Algorithms for solving the inverse problem associated with $KAK = A^{s+1}$

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Abstract

In previous papers, the authors introduced and characterized a class of matrices called $\{K, s+1\}$-potent. Also, they established a method to construct these matrices. The purpose of this paper is to solve the associated inverse problem. Several algorithms are developed in order to find all involutory matrices K satisfying $KA^{s+1}K = A$ for a given matrix $A \in \mathbb{C}^{n\times n}$ and a given natural number s. The cases $s = 0$ and $s \geq 1$ are separately studied since they produce different situations. In addition, some examples are presented showing the numerical performance of the methods.

Keywords: Involutory matrix; $\{K, s+1\}$-potent matrix; Spectrum; Algorithm; Inverse problem

MSC: 15A24, 15A29

1 Introduction

Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, that is $K^2 = I_n$, where I_n denotes the identity matrix of size $n \times n$. In [9], the authors introduced and characterized a new kind of matrices called $\{K, s+1\}$-potent where K is involutory. We

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recall that for an involutory matrix $K \in \mathbb{C}^{n\times n}$ and $s \in \{0, 1, 2, \dots\}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent if $KA^{s+1}K = A$. These matrices generalize all the following classes of matrices: $\{s+1\}$-potent matrices, periodic matrices, idempotent matrices, involutory matrices, centrosymmetric matrices, mirrorsymmetric matrices, 2×2 circulant matrices, etc. Some related classes of matrices are studied in [15, 16, 18, 19, 20].

We emphasize that the role of centrosymmetric matrices is very important in different technical areas. We can mention among them antenna theory, pattern recognition, vibration in structures, electrical networks and quantum physics (see [1, 4, 6, 7, 17, 21, 23]). It is observed that the computational complexity of various algorithms is reduced taking advantage of the structure of these matrices. Also, mirror-symmetric matrices have important applications in studying odd/even-mode decomposition of symmetric multiconductor transmission lines [13]. Additional applications of $\{K, s+1\}$-potent matrices are related to the calculation of high powers of matrices, such as those needed in Markov chains and Graph Theory. Allowing negative values for s, Wikramaratna studied in [24] a new type of matrices for generating pseudo-random numbers. Inspired by this idea, another application, in image processing, has been considered in [12] where algorithms for image blurring/deblurring are designed. The advantage of this method is to avoid the computation of inverses of matrices and it can be applied, for instance, to protect a part of an image.

The class of $\{K, s+1\}$-potent matrices is linked to other kind of matrices such as $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, Hamiltonian matrices, etc. [10]. Moreover, some related results are given in [3] from an algebraic point of view. Furthermore, in [11] the authors developed an algorithm to construct the matrices in this class. This problem is called the direct problem.

The aim of this paper is to solve the inverse problem, that is, to find all the involutory matrices K for which a given matrix A is $\{K, s+1\}$-potent. For this purpose, several algorithms will be developed. The $s = 0$ and $s \geq 1$ cases are separately studied since they produce different situations. In addition, some examples are presented showing the numerical performance of the methods.

In what follows, we will need the spectral theorem:

Theorem 1 ([2]) *Let $A \in \mathbb{C}^{n\times n}$ with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then the matrix A is diagonalizable if and only if there are disjoint projectors*

$P_1, P_2, \ldots, P_k$, *(i.e.,* $P_iP_j = \delta_{ij}P_i$ *for* $i, j \in \{1, 2, \ldots, k\}$*) such that* $A = \sum_{j=1}^{k} \lambda_j P_j$ *and* $I_n = \sum_{j=1}^{k} P_j$.

For a positive integer k, let Ω_k be the set of all k^{th} roots of unity. If we define $\omega = e^{2\pi i/k}$ then $\Omega_k = \{\omega^1, \omega^2, \ldots, \omega^k\}$. The elements of Ω_k will always be assumed to be listed in this order. Define $\Lambda_k = \{0\} \cup \Omega_k = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ so that $\lambda_0 = 0$, and $\lambda_j = \omega^j$ for $1 \leq j \leq k$.

Let $\mathbb{N}_s = \{0, 1, 2, \ldots, (s+1)^2 - 2\}$ for an integer $s \geq 1$. In [9], it was proved that the function $\varphi : \mathbb{N}_s \to \mathbb{N}_s$ defined by $\varphi(j) \equiv j(s+1)$ [mod $((s+1)^2 - 1)$] is a permutation. Moreover, we notice that φ is an involution. It was also shown that the eigenvalues of a $\{K, s+1\}$-potent matrix A are included in the set $\Lambda_{(s+1)^2-1}$ and such a matrix A has associated certain projectors. Specifically, we will consider matrices P_j's satisfying the relations

$$KP_jK = P_{\varphi(j)} \quad \text{and} \quad KP_{(s+1)^2-1}K = P_{(s+1)^2-1}, \tag{1}$$

for $j \in \mathbb{N}_s$, where $P_0, P_1, \ldots, P_{(s+1)^2-1}$ are the eigenprojectors given in Theorem 1. For simplicity in the notation, we are assuming that all of these projectors P_j's are spectral projectors for A. However, for any specific spectral decomposition of A we must consider the (unique) spectral projectors needed for that decomposition. The designed algorithms compute these specific eigenprojectors and, in Section 5, we show some examples with their adequate spectral projectors. Those examples also illustrate all the studied situations throughout the paper.

Theorem 2 ([9]) *Let* $A \in \mathbb{C}^{n \times n}$ *and* $s \geq 1$ *be an integer. Then the following conditions are equivalent:*

(a) A *is* $\{K, s+1\}$*-potent.*

(b) A *is diagonalizable,* $\sigma(A) \subseteq \Lambda_{(s+1)^2-1}$*, and the* P_j*'s satisfy condition (1), where* $\sigma(A)$ *denotes the spectrum of* A*.*

(c) $A^{(s+1)^2} = A$*, and the* P_j*'s satisfy condition (1).*

2 Obtaining the involutory matrices K for $s \geq 1$

It is well known that the Kronecker product is an important tool to solve some matrix problems, as for example the Sylvester and Lyapunov equations. The

Kronecker sum, obtained as a sum of two Kronecker products, is applied, for example, to solve the two-dimensional heat equation, to rewrite the Jacobi iteration matrix, etc. [22]. The notation $\otimes$ used in this paper refers to the Kronecker product; and X^T denotes the transpose of the matrix X [8]. For any matrix $X = [x_{ij}] \in \mathbb{C}^{n\times n}$, let $v(X) = [v_k] \in \mathbb{C}^{n^2\times 1}$ be the vector formed by stacking the columns of X into a single column vector. The expression $[v(X)]_{\{(j-1)n+1,\ldots,(j-1)n+n\}}$, for $j = 1, \ldots, n$, denotes the j^{th} column of X.

In what follows, we will need the following property: if $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ then

$$\operatorname{Ker}(A) \cap \operatorname{Ker}(B) = \operatorname{Ker}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right), \tag{2}$$

which is also valid for a finite number of matrices of suitable sizes, where Ker(.) denotes the null space of the matrix (.).

We recall that when A is a diagonalizable matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_l$, the principal idempotents are given by

$$P_t = \frac{p_t(A)}{p_t(\lambda_t)} \qquad \text{where} \qquad p_t(\eta) = \prod_{\substack{i=1 \\ i\neq t}}^{l} (\eta - \lambda_i). \tag{3}$$

By using the function φ and the projectors introduced in (1), it is possible to construct the matrix

$$M = \begin{bmatrix} (P_0^T \otimes I_n) + (I_n \otimes -P_{\varphi(0)}) \\ (P_1^T \otimes I_n) + (I_n \otimes -P_{\varphi(1)}) \\ \vdots \\ (P_{(s+1)^2-2}^T \otimes I_n) + (I_n \otimes -P_{\varphi((s+1)^2-2)}) \\ (P_{(s+1)^2-1}^T \otimes I_n) + (I_n \otimes -P_{(s+1)^2-1}) \end{bmatrix}. \tag{4}$$

For a given positive integer s, the square, complex matrix A is called a *potential* $\{K, s+1\}$*-potent matrix* if $A^{(s+1)^2} = A$, or equivalently, if A is diagonalizable and $\sigma(A)$ is contained in $\Lambda_{(s+1)^2-1}$. Note that K is completely unspecified here. Also note that it is generally much easier and faster to test that $A^{(s+1)^2} = A$ than is to determine the spectrum of A and to determine that A is diagonalizable.

Algorithm 1

Inputs: An integer $s \geq 1$, and a matrix $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Output: A decision on whether A is potentially $\{K, s+1\}$-potent or not.

Step 1 If either $A^{(s+1)^2} = A$ or, A is diagonalizable and $\sigma(A) \subseteq \Lambda_{(s+1)^2-1}$, then "$A$ is potentially $\{K, s+1\}$-potent." Go to End.

Step 2 "A is not potentially $\{K, s+1\}$-potent, and there is no idempotent matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K, s+1\}$-potent."

End

The algorithm presented below solves the inverse problem stated in the introduction.

ALGORITHM 2

Inputs: An integer $s \geq 1$, and a matrix $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K, s+1\}$-potent matrix, if any such K exist.

Step 1 Apply Algorithm 1 to A. If A is not potentially $\{K, s+1\}$-potent, then no such involutory matrix K exists. Go to End.

Step 2 Compute $\sigma(A)$. Suppose that A has l distinct eigenvalues. Since $\sigma(A) \subseteq \Lambda_{(s+1)^2-1}$, there are l indices j_t with $0 \leq j_1 < j_2 < \cdots < j_l \leq (s+1)^2 - 1$ such that $\sigma(A) = \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_l}\}$.

Step 3 Compute the principal idempotents associated with the eigenvalues of A using (3).

Step 4 Compute $\varphi(j_1), \varphi(j_2), \ldots, \varphi(j_l)$.

Step 5 Compute the submatrix M_A of M given by (4) containing only those rows corresponding to eigenvalues of A.

Step 6 Find the general solution v to $M_A v = \mathbf{0}$. The $n^2 \times 1$ vector v will depend on $d = \dim(\ker(M_A))$ arbitrary parameters.

Step 7 If $v = \mathbf{0}$, or equivalently, if $d = 0$, then go to Step 11.

Step 8 Treating v as $v = v(K)$ for an $n \times n$ complex matrix K containing d parameters, recover K from v.

Step 9 Determine the allowed values for the d arbitrary parameters so that $K^2 = I_n$. If there are no allowed parameter values, then go to Step 11.

Step 10 The output is the set of all of the matrices K whose parameter values are allowed.

Step 11 "There is no involutory matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K, s+1\}$-potent."

End

We will now justify Algorithm 2. Step 2 computes and orders the eigenvalues following the specified notation in the set $\Lambda_{(s+1)^2-1}$. Steps 3 and 4 are justified by Theorem 1.

Next, we focus our attention on solving the nonlinear equations $KP_jK = P_{\varphi(j)}$ in the unknown K that appears in (1), that is, to find the common solutions to

$$KP_j = P_{\varphi(j)}K, \text{ for } j \in \mathbb{N}_s \quad \text{and} \quad KP_{(s+1)^2-1} = P_{(s+1)^2-1}K.$$

For this purpose, we use the Kronecker product and we take into account that

$$v(KP_j) = v(P_{\varphi(j)}K) \iff (P_j^T \otimes I_n)v(K) = (I_n \otimes P_{\varphi(j)})v(K),$$

for $j \in \mathbb{N}_s$, and analogously,

$$v(KP_j) = v(P_jK) \iff (P_j^T \otimes I_n)v(K) = (I_n \otimes P_j)v(K),$$

for $j = (s+1)^2 - 1$. By property (2), it is clear that we have to find (non trivial) solutions $v(K)$ of the null space of the matrix

$$M = \begin{bmatrix} (P_0^T \otimes I_n) + (I_n \otimes -P_{\varphi(0)}) \\ (P_1^T \otimes I_n) + (I_n \otimes -P_{\varphi(1)}) \\ \vdots \\ (P_{(s+1)^2-2}^T \otimes I_n) + (I_n \otimes -P_{\varphi((s+1)^2-2)}) \\ (P_{(s+1)^2-1}^T \otimes I_n) + (I_n \otimes -P_{(s+1)^2-1}) \end{bmatrix}.$$

This last matrix is the expression given in (4). However, observe that only the submatrix M_A is needed to finalize our justification. This reasoning justifies Steps 5, 6, 7 and 8. In Steps 9 and 10 the condition of K being involutory is checked.

3 Alternative methods for obtaining K for $s \geq 1$

The Algorithm 2 has been designed from the relationship established between the projectors P_j's and the matrix K. Now the following algorithm provides a simplified procedure in order to find all the involutory matrices K solving the inverse problem.

Algorithm 3

Inputs: An integer $s \geq 1$, and a matrix $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K, s+1\}$-potent matrix, if any such K exist.

Step 1 Apply Algorithm 1 to A. If A is not potentially $\{K, s+1\}$-potent, then go to Step 7.

Step 2 Construct a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Step 3 Find the general $n \times n$ solution matrix J for the system $JD - D^{s+1}J = \mathbf{0}_{n\times n}$. The set $\mathcal{S}$ of all such matrices J is a subspace of $\mathbb{C}^{n\times n}$.

Step 4 If $J = \mathbf{0}_{n\times n}$, or equivalently $\mathcal{S} = \{\mathbf{0}_{n\times n}\}$, then Go to Step 7.

Step 5 Let $\mathcal{S}_{invol}$ denote the set of all $J \in \mathcal{S}$ such that $J^2 = I_n$. If $\mathcal{S}_{invol} = \emptyset$, then go to Step 7.

Step 6 The set of involutory matrices K such that A is $\{K, s+1\}$-potent is obtained as $\{PJP^{-1} : J \in \mathcal{S}_{invol}\}$. Go to End.

Step 7 "There is no involutory matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K, s+1\}$-potent."

End

Now we give a justification for Algorithm 3. Again, the purpose of the inverse problem is to find all the involutory matrices K such that $KA^{s+1}K = A$. When such a K exists, Theorem 2 shows that A must be diagonalizable with spectrum contained in $\Lambda_{(s+1)^2-1}$. Therefore, it must exist a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ such that $A = PDP^{-1}$. Moreover, we have:

$$(P^{-1}KP)(P^{-1}AP)(P^{-1}KP) = P^{-1}KAKP = P^{-1}A^{s+1}P = (P^{-1}AP)^{s+1},$$

that is, $D = P^{-1}AP$ is $\{P^{-1}KP, s+1\}$-potent provided that $(P^{-1}KP)^2 = I_n$. Thus, the inverse problem can be reformulated as follows: we have to find all the involutory matrices $J \in \mathbb{C}^{n\times n}$ such that D is $\{J, s+1\}$-potent and then calculate $K = PJP^{-1}$. Hence, by definition, we need to solve the matrix equation

$$D^{s+1}J = JD \tag{5}$$

in the unknown J. This reasoning justifies the Step 3. We observe that the linear system (5) is easy to be solved. Among all the matrices J's calculated in Step 3, we have to discard those that are not involutory.

In Algorithms 2 and 3 we use spectral theory corresponding to the matrix A. A variant of the method gives the next algorithm where the diagonalization of the matrix A is not used.

Algorithm 4

Inputs: An integer $s \geq 1$, and a matrix $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K, s+1\}$-potent matrix, if any such K exist.

Step 1 Construct the matrix $Q = A^T \otimes I_n - I_n \otimes A^{s+1}$.

Step 2 Find the general solution v to $Qv = \mathbf{0}$. The $n^2 \times 1$ vector v will depend on $d = \dim(\ker(Q))$ arbitrary parameters.

Step 3 If $v = \mathbf{0}$, or equivalently, if $d = 0$, then go to Step 7.

Step 4 Treating v as $v = v(K)$ for an $n \times n$ complex matrix K containing d parameters, recover K from v.

Step 5 Determine the allowed values for the d arbitrary parameters so that $K^2 = I_n$. If there are no allowed parameter values, then go to Step 7.

Step 6 The output is the set of all of the matrices K whose parameter values are allowed. Go to End.

Step 7 "There is no involutory matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K, s+1\}$-potent."

End

4 $\{K\}$-generalized centrosymmetric matrices ($s = 0$)

We recall that a $\{K, 1\}$-potent matrix ($s = 0$) is also called a $\{K\}$-generalized centrosymmetric matrix [10].

Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with spectral decomposition given by $A = \sum_{i=1}^{k} \lambda_i P_i$ as in Theorem 1. Theorem 15 in [10] assures that if $K \in \mathbb{C}^{n\times n}$ is an involutory matrix such that A is $\{K\}$-generalized centrosymmetric then $KP_i = P_iK$, for all $i \in \{1, 2, \ldots, k\}$. This fact allows us to give the following Algorithm 5. Now, the matrix M defined in (4) has the form

$$M = \begin{bmatrix} P_1^T \otimes I_n + I_n \otimes -P_1 \\ P_2^T \otimes I_n + I_n \otimes -P_2 \\ \vdots \\ P_k^T \otimes I_n + I_n \otimes -P_k \end{bmatrix}. \qquad (6)$$

Remark 1 *Despite the fact that in this section we denote again by λ_i's the eigenvalues of A, it is important to observe that they do not have the property of belonging to $\Lambda_{(s+1)^2-1}$ as it happens for $s \geq 1$.*

ALGORITHM 5

Input: A diagonalizable matrix $A \in \mathbb{C}^{n\times n}$ for some $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K\}$-generalized centrosymmetric matrix, if any such K exist.

Step 1 Compute $\sigma(A)$ and the spectral projectors associated with the eigenvalues of A using (3).

Step 2 Compute M as in (6).

Step 3 Find the general solution v to $Mv = \mathbf{0}$. The $n^2 \times 1$ vector v will depend on $d = \dim(\ker(M))$ arbitrary parameters.

Step 4 If $v = \mathbf{0}$, or equivalently, if $d = 0$, then go to Step 8.

Step 5 Treating v as $v = v(K)$ for an $n \times n$ complex matrix K containing d parameters, recover K from v.

Step 6 Determine the allowed values for the d arbitrary parameters so that $K^2 = I_n$. If there are no allowed parameter values, then go to Step 8.

Step 7 The output is the set of all of the matrices K whose parameter values are allowed.

Step 8 "There is no involutory matrix $K \in \mathbb{C}^{n\times n}$ such that A is

End

Algorithm 3 can be reformulated for the $s = 0$ case as follows.

Algorithm 6

Input: A diagonalizable matrix $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K\}$-generalized centrosymmetric matrix, if any such K exist.

Step 1 Construct a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Step 2 Find the general $n \times n$ solution matrix J for the system $JD - DJ = \mathbf{0}_{n\times n}$. The set $\mathcal{S}$ of all such matrices J is a subspace of $\mathbb{C}^{n\times n}$.

Step 3 Set J be the matrices in Step 2 that satisfy $J^2 = I_n$.

Step 4 If $J = \mathbf{0}_{n\times n}$, or equivalently $\mathcal{S} = \{\mathbf{0}_{n\times n}\}$, then Go to Step 7.

Step 5 Let $\mathcal{S}_{invol}$ denote the set of all $J \in \mathcal{S}$ such that $J^2 = I_n$. If $\mathcal{S}_{invol} = \emptyset$, then go to Step 7.

Step 6 The set of involutory matrices K such that A is $\{K\}$-generalized centrosymmetric is obtained as $\{PJP^{-1} : J \in \mathcal{S}_{invol}\}$. Go to End.

Step 7 'There is no involutory matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K\}$-generalized centrosymmetric'.

End

Notice that the linear system in Step 2 is easy to be solved when all the eigenvalues of the matrix D are distinct. In this case, Step 3 forces the diagonal entries of J's to be in $\{-1, 1\}$. Otherwise, when D has some repeated eigenvalue in consecutive places, the matrices J's will be block diagonal where all the diagonal blocks are involutory.

Remark 2 *We notice that Algorithm 4, designed for $\{K, s+1\}$-potent matrices, also works for $\{K\}$-generalized centrosymmetric matrices.*

In general, $\{K\}$-generalized centrosymmetric matrices are not diagonalizable as the following matrices show:

$$A = \begin{bmatrix} -4 & -2 & -4 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}, \qquad K = \frac{1}{9}\begin{bmatrix} 7 & -4 & 4 \\ -4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}.$$

This fact leads us to consider the Jordan canonical form of the matrix $A \in \mathbb{C}^{n\times n}$ in order to develop a new algorithm to solve the inverse problem. First, we write

$$A = QJ_AQ^{-1} = Q\,\mathrm{diag}(J_{q_1}(\lambda_1), \cdots, J_{q_t}(\lambda_t))\,Q^{-1} \tag{7}$$

where $Q \in \mathbb{C}^{n\times n}$ is nonsingular and $J_{q_i}(\lambda_i)$ is a $q_i \times q_i$ Jordan block corresponding to the eigenvalue λ_i with 1's in the superdiagonal. Now, to find an involutory matrix $K \in \mathbb{C}^{n\times n}$ such that $AK = KA$ is equivalent to find an involutory matrix $W \in \mathbb{C}^{n\times n}$ such that $J_AW = WJ_A$, where $K = QWQ^{-1}$. By Theorem 5.16 in [5] (see also [14]) we can obtain that the general solution W of $J_AW = WJ_A$, partitioned into blocks as $W = [W_{ij}]$ where $W_{ij} \in \mathbb{C}^{q_i\times q_j}$ for every i, j, has the following structure:

(a) if $\lambda_i \neq \lambda_j$ then $W_{ij} = O$

(b) if $\lambda_i = \lambda_j$ then

$$W_{ij} = \begin{cases} T_{ij} & \text{if } q_i = q_j \\ \begin{bmatrix} O_{l_{ij}} & T_{ij} \end{bmatrix} & \text{if } q_i < q_j \\ \begin{bmatrix} T_{ij} \\ O_{l_{ij}} \end{bmatrix} & \text{if } q_i > q_j \end{cases}$$

where $T_{ij} \in \mathbb{C}^{\alpha_{ij} \times \alpha_{ij}}$ is an arbitrary upper triangular Toeplitz matrix with $\alpha_{ij} = \min\{q_i, q_j\}$ and $l_{ij} = |q_i - q_j|$.

Algorithm 7

Inputs: A matrix $A \in \mathbb{C}^{n\times n}$ for some $n \geq 2$.

Outputs: All the involutory matrices $K \in \mathbb{C}^{n\times n}$ such that A is a $\{K\}$-generalized centrosymmetric matrix, if any such K exist.

Step 1 Construct J_A and Q such that $A = QJ_AQ^{-1}$ as in (7).

Step 2 Construct W_{ij} according to the previous items (a) and (b).

Step 3 Let $\mathcal{S}_{invol}$ denote the set of all $W \in \mathcal{S}$ such that $W^2 = I_n$. If $\mathcal{S}_{invol} = \emptyset$, then go to Step 5.

Step 4 The set of involutory matrices K such that A is $\{K\}$-generalized centrosymmetric is obtained as $\{QWQ^{-1} : W \in \mathcal{S}_{invol}\}$. Go to End.

Step 5 'There is no matrix $K \in \mathbb{C}^{n\times n}$ such that A is $\{K\}$-generalized centrosymmetric'.

End

We now explore the expression for the involutory matrix W that has appeared in the analysis previous to Algorithm 7. It is well known that an upper triangular Toeplitz matrix $T \in \mathbb{C}^{n\times n}$ has the form

$$T(a_0, a_1, \ldots, a_{n-1}) = a_0 I_n + a_1 J_n(0) + a_2[J_n(0)]^2 + \cdots + a_{n-1}[J_n(0)]^{n-1},$$

where $J_n(0)$ is an $n \times n$ Jordan block corresponding to the eigenvalue 0. A simple computation indicates that such a matrix T is involutory if and only if $T = \pm I_n$. This fact allows us to simplify the matrix W as $\pm I_{q_1} \oplus \cdots \oplus \pm I_{q_t}$ when all its eigenvalues are distinct. A similar situation occurs when only one of the eigenvalues of A, namely λ_i, has exactly one corresponding Jordan block. In this case, the corresponding block to λ_i in the diagonal of W is $\pm I_{q_i}$. On the other hand, when $J_A = \mathrm{diag}(J_{q_1}(\lambda), J_{q_2}(\lambda))$, we can assure that:

- if $q_1 \neq q_2$, the diagonal entries of the Toeplitz block T_{11} of W are equal to 1 or -1; analogously, T_{22} has 1's or -1's in its diagonal.
- If $q_1 = q_2$ and t_i denotes the diagonal entries of the (diagonal) blocks T_{ii} of W, for $i = 1, 2$, then $t_1 = \pm t_2$.

5 Numerical examples and applications

Our algorithms can easily be implemented on a computer. We have used the MATLAB R2014b package. In this section we present some numerical examples in order to show the performance of our algorithms and demonstrate their applicability. The computational cost of Algorithms 1-5 is $O(n^3)$ while the computational cost of Algorithm 6 is basically given by Step 1.

5.1 Case $s \geq 1$

Example 1 *For $s = 4$ and*

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 5 & -2 \\ 0 & 15 & -6 \end{bmatrix},$$

Algorithm 1 provides the solutions

$$K = I_3, \qquad K = \mathrm{diag}(1, -1, -1), \qquad K = \mathrm{diag}(-1, 1, 1),$$

$$K = -I_3, \qquad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 4 \\ 0 & -30 & 11 \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11 & -4 \\ 0 & 30 & -11 \end{bmatrix},$$

$$K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -11 & 4 \\ 0 & -30 & 11 \end{bmatrix}, \qquad K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 11 & -4 \\ 0 & 30 & -11 \end{bmatrix},$$

with the projectors

$$P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & 15 & -5 \end{bmatrix}, \qquad P_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & -15 & 6 \end{bmatrix}.$$

Example 2 *The same matrix A as in Example 1 is used. The form of the matrices K's obtained from the Algorithm 2 is*

$$K = \begin{bmatrix} z_9 & 0 & 0 \\ 0 & -5z_1 + 6z_5 & 2z_1 - 2z_5 \\ 0 & -15z_1 + 15z_5 & 6z_1 - 5z_5 \end{bmatrix},$$

where $z_1, z_5, z_9 \in \{-1, 1\}$.

Example 3 *For $s = 1$ and the matrix*

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},$$

the form of the matrices K's obtained from the Algorithm 3 is

$$K = \begin{bmatrix} -k_{22} & \frac{k_{22}+\sqrt{4-3k_{22}^2}}{2} \\ \frac{-k_{22}+\sqrt{4-3k_{22}^2}}{2} & k_{22} \end{bmatrix} \text{ or } K = \begin{bmatrix} -k_{22} & \frac{k_{22}-\sqrt{4-3k_{22}^2}}{2} \\ \frac{-k_{22}-\sqrt{4-3k_{22}^2}}{2} & k_{22} \end{bmatrix}$$

for any arbitrary $k_{22} \in \mathbb{C}$. In this case, the eigenvalues of A are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ and its corresponding eigenprojectors are

$$P_1 = \begin{bmatrix} \frac{1}{2} + i\frac{\sqrt{3}}{6} & -i\frac{\sqrt{3}}{3} \\ i\frac{\sqrt{3}}{3} & \frac{1}{2} - i\frac{\sqrt{3}}{6} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} \frac{1}{2} - i\frac{\sqrt{3}}{6} & i\frac{\sqrt{3}}{3} \\ -i\frac{\sqrt{3}}{3} & \frac{1}{2} + i\frac{\sqrt{3}}{6} \end{bmatrix}.$$

5.2 Case $s = 0$

Example 4 *For the matrix*

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 0 & -3 & 10 \\ 0 & -2 & 6 \end{bmatrix}$$

the form of the matrices K's obtained from the Algorithm 4 is $K = \pm I_3$,

$$K = \begin{bmatrix} 1 & -\frac{1}{2}k_{13}+1 & k_{13} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad K = \begin{bmatrix} -1 & -\frac{1}{2}k_{13}-1 & k_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -1 & -\frac{1}{2}k_{13} & k_{13} \\ 0 & 9 & -20 \\ 0 & 4 & -9 \end{bmatrix}, \qquad K = \pm \begin{bmatrix} 1 & -4 & 10 \\ 0 & 9 & -20 \\ 0 & 4 & -9 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & -\frac{1}{2}k_{13} & k_{13} \\ 0 & -9 & 20 \\ 0 & -4 & 9 \end{bmatrix}$$

$$K = \begin{bmatrix} -5k_{32}/2 - k_{33} & \frac{k_{32}(2+5k_{32}+2k_{33})}{2(1-2k_{32}-k_{33})} & \frac{(k_{33}-1)(2+5k_{32}+2k_{33})}{2(1-2k_{32}-k_{33})} \\ -5/2+5k_{32}+5k_{33}/2 & 1+5k_{32}/2 & 5(k_{33}-1)/2 \\ -1+2k_{32}+k_{33} & k_{32} & k_{33} \end{bmatrix}$$

$$K = \begin{bmatrix} -5k_{32}/2 - k_{33} & \frac{k_{32}(2-5k_{32}-2k_{33})}{2(1+2k_{32}+k_{33})} & \frac{(k_{33}+1)(2-5k_{32}-2k_{33})}{2(1+2k_{32}+k_{33})} \\ 5/2+5k_{32}+5k_{33}/2 & -1+5k_{32}/2 & 5(k_{33}+1)/2 \\ 1+2k_{32}+k_{33} & k_{32} & k_{33} \end{bmatrix}$$

for any arbitrary $k_{13}, k_{32}, k_{33} \in \mathbb{C}$.

Example 5 *For the matrix*

$$A = \begin{bmatrix} 7 & 10 & -15 \\ -6 & -10 & 18 \\ -2 & -4 & 8 \end{bmatrix}$$

the form of the matrices K's obtained from the Algorithm 5 is $K = \pm I_3$,

$$K = \begin{bmatrix} -11 & -20 & 30 \\ 12 & 23 & -36 \\ 4 & 8 & -13 \end{bmatrix}, \qquad K = \begin{bmatrix} 11 & 20 & -30 \\ -12 & -23 & 36 \\ -4 & -8 & 13 \end{bmatrix}$$

$$K = \begin{bmatrix} -11 & -18+3k_{23} & 24-9k_{23} \\ 12 & 19-3k_{23} & -24+9k_{23} \\ 4 & 6-k_{23} & -7+3k_{23} \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & -2+3k_{23} & 6-9k_{23} \\ 0 & 5-3k_{23} & -12+9k_{23} \\ 0 & 2-k_{23} & -5+3k_{23} \end{bmatrix}$$

$$K = \begin{bmatrix} -1 & 2+3k_{23} & -6-9k_{23} \\ 0 & -5-3k_{23} & 12+9k_{23} \\ 0 & -2-k_{23} & 5+3k_{23} \end{bmatrix}, \qquad K = \begin{bmatrix} 11 & 18+3k_{23} & -24-9k_{23} \\ -12 & -19-3k_{23} & 24+9k_{23} \\ -4 & -6-k_{23} & 7+3k_{23} \end{bmatrix}$$

$$K = \begin{bmatrix} -5 - 6k_{33} + 2k_{32} & -10 - 8k_{33} + 3k_{32} + 3(1 - k_{33}^2)/k_{32} & 15 + 9k_{33} - 4k_{32} - 9(1 - k_{33}^2)/k_{32} \\ 6 + 6k_{33} - 4k_{32} & 12 + 7k_{33} - 6k_{32} - 3(1 - k_{33}^2)/k_{32} & -18 - 6k_{33} + 8k_{32} + 9(1 - k_{33}^2)/k_{32} \\ 2 + 2k_{33} - 2k_{32} & 4 + 2k_{33} - 3k_{32} - (1 - k_{33}^2)/k_{32} & -6 - k_{33} + 4k_{32} + 3(1 - k_{33}^2)/k_{32} \end{bmatrix}$$

$$K = \begin{bmatrix} 5 - 6k_{33} + 2k_{32} & 10 - 8k_{33} + 3k_{32} + 3(1 - k_{33}^2)/k_{32} & -15 + 9k_{33} - 4k_{32} - 9(1 - k_{33}^2)/k_{32} \\ -6 + 6k_{33} - 4k_{32} & -12 + 7k_{33} - 6k_{32} - 3(1 - k_{33}^2)/k_{32} & 18 - 6k_{33} + 8k_{32} + 9(1 - k_{33}^2)/k_{32} \\ -2 + 2k_{33} - 2k_{32} & -4 + 2k_{33} - 3k_{32} - (1 - k_{33}^2)/k_{32} & 6 - k_{33} + 4k_{32} + 3(1 - k_{33}^2)/k_{32} \end{bmatrix}$$

for any arbitrary $k_{23}, k_{32}, k_{33} \in \mathbb{C}$.

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1	0	0	0	0	0	1	0
-1	0	0	0	0	0	-1	0
1	0	0	a_3	0	a_5	-1	0
-1	0	a_2	a_3	0	a_5	1	0
-1	$-a_7$	a_2	a_3	0	$-\frac{2a_7}{a_2}$	1	a_7
1	$-a_7$	a_2	a_3	0	$\frac{2a_7}{a_2}$	-1	a_7
$-a_6$	$-a_7$	$-\frac{a_6^2-1}{a_4}$	$\frac{a_5a_6^2-a_5-2a_4a_6a_7}{a_4^2}$	a_4	a_5	a_6	a_7

Table 1: Parameters to obtain all the matrices W's.

Example 6 *For the matrix*

$$J_A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

the form of the matrices W's obtained from the Algorithm 6 is

$$W = \begin{bmatrix} T(a_0, a_1) & T(a_2, a_3) \\ T(a_4, a_5) & T(a_6, a_7) \end{bmatrix}$$

where the parameters are listed in Table 1.

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