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On the Sparre–Andersen risk model with different type of interclaim times distributions

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In loving memory of my grandparents

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Resumo

Nesta dissertação trabalhamos com teoria do risco, com especial ênfase sobre dois grandes temas da área, nomeadamente, os modelos de risco e a teoria de ruína. A resolução de equações é uma parte fundamental da matemática e de praticamente qualquer outra ciência. Muito frequentemente somos confrontados com equações que foram formuladas pela observação da natureza durante a resolução de problemas relacionados aos fenómenos que ocorrem na vida real. Em ciências atuariais formulamos modelos de risco, a fim de resolver problemas que surgem na prática atuarial dos seguros.

Em muitas ocasiões, durante a análise de tais modelos, encontramos a equação de Lundberg. Há uma razão para isso: quando estudamos algumas quantidades específicas, como a probabilidade de ruína, chegamos frequentemente a equações integro–diferenciais. Tais equações têm associado algum tipo de equação característica a que chamamos equação de Lundberg.

Neste tese, consideramos o modelo de risco Sparre–Andersen, com três diferentes distribuições de tempo: Erlang(n), Erlang(n) generalizada e Phase–Type(n). Para cada um destes casos, a equação de Lundberg é diferente e, consequentemente, é analisada de um modo único.

Depois, para cada distribuição, estudamos alguns dos mais importantes tópicos de interesse neste modelo, como a ruína e as probabilidades de sobrevivência, a probabilidade de atingir uma barreira superior antes da ruína, a gravidade máxima da ruína e os dividendos descontados esperados. O objetivo desta tese é fornecer novas ferramentas para calcular essas quantidades e uma melhor compreensão delas na prática.

PALAVRAS-CHAVE: modelo de risco Sparre–Andersen; distribuição Erlang(n); distribuição Erlang(n) generalizada; distribuição Phase–Type(n); equação fundamental de Lundberg; equação generalizada de Lundberg; probabilidade de ruína; probabilidade de atingir uma barreira superior antes da ruína; a gravidade máxima da ruína; dividendos descontados esperados antes da ruína.

Abstract

In this dissertation we work with risk theory with particular emphasis on two major topics in the field, namely risk models and ruin theory. Solving equations is a fundamental part of mathematics and of almost any other science. Very often we come across equations that were formulated by observation of the nature, during solving problems related to phenomena that occur in life. In actuarial science, we formulate risk models in order to solve problems that appear in the practice of the insurance business.

Lundberg's equations shows up on many occasions when analyzing such models. There is a reason for this: when we study some particular quantities like, for example, the ruin probability, we often arrive to integro–differential equations. Such integro–differential equations have associated some kind of characteristic equation. The latter is often called the Lundberg's equation.

In this manuscript we consider the Sparre–Andersen risk model with three different interclaim times distributions: Erlang(n), generalized Erlang(n) and Phase–Type(n). For each of these cases the Lundberg's equation is different and therefore it is analyzed in a unique way.

Afterwards, for each distribution, we study some of the most important topics of interest, like the ruin and survival probabilities, the probability of attaining and upper barrier prior to ruin, the maximum severity of ruin and the expected discounted dividends. The aim of this thesis is to provide new tools for computation of those quantities and a better understanding of them in the practice. In the process we give examples to illustrate those methods.

KEYWORDS: Sparre–Andersen risk model; Erlang(n) distribution; generalized Erlang(n) distribution; Phase–Type(n) distribution; fundamental Lundberg's equation; generalized Lundberg's equation; ruin probability; probability of reaching an upper barrier; maximum severity of ruin; expected discounted dividends prior to ruin.

List of Symbols

$B(\mathcal{D})$	differential operator
$C.D.F.$	cumulative distribution function
$J(z;u)$	distribution of the maximum severity of ruin
$L.T.$	Laplace transform
M_u	maximum severity of ruin
$N(t)$	counting process
R	adjustment coefficient
$S(t)$	aggregate claims process
T_r	an operator w.r.t. a complex number r
T_u	time of ruin
$V(u,b)$	expected present value of discounted dividends
$V_m(u,b)$	m–th moment of discounted dividends
W_i	interclaim time i
X_i	claim amount i
$\Phi(u)$	survival probability
$\Psi(u)$	ruin probability
$\chi(u,b)$	probability of attaining the barrier level b from initial surplus u
$\hat{h}(s)$	Laplace transform of h
$\mathbb{I}(A)$	the indicator function of the set A
$\mathcal{D}$	differentiation with respect to u
$\mathcal{I}$	identity operator
c	premium income
$i.i.d.$	independent and identically distributed
$k(t)$	density of interclaim times
$p(x)$	density of claim amounts
u	initial capital

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Chapter 1

Introduction

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is
>
> John von Neumann

Risk theory is a field of mathematics which has its origins in the beginning of twentieth century, when fundamental ideas were published by Lundberg (1903). Risk theory is a synonym for non–life insurance mathematics, which deals with the modeling of claims that arrive in an insurance business and which give insight on how much a premium has to be charged in order to avoid bankruptcy (ruin) of an insurance company. It is based on probability theory, statistics, stochastic processes, renewal theory, functional analysis and optimization theory, and investigates fluctuations shown by incoming claims at an insurance company. Claims are amounts of money to be paid to policy holders, and they are thought to be of uncertain size and coming at uncertain future instants. One of the Lundberg's main contributions is the introduction of a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. By this we mean a portfolio of *contracts* or *policies* for similar risks such as car insurance for a particular kind of car, insurance against theft in households or insurance against water damage of family homes. There are three assumptions in the model:

- Claims happen at the times T_i satisfying $0 \leq T_1 \leq T_2 \leq \cdots$. We call them claim arrivals, claim times or claim arrival times.

- The i–th claim arriving at time T_i causes the claim size or claim severity X_i. The sequence $\{X_i\}$ constitutes an *i.i.d.* sequence of non–negative random variables.

- The claim size process $\{X_i\}$ and the claim arrival process $\{T_i\}$ are mutually independent.

Moreover, Risk theory has been an active research area in Actuarial Science since the 20th century. The heart of risk theory is ruin theory, which discusses how an insurance portfolio may be expected to vary with time. Ruin is said to occur if the insurer's surplus drops under a specific lower bound. The probability that ruins occurs, commonly referred as the ruin probability, is a very important measure of risk.

Much of the literature on ruin theory is concentrated on the classical risk theory, where an insurer starts with an initial surplus u, collects premiums continuously at a constant rate of c, while the aggregate claim process follows a compound Poisson process. The main research interest is the calculation of finite and infinite time ruin probabilities. Later on, actuarial researchers considered more components related to the time of ruin, like the surplus prior to ruin, the severity or deficit at ruin and its maximum, the probability of attaining a given upper barrier before ruin and the expected discounted dividends. Many results involving those quantities have been found during the recent years.

Gerber and Shiu (1998) considers the evaluation of the expected discounted penalty function, giving a unified treatment to the surplus before ruin, the deficit at ruin and the time to ruin.

Great part of the results in the classical risk model, like the results of Gerber *et al.* (1987), Dufresne and Gerber (1988a), Dufresne and Gerber (1988b), Dickson (1992) and Dickson and Egídio dos Reis (1996), are obtained as particular cases when the discount factor is zero, and almost all the previous results in classical ruin theory can be extended to the case with a positive discounting factor.

Lin and Willmot (1999) proposed an approach to solve the defective renewal equation, in which the discounted penalty function is expressed in terms of a compound geometric tail. Lin and Willmot (2000) further used it to derive the moments of the surplus before ruin, the deficit at ruin and the time of ruin.

During the last decades there have been a great interest in more general surplus processes, like surplus models with stochastic premium income processes, classical surplus processes under an economical environment (investment and inflation), surplus processes with dependent claim amounts and claim inter–occurrence times, surplus processes in which aggregate claims come from some classes of dependent or independent businesses, surplus processes with general claim number processes, or classical risk models perturbed by an independent diffusion process.

Sparre-Andersen (1957) in a paper to the International Congress of Actuaries in New York proposed a generalization of the classical (Poisson) risk theory, instead of assuming just exponentially distributed independent inter–occurrence (interclaim) times, he introduced a more general distribution function but retained the assumption of independence. He let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then the Sparre–Andersen model has been studied by many authors. In addition, random walks and queueing theory have provided a more general framework, which has led to explicit results in the case where the waiting times or the claim severities have distributions related to the Erlang (see Borovkov (1976)).

Malinovskii (1998) gives the Laplace transform of the non-ruin probability as a function of a finite time t, if claim sizes are exponentially distributed with parameter α, and waiting times have a general distribution k. Wang and Liu (2002) extends the result to claim sizes that are mixture of two exponential distributions.

Dickson (1998) and Dickson and Hipp (1998) showed how methods that are applied to derive results for the classical risk process can be adapted to derive results for a class of risk process in which the claims occur as a renewal process, namely as an Erlang(2) process. Dickson and Hipp (2001) considered a Sparre–Andersen risk process for which the interclaim times distribution is Erlang(2) with the purpose to find expressions for moments of the time of ruin, given that ruin occurs. They obtain an explicit expression for the Laplace transform of the ruin probability by solving a second order integro–differential equation. More recently, Cheng and Tang (2003) complements the work of Dickson and Hipp (2001), discussing the moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process.

Li and Garrido (2004b) extended the Erlang(2) risk model to Erlang(n) for any integer n. They studied the joint distribution of the time of ruin, the surplus just before ruin and the deficit at ruin and proved that the expected discounted penalty function satisfies an n-th order integro–differential equation. The latter can be solved to obtain a defective renewal equation. Li and Garrido (2004a), Gerber and Shiu (2005), Gerber and Shiu (2003a) and Gerber and Shiu (2003b) further extend the Erlang risk models to generalized Erlangs, in which interclaim times are distributed as the sum of n independent exponential random variables with possible different means.

Li and Dickson (2006) studied the maximum surplus before ruin in a Sparre–Andersen risk process with the inter–claim times being Erlang(n) distributed. The distribution was analyzed through the probability that the surplus process attains a given level from the initial surplus without first going to ruin. It was shown that this probability, viewed as a function of the initial surplus and the given level, satisfied a homogeneous integro–differential equation with certain boundary conditions. Its solution was expressed as a linear combination of n linearly independent particular solutions of the homogeneous integro–differential equation.

Li (2008), continuing this approach, studied the distributions of the maximum surplus before ruin and the maximum severity of ruin for an Erlang(n) risk model, and showed a method to find the particular solutions of the homogeneous integro–differential equation using the roots of the generalized Lundberg's equation.

Dickson and Waters (2004) considered a surplus process modified by the introduction of a constant dividend barrier, which was originally proposed by de Finetti (1957), and extended some results relating to the distribution of the present value of dividend payments until ruin in the classical risk model and show how a discrete time model can be used to provide approximations when analytic results are not available. Later on Albrecher *et al.* (2005) continued the work on dividends for a Sparre–Andersen risk model with generalized Erlang(n) distributed interclaim times and a constant dividend barrier.

Some authors have studied the Sparre–Andersen risk model with Phase–Type interclaim times. Ren (2007) studied this risk model deriving a matrix form expression for the discounted joint density of the surplus prior to ruin and the deficit at ruin when the initial surplus is zero. Li (2008b) analyzed some quantities like the Laplace transform of the recovery time after ruin, the probability that the surplus attains a certain level before ruin and the distribution of the maximum severity of ruin. Ji and Zhang (2011) analyzed the role of the distinct roots of the fundamental Lundberg's equation in the right half of the complex plane and the linear independence of the eigenvectors related to the Lundberg's matrix.

Willmot (1999) considers the ruin probabilities for renewal risk processes where the waiting times have a K_n distribution, for which the associated Laplace–Stieltjes transform is the ratio of a polynomial of degree $m < n$ to a polynomial of degree n. This general class of distributions includes, as special cases, Erlang and Phase–Type distributions, as well as combinations of these.

Stanford *et al.* (2000) presents a recursive method of calculating ruin probabilities for non–Poisson risk processes, by looking at the surplus process embedded at claim instants, in which interclaim times are assumed to be mixtures of exponential and Erlang(n) distributions.

Dufresne (2001) derives the Laplace transform of the integral equation given by Sparre–Andersen, producing the Laplace transform of the non–ruin probability for the wide class of waiting times or severity distributions that admit a rational Laplace transform representation. Lima *et al.* (2002) uses Fourier/Laplace transforms to evaluate numerically quantities of interest in classical and Erlang(2) ruin theory.

Albrecher and Boxma (2005), based on the analysis of the discounted penalty function in a semi–Markovian risk model by means of Laplace–Stieltjes transforms, derived and extended some results in the field. Li and Lu (2007) obtained some results in the dividend payments prior to ruin in a Markov–modulated risk model in which the rate for the Poisson claim arrival process and the distribution of the claim sizes vary in time depending on the state of an underlying (external) Markov jump process.

Outlining this dissertation we start presenting developments in the Sparre–Andersen model for Erlang(n), generalized Erlang(n) interclaim times and PH(n) interclaim times. Our work involves developments in Lundberg's equations, where we search for the possibility of multiple roots. For the models mentioned above, we also present new results in the maximum severity of ruin and expected present value of dividends payable to shareholders prior to ruin.

Chapter 2 reviews the relevant results and techniques in the literature on the classical risk model and the Sparre–Andersen risk model and gives the mathematical preliminaries to the thesis.

In Chapter 3 we consider developments in the Erlang(n) model presenting new theorems regarding the calculation of the maximum severity of ruin as well as a new way of computing expected present value of dividends payable to shareholders prior to ruin.

Chapter 4 is a step forward and studies generalized Erlang(n) interclaim times which are a more general case of the Erlang(n) risk model. We follow the same procedure of Chapter 3 keeping in mind the possibility of multiple roots in the Lundberg's equation.

Chapter 5 deals with the most general case which is PH(n) model. Here we concentrate on analyzing the Lundberg's equation and its roots, the calculation of the survival probability and applications to obtain the maximum severity of ruin.

Chapters 3 to 5 are the main core of this thesis where new developments are presented. Numerical examples are discussed in the parts of the work related to the maximum severity of ruin and dividends.

Finally, some conclusions and comments on further research are set out in Chapter 6.

This thesis is based on the following papers:

1. In Bergel and Egídio dos Reis (2011) we study the maximum severity of ruin in the Erlang(n) risk model. We pay special attention to the multiplicity of the roots of the fundamental Lundberg's equation.

2. In Bergel and Egídio dos Reis (2013a) we consider the Sparre–Andersen risk model when the interclaim times are Erlang(n) distributed. We first address the problem of solving an integro-differential equation that is satisfied by the survival probability and other probabilities, and show an alternative and improved method to solve such equations to that presented by Li (2008). This is done by considering the roots with positive real parts of the generalized Lundberg's equation, and establishing a one–one relation between them and the solutions of the integro–differential equation mentioned before. Afterwards, we apply our findings above in the computation of the distribution of the maximum severity of ruin. This computation depends on the non-ruin probability and on the roots of the fundamental Lundberg's equation. We illustrate and give explicit formulae for Erlang(3) interclaim arrivals with exponentially distributed single claim amounts and Erlang(2) interclaim times with Erlang(2) claim amounts. Finally, considering an interest force, we consider the problem of calculating the expected discounted dividends. Numerical examples are also provided for illustration.

3. In Bergel and Egídio dos Reis (2013c) we propose some new approaches in the Sparre–Andersen risk model when the interclaim times are generalized Erlang(n) distributed. We continued our previous work in Bergel and Egídio dos Reis (2013a), this time considering the cases when all the roots with positive real parts of the Lundberg's equation are single and when there are roots with higher multiplicity. We apply our findings above in the computation of the distribution of the maximum severity of ruin taking into account the cases with multiple roots. Given an interest force, we study the expected discounted dividends prior to ruin, showing an alternative method to that provided by Albrecher *et al.* (2005) for general claim amount distributions.

4. In Bergel and Egídio dos Reis (2013b) we deal with the Sparre–Andersen risk model when the interclaim times follow a Phase–Type distribution, PH(n). First of all we focus our attention on the generalized Lundberg's equation to determine the cases when multiple roots can arise, especially the possibility of double roots. Second, we study the linear independence of the eigenvectors related to the Lundberg's matrix. Afterwards, considering the survival probability we find an integro–differential equation and defective renewal equation. With this equations we obtain expressions for the Laplace transform of the survival probability. Finally, we apply our results to compute the ultimate and finite time ruin probabilities, the probability of arrival to a barrier prior to ruin, severity of ruin and its maximum.

Chapter 2

Risk models and ruin probability

> Obvious is the most dangerous
> word in mathematics
>
> Eric Temple Bell

In this chapter we set out the models and main concepts of risk theory. We use the term “risk” for describing a collection of similar policies, however we also use the term for an individual policy. At the start of a period of insurance cover the insurer does not know how many claims will occur, and if claims occur, what the amounts of these claims will be. It is necessary to talk about a model that takes into account these uncertainties.

In Section 2.1 we describe the collective risk model and denote the aggregate claims as a random variables S. After that we consider the special case when the distribution of S is a compound Poisson random variable.

Section 2.2 considers a risk process known as the Sparre–Andersen renewal risk process, and we introduce some definitions for ruin probabilities, Lundberg’s equation, adjustment coefficient, survival probability and Laplace transforms.

In Section 2.3 we start with useful results concerning the probability that ruin occurs without the surplus process first attaining a specified level. Then we consider the insurer’s deficit when ruin occurs and show the distribution of this deficit. We extend this by considering the insurer’s largest deficit before the surplus process recovers to level zero. After that we talk about the distribution of the time to ruin.

Finally, in Section 2.4 we devote our attention to the problem of calculating the expected discounted dividends.

Many of the definitions and notation of this chapter are taken from Dickson (2005).

2.1 The collective risk model

We define the random variable S to be the aggregate amount of claims for a reference period of time. Let the random variable N denote the number of claims from the risk for the same period, and let the random variable X_i denote the amount of the i–th claim. The aggregate claim amount is the sum of individual claim amounts, so we can write

$$S = \sum_{i=1}^{N} X_i$$

with the assumption that $S = 0$ when $N = 0$. (If there are no claims, the aggregate claims amount are trivially zero).

Our model has claim amounts as non-negative random variables with a positive mean. In this moment we do two important assumptions. First, we consider X_i as a sequence of independent and identically distributed random variables, and, second, we assume that the random variable N is independent of $\{X_i\}_{i=1}^{\infty}$.

These assumption say that the amount of any claim does not depend on the amount of any other claim, and that the distribution of the single claim amounts does not change. They also state that the number of claims has no effect on the amount of claims. Let $P(x) = Pr(X_1 \leq x)$ denote the distribution function of individual claim amounts, $p(x)$ its density and $\mu_k = E[X_1^k]$ for $k = 1, 2, \ldots$ We further assume that $P(0) = 0$, so that all claim amounts are positive. The existence of μ_1 is a basic assumption, higher moments may be required to exist in some parts of this work.

Our risk is a portfolio of insurance policies, and the name collective risk model arises from the fact that we consider the risk as a whole. In particular we are counting the number of claims from the portfolio, and not from individual policies.

When N has a Poisson distribution with parameter λ, we say that S has a compound Poisson distribution with parameters λ and P, and similar terminology applies in the case of other claim number distribution. Since the mean and variance of the $Poisson(\lambda)$ distribution are both λ, then when S has a compound Poisson distribution, with

$$E[S] = \lambda m_1$$

and

$$V[S] = \lambda m_2.$$

Further, the third central moment is

$$E[S] = [(S - \lambda m_1)^3] = \lambda m_3.$$

2.2 Sparre–Andersen risk processes

In a Sparre–Andersen risk process, an insurer's surplus at a fixed time $t > 0$ is determined by three quantities: the amount of surplus at time 0, the amount of premium income received up to time t and the amount paid out in claims up to time t. The only one of these three which is random is the claims outgo, so we start by describing the aggregate claims process, which we denote by $\{S(t)\}_{t\geq 0}$.

Let $\{N(t)\}_{t\geq 0}$ be a counting process for the number of claims, so that for a fixed value $t > 0$, the random variable $N(t)$ denotes the number of claims that occur in the fixed time interval $(0, t]$.

Like before, individual claim amounts are modeled as a sequence of independent and identically distributed random variables $\{X_i\}_{i=1}^{\infty}$, so that X_i denotes the amount of the i–th claim, with cumulative distribution function $P(x)$ and density $p(x)$.

Let the claim inter–occurrence times, or interclaim times, be denoted by the sequence of random variables $\{W_i\}_{i=1}^{\infty}$, that we assume i.i.d. and independent from sequence $\{X_i\}$. Then we have $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$. The cumulative distribution function of the W_i is denoted by $K(t)$ with density $k(t)$.

We say that the aggregate claim amount up to time t, denoted $S(t)$, is

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

when $N(t) = 0$ than $S(t) = 0$.

In the classical risk model it is assumed that $\{N(t)\}_{t\geq 0}$ is a Poisson process and therefore the interclaim times are exponentially distributed. In this case the aggregate claims process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process.

In the Sparre–Andersen renewal risk model the distribution of the interclaim times is not necessarily exponential, and there are no methods to determine the nature of the counting process $\{N(t)\}_{t\geq 0}$ for every possible distribution. For the total claim amount $S(t)$ the expectation can be easily calculated by exploiting the independence of $\{X_i\}$ and $N(t)$, provided $E[N(t)]$ and $E[X_1]$ are finite

$$E[S(t)] = E\left[E\left(\sum_{i=1}^{N(t)} X_i \,|N(t)\right)\right] = E[N(t)E[X_1]] = E[N(t)]E[X_1].$$

The expectation does not tell us too much about the distribution of $S(t)$. We learn more about the order of magnitude of $S(t)$ if we combine the information about $E[S(t)]$ with the variance $Var[S(t)]$.

Assume that $Var[N(t)]$ and $Var[X_1]$ are finite. Conditioning on $N(t)$ and exploiting the independence of $\{X_i\}$ and $N(t)$, we obtain

$$\begin{aligned} Var\left[\sum_{i=1}^{N(t)} (X_i - E[X_1])\,|N(t)\right] &= \sum_{i=1}^{N(t)} Var[X_i|N(t)] \\ &= N(t)Var[X_1|N(t)] = N(t)Var[X_1], \end{aligned}$$

and we can conclude that

$$\begin{aligned} Var[S(t)] &= E[N(t)Var[X_1]] + Var[N(t)E[X_1]] \\ &= E[N(t)]Var[X_1] + Var[N(t)](E[X_1])^2. \end{aligned}$$

Now we can describe the surplus process, denoted by $\{U(t)\}_{t\geq 0}$, as

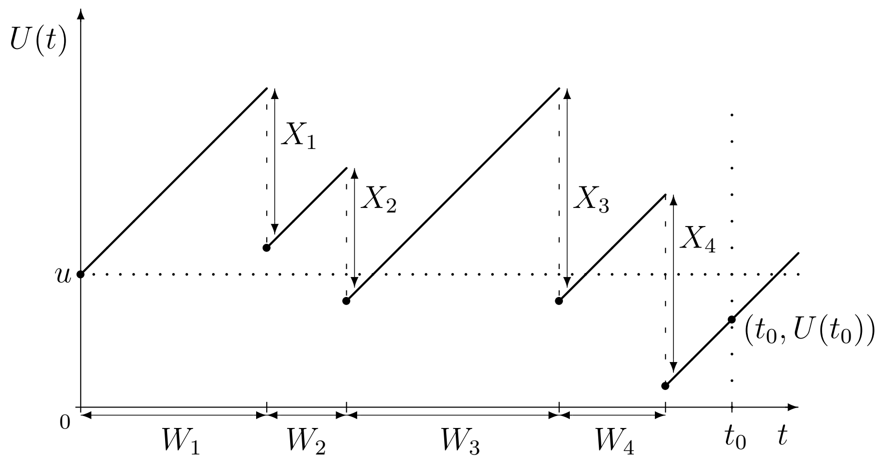
$$U(t) = u + ct - S(t)$$

where u is the insurer's surplus at time 0 and c is the insurer's rate of premium income per unit time, which we assume to be received continuously.

Whenever the moment generating function of X_1 exists, we denote it by M_X and we assume that when it exists, there exists some quantity γ, $0 < \gamma \leq \infty$, such that $M_X(r)$ is finite for all $r < \gamma$ with

$$\lim_{r\to\gamma^-} M_X(r) = \infty$$

Of course this model is a simplification of the reality. Some of the more important simplifications are that we assume that claims are settled in full as soon as they occur, there is no allowance for interest on the insurer's surplus,



$$\mathbf{U(t_0) = u + ct_0 - X_1 - X_2 - X_3 - X_4}, \quad \mathbf{N(t_0) = 4}$$

Figure 2.1: The surplus process

and there is no mention of expenses that an insurer may incur. Nevertheless, this is a useful model which can give us some insight into the stochastic characteristics of an insurance operation. A graphical interpretation of the surplus process is given in Figure 2.1, where we see how the surplus process starts from the initial capital u at time $t = 0$, then grows at the constant rate of premium c paid by the insureds until the time W_1 when the first claim arrives, and continues over time. By the time t_0 the surplus process already had 4 incurred (and settled) claims, so the counting process is equal to 4.

2.3 Some basic definitions

In this moment we introduce the most common equations and definitions for the model. Specifically we talk about the Lundberg's equation, the ruin, survival probabilities and the Laplace transforms.

2.3.1 The ruin probability

The probability of ruin in infinite time, also known as the ultimate ruin probability, is defined as

$$\Psi(u) = Pr(U(t) \leq 0 \text{ for some } t > 0).$$

In words, $\Psi(u)$ is the probability that the insurer's surplus falls below zero at some time in the future, that is when claims outgo exceeds the initial surplus

plus premium income.

We denote the time to ruin, from initial surplus u, as the random variable T_u, so we have $T_u = \inf\{t > 0 : U(t) < 0\}$, $u \geq 0$, and $T_u = \infty$ if and only if $U(t) \geq 0 \ \forall t > 0$. Therefore, we can express the ruin probability as

$$\Psi(u) = Pr(T_u < \infty).$$

Define $\Phi(u) = 1 - \Psi(u)$ to be the probability that ruin never occurs starting from initial surplus u. This probability is also known as the survival or non–ruin probability.

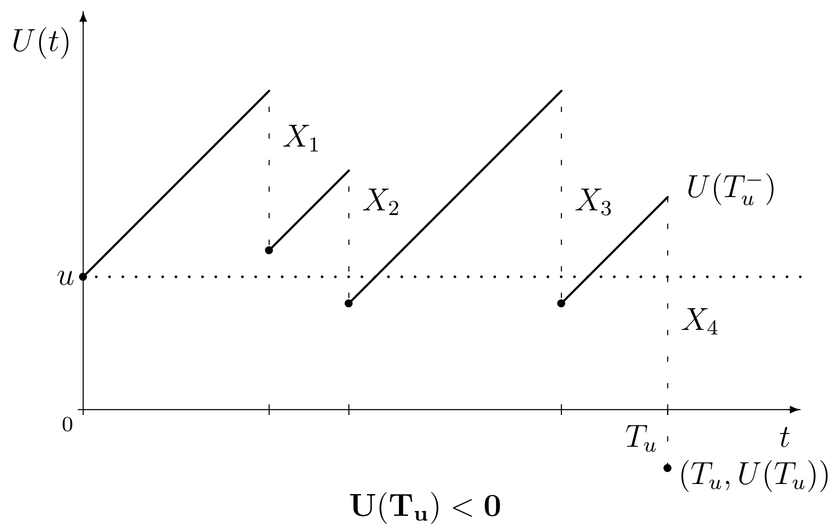


Figure 2.2: The time of ruin

Figure 2.2 represents the time of ruin. In this example we have a surplus process with 4 incurred claims. The surplus immediately prior to ruin, which we denote by $U(T_u^-)$, was smaller than the claim X_4 that arrived at time $t = T_u$.

We also assume the so called net profit condition

$$cE[W_i] > E[X_i], \tag{2.3.1}$$

which means $cE[W_1] > \mu_1$, so that, per unit of time, the premium income exceeds the expected aggregate claim amount. This condition is very important and it brings an economical sense to the model. If this condition does not hold, then $\Psi(u) = 1$ for all $u \geq 0$. It is often convenient to write $cE[W_1] = (1 + \theta)\mu_1$, so that $\theta > 0$ is the premium loading factor. During the interval of time W_i the net income is given by cW_i and the claim outgo is X_i. The net profit $cW_i - X_i$ might be positive or negative, but on average it has to be positive. We show this in the Figure 2.3.

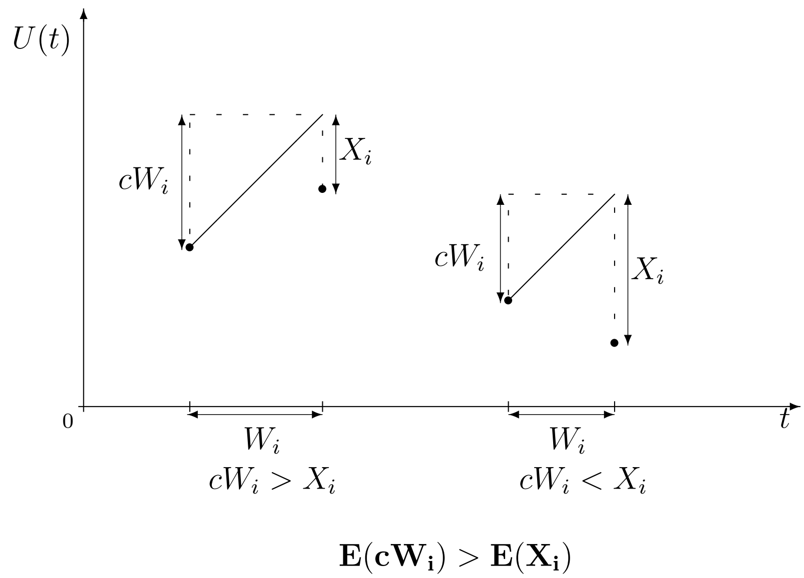


Figure 2.3: The net profit condition

2.3.2 The Lundberg's equation

The adjustment coefficient, which we denote by R, gives a measure of risk for a surplus process. It takes into account two factors in the surplus process: aggregate claims and premium income.

For the classical risk process, the adjustment coefficient is defined to be the unique positive root of

$$\lambda M_X(r) - \lambda - cr = 0,$$

where λ is the Poisson parameter and M_X denotes the moment generating function. Then R is given by

$$M_X(R) = 1 + \frac{c}{\lambda}R.$$

We remark that by writing c as $(1+\theta)\lambda M_X(R)$, we can see that R is independent of the parameter λ.

For a Sparre–Andersen renewal risk process, we also know well the notion of the adjustment coefficient, provided that the moment generating function

M_X of X_1 exists, see Mikosch (2006). The adjustment coefficient R is the unique positive real root of the equation, developed as follows

$$E\left[e^{-r(cW_1-X_1)}\right]=1 \quad\Leftrightarrow\quad E\left[e^{-rcW_1}\right]E\left[e^{rX_1}\right]=1$$
$$\Leftrightarrow\quad M_X(r)=\frac{1}{E\left[e^{-rcW_1}\right]}. \tag{2.3.2}$$

We note that expectation $E\left[e^{rX_1}\right]$ exists at least for $r<0$. The expectation $E\left[e^{-rcW_1}\right]$ can be seen as a Laplace transform. The lefthand side of the starting equation above can be regarded as the expected discounted profit for each *waiting arrival period*. So that the adjustment coefficient R, provided that it exists, makes the expected discounted profit even (considering that premium income and claim costs come together). The constant R can be seen as an interest force. The equation (2.3.2) is known as the fundamental Lundberg's equation.

One of the most important characteristics of the adjustment coefficient is that it provides an upper bound for the ruin probability.

For a Sparre–Andersen renewal risk model with net profit condition (2.3.1) and adjustment coefficient R, the following inequality holds, $\forall u>0$,

$$\Psi(u)\le e^{-Ru}.$$

This inequality is known as the Lundberg's inequality.

For practical purposes we find the Lundberg's equation in the literature written in a different way. We make a change of variable $s=-r$ and extend the domain for $s\in\mathbb{C}$. The advantage of this change is that we get $M_X(r)=M_X(-s)=\hat{p}(s)$, where $\hat{p}$ denotes the Laplace transform of the density p.

Then, the fundamental Lundberg's equation becomes

$$\hat{p}(s)=\frac{1}{E\left[e^{scW_1}\right]}. \tag{2.3.3}$$

We could even go further with this notation and write the fundamental Lundberg's equation as

$$\hat{p}(s)=\frac{1}{\hat{k}(-cs)},\quad \text{or}\quad \hat{k}(-cs)\hat{p}(s)=1. \tag{2.3.4}$$

From now on every time when we refer to the fundamental Lundberg's equation we refer to the equation (2.3.4).

For the barrier and dividend problems that we treat later in this thesis, it is introduced the notion of an interest rate, denoted by $\delta\ge 0$. For such a model with a barrier level, the corresponding Lundberg's equation is called the generalized Lundberg's equation and is defined as follows

$$\hat{p}(s) = \frac{1}{\hat{k}(\delta - cs)}, \quad \text{or} \quad \hat{k}(\delta - cs)\hat{p}(s) = 1. \tag{2.3.5}$$

This last equation can be found in Gerber and Shiu (2005) and Ren (2007).

2.3.3 Laplace transforms

The Laplace transform is an important tool that can be used to solve both differential and integro–differential equations. We will list some of the basic properties of the Laplace transforms.

Let $h(y)$ be a function defined for all $y \geq 0$. Then the Laplace transform of h is defined as

$$\hat{h}(s) = \int_0^\infty e^{-sy} h(y) dy, \quad s \in \mathbb{C}.$$

There are some technical conditions for the existence of $\hat{h}(s)$, but as this hold for our future purpose on this manuscript, we do not discuss them here.

An important property of a Laplace transform is that it uniquely identifies a function, in the same way that a moment generating function uniquely identifies a distribution. The process of going from $\hat{h}$ to h is known as inverting the transform.

The Laplace transform has the following properties:

1. Let h_1 and h_2 be functions whose Laplace transforms exist, and let α_1 and α_2 be constants. Then

$$\int_0^\infty e^{-sy}(\alpha_1 h_1(y) + \alpha_2 h_2(y))dy) = \alpha_1 \hat{h}_1(s) + \alpha_2 \hat{h}_2(s).$$

2. Laplace transform of an integral: let h be a function whose Laplace transform exists and let

$$H(x) = \int_0^x h(y) dy.$$

Then $\hat{H}(s) = \hat{h}(s)/s$.

3. Laplace transform of a derivative: let h be a differentiable function whose Laplace transforms exists. Then

$$\int_0^\infty e^{-sy} \left(\frac{d}{dy} h(y) \right) dy = s\hat{h}(s) - h(0).$$

4. Laplace transform of higher derivatives: let h be a m times differentiable function whose Laplace transforms exists. Then the Laplace transform of $h^{(m)}(y)$ is

$$\int_0^\infty e^{-sy} h^{(m)}(y)dy = s^m \hat{h}(s) - \sum_{i=0}^{m-1} s^{m-1-i} h^{(i)}(0).$$

See Spiegel (1965), page 10.

5. Laplace transform of a convolution: let h_1 and h_2 be as in 1. above, and define

$$h(x) = h_1 * h_2(x) = \int_0^x h_1(y)h_2(x-y)dy.$$

Then $\hat{h}(s) = \hat{h}_1(s)\hat{h}_2(s)$.

6. Laplace transform of a random variable: let $X \sim H$, where $H(0) = 0$. Then

$$E[e^{-sX}] = \int_0^\infty e^{-sy} dH(y).$$

When the distribution is continuous with density function h,

$$E[e^{-sX}] = \hat{h}(s).$$

2.3.4 The survival probability

In this section we define the Laplace transform of Φ and we list some basic properties. We then present general expression for the Laplace transform of Φ, and explain how Φ can be found from this expression. We show the different cases for the classical risk model and for the Sparre–Andersen risk model.

Classical risk process

Recall that in the classical risk model the interclaim times follow an exponential distribution. Let λ be the parameter.

By considering the time and the amount of the first claim, we have the following renewal equation for $\Phi(u)$

$$\Phi(u) = \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} p(x)\Phi(u+ct-x)dxdt, \qquad (2.3.6)$$

noting that if the first claim occurs at time t, its amount must not exceed $u + ct$, since ruin otherwise occurs.

A similar renewal equation can be obtained for the ruin probability $\Psi(u)$

$$\Psi(u) = \int_0^\infty \lambda e^{-\lambda t}\left[\int_0^{u+ct} p(x)\Psi(u+ct-x)dx + \int_{u+ct}^\infty p(x)dx\right]dt.$$

Substituting $s = u + ct$ in (2.3.6) we get

$$\begin{aligned}\Phi(u) &= \frac{1}{c}\int_u^\infty \lambda e^{-\lambda(s-u)/c}\int_0^s p(x)\Phi(s-x)dxds \\ &= \frac{\lambda}{c}e^{\lambda u/c}\int_u^\infty e^{-\lambda s/c}\int_0^s p(x)\Phi(s-x)dxdt. \qquad (2.3.7)\end{aligned}$$

We establish an equation for Φ, known as an integro–differential equation, by differentiating equation (2.3.7), and the resulting equation can be used to derive explicit solutions for Φ. Differentiation gives

$$\begin{aligned}\frac{d}{du}\Phi(u) &= \frac{\lambda^2}{c^2}e^{\lambda u/c}\int_u^\infty e^{-\lambda s/c}\int_0^s p(x)\Phi(s-x)dxds \\ &\quad -\frac{\lambda}{c}\int_0^u p(x)\Phi(u-x)dx \\ &= \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_0^u p(x)\Phi(u-x)dx. \qquad (2.3.8)\end{aligned}$$

We notice that the function Φ appears in three different places in this equation. However, by eliminating the integral term, a differential equation can be created, and solved.

The properties of the Laplace transforms that we gave before can be applied to find the Laplace transform of Φ. Recall equation (2.3.8)

$$\frac{d}{du}\Phi(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_0^u p(x)\Phi(u-x)dx.$$

From Property 3, the Laplace transform of the left-hand side is $s\Phi(s) - \Phi(0)$, and from the properties 1. and 5. the Laplace transform of the second term on the right hand side is $-(\frac{\lambda}{c})\hat{p}(s)\hat{\Phi}(s)$. Hence, we have

$$s\hat{\Phi}(s) - \Phi(0) = \frac{\lambda}{c}\hat{\Phi}(s) - \frac{\lambda}{c}\hat{p}(s)\hat{\Phi}(s),$$

or

$$\hat{\Phi}(s) = \frac{c\Phi(0)}{cs - \lambda(1-\hat{p}(s))} = \frac{-\Phi(0)}{(\frac{\lambda}{c} - s) - \hat{p}(s)}.$$

When $\hat{p}$ is a rational function we can invert $\hat{\Phi}$ to find Φ. Notice that the zeros of the denominator in this expression are the roots of the fundamental Lundberg's equation in the classical risk model. We see later on that this is the case for the Sparre–Andersen model as well.

Sparre–Andersen risk process

We recall that in the Sparre–Andersen model the interclaim times follow a distribution K with density k.

Again, by considering the time and the amount of the first claim, we have the following renewal equations for $\Phi(u)$ and $\Psi(u)$

$$\Phi(u) = \int_0^\infty k(t) \int_0^{u+ct} p(x)\Phi(u+ct-x)dxdt, \tag{2.3.9}$$

$$\begin{aligned} \Psi(u) &= \int_0^\infty k(t) \left[\int_0^{u+ct} p(x)\Psi(u+ct-x)dx \right. \\ &\quad \left. + \int_{u+ct}^\infty p(x)dx \right] dt. \end{aligned} \tag{2.3.10}$$

Replacing $s = u + ct$ in (2.3.9) we get

$$\Phi(u) = \frac{1}{c} \int_u^\infty k\left(\frac{s-u}{c}\right) \int_0^s p(x)\Phi(s-x)dxds. \tag{2.3.11}$$

Depending on the properties of the density function $k(t)$ we follow a method to obtain the Laplace transform of $\Phi(u)$. For the interclaim times distributions that we consider in the following chapters we get an expression of the form

$$\hat{\Phi}(s) = \frac{d_\Phi(s)}{Q(s)},$$

where $d_\Phi(s)$ is a polynomial on s with coefficients that depend on the values of $\Phi(0)$ and the derivatives of Φ at zero, and the zeros of $Q(s)$ are the roots of the fundamental Lundberg's equation. There is no general expression for $Q(s)$, it can only be described when we specify the interclaim times distribution. We will return to this point in the following chapters.

2.4 A barrier problem, severity of ruin

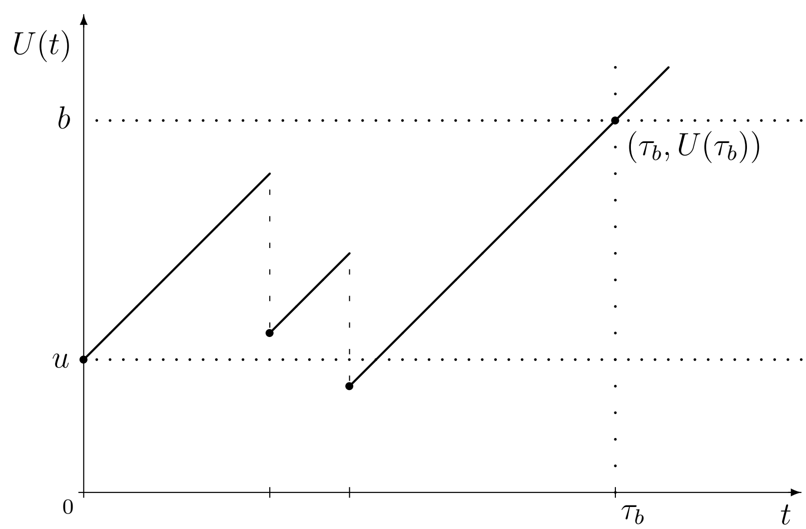
In this section we introduce definition treaties which are standard for the model. We give a brief description of the probability of attaining a given upper level, the maximum severity of ruin and the Laplace transform of the time to ruin.

2.4.1 The probability of attaining a given level

We consider the following question: what is the probability that ruin occurs from initial surplus u without the surplus process reaching level $b > u$ prior to ruin?

We denote this probability by $\xi(u, b)$, and let $\chi(u, b)$ denote the probability that the surplus process attains the level b from initial surplus u without first falling below zero. To find expressions for $\xi(u, b)$ and $\chi(u, b)$, we consider the ruin and survival probabilities respectively in an unrestricted surplus process. Let the random variable τ_b be the time to attain the level b, where it is understood that $\tau_b = \infty$ if the surplus never reaches b. This is shown in Figure 2.4.

If survivals occurs from initial surplus u, then the surplus process must pass through the level $b > u$ at some point in time, as the net profit condition (2.3.1) implies that $U(t) \to \infty$ as $t \to \infty$ with probability one (almost surely).



$\mathbf{U(\tau_b) = b}$

Figure 2.4: The time to attain the level b

In the classical risk model, as the distribution of the time to the next claim from the time the surplus attains b is exponential, the probabilistic behavior of the surplus process once it attains level b is independent of its behavior prior to attaining b. Hence $\Phi(u) = \chi(u,b)\Phi(b)$ or equivalently,

$$\chi(u,b) = \frac{1-\Psi(u)}{1-\Psi(b)}.$$

Similarly, if ruin occurs from initial surplus u, then either the surplus process does or does not attain level b prior to ruin. Hence

$$\Psi(u) = \xi(u,b) + \chi(u,b)\Psi(b),$$

so that

$$\xi(u,b) = \Psi(u) - \frac{1-\Psi(u)}{1-\Psi(b)}\Psi(b) = \frac{\Psi(u)-\Psi(b)}{1-\Psi(b)}.$$

The reason for this is the memoryless property of the exponential distribution.

In a Sparre–Andersen risk model, we no longer have that memoryless property, so the approach to find $\xi(u,b)$ and $\chi(u,b)$ is different.

Considering the amount and the time of arrival of the first claim, we get renewal equations for $\chi(u,b)$ and $\xi(u,b)$ that resemble those we had before for $\Phi(u)$ and $\Psi(u)$

$$\chi(u,b) = \int_0^{\frac{b-u}{c}} k(t) \int_0^{u+ct} p(x)\chi(u+ct-x,b)dxdt + \int_{\frac{b-u}{c}}^{\infty} k(t)dt, \quad (2.4.1)$$

and

$$\begin{aligned} \xi(u,b) &= \int_0^{\frac{b-u}{c}} k(t) \left[\int_0^{u+ct} p(x)\xi(u+ct-x,b)dx \right. \\ &\quad \left. + \int_{u+ct}^{\infty} p(x)dx \right] dt. \end{aligned} \quad (2.4.2)$$

Then it is clear that

$$\lim_{b\to\infty} \chi(u,b) = \Phi(u), \quad \text{and} \quad \lim_{b\to\infty} \xi(u,b) = \Psi(u).$$

Note that $\xi(u,b) + \chi(u,b) = 1$, so that eventually either ruin occurs without the surplus process ever attaining b or the surplus process first attains level b.

In the following chapters we present computations of $\chi(u, b)$ by solving an integro–differential equation that it satisfies, which in turn depends on the nature of $k(t)$.

2.4.2 The severity of ruin and its maximum

In this section we are interested not just in the probability of ruin, but also in the amount of the insurer's deficit at the time of ruin, if ruin occurs.

Given an initial surplus u, recall that we denoted the time to ruin from initial surplus u by T_u. Define

$$G(u, y) = Pr(T_u < \infty \text{ and } U(T_u) \geq -y),$$

to be the probability that ruin occurs and that the insurer's deficit at ruin, or severity of ruin, is at most y. We notice that

$$\lim_{y \to \infty} G(u, y) = \Psi(u),$$

so that

$$\frac{G(u, y)}{\Psi(u)} = Pr(|U(T_u)| \leq y)|T_u < \infty)$$

is proper distribution function. Hence for a given initial surplus u, $G(u, .)$ is a defective distribution with (defective) density

$$g(u, y) = \frac{\partial}{\partial y} G(u, y).$$

We now allow surplus process to continue if ruin occurs, and we consider the insurer's maximum severity of ruin from the time of ruin until the time that the surplus process next attains level 0. As we are assuming the net profit condition (2.3.1), it is certain that the surplus process will attain this level.

We define T'_u to be the time of the first upcrossing of the surplus process through level 0 after ruin occurs and define the random variable M_u by

$$M_u = \sup\{|U(t)|, T_u \leq t \leq T'_u\},$$

so that M_u denotes the maximum severity of ruin. This is shown in Figure 2.5.

Let

$$J(z; u) = Pr(M_u \leq z | T_u < \infty)$$

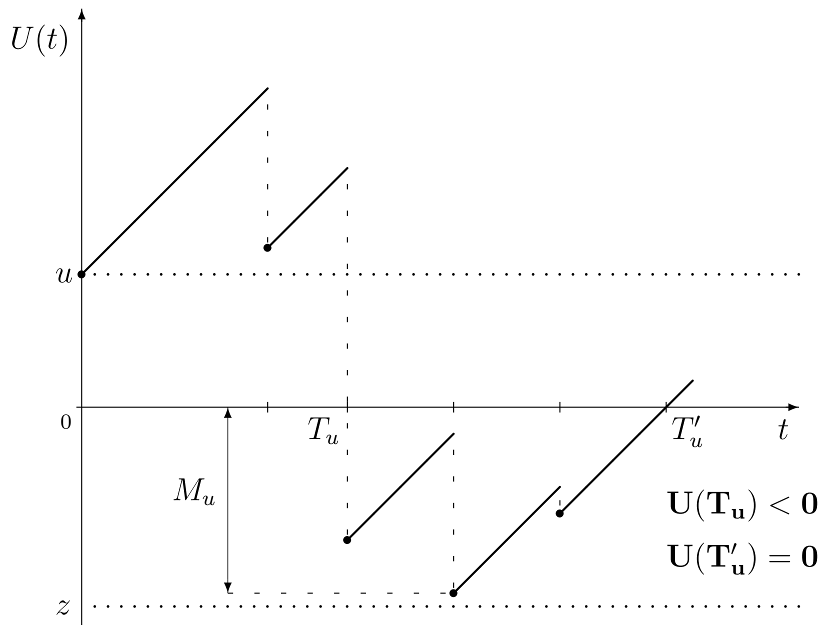


Figure 2.5: The maximum severity of ruin

to be the distribution function of M_u given that ruin occurs. The maximum severity of ruin is no more than z if ruin occurs with a deficit $y \leq z$ and if the surplus does not fall below $-z$ from the level $-y$. The probability of this latter event is $\chi(z-y, z)$ since attaining level 0 form level $-y$ without falling below $-z$ is equivalent to attaining level z from level $z-y$ without falling below 0.

Then

$$J(z; u) = \int_0^z \frac{g(u,y)}{\Psi(u)} \chi(z-y, z) dy. \tag{2.4.3}$$

In the classical risk model we have $\Phi(u) = \chi(u, b)\Phi(b)$ and therefore we evaluate (2.4.3) by nothing that

$$\Psi(u+z) = \int_z^\infty g(u,y)dy + \int_0^z g(u,y)\Psi(z-y)dy. \tag{2.4.4}$$

This follows by noting that if ruin occurs from initial surplus $u+z$, then the surplus process must fall below z at some time in the future.

By conditioning this event according to whether ruin occurs at the time of this fall, the probability of which is given by the first integral, or at a subsequent time, the probability of which is given by the second integral, we obtain equation (2.4.4) for $\Psi(u+z)$.

Noting that $\Psi = 1 - \Phi$ we can write equation (2.4.4) as

$$\begin{aligned}\int_0^z g(u,y)\Phi(z-y)dy &= \int_z^\infty g(u,y)dy + \int_0^z g(u,y)dy - \Psi(u+z)\\ &= \Psi(u) - \Psi(u+z).\end{aligned}$$

Thus,

$$J(z;u) = \frac{\Psi(u) - \Psi(u+z)}{\Psi(u)(1-\Psi(z))}$$

However, this method can not be performed in the same way in the Sparre–Andersen model since, once again, we used the memoryless property in the process.

In the Sparre–Andersen model we work on a method to find $J(z;u)$ in (2.4.3) that involves finding first the probability of attaining an upper barrier level $\chi(u,b)$.

2.4.3 The distribution of the time to ruin

Recall the random variable T_u denoting the time of ruin. The distribution of T_u is important since $Pr(T_u \leq t)$ gives the probability that ruin occurs at or before time t. If we know the distribution of T_u we are able to compute finite time ruin probabilities.

Define a function φ as

$$\varphi(u,\delta) = E[e^{-\delta T_u}\mathbb{I}(T_u < \infty)]$$

where δ is a non-negative parameter which we consider in this section as the parameter of a Laplace transform, and $\mathbb{I}$ is the indicator function, so that $\mathbb{I}(A) = 1$ if the event A occurs and equals 0 otherwise. Notice that

$$\lim_{\delta\to 0}\varphi(u,\delta) = E[\mathbb{I}(T_u < \infty)] = Pr[T_u < \infty] = \Psi(u).$$

In the next section of the expected discounted dividends we consider a function similar to φ, and in that function the interpretation of δ is that it is a force of interest. With this interpretation, $\varphi(u,\delta)$ gives the expected present value of 1 payable at the time of ruin.

We can derive a renewal equation for φ using the technique of conditioning on the time and the amount of the first claim.

In the classical risk model we have

$$\begin{aligned} \varphi(u,\delta) &= \int_0^\infty \lambda e^{-\lambda t} e^{-\delta t} \int_0^{u+ct} p(x)\varphi(u+ct-x,\delta)dxdt \\ &+ \int_0^\infty \lambda e^{-\lambda t} e^{-\delta t} \int_{u+ct}^\infty p(x)dxdt. \end{aligned} \tag{2.4.5}$$

Substituting $s = u + ct$ in equation (2.4.5) gives

$$\begin{aligned} \varphi(u,\delta) &= \frac{\lambda}{c}\int_u^\infty e^{-(\lambda+\delta)(s-u)/c} \int_0^s p(x)\varphi(s-x,\delta)dxds \\ &+ \frac{\lambda}{c}\int_u^\infty e^{-(\lambda+\delta)(s-u)/c} \int_s^\infty p(x)dxds. \end{aligned}$$

Afterwards, we differentiate this equation with respect to u to obtain an integro–differential equation

$$\frac{\partial}{\partial u}\varphi(u,\delta) = \frac{\lambda+\delta}{c}\varphi(u,\delta) - \frac{\lambda}{c}\int_0^u p(u-x)\varphi(x,\delta)dx - \frac{\lambda}{c}(1-P(u)).$$

In the Sparre–Andersen risk model the renewal equation is given by

$$\begin{aligned} \varphi(u,\delta) &= \int_0^\infty k(t) e^{-\delta t} \int_0^{u+ct} p(x)\varphi(u+ct-x,\delta)dxdt \\ &+ \int_0^\infty k(t) e^{-\delta t} \int_{u+ct}^\infty p(x)dxdt. \end{aligned} \tag{2.4.6}$$

Any further developments will depend on the specific characteristics of the density of $k(t)$.

2.5 Expected discounted dividends

We now consider a problem where an insurance portfolio is used to provide dividend income for the insurance company's shareholders. Specifically, let u denote the initial surplus and let $b \geq u$ be a dividend barrier.

Whenever the surplus attains the level b, the premium income is paid to shareholders as dividends until the next claim occurs, so that in this modified surplus process, the surplus never attains a level greater than b, see Figure 2.6.

It is straightforward to show that it is certain that ruin eventually occurs

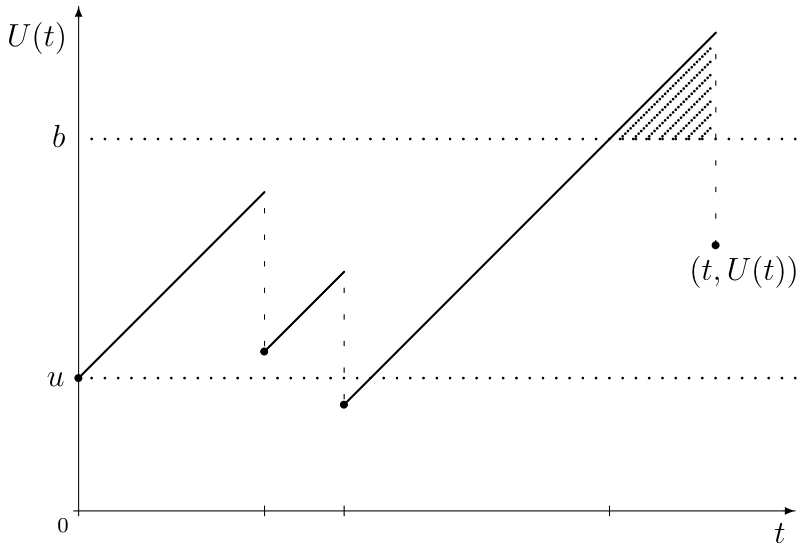


Figure 2.6: The expected discounted dividends

for the modified surplus process. We represent the modified surplus process in Figure 2.7.

Let us assume that the shareholders provide the initial surplus u and pay the deficit at ruin. An interesting question is how the level of the barrier b should be chosen to maximize the expected present value of net income to the shareholders, assuming that there is no further business after the time of ruin. Another interesting question is how the situation changes when we introduce capital injections after the time of ruin.

We define $V(u, b)$ to be the expected present value at force of interest δ of dividends payable to shareholders prior to ruin, $Y_{u,b}$ to be the deficit at ruin and $T_{u,b}$ to be the time of ruin, so that $E[Y_{u,b}\, e^{-\delta T_{u,b}}]$ gives the expected present value of the deficit at ruin. Then we want to choose b such that the following function

$$L(u, b) = V(u, b) - E[Y_{u,b}\, e^{-\delta T_{u,b}}] - u$$

is maximized, and to address this question we must consider the components of $L(u, b)$.

We can find an expression for $V(u, b)$ by the standard technique of conditioning on the time and the amount of the first claim. We note that for $u < b$, if no claim occurs before time $\tau = (b - u)/c$, then the surplus process attains level b at time τ.

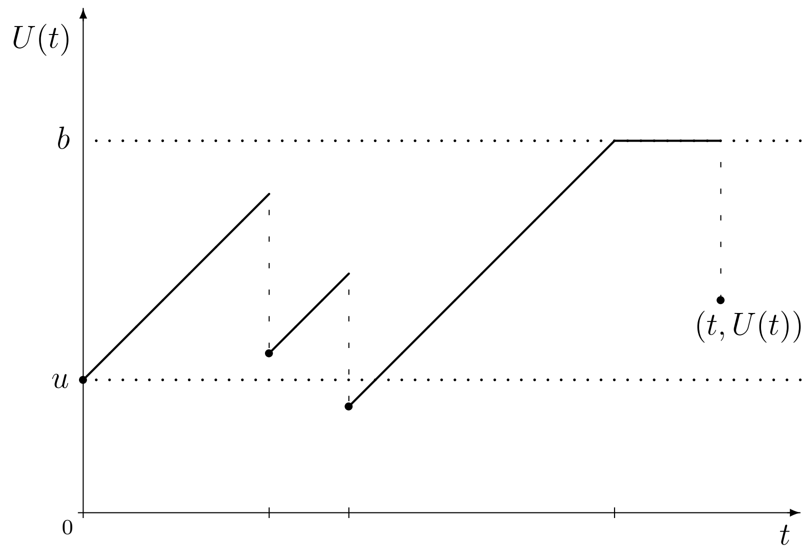


Figure 2.7: The modified surplus

In the classical risk model, for $0 \leq u < b$, the renewal equation is

$$V(u,b) = e^{-(\lambda+\delta)\tau} V(b,b) + \int_0^{\tau} \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct} p(x) V(u+ct-x,b)\,dx\,dt.$$

Substituting $s = u + ct$ we obtain

$$\begin{aligned} V(u,b) &= e^{-(\lambda+\delta)(b-u)/c} V(b,b) \\ &\quad + \frac{\lambda}{c} \int_u^b e^{-(\lambda+\delta)(s-u)/c} \int_0^s p(x) V(s-x,b)\,dx\,ds, \end{aligned}$$

and differentiating with respect to u we get

$$\frac{\partial}{\partial u} V(u,b) = \frac{\lambda+\delta}{c} V(u,b) - \frac{\lambda}{c} \int_0^u p(x) V(u-x,b)\,dx\,dt. \tag{2.5.1}$$

Similarly, by considering dividends payments before and after the first claim, we have

$$\begin{aligned} V(b,b) &= \int_0^{\infty} \lambda e^{-(\lambda+\delta)t} c\,\bar{s}_{\overline{t}|}\,dt \\ &\quad + \int_0^{\infty} \lambda e^{-(\lambda+\delta)t} \int_0^b p(x) V(b-x,b)\,dx\,dt, \end{aligned} \tag{2.5.2}$$

where $\bar{s}_{\overline{t}|} = (e^{\delta t} - 1)/\delta$ is the accumulated amount at time t at force of interest δ of payments at rate 1 per unit time over $(0,t)$.

Integrating out in equation (2.5.2) we obtain

$$V(b,b) = \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^b p(x)V(b-x,b)dx. \tag{2.5.3}$$

From equation (2.5.1) we find that

$$\frac{c}{\lambda+\delta}\,\frac{\partial}{\partial u}V(u,b)\bigg|_{u=b} = V(b,b) - \frac{\lambda}{\lambda+\delta}\int_0^b p(x)V(b-x,b)dx,$$

which, together with equation (2.5.3), gives the boundary condition

$$\frac{\partial}{\partial u}V(u,b)\bigg|_{u=b} = 1.$$

In the Sparre–Andersen risk model the renewal equation for $V(u,b)$ takes the form

$$\begin{aligned} V(u,b) &= \int_{\frac{b-u}{c}}^{\infty} k(t)e^{-\delta t}\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}} + \int_0^b p(x)V(b-x,b)dx\right)dt + \\ &\quad + \int_0^{\frac{b-u}{c}} e^{-\delta t}k(t)\int_0^{u+ct} V(u+ct-x,b)p(x)dx\,dt. \end{aligned} \tag{2.5.4}$$

Furthermore, let the random variable $D_{u,b}$ denote the present value at force of interest $\delta(>0)$ per unit time of dividends payable to shareholders until ruin occurs, and denote m–th moment as $V_m(u,b) = E[D_{u,b}^m],\ m \geq 0$, where $V_0(u,b) \equiv 1$. Then we have $V_1(u,b) = V(u,b)$.

A renewal equation for $V_m(u,b),\ m \geq 1$ is the following

$$\begin{aligned} V_m(u,b) &= \int_{\frac{b-u}{c}}^{\infty} k(t)e^{-m\delta t}\Bigg[\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}}\right)^m + \\ &\quad + \sum_{j=1}^{m}\binom{m}{j}\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}}\right)^{m-j}\int_0^b p(x)V_j(b-x,b)dx\Bigg]dt + \\ &\quad + \int_0^{\frac{b-u}{c}} e^{-m\delta t}k(t)\int_0^{u+ct} V_m(u+ct-x,b)p(x)dx\,dt. \end{aligned} \tag{2.5.5}$$

Depending on the claim amounts distribution $k(t)$, we obtain integro–differential equations together with boundary conditions from the renewal equations (2.5.4) and (2.5.5) that allow us to compute $V_m(u,b), m \geq 1$.

2.6 Final remarks

In this chapter we have learned about the most important quantities of interest in Risk Theory. We have taken a closer look at the renewal risk models, specifically the classical and the Sparre–Andersen risk model. Depending on the choice of the interclaim times distribution we have presented results and formulae. In the next chapter we will discuss how previously presented theory can be applied to deal with a Sparre–Andersen model with Erlang(n) distributed interclaim times.

Chapter 3

The Sparre–Andersen model with Erlang(n) interclaim times

The only way to learn mathematics
is to do mathematics

Paul Halmos

3.1 Introduction

In this chapter we study the Sparre–Andersen risk model under the assumption that the interclaim times are Erlang(n) distributed.

In Section 3.2 we investigate the fundamental and the generalized Lundberg's equations, take a closer look at the roots of these equations and assume the net profit condition. The most important result of this section say that the roots of the equations mentioned above are distinct.

In Section 3.3 we present results from Li and Dickson (2006) and Li (2008b) which are part of the mathematical foundation for the chapter. In particular we begin by describing the problem of finding the survival probability and the probability of attaining an upper barrier prior to ruin.

A new theorem for obtaining the latter is given in section 3.4.

We continue in Section 3.5 applying our theorem for the maximum severity of ruin.

In Section 3.6 we deal with numerical examples that are based on the results of the previous sections. We consider the cases of Erlang(3) distributed interclaim times with exponentially distributed single claim amounts and Erlang(2) interclaim times with Erlang(2) claim amounts.

In the last section of this chapter we focus on dividends, where numerical examples are provided.

3.2 The Lundberg's equation

The density of the interclaim time W_i, which we denote as $k_n(t)$,

$$k_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda > 0, n \in \mathbb{N}^+,$$

and its probability distribution function denoted as

$$K_n(t) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}.$$

The most important properties of this distribution are the following

$$\begin{aligned} \hat{k}_n(s) &= \left(\frac{\lambda}{\lambda + s}\right)^n, \\ k_n'(t) &= \lambda(k_{n-1}(t) - k_n(t)), \\ k_n^{(i)}(0) &= 0, \quad i = 0, \ldots, n-2, \\ k_n^{(n-1)}(0) &= \lambda^n, \end{aligned}$$

which we use later on this chapter.

The fundamental Lundberg's equation from the previous chapter (2.3.4) can be written in the form

$$\left(\frac{\lambda}{c} - s\right)^n = \left(\frac{\lambda}{c}\right)^n \hat{p}(s). \tag{3.2.1}$$

On the other hand we have the generalized Lundberg's equation (2.3.5) expressed as

$$\left(\frac{\lambda + \delta}{c} - s\right)^n = \left(\frac{\lambda}{c}\right)^n \hat{p}(s), \tag{3.2.2}$$

where $\delta > 0$ is the force of interest. We consider this equation in the section of dividends.

We can notice that equation (3.2.2) has exactly n roots with positive real parts, and similarly equation (3.2.1) has $n-1$ roots with positive real parts. The precise proof of this fact uses Rouché's theorem, see Li and Garrido (2004b). Moreover, for both equations, the above mentioned roots are different. We can find in Ji and Zhang (2011) a detailed explanation of this result. By the numbers $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ denote the roots of (3.2.1) which have positive real parts, and $R > 0$ is the adjustment coefficient.

We assume the net profit condition (2.3.1), which in our case becomes

$$cE[W_i] > E[X_i] \Longleftrightarrow c\frac{n}{\lambda} > \mu_1. \tag{3.2.3}$$

3.3 Mathematical background

In the literature of risk theory, it is common to work with integro–differential equations.

We know from Li and Dickson (2006) that the probability of attaining an upper barrier prior to ruin, $\chi(u, b)$, satisfies an order n integro–differential equation with n boundary conditions. This can be written in the form

$$B(\mathcal{D})v(u) = \int_0^u v(u-y)p(y)dy, \quad u \geq 0, \tag{3.3.1}$$

where

$$B(\mathcal{D}) = \left(\mathcal{I} - \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n = \sum_{k=0}^{n}(-1)^k \left(\frac{c}{\lambda}\right)^k \binom{n}{k}\mathcal{D}^k = \sum_{k=0}^{n} B_k \mathcal{D}^k,$$

and $B_k = (-1)^k \left(\frac{c}{\lambda}\right)^k \binom{n}{k}$.

The operator $\mathcal{D} = d/du$ denotes differentiation with respect to u and $\mathcal{I}$ is the identity operator. Thus, $B(\mathcal{D})$ is a differential operator of degree n. If we find n linearly independent particular solutions $v_j(u),\ j = 1, \ldots, n$ for this equation, then we have

$$\chi(u, b) = \overrightarrow{v}(u)[\mathcal{V}(b)]^{-1}\overrightarrow{e}^T, \tag{3.3.2}$$

where $\overrightarrow{v}(u) = (v_1(u), \ldots, v_n(u))$ is a $1 \times n$ vector, $\mathcal{V}(b)$ is a $n \times n$ matrix with entries given by

$$(\mathcal{V}(b))_{ij} = \left.\frac{d^{i-1}v_j(u)}{du^{i-1}}\right|_{u=b}$$

and $\overrightarrow{e} = (1, 0, \ldots, 0)$ which is a $1 \times n$ vector. We can find a complete derivation of this result in Li (2008a).

The equation (3.3.2) gives an expression for $\chi(u, b)$ as linear combination of n linearly independent particular solutions of the integro–differential equation (3.3.1).

In this chapter our aim is to seek for those solutions, which in turn depend on the roots of the fundamental Lundberg's equation. Li (2008a) finds a vector of solutions $\overrightarrow{v}(u)$ for the case when $\rho_1, \rho_2, \ldots, \rho_{n-1}$ are all distinct.

Our work starts by giving an improved version for the expressions presented by Li (2008a) for the functions $v_i(u),\ i = 1, \ldots, n$. This is given in the next section. Further, we apply our results in order to find the distribution of the maximum severity of ruin.

Afterwards, we deal with the dividends problem, we mean the calculation of the moments $V_m(u, b)$. As we mentioned before for a classical risk model, an integro–differential equation for $V(u, b)$ can be found in Dickson (2005), and for $V_m(u, b)$ in Dickson and Waters (2004). For the Erlang(n) model we give the respective integro–differential equations as well as a method of finding the solutions of this equations.

3.4 Solutions for the integro-differential equation

In this part we consider the relation between the roots of the fundamental Lundberg's equation that have positive real parts and the solutions for the integro-differential equation ((3.3.1)). Li (2008a) found that

Theorem 3.4.1 If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \sum_{i=1}^{j-1} a_{i,j} \int_0^u \Phi(u-y) e^{\rho_i y} dy, \quad j = 2, 3, \ldots, n, \end{aligned}$$

where $a_{i,j} = -\dfrac{1}{\prod_{k=1,k\neq i}^{j-1}(\rho_k - \rho_i)}, \quad i = 1, 2, \ldots, j-1.$

We introduce developments, proposing a new version of Theorem 3.4.1:

Theorem 3.4.2 If $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct, then we have the following expressions for the $v_j(u)$'s

$$\begin{aligned} v_1(u) &= \Phi(u), \\ v_j(u) &= \int_0^u \Phi(u-y) e^{\rho_{j-1} y} dy, \quad j = 2, 3, \ldots, n. \end{aligned}$$

Proof: Any solution $v(u)$ of (3.3.1) has Laplace transform

$$\hat{v}(s) = \frac{d_v(s)}{B(s) - \hat{p}(s)},$$

where

$$\begin{aligned} d_v(s) &= \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} \binom{n}{k} \left(\frac{-c}{\lambda} \right)^k v^{(k-1-i)}(0) \right) s^i \\ &= \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} B_k v^{(k-1-i)}(0) \right) s^i. \end{aligned}$$

The survival probability $\Phi(u)$ is a solution of (3.3.1), and its Laplace transform is given by [see Li (2008a)]

$$\hat{\Phi}(s) = -\Phi(0) \left(\frac{c}{\lambda} \right)^n \frac{\prod_{i=1}^{n-1} (\rho_i - s)}{B(s) - \hat{p}(s)},$$

we denote

$$d_\Phi(s) = -\Phi(0) \left(\frac{c}{\lambda} \right)^n \prod_{i=1}^{n-1} (\rho_i - s). \tag{3.4.1}$$

Now we show that any function $v_j(u) = \int_0^u \Phi(u-y) e^{\rho_{j-1} y} dy$, with $j = 2, 3, \ldots, n$, is solution of (3.3.1). We need to prove that

$$B(\mathcal{D}) v_j(u) = d_\Phi(\rho_{j-1}) e^{\rho_{j-1} u} + \int_0^u (B(\mathcal{D}) \Phi(u-t)) e^{\rho_{j-1} t} dt$$

and that

$$\int_0^u v_j(u-y) p(y) dy = \int_0^u (B(\mathcal{D}) \Phi(u-t)) e^{\rho_{j-1} t} dt.$$

For the left hand side, the k-th derivative of $v_j(u)$ gives

$$v_j^{(k)}(u) = \left(\sum_{i=0}^{k-1} \Phi^{(k-1-i)}(0) \rho_{j-1}^i \right) e^{\rho_{j-1} u} + \int_0^u \Phi^{(k)}(u-y) e^{\rho_{j-1} y} dy.$$

Therefore,

$$\begin{aligned}
B(\mathcal{D})v_j(u) &= \sum_{k=0}^{n} B_k v_j^{(k)}(u) = \\
&= \sum_{k=0}^{n} B_k \left(\sum_{i=0}^{k-1} \Phi^{(k-1-i)}(0)\rho_{j-1}^i \right) e^{\rho_{j-1}u} + \\
&\quad \sum_{k=0}^{n} B_k \int_0^u \Phi^{(k)}(u-y)e^{\rho_{j-1}y}dy \\
&= \left(\sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} B_k \Phi^{(k-1-i)}(0) \right) \rho_{j-1}^i \right) e^{\rho_{j-1}u} + \\
&\quad \int_0^u \left(\sum_{k=0}^{n} B_k \Phi^{(k)}(u-y) \right) e^{\rho_{j-1}y}dy \\
&= d_\Phi(\rho_{j-1})e^{\rho_{j-1}u} + \int_0^u \left(\sum_{k=0}^{n} B_k \Phi^{(k)}(u-y) \right) e^{\rho_{j-1}y}dy \\
&= \int_0^u (B(\mathcal{D})\Phi(u-y))e^{\rho_{j-1}y}dy,
\end{aligned}$$

since $d_\Phi(\rho_{j-1}) = 0,\ j = 2, \ldots n$ from (3.4.1).

And for the right hand side we get

$$\begin{aligned}
\int_0^u v_j(u-x)p(x)dx &= \int_0^u \left(\int_0^{u-x} \Phi(u-x-y)e^{\rho_{j-1}y}dy \right) p(x)dx \\
&= \int_0^u \left(\int_0^{u-y} \Phi(u-y-x)p(x)dx \right) e^{\rho_{j-1}y}dy \\
&= \int_0^u (B(\mathcal{D})\Phi(u-y))e^{\rho_{j-1}y}dy.
\end{aligned}$$

We have just proved that the functions $v_j(u)$ are solutions of (3.3.1).

The only remaining part to prove is that those $v_j(u)$'s are linearly independent. We do this in the following way.

Suppose that we have a linear combination such that $\sum_{j=1}^{n} c_j v_j(u) = 0$, $\forall u \geq 0$. Considering the cases (i) and (ii) below.

(i) For $c_1 = 0$. Let $H(t) = \sum_{j=2}^{n} c_j e^{\rho_{j-1} t}$, then

$$\begin{aligned}\sum_{j=1}^{n} c_j v_j(u) &= \sum_{j=2}^{n} c_j \int_0^u \Phi(u-y) e^{\rho_{j-1} y} dy \\ &= \int_0^u \Phi(u-y) \sum_{j=2}^{n} c_j e^{\rho_{j-1} y} dy \\ &= \Phi * H(u) = 0.\end{aligned}$$

The fact that $\Phi * H(u) = 0$, $\forall u \geq 0$ with $\Phi(u) \not\equiv 0$, implies that $H(u) \equiv 0$ almost everywhere. But on the other side $H(t)$ is a continuously differentiable function, this implies that $c_1 = c_2 = \cdots = c_n = 0$.

(ii) For $c_1 \neq 0$. We define $G(t) = \sum_{j=2}^{n} (-c_j/c_1)\, e^{\rho_{j-1} t}$, so $\Phi * G(u) = \Phi(u)$ $\forall u \geq 0$. Not all the remaining coefficients c_j's can be 0, otherwise $G(t) \equiv 0$. But then $\lim_{u \to +\infty} G(u) = \pm\infty$ depending on the sign of the non zero coefficients. As $\Phi(u)$ is a non–decreasing non–negative function with $\lim_{u \to +\infty} \Phi(u) = 1$, we have that $\lim_{u \to +\infty} \Phi * G(u) = \pm\infty$, which is a contradiction and concludes the proof. $\square$

Remark 3.4.1 For any complex root ρ of the fundamental Lundberg's equation the conjugate $\bar{\rho}$ is also a root, we have that $v(u) = \int_0^u \Phi(u-y) e^{\rho y} dy$ and its conjugate $\overline{v(u)} = \int_0^u \Phi(u-y) e^{\bar{\rho} y} dy$ are both solutions of (3.3.1). We take advantage of this fact in Theorem 3.4.2. Thus, for computational purposes this theorem becomes much simpler than Theorem 3.4.1.

3.5 The maximum severity of ruin

In the previous section we have shown how to obtain the solutions of the integro–differential equation. Now, we use these results to obtain the corresponding expressions for the distribution of the maximum severity of ruin. We find an expression for that distribution which only depends on the non-ruin probability $\Phi(u)$ and the claim amounts distribution.

From Dickson (2005) and (3.3.2) we know that the distribution of the maximum severity of ruin $J(z; u)$ can be expressed as:

$$J(z; u) = \frac{1}{1 - \Phi(u)} \int_0^z g(u, y)(v_1(z-y), \ldots, v_n(z-y)) dy [V(z)]^{-1} \vec{e}^{\,T}. \quad (3.5.1)$$

For simplicity we denote by

$$\begin{aligned}
\overrightarrow{h}(z,u) &= \int_0^z g(u,y)(v_1(z-y),\ldots,v_n(z-y))dy \\
&= \left(\int_0^z g(u,y)v_1(z-y)dy,\ldots,\int_0^z g(u,y)v_n(z-y)dy\right) \\
&= (h_1(z,u),\ldots,h_n(z,u)),
\end{aligned}$$

In this moment the only remaining part to find is an expression for every component of $\overrightarrow{h}(z,u)$. We consider the case of the Theorem 3.4.2 like in the previous section. In a similar way as it is done by Li (2008a) we get for $j=1$:

$$\int_0^z g(u,y)v_1(z-y)dy = \Phi(u+z)-\Phi(u), \tag{3.5.2}$$

and for $j=2,\ldots,n$:

$$\begin{aligned}
\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)e^{\rho_{j-1}x}dxdy \\
&= \int_0^z e^{\rho_{j-1}x}[\Phi(u+(z-x))-\Phi(u)]dx.
\end{aligned}$$

Hence we get the desired expression for the maximum severity of ruin.

3.6 Explicit expressions

In this section our aim is to determine explicit expressions for the (existing) moments of the maximum severity of ruin as well as the probability that the maximum severity occurs at ruin. Li (2008a) considered those moments for Erlang(2) interclaim times and exponential claims. We work here with other two cases and present formulae as well as some numerical calculations. Namely, for cases where:

1. Interclaim arrivals are Erlang(3,λ) and single claim amounts are exponential(β) distributed. For simplification we denote this case by Erlang(3)–exponential;

2. Interclaim arrivals are Erlang(2,λ) and single claim amounts are Erlang(2,β) distributed. Similarly, we denote this case by Erlang(2)–Erlang(2).

3.6.1 Erlang(3) – exponential case

Considering the net profit condition (3.2.3) we write $c = (1+\theta)\lambda/3\beta$ with safety loading coefficient $\theta > 0$. The fundamental Lundberg's equation (3.2.1) takes the form

$$\left(1 - \left(\frac{c}{\lambda}\right) s\right)^3 - \frac{\beta}{(s+\beta)} = 0,$$

which has four roots: $0, \rho_1, \rho_2$ and $-R$, where $0 < R < \beta$ is the adjustment coefficient, ρ_1, ρ_2 are complex roots with positive real parts and $\rho_2 = \overline{\rho_1}$. The three solutions for the integro–differential equation (3.3.1) come

$$\begin{aligned}
\Phi(u) &= 1 - \left(1 - \frac{R}{\beta}\right) e^{-Ru}, \\
v_2(u) &= \frac{-1}{\rho_1} + \frac{\beta - R}{\beta(R+\rho_1)} e^{-Ru} + \frac{R(\beta+\rho_1)}{\rho_1\beta(R+\rho_1)} e^{\rho_1 u}, \\
v_3(u) &= \frac{-1}{\rho_2} + \frac{\beta - R}{\beta(R+\rho_2)} e^{-Ru} + \frac{R(\beta+\rho_2)}{\rho_2\beta(R+\rho_2)} e^{\rho_2 u}.
\end{aligned}$$

These solutions were obtained using Theorem 3.4.2.

Distribution and moments of the maximum severity

Continuing work we want to find the corresponding expressions for the distribution and the moments of the maximum severity of ruin. After calculating (3.5.1) we get

$$1 - J(z;u) = \frac{\alpha e^{-Rz}}{1 - \gamma e^{-(\rho_1+R)z} - \epsilon e^{-(\rho_2+R)z} - \eta e^{-Rz}},$$

where

$$\alpha = \frac{R(R+\rho_1)(R+\rho_2)}{\beta(\beta+\rho_1)(\beta+\rho_2)} \qquad \gamma = -\frac{R(\beta-R)(R+\rho_2)}{\rho_1(\beta+\rho_1)(\rho_2-\rho_1)}$$
$$\epsilon = \frac{R(\beta-R)(R+\rho_1)}{\rho_2(\beta+\rho_2)(\rho_2-\rho_1)} \qquad \eta = \frac{(\beta-R)(R+\rho_1)(R+\rho_2)}{\beta\rho_1\rho_2},$$

with $0 < \alpha < 1$, $\epsilon = \overline{\gamma}$ and $0 < \eta = 1 - \alpha - \gamma - \epsilon$. Note that this expression is independent from u.

θ	$n=1$ $m=1$ $E(M_u)$	$s.d.(M_u)$	$n=2$ $m=1$ $E(M_u)$	$s.d.(M_u)$	$n=3$ $m=1$ $E(M_u)$	$s.d.(M_u)$
0.05	3.197	7.324	2.474	5.532	2.236	4.933
0.1	2.638	5.007	2.063	3.805	1.875	3.404
0.15	2.342	4.015	1.848	3.069	1.687	2.754
0.2	2.150	3.443	1.709	2.646	1.567	2.381
0.25	2.012	3.064	1.611	2.368	1.481	2.136
0.3	1.906	2.792	1.536	2.169	1.416	1.962

Table 3.1: Expected values and standard deviations of M_u for $n = 1, 2, 3$ (interclaim times) and $m = 1$ (claim amounts)

The r-th moment of M_u, given that ruin occurs, is given by the formula

$$\begin{aligned} E(M_u^r|T_u < \infty) &= r\int_0^\infty z^{r-1}(1 - J(z; u))dz \\ &= r\alpha\int_0^\infty \frac{z^{r-1}e^{-Rz}}{1 - \gamma e^{-(\rho_1+R)z} - \epsilon e^{-(\rho_2+R)z} - \eta e^{-Rz}}\, dz, \quad (3.6.1) \end{aligned}$$

for $r \geq 1$. Since $|\gamma e^{-(\rho_1+R)z} + \epsilon e^{-(\rho_2+R)z} + \eta e^{-Rz}| < 1$ we can write that

$$1 - J(z; u) = \alpha e^{-Rz}\sum_{k=0}^{\infty}(\gamma e^{-(\rho_1+R)z} + \epsilon e^{-(\rho_2+R)z} + \eta e^{-Rz})^k.$$

Hence,

$$E(M_u^r|T_u < \infty) = \alpha r!\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{k-j}\binom{k}{j}\binom{k-j}{l}\frac{\eta^j\gamma^l\epsilon^{k-j-l}}{(R(k+1) + \rho_1 l + \rho_2(k-j-l))^r}.$$

We choose $\beta = 1$, $\lambda = 3$ and $c = 1 + \theta$ to evaluate formula (3.6.1) for some values of θ with $r = 1$. These values are compared with Li (2008a) results.

Figures are given in Table 3.1. From the table we observe that the mean and the standard deviation of M_u decrease as θ increases for the three cases. This is expected since an increase in θ means an increase in the income unit c, which gives faster growth of the surplus per unit of time. Also, we note that for fixed θ the mean and the standard deviation of M_u decrease as n increases. The reason for this is that for higher values of n with fixed m we are increasing the expected value of the interclaim times, which is given by $E(W_i) = n/\lambda$, so we expect to get claims after longer time intervals.

The probability that the maximum severity occurs at ruin

At the time of ruin T_u, the size of the severity is not necessarily maximal. This means that the probability $\tilde{p}$ that the maximum severity occurs at ruin is not necessarily equal to 1. Therefore it is important to have a measure for this probability. Due to the memoryless property of the exponential distribution we have that $g(u,y) = \Psi(u)p(y)$. Hence

$$\tilde{p} = \Pr(M_u = |U(T)| \mid T < \infty) = \frac{\int_0^\infty g(u,y)\chi(0,y)dy}{\Psi(u)} = \int_0^\infty \chi(0,y)p(y)dy.$$

Now from (3.3.2) we get, for $u = 0$

$$\chi(0,y) = \left(\frac{R}{\beta}\right) \frac{1 + \frac{\rho_1\gamma}{R}e^{-(\rho_1+R)y} + \frac{\rho_2\gamma}{R}e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \epsilon e^{-(\rho_2+R)y} - \eta e^{-Ry}},$$

so

$$\begin{aligned}\tilde{p} &= \left(\frac{R}{\beta}\right) \int_0^\infty \frac{1 + \frac{\rho_1\gamma}{R}e^{-(\rho_1+R)y} + \frac{\rho_2\gamma}{R}e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \epsilon e^{-(\rho_2+R)y} - \eta e^{-Ry}}\, \beta e^{-\beta y}dy \\ &= \int_0^\infty \frac{R + \rho_1\gamma e^{-(\rho_1+R)y} + \rho_2\gamma e^{-(\rho_2+R)y}}{1 - \gamma e^{-(\rho_1+R)y} - \epsilon e^{-(\rho_2+R)y} - \eta e^{-Ry}}\, e^{-\beta y}dy. \qquad (3.6.2)\end{aligned}$$

We choose the same values for λ, β and θ as before and evaluate (3.6.1) to get the figures in Table 3.2. From the table we conclude that the probability that the maximum deficit occurs at ruin increases as θ increases. This means that for bigger premium c is less likely that the surplus will drop to lower levels of deficit after ruin.

θ	0.05	0.1	0.15	0.2	0.25	0.3
$\tilde{p}$	0.735	0.752	0.768	0.782	0.795	0.808

Table 3.2: Probability that the maximum deficit occurs at ruin, for $n = 3$, $m = 1$.

3.6.2 Erlang(2) – Erlang(2) case

Following the previous example we again consider the net profit condition (3.2.3) and we write $c = (1+\theta)\lambda/\beta$ with $\theta > 0$. The fundamental Lundberg's equation (3.2.1) takes the form

$$\left(1-\left(\frac{c}{\lambda}\right)s\right)^2-\frac{\beta^2}{(s+\beta)^2}=0,$$

which has four real roots: $0,-R_1,-R_2$ and ρ, where $0<R_1<\beta$ is the adjustment coefficient and $R_2>\beta$, $\rho>\beta$. The two solutions for the integro–differential equation (3.3.1)

$$\begin{aligned}
\Phi(u) &= 1-\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1u}-\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2u},\\
v_2(u) &= -\frac{1}{\rho}+\frac{R_1R_2(\beta+\rho)^2}{\beta^2\rho(\rho+R_1)(\rho+R_2)}e^{\rho u}+\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)(\rho+R_1)}e^{-R_1u}\\
&\quad +\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)(\rho+R_2)}e^{-R_2u}.
\end{aligned}$$

were obtain following Theorem 3.4.2.

Distribution and moments of the maximum severity

In this case the formula that we get from (3.5.1) is not independent from u, we write it in the following way

$$J(z;u)=\frac{1}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1u}J_1(z;u)+\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2u}J_2(z;u)\right].$$

So,

$$\begin{aligned}
1-J(z;u) &= \frac{1}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1u}(1-J_1(z;u))\right.\\
&\quad \left.+\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2u}(1-J_2(z;u))\right].
\end{aligned}$$

The functions $J_1(z;u)$ and $J_2(z;u)$ are

$$\begin{aligned}
J_1(z;u) &= \frac{1}{J_D(z)}\left(1-\gamma_1e^{-(\rho+R_1)z}-\gamma_2e^{-(\rho+R_2)z}-\right.\\
&\quad \left.(1-\gamma_1)e^{-R_1z}-\tau_1e^{-R_2z}-\omega_1e^{-(\rho+R_1+R_2)z}\right),
\end{aligned}$$

and

$$\begin{aligned}J_2(z;u) &= \frac{1}{J_D(z)}\left(1-\gamma_1 e^{-(\rho+R_1)z}-\gamma_2 e^{-(\rho+R_2)z}-\right.\\ &\quad \left.\tau_2 e^{-R_1 z}-(1-\gamma_2)e^{-R_2 z}-\omega_2 e^{-(\rho+R_1+R_2)z}\right),\end{aligned}$$

where the denominator in both expressions is given by

$$J_D(z)=1-\gamma_1 e^{-(\rho+R_1)z}-\gamma_2 e^{-(\rho+R_2)z}-\delta_1 e^{-R_1 z}-\delta_2 e^{-R_2 z}-\eta e^{-(\rho+R_1+R_2)z},$$

and

$$\gamma_1=-\frac{R_1(\beta-R_1)^2(\rho+R_2)}{\rho(R_2-R_1)(\beta+\rho)^2},\qquad \gamma_2=-\frac{R_2(\beta-R_2)^2(\rho+R_1)}{\rho(R_1-R_2)(\beta+\rho)^2},$$

$$\delta_1=\frac{R_2(\beta-R_1)^2(\rho+R_1)}{\beta^2\rho(R_2-R_1)},\qquad \delta_2=\frac{R_1(\beta-R_2)^2(\rho+R_2)}{\beta^2\rho(R_1-R_2)},$$

$$\tau_1=\frac{R_1(\beta-R_2)^2(\rho+R_2)}{\rho(R_1-R_2)(\beta+\rho)^2},\qquad \tau_2=\frac{R_2(\beta-R_1)^2(\rho+R_1)}{\rho(R_2-R_1)(\beta+\rho)^2},$$

$$\omega_1=-\frac{(\beta-R_2)^2}{(\beta+\rho)^2},\qquad \omega_2=-\frac{(\beta-R_1)^2}{(\beta+\rho)^2},$$

$$\eta=-\frac{(\beta-R_1)^2(\beta-R_2)^2}{\beta^2(\beta+\rho)^2},\qquad \alpha=\frac{R_1R_2(\rho+R_1)(\rho+R_2)}{\beta^2(\beta+\rho)^2},$$

with $0<\alpha<1$ and $\eta=1-\alpha-\gamma_1-\gamma_2-\delta_1-\delta_2$.

In the same way, we compute the conditional moments of M_u, given that ruin occurs,

$$\begin{aligned}E(M_u^r|T<\infty) &= r\int_0^\infty z^{r-1}(1-J(z;u))dz\\ &= \frac{r}{\Psi(u)}\left[\frac{R_2(\beta-R_1)^2}{\beta^2(R_2-R_1)}e^{-R_1 u}\int_0^\infty z^{r-1}(1-J_1(z;u))dz\right.\\ &\quad \left.+\frac{R_1(\beta-R_2)^2}{\beta^2(R_1-R_2)}e^{-R_2 u}\int_0^\infty z^{r-1}(1-J_2(z;u))dz\right],\end{aligned}\tag{3.6.3}$$

for $r\geq 1$.

We choose $\beta=1$, $\lambda=1$ and $c=1+\theta$ to evaluate formula (3.6.3) for some values of θ with $r=1$. Afterwards we compare with the resuts by Li (2008a). As before, Table 3.3 shows figures for $E(M_u)$ and $s.d.(M_u)$. From Table 3.3 we observe that the mean and the standard deviation of M_u decrease as θ increases for all the three cases. This is expected since

θ	$n=2, m=1$ $E(M_u)$	$s.d.(M_u)$	$n=2, m=2$ $E(M_u)$	$s.d.(M_u)$
0.05	2.474	5.532	3.279	7.137
0.1	2.063	3.805	2.759	4.911
0.15	1.848	3.069	2.485	3.959
0.2	1.709	2.646	2.307	3.411
0.25	1.611	2.368	2.179	3.049
0.3	1.536	2.169	2.082	2.791

Table 3.3: Values of $E(M_u)$ and $s.d.(M_u)$ for $n = 2; m = 1$ and $n = m = 2$

an increase in θ means an increase in the premium income c, which gives faster growth of the surplus, per unit of time. Also we note that for a fixed θ the mean and the standard deviation of M_u are higher in the Erlang(2) – Erlang(2) case than in the Erlang(2) – exponential case. The reason for this is that for higher values of m with fixed n we are increasing the expected value of the claim amounts, which is given by $E(X_i) = m/\beta$, so we are increasing the average size of the claims that will be paid.

The probability that the maximum severity occurs at ruin

From (3.3.2) we get, for $u = 0$

$$\chi(0, y) = \left(\frac{R_1 R_2}{\beta^2}\right) \frac{1 + \frac{\rho\gamma_1}{R_1} e^{-(\rho+R_1)y} + \frac{\rho\gamma_2}{R_2} e^{-(\rho+R_2)y}}{d(y)},$$

where

$$d(y) = 1 - \gamma_1 e^{-(\rho+R_1)y} - \gamma_2 e^{-(\rho+R_2)y} - \delta_1 e^{-R_1 y} - \delta_2 e^{-R_2 y} - \eta e^{-(\rho+R_1+R_2)y}.$$

The formula for $P(M_u = |U(T)| \mid T < \infty)$ is obtained in the same way as in equation (3.6.1). Choosing the same values of λ, β and θ as before we evaluate that probability to get the figures in Table 3.4 where $\tilde{p} = P(M_u = |U(T)| \mid T < \infty)$. From the table we conclude that the probability that the maximum deficit occurs at ruin increases along with θ. Like in Section 3.6.1, this means that for a bigger c it is less likely that the surplus drops to lower levels of deficit after the ruin time.

θ	0.05	0.1	0.15	0.2	0.25	0.3
$\tilde{p}$	0.730	0.745	0.759	0.772	0.784	0.795

Table 3.4: Probability that the maximum deficit occurs at ruin, for $n = 2$, $m = 2$.

3.7 Dividends

In this section we consider the dividends problem. We use the method proposed by Dickson and Waters (2004) and present an equation for $V_m(u, b)$ in Erlang(n) risk process. As before, conditioning on the time and the amount of the first claim we get, for $0 \le u < b$ and $m \ge 1$

$$\begin{aligned} V_m(u,b) &= \int_{\frac{b-u}{c}}^{\infty} k_n(t) e^{-m\delta t} \Bigg[\left(c\, \overline{s}_{\overline{t-\frac{b-u}{c}|}} \right)^m + \\ &+ \sum_{j=1}^{m} \binom{m}{j} \left(c\, \overline{s}_{\overline{t-\frac{b-u}{c}|}} \right)^{m-j} \int_0^b p(x) V_j(b-x, b) dx \Bigg] dt + \\ &+ \int_0^{\frac{b-u}{c}} e^{-m\delta t} k_n(t) \int_0^{u+ct} V_m(u+ct-x, b) p(x) dx\, dt. \quad (3.7.1) \end{aligned}$$

In particular, for $m = 1$

$$\begin{aligned} V(u,b) &= \int_{\frac{b-u}{c}}^{\infty} k_n(t) e^{-\delta t} \left(c\, \overline{s}_{\overline{t-\frac{b-u}{c}|}} + \int_0^b p(x) V(b-x, b) dx \right) dt + \\ &+ \int_0^{\frac{b-u}{c}} e^{-\delta t} k_n(t) \int_0^{u+ct} V(u+ct-x, b) p(x) dx\, dt, \quad (3.7.2) \end{aligned}$$

where $\overline{s}_{\overline{t}|} = (e^{\delta t} - 1)/\delta$ in standard actuarial notation.
For an Erlang(n) risk process the integro–differential equations satisfied by the discounted expected dividends are

$$\begin{aligned} \left(\left(1 + \frac{\delta}{\lambda}\right) \mathcal{I} - \frac{c}{\lambda} \mathcal{D} \right)^n V(u,b) &= \int_0^u V(u-x, b) p(x) dx \quad (3.7.3) \\ \left. \frac{d^k V(u,b)}{du^k} \right|_{u=b} &= \left(\frac{\delta}{c} \right)^{k-1}, \; 1 \le k \le n, \end{aligned}$$

and for a general $m \ge 1$,

$$\left(\left(1+\frac{m\delta}{\lambda}\right)\mathcal{I}-\frac{c}{\lambda}\mathcal{D}\right)^n V_m(u,b) = \int_0^u V_m(u-x,b)p(x)dx \tag{3.7.4}$$
$$\left.\frac{d^k V_m(u,b)}{du^k}\right|_{u=b} = \sum_{j=1}^{k} \frac{m!}{(m-j)!}\left\{ {k \atop j} \right\}\left(\frac{\delta}{c}\right)^{k-j} V_{m-j}(b,b),$$

for $1 \le k \le n$, where $\left\{ {k \atop j} \right\} = (1/j!)\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^k$ denotes the Stirling numbers of the second kind. We define for convenience $V_{m-j}(u,b) \equiv 0$, for $m<j$ in the formula (3.7.4).

These equations generalize those proposed by Dickson (2005) and Dickson and Waters (2004) for the classical Poisson risk model, and extend the equations proposed by Albrecher *et al.* (2005).

So far we have shown the equations for the expected discounted dividends and higher moments. In this part our aim is to solve those equations. We follow an argument originally proposed by Bühlman (1970), Section 6.4.9, for a Poisson risk model, and also treated by Zhou *et al.* (2006). For an Erlang(n) risk model we propose $V(u,b)$ in the form

$$V(u,b) = \sum_{i=1}^{n} C_i e^{\rho_i u}\beta_i(u), \tag{3.7.5}$$

where C_i's are constants (that depend on the parameter b), ρ_i's are the n roots with positive real parts of the generalized Lundberg's equation (3.2.2), and the functions $\beta_i(u)$ are solutions of

$$(\lambda_i\mathcal{I} - c\mathcal{D})^n\beta_i(u) = \lambda_i^n\int_0^u \beta_i(u-x)p_i(x)dx, \tag{3.7.6}$$

with $\lambda_i = \lambda\hat{p}^{\frac{1}{n}}(\rho_i)$ and $p_i(x) = e^{-\rho_i x}p(x)/\hat{p}(\rho_i)$.

The constants C_i's we determine using the boundary conditions given in (3.7.3), which gives us a system of n equations with n unknowns

$$\left.\frac{d^k V(u,b)}{du^k}\right|_{u=b} = \sum_{i=1}^{n} C_i \left.\frac{d^k(e^{\rho_i u}\beta_i(u))}{du^k}\right|_{u=b} = \left(\frac{\delta}{c}\right)^{k-1},\ 1\le k\le n, \tag{3.7.7}$$

Equivalently, in matrix form we get,

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} \left.\frac{d(e^{\rho_1 u}\beta_1(u))}{du}\right|_{u=b} & \left.\frac{d(e^{\rho_2 u}\beta_2(u))}{du}\right|_{u=b} & \cdots & \left.\frac{d(e^{\rho_n u}\beta_n(u))}{du}\right|_{u=b} \\ \left.\frac{d^2(e^{\rho_1 u}\beta_1(u))}{du^2}\right|_{u=b} & \left.\frac{d^2(e^{\rho_2 u}\beta_2(u))}{du^2}\right|_{u=b} & \cdots & \left.\frac{d^2(e^{\rho_n u}\beta_n(u))}{du^2}\right|_{u=b} \\ \vdots & \vdots & \ddots & \vdots \\ \left.\frac{d^n(e^{\rho_1 u}\beta_1(u))}{du^n}\right|_{u=b} & \left.\frac{d^n(e^{\rho_2 u}\beta_2(u))}{du^n}\right|_{u=b} & \cdots & \left.\frac{d^n(e^{\rho_n u}\beta_n(u))}{du^n}\right|_{u=b} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \left(\frac{\delta}{c}\right) \\ \vdots \\ \left(\frac{\delta}{c}\right)^{n-1} \end{pmatrix}.$$

We summarize this in the following theorem

Theorem 3.7.1 The solutions of integro-differential equation (3.7.3) are of the form

$$V(u,b) = \sum_{i=1}^{n} C_i e^{\rho_i u}\beta_i(u),$$

where ρ_i's are the roots with positive real parts of the generalized Lundberg's equation (3.2.2), $\beta_i(u)$'s are defined in (3.7.6) and the constants C_i's are defined in (3.7.7).

Proof:

The proof is technical and follows by taking derivatives of $V(u,b)$. In next step of the proof we determine which conditions must be satisfied by the ρ_i's and $\beta_i(u)$'s to get the equality in (3.7.3).

Let ρ be a constant and $\beta(u)$ a function. We apply the integro–differential equation (3.7.3) to the product $e^{\rho u}\beta(u)$. From one side we get

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} - \frac{c}{\lambda}\mathcal{D}\right)^n e^{\rho u}\beta(u) = e^{\rho u}\left(\left(1+\frac{\delta}{\lambda}-\frac{c}{\lambda}\rho\right)\mathcal{I} - \frac{c}{\lambda}\mathcal{D}\right)^n \beta(u),$$

and from the other side

$$\int_0^u e^{\rho(u-x)}\beta(u-x)p(x)dx = e^{\rho u}\int_0^u \beta(u-x)e^{-\rho x}p(x)dx.$$

To obtain an equality we must have

$$\left(\left(1+\frac{\delta}{\lambda}-\frac{c}{\lambda}\rho\right)\mathcal{I} - \frac{c}{\lambda}\mathcal{D}\right)^n \beta(u) = \int_0^u \beta(u-x)e^{-\rho x}p(x)dx.$$

If we assume that ρ is a root of the generalized Lundberg's equation (3.2.2), then

$$((\lambda\hat{p}^{\frac{1}{n}}(\rho))\mathcal{I} - c\mathcal{D})^n\beta(u) = \lambda^n\hat{p}(\rho)\int_0^u \beta(u-x)\frac{e^{-\rho x}p(x)}{\hat{p}(\rho)}dx.$$

Choosing $\tilde{\lambda} = \lambda \hat{p}^{\frac{1}{n}}(\rho)$ and $\tilde{p}(x) = e^{-\rho x} p(x)/\hat{p}(\rho)$ we get the desired result.

Clearly the functions $e^{\rho_i u}\beta_i(u)$ with ρ_i a root of the generalized Lundberg's equation (3.2.2) and $\beta_i(u)$ defined as in (3.7.6) are linearly independent. Therefore, it is possible to obtain the coefficients C_i using the boundary conditions given in (3.7.3) and inverting the corresponding matrix. $\square$

Our method generalizes the results obtained by Albrecher *et al.* (2005). It works for any kind of claim amounts distribution, and not only for the distributions with rational Laplace transforms. Moreover, the same approach can be implemented to find an expression for $V_m(u,b)$, $m \geq 2$ written the form (3.7.5) using the corresponding boundary conditions.

3.7.1 Example

In this section we give one example for the Erlang(2) risk model with Erlang(2) claim amounts. Our aim is to compute $V(u,b)$ and $V_2(u,b)$.

Let the interclaim times W_i and the claim amounts X_i be both Erlang(2,2), let the positive loading $c = 1.1$ and the force of interest $\delta = 0.03$.

Then, for $V(u,b)$ we get the roots $\rho_1 = 0.169$, $\rho_2 = 2.631$, the functions

$$\begin{aligned}\beta_1(u) &= 1 + 0.026e^{-2.954u} - 0.718e^{-0.492u},\\ \beta_2(u) &= 1 + 0.047e^{-5.235u} - 0.108e^{-3.845u},\end{aligned}$$

and the constants

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \left.\dfrac{d(e^{\rho_1 u}\beta_1(u))}{du}\right|_{u=b} & \left.\dfrac{d(e^{\rho_2 u}\beta_2(u))}{du}\right|_{u=b} \\ \left.\dfrac{d^2(e^{\rho_1 u}\beta_1(u))}{du^2}\right|_{u=b} & \left.\dfrac{d^2(e^{\rho_2 u}\beta_2(u))}{du^2}\right|_{u=b} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \left(\dfrac{\delta}{c}\right) \end{pmatrix},$$

Therefore

$$\begin{aligned} C_1 &= C_1(b) = \frac{0.323e^{7.123b} - 0.163e^{8.512b} + 6.849e^{12.358b}}{D(b)},\\ C_2 &= C_2(b) = \frac{-0.205e^{6.942b} + 0.081e^{9.404b} - 0.024e^{9.896b}}{D(b)},\end{aligned}$$

where

$$\begin{aligned} D(b) = {} & 0.002e^{4.337b} - 0.015e^{5.727b} + 0.065e^{6.799b} + 0.057e^{7.291b}\\ & -0.027e^{8.189b} - 0.031e^{8.681b} - 1.039e^{9.572b} + 1.802e^{12.034b}\\ & +1.093e^{12.526b}.\end{aligned}$$

Finally the function $V(u,b)$ gets the form

$$V(u,b) = C_1 e^{\rho_1 u}\beta_1(u) + C_2 e^{\rho_2 u}\beta_2(u).$$

$b\backslash u$	0	1	2	3	4	5	6	7	8	9
0	1.07									
1	0.83	1.80								
2	0.85	1.84	2.84							
3	0.84	1.82	2.81	3.80						
4	0.81	1.72	2.66	3.59	4.57					
5	0.73	1.57	2.42	3.27	4.17	5.14				
6	0.64	1.39	2.15	2.90	3.70	4.57	5.53			
7	0.56	1.21	1.87	2.53	3.22	3.98	4.84	5.79		
8	0.48	1.04	1.61	2.18	2.78	3.43	4.17	5.01	5.96	
9	0.41	0.89	1.38	1.86	2.37	2.93	3.56	4.28	5.11	6.07

Table 3.5: Values of $V(u,b)$ for $0 \leq u, b \leq 9$

From the values in the Table 3.5 we notice that from a cetain initial surplus u, if we increase the level of the barrier b, the values of $V(u,b)$ decrease. This is expected since for a higher barrier the distance $b-u$ is larger and therefore the probability of attaining such barrier from the level u is lower, as well as the expected discounted dividends. On the other hand, for a fixed barrier b if we increase the initial surplus u, with $u \leq b$, we obtain higher values of $V(u,b)$, since the distance $b-u$ is smaller and the probability of attaining b is higher.

In a similar way we get $V_2(u,b)$ with $\rho_1 = 0.273$, $\rho_2 = 2.654$,

$$\begin{aligned} \beta_1(u) &= 1 + 0.033e^{-3.054u} - 0.636e^{-0.673u}, \\ \beta_2(u) &= 1 + 0.047e^{-5.256u} - 0.107e^{-3.873u}, \end{aligned}$$

and the constants

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \left.\dfrac{d(e^{\rho_1 u}\beta_1(u))}{du}\right|_{u=b} & \left.\dfrac{d(e^{\rho_2 u}\beta_2(u))}{du}\right|_{u=b} \\ \left.\dfrac{d^2(e^{\rho_1 u}\beta_1(u))}{du^2}\right|_{u=b} & \left.\dfrac{d^2(e^{\rho_2 u}\beta_2(u))}{du^2}\right|_{u=b} \end{pmatrix}^{-1} \begin{pmatrix} 2V(b,b) \\ 2 + 2\left(\dfrac{\delta}{c}\right)V(b,b) \end{pmatrix}.$$

Finally we get

$$V_2(u,b) = C_1 e^{\rho_1 u} \beta_1(u) + C_2 e^{\rho_2 u} \beta_2(u),$$

and the table with values for $V_2(u,b)$.

$b \backslash u$	0	1	2	3	4	5	6	7	8	9
0	1.71									
1	2.24	5.23								
2	3.51	7.86	12.91							
3	4.19	9.37	15.18	21.97						
4	4.19	9.36	15.17	21.87	30.04					
5	3.76	8.41	13.62	19.63	26.94	36.13				
6	3.16	7.07	11.45	16.50	22.65	30.41	40.30			
7	2.55	5.71	9.25	13.33	18.30	24.57	32.63	42.99		
8	2.01	4.51	7.30	10.52	14.44	19.39	25.75	34.01	44.67	
9	1.56	3.51	5.68	8.18	11.23	15.08	20.03	26.46	34.84	45.69

Table 3.6: Values of $V_2(u,b)$ for $0 \le u, b \le 9$

Analogously to the previous table for $V(u,b)$, a similar reasoning can be done. This means that from a cetain initial surplus u, if we increase the level of the barrier b, the values of $V_2(u,b)$ decrease and, for a fixed b, the values of $V_2(u,b)$ increase as u increases, with $u \le b$. Therefore $V(u,b)$ and $V_2(u,b)$ as well as the standard deviation of the discounted dividends behave in the same way.

3.8 Final remarks

As we have mentioned before, one of the fundamental purposes in insurance mathematics is to provide adequate methods to solve the problems that may appear in the actuarial practice. Throughout this chapter we have considered the Sparre–Andersen risk model with Erlang(n) interclaim times. We have developed new theorems for the computation of two very important quantities in Risk Theory. One of them shows improvements in the calculation of the maximum severity of ruin. The other deals with dividends.

The presented methods can be extended for more general distributions. We will continue our work in the next chapter based on generalized Erlang(n) interclaim times.

Chapter 4

The Sparre–Andersen model with generalized Erlang(n) interclaim times

> Mathematics is the science that
> uses easy words for hard ideas
>
> Edward Kasner

4.1 Introduction

In this chapter we work with the Sparre–Andersen model under the assumption that the interclaim times are generalized Erlang(n) distributed.

In comparison with the last chapter we have more complex developments. We start in Section 4.2 by setting some mathematical background and applying it to the problem of finding the survival probability and the probability of attaining an upper barrier prior to ruin. By investigating the Lundberg's equation we have discovered the possibility of multiple roots.

Taking into account the possibility of multiple roots, we show in Section 4.3 new theorems considering the computation of the probability of attaining an upper barrier prior to ruin. In addition, we find some interesting results about the survival probability and its derivatives.

We continue in Section 4.4 applying new theorems for the maximum severity of ruin.

Finally, in Section 4.5 we apply the developed theory to case of dividends and present examples.

4.2 Mathematical background and notation

In this section we define $k_n(t)$ which is the probability density function of W_i

$$k_n(t) = \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i e^{-\lambda_i t}, \quad n \in \mathbb{N}^+,$$

and the cumulative distribution function

$$K_n(t) = 1 - \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) e^{-\lambda_i t},$$

where $\lambda_1, \ldots, \lambda_n$ are the parameters of the distributions.

As before we assume the net profit condition (2.3.1), which in the case of a Sparre–Andersen model with generalized Erlang(n) distributed interclaim times is

$$cE[W_i] > E[X_i] \Leftrightarrow c \sum_{i=1}^{n} \frac{1}{\lambda_i} > \mu_1. \tag{4.2.1}$$

Gerber and Shiu (2003a) proved that $\chi(u,b)$ satisfies an order n integro–differential equation with n boundary conditions which can be written in the form

$$B(\mathcal{D})v(u) = \int_0^u v(u-y)p(y)dy, \quad u \geq 0, \tag{4.2.2}$$

where

$$B(\mathcal{D}) = \prod_{i=1}^{n} \left(\mathcal{I} - \left(\frac{c}{\lambda_i} \right) \mathcal{D} \right) = \sum_{k=0}^{n} B_k \mathcal{D}^k,$$

and $\mathcal{D}$ is the differential operator, with $B_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left(\frac{(-c)^k}{\lambda_{i_1} \cdots \lambda_{i_k}} \right)$.

Similarly to the Erlang(n) case, if we find n linearly independent particular solutions $v_j(u),\ j = 1, \ldots, n$ for this equation, we have

$$\chi(u,b) = \overrightarrow{v}(u)[\mathcal{V}(b)]^{-1}\overrightarrow{e}^T, \tag{4.2.3}$$

where $\overrightarrow{v}(u) = (v_1(u), \ldots, v_n(u))$ is a $1 \times n$ vector of solutions, $\mathcal{V}(b)$ is a $n \times n$ matrix with entries given by

$$(\mathcal{V}(b))_{ij} = \left. \frac{d^{i-1} v_j(u)}{du^{i-1}} \right|_{u=b}$$

and $\overrightarrow{e} = (1, 0, \ldots, 0)$ is a $1 \times n$ vector.

In the previous chapter we discussed work based on Bergel and Egídio dos Reis (2013a) and Li (2008a), where we found a vector of solutions $\overrightarrow{v}(u)$ for the case when the interclaim times follow an Erlang(n) distribution.

In the present chapter, we start by giving the corresponding version of $\overrightarrow{v}(u)$ when we have generalized Erlang(n) interclaim times. This is given in the next section. We apply results in order to find the corresponding expressions for the distribution of the maximum severity of ruin. Afterwards, we deal with the dividends problem, we mean the calculation of the moments $V_m(u,b)$. For a Poisson model, an integro–differential equation for $V(u,b)$ can be found in Dickson (2005), and for $V_m(u,b)$ in Dickson and Waters (2004). For the generalized Erlang(n) model we give the respective integro–differential equations as well as a method to find their solutions, extending the results of Albrecher *et al.* (2005).

4.2.1 Multiplicity of the roots of the generalized (fundamental) Lundberg's equation

In this section we briefly study the possibility of multiple roots in the Lundberg's equations, specifically double roots.

Recall the fundamental Lundberg's equation given by (2.3.4)

$$\prod_{i=1}^{n}\left(1-\frac{c}{\lambda_i}s\right)=\hat{p}(s). \tag{4.2.4}$$

We denote by the numbers $\rho_1,\rho_2,\ldots,\rho_{n-1}\in\mathbb{C}$, the roots of this equation which have positive real parts, and by $R>0$ the adjustment coefficient. We can write equation (4.2.4) in the form $B(s)=\hat{p}(s)$, where $B(s)=\prod_{i=1}^{n}\left(1-(c/\lambda_i)s\right)=\sum_{k=0}^{n}B_k s^k$.

On the other hand the generalized Lundberg's equation given by (2.3.5) becomes

$$\prod_{i=1}^{n}\left(1+\frac{\delta}{\lambda_i}-\frac{c}{\lambda_i}s\right)=\hat{p}(s), \tag{4.2.5}$$

where $\delta>0$ is the force of interest. This equation has exactly n roots with positive real parts (see Li and Garrido (2004b)) and will be considered in the section of dividends. For simplicity we rewrite equation (4.2.5) in the following form

$$B_\delta(s)=\hat{p}(s),$$

where $B_\delta(s) = \prod_{i=1}^{n} (1 + \delta/\lambda_i - (c/\lambda_i)s) = \sum_{k=0}^{n} B_{k,\delta} s^k$.

The next theorem show the possibility of multiple roots in the generalized Lundberg's equation.

Theorem 4.2.1 Let s_1 and s_2 be two consecutive positive real zeros of $B_\delta(s)$. If $B_\delta(s)$ is positive in the interval (s_1, s_2) then the generalized Lundberg's equation has one of the following:

- Two real roots in the interval.
- A double root in the interval.
- No real roots in this interval.

Proof: The proof is based on a comparison of both sides of the equation (4.2.5). We observe that for $s \in \mathbb{R}^+$, the Laplace transform $\hat{p}(s)$ is a positive and decreasing function of s, with $\hat{p}(0) = 1$ and $lim_{s\to\infty}\hat{p}(s) = 0$. Therefore $\hat{p}(s)$ has no zeros or poles in $s \in \mathbb{R}^+$.

Assume that $B_\delta(s_1) = B_\delta(s_2) = 0$ and $B_\delta(s) > 0$ in the interval (s_1, s_2). The function $B_\delta(s)$ is polynomial and by the mean value theorem it has a global maximum on $s_{max} \in (s_1, s_2)$.

If $B_\delta(s_{max}) < \hat{p}(s_{max})$ there are no real roots of the Lundberg's equation in (s_1, s_2), and if $B_\delta(s_{max}) \geq \hat{p}(s_{max})$ we will have either two real roots or a double root in (s_1, s_2). $\square$

Example 4.2.1 Suppose that the interclaim times W_i follow a generalized Erlang(3) distribution, with parameters $\lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_3 = 2.5$. Then $E[W_i] = 3.067$. Suppose that the claim amounts X_i are exponentially distributed with parameter $\beta \geq 0.5$. Then we choose $c = 1$ to satisfy the positive loading condition and let $\delta = 0.5$.

Notice that the generalized Lundberg's equation becomes

$$B_\delta(s) = \frac{(1-s)(2-s)(3-s)}{1.875} = \frac{\beta}{\beta + s} = \hat{p}(s).$$

The function $B_\delta(s)$ has 3 zeros at $s = 1, 2, 3$, furthermore it is positive in the interval $(2, 3)$. It is easy to verify that the generalized Lundberg's equation has

- Two real roots in $(2, 3)$ for $0.5 \leq \beta < 0.67$.

- A double root 2.61 in $(2, 3)$ for $\beta = 0.67$.
- Two complex conjugate roots, where the real part of them is in $(2, 3)$ for $\beta > 0.67$.

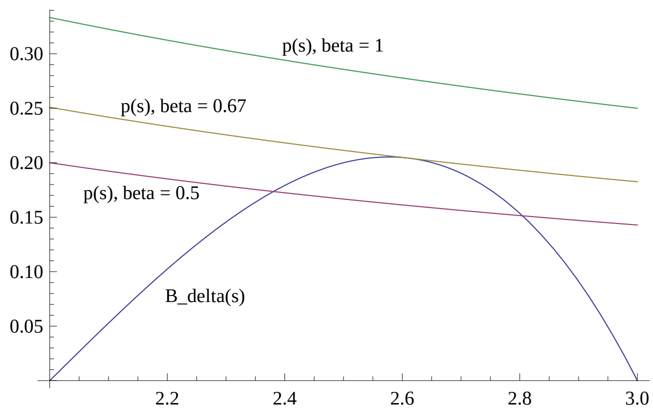


Figure 4.8: Example of the roots of the generalized Lundberg's equation

Corollary 4.2.1 Let s_1 and s_2 be two consecutive positive real zeros of $B(s)$. If $B(s)$ is positive in the interval (s_1, s_2) then the fundamental Lundberg's equation has one of the following:

- Two real roots in the interval.
- A double root in the interval.
- No real roots in this interval.

Remark 4.2.1 So far we have investigated the possibility of double roots in the fundamental and generalized Lundberg's equations for a Sparre–Andersen model with generalized Erlang(n) interclaim times. Empirically, after many numerical tests with different parameters λ_i and different values of n, we haven't found roots of higher order than double. However, the possibility of roots of a higher order is still open, and this is currently one of our lines of research.

4.3 Solutions for the integro–differential equation

In this section we look for n linearly independent particular solutions $v_j(u),\ j = 1, \ldots, n$ of the integro-differential equation (4.2.2). We use the roots of the fundamental Lundberg's equation that have positive real parts ($\rho_i,\ i = 1,\dots,n-1$) and the non–ruin probability $\Phi(u)$ in the following manner.

Theorem 4.3.1 If $\rho_1, \rho_2, \dots, \rho_{n-1} \in \mathbb{C}$ are distinct, then the following functions are linearly independent particular solutions of the integro-differential equation (4.2.2)

$$\begin{aligned} v_j(u) &= \int_0^u \Phi(u-y)e^{\rho_j y}dy, \quad j = 1,2,\dots,n-1, \\ v_n(u) &= \Phi(u). \end{aligned}$$

Proof:

Like in the Erlang(n) case, it can be proven that any solution $v(u)$ of (4.2.2) has Laplace transform

$$\hat{v}(s) = \frac{d_v(s)}{B(s) - \hat{p}(s)},$$

where $d_v(s)$ is a polynomial of degree at most $n-1$ of the form

$$\begin{aligned} d_v(s) &= \sum_{i=0}^{n-1}\left(\sum_{k=i+1}^{n}\left(\sum_{i_1<\dots<i_k}\frac{(-1)^k}{\lambda_{i_1}\cdots\lambda_{i_k}}\right)v^{(k-1-i)}(0)\right)s^i \\ &= \sum_{i=0}^{n-1}\left(\sum_{k=i+1}^{n} B_k v^{(k-1-i)}(0)\right)s^i. \end{aligned} \tag{4.3.1}$$

It is known that $\Phi(u)$ is solution of (4.2.2), its Laplace transform is given by

$$\hat{\Phi}(s) = -\Phi(0)\left(\frac{c^n}{\prod_{i=1}^{n}\lambda_i}\right)\frac{\prod_{i=1}^{n-1}(\rho_i - s)}{B(s)-\hat{p}(s)},$$

Denote by

$$d_\Phi(s) = -\Phi(0)\left(\frac{c^n}{\prod_{i=1}^{n}\lambda_i}\right)\prod_{i=1}^{n-1}(\rho_i - s). \tag{4.3.2}$$

Now we see that any function $v_j(u) = \int_0^u \Phi(u-y)e^{\rho_j y}dy$, with $j = 1,2,\dots,n-1$, is solution of (4.2.2).

We can show that

$$B(\mathcal{D})v_j(u) = d_\Phi(\rho_j)e^{\rho_j u} + \int_0^u (B(\mathcal{D})\Phi(u-t))e^{\rho_j t}dt$$

and that

$$\int_0^u v_j(u-y)p(y)dy = \int_0^u (B(\mathcal{D})\Phi(u-t))e^{\rho_j t}dt,$$

Since the roots of the fundamental Lundberg's equation (4.2.4) are all single, we proceed like we did in the previous chapter.

The remaining part to prove is that those $v_j(u)$'s are linearly independent. The proof is as follows.

Suppose that we have a linear combination such that $\sum_{j=1}^{n} c_j v_j(u) = 0$, $\forall u \geq 0$. We consider the two cases (i) and (ii) below.

(i) $c_n = 0$:

Let $H(t) = \sum_{j=1}^{n-1} c_j e^{\rho_j t}$, then

$$\begin{aligned} \sum_{j=1}^{n} c_j v_j(u) &= \sum_{j=1}^{n-1} c_j \int_0^u \Phi(u-y)e^{\rho_j y}dy \\ &= \int_0^u \Phi(u-y)\sum_{j=1}^{n-1} c_j e^{\rho_j y}dy \\ &= \Phi * H(u) = 0. \end{aligned}$$

The fact that $\Phi * H(u) = 0$, $\forall u \geq 0$ with $\Phi(u) \not\equiv 0$, implies that $H(u) \equiv 0$ almost everywhere. But $H(t)$ is a continuously differentiable function, therefore $c_1 = c_2 = \cdots = c_n = 0$.

(ii) $c_n \neq 0$:

Define $G(t) = \sum_{j=1}^{n-1} (-c_j/c_n) e^{\rho_j t}$, so $\Phi * G(u) = \Phi(u)$ $\forall u \geq 0$. Not all the remaining coefficients c_j's can be 0, otherwise $G(t) \equiv 0$. But then $\lim_{u\to+\infty} G(u) = \pm\infty$ depending on the sign of the non zero coefficients. As $\Phi(u)$ is a non–decreasing non–negative function with $\lim_{u\to+\infty} \Phi(u) = 1$, we have that $\lim_{u\to+\infty} \Phi * G(u) = \pm\infty$, which is a contradiction.

This completes the proof. $\square$

We have shown a set of n linearly independent particular solutions of the integro-differential equation (4.2.2) for the case when the roots $\rho_1, \rho_2, \ldots, \rho_{n-1} \in \mathbb{C}$ are distinct. Since the fundamental Lundberg's equation for the model with generalized Erlang(n) interclaim times has the possibility of multiple roots, we show the corresponding particular solutions for this cases.

First suppose that we have one root with multiplicity $n-1$.

Theorem 4.3.2 If $\rho_1 = \rho_2 = \ldots = \rho_{n-1} = \rho$ then the following functions are linearly independent particular solutions of the integro-differential equation (4.2.2)

$$\begin{aligned} v_j(u) &= \int_0^u \Phi(u-y) y^{j-1} e^{\rho y} dy, \quad j = 1, 2, \ldots, n-1, \\ v_n(u) &= \Phi(u). \end{aligned}$$

Proof: In the same way by taking derivatives of the functions $v_j(u)$'s we get

$$B(\mathcal{D}) v_j(u) = \int_0^u v_j(u-y) p(y) dy, \quad j = 1, 2, \ldots, n.$$

Let $v_j(u) = \int_0^u \Phi(u-y) y^{j-1} e^{\rho y} dy$, for some $1 \leq j \leq n-1$. The k-th derivative of $v_j(u)$ is given by

$$v_j^{(k)}(u) = \left(\sum_{i=0}^{j-1} \binom{j-1}{i} f_k^{(i)}(\rho) u^{j-1-i} \right) e^{\rho u} + \int_0^u \Phi^{(k)}(u-y) y^{j-1} e^{\rho y} dy,$$

where the functions $f_k(s)$ are defined as follows

$$f_k(s) = \sum_{i=0}^{k-1} \Phi^{(k-1-i)}(0) s^i, \quad 0 \leq k \leq n, \quad f_0(s) \equiv 0.$$

Notice that $f_k^{(i)}(s) \equiv 0$ for $i \geq k$. Recalling (4.3.1) and (4.3.2) we obtain

$$\begin{aligned} d_\Phi(s) &= \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} B_k \Phi^{(k-1-i)}(0) \right) s^i \\ &= \sum_{k=0}^{n} B_k \left(\sum_{i=0}^{k-1} \Phi^{(k-1-i)}(0) s^i \right) \\ &= \sum_{k=0}^{n} B_k f_k(s) = -\Phi(0) \left(\frac{c^n}{\prod_{i=1}^n \lambda_i} \right) (\rho - s)^{n-1}. \end{aligned}$$

Therefore, the derivatives of $d_\Phi(s)$ are

$$d_\Phi^{(i)}(s) = \sum_{k=0}^{n} B_k f_k^{(i)}(s),$$

and $d_\Phi^{(i)}(\rho) \equiv 0$ for $0 \le i \le n-2$.

Hence we have, from one side

$$\begin{aligned}
B(\mathcal{D})v_j(u) &= \sum_{k=0}^{n} B_k v_j^{(k)}(u) = \\
&= \sum_{k=0}^{n} B_k \left(\sum_{i=0}^{j-1} \binom{j-1}{i} f_k^{(i)}(\rho) u^{j-1-i} \right) e^{\rho u} + \sum_{k=0}^{n} B_k \int_0^u \Phi^{(k)}(u-y) y^{j-1} e^{\rho y} dy \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} \left(\sum_{k=0}^{n} B_k f_k^{(i)}(\rho) \right) u^{j-1-i} e^{\rho u} + \int_0^u \left(\sum_{k=0}^{n} B_k \Phi^{(k)}(u-y) \right) y^{j-1} e^{\rho y} dy \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} d_\Phi^{(i)}(\rho) u^{j-1-i} e^{\rho u} + \int_0^u \left(B(\mathcal{D})\Phi(u-y) \right) y^{j-1} e^{\rho y} dy \\
&= \int_0^u \left(B(\mathcal{D})\Phi(u-y) \right) y^{j-1} e^{\rho y} dy,
\end{aligned}$$

and from the other side

$$\begin{aligned}
\int_0^u v_j(u-x)p(x)dx &= \int_0^u \left(\int_0^{u-x} \Phi(u-x-y) y^{j-1} e^{\rho y} dy \right) p(x)dx \\
&= \int_0^u \left(\int_0^{u-y} \Phi(u-y-x) p(x) dx \right) y^{j-1} e^{\rho y} dy \\
&= \int_0^u (B(\mathcal{D})\Phi(u-y)) y^{j-1} e^{\rho y} dy.
\end{aligned}$$

This finishes the proof that the functions $v_j(u)$ are solutions of (4.2.2).

To see the linear independence of the $v_j(u)$'s we proceed like in the proof of Theorem 4.3.1. $\square$

Now assume the most general case, when we have k different roots, $\rho_1, \rho_2, \ldots, \rho_k$, where the root ρ_i has multiplicity m_i and $\sum_{i=1}^{k} m_i = n-1$.

Theorem 4.3.3 Under the conditions described above, the following functions are linearly independent particular solutions of the integro-differential equation (4.2.2)

$$\begin{aligned} v_{00}(u) &= \Phi(u), \\ v_{ij}(u) &= \int_0^u \Phi(u-y) y^{j-1} e^{\rho_i y} dy, \quad i = 1, 2, \dots, k, \\ & \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \dots, m_i. \end{aligned}$$

Proof: The proof is based on the Theorems 4.3.1 and 4.3.2. □

4.3.1 A note on the survival probability

In this section we consider the survival probability and its derivatives when $u = 0$. Li and Garrido (2004a) showed that

$$\Phi(0) = \frac{\prod_{i=1}^{n} \lambda_i (c \sum_{i=1}^{n} \frac{1}{\lambda_i} - \mu_1)}{c^n \prod_{i=1}^{n-1} \rho_i}.$$

We write $\Phi(0) = (\bar{\lambda}\ \bar{\mu})/(c^n\ \bar{\rho})$, where $\bar{\lambda} = \prod_{i=1}^{n} \lambda_i$, $\bar{\mu} = c \sum_{i=1}^{n} (1/\lambda_i) - \mu_1$ and $\bar{\rho} = \prod_{i=1}^{n-1} \rho_i$.

First of all we recall an operator of integrable real functions, originally proposed by Dickson and Hipp (2001) and Li and Garrido (2004b). We list some of the most important properties.

Remark 4.3.1 Let f be a real–valued integrable function, and define

$$T_r f(x) = \int_x^{\infty} e^{-r(u-x)} f(u) du, \quad r \in \mathbb{C}, \quad x \geq 0, \tag{4.3.3}$$

where r has a non–negative real part, $Re(r) \geq 0$.

The operator T_r satisfies the following properties:

1. $T_r f(0) = \int_0^{\infty} e^{-ru} f(u) du = \hat{f}(r)$, is the Laplace transform of f.
2. $T_{r_1} T_{r_2} f(x) = T_{r_2} T_{r_1} f(x) = \frac{(T_{r_1} f(x) - T_{r_2} f(x))}{r_2 - r_1}$, for $r_2 \neq r_1$.
3. $T_r^n f(x) = \left(\frac{(-1)^{n-1}}{(n-1)!}\right) \left(\frac{d^{n-1}}{dr^{n-1}}\right) T_r f(x)$ and the corresponding Laplace

transform

$$T_s T_r^n f(0) = \frac{\hat{f}(s)}{(r-s)^n} - \sum_{j=1}^{n} \frac{T_r^j f(0)}{(r-s)^{n+1-j}}, \quad s \in \mathbb{C}.$$

4. If $r_1, r_2, \ldots, r_k$ are distinct complex numbers, then

$$T_{r_k} \cdots T_{r_2} T_{r_1} f(x) = (-1)^{k-1} \sum_{i=1}^{k} \frac{T_{r_i} f(x)}{\tau_k'(r_i)},$$

where $\tau_k(r) = \prod_{i=1}^{k}(r - r_i)$. The corresponding Laplace transform is

$$T_s T_{r_k} \cdots T_{r_2} T_{r_1} f(0) = (-1)^k \left[\frac{\hat{f(s)}}{\tau_k(s)} - \sum_{i=1}^{k} \frac{\hat{f}(r_i)}{(s - r_i)\tau_k'(r_i)} \right], \quad s \in \mathbb{C}. \qquad \square$$

We have mentioned before that

$$\begin{aligned} d_\Phi(s) &= -\Phi(0) \left(\frac{c^n}{\prod_{i=1}^{n} \lambda_i} \right) \prod_{i=1}^{n-1} (\rho_i - s) \\ &= \left(-\frac{\bar{\mu}}{\bar{\rho}} \right) \prod_{i=1}^{n-1} (\rho_i - s) = \sum_{i=0}^{n-1} \tilde{a}_i s^i, \end{aligned} \qquad (4.3.4)$$

where

$$\tilde{a}_i = \left(-\frac{\bar{\mu}}{\bar{\rho}} \right) \left((-1)^j \sum_{i_1 < \cdots < i_{n-1-j}} \rho_{i_1} \cdots \rho_{i_{n-1-j}} \right).$$

On the other hand we have

$$d_\Phi(s) = \sum_{i=0}^{n-1} \left(\sum_{k=i+1}^{n} B_k \Phi^{(k-1-i)}(0) \right) s^i = \sum_{i=0}^{n-1} \tilde{b}_i s^i. \qquad (4.3.5)$$

We compare the coefficient of s^i in (4.3.4) and (4.3.5) to get a system of n equations $\tilde{a}_i = \tilde{b}_i$, $0 \le i \le n-1$, for the unknowns $\Phi^{(k)}(0)$, $k = 0, 1, \ldots, n-1$ (we already know $\Phi(0)$). After solving that system we obtain the following

$$\Phi^{(k)}(0) = A_k \Phi(0), \quad k = 0, 1, \ldots, n-1, \qquad (4.3.6)$$

where the constants A_k are given by $A_0 = 1$ and

$$A_k = \sum_{i_1<\cdots<i_k} (-1)^{k+1} \left[\frac{\lambda_{i_1} \cdots \lambda_{i_k}}{c^k} - \rho_{i_1} \cdots \rho_{i_k} \right] + \sum_{j=1}^{k-1} (-1)^{k+1-j} \left[\frac{\lambda_{i_1} \cdots \lambda_{i_{k-j}}}{c^{k-j}} \right],$$

with $k = 1, \ldots, n-1$. We notice that the higher derivatives of $\Phi(u)$ at $u = 0$ are just multiples of $\Phi(0)$.

Li and Garrido (2004b) found a defective renewal equation for the survival probability $\Phi(u)$

$$\Phi(u) = \int_0^u \Phi(u-y)\eta_0(y)dy + \Phi(0), \tag{4.3.7}$$

where $\eta_0(y) = \frac{\bar{\lambda}}{c^n} T_0 T_{\rho_{n-1}} \cdots T_{\rho_1} p(y)$ is a "defective density".

We compute the derivatives of $\Phi(u)$ at $u = 0$ using equation (4.3.7) and obtain

$$\Phi^{(k)}(0) = \Phi(0) \left[\eta_0^k(0) + \sum_{i=1}^{k-1} \binom{k-1}{i} \eta_0^{k-1-i}(0) \eta_0^{(i)}(0) \right], \; k = 1, \ldots, n-1, \tag{4.3.8}$$

Thus, comparing the expressions for $\Phi^{(k)}(0)$ in (4.3.6) and (4.3.8) we get

$$A_k = \eta_0^k(0) + \sum_{i=1}^{k-1} \binom{k-1}{i} \eta_0^{k-1-i}(0) \eta_0^{(i)}(0).$$

Hence, from the equation above we obtain expressions for the derivatives of $\eta_0(y)$ at $y = 0$

$$\begin{aligned} \eta_0(0) &= A_1, \\ \eta_0^{(k-1)}(0) &= \sum_{j=0}^{k} (-1)^{j+1} \left(\sum_{i_1<\cdots<i_j} \frac{\lambda_{i_1} \cdots \lambda_{i_j}}{c^j} \right) \times \\ &\quad \left(\sum_{i_1 \le \cdots \le i_{k-j}} \rho_{i_1} \cdots \rho_{i_{k-j}} \right), \end{aligned} \tag{4.3.9}$$

for $k = 1, \ldots, n-1$. On the other hand, we compute directly the derivatives of $\eta_0(y)$ at $y = 0$ to get the expression

$$\begin{aligned}\eta_0^{(k-1)}(0) &= -\sum_{i=n-k}^{n}\left(\sum_{1\le j_1<\cdots<j_{n-i}\le n}\left(\prod_{m=1}^{n-i}\left(\rho_{n-(k-1)}-\frac{\lambda_{j_m}}{c}\right)\right)\right)\times\\ &\quad\left(\sum_{1\le j_1\le\cdots\le j_{i-n+k}\le n-k}\left(\prod_{m=1}^{i-n+k}\left(\rho_{j_m}-\rho_{n-(k-1)}\right)\right)\right). \qquad (4.3.10)\end{aligned}$$

Both expressions for $\eta_0^{(k-1)}(0)$ given in (4.3.9) and (4.3.10) are equivalent. From this equivalence we obtain many combinatorial identities, but that belongs to field of Combinatorics and goes beyond the scope of this thesis.

Remark 4.3.2 The defective density $\eta_0(y)$ is a special case of the function $\eta_\delta(y)$ for a force of interest $\delta \ge 0$. This function appears in Li and Garrido (2004b) for the study of the Gerber–Shiu penalty functions. We give a short description of the penalty functions in the Appendix B.

4.4 The maximum severity of ruin

In the previous section we have shown how to obtain the solutions of the integro–differential equation. Now we use these results to obtain the corresponding expressions for the distribution of the maximum severity of ruin. We find an expression for that distribution which only depends on the non-ruin probability $\Phi(u)$ and the claim amounts distribution.

From Dickson (2005) and (4.2.3) we know that the distribution of the maximum severity of ruin $J(z;u)$ can be expressed as

$$J(z;u) = \frac{1}{1-\Phi(u)}\int_0^z g(u,y)(v_1(z-y),\ldots,v_n(z-y))dy[V(z)]^{-1}\overrightarrow{e}^T. \qquad (4.4.1)$$

If we denote by

$$\begin{aligned}\overrightarrow{h}(z,u) &= \int_0^z g(u,y)(v_1(z-y),\ldots,v_n(z-y))dy\\ &= \left(\int_0^z g(u,y)v_1(z-y)dy,\ldots,\int_0^z g(u,y)v_n(z-y)dy\right)\\ &= (h_1(z,u),\ldots,h_n(z,u)),\end{aligned}$$

then we only have to find an expression for every component of $\overrightarrow{h}(z,u)$.

We consider the case of the Theorem 4.3.1 and obtain
For $j = 1, 2, \ldots, n-1$

$$\begin{aligned}\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)e^{\rho_j x}dxdy\\ &= \int_0^z e^{\rho_j x}[\Phi(u+(z-x))-\Phi(u)]dx,\end{aligned}$$

and for $j = n$

$$\int_0^z g(u,y)v_n(z-y)dy = \int_0^z g(u,y)\Phi(z-y)dy = \Phi(u+z)-\Phi(u).$$

In a similar manner when we consider the case of the Theorem 4.3.3 we have
For $i = j = 0$

$$\int_0^z g(u,y)v_{00}(z-y)dy = \int_0^z g(u,y)\Phi(z-y)dy = \Phi(u+z)-\Phi(u),$$

and for $i = 1, \ldots, k; \quad j = 1, \ldots, m_i$

$$\begin{aligned}\int_0^z g(u,y)v_j(z-y)dy &= \int_0^z g(u,y)\int_0^{z-y}\Phi(z-y-x)x^{j-1}e^{\rho_i x}dxdy\\ &= \int_0^z x^{j-1}e^{\rho_i x}\left[\int_0^{z-x} g(u,y)\Phi((z-x)-y)dy\right]dx\\ &= \int_0^z x^{j-1}e^{\rho_i x}[\Phi(u+(z-x))-\Phi(u)]dx.\end{aligned}$$

4.4.1 Example

In this part we present an example. Consider that the interclaim times are generalized Erlang(3,$\lambda_1, \lambda_2, \lambda_3$) distributed, with $\lambda_1 \neq \lambda_2 \neq \lambda_3$, and claim amounts exponential(β) distributed. For simplification we denote this case by generalized Erlang(3) – exponential:

$$W_i \sim \text{generalized Erlang}(3, \lambda), \quad X_i \sim \text{exponential}(\beta)$$

Considering the safety loading $c = \dfrac{(1+\theta)\lambda_1\lambda_2\lambda_3}{\beta(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)}$ with $\theta > 0$, the generalized Lundberg's equation (4.2.3) takes the form

$$\prod_{i=1}^{3}\left(1-\left(\frac{c}{\lambda_i}\right)s\right)-\frac{\beta}{(s+\beta)}=0,$$

which has four roots: $0, \rho_1, \rho_2$ and $-R$, where $0 < R < \beta$ is the adjustment coefficient. Assume that $\rho_1 = \rho_2 = \rho$ is a double (real) root (therefore $\rho > 0$).

Applying Theorem 4.3.2, the 3 solutions for the integro – differential equation (4.2.2) are

$$\begin{aligned}
\Phi(u) &= 1-\left(1-\frac{R}{\beta}\right)e^{-Ru},\\
v_2(u) &= \frac{-1}{\rho}+\frac{\beta-R}{\beta(R+\rho)}e^{-Ru}+\frac{R(\beta+\rho)}{\rho\beta(R+\rho)}e^{\rho u},\\
v_3(u) &= \frac{1}{\rho^2}-\frac{\beta-R}{\beta(R+\rho)^2}e^{-Ru}-\frac{R(2\beta\rho+R\beta+\rho^2)}{\rho^2\beta(R+\rho)^2}e^{\rho u}+\frac{R(\beta+\rho)}{\rho\beta(R+\rho)}ue^{\rho u}.
\end{aligned}$$

We calculate the distribution of the maximum of ruin using Equation (4.4.1) to get

$$J(z;u)=1-\frac{\alpha e^{-Rz}}{1-\gamma e^{-(\rho+R)z}-\delta e^{-(\rho+R)z}z-\eta e^{-Rz}},$$

where

$$\begin{aligned}
\alpha &= \frac{R(R+\rho)^2}{\beta(\beta+\rho)^2}, \qquad \delta = -\frac{R(\rho+R)(\beta-R)}{\rho(\beta+\rho)},\\
\gamma &= -\frac{R(\beta-R)}{\rho^2(\beta+\rho)^2}((R+\rho)(\beta+\rho)+\rho(2\rho+R+\beta)),\\
\eta &= 1-\frac{R}{\beta+\rho}\left[\frac{R+\rho}{\beta}-\frac{(\beta-R)(R+\rho)}{\rho\beta}-\frac{(\beta-R)(R+2\rho)}{\rho^2}\right],
\end{aligned}$$

with $\eta = 1-\alpha-\gamma$.
Observe that the expression for $J(z;u)$ is independent from u.

We obtain formulas for the moments of the maximum severity M_u given that ruin occurs

$$\begin{aligned} E(M_u^r|T<\infty) &= r\int_0^\infty z^{r-1}(1-J(z;u))dz \\ &= r\alpha\int_0^\infty \frac{z^{r-1}e^{-Rz}}{1-\gamma e^{-(\rho+R)z}-\delta z e^{-(\rho+R)z}-\eta e^{-Rz}}\,dz, \end{aligned}$$

for $r \geq 1$.

Choosing $\beta = 1, \lambda_1 = 6.098, \lambda_2 = 2, \lambda_3 = 3, \theta = 0.1$ and $c = 1.103$. We get a double root $\rho = 4.596$, with adjustment coefficient $R = 0.129$ and

$$J(z;u) = 1 - \frac{0.092e^{-0.129z}}{1+0.012e^{-4.724z}+0.021e^{-4.724z}z-0.921e^{-0.129z}}.$$

The expected value and the standard deviation of the maximum severity of ruin are $E(M_u) = 1.932$ and $s.d.(M_u) = 3.528$.

4.5 Dividends

In this section we consider once more the dividends problem. We follow Dickson and Waters (2004) to present an equation for $V_m(u,b)$ in a generalized Erlang(n) risk process. Conditioning on the time and the amount of the first claim we get, for $0 \leq u < b$,

$$\begin{aligned} V_m(u,b) &= \int_{\frac{b-u}{c}}^\infty k_n(t)e^{-m\delta t}\left[\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}}\right)^m + \right. \\ &\left. +\sum_{j=1}^m \binom{m}{j}\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}}\right)^{m-j}\int_0^b p(x)V_j(b-x,b)dx\right] dt + \\ &+\int_0^{\frac{b-u}{c}} e^{-m\delta t}k_n(t)\int_0^{u+ct} V_m(u+ct-x,b)p(x)dx\,dt, \qquad (4.5.1) \end{aligned}$$

for $m \geq 1$. In particular, for $m = 1$,

$$\begin{aligned} V(u,b) &= \int_{\frac{b-u}{c}}^\infty k_n(t)e^{-\delta t}\left(c\,\overline{s}_{\overline{t-\frac{b-u}{c}|}} + \int_0^b p(x)V(b-x,b)dx\right) dt + \\ &+\int_0^{\frac{b-u}{c}} e^{-\delta t}k_n(t)\int_0^{u+ct} V(u+ct-x,b)p(x)dx\,dt, \qquad (4.5.2) \end{aligned}$$

where $\overline{s}_{\overline{t|}} = (e^{\delta t} - 1)/\delta$ in standard actuarial notation.
For a generalized Erlang(n) risk process the integro–differential equations satisfied by the discounted expected dividends are

$$
\begin{aligned}
\prod_{i=1}^{n} \left(\left(1 + \frac{\delta}{\lambda_i} \right) \mathcal{I} - \frac{c}{\lambda_i} \mathcal{D} \right) V(u,b) &= \int_0^u V(u-x,b) p(x) dx \qquad (4.5.3) \\
\left. \frac{d^k V(u,b)}{du^k} \right|_{u=b} &= \left(\frac{\delta}{c} \right)^{k-1}, \; 1 \leq k \leq n,
\end{aligned}
$$

and for a general m

$$
\begin{aligned}
\prod_{i=1}^{n} \left(\left(1 + \frac{m\delta}{\lambda_i} \right) \mathcal{I} - \frac{c}{\lambda_i} \mathcal{D} \right) V_m(u,b) &= \int_0^u V_m(u-x,b) p(x) dx \qquad (4.5.4) \\
\left. \frac{d^k V_m(u,b)}{du^k} \right|_{u=b} &= \sum_{j=1}^{k} \frac{m!}{(m-j)!} \left\{ {k \atop j} \right\} \left(\frac{\delta}{c} \right)^{k-j} V_{m-j}(b,b),
\end{aligned}
$$

for $1 \leq k \leq n$, where $\left\{ {k \atop j} \right\} = (1/j!) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k$ denotes the Stirling numbers of the second kind, as in the previous chapter. For convenience we define $V_{m-j}(u,b) \equiv 0$, for $m < j$ in the formula above.

These equations generalize those proposed by Dickson (2005) and Dickson and Waters (2004) for the classical Poisson risk model, and Albrecher *et al.* (2005) for a Sparre–Andersen risk model.

For our purpose we solve the integro–differential Equations (4.5.3) and (4.5.4) to determine the expected discounted dividends and the higher moments considering that the generalized Lundberg's equation (4.2.5) has k different roots with positive real parts, $\rho_1, \rho_2, \ldots, \rho_k$, and the root ρ_i has multiplicity $m_i \geq 1, \; i = 1, 2, \ldots, k$.

Following an argument originally proposed by Bühlman (1970), Section 6.4.9, for a Poisson risk model, we propose for a generalized Erlang(n) risk model that $V(u,b)$ is

$$
V(u,b) = \sum_{i=1}^{k} \left(\sum_{j=1}^{m_i} C_{ij} \beta_{ij}(u) \right) e^{\rho_i u}, \qquad (4.5.5)
$$

where C_{ij}'s are constants (that depend on the parameter b), and the functions

$\beta_{ij}(u)$ are solutions of the integro–differential equations

$$\prod_{t=1}^{n}\left(\mathcal{I}-\frac{c}{\lambda_{ti}}\mathcal{D}\right)v(u)=\int_0^u v(u-x)p_i(x)dx, \tag{4.5.6}$$

with $\lambda_{ti} = \lambda_t + \delta - c\rho_i$ and $p_i(x) = e^{-\rho_i x}p(x)/\hat{p}(\rho_i)$, for $t = 1,\ldots,n$, $i = 1,\ldots,k$.

Thus, we get the functions $\beta_{ij}(u)$ solving an equation of the same kind as equation (4.2.2) but with different "parameters" and a different "density".

The constants C_{ij}'s are determined with the use of the boundary conditions given in (4.5.3), which gives a system of n equations with n unknowns

$$\begin{aligned}\left.\frac{d^kV(u,b)}{du^k}\right|_{u=b} &= \sum_{i=1}^{k}\sum_{j=1}^{m_i}C_{ij}\left.\frac{d^k(\beta_{ij}(u)e^{\rho_i u})}{du^k}\right|_{u=b}\\ &= \left(\frac{\delta}{c}\right)^{k-1},\ 1\le k\le n.\end{aligned} \tag{4.5.7}$$

We summarize this in the following theorem

Theorem 4.5.1 The solutions of integro-differential equation (4.5.3) are of the form

$$V(u,b)=\sum_{i=1}^{k}\left(\sum_{j=1}^{m_i}C_{ij}\beta_{ij}(u)\right)e^{\rho_i u},$$

where ρ_i's are the roots with positive real parts of the generalized Lundberg's equation (4.2.5), $\beta_{ij}(u)$'s are defined in (4.5.6) and the constants C_{ij}'s are defined in (4.5.7).

Proof:

The proof follows by taking derivatives of $V(u,b)$ like in the Erlang(n) case, and finding the conditions which must be satisfied by the ρ_i's and $\beta_{ij}(u)$'s in order to get the equality in (4.5.3). $\square$

This method generalizes the results of Albrecher *et al.* (2005). It works for any kind of claim amounts distribution, and not only for the distributions with rational Laplace transforms.

We implement the same approach to find the m–th moment $V_m(u,b)$, $m \geq 2$, writing it in the form (4.5.5) and using the corresponding boundary conditions (4.5.4).

4.5.1 Example

We consider the interclaim times generalized Erlang(3) distributed, with parameters $\lambda_1, \lambda_2, \lambda_3$, and the claim amounts exponentially distributed with parameter α. The force of interest is $\delta > 0$.

We get the generalized Lundberg's equation (4.2.5)

$$(\lambda_1 + \delta - cs)(\lambda_2 + \delta - cs)(\lambda_3 + \delta - cs) = \frac{\lambda_1\lambda_2\lambda_3\alpha}{\alpha + s},$$

where $c = \dfrac{(1+\theta)\lambda_1\lambda_2\lambda_3}{\alpha(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)}$ for some $\theta > 0$.

There are three roots with positive real parts. We assume $R > 0$ is the adjustment coefficient and that we have a single root $\rho_1 > 0$ and a double root $\rho_2 > 0$.

Applying Theorem 4.5.1 we write

$$V(u, b) = C_{11}v_{11}(u)e^{\rho_1 u} + (C_{21}v_{21}(u) + C_{22}v_{22}(u))e^{\rho_2 u}.$$

We already know ρ_1 and ρ_2. We must find the constants C_{11}, C_{21}, C_{22} and the functions $v_{11}(u), v_{21}(u), v_{22}(u)$.

To find the functions $v_{11}(u), v_{21}(u)$ and $v_{22}(u)$ we proceed as follows.

Notice that $v_{11}(u)$ is a solution of the integro–differential equation

$$\prod_{t=1}^{3}\left(\mathcal{I} - \frac{c}{\lambda_{t1}}\mathcal{D}\right)v(u) = \int_0^u v(u-x)p_1(x)dx, \tag{4.5.8}$$

where $\lambda_{t1} = \lambda_t + \delta - c\rho_1$ and $p_1(x) = (\alpha + \rho_1)e^{-(\alpha+\rho_1)x} = \alpha_1 e^{-\alpha_1 x}$, for $t = 1, 2, 3$.

Let

$$(\lambda_{11} - cs)(\lambda_{21} - cs)(\lambda_{31} - cs) = \frac{\lambda_{11}\lambda_{21}\lambda_{31}\alpha_1}{\alpha_1 + s}$$

be the associated fundamental Lundberg's equation and $R_1 > 0$ the corresponding adjustment coefficient.

Then we choose $v_{11}(u) = 1 - (1 - \frac{R_1}{\alpha_1})e^{-R_1 u}$, a "survival probability", which is a well known solution of (4.5.8).

The functions $v_{21}(u)$ and $v_{22}(u)$ are both solutions of

$$\prod_{t=1}^{3}\left(\mathcal{I}-\frac{c}{\lambda_{t2}}\mathcal{D}\right)v(u)=\int_0^u v(u-x)p_2(x)dx, \tag{4.5.9}$$

where $\lambda_{t2} = \lambda_t + \delta - c\rho_2$ and $p_2(x) = (\alpha+\rho_2)e^{-(\alpha+\rho_2)x} = \alpha_2 e^{-\alpha_2 x}$, for $t = 1, 2, 3$.

Let

$$(\lambda_{12}-cs)(\lambda_{22}-cs)(\lambda_{32}-cs)=\frac{\lambda_{12}\lambda_{22}\lambda_{32}\alpha_2}{\alpha_2+s}$$

be the associated fundamental Lundberg's equation, $R_2 > 0$ the corresponding adjustment coefficient and ρ_{21}, ρ_{22} the two roots with positive real parts.

Let $\tilde{v}(u) = 1-(1-\frac{R_2}{\alpha_2})e^{-R_2u}$. Thus, we use Theorem 4.3.1 (if $\rho_{21} \neq \rho_{22}$, otherwise we use Theorem 4.3.2), and choose

$$\begin{aligned} v_{21}(u) &= \int_0^u \tilde{v}(u-y)e^{\rho_{21}y}dy \\ &= \frac{-1}{\rho_{21}}+\frac{\alpha_2-R_2}{\alpha_2(R_2+\rho_{21})}e^{-R_2u}+\frac{R_2(\alpha_2+\rho_{21})}{\rho_{21}\alpha_2(R_2+\rho_{21})}e^{\rho_{21}u} \\ v_{22}(u) &= \int_0^u \tilde{v}(u-y)e^{\rho_{22}y}dy \\ &= \frac{-1}{\rho_{22}}+\frac{\alpha_2-R_2}{\alpha_2(R_2+\rho_{22})}e^{-R_2u}+\frac{R_2(\alpha_2+\rho_{22})}{\rho_{22}\alpha_2(R_2+\rho_{22})}e^{\rho_{22}u} \end{aligned}$$

For C_{11}, C_{21}, C_{22} we use the boundary conditions given in (4.5.3) and we get

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \left.\frac{d(e^{\rho_1 u}v_{11}(u))}{du}\right|_{u=b} & \left.\frac{d(e^{\rho_2 u}v_{21}(u))}{du}\right|_{u=b} & \left.\frac{d(e^{\rho_2 u}v_{22}(u))}{du}\right|_{u=b} \\ \left.\frac{d^2(e^{\rho_1 u}v_{11}(u))}{du^2}\right|_{u=b} & \left.\frac{d^2(e^{\rho_2 u}v_{21}(u))}{du^2}\right|_{u=b} & \left.\frac{d^2(e^{\rho_2 u}v_{22}(u))}{du^2}\right|_{u=b} \\ \left.\frac{d^3(e^{\rho_1 u}v_{11}(u))}{du^3}\right|_{u=b} & \left.\frac{d^3(e^{\rho_2 u}v_{21}(u))}{du^3}\right|_{u=b} & \left.\frac{d^3(e^{\rho_2 u}v_{22}(u))}{du^3}\right|_{u=b} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \left(\frac{\delta}{c}\right) \\ \left(\frac{\delta}{c}\right)^2 \end{pmatrix}.$$

The functions $e^{\rho_1 u}v_{11}(u), e^{\rho_2 u}v_{21}(u)$ and $e^{\rho_2 u}v_{22}(u)$ are linearly independent and we can invert the matrix in the above expression to obtain the desired result.

4.6 Final remarks

One of the important goals of the study of risk theory is to find exact numerical techniques which can become popular in insurance practice. In this chapter we have investigated the Lundberg's equations in a Sparre–Andersen risk model with generalized Erlang(n) interclaim times. We have obtained theorems for the calculation of the maximum severity of ruin and the expected discounted dividends which are useful for the problem of the multiple roots. However, the results can be still generalized for other interclaim times distributions. Additionally, the possibility of roots with multiplicities higher than double can be investigated. We will show further developments in the next chapter.

Chapter 5

The Sparre–Andersen model with Phase–Type(n) interclaim times

Pure mathematics is, in its way,
the poetry of logical ideas

Albert Einstein

5.1 Introduction

In this chapter we consider the Sparre–Andersen model under the assumption that the interclaim times are Phase–Type(n) distributed.

In Section 5.2 we give a brief introduction to the Phase–Type(n) distribution family following Asmussen (2000).

We focus on studying the generalized Lundberg's equation in Section 5.3. Our aim is to determine the cases when multiple roots arise. We find an exact expression for the generalized Lundberg's equation involving rational polynomials, using some techniques from linear algebra, which gives us the possibility of analyzing the roots.

We consider in Section 5.4 the survival probability associated to this model. We find an integro–differential equation that is satisfied by the survival probability. Also, we find its Laplace transform and a defective renewal equation. Following a similar procedure like in previous chapters, we propose a method to study the maximum severity of ruin.

In Section 5.5 we introduce the Lundberg's matrix and show new results concerning its eigenvectors.

Finally, in Section 5.6 we apply results to compute the probability of arriving to a barrier prior to ruin.

5.2 Mathematical background and notation

Phase–type distributions are the computational vehicle of much of modern applied probability. Typically, if a problem can be solved explicitly when the relevant distributions are exponentials, then the problem may admit an algorithmic solution involving a reasonable degree of computational effort, if one allows for the more general assumption of phase–type structure, and not in other cases. A proper knowledge of phase–type distributions seems therefore a must for anyone working in an applied probability area like risk theory.

We say that a distribution K on $(0,\infty)$ is Phase–Type(n) if K is the distribution of the lifetime of a terminating continuous time Markov process $\{J(t)\}_{t\geq 0}$ with finitely many states and time homogeneous transition rates. More precisely, we define a terminating Markov process $\{J(t)\}_{t\geq 0}$ with state space $E=\{1,2,\ldots,n\}$ and intensity matrix $\mathbf{B}$ $(n\times n)$ as the restriction to E of a Markov process $\{\bar{J}(t)\}_{0\leq t<\infty}$ on $E_0 = E\cup\{0\}$ where 0 is some extra state which is absorbing, that is, $Pr(\bar{J}(t)=0|\bar{J}(0)=i)=1$ for all $i\in E$ and where all states $i\in E$ are transient. This implies in particular that the intensity matrix for $\{\bar{J}(t)\}$ can be written in block–partitioned form as

$$\left(\begin{array}{c|c} \mathbf{B} & \mathbf{b}^{\mathsf{T}} \\ \hline \mathbf{0} & 0 \end{array}\right) \tag{5.2.1}$$

The $1\times n$ vector $\mathbf{b}=(b_1,\ldots,b_n)$ is the exit rate vector, i. e., the i–th component b_i gives the intensity in state i for leaving E and going to the absorbing state 0.

Note that since (5.2.1) is the intensity matrix of a non–terminating Markov process, the rows sums to zero which in matrix notation can be written as $\mathbf{b}^{\mathsf{T}}+\mathbf{B}\mathbf{1}^{\mathsf{T}}=\mathbf{0}$ where $\mathbf{1}=(1,1,\ldots,1)$ is the column vector with all components equal to one. In particular we have

$$\mathbf{b}^{\mathsf{T}} = -\mathbf{B}\mathbf{1}^{\mathsf{T}}$$

The intensity matrix $\mathbf{B}$ is denoted by $\mathbf{B}=(b_{i,j})_{i,j=1}^{n}$. This matrix satisfies the conditions: $b_{i,i}<0$, $b_{i,j}\geq 0$ for $i\neq j$, and $\sum_{j=1}^{n} b_{i,j}\leq 0$ for $i=1,\ldots,n$.

The vector of entry probabilities is given by $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with

$\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, so $Pr(\bar{J}(0) = i) = \alpha_i$.

We list the most important properties of K.

$$\begin{array}{rrcl} \text{Density} & k(t) & = & \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}}, \quad t \geq 0, \\ C.D.F. & K(t) & = & 1 - \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{1}^{\mathsf{T}}, \quad t \geq 0, \\ L.T. & \hat{k}(s) & = & \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \\ \text{Mean} & E[W_1] & = & -\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}}, \\ & k^{(j)}(0) & = & \boldsymbol{\alpha}\mathbf{B}^{j}\mathbf{b}^{\mathsf{T}}, \quad j \geq 0, \end{array} \tag{5.2.2}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. In the Figure 5.10 below we represent three of the transient states i, j and k as well as their transition rates, exit rates and entry probabilities.

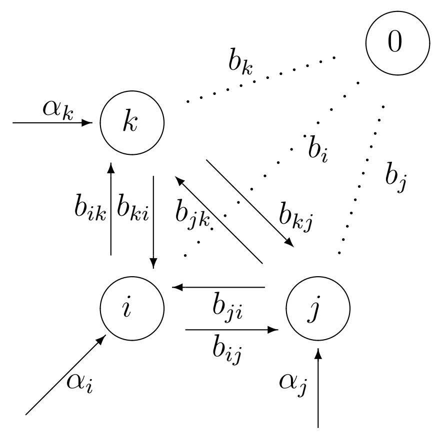


Figure 5.9: Phase–Type distribution

Now we give some examples of some very well known distributions.

Example 5.2.1 Suppose that $n = 1$ and write $\mathbf{B} = (-b_{11}) = (-\beta)$. Then $\boldsymbol{\alpha} = (\alpha_1) = (1)$, $\mathbf{b} = (b_1) = (b_{11}) = (\beta)$ and $k(t) = \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}} = \beta e^{-\beta t}$. Thus, the class of Phase–Type(1) distributions is exactly the class of exponential distributions.

Example 5.2.2 The Erlang distributions. Suppose that $E = \{1, 2, \ldots, n\}$.

Let $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$, $\mathbf{b} = (0, 0, \ldots, \beta)$ and

$$\mathbf{B} = \begin{pmatrix} -\beta & \beta & \cdots & 0 & 0 \\ 0 & -\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & \beta \\ 0 & 0 & \cdots & 0 & -\beta \end{pmatrix}.$$

We compute the density $k(t)$ and obtain

$$k(t) = \boldsymbol{\alpha} e^{\mathbf{B}t} \mathbf{b}^{\mathsf{T}} = \frac{\beta^n t^{n-1} e^{-\beta t}}{(n-1)!}.$$

Therefore K is an Erlang(n) distribution.

Example 5.2.3 The generalized Erlang distributions. Analogously, suppose that $E = \{1, 2, \ldots, n\}$. Let $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$, $\mathbf{b} = (0, 0, \ldots, \beta_n)$ and

$$\mathbf{B} = \begin{pmatrix} -\beta_1 & \beta_1 & \cdots & 0 & 0 \\ 0 & -\beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta_{n-1} & \beta_{n-1} \\ 0 & 0 & \cdots & 0 & -\beta_n \end{pmatrix}.$$

We compute the density $k(t)$ and obtain

$$k(t) = \boldsymbol{\alpha} e^{\mathbf{B}t} \mathbf{b}^{\mathsf{T}} = \sum_{i=1}^{n} \left(\prod_{j=1, j \neq i}^{n} \frac{\beta_j}{\beta_j - \beta_i} \right) \beta_i e^{-\beta_i t}.$$

Therefore K is a generalized Erlang(n) distribution.

In this chapter we assume that the interclaim times W_i follow a Phase–Type(n) distribution K with vector of entry probabilities $\boldsymbol{\alpha}$ and intensity matrix $\mathbf{B}$. Therefore the net profit condition (2.3.1) becomes

$$cE[W_1] > E[X_1] \Longleftrightarrow -c\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}} > \mu_1. \tag{5.2.3}$$

5.3 Lundberg's equation

Our purpose in this section is to find rational expressions for the Lundberg's equations.

Recall that the generalized Lundberg's equation is given by (2.3.5)

$$\hat{p}(s) = \frac{1}{\hat{k}(\delta - cs)}, \quad \text{or} \quad \hat{k}(\delta - cs)\hat{p}(s) = 1.$$

Notice that in order to solve such equation we need to determine an expression for $\hat{k}(\delta - cs)$. We are looking for expressions similar to those given in (3.2.2) and (4.2.5).

In the previous section we mentioned that the Laplace transform of a Phase–Type(n) distribution with density k is

$$\hat{k}(s) = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}.$$

So the main problem is to compute the inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$. Before we go further we give some definitions from linear algebra.

Definition 1 Let $\mathbf{A} = (a_{i,j})_{i,j=1}^{n}$ be a $n \times n$ matrix.

Define, for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$$\mathbf{M}_{i_1,i_2\ldots i_k} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \ldots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ldots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \ldots & a_{i_k,i_k} \end{pmatrix}, \; 1 \leq k \leq n,$$

then

$$tr_k(\mathbf{A}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} det(\mathbf{M}_{i_1,i_2\ldots i_k}).$$

To understand the definition of $tr_k(A)$, we choose k out of n elements from the diagonal of $\mathbf{A}$, compute the determinant of the $k \times k$ minor of $\mathbf{A}$ that has those elements in its diagonal, and finally sum over all the possible choices. The functions $tr_k(A)$ generalize the notion of the trace and the determinant of a matrix A, as we will see in the following example.

Example 5.3.1 For $k = 1$

$$tr_1(\mathbf{A}) = \sum_{i=1}^{n} det(\mathbf{M}_i) = \sum_{i=1}^{n} a_{ii} = tr(\mathbf{A}).$$

For $k = 2$

$$tr_2(\mathbf{A}) = \sum_{1 \le i < j \le n} det(\mathbf{M}_{ij}) = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}).$$

For $k = n - 1$

$$tr_{n-1}(\mathbf{A}) = \sum_{i=1}^{n} det(\mathbf{M}_{1\ldots,i-1,i+1,\ldots,n}) = det(\mathbf{A})tr(\mathbf{A}^{-1}).$$

For $k = n$

$$tr_n(\mathbf{A}) = det(\mathbf{M}_{1\ldots n}) = det(\mathbf{A}).$$

By convention we set $tr_0(\mathbf{A}) = 1$.

We use the functions $tr_k(\mathbf{A})$ in the following lemmas and theorems.

Lemma 5.3.1 The characteristic polynomial of the matrix $\mathbf{B}$ is given by

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B}) s^i.$$

Proof: The matrix $s\mathbf{I} - \mathbf{B}$

$$s\mathbf{I} - \mathbf{B} = \begin{pmatrix} s - \beta_{11} & -\beta_{12} & \cdots & -\beta_{1,n-1} & -\beta_{1,n} \\ -\beta_{21} & s - \beta_{22} & \cdots & -\beta_{2,n-1} & -\beta_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta_{n-1,1} & -\beta_{n-1,2} & \cdots & s - \beta_{n-1,n-1} & \beta_{n-1,n} \\ -\beta_{n,1} & -\beta_{n,2} & \cdots & -\beta_{n,n-1} & s - \beta_{n,n} \end{pmatrix}.$$

The proof follows observing that to get the coefficient of s^i we must choose i elements of the diagonal, eliminate the rows and columns of $s\mathbf{I} - \mathbf{B}$ containing those elements and compute the determinant of the $(n-i) \times (n-i)$ submatrix of $-\mathbf{B}$ that remains. Thus, adding over all the possible choices gives the coefficient $(-1)^{n-i} tr_{n-i}(\mathbf{B})$. □

A very known result from linear algebra, known as the Cayley–Hamilton theorem, states that every square matrix satisfies its own characteristic equation, see Lang (2010). Using the lemma we get

$$det(\mathbf{BI} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B}) \mathbf{B}^i = \mathbf{0}. \tag{5.3.1}$$

Theorem 5.3.1 The inverse matrix $(s\mathbf{I}-\mathbf{B})^{-1}$ has the expression

$$(s\mathbf{I}-\mathbf{B})^{-1} = \frac{N(s,\mathbf{B})}{det(s\mathbf{I}-\mathbf{B})},$$

where the matrix $N(s,\mathbf{B})$ takes the form

$$N(s,\mathbf{B}) = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)s^i.$$

Proof: We prove that $(s\mathbf{I}-\mathbf{B})^{-1}(s\mathbf{I}-\mathbf{B}) = \mathbf{I}$ or, equivalently, that

$$(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.$$

If we denote by

$$a_i = \sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j},$$

then

$$\begin{aligned}(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)s^i \\ &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}a_i s^i \\ &= a_{n-1}s^n + \sum_{i=1}^{n-1}(a_{i-1}-a_i\mathbf{B})s^i - a_0\mathbf{B}.\end{aligned}$$

Now we can easily verify that $a_{n-1} = \mathbf{I}$.
Using (5.3.1) we get $-a_0\mathbf{B} = (-1)^n det(\mathbf{B})\mathbf{I}$ and

$$\begin{aligned}a_{i-1}-a_i\mathbf{B} &= \sum_{j=0}^{n-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-i-j} - \left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)\mathbf{B} \\ &= (-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= \mathbf{I}s^n + \sum_{i=1}^{n-1}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i + (-1)^n det(\mathbf{B})\mathbf{I} \\
&= \sum_{i=0}^{n}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.
\end{aligned}
$$

□

Corollary 5.3.1 The Laplace transform $\hat{k}(s)$ can be written as

$$
\hat{k}(s) = \boldsymbol{\alpha}(s\mathbf{I}-\mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}} = \frac{\boldsymbol{\alpha}N(s,\mathbf{B})\mathbf{b}^{\mathsf{T}}}{det(s\mathbf{I}-\mathbf{B})}.
$$

Example 5.3.2 For $n=1$, $\boldsymbol{\alpha}=(1)$, $\mathbf{B}=(-\beta)$, $\mathbf{1}=(1)$, then

$$
\hat{k}(s) = \frac{\boldsymbol{\alpha}[\mathbf{I}]\mathbf{b}^{\mathsf{T}}}{s-det(\mathbf{B})} = \frac{\beta}{s+\beta}.
$$

For $n=2$

$$
\hat{k}(s) = \frac{\boldsymbol{\alpha}[\mathbf{I}s + (\mathbf{B}-\mathbf{I}tr(\mathbf{B}))]\mathbf{b}^{\mathsf{T}}}{s^2 - tr(\mathbf{B})s + det(\mathbf{B})}.
$$

For $n=3$

$$
\hat{k}(s) = \frac{\boldsymbol{\alpha}[\mathbf{I}s^2 + (\mathbf{B}-\mathbf{I}tr(\mathbf{B}))s + (\mathbf{B}^2 - \mathbf{B}tr(\mathbf{B}) + \mathbf{I}tr_2(\mathbf{B}))]\mathbf{b}^{\mathsf{T}}}{s^3 - tr(\mathbf{B})s^2 + tr_2(\mathbf{B})s - det(\mathbf{B})}.
$$

Example 5.3.3 Consider the Erlang distributions. Following the notation of the Erlang example given before, we notice that since $\boldsymbol{\alpha} = (1,0,\ldots,0)$ and $\mathbf{b} = (0,0,\ldots,\beta)$, the product $\boldsymbol{\alpha}N(s,\mathbf{B})\mathbf{b}^{\mathsf{T}}$ gives the element $(1,n)$ of the matrix $N(s,\mathbf{B})$ (which is equal to β^{n-1}) times β. Thus,

$$
\hat{k}(s) = \frac{\boldsymbol{\alpha}N(s,\mathbf{B})\mathbf{b}^{\mathsf{T}}}{det(s\mathbf{I}-\mathbf{B})} = \frac{\beta^n}{(\beta+s)^n}.
$$

Example 5.3.4 Consider the generalized Erlang distributions. Analogously, since $\boldsymbol{\alpha} = (1,0,\ldots,0)$ and $\mathbf{b} = (0,0,\ldots,\beta_n)$, the product $\boldsymbol{\alpha}N(s,\mathbf{B})\mathbf{b}^{\mathsf{T}}$ gives the element $(1,n)$ of the matrix $N(s,\mathbf{B})$ (which is equal to $\prod_{i=1}^{n-1}\beta_i$) times β_n. Thus,

$$
\hat{k}(s) = \frac{\boldsymbol{\alpha}N(s,\mathbf{B})\mathbf{b}^{\mathsf{T}}}{det(s\mathbf{I}-\mathbf{B})} = \frac{\prod_{i=1}^{n}\beta_i}{\prod_{i=1}^{n}(\beta_i+s)}.
$$

Finally, we get the rational expression for the Lundberg's equations, as desired. The generalized Lundberg's equation for the Phase–Type(n) model becomes

$$\frac{1}{\hat{k}(\delta - cs)} = \frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(\delta - cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \hat{p}(s), \tag{5.3.2}$$

and we obtain the corresponding fundamental Lundberg's equation by setting $\delta = 0$ in equation (5.3.2)

$$\frac{1}{\hat{k}(-cs)} = \frac{det((-cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(-cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \hat{p}(s). \tag{5.3.3}$$

5.3.1 Multiplicity of the roots of the Lundberg's equations

Now, we recall the generalized Lundberg's equation $\hat{k}(\delta - cs)\hat{p}(s) = 1$. Albrecher and Boxma (2005) shows that this equation has n solutions in the right half of the complex plane, which we denote by $\rho_1, \rho_2, \ldots, \rho_n$. Moreover the fundamental Lundberg's equation has only $n-1$ roots with positive real parts, say $\rho_1, \rho_2, \ldots, \rho_{n-1}$, and we have $\rho_n = 0$.

Thus, considering the right half of the complex plane, more specifically the positive real axis, we look for the possibility of having a double real root.

We state the following theorem

Theorem 5.3.2 Let s_1 and s_2, with $s_1 < s_2$, be two real poles of $\hat{k}(\delta - cs)$, and suppose that there is no other real pole or zero of $\hat{k}(\delta - cs)$ in the interval (s_1, s_2). If $\hat{k}(\delta - cs)$ is positive in the interval (s_1, s_2) then the generalized Lundbeg's equation has one of the following:

- Two real roots in the interval.
- A double root in the interval.
- No real roots in this interval.

Proof: We compare both sides of equation (5.3.2).

For $s \in \mathbb{R}^+$, we notice that the Laplace transform $\hat{p}(s)$ is a positive and decreasing function of s, with $p(0) = 1$ and $lim_{s\to\infty}\hat{p}(s) = 0$. Therefore $\hat{p}(s)$ has no zeros or poles in $s \in \mathbb{R}^+$.

On the other hand, the function $\hat{k}(\delta - cs)$ is the quotient of the polynomial $\boldsymbol{\alpha} N(\delta - cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}$, which has degree at most $n - 1$, and the polynomial

$det(s\mathbf{I} - \mathbf{B})$, which has degree n. The poles of $\hat{k}(\delta - cs)$ are the numbers $s = (\delta - \zeta)/c$, where ζ ranges over all the eigenvalues of $\mathbf{B}$.

Now assume that s_1 and s_2, with $s_1 < s_2$, are two real poles of $\hat{k}(\delta - cs)$, and suppose that $\hat{k}(\delta - cs) > 0$ in the interval (s_1, s_2). Then the function $1/\hat{k}(\delta - cs)$ has two consecutive zeros at s_1 and s_2 and it is positive in the interval (s_1, s_2).

Since $1/\hat{k}(\delta - cs)$ is continuous and differentiable in (s_1, s_2) it has a maximum in this interval, and comparing the values of $1/\hat{k}(\delta - cs)$ and $\hat{p}(s)$ at this maximum we get the result.

□

Corollary 5.3.2 Let s_1 and s_2, with $s_1 < s_2$, be two real poles of $\hat{k}(-cs)$, and suppose that there is no other real pole or zero of $\hat{k}(-cs)$ in the interval (s_1, s_2). If $\hat{k}(-cs)$ is positive in the interval (s_1, s_2) then the fundamental Lundbeg's equation has one of the following:

- Two real roots in the interval.
- A double root in the interval.
- No real roots on this interval.

Example 5.3.5 Suppose that the interclaim times W_i follow a Phase–Type(4) distribution, with intensity matrix

$$\mathbf{B} = \begin{pmatrix} -7 & 0 & 1 & 2 \\ 3 & -5 & 1 & 1 \\ 6 & 0 & -8 & 1 \\ 0 & 0 & 2 & -4 \end{pmatrix},$$

$\boldsymbol{\alpha} = (0.2, 0.3, 0.1, 0.4)$ and $\mathbf{b} = (4, 0, 1, 2)$. Then $E[W_i] = 0.475$. Suppose that the claim amounts X_i are exponentially distributed with parameter $\beta \geq 2.11$. Then we choose $c = 1$ to satisfy the net profit condition (5.2.3).

The fundamental Lundberg's equation (5.3.3) becomes

$$\frac{1}{\hat{k}(-cs)} = \frac{810 - 702s + 203s^2 - 24s^3 + s^4}{810 - 317.3s + 40.1s^2 - 1.7s^3} = \frac{\beta}{\beta + s} = \hat{p}(s).$$

The function $\hat{k}(-cs)$ is positive and has no zeros in the interval $(7.646, 9)$. Then it is easy to verify that the fundamental Lundberg's equation has

- Two real roots in $(7.646, 9)$ for $2.11 \leq \beta < 3.239$.

- A double root 8.42 in $(7.646, 9)$ for $\beta = 3.239$.
- Two complex conjugate roots, where the real part of them is in $(7.646, 9)$ for $\beta > 3.239$.

Remark 5.3.1 So far we have investigated the possibility of double roots in the fundamental and generalized Lundberg's equations for a Sparre–Andersen model with Phase–Type(n) interclaim times. Empirically, after many numerical tests with different intensity matrices $\mathbf{B}$ and different vectors of initial probabilities $\boldsymbol{\alpha}$, we only found simple and double roots. Moreover, there were no cases where more than one double root appears. However, the possibility of roots of a higher order is still open, and this is currently one of our lines of research.

5.4 The ruin and survival probabilities

In this section we study the ruin probability associated to this model. We find an integro–differential equation that is satisfied by the ruin probability, the Laplace transform and a defective renewal equation. Following a similar procedure like in previous chapters, we study the maximum severity of ruin.

5.4.1 A differential operator

In Chapter 2 we considered a renewal equation for the survival probability

$$\begin{aligned}\Phi(u) &= \frac{1}{c}\int_u^\infty k\left(\frac{s-u}{c}\right)\int_0^s p(x)\Phi(s-x)dxds \\ &= \frac{1}{c}\int_u^\infty k\left(\frac{s-u}{c}\right)W_\Phi(s)ds, \end{aligned} \tag{5.4.1}$$

where $W_\Phi(s) = \int_0^s p(x)\Phi(s-x)dx$.

Now, we recall the integro–differential equations satisfied by $\Phi(u)$ in the Erlang model (equation (3.3.1))

$$B(\mathcal{D})\Phi(u) = \left(\mathcal{I} - \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n \Phi(u) = \int_0^u \Phi(u-y)p(y)dy = W_\Phi(u), \quad u \geq 0,$$

and the generalized Erlang model (Equation (4.2.2)),

$$B(\mathcal{D})\Phi(u) = \prod_{i=1}^n \left(\mathcal{I} - \left(\frac{c}{\lambda_i}\right)\mathcal{D}\right)\Phi(u) = \int_0^u \Phi(u-y)p(y)dy = W_\Phi(u), u \geq 0.$$

We notice that in both equations there is a differential operator, denoted $B(\mathcal{D})$, that depends entirely on the interclaim times distribution and its parameters. Moreover, the Erlang(n) and generalized Erlang(n) densities satisfy for their respective operators the following equation

$$B(\mathcal{D})k_n\left(\frac{s-u}{c}\right) = 0. \tag{5.4.2}$$

Our objective is to find the differential operator $B(\mathcal{D})$ that corresponds to the Phase–Type(n) model.

Theorem 5.4.1 For the Phase–Type(n) density $k(t) = \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{b}^\mathsf{T}$ the corresponding differential operator is

$$B(\mathcal{D}) = \frac{det(\mathbf{B} + c\mathbf{I}\mathcal{D})}{det(\mathbf{B})} \tag{5.4.3}$$

Proof: We have to show that

$$B(\mathcal{D})k\left(\frac{s-u}{c}\right) = \frac{1}{det(\mathbf{B})} det(\mathbf{B} + c\mathbf{I}\mathcal{D})[\boldsymbol{\alpha} e^{\mathbf{B}(\frac{s-u}{c})}\mathbf{b}^\mathsf{T}] = 0.$$

First of all we write the polynomial form of $B(\mathcal{D})$

$$B(\mathcal{D}) = \frac{det(\mathbf{B} + c\mathbf{I}\mathcal{D})}{det(\mathbf{B})} = \sum_{k=0}^{n} B_k \mathcal{D}^k,$$

with $B_k = c^k \frac{tr_{n-k}(\mathbf{B})}{det(\mathbf{B})}$. Note that $B_0 = 1$.

Then,

$$\begin{aligned}
B(\mathcal{D})k\left(\frac{s-u}{c}\right) &= \sum_{k=0}^{n} B_k \mathcal{D}^k[\boldsymbol{\alpha} e^{\mathbf{B}(\frac{s-u}{c})}\mathbf{b}^\mathsf{T}] = \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \mathcal{D}^k (e^{\mathbf{B}(\frac{s-u}{c})})\right]\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \left(\frac{-1}{c}\right)^k \mathbf{B}^k e^{\mathbf{B}(\frac{s-u}{c})}\right]\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\sum_{k=0}^{n} B_k \left(\frac{-1}{c}\mathbf{B}\right)^k\right] e^{\mathbf{B}(\frac{s-u}{c})}\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[B\left(\frac{-1}{c}\mathbf{B}\right)\right] e^{\mathbf{B}(\frac{s-u}{c})}\mathbf{b}^\mathsf{T} \\
&= \boldsymbol{\alpha}\left[\frac{det(\mathbf{B} + c\mathbf{I}\left(\frac{-1}{c}\mathbf{B}\right))}{det(\mathbf{B})}\right] e^{\mathbf{B}(\frac{s-u}{c})}\mathbf{b}^\mathsf{T} = 0,
\end{aligned}$$

where the last equality comes from the Cayley–Hamilton theorem in (5.3.1). Hence, we get the result. $\square$

Example 5.4.1 For the Erlang and generalized Erlang distributions it is easy to verify that the operator (5.4.3) becomes $\left(\mathcal{I} - \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n$ and $\prod_{i=1}^{n}\left(\mathcal{I} - \left(\frac{c}{\lambda_i}\right)\mathcal{D}\right)$, respectively.

5.4.2 An integro–differential equation for $\Phi(u)$

We apply the differential operator (5.4.3) to the renewal equation (5.4.1), which gives integro–differential for the survival probability in the Phase–Type(n) model.

Using equation (5.4.1) we get the i–th derivative of $\Phi(u)$,

$$\begin{aligned}\Phi^{(i)}(u) &= \frac{1}{c}\int_u^\infty \left(\frac{-1}{c}\right)^i k^{(i)}\left(\frac{s-u}{c}\right) W_\Phi(s)ds \\ &\quad + \sum_{j=0}^{i-1}\left(\frac{-1}{c}\right)^{j+1} k^{(j)}(0) W_\Phi^{(i-1-j)}(u).\end{aligned}$$

Therefore,

$$\begin{aligned}B(\mathcal{D})\Phi(u) &= \sum_{i=0}^{n} B_i \mathcal{D}^i \Phi(u) = \sum_{i=0}^{n} B_i \Phi^{(i)}(u) \\ &= \sum_{i=0}^{n} B_i \left[\frac{1}{c}\int_u^\infty \left(\frac{-1}{c}\right)^i k^{(i)}\left(\frac{s-u}{c}\right) W_\Phi(s)ds \right. \\ &\quad \left. + \sum_{j=0}^{i-1}\left(\frac{-1}{c}\right)^{j+1} k^{(j)}(0) W_\Phi^{(i-1-j)}(u)\right] \\ &= \frac{1}{c}\int_u^\infty \underbrace{B(\mathcal{D})k\left(\frac{s-u}{c}\right)}_{=0} W_\Phi(s)ds + \\ &\quad \sum_{j=0}^{n-1}\left[\sum_{i=j+1}^{n} B_i \left(\frac{-1}{c}\right)^{i-j} k^{(i-1-j)}(0)\right] W_\Phi^{(j)}(u), \quad (5.4.4)\end{aligned}$$

Denote $\tilde{B}_j = \sum_{i=j+1}^{n} B_i \left(\frac{-1}{c}\right)^{i-j} k^{(i-1-j)}(0)$ and define

$$q(\mathcal{D}) = \sum_{j=0}^{n-1} \tilde{B}_j \mathcal{D}^j.$$

Thus, from (5.4.4) we get the integro–differential satisfied by $\Phi(u)$

$$B(\mathcal{D})\Phi(u) = q(\mathcal{D}) W_\Phi(u). \tag{5.4.5}$$

Example 5.4.2 Consider the Erlang distributions.

The Erlang(n) density function $k_n(t)$ satisfies the properties $k_n^{(i)}(0) = 0$, $i = 0, \ldots, n-2$ and $k_n^{(n-1)}(0) = \lambda^n$.

We have $\tilde{B}_j = 0, \quad j = 1, \ldots, n-1$ and

$$\tilde{B}_0 = B_n \left(\frac{-1}{c}\right)^n k^{(n-1)}(0) = \left(\frac{c}{-\lambda}\right)^n \left(\frac{-1}{c}\right)^n \lambda^n = 1.$$

Hence, $q(\mathcal{D}) = \mathcal{I}$ and the integro–differential equation (5.4.5) becomes

$$B(\mathcal{D})\Phi(u) = W_\Phi(u),$$

which corresponds to the Equation (3.3.1).

Example 5.4.3 Consider the generalized Erlang distributions.

The generalized Erlang(n) density function $k_n(t)$ satisfies the properties $k_n^{(i)}(0) = 0$, $i = 0, \ldots, n-2$ and $k_n^{(n-1)}(0) = \prod_{i=1}^{n} \lambda_i$.

Analogously, we have $\tilde{B}_j = 0, \quad j = 1, \ldots, n-1$ and

$$\tilde{B}_0 = B_n \left(\frac{-1}{c}\right)^n k^{(n-1)}(0) = \frac{(-c)^n}{\prod_{i=1}^{n} \lambda_i} \left(\frac{-1}{c}\right)^n \prod_{i=1}^{n} \lambda_i = 1.$$

Then $q(\mathcal{D}) = \mathcal{I}$ and the integro–differential equation (5.4.5) becomes

$$B(\mathcal{D})\Phi(u) = W_\Phi(u),$$

which corresponds to the Equation (4.2.2).

5.4.3 The Laplace transform of $\Phi(u)$

The Laplace transform is a widely used integral transform for solving differential and integro–differential equations. We apply Laplace transforms for the integro–differential equation (5.4.5)

$$\widehat{B(\mathcal{D})\Phi}(s) = q\widehat{(\mathcal{D})W_\Phi}(s) \tag{5.4.6}$$

to find an expression for the Laplace transform of the survival probability.

On the left hand side of (5.4.6) we have

$$\begin{aligned}\widehat{B(\mathcal{D})\Phi}(s) &= \sum_{i=0}^{n} B_i \widehat{\Phi^{(i)}}(s) = \sum_{i=0}^{n} B_i \left[s^i \hat{\Phi}(s) - \sum_{j=0}^{i-1} s^j \Phi^{(i-1-j)}(0) \right] \\ &= B(s)\hat{\Phi}(s) - \sum_{j=0}^{n-1} \left[\sum_{i=j+1}^{n} B_i \Phi^{(i-1-j)}(0) \right] s^j,\end{aligned}$$

and the right hand side gives

$$\begin{aligned}q\widehat{(\mathcal{D})W_\Phi}(s) &= \sum_{i=0}^{n-1} \tilde{B}_i \widehat{W_\Phi^{(i)}}(s) \\ &= \sum_{i=0}^{n-1} \tilde{B}_i \left[s^i \hat{\Phi}(s)\hat{p}(s) - \sum_{j=0}^{i-2} s^j \left(\sum_{m=j+1}^{i-1} p^{(i-1-m)}(0)\Phi^{(m-1-j)}(0) \right) \right] \\ &= q(s)\hat{\Phi}(s)\hat{p}(s) - \sum_{j=0}^{n-3} \left[\sum_{i=j+2}^{n-1} \tilde{B}_i \left(\sum_{m=j+1}^{i-1} p^{(i-1-m)}(0)\Phi^{(m-1-j)}(0) \right) \right] s^j.\end{aligned}$$

Thus,

$$\begin{aligned}B(s)\hat{\Phi}(s) - q(s)\hat{\Phi}(s)\hat{p}(s) &= \sum_{j=0}^{n-1} \left[\sum_{i=j+1}^{n} B_i \Phi^{(i-1-j)}(0) \right] s^j - \\ &\quad \sum_{j=0}^{n-3} \left[\sum_{i=j+2}^{n-1} \tilde{B}_i \left(\sum_{m=j+1}^{i-1} p^{(i-1-m)}(0)\Phi^{(m-1-j)}(0) \right) \right] s^j.\end{aligned}$$

Denoting

$$d_\Phi(s) = \sum_{j=0}^{n-1} \left[\sum_{i=j+1}^{n} B_i \Phi^{(i-1-j)}(0) - \sum_{i=j+2}^{n-1} \tilde{B}_i \left(\sum_{m=j+1}^{i-1} p^{(i-1-m)}(0)\Phi^{(m-1-j)}(0) \right) \right] s^j, \tag{5.4.7}$$

we get

$$(B(s) - q(s)\hat{p}(s))\hat{\Phi}(s) = d_\Phi(s).$$

Finally, the Laplace transform of the survival probability takes the form

$$\hat{\Phi}(s) = \frac{d_\Phi(s)}{B(s) - q(s)\hat{p}(s)}, \tag{5.4.8}$$

where $d_\Phi(s)$ is a polynomial of degree at most $n-1$ defined in (5.4.7).

As a consequence we have the following theorem

Theorem 5.4.2 The fundamental Lundberg's equation (5.3.3) and the equation $B(s) = q(s)\hat{p}(s)$ have identical sets of solutions.

Proof: We need to prove that

$$q(s) = \frac{(-1)^n}{det(\mathbf{B})}\boldsymbol{\alpha}N(-cs, \mathbf{B})\mathbf{b}^\mathsf{T}. \tag{5.4.9}$$

Each side of the equation (5.4.9) represents a polynomial of degree at most $n-1$. On the right hand side the coefficient of s^i is given by

$$\tilde{a}_i = \sum_{j=0}^{n-1-i} (-1)^{n-i-j} c^i \frac{tr_j(\mathbf{B})}{det(\mathbf{B})}\boldsymbol{\alpha}\mathbf{B}^{n-1-i-j}\mathbf{b}^\mathsf{T}, \quad i = 0, \ldots, n-1,$$

while on the left hand side the respective coefficient is equal to

$$\begin{aligned}
\tilde{B}_i &= \sum_{j=i+1}^{n} B_j \left(\frac{-1}{c}\right)^{j-i} k^{(j-1-i)}(0) \\
&= \sum_{j=i+1}^{n} c^j \frac{tr_{n-j}(\mathbf{B})}{det(\mathbf{B})} \left(\frac{-1}{c}\right)^{j-i} \boldsymbol{\alpha}\mathbf{B}^{j-1-i}\mathbf{b}^\mathsf{T} \\
&= \sum_{l=0}^{n-1-i} c^{n-l} \frac{tr_l(\mathbf{B})}{det(\mathbf{B})} \left(\frac{-1}{c}\right)^{n-i-l} \boldsymbol{\alpha}\mathbf{B}^{n-1-i-l}\mathbf{b}^\mathsf{T} \\
&= \sum_{l=0}^{n-1-i} c^i \frac{tr_l(\mathbf{B})}{det(\mathbf{B})}(-1)^{n-i-l}\boldsymbol{\alpha}\mathbf{B}^{n-1-i-l}\mathbf{b}^\mathsf{T}.
\end{aligned}$$

Thus, we get $\tilde{a}_i = \tilde{B}_i$, $i = 0, \ldots, n-1$. This proves equation (5.4.9).

□

5.4.4 A defective renewal equation for the survival probability

In this section we look for a defective renewal equation that is satisfied by the survival probability $\Phi(u)$ in the Phase–Type(n) risk model. Such equation was proposed by Li and Garrido (2004b) and Gerber and Shiu (2003b) for the Erlang(n) model and generalized Erlang(n) models, respectively. Recall that the numbers $\rho_1, \rho_2, \ldots, \rho_{n-1}$ are the $n-1$ roots of the fundamental Lundberg's equation which has positive real parts and $\rho_n = 0$.

Definition 5.4.1 Let $f(s)$ be an arbitrary function. We denote by $f[r_1, r_2, \ldots, r_{k-1}, r_k, s]$ the k–th divided difference of the function $f(s)$ with respect to the points of collocation $r_1, r_2, \ldots, r_{k-1}, r_k$. We give a more detailed definition of the divided differences in Appendix A.

We concentrate our attention on the ruin probability $\Psi(u)$ to find a defective renewal equation that it satisfies. Since $\Phi(u) = 1 - \Psi(u)$ this will give us the desired equation for the survival probability.

Theorem 5.4.3 The ruin probability satisfies the following defective renewal equation

$$
\begin{aligned}
(-1)^n \frac{c^n}{det(\mathbf{B})}\Psi(u) &= (q(\mathcal{D})\Psi) * T_{\rho_n} \cdots T_{\rho_1} p(u) + (-1)^{n-1}\Omega[\rho_1, \ldots, \rho_n, u] \\
&\quad + T_{\rho_n} \cdots T_{\rho_1} \tilde{P}(u), \qquad (5.4.10)
\end{aligned}
$$

where the function $\Omega(s, u)$ takes the values

$$
\Omega(\rho_k, u) = \sum_{i=1}^{n-1} \tilde{B}_i \sum_{j=0}^{i-1} \left[\Psi^{(j)}(0)(T_{\rho_k} p^{(i-1-j)}(u)) - \hat{p}(\rho_k)\rho_k^{i-1-j}\Psi^{(j)}(u) \right],
$$

for $k = 1, \ldots, n$ and $\tilde{P}(u) = 1 - q(\mathcal{D})P(u)$.

Proof: Recall that $B(s) = \frac{det(\mathbf{B}+cs\mathbf{I})}{det(\mathbf{B})} = \sum_{k=0}^{n} B_k s^k$ and $q(s) = \sum_{j=0}^{n-1} \tilde{B}_j s^j$.

We also recall the integral operators T_r defined in Chapter 4, for a real–valued integrable function f

$$
T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u) du, \quad r \in \mathbb{C}, \quad x \geq 0,
$$

It is straightforward to show that the ruin probability $\Psi(u)$ satisfies the integro–differential equation

$$B(\mathcal{D})\Psi(u) = q(\mathcal{D})W_\Psi(u) + \tilde{P}(u). \tag{5.4.11}$$

where $W_\Psi(u) = \int_0^u p(x)\Psi(u-x)dx = \Psi * p(u)$. We consider Equation (5.4.11) as the "stage 0".

To prove the theorem we use an inductive argument as follows.

Let $B(s) = B(\rho_1) + (s - \rho_1)B[\rho_1, s]$. We define the differential operator

$$B(\mathcal{D}) = B(\rho_1)\mathcal{I} - (\rho_1\mathcal{I} - \mathcal{D})B[\rho_1, \mathcal{D}].$$

Applying this operator to $\Psi(u)$ gives

$$\begin{aligned} B(\mathcal{D})\Psi(u) &= B(\rho_1)\Psi(u) - (\rho_1\mathcal{I} - \mathcal{D})B[\rho_1, \mathcal{D}]\Psi(u) \\ &= q(\mathcal{D})W_\Psi(u) + \tilde{P}(u). \end{aligned} \tag{5.4.12}$$

We have $T_{\rho_1} = (\rho_1\mathcal{I} - \mathcal{D})^{-1}$, which means that

$$T_{\rho_1}(\rho_1\mathcal{I} - \mathcal{D})f(u) = (\rho_1\mathcal{I} - \mathcal{D})T_{\rho_1}f(u) = f(u) \;\; \forall \text{ function } f,$$

Thus, we apply the integral operator T_{ρ_1} to the Equation (5.4.12) and we obtain

$$B(\rho_1)T_{\rho_1}\Psi(u) - \underbrace{T_{\rho_1}(\rho_1\mathcal{I} - \mathcal{D})}_{=\mathcal{I}}B[\rho_1, \mathcal{D}]\Psi(u) = T_{\rho_1}q(\mathcal{D})W_\Psi(u) + T_{\rho_1}\tilde{P}(u),$$

so

$$-B[\rho_1, \mathcal{D}]\Psi(u) = T_{\rho_1}q(\mathcal{D})W_\Psi(u) - B(\rho_1)T_{\rho_1}\Psi(u) + T_{\rho_1}\tilde{P}(u). \tag{5.4.13}$$

We need to compute $T_{\rho_1}q(\mathcal{D})W_\Psi(u)$. Using the fact that $T_{\rho_1}\mathcal{D} = \mathcal{D}T_{\rho_1}$, we have

$$\begin{aligned} T_{\rho_1}q(\mathcal{D})W_\Psi(u) &= q(\mathcal{D})T_{\rho_1}W_\Psi(u) = q(\mathcal{D})T_{\rho_1}(\Psi * p)(u) \\ &= q(\mathcal{D})(\Psi * T_{\rho_1}p)(u) + \hat{p}(\rho_1)q(\mathcal{D})T_{\rho_1}\Psi(u) \\ &= (q(\mathcal{D})\Psi) * T_{\rho_1}p(u) + \Omega(\rho_1, u) + B(\rho_1)T_{\rho_1}\Psi(u). \end{aligned}$$

Replacing $T_{\rho_1}q(\mathcal{D})W_\Psi(u)$ in (5.4.13) we get

$$-B[\rho_1, \mathcal{D}]\Psi(u) = (q(\mathcal{D})\Psi) * T_{\rho_1}p(u) + \Omega(\rho_1, u) + T_{\rho_1}\tilde{P}(u). \tag{5.4.14}$$

We consider the Equation (5.4.14) as the "stage 1".

Now we repeat this process. Let $B[\rho_1, s] = B[\rho_1, \rho_2] + (s - \rho_2)B[\rho_1, \rho_2, s]$ and define the differential operator

$$B[\rho_1, \mathcal{D}] = B[\rho_1, \rho_2]\mathcal{I} - (\rho_2\mathcal{I} - \mathcal{D})B[\rho_1, \rho_2, \mathcal{D}].$$

Applying this operator to $\Psi(u)$ and then applying T_{ρ_2} to the resulting equation gives

$$\begin{aligned} -T_{\rho_2}B[\rho_1, \mathcal{D}]\Psi(u) &= -B[\rho_1, \rho_2]T_{\rho_2}\Psi(u) + B[\rho_1, \rho_2, \mathcal{D}]\Psi(u) \\ &= T_{\rho_2}T_{\rho_1}q(\mathcal{D})W_\Psi(u) - B(\rho_1)T_{\rho_2}T_{\rho_1}\Psi(u) + T_{\rho_2}T_{\rho_1}\tilde{P}(u). \end{aligned}$$

We compute $T_{\rho_2}T_{\rho_1}q(\mathcal{D})W_\Psi(u)$

$$\begin{aligned} T_{\rho_2}T_{\rho_1}q(\mathcal{D})W_\Psi(u) &= q(\mathcal{D})T_{\rho_2}T_{\rho_1}W_\Psi(u) = q(\mathcal{D})\left(\frac{T_{\rho_1}W_\Psi(u) - T_{\rho_2}W_\Psi(u)}{\rho_2 - \rho_1}\right) \\ &= (q(\mathcal{D})\Psi) * T_{\rho_2}T_{\rho_1}p(u) - \Omega[\rho_1, \rho_2, u] \\ &\quad + \frac{B(\rho_1)T_{\rho_1}\Psi(u) - B(\rho_2)T_{\rho_2}\Psi(u)}{\rho_2 - \rho_1}, \end{aligned}$$

and similarly we obtain

$$B[\rho_1, \rho_2, \mathcal{D}]\Psi(u) = (q(\mathcal{D})\Psi) * T_{\rho_2}T_{\rho_1}p(u) - \Omega[\rho_1, \rho_2, u] + T_{\rho_2}T_{\rho_1}\tilde{P}(u). \quad (5.4.15)$$

We consider the Equation (5.4.15) as the "stage 2".

Continuing this process until the "stage n" we arrive to the following equation

$$\begin{aligned} (-1)^n B[\rho_1, \ldots, \rho_n, \mathcal{D}]\Psi(u) &= (q(\mathcal{D})\Psi) * T_{\rho_n}\cdots T_{\rho_1}p(u) + \\ &\quad + (-1)^{n-1}\Omega[\rho_1, \ldots, \rho_n, u] + T_{\rho_n}\cdots T_{\rho_1}\tilde{P}(u), \end{aligned}$$

and since $B[\rho_1, \ldots, \rho_n, \mathcal{D}] = B_n\mathcal{I} = \frac{c^n}{det(\mathbf{B})}\mathcal{I}$ we get the desired result. $\square$

Example 5.4.4 We want to calculate the survival probability in the Phase–Type(2) model. For this purpose we use the defective renewal equation satisfied by the ruin probability to derive the respective equation for the survival probability. Then we apply Laplace transforms and inverse Laplace transforms to this equation to obtain $\Phi(u)$.

We have $B(s) = 1 + B_1 s + B_2 s^2$ and $q(s) = 1 + \tilde{B}_1 s$.

The defective renewal equation satisfied by $\Psi(u)$ is

$$\begin{aligned}\frac{c^2}{det(\mathbf{B})}\Psi(u) &= (q(\mathcal{D})\Psi) * T_{\rho_2}T_{\rho_1}p(u) - \Omega[\rho_1,\rho_2,u] \\ &\quad + T_{\rho_2}T_{\rho_1}\tilde{P}(u),\end{aligned} \tag{5.4.16}$$

where

$$-\Omega[\rho_1,\rho_2,u] = \frac{\Omega(\rho_1,u)-\Omega(\rho_2,u)}{\rho_2-\rho_1} = \tilde{B}_1(\Psi(0)T_{\rho_2}T_{\rho_1}p(u) - T_{\rho_2}T_{\rho_1}p(0)\Psi(u)),$$

and

$$T_{\rho_2}T_{\rho_1}\tilde{P}(u) = T_{\rho_2}T_{\rho_1}(1-P(u)-\tilde{B}_1 p(u)).$$

Defining $\eta_0(u) = \frac{det(\mathbf{B})}{c^2}T_{\rho_2}T_{\rho_1}p(u)$ as the "defective density" we can rewrite the Equation (5.4.16) as

$$\Psi(u) = (q(\mathcal{D})\Psi)*\eta_0(u)+\tilde{B}_1(\Psi(0)\eta_0(u)-\eta_0(0)\Psi(u))+\int_u^\infty \eta_0(x)dx-\tilde{B}_1\eta_0(u).$$

Replacing $\psi(u) = 1-\Phi(u)$ in the previous equation we get the defective renewal equation for $\Psi(u)$,

$$\Phi(u) = (q(\mathcal{D})\Phi)*\eta_0(u)+\tilde{B}_1(\Phi(0)\eta_0(u)-\eta_0(0)\Phi(u))+\tilde{B}_1\eta_0(0)+1-\int_0^\infty \eta_0(x)dx. \tag{5.4.17}$$

The next step is to apply Laplace Transforms to the Equation (5.4.17). This gives

$$\hat{\Phi}(s) = \frac{\frac{\Phi(0)}{s}}{1-\hat{\eta_0}(s)-\tilde{B}_1(s\hat{\eta_0}(s)-\eta_0(0))}. \tag{5.4.18}$$

The denominator in the Equation (5.4.18) is given by

$$1-\hat{\eta_0}(s)-\tilde{B}_1(s\hat{\eta_0}(s)-\eta_0(0)) = \frac{B(s)-q(s)\hat{p}(s)}{(c^2/det(\mathbf{B}))s(s-\rho_1)}.$$

Finally, replacing the denominator in (5.4.18) we get

$$\hat{\Phi}(s) = -\Phi(0)\left(\frac{c^2}{det(\mathbf{B})}\right)\frac{\rho_1-s}{B(s)-q(s)\hat{p}(s)}. \tag{5.4.19}$$

If we compare the last expression with Laplace transform of $\Phi(u)$ that was given in (5.4.8) we observe that

$$d_\Phi(s) = -\Phi(0)\frac{c^2}{det(\mathbf{B})}(\rho_1 - s).$$

For the particular case when the claim amounts are exponentially distributed with parameter β, we can invert the Laplace transform in (5.4.19) and results

$$\Phi(u) = 1 - \left(1 - \frac{R}{\beta}\right)e^{-Ru},$$

where R is the adjustment coefficient from the fundamental Lundberg's equation.

5.4.5 Maximum severity of ruin

In order to compute the maximum severity of ruin in the Phase–Type(n) model we proceed in a similar way as in Chapters 3 and 4. We need to

(i) Compute the survival probability $\Phi(u)$.

(ii) Calculate the probability of attaining an upper barrier level $\chi(u, b)$.

(iii) Using the Equation (2.4.3), evaluate $J(z; u)$.

For (i) we can apply Laplace transforms to the defective renewal equation (5.4.10). Then we can invert the Laplace transform of $\Phi(u)$ in (5.4.8) to obtain explicit expressions for the survival probability.

For (ii) we can implement the analogous of Theorems 3.4.2, 4.3.1 and 4.3.2 for the Phase–Type(n) model.

5.5 Lundberg's matrix

The following matrix

$$\mathbf{L}_\delta(s) = \left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{B} + \frac{1}{c}\mathbf{b}^\intercal\alpha\hat{p}(s), \tag{5.5.1}$$

which is called the Lundberg's matrix in Ji and Zhang (2011), have been subject of study in several works, like Albrecher and Boxma (2005), Ren

(2007), Li (2008b), Ji and Zhang (2011). In the expression δ stands for a non negative constant.

According to Ren (2007), the solutions of

$$Det(\mathbf{L}_\delta(s)) = 0, \tag{5.5.2}$$

and the solutions of the generalized Lundberg's equation

$$\hat{k}(\delta - cs)\hat{p}(s) = 1, \tag{5.5.3}$$

as defined in Gerber and Shiu (2005) are identical.

Albrecher and Boxma (2005) show that (5.5.2) has exactly n solutions in the right half of the complex plane using matrix theory, therefore the generalized Lundberg's equation (5.5.3) have exactly the same n solutions in the right half of the complex plane, which we denote by $\rho_1, \rho_2, \ldots, \rho_n$.

In all the papers mentioned before, it is assumed that these roots have distinct values. However, in Section 5.3.1 we show that we can find a great variety of examples where multiple roots arise, specially double roots.

Consider the Lundberg's matrices $\mathbf{L}_\delta(\rho_i)$, $i = 1, 2, \ldots, n$. All those matrices are singular, or equivalently all of them have 0 as an eigenvalue. Let $\mathbf{h}_i$ be an eigenvector of $\mathbf{L}_\delta(\rho_i)$ associated to the eigenvalue 0 or, equivalently, let $\mathbf{h}_i$ be a vector in the null space of $\mathbf{L}_\delta(\rho_i)$.

Theorem 5.5.1 Let $\rho_1, \rho_2, \ldots, \rho_m$ be distinct, $2 \leq m \leq n$. Then the eigenvectors $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m$ are linearly independent.

Proof: By contradiction, suppose that they are linearly dependent. Assume that we can find a subset of l elements of $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_m\}$, with $2 \leq l \leq m$, that is linearly dependent and that every subset with $l-1$ elements or less is linearly independent.

Without loss of generality assume that the dependent subset is $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_l\}$. Then there are constants $c_1, c_2, \ldots, c_l$ not all zero such that

$$c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_l\mathbf{h}_l = \mathbf{0}.$$

Assume that $c_l \neq 0$, then we can write

$$\mathbf{h}_l = \sum_{i=1}^{l-1} \tilde{c}_i\mathbf{h}_i, \quad \tilde{c}_i = -\frac{c_i}{c_l}.$$

Multiplying both sides by $\mathbf{L}_\delta(\rho_l)$ we obtain

$$\begin{aligned}\mathbf{0} &= \mathbf{L}_\delta(\rho_l)\mathbf{h}_l = \mathbf{L}_\delta(\rho_l)\sum_{i=1}^{l-1}\tilde{c}_i\mathbf{h}_i \\ &\sum_{i=1}^{l-1}\tilde{c}_i\mathbf{L}_\delta(\rho_l)\mathbf{h}_i = \sum_{i=1}^{l-1}\tilde{c}_i\tilde{\mathbf{h}}_i,\end{aligned}$$

where $\tilde{\mathbf{h}}_i = \mathbf{L}_\delta(\rho_l)\mathbf{h}_i$, $i = 1,\ldots,l-1$.

Since $\mathbf{h}_i$, $i = 1,\ldots,l-1$ are not eigenvectors of $\mathbf{L}_\delta(\rho_l)$ we have $\tilde{\mathbf{h}}_i \neq \mathbf{0}$, so the vectors $\tilde{\mathbf{h}}_i$ are linearly dependent.

Now the eigenvectors $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{l-1}$ are linearly independent by assumption and they are not in the null space of $\mathbf{L}_\delta(\rho_l)$, therefore $\mathbf{L}_\delta(\rho_l)$ maps them to another set of linearly independent vectors.

However, this means that $\tilde{\mathbf{h}}_i$ are linearly independent and this is a contradiction. $\square$

5.6 The first time the surplus attain a certain level

For a barrier level $b \geq u$ define

$$t_b = min\{t \geq 0 : U(t) = b\},$$

to be the first time the surplus reaches level b. For $\delta \geq 0$ define,

$$T(u,b) = E[e^{-\delta t_b}|U(0) = u],$$

to be the Laplace transform of t_b. Furthermore define,

$$T_{i,j}(u,b) = E_i[e^{-\delta t_b}\mathbb{I}(J(t_b) = j)|U(0) = u],$$

to be the Laplace transform of t_b when the process starts from initial surplus u at state i and reaches the level b at state j. Then,

$$T(u,b) = \boldsymbol{\alpha}\mathbf{T}(u,b)\mathbf{e}^\mathsf{T},$$

where $\mathbf{T}(u,b) = (T_{i,j}(u,b))_{i,j=1}^n$.

It follows from Li (2008b) that

$$\mathbf{T}(u,b)=e^{-\mathbf{K}(b-u)},\quad T(u,b)=\boldsymbol{\alpha}e^{-\mathbf{K}(b-u)}\mathbf{e}^{\mathsf{T}},\quad u\leq b,$$

where $\mathbf{K}$ is a $n\times n$ matrix that satisfies the following equation

$$c\mathbf{K}=(\delta\mathbf{I}-\mathbf{B})-\mathbf{b}^{\mathsf{T}}\boldsymbol{\alpha}\int_0^\infty p(x)e^{-\mathbf{K}x}dx.$$

Assuming that the roots of the generalized Lundberg's equation with positive real parts $\rho_1,\rho_2,\ldots,\rho_n$ are distinct, Li (2008b) shows that

$$\mathbf{K}=\mathbf{H}\boldsymbol{\Delta}\mathbf{H}^{-1},$$

where $\boldsymbol{\Delta}=diag(\rho_1,\rho_2,\ldots,\rho_n)$ and $\mathbf{H}=(\mathbf{h}_1,\mathbf{h}_2,\ldots,\mathbf{h}_n)$. The column vector $\mathbf{h}_i$ is an eigenvector of $L_\delta(\rho_i)$ corresponding to the eigenvalue 0. Then

$$T(u,b)=\boldsymbol{\alpha}\mathbf{H}e^{-\boldsymbol{\Delta}(b-u)}\mathbf{H}^{-1}\mathbf{e}^{\mathsf{T}},\quad u\leq b. \tag{5.6.1}$$

If the roots $\rho_1,\rho_2,\ldots,\rho_n$ are not all distinct then the matrix $\mathbf{H}$ is not invertible and we can not apply formula (5.6.1) to find $T(u,b)$.

In the case of a double root we propose to replace such root by one of the negative roots of the generalized Lundberg's equation. Under our assumptions on this model there is always one negative real root. We denote it by $\rho_0=-R$ where $R>0$ is the adjustment coefficient.

Example 5.6.1 We continue the example 5.3.5 and we consider an interest force of $\delta=0.2$. Choosing $\beta=3.316$ the generalized Lundberg's equation has the following roots

$$\rho_0=-R=-1.534,\rho_1=0.457,\rho_2=5.319,\rho_3=\rho_4=8.62,$$

the corresponding eigenvectors are

$$\mathbf{h}_0=(0.583,0.407,0.503,0.490),\qquad \mathbf{h}_1=(0.484,0.517,0.497,0.501),$$
$$\mathbf{h}_2=(-0.015,-0.986,-0.004,0.167),\qquad \mathbf{h}_3=(-0.008,0.156,-0.905,0.395).$$

Therefore,

$$\mathbf{H}=\begin{pmatrix}0.583&0.484&-0.015&-0.008\\0.407&0.517&-0.986&0.156\\0.503&0.497&-0.004&-0.905\\0.490&0.501&0.167&0.395\end{pmatrix}$$

and we apply formula (5.6.1) to obtain

$$T(u,b) = 0.147e^{1.534(b-u)} + 0.853e^{-0.457(b-u)} - 0.0003e^{-5.319(b-u)} - 0.0003e^{-8.62(b-u)}.$$

Remark 5.6.1 For calculation of the expected discounted future dividends we can apply the same method in the case of double roots, replacing this root by the adjustment coefficient.

Remark 5.6.2 It is still an open problem if the fundamental (generalized) Lundberg's equation can have more than one double root, or if roots with higher multiplicity than double can appear.

5.7 Final remarks

In this chapter we considered Phase–Type(n) interclaim times for a Sparre–Andersen risk model. We studied the Lundberg's equations and we found cases where double roots appeared. For such cases we provided a method to compute the Laplace Transform of the time to reach a barrier level. Moreover, we gave expressions for the defective renewal equation and the integro–differential equation that are satisfied by the survival probability. This is useful, for example, if one is interested in the computation of other quantities, such as the probability of arrival to a barrier level and the maximum severity of ruin. Regarding the Lundberg's Matrix, we showed a proof of the linear independence of the eigenvectors related to different eigenvalues. Like we mentioned before, we can extend the results to determine the possibility of having more than one double root, or roots with higher multiplicity than double.

Chapter 6

Conclusion

> The study of mathematics, like the Nile, begins in minuteness but ends in magnificence
>
> Charles Caleb

It is in general difficult, if not impossible, to achieve all the objective of the work which we want. Scientific research is never a complete perfect product of developments but a long path of new improvements and exploration.

In this chapter we summarize all the work which was achieved during writing this thesis.

In Chapter 3 we analyzed the Sparre–Andersen risk model under the assumption of Erlang(n) distributed interclaim times. We improved techniques that are used to compute the maximum severity of ruin, and we found a new method to calculate the expected discounted dividends. It is always very important to find new calculating tools that can surpass the existing ones.

In Chapter 4 we dealt with a Sparre–Andersen model with generalized Erlang(n) distributed interclaim times, we followed the same approach like in Chapter 3. However we encounter one big difference. In the Erlang(n) model the roots of the fundamental (and the generalized) Lundberg's equation were all different. In the generalized Erlang(n) model that was not the case since we found roots with double multiplicity. We proposed a new way of dealing with the problem of computing the maximum severity of ruin and the expected discounted dividends in the presence of multiple roots. We studied the cases when the double roots can arise and showed with numerical examples how the new methods can be applied. Currently in the literature there were no attempts to find mathematical methods that take into account such multiplicities.

In Chapter 5 we studied the Sparre–Andersen model with Phase–type(n) interclaim times. Needles to say that this was the hardest part in all this manuscript. Our first purpose was to obtain an adequate expression for the Lundberg's equations. Similarly to the case of Chapter 4, we studied the behavior of the roots and determined the possibility of having double roots. We formulated a theorem in this direction. Furthermore, we explored the Lundberg's matrix to obtain other interesting results. Afterwards, we focused our attention in the computation of the survival probability, by means of finding an integro–differential equation and a defective renewal equation. We emulated the same techniques used in the two previous chapters which means that we compute the probability of attaining an upper level from the initial surplus by solving the same integro–differential equation from the survival probability. Finally, we applied those results for computational purposes. As mentioned before every work has the possibility of extension, therefore some results in this thesis can be still generalized. First of all, we can determine the possibility of finding roots of multiplicity higher than double in the Lundberg's equations. We can try to use other interclaims times distributions and find methods to estimate the severity of ruin, to computed dividends in the presence of a threshold strategy, or add a perturbation to the model. Going even further with research we might change the Sparre–Andersen model for other risk models.

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Appendix A

Divided differences

In mathematics, divided differences is a recursive division process. The method can be used to calculate the coefficients in the interpolation polynomial in the Newton form.

Let $f(x)$ be a function and $x_0, x_1, \ldots, x_n$ a set of $n+1$ different points in the complex plane $\mathbb{C}$.

We define recursively the first, second and third divided differences of f with respect to the points $x_0, x_1, \ldots, x_n$ in the following way

$$\begin{aligned} f[x_0] &= f(x_0) \\ f[x_1, x_0] &= \frac{f(x_0)}{(x_0 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)} \\ f[x_2, x_1, x_0] &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned}$$

In general the n–th divided difference is given by

$$f[x_n, \ldots, x_1, x_0] = \sum_{j=0}^{n} \frac{f(x_j)}{\tau_f'(x_j)},$$

where $\tau_f(x) = \prod_{i=0}^{n}(x - x_i)$.

Example

Let $f(x) = p_n(x) = x^n$. Then

$$
\begin{aligned}
p_j[x_n, \ldots, x_1, x_0] &= 0, \quad 0 \le j < n, \\
p_n[x_n, \ldots, x_1, x_0] &= 1, \\
p_{n+1}[x_n, \ldots, x_1, x_0] &= \sum_{i=0}^{n} x_i, \\
p_{n+m}[x_n, \ldots, x_1, x_0] &= \sum_{0 \le i_1 \le \cdots \le i_m \le n} \left(\prod_{j=1}^{m} x_{i_j} \right).
\end{aligned}
$$

Theorem A.0.1 Taylor series for divided differences

$$
f[x_n, \ldots, x_1, x_0] = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} p_j[x_n, \ldots, x_1, x_0].
$$

Appendix B

Gerber–Shiu penalty functions

Consider a Sparre–Andersen surplus process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,\ t \geq 0,$$

and denote by T_u the time to ruin. Let $U(T_u^-)$ be the surplus immediately prior to ruin and $|U(T_u)|$ the deficit at ruin.

For $x, y \geq 0$, let $w(x, y)$ be a non–negative function. For $\delta \geq 0$ define

$$\Phi_\delta(u) = E[e^{-\delta T_u} w(U(T_u^-), |U(T_u)|)\mathbb{I}(T_u < 0)|U(0) = u],\ u \geq 0.$$

The quantity $w(U(T_u^-), |U(T_u)|)$ can be interpreted as a penalty at time of ruin for the surplus $U(T_u^-)$ and the deficit $|U(T_u)|$.

Then $\Phi_\delta(u)$ is the expected discounted penalty with δ being the force of interest. Li and Garrido (2004b) find a defective renewal equation for $\Phi_\delta(u)$ for the case of Erlang(n, λ) interclaim times distributions.

Let $\rho_1, \ldots, \rho_n$ be the n roots of the generalized Lundberg's equation with positive real parts. Then

Theorem B.0.2 $\Phi_\delta(u)$ admits a defective renewal equation representation

$$\Phi_\delta(u) = \int_0^u \Phi_\delta(u - y)\eta_\delta(y)dy + G_\delta(u), \tag{B.0.1}$$

where $\eta_\delta(y) = (\lambda/c)^n T_{\rho_n} \cdots T_{\rho_1} p(y)$, $G_\delta(u) = (\lambda/c)^n T_{\rho_n} \cdots T_{\rho_1} \omega(u)$, with

$\omega(u) = \int_u^\infty w(u, y-u)p(y)dy$.

In particular, for $\delta = 0$ and $w(x,y) \equiv 1$ we have $\Phi_\delta(u) = \Psi(u)$, the ruin probability, and the equation (B.0.1) becomes

$$\Psi(u) = \int_0^u \Psi(u-y)\eta_0(y)dy + \Psi(0),$$

and a similar equation can be derived for the survival probability

$$\Phi(u) = \int_0^u \Phi(u-y)\eta_0(y)dy + \Phi(0).$$