A Note on k-Generalized Projections

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Abstract

In this note, we investigate characterizations for k-generalized projections (i.e., $A^k = A^*$) on Hilbert spaces. The obtained results generalize those for generalized projections on Hilbert spaces in [Hong-Ke Du, Yuan Li, The spectral characterization of generalized projections, Linear Algebra and its Applications, 400, (2005), 313–318] and those for matrices in [J. Benítez, N. Thome, Characterizations and linear combinations of k-generalized projectors, Linear Algebra and its Applications, In Press].

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In [2], it was defined a *generalized projection* as a complex matrix A satisfying $A^2 = A^*$. This concept was extended in [3] for infinite-dimensional Hilbert spaces. For H a Hilbert space, we shall denote

$$\mathcal{B}(H) = \{A/\ A \text{ is linear and bounded operator, } A : H \to H\}.$$

If k is an integer greater than 1, we define a *k-generalized projection* as an element A of $\mathcal{B}(H)$ such that $A^k = A^*$, where A^* is the adjoint operator of A.

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Moreover, the $n \times n$ complex matrices such that $A^k = A^*$ (where A^* denotes its conjugate transpose) were characterized in [1].

We recall that $A \in \mathcal{B}(H)$ is said to be *normal* if $AA^* = A^*A$, it is said to be *orthogonal projection* if $A^2 = A = A^*$, and A is called *k-potent* if $A^k = A$. In particular, A is a *projection* if $A^2 = A$ and A is *tripotent* if $A^3 = A$. In addition, the spectrum of A will be denoted by $\sigma(A)$.

The main purpose of this note is to give characterizations of the k-generalized projections by using the spectral theorem for normal operators on Hilbert spaces (see [4]). We quote this theorem for the sake of completeness.

Theorem 1 ([4]) *Let H be a Hilbert space and $A \in \mathcal{B}(H)$. If A is normal then there exists a unique resolution of the identity E on the Borel subsets of $\sigma(A)$ which satisfies*

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E(\lambda),$$

where $E(\lambda)$ denotes the spectral projection associated with the spectral point $\lambda \in \sigma(A)$ and $E(\lambda) = 0$ if $\lambda \notin \sigma(A)$.

The main result of this note is the following.

Theorem 2 *Let H be a Hilbert space and $A \in \mathcal{B}(H)$. Then the following statements are equivalent.*

(a) A is a k-generalized projection.

(b) A is normal and $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$, where $\sqrt[k+1]{1}$ denotes the unity roots of order $k+1$.

(c) A is normal and $(k+2)$-potent.

In this case, one has

$$A = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda E(\lambda), \tag{1}$$

where $E(\lambda) = 0$ if $\lambda \notin \sigma(A)$ and $\oplus$ stands for the direct sum.

Proof. (a) ⇒ (b). Suppose that $A^k = A^*$. It is evident that $AA^* = A^*A$, i.e., A is normal. Theorem 1 assures that

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E(\lambda) \tag{2}$$

and then $0 = A^k - A^* = \int_{\sigma(A)} (\lambda^k - \overline{\lambda}) \mathrm{d}E(\lambda)$, which implies $\lambda^k - \overline{\lambda} = 0$ for all $\lambda \in \sigma(A)$. The roots of this equation are 0 and $\sqrt[k+1]{1}$ since if $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$, with $r > 0$ and $-\pi \leq \theta < \pi$, then we get $r^k \mathrm{e}^{\mathrm{i}k\theta} = r\mathrm{e}^{-\mathrm{i}\theta}$ and so $r = 1$ and $\mathrm{e}^{\mathrm{i}(k+1)\theta} = 1$, i.e., $\lambda = \mathrm{e}^{\mathrm{i}\theta} \in \sqrt[k+1]{1}$. From (2), it is clear that (1) holds.

(b) ⇒ (c). If A is normal and $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$ then (1) is true from Theorem 1. Now, since $\lambda^{k+2} = \lambda$ for all $\lambda \in \sigma(A)$,

$$A^{k+2} = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda^{k+2} E(\lambda) = \bigoplus_{\lambda \in \sqrt[k+1]{1}} \lambda E(\lambda) = A.$$

(c) ⇒ (a). If A is normal, from Theorem 1 one has that

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E(\lambda). \tag{3}$$

From $A^{k+2} = A$ we get that

$$0 = A^{k+2} - A = \int_{\sigma(A)} (\lambda^{k+2} - \lambda) \mathrm{d}E(\lambda).$$

Hence, $\lambda^{k+2} - \lambda = 0$ for all $\lambda \in \sigma(A)$. Now, it is easy to deduce $\lambda^k = \overline{\lambda}$ for all $\lambda \in \sigma(A)$ and so, from (3) we obtain $A^k = A^*$.

This completes the proof. □

Theorem 2 in [3] and Theorem 2.1 in [1] can be obtained as corollaries of Theorem 2.

Corollary 1 *Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a k-generalized projection.*

(I) If $\sigma(A) \subseteq \mathbb{R}$ and

  (a) k is even then A is a projection.

  (b) k is odd then A is a tripotent operator.

(II) If $\sigma(A) \subseteq \mathrm{i}\mathbb{R}$ and

  (a) k is a multiple of 4 then $A^3 = -A$.

  (b) k is not a multiple of 4 then $A = O$.

Proof. By Theorem 2 we know that A is normal and $\sigma(A) \subseteq \{0\} \cup \sqrt[k+1]{1}$.

(I) By hypothesis, $\sigma(A) \subseteq \{0\} \cup (\sqrt[k+1]{1} \cap \mathbb{R})$. If k is even then $\sigma(A) \subseteq \{0, 1\}$, hence $A^2 = A$. If k is odd then $\sigma(A) \subseteq \{-1, 0, 1\}$, hence $A^3 = A$.

(II) In this case, $\sigma(A) \subseteq \mathrm{i}\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1})$. If k is a multiple of 4 then $i\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1}) = \{0, \mathrm{i}, -\mathrm{i}\}$ and hence $A^3 + A = O$. If k is not a multiple of 4 then $\mathrm{i}\mathbb{R} \cap (\{0\} \cup \sqrt[k+1]{1}) = \{0\}$ and hence $A = O$. This conclude the proof. □

It is well-known that: A is normal and $\sigma(A) \subseteq \mathbb{R}$ if and only if $A = A^*$ (i.e., A is *self-adjoint*). So, the hypothesis that "A is a k-generalized projection and $\sigma(A) \subseteq \mathbb{R}$" is equivalent to "$A$ is a k-generalized projection and $A^* = A$". Analogously, the hypothesis that "A is a k-generalized projection and $\sigma(A) \subseteq \mathrm{i}\mathbb{R}$" is equivalent to "$A$ is a k-generalized projection and $A^* = -A$" (i.e., A is *skew self-adjoint*).

Corollary 2 *Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a k-generalized projection. The range of A (denoted by $\mathcal{R}(A)$) is closed.*

Proof. Since A is a k-generalized projection, by Theorem 2 we get that A is normal and its spectrum is finite, so 0 is not a limited point of the spectrum of the normal operator A, then $\mathcal{R}(A)$ is closed. This completes the proof. □

A similar result to Theorem 2 can be established for matrices and it generalizes Corollary 4 in [3].

Corollary 3 *Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a k-generalized projection. Then A^{k+1} is an orthogonal projection.*

Proof. From Theorem 2, we get $A^{k+2} = A$ and then $(A^{k+1})^2 = A^{k+2}A^k = AA^k = A^{k+1}$. Moreover, A^{k+1} is an orthogonal projection because

$$(A^{k+1})^* - A^{k+1} = (A^k A)^* - A^k A = (A^* A)^* - A^* A = 0,$$

since $A^* A$ is self-adjoint. This completes the proof. □

It is clear that Corollary 2 and Corollary 3 generalize the results given in Corollary 3 in [3].

References

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