The First Chevalley-Eilenberg Cohomology Group of the Lie Algebra on the Transverse Bundle of a Decreasing Family of Foliations

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Abstract

In [Lie Algebra on the Transverse Bundle of a Decreasing Family of Foliations, J. Geom. Phys., 60 (2010) 122-133], we defined the transverse bundle V^k to a decreasing family of k foliations F_i on a manifold M. We have shown that there exists a (1,1) tensor J of V^k such that $J^k \neq 0$, $J^{k+1} = 0$ and we defined by $L_J(V^k)$ the Lie Algebra of vector fields X on V^k such that, for each vector field Y on V^k, $[X, JY] = J[X,Y]$.
In this note, we study the first Chevalley-Eilenberg Cohomology Group i.e. the quotient space of derivations of $L_J(V^k)$ by the subspace of inner derivations, denoted by $H^1\left(L_J(V^k)\right)$.

Keywords: Foliations; Fiber Bundles; Lie Algebra; Derivation; Cohomology group.

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1. Introduction

Let M be a differentiable manifold of dimension m endowed with k foliations $F_1, F_2, ..., F_k$, $k \geq 1$, of respective codimensions $p_1, p_1 + p_2, ..., p_1 + p_2 + ... + p_k$ such that $F_1 \supset F_2 \supset ... \supset F_k$ ($m = p_1 + p_2 + ... + p_k + p_{k+1}$, $p_1 > 0$, $p_i \geq 0$, $2 \leq i \leq k+1$).

In [1], we defined a so-called "order k bundle V^k transverse to the foliations F_i" and we proved that there exists a (1,1) tensor J of V^k such that $J^k \neq 0$, $J^{k+1} = 0$ and for every pair of vector fields X, Y on V^k:

$$[JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y] = 0.$$

Ω being an open set of V^k, we denote by $L_J(\Omega)$ the Lie Algebra of vector fields X defined on Ω such that the Lie derivative $L(X)J$ is equal to zero i.e., for each vector field Y on Ω:

$$[X, JY] = J[X,Y].$$

We define by L_1 a subset of $L_J(V^k)$ constituted by the vector field X on V^k such that $X \in KerJ$. The purpose of this paper is to study the first Chevalley-Eilenberg Cohomology Group of L_J, denoted by $H^1\left(L_J(V^k)\right)$. In [2], J.Lehmann-Lejeune studied the Cohomology on the Transverse Bundle of a Foliation. This paper is organized as follows.

In section 2, we recall some relevant results and notations (cf [1]), more precisely, we define the order *k* bundle V^k and the (1,1) tensor J of V^k, and we remind the most important result showed in [1]: for every $X \in L_1(V^k)$, we can write $X = \sum_i \left[Y_i, Z_i\right]$ where $\sum_i$ is a finite sum and Y_i, Z_i belongs to $L_1(V^k)$.

In section 3, we study the derivations of $L_1(V^k)$. We prove that every derivation of $L_J(V^k)$ restricted to $L_1(V^k)$ is a derivation of $L_1(V^k)$ and also every derivation of $L_1(V^k)$ is local. Moreover, we construct three derivations of $L_1(U)$ witch are not inner derivations, where U is an open set of adapted local coordinates of V^k. On the other hand we show that, for every $x \in V^k$, there exists an open set U containing x such that $\dim H^1\left(L_1(U)\right)$ is infinite.

In section 4, we study the case of foliations defined by submersions and then we show that the dimension of $H^1\left(L_J(V^k)\right)$ is equal to k.

In section 5, we study an example on T^3 with $k=2$ foliations where $\dim H^1\left(L_J(V^k)\right) > k$.

In section 6, we compute $H^1\left(L_1(V^k)\right)$ in the case of the 3- sphere.

2. Preliminaries

Let M be a differentiable manifold of dimension *m* endowed with k foliations $F_1, F_2, ..., F_k$, $k \geq 1$, of respective codimensions $p_1, p_1+p_2, ..., p_1+p_2+...+p_k$ such that $F_1 \supset F_2 \supset ... \supset F_k$ ($m = p_1+p_2+....+p_k+p_{k+1}$, $p_1 > 0$, $p_i \geq 0$, $2 \leq i \leq k+1$).

<u>Notation:</u> we set:

$a(h) = p_1+p_2+....+p_h$	for $1 \leq h \leq k+1$,
$a(h) = 0$	for $h \leq 0$,
$c(t) = a(k+1)+a(k)+....+a(k-t+2)$	for $1 \leq t \leq k+1$,
$c(t) = 0$	for $t \leq 0$

We define a so-called "order *k* bundle V^k transverse to the foliations F_i " (cf [1], p. 123) in the following way. The order *k* tangent bundle of M is the manifold of dimension *(k+1)m* of the $k-$jets of origin 0 of differentiable mappings from IR to M denoted T^kM (cf. [3]).

Let s and h be two integers such that $0 \leq s \leq h \leq k$, $h \geq 1$. On the set of $h-$jets of differentiable mappings of origin 0 from IR to M, we define an equivalence relation. Let φ and ψ be two differentiable mappings from IR to M such that $\varphi(0) = \psi(0)$.

Denote by $(u_1, u_2, ..., u_m)$ the local coordinates of an open set $\hat{U} \subset M$, adapted to the *k* foliations (i.e. $u_1, u_2, ..., u_{a(h)}$ are constants on the leaves of F_h, $1 \leq h \leq k$), such that $\varphi(0) = \psi(0) = x_0 \in \hat{U}$.

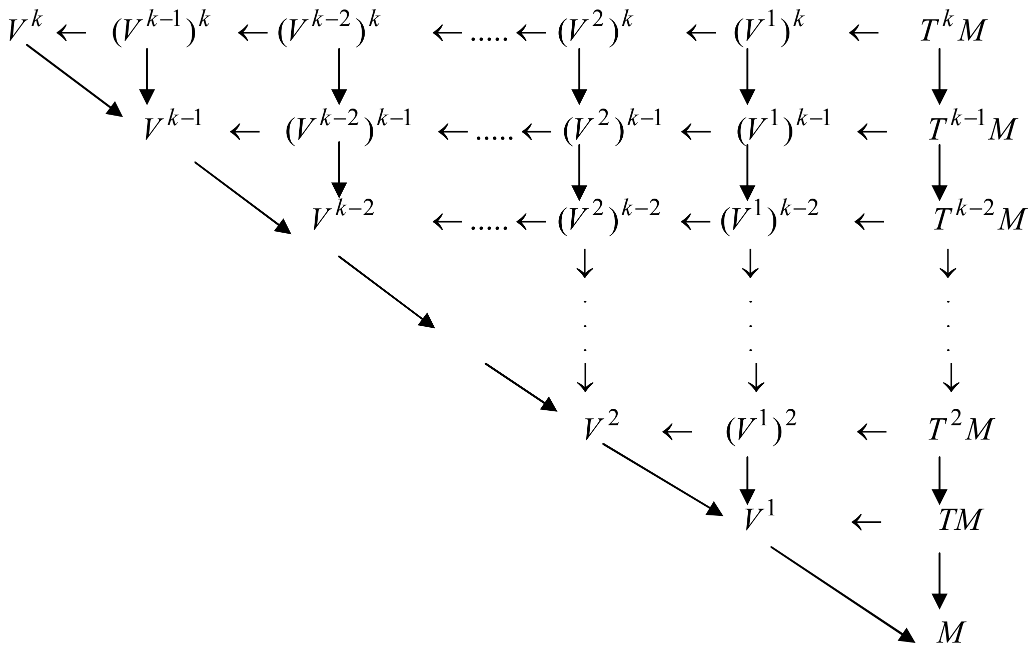
We say that the $h-$jets of φ and ψ are equivalent if: $\frac{d^b\varphi_l}{d\rho^b}(0)=\frac{d^b\psi_l}{d\rho^b}(0)$, $1\le b\le s$, $1\le l\le a(k+1-b)$ and $s+1\le b\le h$, $1\le l\le a(k+1-s)$. This equivalence relation is independent of the open set $\hat{U}$ of coordinates adapted to the k foliations containing x_0.

We denote by $(V^s)^h$ the quotient space of the $h-$jets of differentiable mappings from IR to M endowed with this equivalence relation.

This is a manifold of dimension $\sum_{0\le t\le s} a(k+1-t)+(h-s)a(k+1-s)$.

For $s=h$, $(V^s)^s$ will be denoted, for simplicity, by V^s.

We have the following diagram, where the arrows are the natural projections:



V^k is called order k bundle transverse to the k foliations $F_1,F_2,\ldots,F_k$.

The dimension of V^k is $n=\sum_{0\le t\le k}(t+1)p_{k+1-t}=\sum_{0\le t\le k}a(k+1-t)$.

T^kM (which can be considered as a $(V^s)^k$ with $s=0$) is equipped with an order k nearly tangent structure J_0 of constant range km (cf. [3]). In [1] p. 124, we show that there exists a (1,1) tensor J of V^k which is the projection on V^k of the nearly tangent operator J_0 of order k on T^kM. Its rank is constant and equal to $\sum_{1\le t\le k}a(k+1-t)$: it verifies $J^k\neq 0$, $J^{k+1}=0$ and for every pair of vector fields X, Y on V^k:

$$[JX,JY]-J[JX,Y]-J[X,JY]+J^2[X,Y]=0.$$

Ω being an open set of V^k, we denote by $L_J(\Omega)$ the Lie Algebra of vector fields X defined on Ω such that the Lie derivative $L(X)J$ is equal to zero i.e., for each vector field Y on Ω:

$$[X,JY]=J[X,Y]$$

Let U be an open set of adapted local coordinates $(u_1,\ldots,u_n)$ and X a vector field on U.

X belongs to $L_J(\Omega)$ if and only if, for every open set U of adapted local coordinates $(u_1,....,u_n)$ such that $\Omega\cap U\neq\varnothing$, $X_{|\Omega\cap U}$ is a vector field finite sum $A(s,h,l) = \sum_{0\le q\le s-1} X_{c(h+q)+a(k+1-s-h)+l}\ \partial_{c(h+q)+a(k+1-s-h)+l}$, where $1\le s\le k+1$, $0\le h\le k+1-s$, $1\le l\le p_{k-h-s+2}$ (we set $\partial_i = \dfrac{\partial}{\partial u_i}$).

$X_{c(h)+a(k-s-h+1)+l}$ only depends on $(u_1,....,u_{a(k-s+2)})$ and for $1\le q\le s-1$,

$$X_{c(h+q)+a(k+1-s-h)+l} = \sum \frac{\partial^i X_{c(h)+a(k+1-s-h)+l}}{\partial u_1^{i_1}...\partial u_j^{i_j}...\partial u_r^{i_r}} \prod_{1\le j\le r}\left[\prod_{1\le t\le q}\frac{\left(u_{c(t)+j}\right)^{b_j^t}}{b_j^t\,!}\right] \text{ (cf. [1], Lemma 1).}$$

$A(s,h,l)$ is hence completely determined by its non zero first component $X_{c(h)+a(k-s-h+1)+l}$; if $s=1$, it will be its only one non zero component.

We set: $A_s^h(U)=\sum_{1\le l\le p_{k-h-s+2}} A(s,h,l)$ where $1\le s\le k+1$, $0\le h\le k+1-s$.

Then, for $1\le s\le k+1$, we construct the set $L_s(\Omega)=L_J(\Omega)\cap(KerJ^s_{|\Omega})$ (cf [1], p. 126-127).

We recall the following results (cf [1]):

Theorem 1. *For every $X\in L_1(V^k)$, we can write $X=\sum_i [Y_i,Z_i]$ where $\sum_i$ is a finite sum and Y_i,Z_i belongs to $L_1(V^k)$.*

Lemma 1. *Let U be an open set of adapted local coordinates of V^k and s an integer such that $2\le s\le k+1$ (suppose $p_{k-s+2}\neq 0$). Every element of $L_s(U)$ is a bracket finite sum of elements of $L_s(U)$ which means that: $[L_s(U),L_s(U)]=L_s(U)$.*

3. General study of Derivations

In this section, we suppose that $p_{k+1}\neq 0$.

Proposition 1. *Let D be a derivation of $L_J(V^k)$. Then $D\left(L_1(V^k)\right)\subset L_1(V^k)$ and $D_{|L_1(V^k)}$ is a derivation of $L_1(V^k)$.*

Proof. From theorem 1, for every $X\in L_1(V^k)$, we can write $X=\sum_i [Y_i,Z_i]$ where $\sum_i$ is a finite sum and Y_i,Z_i belongs to $L_1(V^k)$. Thus $D(X)=\sum_i\left([D(Y_i),Z_i]+[Y_i,D(Z_i)]\right)$. Furthermore, $D(Y_i)$ and $D(Z_i)\in L_J(V^k)$. Since $L_1(V^k)$ is an ideal of $L_J(V^k)$ (cf [1], lemma 4), we deduce that $[D(Y_i),Z_i]$ and $[Y_i,D(Z_i)]$ belong to $L_1(V^k)$ and thus $D(X)\in L_1(V^k)$. This completes the proof. □

Proposition 2. *For every derivation D of $L_1(V^k)$ and for every $X\in L_1(V^k)$, $supp\,D(X)\subset supp\,X$; every derivation D of $L_1(V^k)$ is local.*

Proof. Let $X \in L_1(V^k)$ be a vector field on V^k and ω an open set of V^k such that $X_{|\omega} = 0$; setting $\Omega = \pi^{-1}(\pi(\omega))$, we also have $X_{|\Omega} = 0$. For each $x \in \Omega$, there exist open sets Ω_1 and Ω_2 of V^k such that $\Omega_1 \cap \Omega_2 = \varnothing$, $\Omega_i = \pi^{-1}(\pi(\Omega_i))$, i = 1, 2 , *supp* $X \subset \Omega_1$, $x \in \Omega_2$. According to theorem 2 (cf [1], p.128), we can write $X = \sum_i [T_i, Y_i]$, where T_i, Y_i belongs to $L_1(V^k)$ and whose supports are in Ω_1. Since $D(X) = \sum_i \left([D(T_i), Y_i] + [T_i, D(Y_i)] \right)$, we deduce that $D(X)_{|\Omega_2} = 0$, then $D(X)_{|\Omega} = 0$. This completes the proof. □

Proposition 3. *Let U be an open set of adapted local coordinates of V^k and s an integer such that $2 \le s \le k+1$. Suppose $p_{k-s+2} \ne 0$. Let D be a derivation of $L_s(U)$. Then $D(L_{s-1}(U)) \subset L_{s-1}(U)$ and $D_{|L_{s-1}(U)}$ is a derivation of $L_{s-1}(U)$.*

Proof. In fact, according to theorem 1 for *s* = 2 and lemma 6 (cf [1], p. 128) for $3 \le s \le k+1$, for every $X \in L_{s-1}(U)$, we can write $X = \sum_i [Y_i, Z_i]$ where $\sum_i$ is a finite sum and Y_i, Z_i belong to $L_{s-1}(U)$. From lemma 4 (cf [1]), we deduce that $D(X) = \sum_i ([D(Y_i), Z_i] + [Y_i, D(Z_i)])$ belongs to $L_{s-1}(U)$. This completes the proof. □

Lemma 2. *Let U be an open set of adapted local coordinates of V^k and s an integer such that $1 \le s \le k+1$. Suppose $p_{k-s+2} \ne 0$. Let D be a derivation of $L_s(U)$, $X \in L_s(U)$ and $x \in U$ such that $j^3(X)(x) = 0$. Then $D(X)(x) = 0$.*

Proof. This results from lemma 7 (cf [1], p. 128). □

From now on and until the section ends, U is an open set of adapted local coordinates of V^k .

Define a mapping $\Delta : L_1(U) \to L_1(U)$ by:

$$\Delta\left(\sum_{1\le l\le p_{k+1}} X_{a(k)+l}(u_1,\dots,u_{a(k+1)})\partial_{a(k)+l}\right) = \left(\sum_{1\le l\le p_{k+1}} \partial_{a(k)+l} X_{a(k)+l}\right)\left(\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} A_{c(h)+a(k-h)+i}\ \partial_{c(h)+a(k-h)+i}\right)$$

$$\Delta\left(\sum_{\substack{1\le t\le k\\ 1\le j\le p_{k+1-t}}} X_{c(t)+a(k-t)+j}(u_1,\dots,u_{a(k+1)})\partial_{c(t)+a(k-t)+j}\right) = 0$$

where $A_{c(h)+a(k-h)+i}$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$ are C^∞ mappings from U to IR only depending on $u_1,\dots,u_{a(k)}$.

Lemma 3 : *Δ is a derivation of $L_1(U)$, which is not an inner derivation.*

Proof: In fact, we take $X = \sum_{1\le l\le p_{k+1}} X_{a(k)+l} \quad \partial_{a(k)+l}$ and $Y = \sum_{1\le t\le p_{k+1}} Y_{a(k)+t} \quad \partial_{a(k)+t}$

$$[X,Y] = \sum_{1\le t\le p_{k+1}} \left(\sum_l \left(X_{a(k)+l}\partial_{a(k)+l}Y_{a(k)+t} - Y_{a(k)+l}\partial_{a(k)+l}X_{a(k)+t} \right) \right) \ \partial_{a(k)+t}$$

$$\Delta([X,Y]) = \left(\sum_{l,t} \left(X_{a(k)+l}\partial_{a(k)+t}\partial_{a(k)+l}Y_{a(k)+t} - Y_{a(k)+l}\partial_{a(k)+t}\partial_{a(k)+l}X_{a(k)+t} \right) \right) \times$$

$$\times \left(\sum_{\substack{1\le h\le k \\ 1\le i\le p_{k+1-h}}} A_{c(h)+a(k-h)+i} \quad \partial_{c(h)+a(k-h)+i} \right)$$

$$[\Delta(X),Y] + [X,\Delta(Y)] = \left(-\sum_{l,t} Y_{a(k)+t}\partial_{a(k)+l}\partial_{a(k)+t}X_{a(k)+t} + \sum_{l,t} X_{a(k)+l}\partial_{a(k)+l}\partial_{a(k)+t}Y_{a(k)+t} \right) \times$$

$$\times \left(\sum_{\substack{1\le h\le k \\ 1\le i\le p_{k+1-h}}} A_{c(h)+a(k-h)+i} \quad \partial_{c(h)+a(k-h)+i} \right) = \Delta([X,Y]).$$

We now take $Y' = Y'_{c(h)+a(k-h)+j}\partial_{c(h)+a(k-h)+j}$, $1 \le h \le k$, $1 \le j \le p_{k+1-h}$.

$[X,Y'] = \sum_{1\le h\le k} \left(\sum_l X_{a(k)+l}\partial_{a(k)+l}Y'_{c(h)+a(k-h)+j} \right) \ \partial_{c(h)+a(k-h)+j}$, $\Delta([X,Y']) = 0$,

$[\Delta(X),Y'] + [X,\Delta(Y')] = 0 + [X,0] = 0$.

Suppose there exists $Y \in L_1(U)$ such that $\Delta(X) = [Y,X]$: then, for $X = \partial_{a(k)+l}$, $1 \le l \le p_{k+1}$, we shall have: $0 = [Y, \partial_{a(k)+l}]$ and the components of Y will depend only on $u_1,...,u_{a(k)}$. For $X = u_{a(k)+l}\partial_{a(k)+l}$, $1 \le l \le p_{k+1}$, we shall have:

$\sum_{\substack{1\le h\le k \\ 1\le i\le p_{k+1-h}}} A_{c(h)+a(k-h)+i} \quad \partial_{c(h)+a(k-h)+i} = Y_{a(k)+l}\partial_{a(k)+l}$, hence $Y_{a(k)+l} = 0$ and for all h, $1 \le h \le k$, $A_{c(h)+a(k-h)+i} = 0$. This completes the proof. □

In U, we set : $T = \sum_{\substack{1\le t\le k \\ 1\le j\le p_{k+1-t}}} \left(\sum_{\substack{1\le h\le k \\ 1\le i\le p_{k+1-h}}} B^j_{h,i} \quad u_{c(h)+a(k-h)+i} \quad \partial_{c(t)+a(k-t)+j} \right)$,

$B^j_{h,i}$ are C^∞ mappings from U to IR only depending on $u_1,...,u_{a(k)}$. $T \notin L_J(U)$.

We immediately verify that:

Lemma 4. *The mapping from $L_1(U)$ to $L_1(U)$: $X \to [T,X]$ is a derivation of $L_1(U)$ which is not an inner derivation.*

Let Z^0_r, $3 \le r \le k+1$, be the vector fields on U defined by:

$Z^0_r = \sum_{1\le j\le p_{k-r+2}} R_{a(k+1-r)+j}\partial_{a(k+1-r)+j}$, where $R_{a(k+1-r)+j}$, $1 \le j \le p_{k-r+2}$, are C^∞ mappings from U to IR depending on $u_1,...,u_{a(k)}$. $Z^0_r \notin L_J(U)$.

Then we have:

Lemma 5. *The mapping from* $L_1(U)$ *to* $L_1(U)$*:* $X \to \left[\sum_{3\le r\le k+1} Z_r^0, X\right]$ *is a derivation of* $L_1(U)$ *which is not an inner derivation.*

Theorem 2. *Let D be a derivation of* $L_1(U)$*. There exist* $Z_1^h \in A_1^h(U)$*,* $0 \le h \le k$*,* $Z_2^0 \in A_2^0(U)$*,* Z_r^0*,* $3 \le r \le k+1$*, vector fields on U (see lemma 5), a derivation* Δ *(see lemma 3) and a vector field T (see lemma 4), such that for every* $X \in L_1(U)$*:*

$$D(X) = \left[\sum_{0\le h\le k} Z_1^h + \sum_{2\le r\le k+1} Z_r^0, X\right] + \Delta(X) + [T, X].$$

In particular, $\dim H^1(L_1(U)) = +\infty$*.* Z_1^0*,* Z_2^0*,* $\sum_{3\le r\le k+1} Z_r^0$*,* Δ *and T are uniquely determined;* Z_1^h*,* $1 \le h \le k$*, is only determined up to the sum of* $\sum_{1\le j\le p_{k+1-h}} E_{c(h)+a(k-h)+j}\partial_{c(h)+a(k-h)+j}$*, where* $E_{c(h)+a(k-h)+j}$ *only depends on* $u_1,...,u_{a(k)}$.

Proof. 1) First we study the uniqueness: suppose that, for every $X \in L_1(U)$, we also have:

$D(X) = \left[\sum_{0\le h\le k} Z_1'^h + \sum_{2\le r\le k+1} Z_r'^0, X\right] + \Delta'(X) + [T', X]$, where $Z_1'^h \in A_1^h(U)$, $0 \le h \le k$, $Z_2'^0 \in A_2^0(U)$, $Z_r'^0$, $3 \le r \le k+1$, vector fields on U (see lemma 5),

$$\Delta'\left(\sum_{1\le l\le p_{k+1}} X_{a(k)+l}(u_1,...,u_{a(k+1)})\ \partial_{a(k)+l}\right) = \left(\sum_{1\le l\le p_{k+1}} \partial_{a(k)+l}X_{a(k)+l}\right)\left(\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} A'_{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i}\right)$$

$$\Delta'\left(\sum_{\substack{1\le h\le k\\ 1\le j\le p_{k+1-h}}} X_{c(h)+a(k-h)+j}(u_1,...,u_{a(k+1)})\partial_{c(h)+a(k-h)+j}\right) = 0,$$

$$T' = \sum_{\substack{1\le t\le k\\ 1\le j\le p_{k+1-t}}}\left(\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} B'^j_{h,i}\ \ u_{c(h)+a(k-h)+i}\right)\partial_{c(t)+a(k-t)+j},$$ where $A'_{c(h)+a(k-h)+i}$ and $B'^j_{h,i}$,

are C^∞ mappings from U to $I\!R$ only depending on $u_1,...,u_{a(k)}$. We set:

$$Z_1^0 - Z_1'^0 = \sum_{1\le j\le p_{k+1}} b_{a(k)+j}\ \ \partial_{a(k)+j},$$

$$Z_1^h - Z_1'^h = \sum_{1\le i\le p_{k+1-h}} g_{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i},\ \ 1 \le h \le k$$

$$Z_r^0 - Z_r'^0 = \sum_{1\le j\le p_{k-r+2}} d_{a(k+1-r)+j}\ \ \partial_{a(k+1-r)+j},\qquad 2 \le r \le k+1$$

$A_{c(h)+a(k-h)+i} - A'_{c(h)+a(k-h)+i} = A''_{c(h)+a(k-h)+i}$, $B^j_{h,i} - B'^j_{h,i} = B''^j_{h,i}$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$.

For every $X \in L_1(U)$, we have :

$$[\ Z_1^0 - Z_1'^0, X] + [\ \sum_{1\le h\le k}\left(Z_1^h - Z_1'^h\right)\ , X] + [\ \sum_{2\le r\le k+1}\left(Z_r^0 - Z_r'^0\right)\ , X] + (\Delta - \Delta')(X) + [T - T', X] = 0.$$

We deduce that:

$i)$ for $X=\partial_{a(k)+l}$, $1\le l\le p_{k+1}$:

$-\sum_{1\le j\le p_{k+1}}\partial_{a(k)+l}b_{a(k)+j}\ \ \partial_{a(k)+j}-\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}}\partial_{a(k)+l}g_{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i}=0$, then

$\partial_{a(k)+l}b_{a(k)+j}=0$, $1\le j\le p_{k+1}$ and $\partial_{a(k)+l}g_{c(h)+a(k-h)+i}=0$, $1\le h\le k$, $1\le i\le p_{k+1-h}$

$ii)$ for $X=u_{a(k)+l}\partial_{a(k)+l}$, $1\le l\le p_{k+1}$:

$b_{a(k)+l}\ \ \partial_{a(k)+l}\ +\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}}A^{''}_{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i}=0$, then $b_{a(k)+l}=0$, for all l, $1\le l\le p_{k+1}$ and $A^{''}_{c(h)+a(k-h)+i}=0$ for all h, $1\le h\le k$, $1\le i\le p_{k+1-h}$.

$iii)$ for $X=u_{a(k-s)+i}\ \ \partial_{a(k)+l}$, $1\le s\le k$, $1\le i\le p_{k+1-s}$, $1\le l\le p_{k+1}$:

$d_{a(k-s)+i}\ \ \partial_{a(k)+l}=0$ then $d_{a(k-s)+i}=0$, $1\le s\le k$.

$iv)$ for $X=\partial_{c(h)+a(k-h)+i}$, $1\le h\le k$, $1\le i\le p_{k+1-h}$:

$-B^{''j}_{h,i}\ \ \partial_{c(t)+a(k-t)+j}=0$ then $B^{''j}_{h,i}=0$.

2) The existence of Z_1^h, $0\le h\le k$, Z_r^0, $2\le r\le k+1$, Δ and T is induced from the four following lemmas.

Lemma 6. *There exist* $\hat{Z}_1^0\in A_1^0(U)$, $Z_1^h\in A_1^h(U)$, $1\le h\le k$, *such that the mapping from* $L_1(U)$ *to* $L_1(U)$: $X\to D_1(X)=D(X)-[\ \hat{Z}_1^0+\sum_{1\le h\le k}Z_1^h,X]$ *is a derivation of* $L_1(U)$ *which verifies* $D_1(\partial_{a(k)+l})=0$ *for* $1\le l\le p_{k+1}$.

Proof. Setting, for $1\le l\le p_{k+1}$: $D(\partial_{a(k)+l})=\sum_{\substack{0\le h\le k\\ 1\le i\le p_{k+1-h}}}D_{a(k)+l}^{c(h)+a(k-h)+i}\ \ \partial_{c(h)+a(k-h)+i}$,

we have, for $1\le l,f\le p_{k+1}$:

$$D([\ \partial_{a(k)+l},\partial_{a(k)+f}\ ])=0=[\ D(\partial_{a(k)+l}),\partial_{a(k)+f}\ ]+[\ \partial_{a(k)+l},D(\partial_{a(k)+f})\ ]$$

Hence $\partial_{a(k)+f}D_{a(k)+l}^{c(h)+a(k-h)+i}=\partial_{a(k)+l}D_{a(k)+f}^{c(h)+a(k-h)+i}$; thus there exist, in U, C^∞ functions of $u_1,...,u_{a(k+1)}$, $D_{c(h)+a(k-h)+i}$, $0\le h\le k$, $1\le i\le p_{k+1-h}$ such that $\partial_{a(k)+l}D_{c(h)+a(k-h)+i}=D_{a(k)+l}^{c(h)+a(k-h)+i}$. It is sufficient to set:

$\hat{Z}_1^0=-\sum_{1\le i\le p_{k+1}}D_{a(k)+i}\partial_{a(k)+i}$, $\quad Z_1^h=-\sum_{1\le i\le p_{k+1-h}}D_{c(h)+a(k-h)+i}\partial_{c(h)+a(k-h)+i}$, $1\le h\le k$.

This completes the proof. □

Lemma 7. *There exist* $\tilde{Z}_1^0\in A_1^0(U)$, $Z_2^0\in A_2^0(U)$, Z_r^0, $3\le r\le k+1$, *vector fields on* U *(see lemma 5), a derivation* Δ *of* $L_1(U)$ *(see lemma 3) such that the mapping from* $L_1(U)$ *to* $L_1(U)$:

$$X\to D_2(X)=D_1(X)-[\ \textstyle\sum_{2\le r\le k+1}Z_r^0,X]-[\ \tilde{Z}_1^0,X\ ]-\Delta(X)$$
$$=D(X)-[\ \textstyle\sum_{0\le h\le k}Z_1^h+\sum_{2\le r\le k+1}Z_r^0,X\ ]-\Delta(X)$$

is a derivation of $L_1(U)$ *which verifies* $D_2(\partial_{a(k)+l})=0$ *for* $1\le l\le p_{k+1}$, $D_2(u_j\partial_{a(k)+l})=0$ *for* $1\le j\le a(k+1)$, $1\le l\le p_{k+1}$. *(we have set:* $Z_1^0=\hat{Z}_1^0+\tilde{Z}_1^0$*)*

Proof. Setting, for $1 \le j \le a(k+1)$, $1 \le l \le p_{k+1}$:

$$D_1(u_j\partial_{a(k)+l}) = \sum_{\substack{0\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(h)+a(k-h)+i}_{j,a(k)+l}\ \partial_{c(h)+a(k-h)+i},$$ we have, for $1 \le f \le p_{k+1}$:

$$D_1([\ \partial_{a(k)+f}, u_j\partial_{a(k)+l}\ ]) = 0 = [\ \partial_{a(k)+f}, D_1(u_j\partial_{a(k)+l})\ ].$$

We deduce that $D^{c(h)+a(k-h)+i}_{j,a(k)+l}$ only depends on $u_1,...,u_{a(k)}$.

For $1 \le j, r \le a(k+1)$, $1 \le l, f \le p_{k+1}$, we have :

$$D_1([\ u_j\partial_{a(k)+l}, u_r\partial_{a(k)+f}\ ]) = \delta^r_{a(k)+l} D_1(u_j\partial_{a(k)+f}) - \delta^{a(k)+f}_j D_1(u_r\partial_{a(k)+l})$$
$$= [\ D_1(u_j\partial_{a(k)+l}), u_r\partial_{a(k)+f}\ ] + [\ u_j\partial_{a(k)+l}, D_1(u_r\partial_{a(k)+f})\ ]$$

1) Assume $1 \le r \le a(k)$: for $j = a(k)+l = a(k)+f$, we have :
$$-\sum_{\substack{0\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(h)+a(k-h)+i}_{r,a(k)+l}\ \partial_{c(h)+a(k-h)+i} = -D^{a(k)+l}_{r,a(k)+l}\ \partial_{a(k)+l}.$$
We deduce that for $0 \le h \le k$, $1 \le i \le p_{k+1-h}$, $i \ne l$, $D^{c(h)+a(k-h)+i}_{r,a(k)+l} = 0$.
For $j = a(k)+f$, we have : $-D^{a(k)+l}_{r,a(k)+l}\ \partial_{a(k)+l} = -D^{a(k)+f}_{r,a(k)+f}\ \partial_{a(k)+l}$.
We deduce that $D^{a(k)+l}_{r,a(k)+l} = D^{a(k)+f}_{r,a(k)+f}$.

2) Assume $a(k)+1 \le r \le a(k+1)$: for $r \ne a(k)+l = j = a(k)+f$, we have :
$$-\sum_{\substack{0\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(h)+a(k-h)+i}_{r,a(k)+l}\ \partial_{c(h)+a(k-h)+i} = D^r_{a(k)+l,a(k)+l}\ \partial_{a(k)+l} - D^{a(k)+l}_{r,a(k)+l}\ \partial_{a(k)+l}.$$
We deduce that for $0 \le h \le k$, $1 \le i \le p_{k+1-h}$, $i \ne l$, $D^{c(h)+a(k-h)+i}_{r,a(k)+l} = 0$, next $D^r_{a(k)+l,a(k)+l} = 0$. For $r = a(k)+l \ne j = a(k)+f$, we have :
$$D^{a(k)+f}_{a(k)+f,a(k)+f}\ \partial_{a(k)+f} + \sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(h)+a(k-h)+i}_{a(k)+f,a(k)+f}\ \partial_{c(h)+a(k-h)+i} - D^{a(k)+l}_{a(k)+l,a(k)+l}\ \partial_{a(k)+l}$$
$$-\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(h)+a(k-h)+i}_{a(k)+l,a(k)+l}\ \partial_{c(h)+a(k-h)+i} = D^{a(k)+l}_{a(k)+f,a(k)+l}\ \partial_{a(k)+f} - D^{a(k)+f}_{a(k)+l,a(k)+f}\ \partial_{a(k)+l}$$
We deduce: $D^{a(k)+f}_{a(k)+f,a(k)+f} = D^{a(k)+l}_{a(k)+f,a(k)+l}$,
$D^{c(h)+a(k-h)+i}_{a(k)+f,a(k)+f} = D^{c(h)+a(k-h)+i}_{a(k)+l,a(k)+l}$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$.

We set: D_j = common value of $D^{a(k)+l}_{j,a(k)+l}$, $1 \le j \le a(k+1)$,
$A_{c(h)+a(k-h)+i} = D^{c(h)+a(k-h)+i}_{a(k)+l,a(k)+l}$ for $1 \le h \le k$, $1 \le i \le p_{k+1-h}$, which determines Δ,
$Z^0_r = \sum_{1\le i\le p_{k-r+2}} D_{a(k+1-r)+i}\,\partial_{a(k+1-r)+i}$, $2 \le r \le k+1$, $\tilde{Z}^0_1 = \sum_{1\le i\le p_{k+1}} D_{a(k)+i}\,\partial_{a(k)+i}$ and $Z^0_1 = \hat{Z}^0_1 + \tilde{Z}^0_1$. This completes the proof. □

Lemma 8. *There exists a vector field T on U (see lemma 4) such that the mapping from $L_1(U)$ to $L_1(U)$: $X \to D_3(X) = D_2(X) - [T, X]$ is a derivation of $L_1(U)$ which verifies*

$D_3(\partial_{c(h)+a(k-h)+i})=0$, *for* $0\le h\le k$, $1\le i\le p_{k+1-h}$, $D_3(u_j\partial_{a(k)+l})=0$ *for* $1\le l\le p_{k+1}$, $1\le j\le a(k+1)$.

Proof. We set, for $1\le h\le k$, $1\le i\le p_{k+1-h}$:
$D_2(\partial_{c(h)+a(k-h)+i})=\sum_{\substack{0\le r\le k\\ 1\le j\le p_{k+1-r}}} D^{c(r)+a(k-r)+j}_{c(h)+a(k-h)+i}\ \partial_{c(r)+a(k-r)+j}$. For $1\le l\le p_{k+1}$, we have:

$$D_2\left(\left[\ \partial_{a(k)+l},\partial_{c(h)+a(k-h)+i}\ \right]\right)=0=\left[\ \partial_{a(k)+l},D_2(\partial_{c(h)+a(k-h)+i})\ \right].$$

We deduce that $D^{c(r)+a(k-r)+j}_{c(h)+a(k-h)+i}$ only depends on $u_1,...,u_{a(k)}$. We have:

$$D_2\left(\left[\ \partial_{c(h)+a(k-h)+i},\sum_{1\le l\le p_{k+1}} u_{a(k)+l}\partial_{a(k)+l}\ \right]\right)\ =\ 0\ =$$
$$\left[\ \sum_{\substack{0\le r\le k\\ 1\le j\le p_{k+1-r}}} D^{c(r)+a(k-r)+j}_{c(h)+a(k-h)+i}\ \partial_{c(r)+a(k-r)+j},\sum_{1\le l\le p_{k+1}} u_{a(k)+l}\partial_{a(k)+l}\ \right]$$

We deduce that $D^{a(k)+l}_{c(h)+a(k-h)+i}=0$ for $1\le l\le p_{k+1}$. It is enough to set:

$$T=-\sum_{\substack{1\le r\le k\\ 1\le j\le p_{k+1-r}}}\left(\sum_{\substack{1\le h\le k\\ 1\le i\le p_{k+1-h}}} D^{c(r)+a(k-r)+j}_{c(h)+a(k-h)+i}\,u_{c(h)+a(k-h)+i}\right)\partial_{c(r)+a(k-r)+j}.$$

This completes the proof.□

Lemma 9. *For every* $X\in L_1(U)$ *whose components on* $\partial_{c(h)+a(k-h)+i}$, $1\le i\le p_{k+1-h}$, $0\le h\le k$, *are polynomials of variables* u_j, $1\le j\le a(k+1)$, *of degree* ≤ 3, $D_3(X)=0$.

Proof. 1) We take $1\le r,t\le a(k+1)$, $1\le l\le p_{k+1}$:
$D_3(u_ru_t\partial_{a(k)+l})=\sum_{\substack{0\le h\le k\\ 1\le j\le p_{k+1-h}}} D^{c(h)+a(k-h)+j}_{r,t,a(k)+l}\ \partial_{c(h)+a(k-h)+j}$, where the $D^{c(h)+a(k-h)+j}_{r,t,a(k)+l}$ only depends on $u_1,...,u_{a(k+1)}$. For $1\le f\le p_{k+1}$, we have:
$D_3\left(\left[\ \partial_{a(k)+f},u_ru_t\partial_{a(k)+l}\ \right]\right)=0=\left[\ \partial_{a(k)+f},D_3(u_ru_t\partial_{a(k)+l})\ \right]$ then $D^{c(h)+a(k-h)+j}_{r,t,a(k)+l}$ only depends on $u_1,...,u_{a(k)}$.

i) Assume $a(k)+1\le r,t\le a(k+1)$: $\left[\ \sum_{1\le f\le p_{k+1}} u_{a(k)+f}\partial_{a(k)+f},u_ru_t\partial_{a(k)+l}\ \right]=u_ru_t\partial_{a(k)+l}$

Applying D_3 to this, we obtain:

$$-\sum_{1\le f\le p_{k+1}} D^{a(k)+f}_{r,t,a(k)+l}\partial_{a(k)+f}\ =\ \sum_{\substack{0\le h\le k\\ 1\le j\le p_{k+1-h}}} D^{c(h)+a(k-h)+j}_{r,t,a(k)+l}\ \partial_{c(h)+a(k-h)+j}.$$ We deduce:

$2D^{a(k)+j}_{r,t,a(k)+l}=0$ for $1\le j\le p_{k+1}$, $D^{c(h)+a(k-h)+i}_{r,t,a(k)+l}=0$ for $1\le h\le k$, $1\le i\le p_{k+1-h}$, from which it follows that $D_3(u_ru_t\partial_{a(k)+l})=0$.

ii) Assume $1\le r\le a(k)<t\le a(k+1)$: $\left[\ u_r\partial_t,\ u_t^2\partial_{a(k)+l}\ \right]=2u_ru_t\partial_{a(k)+l}$.

From *i*) it follows that $D_3(u_t^2\partial_{a(k)+l})=0$ then $D_3(u_ru_t\partial_{a(k)+l})=0$.

$iii)$ Assume $1 \le r,t \le a(k)$: $[\ u_r \partial_{a(k)+l}, u_t\, u_{a(k)+l} \partial_{a(k)+l}\ ] = u_r u_t \partial_{a(k)+l}$.

From $ii)$ it follows that $D_3(u_t\, u_{a(k)+l} \partial_{a(k)+l}) = 0$ then $D_3(u_r u_t \partial_{a(k)+l}) = 0$.

2) We take $1 \le r,t,s \le a(k+1)$, $1 \le l, f \le p_{k+1}$: from $D_3([\ \partial_{a(k)+f}, u_r u_t u_s \partial_{a(k)+l}\ ]) = 0$ we deduce that $[\ \partial_{a(k)+f}, D_3(u_r u_t u_s \partial_{a(k)+l})\ ] = 0$.

$i)$ Assume $a(k)+1 \le r,t,s \le a(k+1)$:

$[\ \sum_{1 \le f \le p_{k+1}} u_{a(k)+f} \partial_{a(k)+f}, u_r u_t u_s \partial_{a(k)+l}\ ] = 2 u_r u_t u_s \partial_{a(k)+l}$ hence $D_3(u_r u_t u_s \partial_{a(k)+l}) = 0$.

$ii)$ Assume $1 \le r \le a(k) < t,s \le a(k+1)$:

$[\ \sum_{1 \le f \le p_{k+1}} u_{a(k)+f} \partial_{a(k)+f}, u_r u_t u_s \partial_{a(k)+l}\ ] = u_r u_t u_s \partial_{a(k)+l}$ hence $D_3(u_r u_t u_s \partial_{a(k)+l}) = 0$.

$iii)$ Assume $1 \le r,t \le a(k) < s \le a(k+1)$:

$[\ u_r u_t \partial_s, u_s^2 \partial_{a(k)+l}\ ] = 2 u_r u_t u_s \partial_{a(k)+l}$ hence $D_3(u_r u_t u_s \partial_{a(k)+l}) = 0$

$iv)$ Assume $1 \le r,t,s \le a(k)$:

$[\ u_r u_t \partial_{a(k)+l}, u_s u_{a(k)+l} \partial_{a(k)+l}\ ] = u_r u_t u_s \partial_{a(k)+l}$ hence $D_3(u_r u_t u_s \partial_{a(k)+l}) = 0$.

3) $i)$ We set, for $1 \le r \le a(k+1)$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$:

$$D_3(u_r \partial_{c(h)+a(k-h)+i}) = \sum_{\substack{0 \le t \le k \\ 1 \le j \le p_{k+1-t}}} D^{c(t)+a(k-t)+j}_{r,c(h)+a(k-h)+i}\ \partial_{c(t)+a(k-t)+j}.$$

For $1 \le l \le p_{k+1}$, we have:

$D_3([\ \partial_{a(k)+l}, u_r \partial_{c(h)+a(k-h)+i}\ ]) = 0 = [\ \partial_{a(k)+l}, D_3(u_r \partial_{c(h)+a(k-h)+i})\ ]$.

For $1 \le r \le a(k)$, $1 \le l \le p_{k+1}$, we have:

$$\begin{aligned} D_3([\ u_r \partial_{a(k)+l}, u_{a(k)+l} \partial_{c(h)+a(k-h)+i}\ ]) &= D_3(u_r \partial_{c(h)+a(k-h)+i}) \\ &= [\ u_r \partial_{a(k)+l}, D_3(u_{a(k)+l} \partial_{c(h)+a(k-h)+i})\ ] = 0. \end{aligned}$$

Hence, for $1 \le r \le a(k)$, $D_3(u_r \partial_{c(h)+a(k-h)+i}) = 0$.

For $a(k)+1 \le r \le a(k+1)$, we have,

$$D_3(u_r \partial_{c(h)+a(k-h)+i}) = -D^r_{a(k)+l,c(h)+a(k-h)+i} \partial_{a(k)+l}$$

If $r = a(k)+l$, we have, $D^{a(k)+j}_{a(k)+l,c(h)+a(k-h)+i} = 0$ for $j \ne l$, then

$D^{a(k)+l}_{a(k)+l,c(h)+a(k-h)+i} = 0$ and $D^{c(t)+a(k-t)+j}_{a(k)+l,c(h)+a(k-h)+i} = 0$ for $1 \le t \le k$.

If $r \ne a(k)+l$, since $D^r_{a(k)+l,c(h)+a(k-h)+i} = 0$ then $D_3(u_r \partial_{c(h)+a(k-h)+i}) = 0$.

$ii)$ We take now $1 \le t,r,s \le a(k+1)$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$, $1 \le l \le p_{k+1}$:

$[\ u_t u_r \partial_{a(k)+l}, u_{a(k)+l} \partial_{c(h)+a(k-h)+i}\ ] = u_t u_r \partial_{c(h)+a(k-h)+i}$ hence $D_3(u_t u_r \partial_{c(h)+a(k-h)+i}) = 0$.

$[\ u_t u_r u_s \partial_{a(k)+l}, u_{a(k)+l} \partial_{c(h)+a(k-h)+i}\ ] = u_t u_r u_s \partial_{c(h)+a(k-h)+i}$ hence $D_3(u_t u_r u_s \partial_{c(h)+a(k-h)+i}) = 0$.

Let us conclude the demonstration of the theorem by considering any X belonging to $L_1(U)$; for every $x \in U$, there exists $\widetilde{X} \in L_1(U)$ whose components on $\partial_{c(h)+a(k-h)+i}$, $0 \le h \le k$,

$1 \le i \le p_{k+1-h}$, are polynomials of degree ≤ 3 and such that $j^3(X - \widetilde{X})(x) = 0$. By lemma 2 we have $D_3(X - \widetilde{X})(x) = 0$. Since $D_3(\widetilde{X}) = 0$, then $D_3(X)(x) = 0$.
On the other hand, because $Z_1^h - Z_1^{'h} = \sum_{1 \le i \le p_{k+1-h}} g_{c(h)+a(k-h)+i}\, \partial_{c(h)+a(k-h)+i}$, $1 \le h \le k$, with $\partial_{a(k)+l} g_{c(h)+a(k-h)+i} = 0$, $1 \le h \le k$, $1 \le i \le p_{k+1-h}$, $1 \le l \le p_{k+1}$, thus the vector fields Z_1^h are not uniquely determined but determined up to the sum of $\sum_{1 \le j \le p_{k+1-h}} g_{c(h)+a(k-h)+j} \partial_{c(h)+a(k-h)+j}$, where $g_{c(h)+a(k-h)+j}$ only depends on $u_1, ..., u_{a(k)}$.
So the dimension of $H^1\left(L_1(U)\right)$ is infinite for U open set of adapted local coordinates of V^k .
This completes the proof. □

On the other hand, let Z be the vector field on U defined by
$Z^U = \sum_{1 \le h \le k} h \left(\sum_{1 \le j \le a(k+1-h)} u_{c(h)+j} \partial_{c(h)+j} \right)$ (cf [1], p. 124). We showed that, in fact, Z is globally defined. We immediately verify that:

Lemma 10. *The mapping from $L_J(U)$ to $L_J(U)$ (resp. from $L_J(V^k)$ to $L_J(V^k)$): $X \to \left[Z_{|U}, X \right]$ (resp. $X \to [Z, X]$) is a derivation of $L_J(U)$ (resp. $L_J(V^k)$) which is not an inner derivation. So $\dim H^1\left(L_J(U)\right) \ge 1$, $\dim H^1\left(L_J(V^k)\right) \ge 1$.*

The derivations of $L_J(U)$ have been studied by J. Lehmann-Lejeune (cf. [4], th. 1, p. 25). Let us recall the results:

Theorem 3. *For every derivation D of $L_J(U)$ there exist k real constants K_h, $1 \le h \le k$, and an element $Y \in L_J(U)$ such that, for every $X \in L_J(U)$:
$D(X) = [\ \sum_{1 \le h \le k} K_h J^{h-1} Z_{|U} + Y,\ X\ ]$; K_h and Y are uniquely determined; then $\dim H^1\left(L_J(U)\right) = k$.*

4. When the foliations are defined by submersions

In this section, we assume that the k foliations of M are defined by k submersions $\pi_h : M_{h-1} \to M_h$ where $1 \le h \le k$, $M_0 = M$, the M_h are manifolds of dimension $a(k+1-h)$ and $p_1 > 0$, $p_i \ge 0$ $2 \le i \le k+1$. The leaves of each foliation F_{k+1-h} are the connected components of the inverse image by $\pi_h \circ ... \circ \pi_1$ of the points of M_h .
Let $y_0 \in M_0$ be a point of M_0 . Denote by $y_h = \pi_h \circ \pi_{h-1} \circ ... \circ \pi_1(y_0) \in M_h$, $1 \le h \le k$.
For all h , $0 \le h \le k$, there exist $\hat{U}_h$ open sets of local coordinates $(u_1, ..., u_{a(k+1-h)})$, neighborhood of y_h in M_h , such that $\pi_{h+1}(\hat{U}_h) = \hat{U}_{h+1}$ and $\pi_{h+1|\hat{U}_h}$ is a projection : ($u_1, ..., u_{a(k+1-h)}$) $\to$ $(u_1, ..., u_{a(k-h)})$ .Then there exists an open set of local coordinates $U = \pi^{-1}(\hat{U}_0)$ of V^k . This is an ''open set of adapted local coordinates $u_1, ..., u_n$'' which, moreover, is adapted to the submersions.

The automorphisms of the foliations F_{k+1-h} on $M = M_0$, $1 \le h \le k$, defined by $\pi_h \circ \pi_{h-1} \circ ... \circ \pi_1 : M_0 \to M_h$, are projectable vector fields from M_0 to M_h.

Lemma 11. *Let* Ω *be an open set of* V^k *and* $X \in L_s^h(\Omega)$, $1 \le s \le k+1$, $0 \le h \le k+1-s$ *(cf.* [4]*). For every* $x \in \Omega$*, the germ at* x *of* X *is the germ at* x *of an* $X' \in L_J(V^k)$.

Proof. Let Ω be an open set of V^k such that $\Omega = \pi^{-1} \circ \pi(\Omega)$ and $x \in \Omega$. We set $\hat{\Omega} = \pi(\Omega)$, open set of M_0 and $y_0 = \pi(x) \in \hat{\Omega}$. According to lemma 5 (cf [1], p. 127), it is sufficient to show the result for $X \in L_{s+1}^h(\Omega)$, $1 \le s \le k$, $0 \le h \le k-s$.
Let $\hat{X} \in L_{s+1}^h(\hat{\Omega})$ be a vector field on $\hat{\Omega}$, $1 \le s \le k$, $0 \le h \le k-s$, and $X \in L_{s+1}^h(\Omega)$ be the corresponding vector field on Ω (cf. [4]).
$\pi_s^* \circ \pi_{s-1}^* \circ ... \circ \pi_1^*(\hat{X}) = \hat{X}_s$ is a vector field on $\hat{\Omega}_s$, open set of M_s, neighborhood of $y_s = \pi_s \circ \pi_{s-1} \circ ... \circ \pi_1(y_0)$. There exists φ_s, function on M_s, with support contained in $\hat{\Omega}_s$, and equal to 1 in a neighborhood $\hat{\omega}_s$ of y_s. The vector field $\hat{X}^s = \varphi_s \hat{X}_s$ is global on M_s. The germ at y_s of $\hat{X}^s$ is equal to the germ at y_s of $\hat{X}_s$. With the help of a metric on M_0, we can define the lift on M_0 of vector fields defined in M_s. Indeed, let g be a metric on M_0 and y_0 a point of M_0. Denote by S_1 the orthogonal supplementary set relatively to g of $Ker(\pi_1^*)$ to $T_{y_0}M_0$: $T_{y_0}M_0 = Ker(\pi_1^*) \oplus S_1$. Setting $y_1 = \pi_1(y_0)$, S_1 is isomorphic to $T_{y_1}M_1$. For $0 \le h \le k-1$ and $y_h = \pi_h \circ \pi_{h-1} \circ ... \circ \pi_1(y_0)$, assume that the vector space $T_{y_h}M_h$ is endowed with a scalar product; thus $T_{y_h}M_h = Ker(\pi_{h+1}^*) \oplus S_{h+1}$, where S_{h+1} is the orthogonal supplementary set of $Ker(\pi_{h+1}^*)$ in $T_{y_h}M_h$. S_{h+1} is endowed with a scalar product: the restriction of the scalar product on $T_{y_h}M_h$. On the other hand, S_{h+1} is isomorphic to $T_{y_{h+1}}M_{h+1}$; we deduce from this isomorphism a scalar product on $T_{y_{h+1}}M_{h+1}$.
This assertion is true for $h = 0$. Thus it's true for every h, $0 \le h \le k-1$.
We deduce that we can write as an orthogonal direct sum: $T_{y_0}M_0 = \bigoplus_{1 \le r \le k+1} E_r$, where E_r is isomorphic to $Ker(\pi_r^*)$ for $1 \le r \le k$ and E_{k+1} to $T_{y_k}M_k$.
Hence we could lift up a vector field on $M_h, 1 \le h \le k$, into a vector field on M_{h-1}, taking it in S_h. And step by step or gradually, we could lift it on M_0.
Then let $\widetilde{X}_s$ be the lift of $\hat{X}^s$ on M_0. Set $\tilde{X} = P_k\left(J_0^h(R\tilde{X}^s)\right)$ (cf. [4]). It is a vector field globally defined on V^k. Denote by Ω' the open set of V^k such that $\Omega' = (\pi_s \circ \pi_{s-1} \circ ... \circ \pi_1 \circ \pi)^{-1}(\hat{\omega}_s)$. Ω' contains x. The vector field $X_{|\Omega'} - \widetilde{X}_{|\Omega'} \in L_s(\Omega')$.
To show it, we will do an inductive reasoning on s.
For $s = 1$, $X_{|\Omega'} - \tilde{X}_{|\Omega'} \in L_1(\Omega')$. According to lemma 5 (cf [1]), the germ at x of $X_{|\Omega'} - \tilde{X}_{|\Omega'}$ is the germ at x of an $Y \in L_1(V^k)$. $\widetilde{X}$ being global, thus the germ at x of X is the germ at x of $X' = \widetilde{X} + Y \in L_J(V^k)$.

Now, for $1 \le s \le k$, $0 \le h \le k-s$, assume that for every $X \in L_s^h(\Omega)$ the germ at x of X is the germ at x of an $X' \in L_J(V^k)$. Let $X \in L_{s+1}^h(\Omega)$ be a vector field on Ω. Then $X_{|\Omega'} - \tilde{X}_{|\Omega'} \in L_s(\Omega')$. According to the inductive hypothesis, the germ at x of $X_{|\Omega'} - \tilde{X}_{|\Omega'}$ is the germ at x of an $Y \in L_J(V^k)$. $\tilde{X}$ being global, thus the germ at x of X is the germ at x of $X' = \tilde{X} + Y \in L_J(V^k)$. This proves our lemma. □

Proposition 4. *For every derivation D of $L_J(V^k)$ and for every $X \in L_J(V^k)$, $supp\, D(X) \subset supp\, X$; every derivation D of $L_J(V^k)$ is local.*

Proof. Let ω be an open set of V^k such that $\omega = \pi^{-1}(\pi(\omega))$. We set $\hat{\omega} = \pi(\omega)$ open set of M_0. Let $\hat{X} \in L_{s+1}^h(M_0)$ be a vector field on M_0, $0 \le s \le k$, $0 \le h \le k-s$ (cf. [4]) such that $\hat{X}_{|\hat{\omega}} = 0$. (For $s = 0$, $L_{s+1}^h(M_0)$ is the set of the vector fields of M_0, tangent to the leaves of F_{k-h} and orthogonal to the leaves of F_{k+1-h}). Denote by X the corresponding vector field on V^k, $X \in L_{s+1}^h(V^k)$ (cf. [4]). We have: $X_{|\omega} = 0$.

Let $\hat{X}^s$ be the projected of $\hat{X}$ on M_s. For all $y \in \hat{\omega}$, there exist open sets $\hat{\Omega}_1$ and $\hat{\Omega}_2$ of M_s such that $\hat{\Omega}_1 \cap \hat{\Omega}_2 = \varnothing$, $\pi_s \circ \pi_{s-1} \circ \ldots \circ \pi_1(y) = y_s \in \hat{\Omega}_2$, $supp\, \hat{X}^s \subset \hat{\Omega}_1$ (for $s = 0$, $\hat{X}^s = \hat{X}$, $y_s = y \in \hat{\Omega}_2$ and $supp\, \hat{X} \subset \hat{\Omega}_1$). $\hat{X}^s_{|\hat{\Omega}_2} = 0$; in particular $\hat{X}^s$ is zero in a neighborhood of y_s. According to the theorem of A. Lichnerowicz (cf [5], p. 64), we can write $\hat{X}^s = \sum_i \left[\hat{Y}_i^s, \hat{T}_i^s\right]_{M_s}$ where $\hat{Y}_i^s$, $\hat{T}_i^s$ are vector fields on M_s, with support in $\hat{\Omega}_1$: $\hat{Y}^s_{i|\hat{\Omega}_2} = 0$, $\hat{T}^s_{i|\hat{\Omega}_2} = 0$.

Let $\tilde{X}^s$ (respectively $\tilde{Y}_i^s$, $\tilde{T}_i^s$) be the lift of $\hat{X}^s$ (respectively $\hat{Y}_i^s$, $\hat{T}_i^s$) on M_0 (for $s = 0$, $\tilde{X}^s = \hat{X}$, $\tilde{Y}_i^s = \hat{Y}_i^s$, $\tilde{T}_i^s = \hat{T}_i^s$) and $\tilde{X} = P_k\left(J_0^h(R\tilde{X}^s)\right)$ (cf. [3]):

$$\tilde{X} = \sum_i P_k\left(J_0^h\left(R\left[\tilde{Y}_i^s, \tilde{T}_i^s\right]_{M_0}\right)\right).$$

If $\tilde{Y}_i = P_k\left(J_0^h(R\tilde{Y}_i^s)\right)$, $\tilde{T}_i = P_k\left(J_0^h(R\tilde{T}_i^s)\right)$ and $\omega_2 = (\pi_s \circ \pi_{s-1} \circ \ldots \circ \pi_1 \circ \pi)^{-1}(\hat{\Omega}_2)$, open set of V^k containing $x = \pi^{-1}(y)$ (for $s = 0$, $\pi_s = \pi$), we have:

$\left[\tilde{Y}_i, \tilde{T}_i\right] = P_k\left(J_0^h\left(R\left[\tilde{Y}_i^s, \tilde{T}_i^s\right]\right)\right) + R_i$ where $R_i \in L_s(V^k)$ and $R_{i|\omega_2} = 0$. Then $\tilde{X} = \sum_i \left(\left[\tilde{Y}_i, \tilde{T}_i\right] - R_i\right)$. Since $X - \tilde{X} \in L_s(V^k)$, we have: $X = \sum_i \left[\tilde{Y}_i, \tilde{T}_i\right] + R_s$ where $R_s \in L_s(V^k)$ and $R_{s|\omega_2} = 0$. Hence $D(X) = \sum_i \left(\left[D(\tilde{Y}_i), \tilde{T}_i\right] + \left[\tilde{Y}_i, D(\tilde{T}_i)\right]\right) + D(R_s)$.

To conclude, we will do an inductive reasoning on s to show that $D(R_s)_{|\omega_2} = 0$.

For $s = 0$, $R_0 = 0$. Then $D(R_0)_{|\omega_2} = 0$. Thus $D(X)_{|\omega_2} = 0$, since $\tilde{T}_{i|\omega_2} = 0$, $\tilde{Y}_{i|\omega_2} = 0$, hence $D(X)_{|\omega} = 0$. Now we suppose that $D(X)_{|\omega_2} = 0$ for every $X \in L_s^h(V^k)$ such that $X_{|\omega_2} = 0$,

$1 \leq s \leq k$, $0 \leq h \leq k-s$. Let $X \in L_{s+1}^{h}(V^k)$ be a vector field on V^k , $0 \leq h \leq k-s$, $0 \leq s \leq k$. According to the inductive hypothesis, $D(R_s)_{|\omega_2} = 0$, hence $D(X)_{|\omega_2} = 0$, and thus $D(X)_{|\omega} = 0$ .This concludes the proof. □

Theorem 4. *When the k foliations on M are defined by submersions* $dim\, H^1\left(L_J(V^k)\right) = k$.

Proof. Let D be a derivation on $L_J(V^k)$. For every open set Ω of V^k , we have an induced derivation $D_\Omega : L_J(\Omega) \to L_J(\Omega)$. For $X \in L_J(\Omega)$ and $x \in \Omega$, we set :
$D_\Omega(X)(x) = D(X')(x)$ where $X' \in L_J(V^k)$ and coincides with X in an open neighborhood of x (see lemma 11). $D_\Omega(X)(x)$ does not depend on X' according to proposition 4.
Consider now a covering $\left(U_\alpha\right)_{\alpha \in A}$ of V^k by adapted local coordinates open sets. According to theorem 3, for all $\alpha \in A$, there exists $Y_\alpha \in L_J(U_\alpha)$, k constants $K_1^\alpha, ..., K_k^\alpha$ such that for every $X \in L_J(U_\alpha)$:

$$D_{U_\alpha}(X) = [\sum_{0 \leq b \leq k-1} K_{b+1}^\alpha \; J^b Z_{|U_\alpha} + Y_\alpha , \; X \;].$$

Since D_{U_α} and $D_{U_{\alpha'}}$ limited to $U_\alpha \cap U_{\alpha'}$ coincide, Y_α and $Y_{\alpha'}$ limited to $U_\alpha \cap U_{\alpha'}$ are equal and $K_{b+1}^\alpha = K_{b+1}^{\alpha'}$, $0 \leq b \leq k-1$. Thus there exists $Y \in L_J(V^k)$ and k real constants $K_1, ..., K_k$ such that for all $\alpha \in A$, $Y_{|U_\alpha} = Y_\alpha$ and $K_{b+1} = K_{b+1}^\alpha$, $0 \leq b \leq k-1$.
Since for every $X \in L_J(V^k)$, $D(X)_{|U_\alpha} = D_{U_\alpha}\left(X_{|U_\alpha}\right)$, we have for every $X \in L_J(V^k)$:
$D(X) = [\sum_{0 \leq b \leq k-1} K_{b+1} J^b Z + Y , \; X \;]$. This concludes the proof. □

5. Case of the torus endowed with two foliations

We consider the vector fields $X = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}$ and $X' = \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial z}$ in $\mathbb{R}^3$, provided with canonical coordinates (x, y, z) , where α and $\beta \in \mathbb{R} - \mathbb{Q}$. The first integrals of X (rep. X') globally defined are the functions $G(y - \alpha x, z)$ (resp. $G'(z - \beta x, y)$) where G and G' are C^∞ mappings from $\mathbb{R}^2$ to $\mathbb{R}$. Defining an equivalence relation in $\mathbb{R}^3$ by: $(x, y, z) \approx (x', y', z')$ if $x - x' \in \mathbb{Z}$, $y - y' \in \mathbb{Z}$ and $z - z' \in \mathbb{Z}$, we obtain on the torus T^3 the vector fields still denoted by X and X' . The first integrals of X and X' must be periodic C^∞ mappings in x, y and z , of period 1. For a fixed y , $u \to G'(u, y)$ is periodic in u of period 1 and β . Then G' only depends on y . Likewise, for a fixed z , $v \to G'(v, z)$ is periodic in v of period 1 and α . Then G only depends on z .
We endow T^3 with the following two foliations: F_1 is determined by X and X', of codimension 1 and F_2 is determined by X , of codimension 2. We have $F_1 \supset F_2$. The globally defined first integrals associated to the foliation F_1 are the functions $F(x, y, z) = G(y - \alpha x, z) = G'(z - \beta x, y)$. We deduce that the first integrals of F_1 are constant, and those of F_2 are only function of z .

On $M = T^3$, we consider the coordinate change:
$u_1 = -\beta x + \frac{\beta}{\alpha} y + z$, $u_2 = x - \frac{1}{\alpha} y$ and $u_3 = \frac{1}{\alpha} y$ where $u_1 \in]a,b[$, $u_2 \in]a',b'[$, $u_3 \in]a'',b''[$.
(u_1,u_2,u_3) are local coordinates adapted to the foliations F_1 and F_2. We deduce on the transverse bundle V^2, the adapted local coordinates $(u_1,u_2,u_3,u_4,u_5,u_6)$ in the open set $U =]a,b[\times]a',b'[\times]a'',b''[\times \mathbb{R}^3$. We will only consider this kind of open sets of adapted coordinates.
Let U and U' be two such open sets satisfying $U \cap U' \neq \emptyset$. We have, in $U \cap U'$: $u_1 - u_1^{'} = f$, $u_2 - u_2^{'} = g$, $u_3 - u_3^{'} = h$, where f, g and h are locally constant, $u_4 = u_4^{'}$, $u_5 = u_5^{'}$, $u_6 = u_6^{'}$. Then $\partial_1 = \partial_1^{'}$, $\partial_2 = \partial_2^{'}$, $\partial_3 = \partial_3^{'}$, $\partial_4 = \partial_4^{'}$, $\partial_5 = \partial_5^{'}$, $\partial_6 = \partial_6^{'}$. (For simplicity, we have set: $\frac{\partial}{\partial u_i} = \partial_i$).Thus we have six vector fields globally defined on V^2 which realize a parallelism.

We denote by X_2 (resp. X_3) the canonical lifts of X' (resp. X) in V^2. We have:

$$\begin{cases} \partial_1 = \frac{\partial}{\partial z} \\ \partial_2 = \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial z} = X_2, \\ \partial_3 = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} = X_3 \end{cases} \qquad \begin{array}{l} J\partial_1 = \partial_4, \ J\partial_2 = \partial_5, \\ J\partial_3 = 0, \ J\partial_4 = \partial_6, \\ J\partial_5 = 0, \ J\partial_6 = 0, \end{array} \quad \text{and } Z = u_4\partial_4 + u_5\partial_5 + 2u_6\partial_6.$$

We will take as a basis of $T(V^2)$: $\left(\frac{\partial}{\partial z}, X_2, X_3, \partial_4, \partial_5, \partial_6\right)$. For simplicity, we set: $\frac{\partial}{\partial z} = \partial_z$.
Let $Y \in L_J(V^2)$. We set $Y = Y_1\partial_z + Y_2X_2 + Y_3X_3 + Y_4\partial_4 + Y_5\partial_5 + Y_6\partial_6$. For every vector field T in V^2, we have $[Y, JT] = J[Y,T]$. By considering $T = \partial_z$, $T = X_2$, $T = X_3$, $T = \partial_4$, $T = \partial_5$ then $T = \partial_6$, we deduce:

Lemma 12. *Each element of $L_J(V^2)$ is of one of the following types:*
1) $K\partial_z$,
2) $F(z)X_2 + (u_4 + \beta u_5)\partial_z F\partial_5$,
3) $\varphi(x,y,z)X_3$,
4) $G(z)\partial_4 + (u_4 + \beta u_5)\partial_z G\partial_6$,
5) $\psi(x,y,z)\partial_5$,
6) $\phi(x,y,z)\partial_6$ where K is a constant and the mappings F and G from $\mathbb{R}$ to $\mathbb{R}$ (resp. φ,ψ,ϕ from $\mathbb{R}^3$ to $\mathbb{R}$) are 1-periodic in z (resp. 1-periodic in x, y and z). $L_1(V^2)$ is the set of the elements of type 3, 5 and 6. The set of elements of type 2 (resp. 4) is $A_2^0(V^2)$ (resp. $A_2^1(V^2)$) (cf. [2]). We have: $L_J(V^2) = \mathbb{R}\partial_z \oplus A_2^0(V^2) \oplus A_2^1(V^2) \oplus L_1(V^2)$.

Let $Y = Y_1\partial_z + \left(Y_2X_2 + (u_4 + \beta u_5)\partial_z Y_2\partial_5\right) + Y_3X_3 + \left(Y_4\partial_4 + (u_4 + \beta u_5)\partial_z Y_4\partial_6\right) + Y_5\partial_5 + Y_6\partial_6$ an element of $L_J(V^2)$.
We set:

$$\Delta_1(Y) = \left(X_3.Y_3\right)\partial_6, \quad \Delta_2(Y) = Y_4\partial_5 + \beta Y_5\partial_5 + \beta Y_6\partial_6,$$

$\Delta_3(Y)=Y_1(\partial_5-\beta\partial_4)$, $\Delta_4(Y)=Y_2\left(\partial_5-\beta\partial_4\right)$.

It is easy to verify that:

Lemma 13. *Δ_1, Δ_2, Δ_3 and Δ_4 are derivations of $L_J(V^2)$ which are not inner derivations.*

Thus *dim* $H^1\left(L_J(V^2)\right) \geq 6$ since Δ_1, Δ_2, Δ_3, Δ_4, $X\to[Z,X]$ and $X\to[JZ,X]$ are non-inner linearly independent derivations of $L_J(V^2)$.

6. Case of the sphere endowed with two foliations. Study of $H^1\left(L_1(V^2)\right)$.

Let S^3 be the unit 3- sphere defined by $S^3=\left\{(x_1,x_2,x_3,x_4)/\,x_1^2+x_2^2+x_3^2+x_4^2=1\right\}$. We consider $W=S^3-\left\{x_1^2+x_2^2=0,\ x_3^2+x_4^2=0\right\}$. We set $v_1=\sqrt{x_3^2+x_4^2}$, $0<v_1<1$,

$$\begin{cases} x_1=\sqrt{1-v_1^2}\cos v_2 \\ x_2=\sqrt{1-v_1^2}\sin v_2 \\ x_3=v_1\cos v_3 \\ x_4=v_1\sin v_3 \end{cases}$$

where v_2 and v_3 are 2π -periodic and, $X=\dfrac{\partial}{\partial v_2}+v_1^2\dfrac{\partial}{\partial v_3}$ whose first integrals are C^∞ mappings $F(v_1)$.

We have: $\dfrac{dv_1}{dt}=0$, $\dfrac{dv_2}{dt}=1$, $\dfrac{dv_3}{dt}=v_1^2$ thus $v_1=cste$, $t=v_2$ and $v_3=v_1^2v_2+cste$.

We endow W with the following two foliations: F_1 is determined by the foliation of the torus, of codimension 1 and F_2 is determined by X, of codimension 2.

On W, we consider the coordinate change:

$u_1=v_1$, $u_2=v_3-v_1^2v_2$ and $u_3=v_3$ where $u_1\in\left]0,1\right[$, $u_2\in\left]0,2\pi\right[$, $u_3\in\left]0,2\pi\right[$.

(u_1,u_2,u_3) are local coordinates adapted to the foliations F_1 and F_2. We deduce on the transverse bundle V^2, the adapted local coordinates $(u_1,u_2,u_3,u_4,u_5,u_6)$ in the open set $U=\left]0,1\right[\times\left]0,2\pi\right[\times\left]0,2\pi\right[\times\mathbb{R}^3$. We will only consider this kind of open sets of adapted coordinates.

Let U and U' be two such open sets satisfying $U\cap U'\neq\varnothing$. We have, in $U\cap U'$: $u_1=u_1'$, $u_2-u_2'=f$, $u_3-u_3'=g$, where f and g are locally constant on $U\cap U'$, $u_4=u_4'$, $u_5=u_5'$, $u_6=u_6'$. Then $\partial_1=\partial_1'$, $\partial_2=\partial_2'$, $\partial_3=\partial_3'$ $\partial_4=\partial_4'$, $\partial_5=\partial_5'$, $\partial_6=\partial_6'$. (For simplicity, we have set: $\dfrac{\partial}{\partial u_i}=\partial_i$).Thus we have six vector fields globally defined on V^2 which realize a parallelism.

We have:

$$\begin{cases} \dfrac{\partial}{\partial v_1}=\partial_1-2v_1v_2\partial_2 \\ \dfrac{\partial}{\partial v_2}=-v_1^2\partial_2 \\ \dfrac{\partial}{\partial v_3}=\partial_2+\partial_3 \end{cases},\qquad \begin{matrix} J\partial_1=\partial_4,\ \ J\partial_2=\partial_5, \\ J\partial_3=0,\ \ J\partial_4=\partial_6, \\ J\partial_5=0,\ \ J\partial_6=0, \end{matrix}\qquad \text{and } X=u_1^2\partial_3.$$

Let $Y \in L_J(V^2)$. We set $Y = Y_1\partial_1 + Y_2\partial_2 + Y_3X + Y_4\partial_4 + Y_5\partial_5 + Y_6\partial_6$. For every vector field T in V^2, we have $[Y, JT] = J[Y,T]$. By considering $T = \partial_1$, $T = \partial_2$, $T = X$, $T = \partial_4$, $T = \partial_5$ then $T = \partial_6$, we deduce:

Lemma 14. *Each element of $L_J(V^2)$ is of one of the following types:*

1) $F_1(u_1)\partial_1 + u_4\partial_1F_1\partial_4 + \left(\frac{1}{2}u_4^2\partial_1^2F_1 + u_6\partial_1F_1\right)\partial_6$,

2) $F_2(u_1)\partial_2 + u_4\partial_1F_2\partial_5$,

3) $\varphi(u_1,u_2,u_3)X$,

4) $F_4(u_1)\partial_4 + u_4\partial_1F_4\partial_6$,

5) $\psi(u_1,u_2,u_3)\partial_5$,

6) $\phi(u_1,u_2,u_3)\partial_6$.

$L_1(V^2)$ *is the set of the elements of type 3, 5 and 6.*

Let $Y = \varphi(u_1,u_2,u_3)X + \psi(u_1,u_2,u_3)\partial_5 + \phi(u_1,u_2,u_3)\partial_6$ an element of $L_1(V^2)$.
We set: $\Delta(Y) = (X.\varphi)(A(u_1)\partial_5 + B(u_1)\partial_6)$.
Moreover, let $T = (C_5(u_1)u_5 + C_6(u_1)u_6)\partial_5 + (D_5(u_1)u_5 + D_6(u_1)u_6)\partial_6$.
It is easy to verify that Δ and $Y \to [T,Y]$ are derivations of $L_1(V^2)$ which are not inner derivations. Thus we have the following result:

Theorem 5. *Let D be a derivation of $L_1(V^2)$. There exists a unique vector field $S \in L_J(V^2)$ ($S = Z_1 + Z_2 + Z_3$, Z_1 of type 1, Z_2 of type 2 and Z_3 of type 3) such that for every $Y \in L_1(V^2)$: $D(Y) = [S,Y] + \Delta(Y) + [T,Y] + [Z_5,Y] + [Z_6,Y]$. Z_5 and Z_6 are of type 5 and 6 respectively and are determined up to the sum of $\psi(u_1)\partial_5 + \phi(u_1)\partial_6$.*

References.

[1] L. Lebtahi, Lie Algebra on the Transverse Bundle of a Decreasing Family of Foliations, J. Geom. Phys. 60 (2010) 122-133.

[2] J. Lehmann-Lejeune, Cohomologies sur le fibré transverse à un feuilletage, C.R.A.S. Paris, 295 (1982), pp. 495-498.

[3] G. Catz, Sur le fibré tangent d'ordre 2, C.R.A.S. Paris, 278 (1974), 277- 280.

[4] J. Lehmann-Lejeune, Etude locale des automorphismes de la structure - Etude des dérivations, Pub. I.R.M.A de Lille, Vol 2, nº II (1980).

[5] A. Lichnerowicz, Algèbres de Lie attachées à un feuilletage, Ann. Fac. Sc. Toulouse, 1 (1979), pp. 45-76.