

and we will moreover find a computable bound on $l \in \mathbb{N}$. Because by (A) $\{\alpha_n\}$ is decreasing, it is enough to show that

$$\forall m \in \mathbb{N} \ \exists n \ \forall i,j \geq n \ |\alpha_i - \alpha_j| < 2^{-m} \to \forall k \in \mathbb{N} \ \exists l \in \mathbb{N} \ \alpha_l < 2^{-k} \ (!) \ .$$

Note that, in order to derive the quantitative (and effective) information of interest, i.e. the bound Φ, we will apply proof mining to the entire statement (!), and not to just its conclusion. This is because the premise will be weakened to a metastable Cauchy statement in order to apply Proposition 1.

We claim that the negation of (!) will give a contradiction, that is, we claim that

$$\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall i, j \geq n\ |\alpha_i - \alpha_j| < 2^{-m}\ (I)$$
$$\wedge \exists k \in \mathbb{N}\ \forall l \in \mathbb{N}\ \alpha_l \geq 2^{-k}\ (II)$$

will give a contradiction. To show this claim, in (I) above we make the choices [3] $m := k$ where $k \in \mathbb{N}$ is as in (II), $i := n$ and $j := n+1$ i.e. :

$$\exists n\ |\alpha_n - \alpha_{n+1}| = \alpha_n - \alpha_{n+1} < 2^{-k}\ (III)$$

(as $\{\alpha_n\}$ is decreasing by (A)). The assumption (II) for such a k, together with (III) in which k is as in (II) give

$$2^{-k} \leq \alpha_{n+1} < \alpha_n < \alpha_{n+1} + 2^{-k}.$$

Dividing the above by $\alpha_{n+1} > 0$ we have

$$\frac{2^{-k}}{\alpha_{n+1}} \leq 1 < \frac{\alpha_n}{\alpha_{n+1}} < \frac{\alpha_{n+1} + 2^{-k}}{\alpha_{n+1}} = 1 + \frac{2^{-k}}{\alpha_{n+1}} \leq 2$$

so

$$1 < \frac{\alpha_n}{\alpha_{n+1}} < 2$$

which gives

$$[\frac{\alpha_n}{\alpha_{n+1}}] = 1.$$

Now, by the definition of the sequence $\{\alpha_n\}$, substituting the above we obtain

$$\alpha_{n+2} = \alpha_n - [\frac{\alpha_n}{\alpha_{n+1}}]\alpha_{n+1} = \alpha_n - \alpha_{n+1} < 2^{-k}$$

and clearly $\alpha_{n+2} < 2^{-k}$ gives a contradiction to the assumption (II) for $l := n+2$.

Thus the bound Φ on l corresponds to a bound on n shifted by 2. The latter is obtained by applying Proposition 1. Since $\{\alpha_n\} \in (0, \beta] \in [0, \lceil\beta\rceil]$ we obtain

$$\Phi(k) := \lceil\beta\rceil 2^k + 2$$

i.e. we have shown that

$$\forall k \in \mathbb{N}\ \forall n \geq \Phi(k)\ \alpha_n < 2^{-k}$$

with

$$\Phi(k) := \lceil\beta\rceil 2^k + 2.$$

□

[3] this amounts to making the choice $g(k) := 1$ for g as in Proposition 1.

We now show our main result.

Theorem 21. *(Quantitative version of Theorem 16) Let X be a Banach space and let $\{T(t) : t \geq 0\}$ be a one-parameter uniformly equicontinuous semigroup of nonexpansive mappings on a subset C of X with a modulus of uniform equicontinuity ω. Let $\alpha, \beta \in \mathbb{R}^+$ with $2^{-G} < \alpha < \beta$ for some $G \in \mathbb{N}$ and satisfying $\beta/\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with an effective irrationality measure (with domain restricted to $\mathbb{N} \times \mathbb{N}$) $f_{\frac{\beta}{\alpha}}$. Then*

$$\forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall b \in \mathbb{N}\ \forall z \in C_b$$

$$(\|T(\alpha)z - z\| \leq \mathcal{X} \wedge \|T(\beta)z - z\| \leq \mathcal{X} \rightarrow \forall t \in [0, M]\ \|T(t)z - z\| < 2^{-k})$$

with

$$\mathcal{X} = \mathcal{X}(f_{\frac{\beta}{\alpha}}, \lceil\beta\rceil, G, b, M, k, \Phi, \Psi, \omega, \tilde{W}) =$$

$$\frac{\sqrt{5}\frac{2^{-(k+1)}}{6M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}}{((\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1}) \prod_{i=1}^{\tilde{W}-2} \lceil\beta\rceil 2^{\Psi(i+1)}}$$

where

$$\Psi(1) := G, \Psi(2) := G$$

and for $n > 2$

$$\Psi(n) := \sum_{i=2}^{n-2} \lceil \log_2(\max_{l \leq \lceil\beta\rceil 2^{\Psi(i+1)}} \{f_{\frac{\alpha_i}{\alpha_{i+1}}}(l, 1)\}) \rceil + G$$

with

$$f_{\frac{\alpha_1}{\alpha_2}}(p, q) := f_{\frac{\beta}{\alpha}}(p, q)$$

$$f_{\frac{\alpha_{n+1}}{\alpha_{n+2}}}(p, q) := \max_{k \leq \lceil\beta\rceil 2^{\Psi(n+1)}} \{f_{\frac{\alpha_n}{\alpha_{n+1}}}(kp + q, p)\} |\lceil \frac{q}{p} \rceil |,$$

where $\{\alpha_n\}$ is a sequence defined by $\alpha_1 := \beta, \alpha_2 := \alpha$ and $\alpha_{n+2} := \alpha_n - \lfloor \frac{\alpha_n}{\alpha_{n+1}} \rfloor \alpha_{n+1}$,

$$\Phi(k) := \lceil\beta\rceil 2^k + 2$$

and

$$\tilde{W} = \tilde{W}(k, b, M, \lceil\beta\rceil, \Phi, \Psi, \omega) =$$

$$\max\{\Phi(\omega_{b,M+1}(k+1)), \Phi(\omega_{b,\lceil\beta\rceil}(k + 1 + \lceil \log_2(3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}) \rceil))\}.$$

Proof. Define a sequence $\{\alpha_n\} \in (0,\infty)$ as in Lemma 12. For convenience set

$$k_n := [\alpha_n/\alpha_{n+1}].$$

We have

$$\begin{aligned}
\|T(\alpha_{n+2})z - z\| &= \|T(\alpha_{n+2})z - z + T(\alpha_n)z - T(\alpha_n)z\| \\
&\leq \|T(\alpha_{n+2})z - T(\alpha_n)z\| + \|T(\alpha_n)z - z\| \\
&= \|T(\alpha_{n+2})z - T(\alpha_{n+2} + k_n\alpha_{n+1})z\| + \|T(\alpha_n)z - z\| \\
&= \|T(\alpha_{n+2})z - T(\alpha_{n+2})T(k_n\alpha_{n+1})z\| + \|T(\alpha_n)z - z\| \\
&\leq \|z - T(k_n\alpha_{n+1})z\| + \|T(\alpha_n)z - z\| \\
&= \|z - T^{k_n}(\alpha_{n+1})z\| + \|T(\alpha_n)z - z\| \\
&= \|z - T^{k_n}(\alpha_{n+1})z + T(\alpha_{n+1})z - T(\alpha_{n+1})z\| + \|T(\alpha_n)z - z\| \\
&\leq \|T^{k_n}(\alpha_{n+1})z - T(\alpha_{n+1})z\| + \|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\| \\
&\leq \|T^{k_n-1}(\alpha_{n+1})z - z\| + \|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\| \\
&\leq \ldots \\
&\leq k_n\|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\|.
\end{aligned}$$

We have therefore shown that

$$\begin{aligned}
\|T(\alpha_{n+2})z - z\| &\leq k_n\|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\| \\
&\leq k_n(\|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\|).
\end{aligned}$$

(because by Lemma 12

$$0 < \alpha_{n+1} < \alpha_n$$

for all $n \in \mathbb{N}$, thus

$$k_n = [\alpha_n/\alpha_{n+1}] \geq 1$$

for all $n \in \mathbb{N}$).

Now let $b \in \mathbb{N}$ and $z \in C_b$ such that

$$\|T(\alpha_1)z - z\| = \|T(\beta)z - z\| \leq \delta \wedge \|T(\alpha_2)z - z\| = \|T(\alpha)z - z\| \leq \delta$$

for some $\delta > 0$. Let us consider the sequence

$$\{\|T(\alpha_n)z - z\|\}_{n\in\mathbb{N}}$$

defined by

$$\|T(\alpha_{n+2})z - z\| \leq k_n(\|T(\alpha_{n+1})z - z\| + \|T(\alpha_n)z - z\|)$$

and

$$\|T(\alpha_1)z - z\| = \|T(\beta)z - z\| \leq \delta \wedge \|T(\alpha_2)z - z\| = \|T(\alpha)z - z\| \leq \delta$$

for some $\delta > 0$. We will estimate the nth term as follows. Observing the form of the first terms:

$$\|T(\alpha_3)z - z\| \leq k_1 2\delta,$$
$$\|T(\alpha_4)z - z\| \leq k_2(k_1 2\delta + \delta) = \delta(k_2 + 2k_1k_2),$$
$$\|T(\alpha_5)z - z\| \leq k_3(\delta(k_2 + 2k_1k_2) + k_1 2\delta) = \delta(k_3k_2 + 2k_1k_2k_3 + 2k_1k_3),$$
$$\|T(\alpha_6)z - z\| \leq k_4(\delta(k_3k_2 + 2k_1k_2k_3 + 2k_1k_3) + \delta(k_2 + 2k_1k_2))$$
$$= \delta(k_4k_3k_2 + 2k_4k_3k_2k_1 + 2k_4k_1k_3 + k_4k_2 + 2k_4k_2k_1)$$
$$(\leq 2\delta(k_4k_3k_2 + k_4k_3k_2k_1 + k_4k_1k_3 + k_4k_2 + k_4k_2k_1))$$
$$(...)$$

Now note that because, as mentioned above, for all $n \in \mathbb{N}$, $k_n \geq 1$, for $n \leq m$ we have

$$\prod_{i=1}^{n} k_i \leq \prod_{i=1}^{m} k_i.$$

Moreover, note that the number of summands in the respective bound of each term of the above sequence (where each summand is a product of k_is) follows the Fibonnacci sequence, i.e.

	number of summands of products of k_is
$\|T(\alpha_3)z - z\|$	1
$\|T(\alpha_4)z - z\|$	2
$\|T(\alpha_5)z - z\|$	3
$\|T(\alpha_6)z - z\|$	5
(...)	(...)

clearly as each term approximation involves the sum of the two previous term approximations. In particular, the nth term of the sequence $\{\|T(\alpha_n)z - z\|\}_{n\in\mathbb{N}}$ has a factor which is the $n-1$th Fibonnacci number. The nth Fibonnacci number is given by the well-known Binet's formula:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}.$$

Hence, for the nth term of the sequence $\{\|T(\alpha_n)z - z\|\}_{n\in\mathbb{N}}$ we have

$$\|T(\alpha_n)z - z\| \leq \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}} 2\delta \prod_{i=1}^{n-2} k_i.$$

Moreover, note that, for each $i \in \mathbb{N}$

$$k_i = [\alpha_i/\alpha_{i+1}] \leq [\alpha_1/\alpha_{i+1}] = [\beta/\alpha_{i+1}] \leq [\beta 2^{\Psi(i+1)}] \leq \beta 2^{\Psi(i+1)} \leq \lceil \beta \rceil 2^{\Psi(i+1)}.$$

Therefore we may write

$$\|T(\alpha_n)z - z\| \leq \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}} 2\delta \prod_{i=1}^{n-2} \lceil \beta \rceil 2^{\Psi(i+1)}.$$

Thus, for

$$n := \tilde{W}$$

with $\tilde{W}$ as it was previously obtained in our Theorem 20 with $L := \lceil \beta \rceil$, as the above obtained bound on $\|T(\alpha_n)z - z\|$ is nondecreasing on n we have that

$$\|T(\alpha)z - z\| \leq \delta \wedge \|T(\beta)z - z\| \leq \delta \rightarrow \forall n \leq \tilde{W} \|T(\alpha_n)z - z\| \leq$$

$$\leq \frac{(\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1}}{\sqrt{5}} 2\delta \prod_{i=1}^{\tilde{W}-2} \lceil \beta \rceil 2^{\Psi(i+1)}.$$

By choosing $\delta > 0$ to be such that

$$\frac{(\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1}}{\sqrt{5}} 2\delta \prod_{i=1}^{\tilde{W}-2} \lceil \beta \rceil 2^{\Psi(i+1)} \leq W$$

where W is the bound extracted in Theorem 20, i.e. by choosing

$$\delta \leq \frac{\sqrt{5} \frac{2^{-(k+1)}}{6M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}}{((\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1}) \prod_{i=1}^{\tilde{W}-2} \lceil \beta \rceil 2^{\Psi(i+1)}}$$

the premise of Theorem 20 is now fulfilled, and therefore by Theorem 20 we obtain that

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \ \forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall t \in [0, M]$$

$$(\|T(\alpha)z - z\| \leq \mathcal{X} \wedge \|T(\beta)z - z\| \leq \mathcal{X} \rightarrow \|T(t)z - z\| < 2^{-k})$$

with

$$\mathcal{X} := \frac{\sqrt{5} \frac{2^{-(k+1)}}{6M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}}{((\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1}) \prod_{i=1}^{\tilde{W}-2} \lceil \beta \rceil 2^{\Psi(i+1)}}$$

where Φ, Ψ for the particular sequence $\{\alpha_n\}$ are as extracted in Lemma 12. □

Remark 13. *Corollary to the proof.* *It would be possible to remove the equicontinuity assumption for the semigroup $\{T(t) : t \geq 0\}$. Then, the modulus of continuity of $\{T(t) : t \geq 0\}$, and thus also the final bound, would depend on $z \in C$ instead of the input $b \in \mathbb{N}$ so that $C_b := \{z \in C\ :\ z \leq \|b\|\}$. That would, strictly speaking, constitute a direct quantitative version of Suzuki's result Theorem 16. However, omitting our supplementary equicontinuity assumption would have the following disadvantages:*

- *Clearly the result would be less uniform,*
- *it would not be possible to derive the corollary on asymptotic regularity, that we will derive here in the end of this section.*

We can compare the result of Theorem 21 of this second approach with the result of Theorem 18 of the first approach.

Let us assume that we have $z \in C_b$ so that, for some $\delta > 0$,

$$\|T(\alpha)z - z\| \leq \delta \wedge \|T(\beta)z - z\| \leq \delta.$$

Then

$$\begin{aligned}\|(\lambda T(\alpha) + (1-\lambda)T(\beta))z - z\| &= \|(\lambda T(\alpha) + (1-\lambda)T(\beta))z - z + \lambda z - \lambda z\| \\ &\leq \lambda\|T(\alpha)z - z\| + (1-\lambda)\|T(\beta)z - z\| < \delta.\end{aligned}$$

In Theorem 18 let us make the choice $\lambda := \frac{1}{2} \in (0,1)$ and $\Lambda := 2$. Therefore the bound could be stated as:

$$\forall b \in \mathbb{N}\ \forall z \in C_b\ \forall M \in \mathbb{N}\ \forall m \in \mathbb{N}$$

$$((\|T(\alpha)z - z\| \leq \Psi \wedge \|T(\beta)z - z\| \leq \Psi) \rightarrow \forall t \in [0, M]\ \|T(t)z - z\| < 2^{-m})$$

with

$$\Psi = \Psi(m, M, N, D, b, f_\gamma, \omega) = \frac{2^{-m}}{4(\sum_{i=1}^{\phi(k,f_\gamma)-1} 2^i + 1)(1 + MN)}$$

where $k \in \mathbb{N}$ and $\phi(k, f) \in \mathbb{N}$ are as in Theorem 18.

Comparing the bound Ψ that would follow from Theorem 18 from the first approach to the bound $\mathcal{X}$ obtained in Theorem 21 from the second approach we make the interesting observation that proof mining on Suzuki's two completely different proofs of essentially the same statement gave us a completely different result. Also note that both proofs by Suzuki (in [80] and [79] respectively) although completely different to each other, both used an irrationality assumption on the ratio of α and β thus both the quantitative analyses presented for the first and second approach made use of the notion of effective irrationality measure for an irrational number. (Recall that within the first approach the effective irrationality measure for reasons of simplicity was made to depend only on one variable).

Finally, under the assumption that the Banach space X is moreover uniformly convex, we will now give a corollary to Theorem 21 using a result by Khan and Kohlenbach in [35] on the asymptotic regularity of the semigroup $\{T(t) : t \geq 0\}$ with respect to a classical iteration schema introduced by Kuhfittig in 1981 ([55]) :

Definition 27. *(Kuhfittig ([55])) Let C be a nonempty convex subset of a Banach space X and let $\{T_i : 1 \leq i \leq k\}$ be a finite family of nonexpansive self-mappings. Let $U_0 := I$ where I denotes the identity mapping. Let $\lambda \in (0,1)$. Consider the mappings:*

$$U_1 = (1-\lambda)I + \lambda T_1 U_0$$

$$U_2 = (1-\lambda)I + \lambda T_2 U_1$$

$$...$$

$$U_k = (1-\lambda)I + \lambda T_k U_{k-1}.$$

Define

$$x_0 \in C,\ x_{n+1} := (1-\lambda)x_n + \lambda T_k U_{k-1} x_n, n \geq 0.$$

By proof mining on a the proof of a theorem by Kuhfittig (implicit) in [55] (also see Theorem 1.2 in [35]), Khan and Kohlenbach showed in [35] the following Theorem 22 which is a quantitative version of Kuhfittig's theorem. Note that in [35] Theorem 22 is actually shown in the more general context of UCW-hyperbolic spaces but here we state it adapted to the special case of Banach spaces :

Theorem 22. *(Khan and Kohlenbach (Theorem 3.2 in [35])) Let C be a nonempty convex subset of a uniformly convex Banach space X with a modulus of uniform convexity η and let $\{T_i : 1 \leq i \leq k\}$ be a finite family of nonexpansive self-mappings of C with $\cap_{i=1}^k F(T_i) \neq \emptyset$. Let $p \in \cap_{i=1}^k F(T_i)$ and $D > 0$ such that $\|x_0 - p\| \leq D$ for some $x_0 \in C$. Then for the sequence $\{x_n\}$ generated by the iteration schema of Definition 27, we have, for all $1 \leq i \leq k$*

$$\forall \epsilon \in (0,2]\ \forall n \geq \Theta_i(D, \epsilon, N, \eta)\ (\|T_i x_n - x_n\| \leq \epsilon)$$

with a rate of asymptotic regularity

$$\Theta_i := \theta(\hat{\eta}^{(k-i+\min(1,k-1))}(\frac{\epsilon}{2})),$$

where $N \in \mathbb{N}$ is such that $\frac{1}{N} \leq \lambda(1-\lambda)$,

$$\theta(\epsilon) := \lceil \frac{D}{\hat{\eta}(\epsilon)} \rceil,$$

$$\hat{\eta}(\epsilon) := \frac{1}{N}\eta(\frac{\epsilon}{D+1})\epsilon.$$

Remark 14. *In the case where the Banach space has a modulus of convexity η that can be written as $\eta(\epsilon) = \epsilon\tilde{\eta}(\epsilon)$ where $\tilde{\eta}(\epsilon)$ is increasing, (for instance, in the case of the Banach spaces L_p, that, for $p \geq 2$ have an asymptotically optimal modulus of convexity $\frac{\epsilon^p}{p2^p}$) then η can be replaced with $\tilde{\eta}$ in the bound (see Remark 1 as well as Remark 3.3 in [35]).*

We show the following corollary to Theorem 21 by making use of the above Theorem 22.

Corollary 5. *Let C be a nonempty convex subset of a uniformly convex Banach space X with a modulus of uniform convexity η and let $\{T(t) : t \geq 0\}$ be a one-parameter uniformly equicontinuous semigroup of nonexpansive mappings on C with a modulus of uniform equicontinuity ω. Let $\alpha, \beta \in \mathbb{R}^+$ with $2^{-G} < \alpha < \beta$ for some $G \in \mathbb{N}$ and satisfying $\beta/\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with effective irrationality measure (with domain restricted to $\mathbb{N} \times \mathbb{N}$) $f_{\frac{\beta}{\alpha}}$ and let $F(T(\alpha)) \cap F(T(\beta)) \neq \emptyset$. Let $p \in F(T(\alpha)) \cap F(T(\beta))$ and let $D > 0$ such that $\|x_0 - p\| \leq D$ for some $x_0 \in C$. Then for the sequence $\{x_n\}$ generated by the iteration schema of Definition 27, we have*

$$\forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall b \in \mathbb{N}\ \forall n \geq \tilde{\Theta}\ \forall x_n \in C_b\ \forall t \in [0, M]\ \|T(t)x_n - x_n\| \leq 2^{-k}$$

with a rate of asymptotic regularity

$$\tilde{\Theta} := \max_{i=1,2}\{\Theta_i\},$$

where

$$\Theta_i = \Theta_i(D, N, \eta, f_{\frac{\beta}{\alpha}}, G, \lceil\beta\rceil, \omega, b, M, k) := \theta(\hat{\eta}^{(3-i))}(\frac{\mathcal{X}}{2})),$$

where

$$\theta(\epsilon) := \lceil \frac{D}{\hat{\eta}(\epsilon)} \rceil,$$

$$\hat{\eta}(\epsilon) := \frac{1}{N}\eta(\frac{\epsilon}{D+1})\epsilon,$$

$N \in \mathbb{N}$ is such that $\frac{1}{N} \leq \lambda(1-\lambda)$, and

$$\mathcal{X} = \mathcal{X}(f_{\frac{\beta}{\alpha}}, G, \lceil\beta\rceil, \tilde{W}, \Psi, \Phi, \omega, b, M, k) =$$

$$= \frac{\sqrt{5}\frac{2^{-(k+1)}}{6M\sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)}}}{((\frac{1+\sqrt{5}}{2})^{\tilde{W}-1} - (\frac{1-\sqrt{5}}{2})^{\tilde{W}-1})\prod_{i=1}^{\tilde{W}-2}\lceil\beta\rceil 2^{\Psi(i+1)}}$$

where

$$\Psi(1) := G, \Psi(2) := G$$

and for $n > 2$

$$\Psi(n) := \sum_{i=2}^{n-2}\lceil \log_2(\max_{l \leq \lceil\beta\rceil 2^{\Psi(i+1)}}\{f_{\frac{\alpha_i}{\alpha_{i+1}}}(l, 1)\})\rceil + G$$

with

$$f_{\frac{\alpha_1}{\alpha_2}}(p, q) := f_{\frac{\beta}{\alpha}}(p, q)$$

$$f_{\frac{\alpha_{n+1}}{\alpha_{n+2}}}(p,q) := \max_{k \leq \lceil \beta \rceil 2^{\Psi(n+1)}} \{f_{\frac{\alpha_n}{\alpha_{n+1}}}(kp+q,p)\} | \lceil \frac{q}{p} \rceil |,$$

where $\{\alpha_n\}$ *is a sequence defined by* $\alpha_1 := \beta, \alpha_2 := \alpha$ *and* $\alpha_{n+2} := \alpha_n - [\frac{\alpha_n}{\alpha_{n+1}}]\alpha_{n+1}$,

$$\tilde{W} = \tilde{W}(k,b,M,\lceil \beta \rceil, \Phi, \Psi, \omega) =$$

$$\max\{\Phi(\omega_{b,M+1}(k+1)), \Phi(\omega_{b,\lceil \beta \rceil}(k+1+\lceil \log_2(3M \sum_{i=1}^{\Phi(\omega_{b,M+1}(k+1))} 2^{\Psi(i)})\rceil))\}.$$

and

$$\Phi(k) := \lceil \beta \rceil 2^k + 2.$$

Proof. By Theorem 22, for $k = 2$ and setting $T_1 := T(\alpha_1) = T(\alpha)$, $T_2 := T(\alpha_2) = T(\beta)$ we have

$$\forall \epsilon \in (0,2]\ \forall n \geq \Theta_1(D,\epsilon,N,\eta)\ (\|T(\alpha)x_n - x_n\| \leq \epsilon),$$

$$\forall \epsilon \in (0,2]\ \forall n \geq \Theta_2(D,\epsilon,N,\eta)\ (\|T(\beta)x_n - x_n\| \leq \epsilon)$$

with

$$\Theta_1 := \theta(\hat{\eta}^{(2)}(\frac{\epsilon}{2})), \Theta_2 := \theta(\hat{\eta}(\frac{\epsilon}{2})),$$

where $N \in \mathbb{N}$ is such that $\frac{1}{N} \leq \lambda(1-\lambda)$,

$$\theta(\epsilon) := \lceil \frac{D}{\hat{\eta}(\epsilon)} \rceil,$$

$$\hat{\eta}(\epsilon) := \frac{1}{N}\eta(\frac{\epsilon}{D+1})\epsilon.$$

We may therefore write

$$\forall \epsilon \in (0,2]\ \forall n \geq \max_{i=1,2}\{\Theta_i(D,\epsilon,N,\eta)\}\ (\|T(\alpha)x_n - x_n\| \leq \epsilon \wedge \|T(\beta)x_n - x_n\| \leq \epsilon).$$

By setting $\epsilon := \mathcal{X} \in (0,2]$ in the above, where $\mathcal{X}$ is as in Theorem 21, the premise of Theorem 21 is fulfilled, and we thus directly obtain:

$$\forall k \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall b \in \mathbb{N}\ \forall n \geq \max_{i=1,2}\{\Theta_i(D,\mathcal{X},N,\eta)\}$$

$$\forall x_n \in C_b\ \forall t \in [0,M] \|\ T(t)x_n - x_n\| \leq 2^{-k})$$

where $\mathcal{X} = \mathcal{X}(f_{\frac{\beta}{\alpha}}, G, \lceil \beta \rceil, \tilde{W}, b, M, k, \Phi, \Psi, \omega)$ is as in Theorem 21.

□

Chapter 4

A Short Comment on Future Work

In this thesis we have presented several case studies of the proof mining program that involved applications in nonlinear analysis. As we saw, proof mining can be considered as a practice within the generalized Hilbert's program. Another program of proof theory that can also be of service to analysis and, like proof mining, seeks in a wider sense to reduce mathematics to constructive reasoning, is reverse mathematics. This program fits rather in the category of relativized Hilbert programs as it concentrates on so-called proof-theoretic reductions of systems of classical mathematics to more restricted systems, the reductions being carried out using finitistic means. Essentially the question examined is how much of classical mathematics can be reduced to finitary mathematics and to this end the program seeks to determine which axiomatic systems are required to prove mathematical theorems. Harvey Friedman, who originally initiated the program ([19]), has pointed out the empirical fact that "*when the theorem is proved from the right axioms, the axioms can be proved from the theorem*" ([76]).

Some interesting comments on the philosophical and foundational significance of the program, especially with respect to Hilbert's program, can be found in [75], also see Section 4.3 in [89].

Stephen Simpson, a pioneer of reverse mathematics, summarizes the main scope of the program in the following question :

Question 3. *([76]) "Given a theorem τ of ordinary mathematics, which is the weakest natural subsystem of second order arithmetic in which τ is provable?"*

For a future project, as suggested by U. Kohlenbach to the author, it would be of interest to explore Question 3 for τ being the theorem that asserts the existence of a weak solution for the Navier-Stokes equations.

In the following we briefly explain the motivation behind this idea.

The Navier-Stokes equations are nonlinear partial differential equations describing the time evolution of the velocity of a fluid. They are of significant interest as they model many physical and engineering systems. A comprehensive reference on Navier-Stokes is for instance [85]. In the known literature all proofs for the existence of (weak) solution for the Navier-Stokes equations from a foundational point of view require a theory of sequential compactness which is provided by the axiom system ACA_0 (Arithmetical Comprehension)- in particular the Ascoli lemma is needed which is equivalent to ACA_0 within RCA_0 (see [76]). For example, a proof of the existence of a weak solution for Navier-Stokes is given in the classical paper by Masuda ([60]). Another proof, via nonstandard analysis, is given in [12], also see [16].

The existence of a solution for ordinary differential equations (also known as Peano existence theorem) has been shown within the axiom system WKL_0. The latter consists of the axioms of RCA_0 (Recursive Comprehension) plus WKL. WKL_0 is a mathematically stronger natural subsystem of second order arithmetic than RCA_0 and weaker than ACA_0. We refer to Simpson's proof ([76]) and to [74] as well as Tanaka's much shorter proof via nonstandard analysis ([82]). We also refer to [73]. Note that in [76] the converse is also shown, i.e. it is shown that the Peano existence theorem is equivalent to WKL within RCA_0.

Motivated by the above results, we would like to attempt to give a proof for the existence of a weak solution for the Navier-Stokes equations formalized within the axiom system WKL_0, while all the proofs in the literature seem to be carried out within the mathematically stronger axiom system ACA_0 (though an actual formalization within ACA_0 or any other system has in fact not been carried out yet). In particular, motivated by Tanaka's proof via nonstandard analysis ([82]) for the Peano existence theorem, as well as by the nonstandard proofs for Navier-Stokes in [12], [16], it would be reasonable to attempt this within the setting of nonstandard analysis, especially as Keita Yokoyama has been developing in recent years a framework for second order arithmetic within nonstandard analysis, see [88], [31], [87].

It is however at the moment not clear whether/how the aforementioned statement and some proof of it could be described within a formal framework for reverse mathematics. If such a formalization were possible, one would suggest to attempt to show the following:

Conjecture 1. *The existence of a weak solution for the Navier-Stokes equations can be shown within the axiom system WKL_0.*

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