The inverse eigenvalue problem for a Hermitian reflexive matrix and the optimization problem

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Abstract

The inverse eigenvalue problem and the associated optimal approximation problem for Hermitian reflexive matrices with respect to a normal $\{k+1\}$-potent matrix are considered. First, we study the existence of the solutions of the associated inverse eigenvalue problem and present an explicit form for them. Then, when such a solution exists, an expression for the solution to the corresponding optimal approximation problem is obtained.

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1. Introduction

In this paper the set of $m \times n$ complex matrices will be denoted by $\mathbb{C}^{m\times n}$ and I_n will stand for the $n \times n$ identity matrix. We will consider the inner product in $\mathbb{C}^{m\times n}$ given by

$$\langle A, B\rangle = \operatorname{trace}(B^*A), \qquad \text{for all } A, B \in \mathbb{C}^{m\times n},$$

where B^* denotes the conjugate transpose of the matrix B. As usual, $\|A\|_F = \sqrt{\langle A, A\rangle}$ stands for the Frobenius norm of A. We recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called reflexive with respect to a certain matrix $J \in \mathbb{C}^{n\times n}$ if $A = JAJ$.

From now on, we will consider a $\{k+1\}$-potent normal matrix $J \in \mathbb{C}^{n\times n}$ (i.e., $JJ^* = J^*J$ and $J^{k+1} = J$, $k \in \mathbb{N}$). The set of all Hermitian matrices that are reflexive with respect to J will be denoted by $\mathcal{H}J^{n\times n}$, that is,

$$\mathcal{H}J^{n\times n} = \{A \in \mathbb{C}^{n\times n} : \ A^* = A = JAJ\}.$$

In this paper, we investigate the inverse eigenvalue problem for Hermitian matrices that are reflexive with respect to a $\{k+1\}$-potent normal matrix J. Specifically, we will solve the following two problems:

> **Inverse eigenvalue problem:** Find all matrices $A \in \mathcal{H}J^{n\times n}$ such that $AX = XD$ for a given matrix $X \in \mathbb{C}^{n\times m}$ and a given diagonal matrix $D \in \mathbb{R}^{m\times m}$.

In other words, if we solve this inverse eigenvalue problem, we are obtaining all matrices in $\mathcal{H}J^{n\times n}$ with a prescribed eigenstructure.

Let $\mathcal{S}$ denote the set of all solutions of the previous inverse eigenvalue problem.

> **Procrustes optimization problem:** If $\mathcal{S} \neq \emptyset$, for a given matrix $B \in \mathbb{C}^{n\times n}$, we look for $\hat{A} \in \mathcal{S}$ such that
>
> $$\min_{A\in\mathcal{S}} \|A - B\|_F = \|\hat{A} - B\|_F.$$

In other words, this problem finds the closest matrix $\hat{A}$ (in the set $\mathcal{S}$) to a given matrix B.

The inverse eigenvalue problem has been applied in a wide range of areas such as control theory, mechanic engineering, quantic physics and electromagnetism, etc. [1, 2, 12, 15]. In the literature the solution of the Procrustes problems have been found for a variety of classes of matrices. For instance, the problem for Hermitian matrices anti-reflexive with respect to a generalized reflection ($J^2 = I_n$ and $J^* = J$) was solved in [14]. The optimization problem related to reflexive matrices with respect to a pair of generalized reflections was studied in [4]. The inverse eigenvalue problem for Hermitian reflexive (anti-reflexive) matrices with respect to a Hermitian tripotent matrix was analyzed in [9]. Also, for structured matrices such as Toeplitz and generalized K-centrohermitian, problems like these ones have been studied in [6, 7]. For a left and right inverse eigenvalue problem with reflections we can refer to [11]. The problem treated in this paper extends all these known cases in the literature related to reflexivity.

We recall that for a given matrix $A \in \mathbb{C}^{m\times n}$, its Moore-Penrose inverse is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$. It always exists and it is unique [3]. We will denote $W^{(l)}(A) = I - A^\dagger A$ and $W^{(r)}(A) = I - AA^\dagger$. It is remarkable that $W^{(l)}(A^*) = W^{(r)}(A)$, $W^{(l)}(A)A^\dagger = O$ and $A^*W^{(r)}(A) = O$.

This paper is organized as follows. In Section 2, the structure of the set $\mathcal{H}J^{n\times n}$ is given. We further analyze necessary and sufficient conditions for the inverse eigenvalue problem to have a solution and an explicit solution is also presented. In Section 3, after analyzing the existence and uniqueness of the Procrustes problem, we find the solution of the optimization problem provided that the set $\mathcal{S}$ is not empty.

2. Inverse eigenvalue problem

Given a matrix $X \in \mathbb{C}^{n\times m}$ and a diagonal matrix $D \in \mathbb{R}^{m\times m}$, we look for solutions of the matrix equation

$$AX = XD \tag{1}$$

satisfying that $A \in \mathbb{C}^{n\times n}$ is Hermitian reflexive with respect to a $\{k+1\}$-potent normal matrix $J \in \mathbb{C}^{n\times n}$.

Notice that the diagonal matrix D has only real entries because A is Hermitian. Since J is normal, it is unitarily diagonalizable. The condition $J^{k+1} = J$ implies that the spectrum of J is included in $\{0\} \cup \Omega_k$, where Ω_k is the set of roots of unity of order k [8]. Then, there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$J = U\,\mathrm{diag}(\omega_1 I_{r_1}, \ldots, \omega_t I_{r_t}, O_{r_{t+1}})\,U^*, \tag{2}$$

with $\omega_i \in \Omega_k$, $i = 1, \ldots, t$, $r_1 + \cdots + r_t = \mathrm{rank}(J)$ and $r_{t+1} = n - \mathrm{rank}(J)$.

In order to find the structure of the matrix A we partition U^*AU in blocks, of adequate size according to the blocks of the partition of J, as follows:

$$U^*AU = \begin{bmatrix} A_{1,1} & \ldots & A_{1,t} & A_{1,t+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_{t,1} & \ldots & A_{t,t} & A_{t,t+1} \\ A_{t+1,1} & \ldots & A_{t+1,t} & A_{t+1,t+1} \end{bmatrix}. \tag{3}$$

From (2) and (3), the equality $A = JAJ$ yields

$$\begin{cases} A_{t+1,j} = O & \text{for} \quad j \in \{1, \ldots, t+1\} \\ A_{j,t+1} = O & \text{for} \quad j \in \{1, \ldots, t\} \\ A_{i,j} = \omega_i\omega_j A_{i,j} & \text{for} \quad i,j \in \{1, \ldots, t\} \end{cases}.$$

It then follows that $\omega_i\omega_j = 1$ or $A_{i,j} = O$ with $i,j \in \{1, \ldots, t\}$. Thus, for each $i,j \in \{1, \ldots, t\}$ we get $\omega_i = \overline{\omega}_j$ or the blocks $A_{i,j}$ and $A_{j,i}$ are both zero. We observe that the form of the matrix U^*AU depends on the roots of unity that appear in the decomposition of the matrix J. We will assume that the eigenvalues of the matrix J in (2) are arranged as

$$1, -1, \omega_3, \overline{\omega}_3, \ldots, \omega_{p-1}, \overline{\omega}_{p-1}, 0$$

(when 1 and -1 appear). Then, the matrix A has the form

$$A = U\,\mathrm{diag}(A_{1,1}, A_{2,2}, \tilde{A}_{3,4}, \ldots, \tilde{A}_{p-1,p}, O)\,U^* \tag{4}$$

where

$$\tilde{A}_{s,s+1} = \begin{bmatrix} O & A_{s,s+1} \\ A_{s+1,s} & O \end{bmatrix} \qquad \text{with } s \in \{3, 5, \ldots, p-1\} \tag{5}$$

for $p \geq 4$ being an adequate positive integer.

In (4), block $A_{1,1}$ is associated with the eigenvalue 1 and block $A_{2,2}$ with -1 (when they appear in J). Also, each block $\tilde{A}_{s,s+1}$ as in (5) is associated with the eigenvalues ω_s and $\overline{\omega}_s$.

Since A has to be Hermitian, $A^*_{i,i} = A_{i,i}$ for $i = 1, 2$ and $A^*_{s,s+1} = A_{s+1,s}$ for all $s \in \{3, 5, \ldots, p-1\}$, where p is as in (5). The explicit solution of the inverse eigenvalue problem is given in the following result.

Theorem 1. *Let $X \in \mathbb{C}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ be a diagonal matrix and $J \in \mathbb{C}^{n\times n}$ be a $\{k+1\}$-potent normal matrix as in (2). Consider the partition (of adequate sizes)*

$$X = U \begin{bmatrix} X_1^* & X_2^* & \tilde{X}_3^* & \dots & \tilde{X}_{p-1}^* & X_{p+1}^* \end{bmatrix}^* \tag{6}$$

with $\tilde{X}_s^* = \begin{bmatrix} X_s^* & X_{s+1}^* \end{bmatrix}$, $s \in \{3, 5, \dots, p-1\}$. Then there is a matrix $A \in \mathcal{H}J^{n\times n}$ such that $AX = XD$ if and only if $X_i D W^{(l)}(X_i) = O$,

$$X_i D = X_i X_i^\dagger (X_i^\dagger)^* D X_i^* X_i \tag{7}$$

hold for $i = 1, 2$,

$$X_s D W^{(l)}(X_{s+1}) = O, \qquad W^{(r)}(X_s^*) D X_{s+1}^* = O, \qquad X_s^* X_s D = D X_{s+1}^* X_{s+1} \tag{8}$$

hold for $s \in \{3, 5, \dots, p-1\}$ and $X_{p+1} D = O$. In this case, the general solution is given by

$$A = U \operatorname{diag}\left(X_1 D X_1^\dagger + Y_1 W^{(r)}(X_1), X_2 D X_2^\dagger + Y_2 W^{(r)}(X_2), \tilde{A}_{3,4}, \dots, \tilde{A}_{p-1,p}, O \right) U^*, \tag{9}$$

with

$$\tilde{A}_{s,s+1} = \begin{bmatrix} O & (X_s^*)^\dagger D X_{s+1}^* + W^{(r)}(X_s) Y_s W^{(r)}(X_{s+1}) \\ X_{s+1} D^* X_s^\dagger + W^{(r)}(X_{s+1}) Y_s^* W^{(r)}(X_s) & O \end{bmatrix}$$

where Y_1, Y_2 and Y_s are arbitrary matrices of suitable sizes for $s \in \{3, 5, \dots, p-1\}$.

Proof. We first assume that there is a Hermitian matrix A reflexive with respect to J such that $AX = XD$. By a similar reasoning as before, the form of the matrix A is given by (4). Substituting partition (6) in $AX = XD$, it is obtained that

$$\begin{bmatrix} A_{1,1} & O & O & \dots & O & O \\ O & A_{2,2} & O & \dots & O & O \\ O & O & \tilde{A}_{3,4} & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & \tilde{A}_{p-1,p} & O \\ O & O & O & \dots & O & O \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \tilde{X}_3 \\ \vdots \\ \tilde{X}_{p-1} \\ X_{p+1} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \tilde{X}_3 \\ \vdots \\ \tilde{X}_{p-1} \\ X_{p+1} \end{bmatrix} D.$$

Some block manipulations lead to

$$A_{i,i} X_i = X_i D,\ i = 1, 2; \qquad \tilde{A}_{s,s+1} \tilde{X}_s = \tilde{X}_s D,\ s \in \{3, 5, \dots, p-1\}; \qquad X_{p+1} D = O.$$

In order to solve the two first equations it is necessary to use generalized inverses. Each equation $A_{i,i} X_i = X_i D$ for $i = 1, 2$ has a solution $A_{i,i}$ if and only if $X_i D X_i^\dagger X_i = X_i D$. In these cases, the general solution is given by

$$A_{i,i} = X_i D X_i^\dagger + Y_i (I - X_i X_i^\dagger), \quad \text{for} \quad i = 1, 2, \tag{10}$$

where Y_1 and Y_2 are arbitrary matrices of adequate sizes [3]. The conditions $A_{i,i}^* = A_{i,i}$ are equivalent to $X_i D X_i^\dagger + Y_i(I - X_i X_i^\dagger) = (X_i^\dagger)^* D X_i^* + (I - X_i X_i^\dagger) Y_i^*$. Pre and post-multiplying by $X_i X_i^\dagger$ the previous equality we arrive at $X_i D X_i^\dagger = X_i X_i^\dagger (X_i^\dagger)^* D X_i^* X_i X_i^\dagger$. Finally, post-multiplying by X_i and using the condition $X_i D X_i^\dagger X_i = X_i D$ we obtain (7). By using the notations (5) and (6) and the fact that $A_{s+1,s}^* = A_{s,s+1}$, the equations given by $\tilde{A}_{s,s+1}\tilde{X}_s = \tilde{X}_s D$ yield to the matrix system

$$\left\{ \begin{array}{rcl} A_{s,s+1} X_{s+1} & = & X_s D \\ X_s^* A_{s,s+1} & = & D X_{s+1}^* \end{array} \right.$$

for each $s \in \{3, 5, \ldots, p-1\}$. The conditions given in (8) guarantee the existence of the solution of the previous matrix system [3]. Its solution can be expressed as

$$A_{s,s+1} = (X_s^*)^\dagger D X_{s+1}^* + W^{(l)}(X_s^*) Y_s W^{(r)}(X_{s+1}), \tag{11}$$

because $W^{(l)}(X_s^*) X_s D X_{s+1}^\dagger = O$. Finally, from (10) and (11), the general solution of the problem is given by (9). The converse is evident. ∎

The following result gives sufficient conditions for the third equality in (8).

Lemma 1. *Under the notation of Theorem 1, for each $i = 1, 2, \ldots, r_s$ we denote by $x_{i,\cdot}^{(s)} = \left[\begin{array}{ccc} x_{i1}^{(s)} & \ldots & x_{im}^{(s)} \end{array} \right]$ the rows of the matrix $X_s \in \mathbb{C}^{r_s \times m}$ and $|x_{i,\cdot}^{(s)}| = \left[\begin{array}{ccc} |x_{i1}^{(s)}| & \ldots & |x_{im}^{(s)}| \end{array} \right]$. If*

$$X_s^* X_s D = D X_{s+1}^* X_{s+1} \tag{12}$$

then the following conditions hold:

(a) $\det(D) = 0$ *or* $\prod_{i=1}^m \sigma_i(X_s) = \pm \prod_{i=1}^m \sigma_i(X_{s+1})$, *for all* $s = 3, 4, \ldots, p$.

(b) $\left\langle \sum_{i=1}^{r_s} |x_{i,\cdot}^{(s)}|^2 - \sum_{j=1}^{r_{s+1}} |x_{j,\cdot}^{(s+1)}|^2, \operatorname{diag}(D) \right\rangle = 0$, *for all* $s = 3, 4, \ldots, p$.

Proof. From (6), it is clear that $X_s^* X_s \in \mathbb{C}^{m \times m}$. By taking determinants in (12) we get $\det(D) = 0$ or $\det(X_s^* X_s) = \det(X_{s+1}^* X_{s+1})$. Since the determinant of a square matrix is the product of its eigenvalues, if $\sigma_i(.)$ denotes the singular values of (.), we have $\prod_{i=1}^m (\sigma_i(X_s))^2 = \prod_{i=1}^m (\sigma_i(X_{s+1}))^2$, from which (a) holds.

On the other hand, by using properties of the trace we get $\operatorname{tr}((X_s^* X_s - X_{s+1}^* X_{s+1}) D) = 0$. Rewriting the trace of the product of a matrix by a diagonal matrix as an inner product of the diagonals of each matrix, we derive condition (b). ∎

In what follows we analyze an interesting model useful in engineering modeling a certain mechanical or civil structure.

Example 1. *The matrix representation of a lineal undamped multi-degree-of-freedom vibration system is*

$$M\ddot{X} + KX = 0 \tag{13}$$

where the $n \times n$ real matrices K, M are the stiffness and mass matrices, respectively. The solution vector X must satisfy $\ddot{X} = -\omega^2 X$ provided that we are attempting a vector with components $x_i(t) = X_i\, sin(\omega t + \varphi_i)$, for $i = 1, \ldots, n$. Substituting in the equation (13) we get

$$-\omega^2 MX + KX = 0.$$

Since in real situations M is generally a positive definite matrix, it is nonsingular, and pre-multiplying the last equation by M^{-1} we get

$$M^{-1}KX = \omega^2 X.$$

When the eigenstructure given by X and ω and the mass matrix M are available, the problem of finding some matrix K can be regarded as an inverse eigenvalue problem. In this case, the general matrix equation becomes $AX = XD$ for $A = M^{-1}K$ and D being a diagonal matrix with all entries equal ω^2. Let us consider the inputs

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ X = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},\ J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to see that -1 is a double eigenvalue of J and $U = I_2$. So, $r_1 = 2$. In this case $U^*X = X_1$. Moreover, we have $X^\dagger = X$ and all the conditions in Theorem 1 are satisfied. According to (9),

$$K = A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

It is easy to see that $A \in \mathcal{H}J^{2\times 2}$ and both matrices M and K are symmetric and positive-definite.

3. Procrustes optimization problem

In this section the optimal approximation problem is treated. We recall that $\mathcal{S}$ is the set of all the matrices $A \in \mathcal{H}J^{n\times n}$ such that $AX = XD$. For a given matrix $B \in \mathbb{C}^{n\times n}$, if $\mathcal{S} \neq \emptyset$, we look for $\hat{A} \in \mathcal{S}$ such that

$$\min_{A\in\mathcal{S}} \|A - B\|_F = \|\hat{A} - B\|_F. \tag{14}$$

It is easy to see that $\mathcal{S}$ is a closed convex set. The uniqueness of the solution of problem (14) is assured by the fact that $\mathbb{C}^{n\times n}$ is a uniformly convex Banach space under the Frobenius norm (see, for example, [5], pp. 22).

In order to find the solution $\hat{A}$, we first transform the problem into a simpler one.

3.1. *The simplified problem*

Let U be the unitary matrix that diagonalizes J as in (2) and let $B \in \mathbb{C}^{n\times n}$ be an arbitrary given matrix partitioned as

$$B = U \begin{bmatrix} B_{1,1} & \dots & B_{1,p} & B_{1,p+1} \\ \vdots & \ddots & \vdots & \vdots \\ B_{p,1} & \dots & B_{p,p} & B_{p,p+1} \\ B_{p+1,1} & \dots & B_{p+1,p} & B_{p+1,p+1} \end{bmatrix} U^*,$$

where $B_{i,j} \in \mathbb{C}^{m_i \times t_j}$ and $m_1 + \cdots + m_p + m_{p+1} = t_1 + \cdots + t_p + t_{p+1} = n$.

Since Frobenius norm is unitarily invariant, by using the solution (9), it is obtained

$$\begin{aligned} \|A - B\|_F^2 &= \|U^*AU - U^*BU\|_F^2 \\ &= \|A_{1,1} - B_{1,1}\|_F^2 + \|A_{2,2} - B_{2,2}\|_F^2 + \|A_{3,4} - B_{3,4}\|_F^2 + \|A_{4,3} - B_{4,3}\|_F^2 + \\ &\quad + \cdots + \|A_{p-1,p} - B_{p-1,p}\|_F^2 + \|A_{p,p-1} - B_{p,p-1}\|_F^2 + \sum_{i=3}^{p+1} \|B_{i,i}\|_F^2 + \\ &\quad + \sum_{i=2}^{p+1} \|B_{i,1}\|_F^2 + \sum_{j=2}^{p+1} \|B_{1,j}\|_F^2 + \sum_{\substack{i=1 \\ i\neq 2}}^{p+1} \|B_{i,2}\|_F^2 + \sum_{\substack{j=1 \\ j\neq 2}}^{p+1} \|B_{2,j}\|_F^2 + \\ &\quad + \sum_{i\in J_1} \sum_{j=i+2}^{p+1} (\|B_{i,j}\|_F^2 + \|B_{i+1,j}\|_F^2) + \sum_{j\in J_1} \sum_{i=j+2}^{p+1} (\|B_{i,j}\|_F^2 + \|B_{i,j+1}\|_F^2) + \\ &\quad + \|B_{p-1,p+1}\|_F^2 + \|B_{p,p+1}\|_F^2 + \|B_{p+1,p-1}\|_F^2 + \|B_{p+1,p}\|_F^2 \end{aligned}$$

where $J_1 = \{3, 5, 7, \dots, p-3\}$.

Now, the problem (14) leads equivalently to compute

$$\begin{aligned} \min\nolimits_{A\in\mathcal{S}} \quad & (\|A_{1,1} - B_{1,1}\|_F^2 + \|A_{2,2} - B_{2,2}\|_F^2 + \|A_{3,4} - B_{3,4}\|_F^2 + \|A_{4,3} - B_{4,3}\|_F^2 + \cdots + \\ & + \|A_{p-1,p} - B_{p-1,p}\|_F^2 + \|A_{p,p-1} - B_{p,p-1}\|_F^2) \end{aligned} \tag{15}$$

because all the remaining elements are fixed. Now, we are going to solve this last problem.

3.2. *Explicit form of the solution*

We need some previous results in order to minimize some special structures of matrices.

Lemma 2. *Let $M_1 \in \mathbb{C}^{m\times r}$, $M_2 \in \mathbb{C}^{n\times t}$, $R_1 \in \mathbb{C}^{r\times n}$ and $R_2 \in \mathbb{C}^{r\times n}$. Then the solutions of the minimization problem*

$$\min_{Y\in\mathbb{C}^{r\times n}} \|W^{(l)}(M_1) Y W^{(r)}(M_2) - R_1\|_F^2 + \min_{Y\in\mathbb{C}^{r\times n}} \|W^{(l)}(M_1) Y W^{(r)}(M_2) - R_2\|_F^2 \tag{16}$$

are the matrices $\hat{Y} \in \mathbb{C}^{r\times n}$ given by

$$\hat{Y} = \frac{1}{2}(R_1 + R_2) + G M_2 M_2^\dagger + M_1^\dagger M_1 G - M_1^\dagger M_1 G M_2 M_2^\dagger$$

for arbitrary $G \in \mathbb{C}^{r\times n}$.

Proof. Properties of the Moore-Penrose inverse, inner product, and trace of a matrix allow us to make the following calculations:

$$\begin{aligned}
& \|(I_r - M_1^\dagger M_1)Y(I_n - M_2M_2^\dagger) - R_1\|_F^2 + \|(I_r - M_1^\dagger M_1)Y(I_n - M_2M_2^\dagger) - R_2\|_F^2 = \\
= \;& 2\|(I_r - M_1^\dagger M_1)Y(I_n - M_2M_2^\dagger) - \frac{1}{2}(R_1 + R_2)\|_F^2 + \frac{1}{2}\|R_1 - R_2\|_F^2 \\
= \;& 2\|(I_r - M_1^\dagger M_1)[Y(I_n - M_2M_2^\dagger) - \frac{1}{2}(R_1 + R_2)] - \frac{1}{2}M_1^\dagger M_1(R_1 + R_2)\|_F^2 + \frac{1}{2}\|R_1 - R_2\|_F^2 \\
= \;& 2\|(I_r - M_1^\dagger M_1)[Y(I_n - M_2M_2^\dagger) - \frac{1}{2}(R_1 + R_2)]\|_F^2 + \frac{1}{2}\|M_1^\dagger M_1(R_1 + R_2)\|_F^2 + \\
& + \frac{1}{2}\|R_1 - R_2\|_F^2 \\
= \;& 2\|(I_r - M_1^\dagger M_1)[Y - \frac{1}{2}(R_1 + R_2)](I_n - M_2M_2^\dagger)\|_F^2 + \frac{1}{2}\|M_1^\dagger M_1(R_1 + R_2)\|_F^2 + \\
& + \frac{1}{2}\|(I_r - M_1^\dagger M_1)(R_1 + R_2)M_2M_2^\dagger\|_F^2 + \frac{1}{2}\|R_1 - R_2\|_F^2.
\end{aligned}$$

Now, to find

$$\min_{Y\in\mathbb{C}^{r\times n}} \|W^{(l)}(M_1)YW^{(r)}(M_2) - R_1\|_F^2 + \min_{Y\in\mathbb{C}^{r\times n}} \|W^{(l)}(M_1)YW^{(r)}(M_2) - R_2\|_F^2$$

is equivalent to find

$$\min_{Y\in\mathbb{C}^{r\times n}} \|(I_r - M_1^\dagger M_1)[Y - \frac{1}{2}(R_1 + R_2)](I_n - M_2M_2^\dagger)\|_F^2. \tag{17}$$

By [3], we have that the solution of (17) is $\hat{Y} = \frac{1}{2}(R_1 + R_2) + GM_2M_2^\dagger + M_1^\dagger M_1 G - M_1^\dagger M_1 G M_2 M_2^\dagger$, where $G \in \mathbb{C}^{r\times n}$ is an arbitrary matrix. Then, the solution of (16) is expressed as $\hat{Y} = \frac{1}{2}(R_1 + R_2) + GM_2M_2^\dagger + M_1^\dagger M_1 G - M_1^\dagger M_1 G M_2 M_2^\dagger$ as desired. ■

Remark 1. *It is easy to see that, in Lemma 2,*

$$\begin{aligned}
& \|W^{(l)}(M_1)\hat{Y}W^{(r)}(M_2) - R_1\|_F^2 + \|W^{(l)}(M_1)\hat{Y}W^{(r)}(M_2) - R_2\|_F^2 = \\
= & \frac{1}{2}\|R_1 - R_2\|_F^2 + \|M_1^\dagger M_1(R_1 + R_2)\|_F^2 + \|W^{(l)}(M_1)(R_1 + R_2)M_2M_2^\dagger\|_F^2,
\end{aligned}$$

which is invariant for any choice of G.

A particular case will be needed to obtain the general solution.

Corollary 1. *Let $M \in \mathbb{C}^{n\times m}$ and $R \in \mathbb{C}^{r\times n}$. Then there is a matrix $\hat{Y} \in \mathbb{C}^{r\times n}$ such that*

$$\min_{Y\in\mathbb{C}^{r\times n}} \|Y\left(I_n - MM^\dagger\right) - R\|_F = \|\hat{Y}(I_n - MM^\dagger) - R\|_F \tag{18}$$

where $\hat{Y} = R + GMM^\dagger$ for an arbitrary matrix $G \in \mathbb{C}^{r\times m}$.

Proof. It is a direct consequence of Lemma 2 by taking $M_1 = O$, $M_2 = M$ and $R_1 = R_2 = R$. ■

Remark 2. *In Corollary 1, the matrix $\hat{Y}$ can be simplified as follows: If we substitute $M_1 = O$, $M_2 = M$ and $R_1 = R_2 = R$ into (17), we get*

$$\min_{Y \in \mathbb{C}^{r\times n}} \| (Y - R) \left(I_n - MM^\dagger\right) \|_F. \tag{19}$$

Now we have that $(Y - R)\left(I_n - MM^\dagger\right) = O$ if and only if $\mathcal{R}(I_n - MM^\dagger) \subseteq \mathcal{N}(Y - R)$. Since $I_n - MM^\dagger$ is a projector, we have $\mathcal{R}(I_n - MM^\dagger) = \mathcal{N}(MM^\dagger) = \mathcal{N}(M^\dagger)$ because $M^\dagger MM^\dagger = M^\dagger$. It is not hard to show that $\mathcal{N}(M^\dagger) \subseteq \mathcal{N}(Y - R)$ is equivalent to the existence of a matrix $G \in \mathbb{C}^{r\times m}$ such that $Y - R = GM^\dagger$. Then a new expression for $\hat{Y}$ is $\hat{Y} = R + GM^\dagger$.

Remark 3. *In Corollary 1, it is easy to see that $\|\hat{Y}\left(I_n - MM^\dagger\right) - R\|_F = \|RMM^\dagger\|_F$ holds. That is, again the value of $\|\hat{Y}\left(I_n - MM^\dagger\right) - R\|_F$ is invariant for any choice of G.*

Now, we are ready to give the explicit solution of problem (14).

Theorem 2. *Let $B \in \mathbb{C}^{n\times n}$ be a partitioned matrix as*

$$U^*BU = \begin{bmatrix} B_{1,1} & \dots & B_{1,p} & B_{1,p+1} \\ \vdots & \ddots & \vdots & \vdots \\ B_{p,1} & \dots & B_{p,p} & B_{p,p+1} \\ B_{p+1,1} & \dots & B_{p+1,p} & B_{p+1,p+1} \end{bmatrix} \tag{20}$$

where $B_{i,j} \in \mathbb{C}^{m_i \times t_j}$, $m_1 + m_2 + \cdots + m_{p+1} = t_1 + t_2 + \cdots + t_{p+1} = n$ and U is as in (2). Under the conditions (and notations) of Theorem 1, if $\mathcal{S} \neq \emptyset$, then the problem (14) has a unique solution given by

$$\hat{A} = U \operatorname{diag}(A_{1,1}, A_{2,2}, \tilde{A}_{3,4}, \dots, \tilde{A}_{p-1,p}, O)\, U^*, \tag{21}$$

where $A_{i,i} = X_i D X_i^\dagger + B_{i,i} W^{(r)}(X_i)$ for $i = 1, 2$, and

$$\tilde{A}_{s,s+1} = \begin{bmatrix} O & (X_s^*)^\dagger D X_{s+1}^* \\ X_{s+1} D^* X_s^\dagger & O \end{bmatrix} + \\ + \ \frac{1}{2}\begin{bmatrix} O & W^{(r)}(X_s)(B_{s,s+1} + B_{s+1,s}^*)W^{(r)}(X_{s+1}) \\ W^{(r)}(X_{s+1})(B_{s,s+1}^* + B_{s+1,s})W^{(r)}(X_s) & O \end{bmatrix},$$

for $s \in \{3, 5, \dots, p-1\}$.

Proof. By using the expressions of $A_{1,1}$, $A_{2,2}$ given by (10) and the relationship $A_{s,s+1} = A^*_{s+1,s}$ given by (11) for $s \in \{3, 5, \ldots, p-1\}$ we have that

$$\|A_{i,i} - B_{i,i}\|_F^2 = \|Y_i(I_{t_i} - X_i X_i^\dagger) - (B_{i,i} - X_i D X_i^\dagger)\|_F^2 \quad \text{for} \quad i = 1, 2,$$

and

$$\begin{aligned} \|A_{s,s+1} - B_{s,s+1}\|_F^2 + \|A_{s+1,s} - B_{s+1,s}\|_F^2 &= \|A_{s,s+1} - B_{s,s+1}\|_F^2 + \|A_{s,s+1} - B^*_{s+1,s}\|_F^2 \\ &= \|W^{(l)}(X_s^*) Y_s W^{(r)}(X_{s+1}) - R_s\|_F^2 + \|W^{(l)}(X_s^*) Y_s W^{(r)}(X_{s+1}) - R_{s+1}\|_F^2 \end{aligned}$$

where R_s and R_{s+1} are given by

$$\begin{aligned} R_s &= B_{s,s+1} - (X_s^*)^\dagger D X_{s+1}^* - W^{(l)}(X_s^*) X_s D X_{s+1}^\dagger, \\ R_{s+1} &= B^*_{s+1,s} - (X_s^*)^\dagger D X_{s+1}^* - W^{(l)}(X_s^*) X_s D X_{s+1}^\dagger, \end{aligned}$$

for $s \in \{3, 5, \ldots, p-1\}$. Since (14) and (15) are equivalent problems, we have to find

$$\min_{Y_i} \|Y_i(I_{t_i} - X_i X_i^\dagger) - (B_{i,i} - X_i D X_i^\dagger)\|_F^2$$

for $i = 1, 2$, and

$$\min_{Y_s} \|W^{(l)}(X_s^*) Y_s W^{(r)}(X_{s+1}) - R_s\|_F^2 \ + \ \min_{Y_s} \|W^{(l)}(X_s^*) Y_s W^{(r)}(X_{s+1}) - R_{s+1}\|_F^2$$

for $s \in \{3, 5, \ldots, p-1\}$.

From Corollary 1 we can deduce that there exist matrices $Y_i \in \mathbb{C}^{m_i \times t_i}$, such that

$$Y_i = B_{i,i} - X_i D X_i^\dagger + G_i X_i^\dagger$$

where $G_i \in \mathbb{C}^{m_i \times m_i}$ are arbitrary matrices for $i = 1, 2$. From Lemma 2, there exist matrices $Y_s \in \mathbb{C}^{m_s \times t_{s+1}}$ such that

$$Y_s = \frac{1}{2}(R_s + R_{s+1}) - (X_s^*)^\dagger X_s^* G_s X_{s+1} X_{s+1}^\dagger + G_s X_{s+1} X_{s+1}^\dagger + (X_s^*)^\dagger X_s^* G_s,$$

where $G_s \in \mathbb{C}^{m_s \times t_{s+1}}$ are arbitrary for $s \in \{3, 5, \ldots, p-1\}$. Substituting Y_1, Y_2 and Y_s, $s \in \{3, 5, \ldots, p-1\}$, in (9), we obtain that the unique solution of the problem (14) is given by (21). ∎

A similar reasoning as in this paper allows us to give the corresponding results for matrices A which are anti-reflexive with respect to a $\{k+1\}$-potent normal matrix J.

4. Algorithm and numerical example

In this section, we first give an algorithm to solve the optimal approximation problem.
Algorithm

Inputs: $B \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$ diagonal, $J \in \mathbb{C}^{n \times n}$ $\{k+1\}$-potent normal.
Output: $\hat{A}$.

Step 1 Compute $r_1, \cdots, r_{t+1}$ and U as in (2).

Step 2 Partition U^*X to obtain $X_1, X_2, \ldots, X_p, X_{p+1}$ as in (6).

Step 3 Perform $X_1^\dagger, X_2^\dagger, \ldots, X_p^\dagger$.

Step 4 If $X_i D W^{(l)}(X_i) = O$, $X_i D = X_i X_i^\dagger (X_i^\dagger)^* D X_i^* X_i$ for $i = 1, 2$, $X_s D W^{(l)}(X_{s+1}) = O$, $W^{(r)}(X_s^*) D X_{s+1}^* = O$, $X_s^* X_s D = D X_{s+1}^* X_{s+1}$ for $s \in \{3, 5, \ldots, p-1\}$ and $X_{p+1} D = O$, then go to Step 5. Otherwise, Stop.

Step 5 Partition $B_{i,j}$, $i, j \in \{1, 2, \ldots, p+1\}$ as in (20).

Step 6 Compute $A_{i,i} = X_i D X_i^\dagger + B_{i,i} W^{(r)}(X_i)$ for $i = 1, 2$.

Step 7 Compute $A_{s,s+1} = (X_s^*)^\dagger D X_{s+1}^* + \frac{1}{2} W^{(r)}(X_s)(B_{s,s+1} + B_{s+1,s}^*) W^{(r)}(X_{s+1})$ for $s \in \{3, 5, \ldots, p-1\}$.

Step 8 Compute $A_{s+1,s} = A_{s,s+1}^*$ for $s \in \{3, 5, \ldots, p-1\}$.

Step 9 Set $\tilde{A}_{s,s+1}$ for $s \in \{3, 5, \ldots, p-1\}$ as in (5).

Step 10 Perform $\hat{A} = U \operatorname{diag}(A_{1,1}, A_{2,2}, \tilde{A}_{3,4}, \ldots, \tilde{A}_{p-1,p}, O)\, U^*$.

End

Our algorithm can be easily implemented on a computer using MATLAB R2013b package.

Next numerical result shows the performance of our algorithm.

Example 2. *Let us consider the inputs*

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; X = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ -\frac{1+i}{2} & \frac{1+i}{2} & -\frac{1+i}{2} & 0 & 0 \\ i & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$J = \begin{bmatrix} \frac{1}{4} + i\frac{\sqrt{3}}{4} & 0 & \frac{3\sqrt{2}-\sqrt{6}}{8} - i\frac{3\sqrt{2}+\sqrt{6}}{8} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \frac{3\sqrt{2}+\sqrt{6}}{8} + i\frac{3\sqrt{2}-\sqrt{6}}{8} & 0 & \frac{1}{4} + i\frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be seen that $J = UD_JU^*$ where

$$D_J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1+i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0.5+0.5i & 0 & -0.5-0.5i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and, moreover, $JJ^* = J^*J$ and $J^7 = J$. Notice that J is singular (the case for non-singular matrices was solved in [10]). So, $r_1 = r_2 = r_3 = r_4 = r_5 = 1$ and this is Step 1. After computing U^*X we obtain

$$X_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Moore-Penrose of these matrices required in Step 3 are

$$X_1^\dagger = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^*, \qquad X_2^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^*,$$

$$X_3^\dagger = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \end{bmatrix}^*, \qquad X_4^\dagger = \begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 \end{bmatrix}^*, \quad X_5^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*.$$

All the equalities in Step 4 are satisfied. We now compute

$$U^*BU = \begin{bmatrix} 0.8536+0.8536i & -0.7071i & 0.1464-0.8536i & 0.5-0.2071i & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.8536-0.1464i & -0.7071i & 0.1464+0.1464i & -0.5-1.2071i & 0 \\ -0.5000+1.2071i & 0 & 0.5000+0.2071i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{i,j} \end{bmatrix}_{1 \le i,j \le 5}.$$

Since $W^{(r)}(X_i) = O$ for $i = 1, 2$, we get $A_{1,1} = 2$ and $A_{2,2} = 1$. In Step 7, $A_{3,4} = -1$. Thus, $A_{4,3} = -1$. Finally, the matrix in Step 10 is

$$\begin{aligned} \hat{A} &= U \operatorname{diag}(A_{1,1}, A_{2,2}, \tilde{A}_{3,4}, O)\, U^* \\ &= \begin{bmatrix} 1.0000 & 0 & 0.7071-0.7071i & -0.7071i & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.7071+0.7071i & 0 & 1 & -0.5000+0.5000i & 0 \\ 0.7071i & 0 & -0.5000-0.5000i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

In this case, $\|\hat{A} - B\|_F = 3.2536$.

5. Conclusions

The interest of inverse eigenvalue problems is remarkable for their applications in engineering [1, 2, 6, 7, 12, 15]. Recently, some specific problems have been tackled [9, 10, 11, 14]. In particular, a similar problem has been treated in [13] solved by different techniques than those used in this paper. Our approach extends some of the sets $\mathcal{S}$ on which the solution is found. While the common sets include Hermitian, skew-Hermitian or unitary matrices we consider the case of normal matrices covering all the mentioned cases. On the other hand, a matrix J that gives the reflexivity is needed. In general, the common studies were done for generalized reflexions ($J^2 = I$ and $J^* = J$) that are obviously nonsingular matrices. In our study, the case of general powers of J are considered, besides allowing singularity on the matrix J. An algorithm has been designed for solving the Procrustes problem considered. Two applications have been given illustrating the applicability of our results. The first one gives a real application to mechanical and/or civil structure and the second one considers the case of a singular matrix J.

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