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### The Philosophies Of Mathematics

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## Reviews

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# REVIEWS

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## The Philosophies of Mathematics

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**Alan Baker**

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What does it mean to assert a mathematical claim, for example that there is a prime between 5 and 10? If the claim is true, then what makes it true? And how do we come to know it in the first place? It is apparently basic questions such as these that drive the field of philosophy of mathematics. That these questions arise for even the most elementary mathematical propositions makes the philosophical project to elucidate the nature of mathematics accessible to nonspecialists. It also makes it frustratingly inconclusive.

Before delving into contemporary philosophy of mathematics, let us begin by casting a glance back one hundred years to the early part of the twentieth century. At this time, philosophers of mathematics were focused on the following question

- (i) Are the central claims of our core mathematical theories *true*? If so, what makes them true?   **[foundation]**

Interestingly, the project of addressing the foundation question was taken up not just by philosophers, but also by a number of prominent mathematicians. Many philosophers view this period as the “golden age” of philosophy of mathematics. Although the foundation of mathematics went through a series of crises during this time, the issues being addressed were of interest to the wider mathematical community. The “Big Four” philosophical views on the nature of mathematics that emerged during this period were logicism, intuitionism, formalism, and platonism.

According to logicism, the truths of mathematics are ultimately truths of logic. Once appropriate definitions of the basic terms are given, statements such as “ $2 + 2 = 4$ ” can be seen to be true solely by virtue of the meanings of the expressions involved, as is sometimes the case with nonmathematical claims, such as “All bachelors are unmarried.” Logicism began with the work of the German philosopher Gottlob Frege in the late nineteenth century, was taken up by Bertrand Russell in the early twentieth century, and culminated with the massive (and massively impenetrable) three-volume work, *Principia Mathematica*, published in 1910 by Russell and Whitehead.

According to intuitionism, which was championed by the Dutch mathematicians Brouwer and Heyting, and which took its inspiration from the philosophy of Immanuel Kant, mathematical entities such as numbers are created by the mental acts of mathematicians. Intuitionism is a form of antirealism, since it denies that there is a preexisting universe of mathematical entities waiting to be described. However, mathematical claims can be true if they are proved in the right way. In particular, proofs of the existence of some particular mathematical entity must proceed by giving an explicit “recipe” for constructing the given entity.

According to formalism, mathematical claims are meaningless strings of symbols that are manipulated according to explicitly stated formal rules. This is a more radical

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form of antirealism, since it denies that mathematical claims are even true (or false either)! The most famous defender of formalism was David Hilbert. Hilbert was not a formalist about the whole of mathematics, only the part that deals with infinite totalities. For Hilbert, a mathematical claim such as, “There is a prime number between 10 and 20” can be expressed as a finite string of claims about finite numbers (i.e., “Either 10 is prime or 11 is prime or . . . or 20 is prime.”) and thus is meaningful. By contrast, a claim such as, “There is no largest prime number,” cannot be expressed as a finite string of claims about finite numbers, and hence, on Hilbert’s view, is not capable of being true or false. For Hilbert, infinitary statements are merely vehicles for moving between (meaningful) finitary claims. The use of infinitary claims is permissible provided that we can show that their use never results in inconsistency.

Famously, all three of these philosophical views of mathematics ran into technical difficulties. Frege’s version of logicism was dealt a fatal blow by Russell’s paradox. On Frege’s view, every property determines a set of things that have that property. Russell’s paradox asks about the property, which we shall denote by  $p$ , of not being self-membered. Is the set  $S$  determined by  $p$  a member of itself? If so, then it does have the determining property  $p$ , implying it is not a member of itself. But if it is not a member of itself, then it has property  $p$ , implying that it *is* a member of itself! Russell’s own response to this paradox was to build a version of logicism that separates objects and sets made up of those objects into different levels. This paved the way to the development of modern set theory. While set theory seems to make an excellent foundation for the rest of mathematics, it does not vindicate logicism because set theory is not logic.

The main technical problem with intuitionism is that it requires an underlying logic that rejects the Law of the Excluded Middle. This is the principle that, for any statement  $p$ , either  $p$  or not- $p$  is true. If  $p$  is a mathematical existence claim, then one way to prove  $p$ , in classical mathematics, is to show that the assumption that not- $p$  leads to a contradiction. The intuitionist rejects this form of *reductio ad absurdum* proof, since what the intuitionist requires in order to establish  $p$  is the construction of a particular example that fits the existence claim. For example, if  $p$  is the claim that there exist irrational numbers which, when raised to irrational powers, are rational, then the intuitionist will demand at least one specific example of such a number. This feature of intuitionism is not contradictory, as was the case with Frege’s logicism, but it does conflict with mainstream mathematical practice. David Hilbert famously complained that “taking the Principle of the Excluded Middle from the mathematician . . . is the same as . . . prohibiting the boxer the use of his fists.”

Hilbert’s own preferred philosophy of mathematics, formalism, ran into its own roadblock in the formidable shape of Gödel’s celebrated incompleteness theorems. A corollary of these theorems is that a consistent system strong enough for arithmetic cannot be used to probe its own consistency. This means that the finitary part of mathematics cannot be relied upon to guarantee the consistency of the infinitary—and, for Hilbert, meaningless—parts.

There was also a fourth philosophical view floating around in the early twentieth century, one with much older roots and with some well-known proponents, including such luminaries as G. H. Hardy and Kurt Gödel. Here is a characteristic passage from Hardy’s *A Mathematician’s Apology*:

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our “creations,” are simply our notes to our observations. This view has been held, in one form or another, by many philosophers of high

reputation from Plato onwards, and I shall use the language which is natural to a man who holds it.

While not a view about the foundations of mathematics *per se*, platonism is in an important sense the most straightforwardly realist position of all: mathematicians are exploring and describing an abstract landscape that exists independently of us. While not subject to the technical problems that afflicted the preceding three views, platonism runs into severe difficulties answering a second philosophical question:

- (ii) How do we come to *know* the truth of the central claims of our core mathematical theories?     **[knowledge]**

Note that Hardy's talk of "observations" in the above passage is at best metaphorical. No mathematician has ever literally observed a mathematical object.

What does philosophy of mathematics look like today? Fast-forwarding a hundred years from the foundational controversies of the early twentieth century, we can see successors of each of the "Big Four" philosophical views on the nature of mathematics. In addition to a shift from the foundation question to the knowledge question, a third question has also come to increasing prominence in contemporary debates:

- (iii) What explains the usefulness of mathematics in science, and its *applicability* more generally to the world?     **[application]**

In the remainder of this review, I shall briefly outline the four most prominent current philosophies of mathematics, and suggest in each case a book that explores the given position in more detail.

First up is *neologicism*. For decades after Frege's logicism was torpedoed by Russell's paradox, it was assumed that this dealt a fatal blow to logicism more generally. It was not until the early 1980's that philosophers noticed a relatively straightforward way to salvage the core aspects of Frege's approach while avoiding Russell's paradox. In his original work, Frege proposes an axiom that he calls "Basic Law V." One implication of Basic Law V is that for every property there is a set of objects that fall under that property. Frege uses Basic Law V to prove a key foundational result, that the number of *F*s is equal to the number of *G*s if and only if the *F*-objects can be put into one-to-one correspondence with the *G*-objects. This latter result has come to be known as "Hume's Principle." Basic Law V is what gives rise to Russell's paradox. (Consider the property of not being self-membered. According to Basic Law V, there is a set *S* of objects with this property. Is *S* a member of itself? Either answer leads to contradiction.) However, Basic Law V plays a very little role in Frege's system other than to prove Hume's principle, and Hume's principle itself does not fall prey to Russell's paradox. Neologicism proposes to jettison Basic Law V and instead use Hume's principle, plus logic, as the foundation of arithmetic. Hume's principle is one example of a family of principles known as *abstraction principles*. Another goal of neologicism is to find a way of distinguishing "good" (i.e., consistent) abstraction principles from "bad" (i.e., inconsistent) abstraction principles and to find foundational abstraction principles for other areas of mathematics. (An example from geometry is the principle that the direction of line *M* is equal to the direction of line *N* if and only if *M* is parallel to *N*.) An excellent book-length summary of neologicism, including both the philosophical motivations and the technical results, is *Fixing Frege* by John Burgess [1].

The second of our four contemporary philosophies of mathematics is *structuralism*. While not a direct successor of intuitionism (in the way that neologicism is a direct successor of logicism), structuralism shares with the older intuitionist position a down-playing of the status of mathematical objects as mind-independent entities.

For the structuralist, mathematics is about structure, not objects. Indeed, any collection of objects with the right structure can serve to instantiate a given mathematical theory. Take the natural numbers, for example. According to structuralism, numbers are simply places in the natural number structure. There is no independent object that is the number 17. Structuralism helps to address all three of the issues that preoccupy philosophers of mathematics. It helps with the knowledge question, since knowledge of structures seems more tractable than knowledge of abstract objects. It helps with the applicability question, since structures are by their nature realizable in multiple ways by different physical phenomena. And it fits well with the way that mathematicians talk about mathematics, and with research into structure-centered foundations for mathematics such as category theory. Philosopher Stewart Shapiro has done a lot to articulate and defend structuralism and his book *Philosophy of Mathematics: Structure and Ontology* [4] is an excellent overview of this position.

Formalism not only falls foul of Gödel's results, it also flies in the face of mathematical practice. When mathematicians describe themselves as formalists, they tend to use this label merely to emphasize their view of the importance of rigor and proof. Few actually believe—as formalism dictates—that mathematical claims are meaningless strings of symbols. The next philosophical position I shall discuss is *fictionalism*. According to the fictionalist, core mathematical claims are meaningful, but they are false. A claim such as “7 is a prime number” is akin to a claim about a fictional character, such as “Sherlock Holmes is a pipe-smoking detective.” Each is an acceptable claim to make, in the right context, yet each is, strictly speaking, false. Sherlock Holmes does not exist, and nor does the number 7. On the fictionalist view, what mathematicians are doing is setting out fictional scenarios and then exploring their consequences. Thus, for example, the story of arithmetic might begin: “Once upon a time there was a number, 0, that was the successor of no number, and it had a successor, 1, and . . .” Fictionalism does well on the knowledge question: we make up our mathematical fictions, so there is no problem explaining how we know about what happens in them. More of a problem is the applicability question. The Sherlock Holmes stories may be entertaining, but they are not particularly *useful*. What makes our mathematical fictions so invaluable for theorizing about the physical world? Mary Leng develops and defends a broadly fictionalist position in her book *Mathematics and Reality* [3].

This brings us to the fourth and final philosophical position, known as *indispensabilist platonism* (IP). The strategy underlying IP is to use the applicability of mathematics to answer the knowledge question for platonism. We begin by noting that science makes reference to a variety of theoretical entities such as electrons, genes, and black holes. We believe in the existence of these entities because they are part of our best scientific theories. But science also refers to a variety of mathematical entities such as numbers, sets, and functions. Moreover, these mathematical entities are seemingly *indispensable* to science: we do not know how to formulate our theories without them. IP argues that this provides sufficient grounds for believing in the existence of numbers, sets, and functions. In brief, we ought to believe in the literal truth of mathematics because we believe our best scientific theories and we need mathematics for our best scientific theories. The pros and cons of indispensabilist platonism have been much discussed over the past two decades. Mark Colyvan's book, *The Indispensability of Mathematics* [2] provides a nice overview of this position.

Philosophers of mathematics have traditionally focused their attention on a very narrow selection of core areas of mathematics such as arithmetic, geometry, and set theory. This has changed over the past several decades, with philosophers now routinely bringing in a more diverse array of examples from fields such as topology, group theory, linear algebra, and knot theory. This broadening of perspective has gone hand

in hand with a greater sensitivity to the details of actual mathematical practice, as opposed to some philosophical idealization of mathematics. It is ironic, therefore, that these developments have occurred just as philosophy of mathematics has fallen off most mathematicians' radar. I end with one final book recommendation, this time for a more general overview of philosophy of mathematics for the mathematically informed nonspecialist: Stewart Shapiro's book *Thinking About Mathematics: The Philosophy of Mathematics* [5]. Promoting ongoing dialogue between mathematicians and philosophers of mathematics is surely beneficial to both fields.

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