Relations between $\{K, s+1\}$-Potent Matrices and Different Classes of Complex Matrices

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Abstract

In this paper, $\{K, s+1\}$-potent matrices are considered. A matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent when $KA^{s+1}K = A$ where K is an involutory matrix and $s \in \{1, 2, 3, \dots\}$. Specifically, $\{K, s+1\}$-potent matrices are analyzed considering their relations to different classes of complex matrices. These classes of matrices are: $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, and matrices $B \in \mathbb{C}^{n\times n}$ (anti-)commuting with K or such that KB is involutory, Hermitian or normal. In addition, some new relations for K-generalized centrosymmetric matrices have been derived.

Keywords: Involutory matrix; idempotent matrix; $\{K, s+1\}$-potent matrix; normal matrix, centrosymmetric matrix; $\{s+1\}$-potent matrix.
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1 Introduction and background

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be (skew-)centrosymmetric if $JAJ = A$ ($JAJ = -A$), where $J \in \mathbb{C}^{n\times n}$ is the exchange matrix with ones on the anti-diagonal (lower left to upper right) and zeros elsewhere. These matrices have been widely studied and have applications in differential equations, signal processing, Markov processes, engineering problems, etc. (see, for example, [1, 2, 8, 14, 21]).

In [17], Stuart gave a generalization of a centrosymmetric matrix called a P-commutative matrix where P is a permutation. In [13], Li and Feng analyzed mirror matrices and exchange matrices, and that work is a special case of the generalizations that we consider in this paper. They have applications on multi-conductor transmission lines. Further results related to this class of matrices can be found, for instance, in [7, 9, 10, 15, 16, 18, 19, 20].

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The common property that matrices in these classes (exchange, permutation, etc.) share is that they are all involutory. This leads to the introduction of the so-called K-generalized centrosymmetric matrices, which are the matrices $A \in \mathbb{C}^{n\times n}$ satisfying $A = KAK$ where $K \in \mathbb{C}^{n\times n}$ is an involutory matrix [22].

In recent years, matrices A such that $A^{s+1} = A$ for some positive integer s have been studied as an extension of the idempotent matrices; such matrices are called $\{s+1\}$-potent. In addition, the relation between the $\{s+1\}$-potent matrix A and its group inverse has been given [5, 6].

All these ideas motivate the following definition. For a given positive integer s and a given involutory matrix $K \in \mathbb{C}^{n\times n}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent when $KA^{s+1}K = A$. In [12], the authors introduce $\{K, s+1\}$-potent matrices, and investigate their spectral and structural properties, as well as give a variety of equivalent characterizations of these matrices. In particular, it has been established that a matrix $A \in \mathbb{C}^{n\times n}$ is $\{K, s+1\}$-potent if and only if any of the following equivalent conditions holds: $KAK = A^{s+1}$, $KA = A^{s+1}K$, or $AK = KA^{s+1}$. In this paper the relation among different classes of matrices is analyzed.

Let Ω_k be the set of all k^{th} roots of unity with k a positive integer. If we define $\omega = e^{2\pi i/k}$ then $\Omega_k = \{\omega^1, \omega^2, \ldots, \omega^k\}$. Moreover, the following set will be used:

$$\mathbf{\Omega}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \sigma(A) \subseteq \{0\} \cup \Omega_{s+1}\}, \tag{1}$$

where $\sigma(A)$ denotes the set of all eigenvalues of A, that is, the spectrum of A.

Throughout this paper, we will assume that the matrix $K \in \mathbb{C}^{n\times n}$ is involutory and that $s \in \{1, 2, 3 \ldots\}$. Moreover,

$$\mathbf{P}^{(K,s)} = \{A \in \mathbb{C}^{n\times n} : KA^{s+1}K = A\} \tag{2}$$

In addition, the notations $K\mathbf{S} = \{KB : B \in \mathbf{S}\}$ and $\mathbf{S}K = \{BK : B \in \mathbf{S}\}$ will be useful where K is the above fixed matrix and $\mathbf{S}$ is a prescribed subset of $\mathbb{C}^{n\times n}$.

A matrix $A \in \mathbb{C}^{n\times n}$ with the property $KA^*K = A$ is called $\{K\}$-Hermitian. The equality $KA^*K = A$ is equivalent to $KAK = A^*$ because $K^2 = I_n$, where I_n denotes the identity matrix. In [11], Hill and Waters emphasize κ, a fixed product of disjoint transposition in the set of all permutation, and they called κ-Hermitian matrices $\{K\}$-Hermitian matrices, whereas in this paper, we stress the matrix K for coherence with the remaining definitions.

Moreover, some of the following results have been established in [12].

Lemma 1 *If $A \in \mathbf{P}^{(K,s)}$ then the following properties hold:*

(a) $A^j \in \mathbf{P}^{(K,s)}$ for all integer $j \geq 1$.

(b) $(KA)^2 = A^{s+2}$.

(c) $(KA)^{2s+1} = KA$ and $(AK)^{2s+1} = AK$.

(d) If K is Hermitian (that is, $K^* = K$) then $A^* \in \mathbf{P}^{(K,s)}$.

(e) If A is a nonsingular matrix then $A^{-1} \in \mathbf{P}^{(K,s)}$.

Proof. By induction, let us suppose that $KAK = A^{s+1}$ and $KA^jK = A^{j(s+1)}$ hold for some $j > 1$. Since $K^2 = I_n$,

$$KA^{j+1}K = (KA^jK)(KAK) = A^{j(s+1)}A^{s+1} = A^{(j+1)(s+1)}.$$

Then, property (a) is shown. Property (b) follows directly by definition. The first part of property (c) can be demonstrated as follows

$$\begin{aligned}(KA)^{2s+1} &= (KA)((KA)^2)^s = (KA)(A^{s+2})^s = KA^{(s+1)^2} \\ &= K(A^{s+1})^{s+1} = KKAK^{s+1} = K(KA^{s+1}K) = KA.\end{aligned}$$

The second part is similar. Properties (d) and (e) can be directly obtained by definition. ■

In what follows we will need the following spectral theorem.

Theorem 1 ([3]) *Let $A \in \mathbb{C}^{n\times n}$ with k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then A is diagonalizable if and only if there exist disjoint projectors $P_1, P_2, \ldots, P_k$, (i.e., $P_iP_j = \delta_{ij}P_i$ for $i, j \in \{1, 2, \ldots, k\}$) such that $A = \sum_{j=1}^{k} \lambda_j P_j$ and $I_n = \sum_{j=1}^{k} P_j$. Moreover, when A is a normal matrix, the projectors $P_1, P_2, \ldots, P_k$ are orthogonal (i.e., $P_i^* = P_i$ for $i \in \{1, 2, \ldots, k\}$).*

In section 2, we derive relations between $\{K, s+1\}$-potent matrices and $\{K\}$-Hermitian matrices, $\{s+1\}$-generalized projectors, unitary matrices, and matrices $B \in \mathbb{C}^{n\times n}$ such that KB is involutory. In section 3, relations with normal matrices and matrices $B \in \mathbb{C}^{n\times n}$ such that KB is Hermitian or normal are analyzed. In section 4, Hermitian and skew-Hermitian matrices, and matrices $B \in \mathbb{C}^{n\times n}$ (skew-)commuting with K are related to $\{K, s + 1\}$-potent matrices. In all the cases, the class equivalences are established, or in case of obtaining only one inclusion, the necessary examples are presented showing that they are proper subsets. In addition, a set of new properties is derived for centrosymmetric matrices.

2 Analysis of $\mathbf{P}^{(K,s)}$ through different sets of matrices

In [12], it was proven that if $\varphi : \{0, 1, 2, \ldots, (s + 1)^2 - 2\} \to \{0, 1, 2, \ldots, (s + 1)^2 - 2\}$ is the function given by $\varphi(j) = b_j$ where b_j is the smallest nonnegative integer such that $b_j \equiv j(s + 1)$ [mod $((s + 1)^2 - 1)$] then φ is bijective. For the sake of completeness we report a simpler proof suggest by one referee. In fact, it is sufficient to show that if q is an arbitrary positive integer and $\gcd(r, q) = 1$, then the mapping $\phi(l) = rl$[mod (q)], $0 \le l \le q - 1$ is a permutation of $\{0, 1, \ldots, q - 1\}$. To see this, suppose $rl \equiv rm$[mod (q)] with $r, l \in \{0, 1, \ldots, q-1\}$. Then $r(l-m)$ is a multiple of q. Since $\gcd(r, q) = 1$ this means that $l - m = tq$ for some integer t. Since $|l - m| < q$, $t = 0$; hence $l = m$. In particular, the remark for the mapping φ follows directly setting $r = s + 1$ and $q = (s + 1)^2 - 1$.

Moreover, in that paper the authors have shown that the eigenvalues of a $\{K, s+1\}$-potent matrix are included in the set constituted by $0, \omega^1, \ldots, \omega^{p-1}, 1$, where $p = (s+1)^2 - 1$ and $\omega = e^{2\pi i/p}$, and such a matrix A has associated certain projectors. Specifically, we will consider matrices P_j's satisfying the relations

$$KP_jK = P_{j(s+1)} \text{ and } KP_pK = P_p \tag{3}$$

for $j \in \{0, 1, \ldots, p-1\}$ where $P_0, P_1, \ldots, P_p$ are the projectors appearing in the spectral decomposition of A given in Theorem 1 associated to the previous eigenvalues, respectively. All the subscripts are interpeted modulo p since $j \to j(s+1)[\text{mod } (p)]$ is a permutation of $\{0, 1, \ldots, p-1\}$.

It is easy to see that when $KA^{s+1}K = A$ then $\text{rank}(A^2) = \text{rank}(A)$, hence there exists the group inverse of A.

Theorem 2 ([12]) *Let $A \in \mathbb{C}^{n\times n}$ and $p = (s+1)^2 - 1$. Then the following conditions are equivalent:*

(a) $A \in \mathbf{P}^{(K,s)}$.

(b) A is diagonalizable, $\sigma(A) \subseteq \{0\} \cup \Omega_p$, and the P_j's satisfy condition (3).

(c) $A^{p+1} = A$, and the P_j's satisfy condition (3).

(d) $A^{\#} = A^{p-1}$, and the P_j's satisfy condition (3).

Considering the sets

$$\mathbf{D} = \{A \in \mathbb{C}^{n\times n} : \ A \text{ is diagonalizable}\},$$
$$\mathbf{\Phi} = \{A \in \mathbb{C}^{n\times n} : \ \text{ projectors } P_j \text{ 's satisfy condition (3)}\},$$
$$\mathbf{P}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = A\},$$
$$\mathbf{G}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{\#} = A^{s+1}\},$$

we can rewrite Theorem 2 as follows:

$$\mathbf{P}^{(K,s)} = \mathbf{D} \cap \mathbf{\Omega}^{(s+1)^2-2} \cap \mathbf{\Phi} = \mathbf{P}^{((s+1)^2-1)} \cap \mathbf{\Phi} = \mathbf{G}^{((s+1)^2-3)} \cap \mathbf{\Phi}.$$

According to the above notation, we have $\mathbf{P}^{(I_n,s)} = \mathbf{P}^{(s)}$. Moreover, $\mathbf{P}^{(1)}$ is the set of all $n \times n$ idempotent matrices. In addition, $\mathbf{P}^{(s+2)} = \mathbf{G}^{(s)}$.

We recall that a matrix $A \in \mathbb{C}^{n\times n}$ with the property $A^{s+1} = A^*$ for some positive integer s is called an $\{s+1\}$-generalized projector [4, 9], and also that A is called $\{K\}$-Hermitian when $KA^*K = A$. The following sets will be useful later:

$$\mathbf{GP}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = A^*\},$$

$$\mathbf{H}^{(K)} = \{A \in \mathbb{C}^{n\times n} : \ KA^*K = A\}.$$

In the following result we derive the relationship between the known concepts of an $\{s+1\}$-generalized projector and of a $\{K\}$-Hermitian matrix, and the newer concept of a $\{K, s+1\}$-potent matrix.

The matrix

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ 1+i & 1+i \end{bmatrix} \tag{4}$$

is a $\{3\}$-generalized projector for $K = I_2$. However, A is neither $\{K, 3\}$-potent nor $\{K\}$-Hermitian. On the other hand, the matrix

$$A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{5}$$

is $\{K\}$-Hermitian for $K = I_2$. Nevertheless, A is neither $\{K, s+1\}$-potent nor an $\{s+1\}$-generalized projector for any odd positive integer s. Finally, the matrix

$$A = \begin{bmatrix} -3 & -2\sqrt{7} \\ \frac{\sqrt{7}}{2} & 2 \end{bmatrix} \tag{6}$$

is $\{K, 2\}$-potent matrix for

$$K = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

However, A is neither $\{K\}$-Hermitian nor a $\{2\}$-generalized projector.

With these three examples, we tried to find some relations between $\{s+1\}$-generalized projectors, $\{K, s+1\}$-potent matrices and $\{K\}$-Hermitian matrices. In order to satisfy one of them, it is easy to see that the other two are required.

As before, we will assume that $K \in \mathbb{C}^{n\times n}$ is an involutory matrix and $s \in \{1, 2, 3, \dots\}$ until the end of this section.

Theorem 3 *The following inclusions hold:*

(a) $\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \subseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(s+2)} = \mathbf{GP}^{(s)}$.

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} \subseteq \mathbf{H}^{(K)} \cap \mathbf{P}^{(s+2)}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(K,s)} \cap \mathbf{P}^{(s+2)}$.

Proof. Let $A \in \mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)}$. Then, $KAK = A^{s+1}$ and $KAK = A^*$. So, $A \in \mathbf{GP}^{(s)}$. From Theorem 2.1 in [4] we have that $A^{s+3} = A$, thus $A \in \mathbf{P}^{(s+2)}$. In addition, we have obtained that $\mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(s+2)}$, then $\mathbf{GP}^{(s)} \cap \mathbf{P}^{(s+2)} = \mathbf{GP}^{(s)}$. Thus, (a) is shown.

By using the definitions, we can easily see that both intersections $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)}$ and $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)}$ are included in $\mathbf{H}^{(K)}$ and $\mathbf{P}^{(K,s)}$, respectively. Again, $\mathbf{GP}^{(s)} \subseteq \mathbf{P}^{(s+2)}$ leads to properties (b) and (c). ■

The examples preceding Theorem 3 show that none of the conditions imply the other two, and that, in general, all three inclusions are proper.

Note that Theorem 3 links one set that depends only on K to two others that depend only on s, and to another set that depends on both K and s. Furthermore, the three smaller sets in the inclusions are non-empty because all of them contain the identity matrix.

A particularly important case is when the matrices A are required to be nonsingular. In this case, the relations with unitary matrices appear. Let

$$\mathbf{GL} = \{A \in \mathbb{C}^{n\times n} : \ A \text{ is nonsingular}\},$$

$$\mathbf{U} = \{A \in \mathbb{C}^{n\times n} : \ AA^* = A^*A = I_n\}.$$

Theorem 4 *The following inclusion holds:*

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GL} \subseteq \mathbf{U}.$$

Proof. Let A be both $\{K, s+1\}$-potent and $\{K\}$-Hermitian. Item (a) of Theorem 3 implies that $A^{s+3} = A$. Since A is nonsingular, we get $A^{s+2} = I_n$ and then $AA^* = AA^{s+1} = I_n$, which shows that A is unitary. The proof is then completed. ■

We can say even more. In general,

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GL} \subsetneq \mathbf{U} \tag{7}$$

because the matrix given in (4) is unitary and belongs to $\mathbf{P}^{(K,4)}$ but it does not belong to $\mathbf{H}^{(K)}$ for $K = I_2$. Also, the matrix given in (5) is unitary and $\{K\}$-Hermitian but it does not belong to $\mathbf{P}^{(K,s)}$ when s is odd for $K = I_2$.

Several interesting observations are obtained as direct consequences of Theorem 3 and Theorem 4. The first one is:

$$\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} = \mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} = \mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} = \mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)}, \tag{8}$$

that is, if an element belongs to two of the three sets considered, then it belongs to the remaining one.

Another general observation based on (7) and (8) is that the following proper inclusions hold:

(a) $\mathbf{P}^{(K,s)} \cap \mathbf{H}^{(K)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GP}^{(s)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{GP}^{(s)} \cap \mathbf{GL} \subsetneq \mathbf{U}$.

Even more, from Theorem 2.1 in [4] it can be shown a more general inclusion holds.

Theorem 5 *The following inclusion holds:*

$$\mathbf{GP}^{(s)} \cap \mathbf{GL} \subseteq \mathbf{U}.$$

Example (5) shows that the last inclusion is strict: s must be odd.

In (7) we have presented an inclusion involving the set of unitary matrices. What happens with $\supseteq$ if we intersect $\mathbf{U}$ with another set?

Let

$$\mathbf{I}^{(s)} = \{A \in \mathbb{C}^{n\times n}: \ A^{s+1} = I_n\}.$$

Note that, when $s = 1$ the set $\mathbf{I}^{(1)}$ corresponds to the involutory matrices, and moreover, it is clear that $\mathbf{I}^{(s)} \subsetneq \mathbf{P}^{(s+1)}$ and $\mathbf{I}^{(s)} = \mathbf{P}^{(s+1)} \cap \mathbf{GL}$.

Theorem 6 *The following inclusions hold:*

(a) $\mathbf{U} \cap K\mathbf{I}^{(1)} \subseteq \mathbf{H}^{(K)}$.

(b) $\mathbf{U} \cap \mathbf{H}^{(K)} \subseteq K\mathbf{I}^{(1)}$.

(c) $\mathbf{U} \cap \mathbf{P}^{(s+2)} \subseteq \mathbf{GP}^{(s)}$.

(d) $K\mathbf{I}^{(1)} \cap \mathbf{H}^{(K)} \subseteq \mathbf{U}$.

Proof. Suppose that A is unitary and there is $B \in \mathbf{I}^{(1)}$ such that $A = KB$. Then $KAKA = (KA)^2 = B^2 = I_n$ and so, $KAK = A^{-1} = A^*$. Thus, (a) is proved.

In order to prove item (b), post-multiplying by A equality $KAK = A^*$ we obtain $(KA)^2 = I_n$ because $A \in \mathbf{U}$. Then, $KA \in \mathbf{I}^{(1)}$ and so $A \in K\mathbf{I}^{(1)}$.

Let $A \in \mathbf{U} \cap \mathbf{P}^{(s+2)}$. Since A is nonsingular, $A^{s+3} = A$ implies that $A^{s+2} = I_n$ which is equivalent to $A^{s+1} = A^{-1} = A^*$. Thus, $A \in \mathbf{GP}^{(s)}$.

Finally, if we suppose that $KAK = A^*$ and $A = KB$ with $B^2 = I_n$, then item (d) follows from $A^*A = (KAK)A = (KA)^2 = B^2 = I_n$. Thus, the proof is completed. ■

In order to see that, in general, the inclusions in Theorem 6 are proper we can consider the following examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2i \\ -\frac{1}{2}i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ with } s \in \{1, 2, 3, \dots\}, \text{ and the matrix given by (4),}$$

all for $K = I_2$, corresponding respectively to items $(a) - (d)$.

For the set $K\mathbf{I}^{(1)}$ we can also obtain the following inclusions:

(a) $K\mathbf{I}^{(1)} \cap \mathbf{P}^{(K,s)} \subseteq \mathbf{I}^{(s+1)}$.

(b) $\mathbf{GP}^{(s)} \cap K\mathbf{I}^{(1)} \cap \mathbf{GL} \subseteq \mathbf{P}^{(K,s)}$.

(c) $\mathbf{P}^{(s+2)} \cap K\mathbf{I}^{(1)} \cap \mathbf{H}^{(K)} \subseteq \mathbf{P}^{(K,s)}$.

The proof requires property (b) in Lemma 1 and Theorem 2.1 in [4]. In general, all the inclusions are proper as the following respective examples show: the same last matrix A for (a), and the matrix given in (6) for (b) and (c).

Again, what happens in (7) with $\supseteq$ if we intersect $\mathbf{U}$ with $\mathbf{P}^{(K,s)}$? The following result can be given.

Proposition 1 *Let* $(\mathbf{P}^{(K,s)})^* = \{A^* : A \in \mathbf{P}^{(K,s)}\}$*. Then the following property holds:*

$$\mathbf{U} \cap \mathbf{P}^{(K,s)} = \mathbf{U} \cap (\mathbf{P}^{(K,s)})^*.$$

Proof. Let $A \in \mathbf{U}$. We have to prove that $A \in \mathbf{P}^{(K,s)}$ if and only if $A^* \in \mathbf{P}^{(K,s)}$. In fact, if $A \in \mathbf{P}^{(K,s)}$ by using property (e) from Lemma 1 we have

$$KA^*K = KA^{-1}K = (A^{-1})^{s+1} = (A^*)^{s+1}.$$

Thus, $A^* \in \mathbf{P}^{(K,s)}$ and so $A \in (\mathbf{P}^{(K,s)})^*$. The converse can be obtained in a similar way. ■

What happens with the equality $\mathbf{P}^{(K,s)} = (\mathbf{P}^{(K,s)})^*$ when $A \notin \mathbf{U}$? In general, it is not satisfied. In fact, the matrix given in (6) belongs to $\mathbf{P}^{(K,1)}$ and $A \notin \mathbf{U}$, but A does not belong to $(\mathbf{P}^{(K,1)})^*$.

However, the equality remains valid when matrix K is Hermitian. The following result extends the properties (d) and (e) presented in Lemma 1.

Lemma 2 *Let* $(\mathbf{P}^{(K,s)})^{-1} = \{A^{-1} : A \in \mathbf{P}^{(K,s)} \cap \mathbf{GL}\}$*. Then*

(a) $\mathbf{P}^{(K^*,s)} = (\mathbf{P}^{(K,s)})^*$*. In particular,* $\mathbf{P}^{(K,s)} = (\mathbf{P}^{(K,s)})^*$ *when* K *is Hermitian.*

(b) $\mathbf{P}^{(K,s)} \cap \mathbf{GL} = (\mathbf{P}^{(K,s)})^{-1}$.

In particular, $A \in (\mathbf{P}^{(K,s)})^{-1}$ if and only if $A^{-1} \in (\mathbf{P}^{(K,s)})^{-1}$.

Furthermore, in general, the equality $\mathbf{P}^{(K,s)} = \mathbf{P}^{(K^*,s)}$ does not hold when K is not Hermitian. This can be checked using the matrix of example (6).

In addition, we can assure that in general

$$\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)} \cap \mathbf{GL} \nsubseteq \mathbf{U},$$

as the $\{K, 4\}$-potent matrix

$$A = \begin{bmatrix} 1 & -3 + i\sqrt{3} \\ 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}$$

shows for $K = I_2$.

Note that $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)} \neq \emptyset$ because it contains the zero matrix.

In order to see that in general $\mathbf{U} \nsubseteq \mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$, we can consider $A = -I_n$, any positive odd number s, and any involutory matrix $K \in \mathbb{C}^{n \times n}$.

We close this section by analyzing the last considered set $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$. We recall that K^*K is a Hermitian matrix. Hence, there is a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that

$$K^*K = UDU^* \tag{9}$$

Theorem 7 *Let us consider a unitary matrix* U *and a real diagonal matrix*

$$D = \mathrm{diag}(\lambda_1 I_{r_1}, \lambda_2 I_{r_2}, \ldots, \lambda_t I_{r_t})$$

as in (9) with $r_1+r_2+\cdots+r_t=n$ and $\lambda_i \neq \lambda_j$ for $i\neq j$. Then the set $\mathbf{P}^{(K,s)} \cap \mathbf{P}^{(K^*,s)}$ is given by

$$\left\{U\tilde{D}U^*:\ \tilde{D}=\operatorname{diag}(A_{11},A_{22},\ldots,A_{tt})\in \mathbf{P}^{(U^*KU,s)}\cap \mathbf{P}^{(U^*K^*U,s)}, A_{ii}\in\mathbb{C}^{r_i\times r_i}, i=1,\ldots,t\right\}.$$

Proof. Let $A\in \mathbf{P}^{(K,s)}\cap \mathbf{P}^{(K^*,s)}$. Then $A^{s+1}=KAK$ and $(A^*)^{s+1}=KA^*K$. By making some algebraic manipulations we get $K^*KA=AK^*K$. Since K^*K is Hermitian, we can suppose that $K^*K=UDU^*$ with U and D as in the statement. Then $U^*AUD=DU^*AU$. Now, we partition matrix U^*AU as follows

$$\tilde{A}=U^*AU=\begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1t}\\ A_{21} & A_{22} & \ldots & A_{2t}\\ \vdots & \vdots & \ddots & \vdots\\ A_{t1} & A_{t2} & \ldots & A_{tt}\end{bmatrix}$$

according to the sizes of the blocks of D. Thus, $\tilde{A}D=D\tilde{A}$ yields to $A_{ij}=O$ for all $i,j\in\{1,2,\ldots,t\}$ with $i\neq j$, and consequently

$$A=U\operatorname{diag}(A_{11},A_{22},\ldots,A_{tt})U^*.$$

Since $A\in\mathbf{P}^{(K,s)}$, we have that

$$U^*KU\begin{bmatrix} A_{11}^{s+1} & O & \ldots & O\\ O & A_{22}^{s+1} & \ldots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \ldots & A_{tt}^{s+1}\end{bmatrix}U^*KU=\begin{bmatrix} A_{11} & O & \ldots & O\\ O & A_{22} & \ldots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \ldots & A_{tt}\end{bmatrix}.$$

Then $\tilde{D}=\operatorname{diag}(A_{11},A_{22},\ldots,A_{tt})$ is $\{U^*KU,s+1\}$-potent because U^*KU is involutory. A similar reasoning is valid for $A\in\mathbf{P}^{(K^*,s)}$. Then, $\tilde{D}\in\mathbf{P}^{(U^*KU,s)}\cap\mathbf{P}^{(U^*K^*U,s)}$. The other inclusion is obvious. ■

The previous theorem allows us to obtain another representation for the set $\mathbf{P}^{(K,s)}$ (see Theorem 2). Its advantage is that the diagonalization is given by a unitary matrix U, however, unlike Theorem 2, the diagonal matrix $\tilde{D}$ is given by blocks.

Note that, with respect to the spectrum of every A_{ii}, for $i=1,2,\ldots,t$, we have

$$\sigma(A_{ii})\subseteq\sigma(\tilde{D})\subseteq\{0\}\cup\Omega_{(s+1)^2-1}.$$

3 Analysis of $\mathbf{P}^{(K,s)}$ through $\mathbf{GP}^{(s)}$ and normality

More relations can be obtained when Hermitian matrices are involved. Let

$$\mathbf{H}=\{A\in\mathbb{C}^{n\times n}:\ A^*=A\}. \tag{10}$$

Note that $\mathbf{H}=\mathbf{H}^{(I_n)}$. When $K\in\mathbf{H}$ it can be shown that $\mathbf{H}=K\mathbf{H}^{(K)}=\mathbf{H}^{(K)}K$, or equivalently, $\mathbf{H}^{(K)}=K\mathbf{H}=\mathbf{H}K$.

We recall that, until the end of this section, K will be involutory. When an extra hypothesis is needed, it will be explicitly mentioned.

Theorem 8 *The following inclusion holds:*

$$K\mathbf{H} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{P}^{(2)}.$$

Proof. Let $A \in K\mathbf{H} \cap \mathbf{P}^{(K,s)}$. It is well-known that the spectrum of the Hermitian matrix KA is real and also that it is included in $\{0\} \cup \Omega_{2s}$ when $A \in \mathbf{P}^{(K,s)}$ by using property (c) of Lemma 1. Hence, KA has its spectrum included in $\{0, 1, -1\}$. Since KA is diagonalizable, it is tripotent which ends the proof. ∎

In general, the last inclusion is strict as the following matrices show:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix} = KB$$

where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}.$$

However, the additional condition on K that is Hermitian allows us to obtain a more interesting result.

Theorem 9 *Let $K \in \mathbf{H}$. Then*

$$K\mathbf{H} \cap \mathbf{P}^{(K,s)} = \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in K\mathbf{H}$ if and only if $A \in \mathbf{GP}^{(s)}$. In fact, if there is $B \in \mathbf{H}$ such that $A = KB$ then it is obvious that $KA = B$ and so $KA \in \mathbf{H}$. Moreover, the equalities

$$KA = (KA)^* = A^*K$$

allow us to conclude that $A^{s+1} = KAK = A^*$. Thus $A \in \mathbf{GP}^{(s)}$.

Conversely, if $A \in \mathbf{GP}^{(s)}$, by definition $A^{s+1} = A^*$. Then, $KAK = A^*$ and $KA = A^*K$. So, we get $A = K(A^*K)$, that is, $A = KB$ where $B = A^*K$ satisfies

$$B^* = K^*A = KA = A^*K = B.$$

Thus, $B \in \mathbf{H}$ and finally $A \in K\mathbf{H}$. ∎

We now consider the set of normal matrices:

$$\mathbf{N} = \{A \in \mathbb{C}^{n\times n} : \ AA^* = A^*A\}.$$

In [4], the equivalences between the following statements are established:

(i) $A^{s+1} = A^*$.

(ii) A is a normal matrix and $\sigma(A) \subseteq \Omega_{s+2}$.

(iii) A is a normal matrix and $A^{s+3} = A$.

From these equivalences and Theorem 9 we derive a relationship between $\{K, s+1\}$-potent matrices and normal matrices, $\{s+1\}$-generalized projectors, and $\{s+1\}$-potent matrices.

Proposition 2 *Let $K \in \mathbf{H}$. Then*

$$\mathbf{N} \cap \mathbf{\Omega}^{(s+1)} \cap \mathbf{P}^{(K,s)} = \mathbf{N} \cap \mathbf{P}^{(s+2)} \cap \mathbf{P}^{(K,s)} = K\mathbf{H} \cap \mathbf{P}^{(K,s)}. \tag{11}$$

In particular, in Proposition 2 we establish that

$$\mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)} \subseteq \mathbf{N} \cap \mathbf{P}^{(K,s)}$$

holds. However, in general

$$\mathbf{N} \cap \mathbf{P}^{(K,s)} \nsubseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$$

as the following example shows. In fact, it is clear that the matrix $A = iI_2$ is normal, and considering $K = I_2$, we have that A is $\{K, 5\}$-potent and $A^5 \neq A^*$.

Furthermore, in general

(a) $\mathbf{N} \cap \mathbf{\Omega}^{(s+1)} \nsubseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as matrix (4) shows by setting $s = 2$).

(b) $\mathbf{P}^{(s+2)} \cap \mathbf{P}^{(K,s)} \nsubseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{12}$$

shows by considering $K = I_2$ and $s = 1$).

(c) $\mathbf{\Omega}^{(s+1)} \cap \mathbf{P}^{(K,s)} \nsubseteq \mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)}$ (as the same example used in item (b) shows).

(d) $\mathbf{GP}^{(s)} \nsubseteq K\mathbf{H} \cap \mathbf{P}^{(K,s)}$ (as matrix (4) shows by setting $s = 2$).

A relationship between $\mathbf{N}$ and $K\mathbf{N}$ is established for matrices in $\mathbf{P}^{(K,s)}$.

Theorem 10 *Let $K \in \mathbf{H}$. Then $\mathbf{N} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{N}$.*

Proof. Let $A \in \mathbf{N} \cap \mathbf{P}^{(K,s)}$. Recalling that $p = (s+1)^2 - 1$, Theorems 1 and 2 assure that

$$A = \sum_{j=1}^{p} \omega^j P_j$$

where P_j's are orthogonal disjoint projectors satisfying condition (3). Note that $P_{j_0} = O$ if there exists $j_0 \in \{1, 2, \dots, p\}$ such that $\omega^{j_0} \notin \sigma(A)$ and moreover, $P_0 = O$ when $0 \notin \sigma(A)$. Then,

$$\begin{aligned} KA(KA)^* &= \left(\sum_{j=1}^{p} \omega^j K P_j\right)\left(\sum_{j=1}^{p} \omega^{-j} P_j K\right) \\ &= \sum_{j=1}^{p} K P_j K = \sum_{j=1}^{p-1} P_{j(s+1)} + P_p = \sum_{i=1}^{p} P_i. \end{aligned}$$

In the last equality we have used relation (3). A similar computation yields

$$(KA)^* KA = \sum_{j=1}^{p} P_j.$$

Thus, $KA \in \mathbf{N}$. ■

Nevertheless, $\mathbf{N}$ is not closed under multiplication by K. In general, when K is Hermitian, $\mathbf{N} \cap \mathbf{P}^{(K,s)} \subsetneq K\mathbf{N}$. In fact, the matrix

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$$

is not normal and $KA \in \mathbf{N}$ where

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Next, we characterize the $\{K, s+1\}$-potent matrices that are also normal matrices.

Theorem 11 *The following equality holds:*

$$\mathbf{N} \cap \mathbf{P}^{(K,s)} = \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in \mathbf{N}$ if and only if $A \in \mathbf{GP}^{((s+1)^2-3)}$. In fact, if $A \in \mathbf{N}$, by Theorem 1 and Theorem 2, we have that all the distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of A belong to $\{0\} \cup \Omega_{(s+1)^2-1}$ and there exist disjoint orthogonal projectors $P_1, P_2, \dots, P_k$ such that $A = \sum_{j=1}^{k} \lambda_j P_j$. Under these conditions, it is easy to see that

$$A^* = \sum_{j=1}^{k} \bar{\lambda}_j P_j \quad \text{and} \quad A^m = \sum_{j=1}^{k} \lambda_j^m P_j$$

for each $m \in \{1, 2, 3, \dots\}$. For all $\lambda_j \in \Omega_{(s+1)^2-1}$, we get $\lambda_j \bar{\lambda}_j = 1 = \lambda_j \lambda_j^{(s+1)^2-2}$ and then

$$A^* = \sum_{j=1}^{k} \lambda_j^{-1} P_j = \sum_{j=1}^{k} \lambda_j^{(s+1)^2-2} P_j = A^{(s+1)^2-2}.$$

This proves that $A \in \mathbf{GP}^{((s+1)^2-3)}$.

The converse is obvious because A^* is a power of A. ■

Theorems 9 and 11 and the equality (11) imply

$$\mathbf{GP}^{(s)} \cap \mathbf{P}^{(K,s)} = \mathbf{\Omega}^{(s+1)} \cap \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} = \mathbf{P}^{(s+2)} \cap \mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)}.$$

The following result is a consequence of Theorems 10 and 11.

Corollary 1 *Let $K \in \mathbf{H}$. Then*

$$\mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{N} \cap \mathbf{P}^{(K,s)}.$$

Observe that, in general, the opposite inclusion in Corollary 1 is not true. Consider the matrices A and K given in (12); $A \in \mathbf{P}^{(K,1)}$ and $A \in K\mathbf{N}$, but $A \notin \mathbf{GP}^{(1)}$. In general,

$$\mathbf{GP}^{((s+1)^2-3)} \cap \mathbf{P}^{(K,s)} \subsetneq K\mathbf{N} \cap \mathbf{P}^{(K,s)}.$$

We now derive a relation between $K\mathbf{N}$ and $K\mathbf{GP}^{(m)}$.

Theorem 12 *The following equality holds:*

$$K\mathbf{N} \cap \mathbf{P}^{(K,s)} = K\mathbf{GP}^{(2s-2)} \cap \mathbf{P}^{(K,s)}.$$

Proof. Let $A \in \mathbf{P}^{(K,s)}$. We have to prove that $A \in K\mathbf{N}$ if and only if $A \in K\mathbf{GP}^{(2s-2)}$. In fact, if $KA \in \mathbf{N}$, by Theorems 1, 2 and Lemma 1 (c), we have that all the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of KA belong to $\{0\} \cup \Omega_{2s}$ and there exist disjoint orthogonal projectors $P_1, P_2, \ldots, P_k$ such that $KA = \sum_{j=1}^{k} \lambda_j P_j$. Under these conditions, it is easy to see that $(KA)^* = \sum_{j=1}^{k} \bar{\lambda}_j P_j$ and $(KA)^m = \sum_{j=1}^{k} {\lambda_j}^m P_j$ for all $m \in \{1, 2, 3, \ldots\}$. For all $\lambda_j \in \Omega_{2s}$, we get $\lambda_j \bar{\lambda}_j = 1 = \lambda_j^{2s}$ and then

$$(KA)^* = \sum_{j=1}^{k} \bar{\lambda}_j P_j = \sum_{j=1}^{k} \lambda_j^{-1} P_j = \sum_{j=1}^{k} \lambda_j^{2s-1} P_j = (KA)^{2s-1}.$$

This proves that $KA \in \mathbf{GP}^{(2s-2)}$.

The converse is obvious because $(KA)^*$ is a power of KA. ■

Remark 1 *We note that $\mathbf{GP}^{(0)} = \mathbf{H}$. From Theorem 12 we can deduce that*

$$K\mathbf{N} \cap \mathbf{P}^{(K,1)} = K\mathbf{H} \cap \mathbf{P}^{(K,1)}.$$

4 Further results

In this section we present some properties similar to those studied so far, but we want to focus on the (anti-)commutative matrices, skew-Hermitian matrices, and matrices A for which $A^{s+1} = -A$ for some positive integer s.

4.1 Analysis of $\mathbf{P}^{(K,s)}$ through H, SH, $\mathbf{C}^{(K,m)}$, and $\mathbf{SP}^{(s)}$

We start this subsection by defining the set of skew-Hermitian matrices and the set of all matrices that commute (anti-commute) with the matrix K:

$$\mathbf{SH} = \{A \in \mathbb{C}^{n\times n} : \ A^* = -A\}$$

$$\mathbf{C}^{(K,m)} = \{A \in \mathbb{C}^{n\times n} : \ AK = mKA\} \text{ where } m \in \{+,-\}.$$

We also introduce the set given by:

$$\mathbf{SP}^{(s)} = \{A \in \mathbb{C}^{n\times n} : \ A^{s+1} = -A\}.$$

Now, we can state the following results.

Theorem 13 *The following equalities hold:*

(a) $\mathbf{H} \cap \mathbf{P}^{(K,s)} = \mathbf{C}^{(K,+)} \cap \mathbf{T}$ *where*

$$\mathbf{T} = \begin{cases} \mathbf{P}^{(1)} \cap \mathbf{GP}^{(1)} & \textit{when } s \textit{ is odd,} \\ \mathbf{P}^{(2)} \cap \mathbf{GP}^{(2)} & \textit{when } s \textit{ is even.} \end{cases}$$

(b) *(i)* $\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \{O\}$ *when s is odd.*

(ii) $\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \mathbf{SP}^{(2)} \cap \mathbf{GP}^{(2)} \cap \mathbf{T}$ *where*

$$\mathbf{T} = \begin{cases} \mathbf{C}^{(K,+)} & \textit{when } s = 4t, \\ \mathbf{C}^{(K,-)} & \textit{when } s = 4t+2, \end{cases}$$

for $t \in \{1, 2, 3, \dots\}$.

Proof. In order to show (a) we suppose that $A^* = A$ and $KA^{s+1}K = A$. Then

$$\sigma(A) \subseteq \mathbb{R} \cap (\{0\} \cup \Omega_{(s+1)^2-1}) = \begin{cases} \{0,1\} & \text{when } s \text{ is odd,} \\ \{0,1,-1\} & \text{when } s \text{ is even,} \end{cases}$$

so,

$$A = \begin{cases} A^2 & \text{when } s \text{ is odd,} \\ A^3 & \text{when } s \text{ is even,} \end{cases}$$

and thus, we get $KAK = A^{s+1} = A$. The converse can be easily checked.

Item (b) follows trivially when s is odd. If s is even and $A \in \mathbf{SH} \cap \mathbf{P}^{(K,s)}$ then $\sigma(A) \subseteq i\mathbb{R} \cap (\{0\} \cup \Omega_{s(s+2)}) = \{0, i, -i\}$. Hence $A^3 = -A$, and so, when $s = 4t$ then

$$KAK = A^{s+1} = A^{4t+1} = -A^3 = A,$$

for every $t \in \{1, 2, 3, \dots\}$. In a similar way, the case $s = 4t+2$ follows from $A^{4t} = -A^2$. As before, the converse can be easily checked. ∎

Furthermore, the $\{K, s+1\}$-potent matrices which are also in $K\mathbf{SH}$ satisfy the following relations.

Theorem 14 *The following statements hold:*

(a) $K\mathbf{SH} \cap \mathbf{P}^{(K,s)} \subseteq K\mathbf{SP}^{(2)}$ *when s is even.*

(b) $K\mathbf{SH} \cap \mathbf{P}^{(K,s)} = \{O\}$ *when s is odd.*

Proof. Let $A \in K\mathbf{SH}$. We first observe that $KA \in \mathbf{SH}$ and thus $\sigma(KA) \subseteq i\mathbb{R}$. Now, for $A \in \mathbf{P}^{(K,s)}$ we get $(KA)^{2s+1} = KA$ and so, $\sigma(KA) \subseteq \{0\} \cup \Omega_{2s}$. Hence, the case for s odd follows trivially. When s is even, we can see that $\sigma(KA) \subseteq \{0, i, -i\}$. Since KA is diagonalizable, we obtain that $(KA)^3 = -KA$, which means that $A \in K\mathbf{SP}^{(2)}$. ■

Note that, in general, the inclusion in (a) is strict as can be checked by using the example given in (12).

We can also observe that $K\mathbf{SP}^{(2)} \subseteq \mathbf{SP}^{(s+2)}$ because $(KA)^3 = -KA$ implies that $A^{s+3} = -A$. In general, in this case the inclusion is strict because the 1×1 matrix defined by $A = \begin{bmatrix} \omega \end{bmatrix}$ satisfies the required conditions assuming that $\omega^3 = -1 \neq \omega^2$ holds for $K = \begin{bmatrix} 1 \end{bmatrix}$.

Finally, when $K^* = K$ we can observe that the following conditions are equivalent:

(a) $KA \in \mathbf{SH}$.

(b) $KA^*K = -A$.

(c) $AK \in \mathbf{SH}$.

Also, the condition $A \in \mathbf{SH}$ is equivalent to $K(KA)^*K = -KA$.

In [12], $\{K, s+1\}$-potent matrices were only defined for positive integers s, and $s = 0$ was excluded. The reason is that Theorem 2 does not give any information for the case $s = 0$. So, the next subsection provides with some results valid for $s = 0$.

4.2 New results on K-generalized centrosymmetric matrices

First, we observe that $\mathbf{P}^{(K,0)} = \mathbf{C}^{(K,+)}$, which is the set of the K-generalized centrosymmetric matrices. We can easily deduce the following result for the particular class of diagonalizable matrices (where the spectral decomposition can certainly be applied).

Theorem 15 *Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with spectral decomposition given by $A = \sum_{i=1}^{k} \lambda_i P_i$ as in Theorem 1, and let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix. Then $A \in \mathbf{C}^{(K,+)}$ if and only if $KP_i = P_iK$, for all $i \in \{1, 2, \dots, k\}$.*

We now derive some new results for the K-generalized centrosymmetric matrices from those established in sections 2 and 3. Henceforth, we can get these new relations by setting $s = 0$ in all the corresponding results after doing simple verifications. In what follows, we present the results deduced from section 2.

Theorem 16 *The following statements hold:*

(a) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \subsetneq \mathbf{H}$.

(b) $\mathbf{C}^{(K,+)} \cap \mathbf{H} \subsetneq \mathbf{H}^{(K)}$.

(c) $\mathbf{H}^{(K)} \cap \mathbf{H} \subsetneq \mathbf{C}^{(K,+)}$.

(d) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \cap \mathbf{H} = \mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} = \mathbf{C}^{(K,+)} \cap \mathbf{H} = \mathbf{H}^{(K)} \cap \mathbf{H}$.

(e) $\mathbf{C}^{(K,+)} \cap \mathbf{H}^{(K)} \cap \mathbf{I}^{(1)} \subsetneq \mathbf{U}$.

(f) $\mathbf{C}^{(K,+)} \cap \mathbf{H} \cap \mathbf{I}^{(1)} \subsetneq \mathbf{U}$.

(g) $\mathbf{C}^{(K,+)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{I}^{(1)}$.

(h) $\mathbf{I}^{(1)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{C}^{(K,+)}$.

(i) $\mathbf{P}^{(2)} \cap K\mathbf{I}^{(1)} \subsetneq \mathbf{C}^{(K,+)}$.

(j) $\mathbf{U} \cap \mathbf{C}^{(K,+)} = \mathbf{U} \cap (\mathbf{C}^{(K,+)})^*$.

The matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

allows us to show that the inclusion in item (e) is strict for $K = I_2$.

The results deduced from section 3 are the following.

Theorem 17 *Let* $K \in \mathbf{H}$. *Then*

(a) $\mathbf{H} \cap \mathbf{C}^{(K,+)} = K\mathbf{H} \cap \mathbf{C}^{(K,+)}$.

(b) $\mathbf{N} \cap \mathbf{\Omega}^{(1)} \cap \mathbf{C}^{(K,+)} = \mathbf{N} \cap \mathbf{P}^{(2)} \cap \mathbf{C}^{(K,+)} = K\mathbf{H} \cap \mathbf{C}^{(K,+)}$.

(c) $\mathbf{N} \cap \mathbf{C}^{(K,+)} \subsetneq K\mathbf{N}$.

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