Characterizations of $\{K, s+1\}$-Potent Matrices and Applications

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Abstract

Recently, situations where a matrix coincides with some of its powers have been studied. This kind of matrices is related to the generalized inverse matrices. On the other hand, it is possible to introduce another class of matrices that involve an involutory matrix, generalizing the well-known idempotent matrix, widely useful in many applications. In this paper, we introduce a new kind of matrices called $\{K, s+1\}$-potent, as an extension of the aforementioned ones. First, different properties of $\{K, s+1\}$-potent matrices have been developed. Later, the main result developed in this paper is the characterization of this kind of matrices from a spectral point of view, in terms of powers of the matrix, by means of the group inverse and, via a block representation of a matrix of index 1. Finally, an application of the above results to study linear combinations of $\{K, s+1\}$-potent matrices is derived.

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1 Introduction

An involutory matrix $K \in \mathbb{C}^{n\times n}$ is a matrix that is its own inverse, that is, when applied twice, brings one back to the starting point. Algebraically that is $K^2 = I_n$, where I_n denotes the identity matrix of size $n \times n$. Some examples of involutory matrices are: one of the three classes of elementary matrix (namely the row-interchange elementary matrix) which corresponds to permutation matrices, the signature matrices, an orthogonal matrix which is also symmetric, etc. Involutions have been applied to different areas. For example, Euclidean geometry (e.g., reflection against a plane), group theory (e.g., classification of finite simple groups), ring theory (e.g., taking the transpose in a matrix ring), and so on.

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On the other hand, in cryptography, it was suggested to use an involutory matrix as a key while encrypting with the Hill Cipher. In this case, the use of involutory matrices to eliminate the computation of matrix inverses for Hill decryptions is very useful. This means that the same tool could be used for both encryption and decryption of messages [7].

It is well-known that an exchange matrix $J = [j_{k,t}] \in \mathbb{C}^{n\times n}$ is defined as a matrix with 1's along the cross-diagonal (i.e., $j_{k,n-k+1} = 1$, $1 \leq k \leq n$) and 0's everywhere else. Matrices $A \in \mathbb{C}^{n\times n}$ such that $JA = AJ$ are called centrosymmetric and have been widely studied because of their applications in wavelets, partial differential equations, and other areas [6]. A matrix $A \in \mathbb{C}^{n\times n}$ is centrosymmetric with respect to an involutory matrix K if it satisfies $KA = AK$ and this class of matrices has been studied in [9].

In previous years, situations where a square matrix A equals some of its powers $s+1$ were studied [2]. In particular, $A^{s+1} = A$ for some $s = 2, 3, \ldots$ if and only if $A^{\#} = A^{s-1}$, where $A^{\#}$ represents the group inverse of the matrix $A \in \mathbb{C}^{n\times n}$ [3]. We recall that for a given matrix $A \in \mathbb{C}^{n\times n}$, we call the group inverse of A, and we denote by $A^{\#}$, the matrix satisfying the following conditions $AA^{\#}A = A$, $A^{\#}AA^{\#} = A^{\#}$, and $AA^{\#} = A^{\#}A$. The matrix $A^{\#}$ exists if and only if A and A^2 have the same rank. If it exists, it is unique [1].

Motivated by all these ideas we introduce the following definition.

Definition 1 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix and $s \in \{1, 2, 3, \ldots\}$. A matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent if it satisfies*

$$KA^{s+1}K = A. \tag{1}$$

Clearly, this definition includes some extensions of the situations mentioned above. It is clear that when $K = I_n$ the concept of $\{K, s+1\}$-potent coincides with that of $\{s+1\}$-potent matrix (that is, $A^{s+1} = A$), which has been already considered and it is not relevant in this paper [2].

We will denote by Ω_k the set of all k^{th} roots of unity with k a positive integer which is a multiplicative group. If we define $\omega_k = e^{2\pi i/k}$ then $\Omega_k = \{\omega_k, \omega_k^2, \ldots, \omega_k^k\}$.

This paper is organized as follows. In Section 2, properties of $\{K, s+1\}$-potent matrices are obtained. Specifically, we show that for each positive integer n there exists at least one matrix of size $n \times n$ belonging to this class. We also present a method to construct an infinite number of $\{K, s+1\}$-potent matrices from only one of them. Furthermore, sums, products, direct sums and inverses of $\{K, s+1\}$-potent matrices have been studied. In Section 3, characterization of $\{K, s+1\}$-potent matrices are presented from a spectral point of view, in terms of powers of the matrix, by means of the group inverse and, via a block representation of a matrix of index 1. Finally, in Section 4, as an application we give conditions under which a linear combination of two commuting $\{K, s+1\}$-potent matrices is $\{K, s+1\}$-potent.

2 On the existence and properties of $\{K, s+1\}$-potent matrices

The first question is related to the existence of the $\{K, s+1\}$-potent matrices and it is analyzed in the following result.

Theorem 1 *For each $n \in \{1, 2, 3, \dots\}$, there exists at least one matrix $A \in \mathbb{C}^{n\times n}$ such that A is $\{K, s+1\}$-potent for each involutory matrix K and for each $s \in \{1, 2, 3, \dots\}$.*

Proof. Let $n \in \{1, 2, 3, \dots\}$. If we consider the matrix $A = \omega I_n$ being $w \in \Omega_s$, one has that $A^{s+1} = w^{s+1} I_n = w I_n = A$ and then $KA^{s+1}K = KAK = wK^2 = A$. ■

We now establish properties concerning $\{K, s+1\}$-potent matrices.

Lemma 1 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3\dots\}$, and $A \in \mathbb{C}^{n\times n}$.*

(I) Then the following conditions are equivalent:

- *(a) A is a $\{K, s+1\}$-potent matrix.*
- *(b) $KAK = A^{s+1}$.*
- *(c) $KA = A^{s+1}K$.*
- *(d) $AK = KA^{s+1}$.*

(II) If $A \in \mathbb{C}^{n\times n}$ is a $\{K, s+1\}$-potent matrix then $A^{(s+1)^2} = A$.

Proof. From $K^2 = I_n$, multiplying both sides of the equality $KA^{s+1}K = A$ by K, we get $K^2A^{s+1}K^2 = KAK$ and then $A^{s+1} = KAK$. The converse is similar and then the equivalence between (a) and (b) in (I) is proved. The other equalities of (I) follow directly taking into account that $K^{-1} = K$.

By (I) (b) and the definition we have that $A^{(s+1)^2} = (A^{s+1})^{s+1} = (KAK)^{s+1} = KA^{s+1}K = A$ which shows (II) and thus the lemma has been proved. ■

In addition, we present more properties showing when the set of $\{K, s+1\}$-potent matrices is closed under certain operations.

Lemma 2 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3\dots\}$, and $A, B \in \mathbb{C}^{n\times n}$ be two $\{K, s+1\}$-potent matrices. The following properties hold.*

(a) If $s = 1$ then $AB = -BA$ if and only if $A + B$ is a $\{K, 2\}$-potent matrix.

(b) If $AB = BA = O$ then $A + B$ is a $\{K, s+1\}$-potent matrix.

(c) If $AB = BA$ then AB is a $\{K, s+1\}$-potent matrix.

(d) If $t \in \{0\} \cup \Omega_s$ then tA is a $\{K, s+1\}$-potent matrix.

(e) If A is a nonsingular matrix then A^{-1} is a $\{K, s+1\}$-potent matrix.

(f) If K is Hermitian (that is, $K^* = K$) then A^* is $\{K, s+1\}$-potent.

(g) If $W \in \mathbb{C}^{n\times n}$ is nonsingular and $KW = WK$ then WAW^{-1} is $\{K, s+1\}$-potent.

Proof. Set $s = 1$. Since $KA^2K = A$ and $KB^2K = B$, one gets that the condition $K(A+B)^2K = A+B$ is equivalent to $AB = -BA$. Item (a) is then shown. Item (b) is similar to (a). Item (c) follows directly from $K(AB)^{s+1}K = (KA^{s+1}K)(KB^{s+1}K) = AB$ because $AB = BA$. The assumption $t^{s+1} = t$ implies that $K(tA)^{s+1}K = t^{s+1}KA^{s+1}K = tA$. The property related to the nonsingularity of A is true because $A^{-1} = (KA^{s+1}K)^{-1} = K^{-1}(A^{s+1})^{-1}K^{-1} = K(A^{-1})^{s+1}K$. In a similar way as before we get $A^* = (KA^{s+1}K)^* = K(A^{s+1})^*K = K(A^*)^{s+1}K$, because K is Hermitian. Finally, by definition and the assumption the equality $K(WAW^{-1})^{s+1} = WAW^{-1}$ follows directly. ■

By using unitary similarity the next results allow us to construct more examples from some initial ones. Even, Corollary 1 allows to construct infinite $\{K, s+1\}$-potent matrices starting from only one of them where K is Hermitian and commutes with a unitary matrix.

Corollary 1 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory and Hermitian matrix, $s \in \{1, 2, 3 \dots\}$, and $A \in \mathbb{C}^{n\times n}$ be a $\{K, s+1\}$-potent matrix. If $U \in \mathbb{C}^{n\times n}$ is a unitary matrix such that $KU = UK$ then UAU^* is $\{K, s+1\}$-potent.*

Proof. Since $KA^{s+1}K = A$ and $K^{-1} = K = K^*$ we get

$$K(UAU^*)^{s+1}K = KUA^{s+1}U^*K^* = UKA^{s+1}K^*U^* = UAU^*,$$

and then the result has been shown. ■

Example 1 *For the matrices*

$$A = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

it is easy to see that A is a $\{K, 2\}$-potent matrix. The most general form for the unitary matrix U where $UK = KU$ is

$$U = \begin{bmatrix} a & b & b \\ d & e & f \\ d & f & e \end{bmatrix}$$

being $a, b, d, e, f \in \mathbb{C}$ that satisfy $|a|^2 + 2|b|^2 = 1$, $a\bar{d} + b\bar{e} + b\bar{f} = 0$, $|d|^2 + |e|^2 + |f|^2 = 1$, and $|d|^2 + e\bar{f} + f\bar{e} = 0$. For example, setting $e = 0$ we get $d = 0$, $|a| = |f| = 1$, and $b = 0$. In this case we obtain the infinite $\{K, 2\}$-potent matrices

$$UAU^* = \begin{bmatrix} 0 & -ia\bar{f} & 0 \\ 0 & 0 & 1 \\ if\bar{a} & 0 & 0 \end{bmatrix}$$

where $|a| = |f| = 1$.

We would like to remark that Corollary 1 has a simple proof. However, it is interesting to construct a big collection of examples from only one. Specifically, this example shows that with only one $\{K, s+1\}$-potent matrix we can construct infinite examples. In order to answer the question: Does any $\{K, s+1\}$-potent matrix has this form? (that is, for two given $\{K, s+1\}$-potent matrices A and B of the same size, is it possible to find a unitary matrix U such that $B = UAU^*$ and $KU = UK$?). The answer is negative as shown by the following counterexample: the matrices

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

are $\{K, 5\}$-potent for $K = I_2$, but it is easy to check that there is no unitary matrix U such that $B = UAU^*$. This example also solves the same problem for nonsingular matrices, that is, the corresponding to case (g) of Lemma 2.

Now, we present some properties related to $\{K, s+1\}$-potency by using block matrices.

Lemma 3 *Let $s \in \{1, 2, 3 \dots\}$, and $\{K_1, K_2, \dots, K_t\}$ and $\{A_1, A_2, \dots, A_t\}$ be two sets of matrices such that $K_i, A_i \in \mathbb{C}^{n_i \times n_i}$ being K_i an involutory matrix for $i = 1, 2, \dots, t$. If for each $i = 1, 2, \dots, t$, every matrix A_i is $\{K_i, s+1\}$-potent, then defining the direct sums*

$$A = \bigoplus_{i=1}^{t} A_i \qquad \textit{and} \qquad K = \bigoplus_{i=1}^{t} K_i,$$

the matrix A is $\{K, s+1\}$-potent.

Proof. As $A = \bigoplus_{i=1}^{t} A_i$ and $K = \bigoplus_{i=1}^{t} K_i$, performing the following product $KA^{s+1}K$ by blocks it is easy to see that it is equal to A. Note that for all matrices A_i, $i = 1, 2, \dots, t$, it is necessary to use the same power $s+1$. ■

3 Characterization of $\{K, s+1\}$-potent matrices

We start this section with the following known result.

Lemma 4 ([1]) *Let $A \in \mathbb{C}^{n \times n}$ with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is diagonalizable if and only if there exist disjoint projectors $P_1, P_2, \dots, P_k$, that is $P_i P_j = \delta_{ij} P_i$ for $i, j \in \{1, 2, \dots, k\}$, such that $A = \sum_{j=1}^{k} \lambda_j P_j$ and $I_n = \sum_{j=1}^{k} P_j$.*

A special relation between the elements of the set $\{0, 1, 2, \dots, (s+1)^2 - 2\}$ will be necessary in what follows.

Lemma 5 *Let $s \in \{1, 2, 3, \dots\}$ and $\varphi : \{0, 1, 2, \dots, (s+1)^2 - 2\} \to \{0, 1, 2, \dots, (s+1)^2 - 2\}$ be the function defined by $\varphi(j) = b_j$ where b_j is the smallest nonnegative integer such that $b_j \equiv j(s+1)$ [mod $((s+1)^2 - 1)$]. Then φ is a bijective function.*

Proof. It is evident that the function φ is well-defined. Now, we define the following sets:

$$B_1 = \{0, s+1, 2(s+1), \dots, s(s+1)\}$$

and, by recurrence, for each $k \in \{1, 2, \dots, s-1\}$, the sets

$$B_{k+1} = \{1\} + B_k$$

and, finally,

$$B_{s+1} = \{s, (s+1)+s, 2(s+1)+s, \dots, (s-1)(s+1)+s\}.$$

Calling B the joint of all of them, by construction we have

$$B = \bigcup_{k=1}^{s+1} B_k = \{0, 1, 2, \dots, (s+1)^2 - 2\}.$$

We will show that for every $b \in B$, there is a unique value $j \in \{0, 1, 2, \dots, (s+1)^2 - 2\}$ such that $b \equiv j(s+1)$ [mod $((s+1)^2 - 1)$] where b is the smallest nonnegative integer satisfying these conditions. For that, we first construct the following sets:

$$J_1 = \{0, 1, 2, \dots, s\}$$

and, by recurrence, for each $i \in \{1, 2, \dots, s-1\}$ the sets

$$J_{i+1} = \{s+1\} + J_i$$

and, finally,

$$J_{s+1} = \{s(s+1), s(s+1)+1, s(s+1)+2, \dots, s(s+1)+(s-1)\}.$$

Set $b \in B$. Then there exists $k \in \{1, 2, \dots, s+1\}$ such that $b \in B_k$.

If $k = 1$ then $b \in B_1$, and it is clear that there exists a unique $j \in J_1$ such that $b = j(s+1)$ and so $b \equiv j(s+1)$ [mod $((s+1)^2 - 1)$].

If $k \in \{2, 3, \dots, s\}$ then

$$b \in B_k = \{k-1, (s+1)+(k-1), 2(s+1)+(k-1), \dots, s(s+1)+(k-1)\},$$

then it is clear that there exists a unique $j \in J_k$ such that $j(s+1) = b + (k-1)((s+1)^2 - 1)$, and so, $b \equiv j(s+1)$ [mod $((s+1)^2 - 1)$].

Finally, if $b \in B_{s+1}$ then it is clear that there exists a unique $j \in J_{s+1}$ such that $j(s+1) = b + s((s+1)^2 - 1)$ and so $b \equiv j(s+1)$ [mod $((s+1)^2 - 1)$].

Moreover, by construction

$$\bigcup_{i=1}^{s+1} J_i = \{0, 1, 2, \dots, (s+1)^2 - 2\}$$

and thus the previous reasoning proves the existence. Finally, it is clear that $\#(J_i) = s+1$ for $i \in \{1, 2, \dots, s\}$, $\#(J_{s+1}) = s$ and the condition

$$J_i \cap J_k = \emptyset \qquad \text{holds for every } i, k \in \{1, 2, \dots, s+1\} \quad \text{with } i \neq k,$$

which guarantees the uniqueness. The prove is complete. ■

The spectral theory is a suitable approach to obtain characterizations of different classes of matrices that involve powers [5]. We will use this theory in order to characterize $\{K, s+1\}$-potent matrices.

Theorem 2 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3 \dots\}$, and $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent.*

(a) A is $\{K, s+1\}$-potent.

(b) A is diagonalizable, $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}$, $KP_jK = P_{\varphi(j)}$, where $j \in \{0, 1, \dots, (s+1)^2 - 2\}$ and $KP_{(s+1)^2-1}K = P_{(s+1)^2-1}$ being φ the bijection defined in Lemma 5 and $P_0, P_1, \dots, P_{(s+1)^2-1}$ the projectors appearing in the spectral decomposition of A given in Lemma 4 associated to the eigenvalues $0, w^1_{(s+1)^2-1}, \dots, w^{(s+1)^2-2}_{(s+1)^2-1}, 1$, respectively.

(c) $A^{(s+1)^2} = A$, $KP_jK = P_{\varphi(j)}$, where $j \in \{0, 1, \dots, (s+1)^2 - 2\}$ and $KP_{(s+1)^2-1}K = P_{(s+1)^2-1}$ being φ the bijection defined in Lemma 5 and $P_0, P_1, \dots, P_{(s+1)^2-1}$ the projectors appearing in the spectral decomposition of A given in Lemma 4 associated to the eigenvalues $0, w^1_{(s+1)^2-1}, \dots, w^{(s+1)^2-2}_{(s+1)^2-1}, 1$, respectively.

Proof.

(a) $\Longrightarrow$ *(b)* Since A is $\{K, s+1\}$-potent, property (II) of Lemma 1 implies that $A^{(s+1)^2} = A$. As the polynomial $q(t) = t^{(s+1)^2} - t$ is a multiple of the minimal polynomial $q_A(t)$ of A and by using that every root of $q_A(t)$ has multiplicity 1 then A is diagonalizable and moreover, it is clear that $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}$.

On the other hand, by Lemma 4, there exist disjoint projectors $P_0, P_1, \dots, P_{(s+1)^2-1}$ such that

$$A = \sum_{j=1}^{(s+1)^2-1} \omega^j_{(s+1)^2-1} P_j \qquad \text{and} \qquad \sum_{j=0}^{(s+1)^2-1} P_j = I_n, \tag{2}$$

where we must understand that $P_{j_0} = O$ if there exists $j_0 \in \{1, 2, \dots, (s+1)^2 - 1\}$ such that $w^{j_0}_{(s+1)^2-1} \notin \sigma(A)$ and moreover that $P_0 = O$ when $0 \notin \sigma(A)$.

Pre and postmultiplying the previous expressions by matrix K we have

$$KAK = \sum_{j=1}^{(s+1)^2-1} \omega^j_{(s+1)^2-1} KP_jK$$

and

$$\sum_{j=0}^{(s+1)^2-1} KP_jK = I_n \tag{3}$$

since $K^2 = I_n$. Therefore, as projectors P_j are disjoint, it is evident that KP_jK are also disjoint projectors for all $j = 0, 1, \ldots, (s+1)^2 - 1$. Again, by using the fact that P_j are disjoint projectors, from (2) we deduce by recurrence that

$$A^{s+1} = \sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{j.(s+1)} P_j$$

and because of $\varphi(j) \equiv j(s+1)$ [mod $((s+1)^2-1)$] for all $j = 1, 2, \ldots, (s+1)^2 - 2$ we arrive to

$$A^{s+1} = \sum_{j=1}^{(s+1)^2-2} \omega_{(s+1)^2-1}^{\varphi(j)} P_j + P_{(s+1)^2-1}.$$

Using the hypothesis and Lemma 1 (I) (b), and equating the expressions KAK and A^{s+1} we get

$$\sum_{i=1}^{(s+1)^2-2} \omega_{(s+1)^2-1}^{i} KP_iK + KP_{(s+1)^2-1}K = \sum_{j=1}^{(s+1)^2-2} \omega_{(s+1)^2-1}^{\varphi(j)} P_j + P_{(s+1)^2-1}.$$

Since φ is a bijection, for every $i \in \{1, 2, \ldots, (s+1)^2 - 2\}$, there exists a unique $j \in \{1, 2, \ldots, (s+1)^2 - 2\}$ such that $i = \varphi(j)$. From the uniqueness of the spectral decomposition we obtain that for every $i \in \{1, 2, \ldots, (s+1)^2 - 2\}$, there exists a unique $j \in \{1, 2, \ldots, (s+1)^2 - 2\}$ such that $KP_{\varphi(j)}K = KP_iK = P_j$ and then $KP_jK = P_{\varphi(j)}$ holds. It is clear that such a uniqueness also implies that $KP_{(s+1)^2-1}K = P_{(s+1)^2-1}$. Finally, from (2) we get

$$P_0 = I_n - \sum_{j=1}^{(s+1)^2-1} P_j$$

and taking into account (3) and the definition of bijection φ we get

$$\begin{aligned} KP_0K &= I_n - \sum_{i=1}^{(s+1)^2-2} KP_iK - KP_{(s+1)^2-1}K = I_n - \sum_{j=1}^{(s+1)^2-2} KP_{\varphi(j)}K - KP_{(s+1)^2-1}K \\ &= I_n - \sum_{i=1}^{(s+1)^2-2} P_i - P_{(s+1)^2-1} = P_0. \end{aligned}$$

We must observe that in the case where $j_0 \in \{1, 2, \ldots, (s+1)^2 - 2\}$ such that $w_{(s+1)^2-1}^{j_0} \notin \sigma(A)$ exists, it has been indicated that we must consider $P_{j_0} = O$. In this situation, $P_{\varphi(j_0)} = KP_{j_0}K = O$ must also be true.

$(b) \Longrightarrow (a)$ By hypothesis and Lemma 4, it is clear that

$$A = \sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{j} P_j \tag{4}$$

and by recurrence one has

$$A^{s+1} = \sum_{j=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{j.(s+1)} P_j.$$

By a similar reasoning as in the implication $(a) \Longrightarrow (b)$ and by using the hypothesis it is clear that

$$\begin{aligned} A^{s+1} &= \sum_{j=1}^{(s+1)^2-2} \omega_{(s+1)^2-1}^{\varphi(j)} P_j + P_{(s+1)^2-1} = \sum_{j=1}^{(s+1)^2-2} \omega_{(s+1)^2-1}^{\varphi(j)} K P_{\varphi(j)} K + K P_{(s+1)^2-1} K \\ &= \sum_{i=1}^{(s+1)^2-1} \omega_{(s+1)^2-1}^{i} K P_i K = KAK, \end{aligned}$$

where in the last step we have used (4) again.

$(b) \Longleftrightarrow (c)$ It is sufficient to show that the fact that A is diagonalizable and $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}$ is equivalent to $A^{(s+1)^2} = A$.

It is clear that if A is diagonalizable then $A = PDP^{-1}$ where D is a diagonal matrix with diagonal entries belonging to the set $\{0\} \cup \Omega_{(s+1)^2-1}$. A direct computation shows that $A^{(s+1)^2} = A$. Conversely, if $A^{(s+1)^2} = A$, a similar reasoning as before in $(a) \Longrightarrow (b)$ allows to obtain the conclusion. This ends the proof. ■

Remark 1 *As we can see in the proof of Theorem 2, for each* $j_0 \in \{0, 1, 2, \ldots, (s+1)^2-2\}$, *the projectors* P_{j_0} *and* $P_{\varphi(j_0)}$ *must be both zero or both nonzero matrices in the spectral decomposition of* A *since* $KP_{j_0}K = P_{\varphi(j_0)}$. *In addition, the projector* $P_{(s+1)^2-1}$ *appears or not in that decomposition satisfying the relations indicated in items* (b) *and* (c) *depending on whether the eigenvalue* 1 *belongs or not to the spectrum of* A. *For example, if* $s = 2$ *(that is* $(s+1)^2 - 1 = 8$*) and* ω *is a primitive* 8^{th} *root of unity, then* $A = \omega^7 I_n$ *can not be a* $\{K, 3\}$*-potent matrix for any involutory matrix* $K \in \mathbb{C}^{n\times n}$. *In fact, the spectrum of the matrix* A *should contain the value* ω^5 *(because* $KP_7K = P_{\varphi(7)} = P_5$*), which yields to a contradiction. We can also see it directly:* $KA^3K = \omega^5 K^2 = \omega^{-2} A \neq A$.

As we have shown, $A^{(s+1)^2} = A$ holds for every $\{K, s+1\}$-potent matrix A. Could the power k of A such that $A^k = A$ be less than $(s+1)^2$? The answer is given in the following result.

Corollary 2 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3 \dots\}$, $A \in \mathbb{C}^{n\times n}$, and let's assume that there exists a positive integer $1 < k < (s+1)^2$ such that $A^k = A$. Then A is a $\{K, s+1\}$-potent matrix if and only if $k-1$ divides $(s+1)^2 - 1$, A is diagonalizable,*

$$\sigma(A) = \{\lambda_0, \lambda_1, \dots, \lambda_t\} \subseteq \{0\} \cup \Omega_{k-1} \tag{5}$$

(being $\lambda_i \neq \lambda_j$ for $i \neq j$, for $i, j \in \{0, 1, \dots, t\}$), and for each $i \in \{0, 1, \dots, t\}$ there exists a unique $j \in \{0, 1, \dots, t\}$ such that $\lambda_i = \lambda_j^{s+1}$ and $P_i = KP_jK$ where $P_0, P_1, \dots, P_t$ are the projectors appearing in the spectral decomposition of Lemma 4 of A associated to the eigenvalues given in (5), respectively.

Proof. Since A is $\{K, s+1\}$-potent, by Theorem 2, the matrix A is diagonalizable and $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^2-1}$. The assumption $A^k = A$ yields to $\sigma(A) \subseteq \{0\} \cup \Omega_{k-1}$ and so $\sigma(A) \subseteq \{0\} \cup \Omega_{\gcd(k-1,(s+1)^2-1)} = \Omega_{k-1}$, because $k-1$ divides $(s+1)^2 - 1$. From $A = KA^{s+1}K$ and the uniqueness assured in the spectral decomposition, we arrive to the correspondence $\lambda_i = \lambda_j^{s+1}$ and $P_i = KP_jK$ as indicated. The converse can be shown in a similar way as in Theorem 2. ■

Example 2 *The matrix*

$$A = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is $\{K, 2\}$-potent for the involutory matrix

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and it satisfies $A^4 = A$. It is possible to check that when $\omega = \frac{-1+i\sqrt{3}}{2}$,

$$\sigma(A) = \{\omega, \omega^2, \omega^3\} = \left\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}, 1\right\}$$

The corresponding projectors are

$$P_1 = \frac{1}{3}\begin{bmatrix} 1 & -i\,\omega & -i\,\overline{\omega} \\ i\,\overline{\omega} & 1 & \omega \\ i\,\omega & \overline{\omega} & 1 \end{bmatrix}, \quad P_2 = \frac{1}{3}\begin{bmatrix} 1 & -i\,\overline{\omega} & -i\,\omega \\ i\,\omega & 1 & \overline{\omega} \\ i\,\overline{\omega} & \omega & 1 \end{bmatrix}, \quad P_3 = \frac{1}{3}\begin{bmatrix} 1 & -i & -i \\ i & 1 & 1 \\ i & 1 & 1 \end{bmatrix}$$

and, in this case, $KP_1K = P_2$ and $KP_3K = P_3$.

On the other hand, it is easy to verify that any $\{K, s+1\}$-potent matrix has a group inverse matrix. Furthermore, another equivalence of the $\{K, s+1\}$-potent matrices can be obtained by means of this kind of generalized inverse matrices.

Corollary 3 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1,2,3,\dots\}$, and $A \in \mathbb{C}^{n\times n}$. Then A is $\{K, s{+}1\}$-potent if and only if $A^{\#} = A^{(s+1)^2-2}$, $KP_jK = P_{\varphi(j)}$, $j \in \{0,1,2,\dots,(s+1)^2-2\}$ and $KP_{(s+1)^2-1}K = P_{(s+1)^2-1}$ where φ is the bijection defined in Lemma 5 and $P_0, P_1, \dots, P_{(s+1)^2-1}$ the projectors appearing in the spectral decomposition of A given in Lemma 4 associated to the eigenvalues $0, w^1_{(s+1)^2-1}, \dots, w^{(s+1)^2-2}_{(s+1)^2-1}, 1$, respectively.*

Proof. In [3] the equivalence between the conditions $A^{\#} = A$ and $A^3 = A$ was proved and a direct extension allows to establish that the equalities $A^{\#} = A^{(s+1)^2-2}$ and $A^{(s+1)^2} = A$ are two equivalent conditions. The proof ends by the application of property (c) of Theorem 2. ■

Again, by using the fact that any $\{K, s+1\}$-potent matrix has a group inverse matrix, we give another characterization of $\{K, s+1\}$-potent matrices. Actually, we will use the following fact: a square matrix has index 1 if and only if A and A^2 have the same rank. We recall that the index of a square matrix A is the smallest nonnegative integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ [4].

Theorem 3 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1,2,3\dots\}$, and $A \in \mathbb{C}^{n\times n}$. Then A is a $\{K, s+1\}$-potent matrix if and only if there are nonsingular matrices $Q, P \in \mathbb{C}^{n\times n}$ such that*

$$A = Q \begin{bmatrix} D & O \\ O & O \end{bmatrix} Q^{-1}, \qquad K = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^{-1}, \qquad P^{-1}Q = \begin{bmatrix} W & O \\ O & I_{n-r} \end{bmatrix},$$

where $D = [d_{ij}] \in \mathbb{C}^{r\times r}$ is diagonal, $r = \operatorname{rank}(A)$, and $d_{ii} \in \{0\} \cup \Omega_{(s+1)^2-1}$; $X \in \mathbb{C}^{r\times r}$ and $T \in \mathbb{C}^{(n-r)\times(n-r)}$ are involutory matrices, and $W \in \mathbb{C}^{r\times r}$ is a nonsingular matrix such that $C = WDW^{-1} \in \mathbb{C}^{r\times r}$ is $\{X, s+1\}$-potent.

Proof. If A is $\{K, s+1\}$-potent then A has index 1. Hence, it is possible to write A (of rank r) in the form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \tag{6}$$

where $C \in \mathbb{C}^{r\times r}$ is nonsingular. Using this expression, performing A^{s+1} and KAK, and equating we get

$$P \begin{bmatrix} C^{s+1} & O \\ O & O \end{bmatrix} P^{-1} = KP \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}K^{-1}. \tag{7}$$

Now, we consider the partitioning of $P^{-1}KP$ given by

$$P^{-1}KP = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}$$

with X, Y, Z, T of adequate sizes according to the partition of A given in (6). Substituting this matrix in (7) and equating we obtain

$$C^{s+1}X = XC, \qquad C^{s+1}Y = O, \qquad \text{and} \qquad ZC = O.$$

As C is nonsingular, $Y = O$, $Z = O$ and so

$$K = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^{-1}.$$

Since $K^2 = I_n$, this last expression implies that X and T are involutory matrices, thus $XC^{s+1}X = C$, meaning that C is $\{X, s+1\}$-potent. By Theorem 2, there are a nonsingular matrix W and a diagonal matrix D such that $C = WDW^{-1}$ where the diagonal elements of D belong to $\{0\} \cup \Omega_{(s+1)^2-1}$. Substituting in (6) and denoting

$$Q = P \begin{bmatrix} W & O \\ O & I_{n-r} \end{bmatrix}$$

we get the required form for the matrix A.

The converse is straightforward. This completes the proof. ■

Remark 2 *In order to have the same similarity matrix in both matrices A and K, a similar reasoning as in the above theorem gives the following result: A is a $\{K, s+1\}$-potent matrix if and only if there are nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ such that*

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \qquad K = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^{-1}$$

where $r = \operatorname{rank}(A)$, $X \in \mathbb{C}^{r\times r}$ and $T \in \mathbb{C}^{(n-r)\times(n-r)}$ are involutory matrices, and C is a $\{X, s+1\}$-potent matrix.

By exploiting the structure of the involutory matrix K we can give the following result.

Theorem 4 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix and let $P \in \mathbb{C}^{n\times n}$ be such that*

$$K = P \begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix} P^{-1}.$$

Let assume that

$$A = P \begin{bmatrix} B & C \\ O & D \end{bmatrix} P^{-1}$$

where $B \in \mathbb{C}^{p\times p}$, $C \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{q\times q}$. Then A is $\{K, s+1\}$-potent if and only if B and D are $\{s+1\}$-potent and

$$C + \sum_{i=0}^{s} B^{s-i} C D^i = O.$$

Proof. It is easy to see that

$$A^{s+1} = P \left[\begin{array}{cc} B^{s+1} & \sum_{i=0}^{s} B^{s-i} C D^{i} \\ O & D^{s+1} \end{array} \right] P^{-1}.$$

Now, after doing some algebraic manipulations one gets that $KA^{s+1}K = A$ if and only if B and D are $\{s+1\}$-potent and $C + \sum_{i=0}^{s} B^{s-i} C D^{i} = O$, which ends the proof. ■

A similar result to this last one can be obtained by assuming that matrix A is similar to a block lower triangular matrix by means of the similarity matrix P. However, when this special form of A is not assumed, the conclusion is, in general, not satisfied. For example, let consider

$$K = \left[\begin{array}{cc} 0 & 2 \\ \frac{1}{2} & 0 \end{array} \right] = P \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] P^{-1}$$

and

$$A = \left[\begin{array}{cc} -3 & -2\sqrt{7} \\ \frac{\sqrt{7}}{2} & 2 \end{array} \right] = P \left[\begin{array}{cc} -\frac{1}{2} & \frac{5}{2} - \sqrt{7} \\ \frac{5}{2} + \sqrt{7} & -\frac{1}{2} \end{array} \right] P^{-1}$$

a $\{K, 2\}$-potent matrix where

$$P = \left[\begin{array}{cc} 2 & -2 \\ 1 & 1 \end{array} \right].$$

By using the same notation as in Theorem 4, it is clear that $B = D = -\frac{1}{2}$ are not $\{2\}$-potent matrices and then the conclusion is not valid.

Other characterization for A to be $\{K, s+1\}$-potent matrix is presented in next theorem (we can compare with Theorem 2).

Theorem 5 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3, \dots\}$, and $A \in \mathbb{C}^{n\times n}$ with spectrum*

$$\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_t\} \tag{8}$$

(being $\lambda_i \neq \lambda_j$ for $i \neq j$ and $i, j \in \{1, 2, \dots, t\}$). Then A is $\{K, s+1\}$-potent if and only if A is diagonalizable and for each $i \in \{1, 2, \dots, t\}$ there is a unique $j \in \{1, 2, \dots, t\}$ such that $\lambda_i = \lambda_j^{s+1}$ and $P_i = KP_jK$ where $P_1, P_2, \dots, P_t$ are the projectors appearing in the spectral decomposition of Lemma 4 of A associated to the eigenvalues given in (8), respectively.

Proof. Since A is $\{K, s+1\}$-potent, by Theorem 2, A is diagonalizable. Since $A = KA^{s+1}K$, we get that A is similar to A^{s+1}. The uniqueness assured in the spectral decomposition allows to state the correspondence: for every $i \in \{1, 2, \dots, t\}$, there is a unique $j \in \{1, 2, \dots, t\}$ such that $\lambda_i = \lambda_j^{s+1}$ and $P_i = KP_jK$ as indicated in Theorem 2 because

$$A = \sum_{i=1}^{t} \lambda_i P_i, \qquad \text{and} \qquad KA^{s+1}K = \sum_{j=1}^{t} \lambda_j^{s+1} KP_jK.$$

The converse can be shown by means of the spectral theorem in a similar way. ■

Note that, under the notation of Theorem 5, we can define the function $\theta : \{1, 2, \dots, t\} \to \{1, 2, \dots, t\}$ by means of $\theta(i) = j$ where j is the unique element in $\{1, 2, \dots, t\}$ such that $\lambda_i = \lambda_j^{s+1}$. Then, it can be seen that θ is an involution (in particular, bijective). This function allows us to write $\lambda_i = \lambda_{\theta(i)}^{s+1}$ and $P_i = KP_{\theta(i)}K$ in Theorem 5.

Another necessary condition for A to be a $\{K, s+1\}$-potent matrix is given in the following result where eigenvectors are involved.

Corollary 4 *Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, $s \in \{1, 2, 3, \dots\}$, and $A \in \mathbb{C}^{n\times n}$ such that $Ax_i = \lambda_i x_i$ being $\lambda_i \neq \lambda_j$ for $i \neq j$, $x_i \neq 0$ for $i, j \in \{1, 2, \dots, n\}$. If there exists $i_0 \in \{1, 2, \dots, n\}$ such that $A^{s+1}x_{\theta(i_0)} \neq \lambda_{i_0} x_{\theta(i_0)}$ then A is not $\{K, s+1\}$-potent.*

Proof. If we assume that A is $\{K, s+1\}$-potent then $A = P\mathrm{diag}(\lambda_1, \dots, \lambda_n)P^{-1}$ and then Theorem 5 yields to

$$\begin{aligned} A^{s+1} &= P\mathrm{diag}(\lambda_1^{s+1}, \dots, \lambda_n^{s+1})P^{-1} = P\mathrm{diag}(\lambda_{\theta(1)}, \dots, \lambda_{\theta(n)})P^{-1} \\ &= P\Sigma\mathrm{diag}(\lambda_1, \dots, \lambda_n)\Sigma^T P^{-1} = (P\Sigma)\mathrm{diag}(\lambda_1, \dots, \lambda_n)(P\Sigma)^{-1}, \end{aligned}$$

where Σ represents a (product of) permutation matrix that reorder the eigenvalues. This fact is contrary to the hypothesis. ■

4 Application: Linear combinations of $\{K, s+1\}$-potent matrices

As an application of the results presented in Section 3 we give the following theorem. Recall that two commuting diagonalizable matrices share a basis of eigenvectors, so they are simultaneously diagonalizable [8].

Theorem 6 *Let c_1, c_2 be nonzero complex numbers and $A, B \in \mathbb{C}^{n\times n}$ nonzero $\{K, s+1\}$-potent matrices such that $AB = BA$. Assume that $C = c_1A + c_2B$ is a linear combination of A and B which is $\{K, s+1\}$-potent. Then any of the following conditions holds.*

a) $c_1, c_2 \in \Omega_{(s+1)^2-1}$.

b) $c_1 \in \Omega_{(s+1)^2-1}$ *and there is* $r \in \{0, 1, \dots, (s+1)^2 - 2\}$ *such that* $\omega_{(s+1)^2-1}^r c_1 + c_2 \in \{0\} \cup \Omega_{(s+1)^2-1}$.

c) $c_2 \in \Omega_{(s+1)^2-1}$ *and there is* $t \in \{0, 1, \dots, (s+1)^2 - 2\}$ *such that* $c_1 + \omega_{(s+1)^2-1}^t c_2 \in \{0\} \cup \Omega_{(s+1)^2-1}$.

d) *There are* $r, t \in \{0, 1, \dots, (s+1)^2 - 2\}$ *such that* $r + t$ *is not a multiple of* $(s+1)^2 - 1$ *and there are* $\zeta_1, \zeta_2 \in \{0\} \cup \Omega_{(s+1)^2-1}$, *with* $\zeta_1 \neq 0$ *or* $\zeta_2 \neq 0$, *and*

$$c_1 = \frac{\zeta_1\omega_{(s+1)^2-1}^t - \zeta_2}{\omega_{(s+1)^2-1}^{r+t} - 1}, \qquad c_2 = \frac{\zeta_2\omega_{(s+1)^2-1}^r - \zeta_1}{\omega_{(s+1)^2-1}^{r+t} - 1}.$$

e) $c_1 + c_2 \in \{0\} \cup \Omega_{(s+1)^2-1}$.

f) *There is* $t \in \{0, 1, \ldots, (s+1)^2 - 2\}$ *such that* $\omega_{(s+1)^2-1}^{-t} c_1 + c_2 \in \{0\} \cup \Omega_{(s+1)^2-1}$.

Proof. Since A and B are $\{K, s+1\}$-potent matrices such that $AB = BA$, there exist (Theorem 1.3.19 [8] and Theorem 2) a nonsingular matrix P and diagonal matrices D_A and D_B such that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$. Then $C = c_1A + c_2B = P(c_1D_A + c_2D_B)P^{-1}$. Denote $D_A := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $D_B := \mathrm{diag}(\mu_1, \ldots, \mu_n)$.

The eigenvalues of D_A, D_B and $c_1D_A + c_2D_B$ are elements in $\{0\} \cup \Omega_{(s+1)^2-1}$ because A, B, and C are $\{K, s+1\}$-potent matrices (Theorem 2). Thus,

$$c_1\lambda_i + c_2\mu_i \in \{0\} \cup \Omega_{(s+1)^2-1}, \qquad \text{for all } \; i = 1, \ldots, n. \tag{9}$$

Since $D_A \neq O$, there exists $i_0 \in \{1, \ldots, n\}$ such that $\lambda_{i_0} \neq 0$ and so $\lambda_{i_0} \in \Omega_{(s+1)^2-1}$. From (9),

$$c_1 + c_2\frac{\mu_{i_0}}{\lambda_{i_0}} \in \{0\} \cup \Omega_{(s+1)^2-1},$$

and moreover $\mu_{i_0}/\lambda_{i_0} \in \{0\} \cup \Omega_{(s+1)^2-1}$ because $\Omega_{(s+1)^2-1}$ is a multiplicative group. Analogously, there is $j_0 \in \{1, \ldots, n\}$ such that $\mu_{j_0} \in \Omega_{(s+1)^2-1}$ because $D_B \neq O$. Again, from (9) we get

$$\frac{\lambda_{j_0}}{\mu_{j_0}}c_1 + c_2 \in \{0\} \cup \Omega_{(s+1)^2-1},$$

If $\mu_{i_0} = 0 = \lambda_{j_0}$ then $c_1, c_2 \in \Omega_{(s+1)^2-1}$ (because $c_1 \neq 0 \neq c_2$) and so case (a) is obtained.

If $\mu_{i_0} = 0$ and $\lambda_{j_0} \in \Omega_{(s+1)^2-1}$ then $c_1 \in \Omega_{(s+1)^2-1}$ and there is $r \in \{0, 1, \ldots, (s+1)^2-2\}$ such that $\omega_{(s+1)^2-1}^{r}c_1 + c_2 \in \{0\} \cap \Omega_{(s+1)^2-1}$. Case (b) is thus obtained.

If $\mu_{i_0} \neq 0$ and $\lambda_{j_0} = 0$, case (c) can be obtained in a similar way as case (b).

Finally, if $\mu_{i_0}, \lambda_{j_0} \in \Omega_{(s+1)^2-1}$ then we have to solve the linear system $\omega_{(s+1)^2-1}^{r}c_1 + c_2 = \zeta_1$ and $c_1 + \omega_{(s+1)^2-1}^{t}c_2 = \zeta_2$ in the unknowns c_1 and c_2 where $\zeta_1, \zeta_2 \in \{0\} \cap \Omega_{(s+1)^2-1}$. By Gaussian elimination it is easy to see that

$$c_1 = \frac{\zeta_1\omega_{(s+1)^2-1}^{t} - \zeta_2}{\omega_{(s+1)^2-1}^{r+t} - 1}, \qquad c_2 = \frac{\zeta_2\omega_{(s+1)^2-1}^{r} - \zeta_1}{\omega_{(s+1)^2-1}^{r+t} - 1}$$

in case that $\omega_{(s+1)^2-1}^{r+t} \neq 1$, that is, when $r + t$ is not a multiple of $(s+1)^2 - 1$ (case (d)). When $\omega_{(s+1)^2-1}^{r+t} = 1$, we have two possibilities: $r = t = 0$ or $r + t = (s+1)^2 - 1$. The first yields to case (e) and the second one gives case (f) because $\omega_{(s+1)^2-1}^{r} = \omega_{(s+1)^2-1}^{-t}$. The proof is then finished. ∎

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References

[1] A. Ben-Israel, T. Greville. Generalized Inverses: Theory and Applications, John Wiley & Sons, 2nd Edition, 2003.

[2] J. Benítez, N. Thome, $\{k\}$-group periodic matrices. *SIAM J. Matrix Anal. Appl.* 28, 1 (2006) 9–25.

[3] R. Bru, N. Thome. Group inverse and group involutory matrices, *Linear and Multilinear Algebra,* 45 (2-3), 207–218, 1998.

[4] S.L. Campbell, C.D. Meyer, Jr.. Generalized Inverses of Linear Transformations. Dover, New York (1979).

[5] Hong-Ke Du, Yuan Li, The spectral characterization of generalized projections, *Linear Algebra and its Applications*, 400, (2005) 313–318.

[6] C. R. H. Hanusa, Pseudo-centrosymmetric matrices, with applications to counting perfect matchings, *Linear Algebra and its Applications*, 427, (2-3), 206–217, 2007.

[7] N. J. Higham. Functions of Matrices, SIAM, Philadelphia, 2008.

[8] R. A. Horn, C. R. Johnson. Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[9] D. Tao, M. Yasuda, A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices, *SIAM J. Matrix Anal. Appl.*, 23, 3, 885-895, 2001/02, (electronic), http://epubs.siam.org/sambin/dbq/article/38673.