On a matrix group constructed from an $\{R, s+1, k\}$-potent matrix

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Abstract

For a $\{k\}$-involutory matrix $R \in \mathbb{C}^{n\times n}$ (that is, $R^k = I_n$) and $s \in \{0, 1, 2, 3, \dots\}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{R, s+1, k\}$-potent if A satisfies $RA = A^{s+1}R$. In this paper, a matrix group corresponding to a fixed $\{R, s+1, k\}$-potent matrix is explicitly constructed and properties of this group are derived and investigated. This constructed group is then reconciled with the classical matrix group G_A that is associated with a generalized group invertible matrix A.

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1 Introduction

For a matrix $A \in \mathbb{C}^{n\times n}$, the *group inverse*, if it exists, is the unique matrix $A^\#$ satisfying the matrix equations

$$AA^\#A = A, \quad A^\#AA^\# = A^\#, \; AA^\# = A^\#A. \tag{1}$$

It is well known that $A^\#$ exists if and only if rank A^2 = rank A. Further information on group inverses and their applications can be found in [4], and a collection of results on the importance of group inverses of certain classes of singular matrices in several application areas can be found in the recent book [5]. Theorem 7.2.5 in [4, pp. 124] states that a

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square matrix A of rank $r > 0$ belongs to a (multiplicative) matrix group G_A if and only if rank $A^2 =$ rank A. In this case, $A \in \mathbb{C}^{n\times n}$ has the canonical form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \tag{2}$$

where $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ are nonsingular matrices. The matrix group G_A corresponding to A is then given by

$$G_A = \left\{ P \begin{bmatrix} X & O \\ O & O \end{bmatrix} P^{-1} : \ X \in \mathbb{C}^{r\times r}, \ \mathrm{rank}(X) = r \right\}. \tag{3}$$

The identity element in G_A is

$$E = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^{-1},$$

where $I_r \in \mathbb{C}^{r\times r}$ is the identity matrix, and the inverse of A in this group is

$$A^g = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}.$$

Some results related to matrix groups on nonnegative matrices can be found in [1].

Note that the inverse A^g of A in G_A satisfies the matrix equations in (1), and by uniqueness, $A^g = A^{\#}$; the identity element E in G_A satisfies $E = AA^{\#} = A^{\#}A$.

For $p \in \{2, 3, \dots\}$, a matrix A is called $\{p\}$-*group involutory* if the group inverse of A exists and satisfies $A^{\#} = A^{p-1}$; in such a case, an equivalent condition is that $A^{p+1} = A$ (see [2, 3]).

Throughout this paper we will use matrices $R \in \mathbb{C}^{n\times n}$ such that $R^k = I_n$ where $k \in \{2, 3, 4, \dots\}$. These matrices R are called $\{k\}$-*involutory* [11, 12, 14], and they generalize the well-studied *involutory matrices* ($k = 2$). Note that the definition given in [11, 12] differs from that in [14]; in this paper we adopt the definition given in [14], namely that R is $\{k\}$-involutory does not require that k be minimal with respect to $R^k = I$.

Let $R \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix and $s \in \{0, 1, 2, 3, \dots\}$. A matrix $A \in \mathbb{C}^{n\times n}$ is called $\{R, s+1, k\}$-*potent* if it satisfies

$$RA = A^{s+1}R. \tag{4}$$

These matrices generalize *centrosymmetric matrices* (that is, matrices $A \in \mathbb{C}^{n\times n}$ such that $AJ = JA$ where J is the $n \times n$ antidiagonal matrix; see [13]), the matrices $A \in \mathbb{C}^{n\times n}$ such that $AP = PA$ where P is an $n \times n$ permutation matrix (see [10]), and $\{K, s+1\}$-*potent matrices* (that is, matrices $A \in \mathbb{C}^{n\times n}$ for which $KAK = A^{s+1}$ where $K^2 = I_n$; see [7, 8]). For a study of $\{R, s+1, k\}$-potent matrices we refer the reader to [6] where, in particular, the following characterization was given.

Theorem 1. *[6, Theorem 1] Let $R \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix, $s \in \{1, 2, 3, \dots\}$, $n_{s,k} = (s+1)^k - 1$, and $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) A *is* $\{R, s+1, k\}$*-potent.*

(b) A *is an* $\{n_{s,k}\}$*-group involutory matrix and there exist disjoint projectors* $P_0, P_1, \ldots, P_{n_{s,k}}$ *with*

$$A = \sum_{j=1}^{n_{s,k}} \omega^j P_j \qquad \text{and} \qquad \sum_{j=0}^{n_{s,k}} P_j = I_n,$$

where $\omega = e^{\frac{2\pi i}{n_{s,k}}}$*, and* $P_j = O$ *when* $\omega^j \notin \sigma(A)$ *and* $P_0 = O$ *when* $0 \notin \sigma(A)$*, and such that the projectors* $P_0, P_1, \ldots, P_{n_{s,k}}$ *satisfy*

(i) For each $i \in \{1, \ldots, n_{s,k} - 1\}$*, there exists a unique* $j \in \{1, \ldots, n_{s,k} - 1\}$ *such that* $RP_iR^{-1} = P_j$*,*

(ii) $RP_{n_{s,k}}R^{-1} = P_{n_{s,k}}$*, and*

(iii) $RP_0R^{-1} = P_0$*.*

(c) A *is diagonalizable and there exist disjoint projectors* $P_0, P_1, \ldots, P_{n_{s,k}}$ *satisfying conditions (i), (ii), and (iii) given in (b).*

In [9], a matrix group constructed from a given $\{K, s+1\}$-potent matrix was presented and studied. The goal of this paper is to construct a matrix group corresponding to a given $\{R, s+1, k\}$-potent matrix. We then reconcile this constructed group with the matrix group G_A given in (3).

2 First results

In this section we assume $s \geq 1$. We now establish properties of $\{R, s+1, k\}$-potent matrices.

Lemma 1. *Suppose that* $A \in \mathbb{C}^{n \times n}$ *is an* $\{R, s+1, k\}$*-potent matrix. Then the following properties hold.*

(a) $A^{(s+1)^k} = A$*.*

(b) $A^\# = A^{(s+1)^k - 2}$ *and the group projector* $AA^\#$ *satisfies* $AA^\# = A^{(s+1)^k - 1}$*.*

(c) $(A^{(s+1)^k - 1})^j = A^{(s+1)^k - 1}$ *for every* $j \in \{1, 2, 3, \ldots\}$*.*

(d) $R^p A^j = A^{j(s+1)^p} R^p$ *for every* $p \in \{1, 2, \ldots, k\}$*,* $j \in \{1, 2, \ldots, (s+1)^k - 1\}$*. In particular,* R^p *and* $A^{(s+1)^k - 1}$ *commute, the matrices* A^j *are* $\{R, s+1, k\}$*-potent and* A *is* $\{R^p, (s+1)^p - 1, k\}$*-potent.*

(e) $(A^j R^p)^m = A^{j[(s+1)^{mp} - 1]/[(s+1)^p - 1]} R^{mp}$*, for every* $j \in \{1, 2, \ldots, (s+1)^k - 1\}$*,* $p \in \{1, 2, \ldots, k\}$*,* $m \in \{1, 2, \ldots, k\}$*. In particular,*

(e)' $(A^s R)^m = A^{(s+1)^m - 1} R^m$ *for every* $m \in \{1, 2, \ldots, k\}$*.*

(f) For every $j, \ell \in \{1, 2, \ldots, (s+1)^k - 1\}$*,* $p, m \in \{1, 2, \ldots, k\}$*,* $(A^j R^p)(A^\ell R^m) = A^{\ell'} R^{p'}$*, where* $\ell' \equiv \ell(s+1)^p + j \; [\text{mod } ((s+1)^k - 1)]$ *and* $p' \equiv p + m \; [\text{mod } (k)]$*.*

(g) $(A^j R^p) A^{(s+1)^k - 1} = A^{(s+1)^k - 1}(A^j R^p) = A^j R^p$*, for every* $j \in \{1, 2, \ldots, (s+1)^k - 1\}$*,* $p \in \{1, 2, \ldots, k\}$*.*

(h) For every $j \in \{1, 2, \ldots, (s+1)^k - 1\}$*,* $p \in \{1, 2, \ldots, k\}$*, the following equalities hold:* $(A^\ell R^{k-p})(A^j R^p) = (A^j R^p)(A^\ell R^{k-p}) = A^{(s+1)^k - 1}$*, where* ℓ *is the unique element of* $\{1, 2, \ldots, (s+1)^k - 1\}$ *such that* $\ell \equiv -j(s+1)^{k-p} \; [\text{mod } ((s+1)^k - 1)]$*.*

(i) $(AR)^{ks+1} = AR$*.*

Proof. Statements (a) and (b) were proved in [6]. Using (a),

$$(A^{(s+1)^k - 1})^2 = A^{(s+1)^k} A^{(s+1)^k - 2} = A A^{(s+1)^k - 2} = A^{(s+1)^k - 1},$$

and now (c) follows by induction.

We next prove (d). First note that

$$RAR^{-1} = A^{s+1} \tag{5}$$

implies $RA^jR^{-1} = A^{j(s+1)}$, for all $j \geq 1$. Thus, if A is $\{R, s+1, k\}$-potent then so is A^j for all $j \geq 1$. In particular, let $j = s+1$. Then

$$RA^{s+1}R^{-1} = A^{(s+1)^2}, \tag{6}$$

and (5) and (6) gives $R^2AR^{-2} = A^{(s+1)^2}$. By induction, $R^pAR^{-p} = A^{(s+1)^p}$ for all $p \geq 1$. Since for all $j > 1$, A^j is also $\{R, s+1, k\}$-potent, it follows that $R^pA^jR^{-p} = A^{j(s+1)^p}$ for all $j \geq 1$ and all $p \geq 1$. This proves (d).

For (e), the equality is clear for $m = 1$. For $m = 2$, we have

$$\begin{aligned} (A^j R^p)^2 &= A^j R^p A^j R^p \\ &= A^j A^{j(s+1)^p} R^{2p}, \text{ by } (d) \\ &= A^{j(1+(s+1)^p)} R^{2p}. \end{aligned}$$

The general case $(A^jR^p)^m = A^{j[1+(s+1)^p+(s+1)^{2p}+\ldots+(s+1)^{(m-1)p}]}R^{mp}$ follows by induction. The identity $[(s+1)^p - 1][(s+1)^{(m-1)p} + \cdots + (s+1)^p + 1] = (s+1)^{mp} - 1$ yields the result. For the proof of $(e)'$, it is enough to set $j = s$ and $p = 1$ in (e).

Statement (f) follows easily from (d). Next, by using (c) and (d),

$$(A^jR^p)A^{(s+1)^k-1} = A^jA^{(s+1)^k-1}R^p = A^{j-1}A^{(s+1)^k}R^p = A^{j-1}AR^p = A^jR^p$$

for every $j \in \{1, 2, \ldots, (s+1)^k - 1\}$ and $p \in \{1, 2, \ldots, k\}$. This proves one equality in (g). The other equality can be directly shown as

$$A^{(s+1)^k-1}(A^jR^p) = A^{(s+1)^k}A^{j-1}R^p = A^jR^p.$$

For the proof of (h), let $j \in \{1, 2, \ldots, (s+1)^k - 1\}$. By (d), there exists ℓ such that $(A^\ell R^{k-p})(A^j R^p) = A^{(s+1)^k-1}$ if and only if $A^{\ell + j(s+1)^{k-p}} = A^{(s+1)^k-1}$. This last equality holds if and only if $\ell \equiv -j(s+1)^{k-p}$ [mod $((s+1)^k - 1)$]. Using this value of ℓ we can get $\ell(s+1)^p \equiv -j(s+1)^k$ [mod $((s+1)^k - 1)$]. Now,

$$(A^j R^p)(A^\ell R^{k-p}) = A^j A^{\ell(s+1)^p} R^p R^{k-p} = A^{j(s+1)^k} A^{\ell(s+1)^p} = A^{j(s+1)^k + \ell(s+1)^p} = A^{(s+1)^k-1},$$

which leads to (h). Observe that $\ell \equiv -j(s+1)^{k-p}$ [mod $((s+1)^k - 1)$] is equivalent to $j(s+1)^k \equiv -\ell(s+1)^p$ [mod $((s+1)^k - 1)$].

Finally, by setting $j = p = 1$ and $m = k$ in (e), we obtain

$$(AR)^{ks+1} = [(AR)^k]^s AR = \left[A^{\frac{(s+1)^k-1}{s}}\right]^s AR = A^{(s+1)^k-1} AR = AR,$$

where the last equality follows from (a). This proves statement (i), and completes the proof of Lemma 1. □

3 Construction of the matrix group

Using Lemma 1, we construct, from a given $\{R, s+1, k\}$-potent matrix, a matrix group containing a cyclic subgroup of $\{R, s+1, k\}$-potent matrices. Throughout this section we assume $s \geq 1$.

Theorem 2. *Suppose $A \in \mathbb{C}^{n \times n}$ is an $\{R, s+1, k\}$-potent matrix, and assume that $A^i \neq A^j$ for all distinct $i, j \in \{1, 2, \ldots, (s+1)^k - 1\}$. Then the set*

$$G = \{A^j R^p : \ j \in \{1, 2, \ldots, (s+1)^k - 1\},\ p \in \{1, 2, \ldots, k\}\}$$

is a group under matrix multiplication, and the following statements hold.

(a) A is an element of order $(s+1)^k - 1$, and the set

$$S_A = \{A^j,\ j \in \{1, 2, \ldots, (s+1)^k - 1\}\} \tag{7}$$

is a cyclic subgroup of G. Moreover, S_A is the smallest (in the inclusion sense) subgroup of G that contains A, $A^\#$, and $AA^\#$.

(b) $A^s R$ and $A^{(s+1)^k-1} R^{k-1}$ are elements of order k of G.

(c) $(A^s R) A (A^s R)^{k-1} = A^{s+1}$.

(d) The set S_A is a normal subgroup of G and all its elements are $\{R, s+1, k\}$-potent matrices.

(e) The order of G is $k((s+1)^k - 1)$ and G is not commutative.

Proof. Properties $(f)-(h)$ in Lemma 1 show that G is a group under multiplication with identity element $A^{(s+1)^k-1}$.

Statement (a) follows from properties $(a)-(c)$ in Lemma 1 and the assumption that the powers A^i are distinct for $i \in \{1, 2, \ldots, (s+1)^k - 1\}$.

By setting $m = k$ in property $(e)'$ in Lemma 1, we obtain $(A^sR)^k = A^{(s+1)^k-1}$. On the other hand, since $A^{(s+1)^k-1}$ and R^{k-1} commute by property (d) in Lemma 1,

$$(A^{(s+1)^k-1}R^{k-1})^k = (A^{(s+1)^k-1})^k(R^k)^{k-1} = A^{(s+1)^k-1},$$

proving statement (b).

By setting $m = k-1$ in property $(e)'$ in Lemma 1, we obtain

$$(A^sR)A(A^sR)^{k-1} = A^sRA^{(s+1)^{k-1}}R^{k-1} = A^sA^{(s+1)^{k-1}(s+1)}RR^{k-1} = A^{s+1}.$$

proving statement (c).

For the proof of statement (d), let $j, t \in \{1, 2, \ldots, (s+1)^k - 1\}$, $p \in \{1, 2, \ldots, k\}$, and $\ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ such that $j(s+1)^k \equiv -\ell(s+1)^p$ [mod $((s+1)^k - 1)$]. Using property (d) of Lemma 1, we obtain

$$(A^jR^p)A^t(A^\ell R^{k-p}) = A^jA^{t(s+1)^p}R^pA^\ell R^{k-p} = A^jA^{t(s+1)^p}A^{\ell(s+1)^p}R^pR^{k-p} = A^{t(s+1)^p}.$$

Hence, S_A is a normal subgroup of G, and by setting $p = 1$ in property (d) in Lemma 1, we find that the elements of S_A are $\{R, s+1, k\}$-potent matrices.

For the proof of statement (e), we show that the elements A^jR^p, $j \in \{1, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k\}$, are pairwise distinct.

First we show that for fixed $p \in \{1, \ldots, k-1\}$, $AR^p \neq A^j$ for any $j \in \{1, \ldots, (s+1)^k - 1\}$. Otherwise, $AR^pA = A^{j+1}$, and using property (d) in Lemma 1, $A(R^pA) = A(A^{(s+1)^p}R^p) = A^{(s+1)^p}(AR^p) = A^{(s+1)^p+j}$. But then, $A^{j+1} = A^{(s+1)^p+j}$, contradicting the assumption that the powers A^i are pairwise distinct for $i \in \{1, \ldots, (s+1)^k - 1\}$. Next, since for $p \in \{1, \ldots, k-1\}$, $AR^p \neq A^j$ for any $j \in \{1, \ldots, (s+1)^k - 1\}$, it follows that for any $\ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k-1\}$, $A^\ell R^p \neq A^j$ for any $j \in \{1, 2, \ldots, (s+1)^k - 1\}$. Finally, if $A^jR^p = A^\ell R^m$ for some $j, \ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ and $p, m \in \{1, \ldots, k\}$ with $(j, p) \neq (\ell, m)$, then $A^jR^{p-m} = A^\ell$, contradicting the previous assertion. Thus, the elements A^jR^p, $j \in \{1, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k\}$, are pairwise distinct, and the order of G is $k[(s+1)^k - 1]$. In order to show that G is not commutative, it is enough to see that $(AR)(A^{s+1}R^{k-1}) = (A^{s+1}R^{k-1})(AR)$ gives $A^{(s+1)^2+1} = A^{(s+1)^{k-1}+s+1}$ which leads to a contradiction. □

Theorem 3.1 (e) in [9] states that for a $\{K, s+1\}$-potent matrix, the associated matrix group G either has order $(s+1)^2 - 1$ and is commutative, or has order $2((s+1)^2 - 1)$ and is not commutative; Theorem 2 (e) now asserts that the former case does not occur.

We have shown that A, $A^\#$, and $AA^\#$ belong to S_A. Is $I_n - AA^\#$ also an element of the group G?

Proposition 1. *If $A \in \mathbb{C}^{n\times n}$ is a nonzero $\{R, s+1, k\}$-potent matrix then the eigenprojection at zero does not belong to G, that is,*

$$I_n - AA^{\#} \notin G.$$

Proof. If we suppose that $I_n - AA^{\#} \in G$ then there exist $j \in \{1, 2, \ldots, (s+1)^k - 1\}$, $p \in \{1, 2, \ldots, k\}$ such that $I_n - AA^{\#} = A^j R^p$. Pre-multiplying by A we get $A^{j+1} = O$, that is, A is nilpotent. Since A is diagonalizable, we arrive at $A = O$, which is a contradiction. □

Let H be the set defined by

$$H = \{A^{(s+1)^k - 1} R^p : \; p \in \{1, 2, \ldots, k\}\}.$$

Then under matrix multiplication, H is a cyclic subgroup of G that is not normal because if $g = A^{(s+1)^k - 2}$ and $h = A^{(s+1)^k - 1} R^p$ for $p \in \{1, 2 \ldots, k-1\}$ then $ghg^{-1} \notin H$.

Corollary 1. *The group G is a semidirect product of H acting on S_A.*

Proof. Every element $A^j R^p$ of G can be written as a product of an element of S_A and an element of H as $A^j R^p = A^j (A^{(s+1)^k - 1} R^p)$ and this representation is unique. This uniqueness follows from the fact that G has order $k((s+1)^k - 1)$. □

Observe that $H \simeq \mathbb{Z}_k$, $S_A \simeq \mathbb{Z}_{(s+1)^k - 1}$, and another way to see that G is isomorphic to a semidirect product of $\mathbb{Z}_k$ acting on $\mathbb{Z}_{(s+1)^k - 1}$ is by considering its representation in the form $\langle a, b |\, a^k = e, b^r = e, aba = b^m \rangle$ where m, r are coprime. Here $r = (s+1)^k - 1$, $a = A^s R$, $b = A$, $m = s + 1$.

Moreover, notice that the result presented in Corollary 1 describes the quotient group G/S_A. In fact, the natural embedding $\iota : H \hookrightarrow G$, composed with the natural projection $\pi : G \to G/S_A$, gives an isomorphism between G/S_A and H, which is represented in (8).

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/S_A \\ {\scriptstyle \iota}\uparrow & \nearrow{\scriptstyle g} & \\ H & & \end{array} \tag{8}$$

We next reconcile the matrix group G given in Theorem 2 that is constructed from an $\{R, s+1, k\}$-potent matrix A, and the matrix group G_A given in (3). We begin with the following lemma.

Lemma 2. *Suppose that $R \in \mathbb{C}^{n\times n}$ is $\{k\}$-involutory, $s \in \{1, 2, 3, \ldots\}$, and $A \in \mathbb{C}^{n\times n}$ has rank $r > 0$. Then A is $\{R, s+1, k\}$-potent if and only if there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \qquad R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1}, \tag{9}$$

where $R_1 \in \mathbb{C}^{r\times r}$, $R_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ are $\{k\}$-involutory, and $C \in \mathbb{C}^{r\times r}$ is nonsingular and $\{R_1, s+1, k\}$-potent.

Proof. Suppose that A is $\{R, s+1, k\}$-potent. Then A has index at most 1 and so it has the form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \tag{10}$$

where $C \in \mathbb{C}^{r\times r}$ is nonsingular. We now partition R conformable to A as follows

$$R = P \begin{bmatrix} R_1 & R_3 \\ R_4 & R_2 \end{bmatrix} P^{-1}. \tag{11}$$

Using expressions (10) and (11) we have that

$$A^{s+1}R = P \begin{bmatrix} C^{s+1}R_1 & C^{s+1}R_3 \\ O & O \end{bmatrix} P^{-1}$$

and

$$RA = P \begin{bmatrix} R_1C & O \\ R_4C & O \end{bmatrix} P^{-1}.$$

Equating blocks,

$$C^{s+1}R_1 = R_1C, \qquad C^{s+1}R_3 = O, \qquad \text{and} \qquad R_4C = O.$$

Since C is nonsingular, $R_3 = O$, $R_4 = O$, and so

$$R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1}.$$

Using $R^k = I_n$, this last expression implies that R_1 and R_2 are both $\{k\}$-involutory. Hence, C is $\{R_1, s+1, k\}$-potent.

The converse is trivial. □

Recall that the elements of G_A have a canonical form as given in (3).

Theorem 3. *Suppose $A \in \mathbb{C}^{n\times n}$ is an $\{R, s+1, k\}$-potent matrix, and suppose that $A^i \neq A^j$ for all pairwise distinct $i, j \in \{1, 2, \ldots, (s+1)^k - 1\}$. If A and R are expressed as in (9) then*

$$G = \left\{ P \begin{bmatrix} C^jR_1^p & O \\ O & O \end{bmatrix} P^{-1} : \ j \in \{1, 2, \ldots, (s+1)^k - 1\}, \, p \in \{1, 2, \ldots, k\} \right\}.$$

Moreover, G is a subgroup of G_A.

Proof. The description of the elements of G follows from Theorem 2 and Lemma 2. It is clear that $G \subseteq G_A$. Since C is $\{R_1, s+1, k\}$-potent, G is closed, hence G is a subgroup of G_A. □

4 Final remarks: the case $s = 0$

For the case $s = 0$ in (4), the matrix A satisfies $AR = RA$ where $R^k = I_n$. Notice that property (a) in Lemma 1 does not give any information. However, if there exists some positive integer t such that $A^{t+1} = A$ and t is the smallest positive integer satisfying this property, then we can construct the group $G = \{A^j R^p, j \in \{1, 2, \ldots, t\}, p \in \{1, 2, \ldots, k\}\}$ having similar properties as in the case $s \geq 1$. If such an integer t does not exist, it is impossible to construct the corresponding group, as the following example shows.

Example 1. Consider the matrices

$$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $\alpha \in \mathbb{R}$, we have that $R^4 = I_3$, $AR = RA$ and

$$A^m = \begin{bmatrix} \cos(m\alpha) & \sin(m\alpha) & 0 \\ -\sin(m\alpha) & \cos(m\alpha) & 0 \\ 0 & 0 & 2^m \end{bmatrix} \quad \text{for all } m \geq 2.$$

In general, when $s = 0$ there is no relation between the existence of the group inverse of A and of A being $\{R, 1, k\}$-potent. In Example 1 we have a $\{R, 1, 4\}$-potent matrix that is nonsingular whereas in Example 2 below the given $\{R, 1, 4\}$-potent matrix does not have a group inverse.

Example 2. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, $AR = RA$, $R^4 = I_3$, but the group inverse of A does not exist.

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