Matrices A such that $RA = A^{s+1}R$ when $R^k = I$

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Abstract

This paper examines matrices $A \in \mathbb{C}^{n\times n}$ such that $RA = A^{s+1}R$ where $R^k = I$, the identity matrix, and where s and k are nonnegative integers with $k \geq 2$. Spectral theory is used to characterize these matrices. The cases $s = 0$ and $s \geq 1$ are considered separately since they are analyzed by different techniques.

Keywords: Potent matrix; idempotent matrix; spectrum; Jordan form; involutory matrix.
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1 Introduction and Preliminaries

Let R_1 be the square matrix with ones on the cross diagonal and zeros elsewhere; note that R_1 is often called the *centrosymmetric permutation matrix*. A matrix A_1 that commutes with R_1 is called a *centrosymmetric matrix* [12]. Any square matrix R_2 satisfying $R_2^2 = I$, where I is the identity matrix, is called an *involution* or an *involutory matrix*. The real eigenvalues of nonnegative matrices that commute with a real involution were studied in [13]. It is well-known that if P is a permutation matrix, then $P^k = I$ for some positive integer k. Matrices that commute with a permutation matrix P were studied in [8]. A well-known and important class of matrices that commute with a permutation matrix are the *circulant matrices* [3, 6], consisting of all matrices that commute with R_3, where R_3 is the irreducible permutation matrix with ones on the first superdiagonal, a one in the lower left-hand corner, and zeros elsewhere. If A is an $n \times n$ circulant matrix, then $R_3A = AR_3$ can be expressed as $R_3AR_3^{n-1} = A$ since $R_3^n = I_n$, the $n \times n$ identity matrix.

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A matrix $R \in \mathbb{C}^{n\times n}$ such that $R^k = I_n$ for some positive integer k with $k \geq 2$ is called a $\{k\}$-*involutory matrix* [10, 11]. Throughout this paper, all matrices R will be $\{k\}$-involutory. It is clear that when $k = 2$, such an R is either $\pm I_n$, or else a nontrivial involution. Also, $k = n$ is the smallest positive integer for which R_3 is $\{k\}$-involutory, and this guarantees that there are nontrivial, nondiagonal $\{k\}$-involutory matrices for all integers k and n with $n \geq 2$ and $2 \leq k \leq n$. The matrix $\exp(\frac{2\pi i}{k})\, I_n$ is $\{k\}$-involutory for all positive integers n and all integers $k \geq 2$, and, thus, it should be clear that there are $\{k\}$-involutory matrices for which $k > n$ must occur. Finally, we always assume that $R \neq I_n$, and hence, if $R^k = I_n$, then $k \geq 2$.

This paper is focused on the study of the $\{R, s+1, k\}$-potent matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is called an $\{R, s+1, k\}$-*potent matrix* if $RA = A^{s+1}R$ for some nonnegative integer s and some $\{k\}$-involutory matrix R. Note that the cases, $k = 2$ and $s \geq 1$, and $k \geq 2$ and $s = 0$, have already been analyzed in [7, 14], respectively. Spectral properties of matrices related to the $\{R, s+1, k\}$-potent matrices are presented in [4, 9]. Other similar classes of matrices and their spectral properties have been studied in [5, 9, 10, 11].

In this paper characterizations of $\{R, s+1, k\}$-potent matrices are given, with the cases $s \geq 1$ and $s = 0$ treated separately. In the first case, the concept of $\{t+1\}$-group involutory matrix will be used. These matrices were introduced in [2] for $t = 2$, and the definition can be extended for any integer $t > 2$ as follows: A matrix $A \in \mathbb{C}^{n\times n}$ is called a $\{t+1\}$-*group involutory matrix* if $A^{\#} = A^{t-1}$, where $A^{\#}$ denotes the group inverse of A. We recall that the group inverse of a square matrix A is the only matrix $A^{\#}$ (when it exists) satisfying: $AA^{\#}A = A$, $A^{\#}AA^{\#} = A^{\#}$, $AA^{\#} = A^{\#}A$. Moreover, $A^{\#}$ exists if and only if $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ [1].

2 Main results

Clearly, I_n and $n \times n$ zero matrix O are always $\{R, s+1, k\}$-potent matrices. For any given positive integers n, s and k (with $k \geq 2$), and for any given $n \times n$ $\{k\}$-involutory matrix R, there exists a nontrivial $\{R, s+1, k\}$-potent matrix. Consider $A = \omega I_n$ where ω is a primitive s^{th} root of unity. Note that when $s = 0$, $A = R$ is an $\{R, 1, k\}$-potent matrix that is nontrivial when R is nontrivial.

The question arising in this paper follows from the observation that if $A \in \mathbb{C}^{n\times n}$ is an $\{R, s+1, k\}$-potent matrix, then $A^{(s+1)^k} = A$. To see this, note that from $RA = A^{s+1}R$, it follows that $R^2A = R(AA^sR) = A^{s+1}RA^sR = A^{s+1}A^{s+1}RA^{s-1}R = \cdots = A^{(s+1)(s+1)}R^2$, and similarly, $R^kA = A^{(s+1)^k}R^k$. (The equality in the observation is uninformative when $s = 0$; the $s = 0$ case will be addressed in Subsection 2.2.) The necessity of $A^{(s+1)^k} = A$ is clear, but is this condition sufficient to guarantee that a matrix A is an $\{R, s+1, k\}$-potent matrix for an arbitrary $\{k\}$-involution R? Not surprisingly, since R does not appear in the equality, the condition is not sufficient as the following example demonstrates:

$$A = \exp\left(\frac{2\pi i}{3}\right) I_2, \qquad R = \operatorname{diag}(i, -1), \quad s = 1, \quad k = 4.$$

Consequently, we seek a complementary condition that in conjunction with $A^{(s+1)^k} = A$ implies A is an $\{R, s+1, k\}$-potent matrix.

2.1 The case $s \geq 1$

Assume that A is an $\{R, s+1, k\}$-potent matrix. Let $n_s = (s+1)^k - 1$. Since $A^{(s+1)^k} = A$, the polynomial $t^{(s+1)^k} - t$, whose roots all have multiplicity 1, is divisible by the minimal polynomial of A. Thus, A is diagonalizable with spectrum $\sigma(A) \subseteq \{0\} \cup \{\omega^1, \omega^2, \ldots, \omega^{n_s-1}, \omega^{n_s} = 1\}$ where $\omega := \exp\left(\frac{2\pi i}{n_s}\right)$. Hence, the spectral theorem [1] assures that there exist disjoint projectors

$$P_0, P_1, P_2, \ldots, P_{n_s-1}, P_{n_s}$$

such that

$$A = \sum_{j=1}^{n_s} \omega^j P_j \qquad \text{and} \qquad \sum_{j=0}^{n_s} P_j = I_n, \tag{1}$$

where $P_{j_0} = O$ if there exists $j_0 \in \{1, 2, \ldots, n_s\}$ such that $\omega^{j_0} \notin \sigma(A)$ and moreover that $P_0 = O$ when $0 \notin \sigma(A)$.

Pre-multiplying the previous expressions given in (1) by the matrix R and post-multiplying by R^{-1} gives

$$RAR^{-1} = \sum_{j=1}^{n_s} \omega^j R P_j R^{-1}$$

and

$$\sum_{j=0}^{n_s} R P_j R^{-1} = I_n. \tag{2}$$

It is clear that the nonzero RP_jR^{-1} are disjoint projectors for each $j = 0, 1, \ldots, n_s$. From (1),

$$A^{s+1} = \sum_{j=1}^{n_s} \omega^{j(s+1)} P_j$$

because the nonzero P_j are disjoint projectors.

Let $S = \{1, 2, \ldots, n_s - 1\}$. Now consider $\varphi : S \cup \{0\} \to S \cup \{0\}$ as the function defined by $\varphi(j) = b_j$, where b_j is the smallest nonnegative integer such that $b_j \equiv j(s+1)$ [mod n_s]. Then φ is a bijection [7]. It follows that

$$A^{s+1} = \sum_{j=1}^{n_s-1} \omega^{\varphi(j)} P_j \; + \; P_{n_s}$$

and since A is an $\{R, s+1, k\}$-potent matrix,

$$A^{s+1} = RAR^{-1}.$$

Hence,

$$\sum_{i=1}^{n_s-1} \omega^i RP_iR^{-1} + RP_{n_x}R^{-1} = \sum_{j=1}^{n_s-1} \omega^{\varphi(j)} P_j + P_{n_s}.$$

Since φ is a bijection, for each $i \in S$, there exists a unique $j \in S$ such that $i = \varphi(j)$. From the uniqueness of the spectral decomposition, it follows that for every $i \in S$, there exists a unique $j \in S$ such that

$$RP_iR^{-1} = RP_{\varphi(j)}R^{-1} = P_j. \tag{3}$$

It is clear that uniqueness also implies that

$$RP_{n_s}R^{-1} = P_{n_s}. \tag{4}$$

Finally, from (1)

$$P_0 = I_n - \sum_{j=1}^{n_s} P_j.$$

Taking into account (2) and the definition of the bijection φ,

$$RP_0R^{-1} = P_0 \tag{5}$$

because of the uniqueness of the spectral decomposition. Observe that in the case where there exists $j_0 \in S$ such that $\omega^{j_0} \notin \sigma(A)$, it has been indicated that $P_{j_0} = O$. In this situation, $P_{\varphi(j_0)} = RP_{j_0}R^{-1} = O$ is also true.

Conversely, assuming $A^{(s+1)^k} = A$ and that the relationships on the projectors obtained in (3), (4), and (5) hold, we can consider

$$A = \sum_{j=1}^{n_s} \omega^j P_j \tag{6}$$

It is now easy to check that $A^{s+1} = RAR^{-1}$.

The matrices P_j's satisfying relations (3), (4), and (5) where

$$P_0, P_1, \dots, P_{n_s}$$

are the projectors appearing in the spectral decomposition of A associated to the eigenvalues

$$0, \omega^1, \dots, \omega^{n_x-1}, 1,$$

are said to satisfy condition $(\mathcal{P})$. Then, the complementary condition we were looking for is condition $(\mathcal{P})$.

These results are summarized in what follows. Before that, note

$$\operatorname{rank}(A) = \operatorname{rank}(A^{(s+1)^k}) \leq \operatorname{rank}(A^2) \leq \operatorname{rank}(A)$$

when $A^{(s+1)^k} = A$. Then, in this case, the group inverse of A exists, and it is easy to check that $A^{\#} = A^{(s+1)^k-2}$, that is, A is a $\{(s+1)^k\}$-group involutory matrix.

The main result of this subsection is now stated.

Theorem 1 *Let $R \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix, $s \in \{1,2,3,\dots\}$, $n_s = (s+1)^k - 1$, and, $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

1. *A is $\{R, s+1, k\}$-potent.*
2. *$A^{(s+1)^k} = A$ and there exist $P_0, P_1, P_2, \dots, P_{n_s}$ satisfying condition ($\mathcal{P}$).*
3. *A is diagonalizable,*
$$\sigma(A) \subseteq \{0\} \cup \{\omega^1, \omega^2, \dots, \omega^{n_s} = 1\},$$
with $\omega = \exp\left(\frac{2\pi i}{n_s}\right)$, and there exist $P_0, P_1, P_2, \dots, P_{n_s}$ satisfying condition ($\mathcal{P}$).
4. *A is an $\{(s+1)^k\}$-group involutory matrix and there exist $P_0, P_1, P_2, \dots, P_{n_x}$ satisfying condition ($\mathcal{P}$).*

From the definition of an $\{R, s+1, k\}$-potent matrix, if A is $\{R, s+1, k\}$-potent, then A is similar to A^{s+1}. Hence, the uniqueness of the spectral decomposition of A allows us to state the correspondence between the distinct eigenvalues of A as well as between their corresponding projectors. Specifically:

Corollary 2 *Let $R \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix, $s \in \{1,2,3,\dots\}$, and $A \in \mathbb{C}^{n\times n}$ with spectrum*
$$\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}, \qquad \text{with } m \geq 1$$
where the λ_h are the distinct eigenvalues of A. Then A is $\{R, s+1, k\}$-potent if and only if A is diagonalizable and for each $i \in \{1,2,\dots,m\}$ there is a unique $j \in \{1,2,\dots,m\}$ such that $\lambda_i = \lambda_j^{s+1}$ and $P_iR = RP_j$ where $P_1, P_2, \dots, P_m$ are the projectors satisfying condition ($\mathcal{P}$).

Note that from condition (c) in Theorem 1 we know if $\sigma(A) \nsubseteq \{0\} \cup \{\omega^0, \omega^1, \dots, \omega^{(s+1)^k-2}\}$ then A is not $\{R, s+1, k\}$-potent. Even more, Corollary 2 gives us another simple sufficient condition for A to not be $\{R, s+1, k\}$-potent. The following example illustrates this situation. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is obvious that the eigenvalues of A are its diagonal elements. Then, we can conclude that A is not $\{R, 3, 2\}$-potent because cubing the eigenvalue $-i$ of A gives the value i which is not an eigenvalue.

The general situation is given in the following result.

Corollary 3 *Let $R \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix and $s \in \{1,2,3,\dots\}$. If the matrix $A \in \mathbb{C}^{n\times n}$ has an eigenvalue λ such that one of the following conditions holds:*

1. $\lambda^{s+1} \notin \sigma(A)$
2. $\lambda^{(s+1)^k} \neq \lambda$

then A is not $\{R, s+1, k\}$-potent.

Up to now we have considered $s \in \{1,2,3,\dots\}$ where the diagonalizability of A is a consequence of the fact that A is $\{R, s+1, k\}$-potent. The case $s = 0$ is now examined.

2.2 The case $s = 0$

This situation corresponds to those matrices $A \in \mathbb{C}^{n\times n}$ such that $RA = AR$ and $R^k = I_n$. Such matrices are called $\{R,k\}$-generalized centrosymmetric matrices or, for consistency, $\{R,1,k\}$-potent matrices. These matrices are in general not diagonalizable, as is shown by the following example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad k = 2.$$

When the diagonalizability is assumed, the uniqueness of the spectral decomposition (see [1], pp. 62) gives the following result.

Theorem 4 *Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with m distinct eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_m$, and spectral decomposition $A = \sum_{i=1}^{m} \lambda_i P_i$. Suppose that $R \in \mathbb{C}^{n\times n}$ is $\{k\}$-involutory for some integer $k \geq 2$. Then A is an $\{R,k\}$-generalized centrosymmetric matrix if and only if $RP_i = P_iR$ for all $i \in \{1,2,\dots,m\}$.*

Note that all of the cases $k < m$, $k = m$, and $k > m$ can occur as the following examples show:

1. If $A = \mathrm{diag}(1,2,3,2,1)$ and R is the 5×5 centrosymmetric permutation matrix then $AR = RA$ and $k = 2 < 3 = h$.
2. If $A = \mathrm{diag}(1,2)$ and $R = \mathrm{diag}(1,-1)$ then $AR = RA$ and $k = 2 = h$.
3. If $A = I_2$ and $R = \exp\left(\frac{2\pi i}{25}\right) I_2$ then $AR = RA$ and $k = 25 > 1 = h$.

Suppose $R \in \mathbb{C}^{n\times n}$, $R^k = I$, and R has n distinct eigenvalues. Then $k \geq n$, and R is diagonalizable. Further, $AR = RA$ exactly when R and A are simultaneously diagonalizable. Consequently, if A is an $\{R,k\}$-potent matrix then A is diagonalizable. Further, when $k = n$, the spectrum of R is the complete set of n^{th} roots of unity, so R is similar to the $n\times n$ circulant permutation matrix

R_3. That is, there is a nonsingular matrix Q such that $QRQ^{-1} = R_3$. Further, $AR = RA$ exactly when QAQ^{-1} is a circulant matrix (see for example Theorem 3.1.1 in [3]). Next, we investigate the cases where R does not have n distinct eigenvalues.

First, we present a classic result, and we include its proof for the sake of completeness.

Lemma 5 *For each $\{k\}$-involutory matrix $R \in \mathbb{C}^{n\times n}$, there exists an integer t with $1 \le t \le n$ and a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that the Jordan form of R, $J_R = Q^{-1}RQ$ is the diagonal matrix $J_R = \operatorname{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}, \ldots, \omega_t I_{n_t})$, where the ω_i are distinct k^{th} roots of unity and $n_1 + n_2 + \cdots + n_t = n$.*

Proof. Assume that $k > 1$. Let $\omega = \exp\left(\frac{2\pi i}{k}\right)$. Since $R^k = I_n$, the minimum polynomial $m_R(\lambda)$ of R divides $\lambda^k - 1 = \prod_{j=1}^{k}\left(\lambda - \omega^j\right)$, and consequently, every factor of $m_R(\lambda)$ must be a distinct linear factor. It follows that R is diagonalizable, and hence, that J_R has the specified form where the ω_j are distinct elements from $\{\omega^1, \omega^2, \ldots, \omega^k\}$ whose sum of multiplicities is n. ■

Theorem 6 *Suppose that $R \in \mathbb{C}^{n\times n}$ is a $\{k\}$-involutory matrix with nonsingular matrix Q and Jordan form J_R as given in the preceding lemma. Then $AR = RA$ for $A \in \mathbb{C}^{n\times n}$ if and only if the blocks of $Y = Q^{-1}AQ$ satisfy $Y_{ij} = O$ when $i \ne j$, and $Y_{ii} \in \mathbb{C}^{n_i\times n_i}$ is arbitrary for $1 \le i, j \le t$. The matrices Y contain exactly*

$$d = \sum_{j=1}^{t} n_i^2$$

arbitrary parameters, so $C(R) = \{A \in \mathbb{C}^{n\times n} : RA = AR\}$ is a vector space of dimension d. Further,

$$C(R) \simeq \bigoplus_{i=1}^{t} \mathbb{C}^{n_i\times n_i}$$

where $\mathbb{C}^{n_i\times n_i}$ is the full matrix algebra of $n_i \times n_i$ matrices over the complex field and where the isomorphism sends A to $Q^{-1}AQ$.

Proof. $AR = RA$ if and only if $Y = Q^{-1}AQ$ satisfies $YJ_R = J_R Y$. For $1 \le i, j \le t$,

$$Y_{ij}\left(\omega_j I_{n_j}\right) = \left(\omega_i I_{n_i}\right) Y_{ij}.$$

Since $\omega_i \ne \omega_j$ when $i \ne j$, $Y_{ij} = O$. When $i = j$, Y_{ij} is an arbitrary $n_i \times n_i$ matrix. Thus, Y is a direct sum of arbitrary submatrices containing $\sum_{j=1}^{t} n_i^2$ arbitrary entries. ■

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