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SOME RESULTS ON LOCALLY FINITE GROUPS

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I, FRANCESCA SPAGNUOLO, declare that this *Dissertation* entitled, **“Some results on locally finite groups ”** and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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DECLARE:

That this *Dissertation* entitled, **"Some results on locally finite groups "** presented by the *B.Sc.* FRANCESCA SPAGNUOLO has been done under our supervision at the Department of Mathematics, University of Valencia and the Department of Mathematics, University Federico II of Naples. We would also like to state that this *Dissertation*, and the work included in it correspond to the Thesis Project approved by these institutions and that this project satisfies all the requisites to be presented to obtain the scientific degree of International Doctor in Mathematics by the University of Valencia.

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"Mathematics is the only infinite human activity. It is conceivable that humanity could eventually learn everything in physics or biology. But humanity certainly won't ever be able to find out everything in mathematics, because the subject is infinite."

Paul Erdős

Agradecimientos

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Ai miei genitori, pilastri della mia vita.
A mio fratello, primo compagno ed amico.
A Vladimir, mi gran amor.

Thesis Structure

The main goal of this thesis is to present some results on p-nilpotency and permutability in locally finite groups. It is organised in five chapters.

Well known definitions and results which are widely used in the thesis are collected in the first chapter. They are stated with suitable references. No proofs are included.

Chapter two is devoted to the study of p-nilpotency of hyperfinite groups, p a prime. The results are published in the paper:

Ballester-Bolinches, A.; Camp-Mora, S.; Spagnuolo, F., **On p-nilpotency of hyperfinite groups**. *Monatshefte für Mathematik, 176, no. 4, 497-502, 2015.*

A group is p-nilpotent if it has a normal Hall p'-subgroup. In finite groups, a group is p-nilpotent if and only if every Sylow p-subgroup has a normal p-complement. In this chapter we study which properties of the Sylow p-subgroups determine the p-nilpotency of the group by the p-nilpotency of their normalisers. For example, a classical result of Burnside states that a finite group G with an abelian Sylow p-subgroup P is p-nilpotent if and only if the normalizer of P in G is p-nilpotent. A class of p-groups $\mathcal{X}$ determines p-nilpotency locally if every finite group G with a Sylow p-subgroup P in $\mathcal{X}$ is p-nilpotent if and only if the normalizer of P in G is p-nilpotent. The main results of chapter 2 extend some known results of p-nilpotency of finite groups to hyperfinite groups. We prove:

- If $\mathcal{X}$ is a subgroup closed class of p-groups closed under taking epimorphic images that determines p-nilpotency locally and G is a hyperfinite group with a pronormal Sylow p-subgroup P in the class $\mathcal{X}$, then G is p-nilpotent if and only if the normalizer of P in G is p-nilpotent (Theorem 2.4).
- It $\mathcal{X}$ is a subgroup closed class of p-groups that determines p-nilpotency locally and G is a hyperfinite locally p-soluble group with a Sylow p-subgroup P in the class $\mathcal{X}$, then G is p-nilpotent if and only if the normalizer of P in G is p-nilpotent (Theorem 2.5).

In chapters 3 and 4 we study the structural influence of the subgroups of infinite rank. The results are collected in the following two papers.

Ballester-Bolinches, A.; Camp-Mora, S.; Kurdachenko, L.A.; Spagnuolo, F., **On groups whose subgroups of infinite rank are Sylow permutable**. *Annali di Matematica Pura ed Applicata (4), 195, no. 3, 717-723, 2016,*

Ballester-Bolinches, A.; Camp-Mora, S.; Dixon, M.R.; Ialenti, R.; Spagnuolo, F., **On locally finite groups whose subroups of infinite rank have some permutable property**. Submitted.

Recall that a group G has finite rank equal to r if every finitely generated subgroup of G is generated by at most r elements and r is the least integer with this property. If such an integer r does not exist then we say that G has infinite rank. Furthermore, G has finite section p-rank equal to r if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section of G of order exactly p^r. As before, if such an integer r does not exist then G has infinite section p-rank.

The properties of subgroups we consider includes permutability, S-permutability, semipermutability and S-semipermutability. The main results of Chapter 3 are:

- If in a hyper-(abelian or finite) group G with infinite section p-rank all subgroups with infinite section p-rank are S-permutable, then G is locally nilpotent (Theorem 3.1).
- If in a locally finite group G with infinite rank all subgroups of infinite rank are S-permutable, then G is locally nilpotent (Theorem 3.3).
- If in a locally finite group G with infinite section p-rank all subgroups with infinite section p-rank are permutable, then G is an Iwasawa group (Theorem 3.4).

We prove some known results as a consequence of the main theorems.

In chapter 4, the properties considered are semipermutability and S- semipermutability. The main results are:

- In locally finite groups with infinite section p-rank whose subgroups with infinite section p-rank are semipermutable, all subgroups are semipermutable (Theorem 4.5).
- For S-semipermutability, it is proved that in a locally finite group with the minimal condition on p-subgroups for every prime p, if all subgroups with infinite rank are S-semipermutable then all subgroups are S-semipermutable (Theorem 4.9).

It is presented a counterexample that shows that the minimal condition on the p-subgroups cannot be omitted (Proposition 4.10).

In chapter 5 all groups considered are finite. We study the immersion of semimodular subgroups of odd order in a finite group. The results presented can be found in the following paper, submitted to a scientific journal:

Ballester-Bolinches, A.; Heineken, H.; Spagnuolo, F., **On semipermutable subgroups of finite groups**. Submitted.

Some of the results of this chapter are:

- A finite group, product of a normal supersoluble subgroup and a subnormal semimodular subgroup of odd order, is supersoluble (Theorem 5.1).
- If a finite group G has a normal supersoluble subgroup N and a subnormal semimodular subgroup of odd order S, then the product NS^G is supersoluble (Theorem 5.2).

An interesting consequence of the last result is that the normal closure of a subnormal semimodular subgroup of odd order is supersoluble.

Resumen

En esta tesis se presentan algunos resultados sobre p-nilpotencia y permutabilidad en grupos localmente finitos. Está estructurada en cinco capítulos.

El primer capítulo, que tiene carácter introductorio: contiene definiciones y resultados conocidos que serán utilizados en los capítulos sucesivos. Por tratarse de resultados ya conocidos, se introducen con referencias y sin demostraciones.

En el capítulo 2 se trata la p-nilpotencia en grupos hiperfinitos, donde p es un primo. Los resultados presentados se encuentran publicados en el siguiente artículo

Ballester-Bolinches, A.; Camp-Mora, S.; Spagnuolo, F., **On p-nilpotency of hyperfinite groups**. *Monatshefte für Mathematik*, **176**, no. 4, 497–502, 2015 [10].

Un grupo se dice p-nilpotente si tiene un p'-subgrupo de Hall normal. En el caso de grupos finitos, se tiene que un grupo es p-nilpotente si y solo si todo p-subgrupo de Sylow tiene un p-complemento normal. En este capítulo se estudian propiedades de los p-subgrupos de Sylow de un grupo que garanticen que el grupo es p-nilpotente si y solo si el normalizador de un p-subgrupo de Sylow es p-nilpotente. Por ejemplo, un resultado clásico de Burnside establece que un grupo finito con p-subgrupos de Sylow abelianos es p-nilpotente si y solo si el normalizador de un p-subgrupo de Sylow es p-nilpotente. Para ello, se define la propiedad de determinación local de p-nilpotencia: una clase de p-grupos $\mathcal{X}$ *determina la p-nilpotencia localmente* si todo grupo finito G con un p-subgrupo de Sylow P en la clase $\mathcal{X}$ es p-nilpotente si y solo si el normalizador de P en G es p-nilpotente.

Los resultados principales del capítulo 2 extienden varios resultados conocidos sobre p-nilpotencia de grupos finitos. Se prueba que:

- Si $\mathcal{X}$ es una clase de p-grupos cerrada para subgrupos e imágenes epimorfas que determina la p-nilpotencia localmente y G es un grupo hiperfinito con un p-subgrupo de Sylow pronormal P en la clase $\mathcal{X}$, entonces G es p-nilpotente si y solo si el normalizador de P en G es p-nilpotente (teorema 2.4).

- Si $\mathcal{X}$ es una clase de p-grupos cerrada para subgrupos que determina la p-nilpotencia localmente y G es un grupo hiperfinito localmente p-resoluble con un p-subgrupo de Sylow P en la clase $\mathcal{X}$, entonces G es p-nilpotente si y solo si el normalizador de P en G es p-nilpotente (teorema 2.5).

En los capítulos 3 y 4 se estudian grupos de rango infinito en los que el comportamiento de los subgrupos de rango infinito respecto a cierta propiedad determina la estructura del grupo. Los resultados de estos capítulos aparecen en el artículo

Ballester-Bolinches, A.; Camp-Mora, S.; Kurdachenko, L.A.; Spagnuolo, F., **On groups whose subgroups of infinite rank are Sylow permutable**. *Annali di Matematica Pura ed Applicata (4)*, **195**, no. 3, 717–723, 2016 [8],

y en el trabajo enviado para su posible publicación en una revista científica

Ballester-Bolinches, A.; Camp-Mora, S.; Dixon, M.R.; Ialenti, R.; Spagnuolo, F., **On locally finite groups whose subroups of infinite rank have some permutable property**, enviado [7].

Recordemos que un grupo G tiene rango finito e igual a r si los subgrupos finitamente generados de G están generados por como máximo r elementos y r es el menor entero con tal propiedad. Si no existe un tal r, se dice que G tiene rango infinito. Se dice que G tiene p-rango de sección finito e igual a r si toda sección p-elemental abeliana de G es finita y tiene orden como máximo p^r y hay una sección p-elemental abeliana con orden exactamente p^r. Igualmente, si no existe un tal r, se dice que G tiene p-rango de sección infinito.

Las propiedades de los subgrupos de rango infinito que se consideran son la permutabilidad y algunas generalizaciones de la permutabilidad. En particular, en el capítulo 3 los principales resultados obtenidos se refieren a las propiedades de permutabilidad y permutabilidad con los subgrupos de Sylow. Se prueban los siguientes resultados.

- Si en un grupo G hiper-(abeliano o finito) con p-rango de sección infinito todo subgrupo con p-rango de sección infinito es S-permutable, entonces G es localmente nilpotente (teorema 3.1).

- Si en un grupo G localmente finito con rango infinito todo subgrupo de rango infinito es S-permutable, entonces G es localmente nilpotente (teorema 3.3).

- Si en un grupo G localmente finito con p-rango de sección infinito todo subgrupo con p-rango de sección infinito es permutable, entonces G es un grupo de Iwasawa (teorema 3.4).

Como consecuencia de los resultados principales se recuperan algunos resultados ya conocidos.

En el capítulo 4 se consideran las propiedades de semipermutabilidad y S- semipermutabilidad. Se prueban los siguientes resultados:

- En los grupos localmente finitos con p-rango de sección infinito cuyos subgrupos con p-rango de sección infinitos son semipermutables, todos los subgrupos son semipermutables (teorema 4.5).
- Para la S-semipermutabilidad, se prueba que en un grupo localmente finito con condición minimal sobre los p-subgrupos para todos los primos p, si todos los subgrupos de rango infinito son S-semipermutables, entonces todos los subgrupos son S-semipermutables (teorema 4.9).

Se presenta un contraejemplo que muestra que en el último resultado no se puede eliminar la hipótesis de que el grupo tenga la condición minimal sobre los p-subgrupos para todo primo p (proposición 4.10).

En el capítulo 5 se consideran únicamente grupos finitos y se estudia la inmersión de los subgrupos semimodulares de orden impar en un grupo finito. Un grupo se dice *semimodular* si todos sus subgrupos son semipermutables. Los resultados forman parte del siguiente trabajo, que ha sido enviado para su posible publicación en una revista científica:

Ballester-Bolinches, A.; Heineken, H.; Spagnuolo, F., **On semipermutable subgroups of finite groups**, enviado. [13].

Algunos de los resultados de este capítulo son los siguientes:

- Un grupo finito que es producto de un subgrupo normal superresoluble y de un subgrupo subnormal semimodular es superresoluble (teorema 5.1).
- Si un grupo finito G tiene un subgrupo normal N superresoluble y un subgrupo S subnormal semimodular de orden impar, entonces el producto NS^G es superresoluble (teorema 5.2).

Una consecuencia interesante del último resultado es que la clausura normal de un subgrupo subnormal semimodular de orden impar es superresoluble.

CHAPTER 1

Preliminaries

Contents

In this chapter, we collect together a number of results. Many are well known and are included here for convenient reference, because of their extensive use in later chapters.

To begin with, we recall that if $\mathcal{X}$ is a class of groups, a group G is said to be locally $\mathcal{X}$ if every finitely generated subgroup of G belongs to $\mathcal{X}$. In this work, we are interested in locally finite groups, or groups which are locally $\mathcal{X}$ for the class $\mathcal{X}$ of all finite groups.

In the first section of this chapter we are concerned with a finiteness condition that has been playing an important role in ring theory and group theory: the minimal condition.

1.1 Minimal condition and Černikov groups

Recall first that a subgroup theoretical property $\mathcal{P}$ is a property that certain subgroups of a group possess that is invariant under isomorphisms.

Definition 1.1. Let $\mathcal{P}$ be a subgroup theoretical property. A group G is said to satisfy the *minimal condition* on *$\mathcal{P}$-subgroups* (min-$\mathcal{P}$ for short) if all sets of $\mathcal{P}$-subgroups of G partially ordered by inclusion have a minimal element.

In the particular case when $\mathcal{P}$ is the property of being a subgroup, then the condition min-$\mathcal{P}$ is called the *minimal condition*, or simply, *min*. Condition *min-p*, p a prime, denotes the case when $\mathcal{P}$ is the property of being a p-subgroup.

The following result is elementary.

Lemma 1.2. *A group G satisfies min-$\mathcal{P}$ if and only if every descending chain of $\mathcal{P}$-subgroups of G terminates in finitely many steps.*

Some useful properties of the groups with the minimal condition are gathered in the following result.

Proposition 1.3. *1. Every group satisfying the minimal condition is periodic.*

2. *([25, Lemma 1.5.3]) The class of groups with the minimal condition is closed under taking subgroups and homomorphic images.*

3. *([25, Lemma 1.5.3]) If L is a normal subgroup of a group G such that L and G/L satisfy the minimal condition, then G satisfies the minimal condition. In particular, the class of groups with the minimal condition is closed under taking finite normal products.*

4. *([25, Lemma 1.5.4]) The group G satisfies the minimal condition if and only if every countable subgroup of G satisfies the minimal condition.*

The most important examples of infinite groups with the minimal condition are the Prüfer p-groups of type p^∞ (Prüfer p-group for short), p a prime. These groups, denoted by C_{p^∞}, play a very important role in group theory and can be thought of as the multiplicative group of complex p^n-th roots of unity, or as the set of elements of p-power order of the additive abelian group $\mathbb{Q}/\mathbb{Z}$. These groups are also the main examples of infinite groups with every proper subgroup finite. They also are canonical examples of divisible abelian groups.

Recall that a *radicable group* G is a group in which we can solve the equation $y^n = x$ for every element $x \in G$ and every integer n. An abelian radicable group is called *divisible*.

Another example of a divisible abelian group is the additive group of the rational numbers $\mathbb{Q}$. These two examples of abelian divisible groups are central since an abelian group is divisible if and only if it is a direct sum of isomorphic copies of $\mathbb{Q}$ and Prüfer groups (see [49, 4.1.5]).

The following property of abelian groups should be noted.

Theorem 1.4. *[49, 4.1.4] If G is an abelian group, there exists a unique largest divisible subgroup D of G. Moreover, $G = D \oplus R$, where R is a reduced subgroup, that is, R has no nontrivial divisible subgroups.*

The structure of groups with the minimal condition is not as well understood as the corresponding structure for rings. Nevertheless, there are some interesting positive results. The following theorem, due to Kuroš, characterises the abelian groups in the class and it is a nice application of Theorem 1.4.

Theorem 1.5. *[25, 1.5.5] Let G be an abelian group. Then G has the minimal condition if and only if G is a finite direct product of Prüfer groups and finite cyclic groups.*

Definition 1.6. A finite extension of an abelian group with the minimal condition on subgroups is called *Černikov* group.

Such groups were named in honour of S. N. Černikov, who studied the class of groups with minimal condition (see for example [16, 17]).

Note that from Theorem 1.5 it follows that a group G is Černikov if and only if G has a normal divisible abelian subgroup D of finite index and D is a direct product of finitely many Prüfer groups. Moreover, G is Černikov if and only if G has a finite series whose factors are finite or Prüfer.

The following proposition contains some useful properties of Černikov groups.

Proposition 1.7.
1. *The class of Černikov groups is closed under taking subgroups and epimorphic images.*
2. *If G is a group and L is a normal subgroup of G such that L and G/L are divisible Černikov groups, then G is divisible and hence abelian Černikov group.*
3. *If G is a group and L is a normal subgroup of G such that L and G/L are Černikov groups, then G is a Černikov group.*

Recall that if G is a group and $\mathfrak{X}$ is a class of groups, the $\mathfrak{X}$-*residual* of G is the subgroup

$$G^{\mathfrak{X}} = \bigcap \{N \lhd G | G/N \in \mathfrak{X}\}.$$

Lemma 1.8. *[25, 1.5.8] Let G be a group and suppose that the set of normal subgroups of finite index of G has the minimal condition. Then the finite residual of G has finite index in G.*

In the case of a Černikov group G the finite residual $G^{\mathfrak{F}}$ is the largest normal divisible subgroup of G.

The periodic group of automorphisms of a Černikov group studied by Baer [3] and Polovickiĭ [48] will be also used in our work.

Theorem 1.9. *[25, 1.5.16] Let G be a Černikov group. If A is a periodic group of automorphisms of G, then A is a Černikov group.*

We bring this subsection to a close with a nice consequence of Theorem 1.9.

Corollary 1.10. *[25, 1.5.17] Let G be a nilpotent Černikov group. If A is a periodic group of automorphisms of G, then A is finite.*

1.2 Ranks of a group

In this section, we collect some results concerning some numerical invariants of a group which are important in our work. They are motivated by the concept of dimension of vector spaces.

We begin with the abelian case and consider an analoge of linearly independent subsets of a vector space.

Definition 1.11. Let G be an abelian group. A nonempty subset X of G is called *linearly independent*, or independent for short, if given $x_1, \dots, x_m$ distinct elements of X and integers $n_1, \dots, n_m$, the relation $n_1x_1 + \cdots + n_mx_m = 0$ implies that $n_ix_i = 0$ for all i.

Zorn's Lemma shows that every independent subset of G is contained in a maximal independent subset. Moreover, if we restrict the attention on independent subsets consisting of elements of infinite order or of elements of order some power of a fixed prime, we obtain maximal independent subsets consisting of elements of these types. Thus, we have:

Theorem 1.12. *[49, 4.2.1] Let G be an abelian group. Then two maximal linearly independent subsets of G consisting of elements of infinite order (respectively elements of order some power of a fixed prime) have the same cardinality.*

Bearing in mind the above result, the following definition turns out to be natural.

Definition 1.13. Let G be an abelian group. The cardinality $r_0(G)$ of a maximal independent subset consisting of elements of infinite order is called the 0-*rank* of G; if p is a prime, the p-*rank* of G, $r_p(G)$, is defined as the cardinality of a maximal independent subset consisting of elements of p-power order.

If $T(G)$ is the *torsion subgroup* of G, that is, the subgroup composed of all elements of finite order, then $r_0(G) = r_0(G/T(G))$ and $r_p(G) = r_p(T(G))$.

We define the *special rank* $r(G)$, often just called the rank, of an abelian group G as $r(G) = r_0(G) + max_p\{r_p(G)\}$.

This motivates the following:

Definition 1.14. A group G has *finite special rank* (or just rank) rk$(G) = r$ if every finitely generated subgroup of G is generated by at most r elements and r is the least integer with this property. If such an integer r does not exist then we say that G has infinite special rank.

It is clear that the additive group of $\mathbb{Q}$ is of rank 1. Moreover, we have:

Lemma 1.15. *Let G be a periodic abelian group. Then G satisfies the minimal condition if and only if G has finite rank. Furthermore, in this case G is a Černikov group and it is a direct sum $G = D \oplus F$ for some divisible subgroup D and finite subgroup F.*

A group of finite rank with the minimal condition need not to be Černikov: the Tarski monster constructed by Ol'shanskii [46] is an infinite simple 2-generator group with all proper subgroups cyclic of prime order.

The concept of rank of groups was largely studied in last seventy years (see for example [4, 18, 26, 28, 30, 31, 34, 38, 42–45, 50, 51, 58, 62]).

Some useful properties of groups of finite rank are contained in the following proposition:

Proposition 1.16. *Let G be a group and N a subgroup of G.*

1. *If G has finite rank $r(G) = r$, then N has rank at most r. Furthermore, if N is a normal subgroup of G, G/N has rank at most r.*
2. *If N is normal in G, N has rank r and G/N has rank s, then G has rank at most $r + s$.*
3. *G has finite rank if and only if every countable subgroup of G has finite rank.*

Note that an abelian p-group P, p a prime, has finite rank r if and only if every elementary abelian section U/V of P has finite order at most p^r, and P has an elementary abelian section A/B of order p^r.

This motivates the following numerical invariant associated to the prime p.

Definition 1.17. Let p be a prime. A group G has *finite section p-rank* $sr_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

Observe that if G is an abelian group, then $r_p(G) \leq sr_p(G)$ but the equality does not hold in general: the group $C_p \times \mathbb{Z}$ has p-rank 1, but the elementary p-section $(C_p \times \mathbb{Z})/p\mathbb{Z}$ has order p^2, so the section p-rank of $\mathcal{C}_p \times \mathbb{Z}$ is (at least) 2.

Note that if a group G has an element g of infinite order, then G has a section $\langle g \rangle / \langle g^p \rangle$ of order p for every prime p. Then if G is not periodic, $sr_p(G) \geq 1$ for every prime p. If G is a periodic group that contains no p-elements, then $sr_p(G) = 0$. Otherwise, $sr_p(G) \geq 1$.

The section p-rank version of Proposition 1.16 is the following:

Proposition 1.18. *Let G be a group and p be a prime number.*

(i) If G has finite section p-rank, then

1. *if K is a subgroup of G and H is a normal subgroup of K, $sr_p(K/H) \leq sr_p(G)$;*
2. *if H is a normal subgroup of G, $sr_p(G) \leq sr_p(H) + sr_p(G/H)$;*
3. *if H is a normal periodic subgroup of G such that $p \notin \pi(H)$, $sr_p(G/H) = sr_p(G)$.*

(ii) If H is a normal subgroup of G and H and G/H have finite section p-rank, then G has finite section p-rank.

1.3 Groups with min-p

In this section, we collect some results about the interesting class of groups with min-p, p a prime.

We begin with the following lemma.

Lemma 1.19. *[25, 2.5.1] Let G be a locally finite group. Then G satisfies min-p for the prime p if and only if G contains a maximal elementary abelian p-subgroup which is finite.*

Definition 1.20. Let G be a locally finite group and let p be a prime. A maximal p-subgroup P of G is called a *Wehrfritz p-subgroup* if P contains an isomorphic copy of every p-subgroup of G.

Theorem 1.21. *[25, 2.5.4] Let G be a group with min-p for some p. Then G contains Wehrfritz p-subgroups and every finite p-subgroup lies in at least one of these.*

Lemma 1.22. *[25, 2.5.3] Let P and Q be Černikov groups. If Q contains an isomorphic copy of every finite subgroup of P then Q contains a subgroup isomorphic to P.*

Lemma 1.23. *[25, 2.5.5] Let G be a group with min-p for the prime p. If P is a p-subgroup of G and Q is a Wehrfritz p-subgroup of G then the following are equivalent:*

(i) P is a Wehrfritz p-subgroup of G;

(ii) P contains an isomorphic copy of every finite p-subgroup of G;

(iii) $P \cong Q$.

Corollary 1.24. *[25, 2.5.8] Let G be a group with min-p for the prime p. Then the p-subgroup P is a Wehrfritz p-subgroup of G if and only if P contains a conjugate of every finite p-subgroup of G.*

The following theorem is due to Kargapolov [38]. Recall that a group G involves a group H if H is isomorphic to a section of G.

Theorem 1.25. *Let G be a locally finite group that satisfies min-p for a prime p. Then $G/O_{p',p}(G)$ is finite if and only if G does not involve any infinite simple group containing elements of order p.*

Corollary 1.26. *[25, 2.5.13] Suppose G is a locally soluble group satisfying min-p for all primes p. Let π be a finite set of primes. Then $G/O_{\pi'}(G)$ is a soluble Černikov group.*

Theorem 1.27. *[25, 2.5.14] Let G be a locally soluble group with min-p for all primes p. Then G contains a radicable abelian normal subgroup G^0 such that G/G^0 is residually finite and the Sylow p-subgroups of G/G^0 are finite for each prime p.*

The following theorem of Belyaev [15] quite useful. Recall that a group is almost locally soluble if it has a locally soluble subgroup with finite index.

Theorem 1.28. *If G is a locally finite group satisfying min-p for every p, then G is almost locally soluble.*

1.4 Permutability properties of subgroups

A subgroup H of a group G is said to be *permutable in G* (or quasi-normal in G), if $HK = KH$ for every subgroup K of G. This concept arises as a generalization of that of normal subgroup and was introduced by Ore in 1939 ([47]). Permutable subgroups are subnormal in the finite case by a result of Ore, and ascendant, non-necessarily subnormal, in the general one by a result of Stonehewer ([57]).

Another interesting subgroup embedding property which can be defined in a periodic group is the Sylow permutabilty.

Let $\pi(G)$ denote the set of primes dividing the order of some element of a periodic group G. Let π be a subset of $\pi(G)$. Following Kegel ([39]), we say that a subgroup H of G is said to be *π–permutable* (or *π–quasinormal)* in G, if H permutes with every Sylow p–subgroup of G for every $p \in \pi$. A subgroup H of G is said to be *Sylow–permutable* (or *S–permutable*) in G if H is π–permutable, for $\pi = \pi(G)$.

According to a result of Kegel ([39], [11, Lemma 1.2.8, Theorem 1.2.14]), an S-permutable subgroup of a finite group is subnormal. Unfortunately, this result does not remain true in the locally finite universe: every subgroup of a locally nilpotent group is S-permutable, and there are many examples of locally nilpotent groups with non-subnormal subgroups. Moreover, there exist locally finite p-groups with no ascendant subgroups (see for example [16, 56]). Therefore, S-permutable subgroups are not ascendant in general. However, they are ascendant in the hyperfinite universe ([14, Proposition 2.4]). In addition, if G is a locally finite group with Černikov Sylow subgroups, every S-permutable subgroup of G is ascendant ([9, Theorem 11]).

The following lemma is very useful.

Lemma 1.29. *Let G be a locally finite group.*

(i) Let L be a normal subgroup of G such that G/L is countable.

1. *[25, Lemma 2.3.9] If P/L is a Sylow p-subgroup of G/L, then G has a Sylow p-subgroup S such that $P/L = SL/L$.*
2. *If H is an S-permutable subgroup of G, then HL/L is S-permutable in G/L.*

(ii) Let H be an S-permutable subgroup of G. If K is a subgroup of G containing H, then H is S-permutable in K.

Others interesting extensions of the permutability which have been studied intensively in recent years are the semipermutability and S-semipermutability introduced by Chen in [19] in the finite case.

Definition 1.30. A subgroup H of a periodic group G is said to be *semipermutable* (respectively, *S-semipermutable*) provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $\pi(H) \cap \pi(K) = \emptyset$.

Unfortunately semipermutable subgroups are not subnormal in general. It is enough to consider a Sylow 2-subgroup of the symmetric group of degree 3.

Semipermutability and S-semipermutability are not closed under homomorphic images either ([6]).

We recall the following easy fact.

Lemma 1.31. *Let G be a periodic group and H a subgroup of G. If H is semipermutable (respectively S-semipermutable) in G and $H \leq K \leq G$, then H is semipermutable (respectively, S-semipermutable) in K.*

Clearly the extent to which a subnormal subgroup of a finite group can differ from being S-permutable is of interest and so the description of the periodic groups in which S-permutability is transitive could help.

Definition 1.32. A periodic group G is a PST*-group* if S-permutability is a transitive relation in G, that is, if $H \leq K \leq G$ and H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

According to Kegel's result, a finite group G is a PST-group if and only if every subnormal subgroup is S-permutable in G. The class of finite PST-groups has been extensively investigated with a lot of nice results available. The reader is referred to [11, Chapter 2] for basic results about this class of groups.

In particular, Agrawal ([11, 2.1.8]) characterised soluble PST-groups.

Theorem 1.33. *A finite group G is a soluble PST-group if and only if the nilpotent residual of G is a Hall subgroup of odd order acted upon by conjugation as a group of power automorphisms by G.*

In particular, the class of finite soluble PST-groups is subgroup-closed.

The structure theorem for finite PST-groups was showed by Robinson in [52] (see [11, 2.1.19]).

The finiteness does not tend to be unnecessary in much of this theory. For instance, there exist locally finite non-PST-groups in which every subnormal subgroup is S-permutable (see [14, example 2.8]).

The problems to deal with infinite PST-groups are due to the bad behaviour of Sylow subgroups, in particular from the difficulty to form quotients and the deficiency of Sylow p-subgroups to be conjugate.

In [53], Robinson proved the following characterisation of locally finite groups with all finite subgroups PST:

Theorem 1.34. *[53, Theorem PST] Let G be a locally finite group. Then the following statements are equivalent:*

1. *all finite subgroups of G are PST-groups;*
2. *there is an abelian normal subgroup L containing no involutions such that G/L is locally nilpotent, $\pi(L)\cap\pi(G/L) = \emptyset$ and elements of G induce power automorphisms in L;*
3. *in each section of G the serial subgroups and the S-permutable subgroups coincide;*
4. *every section of G is a PST-group.*

Clearly S-semipermutability is not transitive either. Hence it is natural to consider the following class of groups.

Definition 1.35. A group G is called a BT*-group* if S-semipermutability is a transitive relation in G, that is, if $H \leq K \leq G$ and H is S-semipermutable in K and K is S-semipermutable in G, then H is S-semipermutable in G.

This class was introduced and characterized by Wang, Li and Wang in [59]. Further contributions were presented in [1].

The following important theorem shows that soluble BT-groups are a subclass of PST-groups:

Theorem 1.36 ([59])**.** *Let G be a finite group with nilpotent residual L. The following statements are equivalent:*

1. *G is a soluble BT-group;*
2. *every subgroup of G of prime power order is S-semipermutable;*
3. *every subgroup of G of prime power order is semipermutable;*
4. *every subgroup of G is semipermutable;*
5. *G is a soluble PST-group and if p and q are distinct primes not dividing the order of L with G_p a Sylow p-subgroup of G and G_q a Sylow q-subgroup of G, then $[G_p, G_q] = 1$.*

Corollary 1.37. *[59, 3.3] If G is a finite group in which all subgroups are semipermutable, then G is a PST-group.*

The soluble BT-groups are a proper subclass of soluble PST-groups ([5]).

Lemma 1.38. *Let G be a soluble finite BT-group. Then:*

1. *G is supersoluble.*

2. *G/K is BT-group for all normal subgroups K of G.*

Proof. Statement 1 follows from Corollary 1.37 and Theorem 1.33.

Assume that K is a normal subgroup of G. If A, B are two subgroups of G containing K, we know about minimal supplements A^+, B^+ of K in A, B that their intersections with K in A, B are contained in the Frattini subgroups $\Phi(A^+), \Phi(B^+)$ respectively. Hence if $A/K = A^+K/K \cong A^+/(A^+ \cap K)$ is a π-subgroup, so is A^+, and $gcd(|A/K|, |B/K|) = 1$ yields $\gcd(|A^+|, |B^+|) = 1$. Now $A^+B^+ = B^+A^+$ since G is a soluble BT-group (by Theorem 1.36) and $AB = KA^+B^+ = KB^+A^+ = BA$, further $(A/K)(B/K) = (B/K)(A/K)$. This proves Statement 2. □

It is easy to see that semipermutability and S-semipermutability are not closed under taking subgroups and homomorphic images. However the following property is true in finite groups

Lemma 1.39. *[63, Property 1, Property 2] Let H be a S-semipermutable subgroup of a finite group G. Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is S-semipermutable in G/N.*

CHAPTER 2

On p-nilpotency of hyperfinite groups

Contents

2.1 Introduction

The results presented in this chapter are the contents of the paper:

Ballester-Bolinches, A.; Camp-Mora, S.; Spagnuolo, F., **On p-nilpotency of hyperfinite groups**. *Monatshefte für Mathematik, 176, no. 4, 497-502, 2015.*

In this chapter all groups considered are periodic; p will denote a prime number.

We say that a group G is p-*nilpotent* if G has a normal Hall p'-subgroup (or Sylow p'-subgroup in the terminology of [25]). In this case $O_{p'}(G)$, the largest normal p'-subgroup of G, is a Sylow p'-subgroup of G. If G is finite, the classical Schur–Zassenhaus theorem shows that $O_{p'}(G)$ is complemented in G by a Sylow p-subgroup of G. Hence a finite group is p-nilpotent if and only if every Sylow p-subgroup has a normal complement. In the case of infinite groups, the situation is much more complicated since there exist groups with non-complemented normal Sylow p'-subgroups (see [25]).

The p-nilpotency of a finite group G is a property which can be read off from the structure of the Sylow p-subgroups of G and the way in which they are embedded in G. For instance, if a Sylow p-subgroup P of a finite group is abelian, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent. This is a classical result of Burnside ([36, Chap.IV, 2.6]), which was extended to modular Sylow p-subgroups, i. e. groups with modular subgroup lattice, by Ballester-Bolinches and Esteban-Romero (see [11, Theorem 2.2.5]). The latter result remains true for hyperfinite groups, i. e. a group with an ascending series of normal subgroups whose factors are finite, and pronormal Sylow p-subgroups as Kurdachenko and Otal showed in [40].

Theorem 2.1 ([40, Main Theorem])**.** *Let P be a Sylow p-subgroup of a hyperfinite group. If P is modular and pronormal in G, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

For our purposes, it seems worthwhile introducing the following definition ([60]).

Definition 2.2. A class of p-groups $\mathcal{X}$ *determines p-nilpotency locally* if every finite group G with a Sylow p-subgroup P in $\mathcal{X}$ is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.

The present chapter deals with the problem of determining when a hyperfinite group G is p-nilpotent, and can be considered as a continuation of [40].

The above-mentioned results show that the class of all abelian p-groups and the class of all modular p-groups are both examples of subgroup-closed classes of p-groups determining p-nilpotency locally. Regular p-groups ([36, Chap.III, Section 10]) also constitute a subgroup-closed class of p-groups which determines p-nilpotency locally by virtue of a result of Hall and Wielandt ([36, Chap.IV, 8.1]). All finite p-groups of nilpotency class less or equal to $p-1$ and all finite p-groups of exponent p are regular. Therefore the class of all p-groups of nilpotency class at most $p-1$ and the class of all p-groups of exponent p are subgroup-closed classes determining p-nilpotency locally.

Weigel [60] proved that if p is odd, there exists a subgroup-closed class of p-groups which determines p-nilpotency locally and contains every subgroup-closed class of finite p-groups with this property. It is defined as follows.

Definition 2.3. Let $E = \langle g_1, g_2, \ldots, g_p\rangle$ be an elementary abelian group of order p^p. Let $C = \langle x\rangle$ be a cyclic group of order p^m acting on E, where m is a natural number, in such a way $g_i^x = g_{i+1}$ for $1 \leq i \leq p-1$ and $g_p^x = g_1$. Let $Y_p(m) = E \rtimes C$ be the corresponding semidirect product.

Note that $Y_p(1)$ is just the regular wreath product $C_p \wr C_p$.

We say that a p-group P is *slim* if no subgroup $Y_p(m)$ for $m \geq 1$ can be embedded in P.

By [60, Main Theorem] (see also [12]), the class of all slim p-groups, p odd, determines p-nilpotency locally and it contains every subgroup-closed class finite of p-groups which determines p-nilpotency locally ([60, Proposition4.3]).

Our main results show that Theorem 2.1 is not accidental and it is a consequence of a general completeness property of certain classes of p-groups.

Theorem 2.4. *Suppose that $\mathcal{X}$ is a subgroup-closed class of p-groups which is closed under taking epimorphic images. If $\mathcal{X}$ determines p-nilpotency locally and G is a hyperfinite group with a pronormal Sylow p-subgroup P in $\mathcal{X}$, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

For classes of p-groups which are not closed under taking epimorphic images, we need to impose local p-solubility. In this case, the hypothesis of pronormality of the Sylow p-subgroup can be removed.

Theorem 2.5. *Suppose that $\mathcal{X}$ is a subgroup-closed class of p-groups. If $\mathcal{X}$ determines p-nilpotency locally and G is a hyperfinite locally p-soluble group with a Sylow p-subgroup P in $\mathcal{X}$, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

2.2 Proofs

Most of the labour of the proof of Theorem 2.4 resides in establishing the following key lemmas.

Lemma 2.6. *Let $\mathcal{X}$ be a class of p-groups which is closed under taking subgroups. Assume that $G = PN$ is a locally finite group which is the product of a Sylow p-subgroup $P \in \mathcal{X}$ and a finite normal subgroup N. If $\mathcal{X}$ determines p-nilpotency locally, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

Proof. Since P is has finite index in G, it follows that $\mathrm{Core}_G(P)$ has finite index in G. Then $O_p(G)$, the largest normal p-subgroup of G, has finite index in G and so $G/O_p(G)$ is a finite group. Therefore the Sylow p-subgroups of G are conjugate.

It is clear that only the sufficiency of the condition is in doubt. Suppose that $N_G(P)$ is p-nilpotent. We see that $G/O_{p'}(G)$ satisfies the hypothesis of the lemma. Let $T/O_{p'}(G)$ be a p-subgroup of $G/O_{p'}(G)$ containing $PO_{p'}(G)/O_{p'}(G)$. Applying [25, Theorem 2.4.5], there exists a Sylow p-subgroup A of T such that $T = AO_{p'}(G)$. Since $O_{p'}(G)$ is finite, it follows that A has finite index in T and so the Sylow p-subgroups of T are conjugate. In particular, P and A are conjugate in T. Hence $T = PO_{p'}(G)$ and $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Moreover, the conjugacy of the Sylow p-subgroups of G yields $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$. Thus $G/O_{p'}(G)$ satisfies the hypothesis of the lemma. If $G/O_{p'}(G)$ is p-nilpotent, then G is p-nilpotent. Therefore we may assume that $O_{p'}(G) = \{1\}$.

We prove that G is p-nilpotent by induction on $|G : P|$, the index of P in G. Suppose that $N_G(P \cap N)$ is a proper subgroup of G. Then P is a Sylow p-subgroup of $N_G(P \cap N)$

and $|N_G(P\cap N) : P| < |G : P|$. By induction, $N_G(P\cap N)$ is p-nilpotent. Since N is finite and the Sylow p-subgroups of G are conjugate, we have that $P\cap N$ is a Sylow p-subgroup of N. Moreover, $P\cap N$ belongs to $\mathcal{X}$. Hence N is p-nilpotent and so N is a p-group since $O_{p'}(N) \leq O_{p'}(G) = \{1\}$. Thus G is a p-group and the lemma is proved. Hence we may assume that $P \cap N$ is a normal subgroup of G. Let X be a finite subgroup of P such that $P\cap N$ is contained in X and $P = O_p(G)X$ (note that $O_p(G)$ is a proper subgroup of P). Then X is a p-subgroup of NX and $|NX : X| = |N : N \cap X|$ is a p'-number. Hence X is a Sylow p-subgroup of NX and $X \in \mathcal{X}$. Since $N_{NX}(X)$ is contained in $N_G(P)$, it follows that $N_{NX}(X)$ is p-nilpotent. The hypothesis on $\mathcal{X}$ implies that XN is p-nilpotent. Then N is p-nilpotent and so N is a p-group since $O_{p'}(N) = \{1\}$. The proof of the lemma is complete. □

Lemma 2.7. *Let $\mathcal{X}$ be a class of p-groups which is closed under taking subgroups and epimorphic images. Assume that G is a hyperfinite group with a Sylow p-subgroup $P \in \mathcal{X}$ of finite index in G. If $\mathcal{X}$ determines p-nilpotency locally, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

Proof. As in Lemma 2.6, we have that the Sylow p-subgroups of G are conjugate. Suppose that $N_G(P)$ is p-nilpotent. We argue that G is p-nilpotent by induction on $|G : P|$. We may assume $O_{p'}(G) = \{1\}$. Since $G/\operatorname{Core}_G(P)$ is a finite group and P is a proper subgroup of G, there exists a maximal subgroup M of G containing P. Then M satisfies the hypotheses of the lemma and $|M : P| < |G : P|$. By induction, M is p-nilpotent. If P is a proper subgroup of M, then $\{1\} \neq O_{p'}(M) \trianglelefteq M$. Therefore $M \leq N_G(O_{p'}(M))$. The maximality of M implies that either $N_G(O_{p'}(M)) = G$ or $N_G(O_{p'}(M)) = M$. If $N_G(O_{p'}(M)) = G$, then $O_{p'}(M) \trianglelefteq G$ and $O_{p'}(M) \leq O_{p'}(G) = \{1\}$, contrary to supposition. Thus $N_G(O_{p'}(M)) = M$, and $|G : N_G(O_{p'}(M))|$ is finite. In particular, $O_{p'}(M)$ has a finitely many G-conjugates. Since $G/\operatorname{Core}_G(P)$ is a finite group and $O_{p'}(M) \cap \operatorname{Core}_G(P) = \{1\}$, it follows that $O_{p'}(M)$ is finite. Consequently, the normal closure N of $O_{p'}(M)$ in G is finite. By Lemma 2.6, it follows that PN is p-nilpotent. Then N is a normal p'-subgroup of G and so $N \leq O_{p'}(G) = \{1\}$. This contradiction yields $M = P$ and so $G/O_p(G)$ is a primitive group (see [36, Chap. II, Section 3]). Since $P/O_p(G)$ is in $\mathcal{X}$, it follows that $G/O_p(G)$ is p-nilpotent. Then G is locally p-soluble and $\operatorname{Soc}(G/O_p(G))$ is a p'-group. In particular, every chief factor of $G/O_p(G)$ whose order is divisible by p is central in G. Let H/K be a chief factor of G such that $H \leq O_p(G)$. Since G is hyperfinite, H/K is finite and then $O_p(G) \leq C_G(H/K)$ by [25, Lemma 1.7.11]. Let X be a finite supplement of $O_p(G)$ in $G = XO_p(G)$ containing a transversal of K in H. Let S be a Sylow p-subgroup of X, then $SO_p(G)$ is a Sylow p-subgroup of G. Since $N_G(S)$ is contained in $N_G(SO_p(G))$, it follows that is $N_G(S)$ is p-nilpotent. Since $S \in \mathcal{X}$, we have that X is p-nilpotent. Since $O_p(G)$ centralises H/K, it follows that H/K is a chief factor of the p-nilpotent group XK/K. Then $H/K \leq Z(XK/K)$. Hence H/K is central in G. The proof of [25, Lemma 6.2.3] can be adapted to conclude that G is p-nilpotent.

The converse is clear. □

Note that if $\mathcal{X}$ is not closed under taking epimorphic images, we cannot guarantee that the primitive group $G/O_p(G)$ is p-nilpotent. Hence we have to impose local p-solubility to G to get the same result.

Lemma 2.8. *Let $\mathcal{X}$ be a class of p-groups which is closed under taking subgroups. Assume that G is a hyperfinite locally p-soluble group with a Sylow p-subgroup $P \in \mathcal{X}$ of finite index in G. If $\mathcal{X}$ determines p-nilpotency locally, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

Lemma 2.9. *If P be a hyperfinite p-group, then P is hypercentral.*

Proof. Assume that P is not hypercentral. Let $\bar{Z} = \bar{\mathrm{Z}}(P)$ be the hypercentre of P, then $\bar{P} = P/\bar{Z}$ is not trivial, but $\mathrm{Z}(\bar{P})$ is trivial. Since P is hyperfinite, $\bar{P}$ is also hyperfinite. In particular, $\bar{P}$ has a non-trivial finite normal subgroup $\bar{N}$. Let $p^r = |\bar{N}|$. Since $\bar{P}$ induces a group of automorphisms of $\bar{N}$ by conjugation, $\tilde{P} = \bar{P}/\mathrm{C}_{\bar{P}}(\bar{N})$ is finite. The action of $\tilde{P}$ induced by the action of $\bar{P}$ on the finite subgroup $\bar{N}$ decomposes $\bar{N}$ as a union of orbits. Since $\tilde{P}$ is a finite p-group, each of these orbits is of length a power of p. The orbit of the identity element 1 of $\bar{N}$ is $\{1\}$. Since $\mathrm{Z}(\bar{P}) = \{1\}$, there cannot be more orbits of length 1 in this action, since all these orbits correspond to central elements of $\bar{P}$. In particular, all other orbits have length divisible by p. But then $|\bar{N}| = p^r = 1 + pt$ for a certain natural number t, that is, 1 is a multiple of p. This is a contradiction. Therefore P must be hypercentral. □

Common arguments. It seems desirable to collect the arguments common to our main theorems.

It is clear that only the sufficiency of the condition of p-nilpotency of $N_G(P)$ in Theorems 2.4, 2.5 is in doubt. Assume that $N_G(P)$ is p-nilpotent. Let P be a Sylow p-subgroup of G in $\mathcal{X}$, and let

$$1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq H_{\alpha+1} \leq \cdots \leq H_\gamma = G$$

be an ascending series of normal subgroups of G with finite factors.

Since H_1 is a finite normal subgroup of G, we can apply Lemma 2.6 to conclude that PH_1 is p-nilpotent. Since the Sylow p-subgroups of PH_1 are conjugate we have that $PH_1 = PO_{p'}(PH_1)$. Now $O_{p'}(PH_1)$ is contained in $O_{p'}(H_1)$, so that $PH_1 = PO_{p'}(H_1)$.

We argue by transfinite induction that $PH_\alpha = PO_{p'}(H_\alpha)$ for all ordinals $\alpha \leq \gamma$. If this is false, there is a first ordinal α such that $PO_{p'}(H_\alpha)$ is a proper subgroup of PH_α. Then $\alpha > 1$ and $PH_\beta = PO_{p'}(H_\beta)$ for all ordinals $\beta < \alpha$. If α is a limit ordinal, $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, $O_{p'}(H_\alpha) = \bigcup_{\beta<\alpha} O_{p'}(H_\beta)$ and $PH_\alpha = PO_{p'}(H_\alpha)$. Hence α cannot be a limit ordinal, $\delta = \alpha - 1$ exists and $PH_\delta = PO_{p'}(H_\delta)$. Write $N = H_\delta$ and $H = H_\alpha$. Then $N = (P \cap N)O_{p'}(N)$. Write $M = O_{p'}(N)$ and $X = HP$. Then $X/M = (H/M)(PM/M)$. Let R be a finite subgroup of H such that $H = NR$. Then $X = (PM)R$. Therefore PM/M is of finite index in X/M. Let T/M be a Sylow p-subgroup of X/M containing PM/M. Since T/M is hypercentral by Lemma 2.9, it follows that PM/M is ascendant in T/M.

If PM/M is a proper subgroup of T/M, then there exists a subgroup Y/M of T/M such that PM/M is a proper normal subgroup of Y/M.

Assume that PM/M is a Sylow p-subgroup of X/M. Then we apply Lemma 2.7 if $\mathcal{X}$ is closed under taking epimorphic images and Lemma 2.8 if G is locally p-soluble to conclude that X/M is p-nilpotent. Hence X is p-nilpotent and $X = PO_{p'}(H)$. This contradiction shows that PH_α is p-nilpotent for all ordinals $\alpha \leq \gamma$. In particular, $G = PH_\gamma$ is p-nilpotent. Therefore Theorem 2.4 and Theorem 2.5 hold in this case. □

Proof of Theorem 2.4. Assume that P is pronormal in G. We need only to prove that PM/M is a Sylow p-subgroup of G/M. Suppose that PM/M is a proper subgroup of T/M. Then $Y/M \leq N_{G/M}(PM/M) = N_G(P)M/M$. Since PM/M contains every p-element of $N_G(P)M/M$, it follows that Y/M should be contained in PM/M. This is a contradiction. Hence PM/M is a Sylow p-subgroup of X/M and the theorem holds. □

Proof of Theorem 2.5. Assume that G is locally p-soluble. Then we may also assume that $H_{\beta+1}/H_\beta$ is either a p-group or a p'-group for all $\beta \leq \gamma$. If H/N is a p-group, then H/M is a p-group and so is X/M. Then X is p-nilpotent. This contradiction implies that H/N and H/M are p'-groups. In this case, PM/M is a Sylow p-subgroup of X/M, and the proof of the theorem is complete. □

CHAPTER 3

On groups whose subgroups of infinite rank are Sylow permutable

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Contents

3.1 Introduction

The results presented in this chapter are further contributions to the research project that analyses the influence of the embedding of subgroups of infinite rank on the structure of a periodic group (see for example [20–24, 27, 29, 33, 41, 55]). They appear in the articles:

- Ballester-Bolinches, A.; Camp-Mora, S.; Kurdachenko, L.A.; Spagnuolo, F., **On groups whose subgroups of infinite rank are Sylow permutable**. *Annali di Matematica Pura ed Applicata (4), 195, no. 3, 717-723, 2016*,
- Ballester-Bolinches, A.; Camp-Mora, S.; Dixon, M.R.; Ialenti, R.; Spagnuolo, F., **On locally finite groups whose subroups of infinite rank have some permutable property**. Submitted,

and show that for normality, permutability and S-permutability, the behaviour of the subgroups of finite rank with respect to that embedding property can be ignored.

Recall that a group G is said to be *hyper-(abelian or finite)* if it has an ascending series of normal subgroups whose factors are abelian or finite.

In the sequel, *all groups considered are periodic*; p will denote a prime number.

Our first main result analyses hyper-(abelian or finite) groups in which the subgroups of infinite section p-rank are S-permutable.

Theorem 3.1. *Let G be a hyper-(abelian or finite) group of infinite section p-rank. If every subgroup of infinite section p-rank is S-permutable in G, then G is locally nilpotent.*

As an immediate deduction we have the following.

Corollary 3.2. *Let G be a hyper-(abelian or finite) group of infinite section rank. If every subgroup of infinite section rank is S-permutable in G, then G is locally nilpotent.*

For (special) rank, in [7] Theorem 3.1 is extended to the case of locally finite groups, the second main result.

Theorem 3.3. *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are S-permutable. Then G is locally nilpotent.*

Our next result is a consequence of Theorem 3.1. Recall that a group G is called quasihamiltonian or Iwasawa if all its subgroups are permutable. The structure of quasihamiltonian groups have been completely described by Iwasawa (see [54, Chapter 2]): they are exactly the locally nilpotent groups with modular subgroup lattice. Dixon and Karatas ([29]), motivated by the papers [33, 41], proved that if a (generalised) soluble group G of infinite rank has all its subgroups of infinite rank permutable, then G is quasihamiltonian.

The following theorem is an extension of Dixon and Karatas' result in the locally finite case.

Theorem 3.4. *Let G be a locally finite group of infinite section p-rank. If every subgroup of infinite section p-rank is permutable in G, then G is an Iwasawa group.*

As a consequence we have.

Corollary 3.5. *Let G be a locally finite group of infinite section rank. If every subgroup of infinite section rank is permutable in G, then G is an Iwasawa group.*

Corollary 3.6 ([29])**.** *Let G be a locally finite group with infinite rank. If every subgroup of infinite rank is permutable in G, then G is locally nilpotent.*

Evans and Kim [33] proved that a (generalised) soluble group of infinite rank is Dedekind (all its subgroups are normal) if its subgroups of infinite rank are normal. The following extension in the locally finite universe follows from Theorem 3.4.

Theorem 3.7. *Let G be a locally finite group of infinite section p-rank. If every subgroup of infinite section p-rank is normal in G, then G is a Dedekind group.*

Corollary 3.8. *Let G be a locally finite group of infinite section rank. If every subgroup of infinite section rank is normal in G, then G is a Dedekind group.*

Corollary 3.9 ([33]). *Let G be a locally finite group with infinite rank. If every subgroup of infinite rank is normal in G, then G is a Dedekind group.*

3.2 Groups with S-permutable subgroups of infinite section rank

The first main result of this section analyses hyper-(abelian or finite) groups in which the subgroups of infinite section p-rank are S-permutable. From now on we establish a number of results that, when taken together, give a good picture of hyper-(abelian or finite) groups of infinite section p-rank with all subgroups of infinite section p-rank S-permutable.

Start recalling that if a locally finite group G is locally nilpotent, then G has all Sylow subgroups normal. Hence every subgroup of G is S-permutable in G. Next lemma shows that the converse is also true.

Lemma 3.10. *Let G be a locally finite group. Then G is locally nilpotent if and only if every subgroup of G is S-permutable in G.*

Proof. Assume that every subgroup of G is S-permutable in G, and let F be a finite subgroup of G. Then, by Lemma 1.29, every subgroup of F is S-permutable in F. By [39], F is nilpotent. Thus G is locally nilpotent. □

Observing the structure of locally finite groups with infinite section p-rank, we proved that they have at least one Sylow p-subgroup of infinite rank.

Lemma 3.11. *Let G be a locally finite group. If G has infinite section p-rank, then G has a Sylow p-subgroup S which is not a Černikov subgroup.*

Proof. Assume that G has two subgroups U and V such that U is a normal subgroup of V and V/U is an infinite elementary abelian p-group. Then V/U has an infinite countable elementary abelian p-subgroup, X/U say. By Lemma 1.29, X has a p-subgroup R such that $X = UR$. Since $R/(R \cap U)$ is an infinite elementary abelian p-section of R, it follows that R is not a Černikov group. Therefore every Sylow p-subgroup of G containing R is not a Černikov group, and lemma holds in this case.

Suppose now that every elementary abelian p-section of G is finite. Then, by Lemma 1.19, G satisfies min-p and so the p-subgroups of G are all Černikov. Applying Theorem 1.21, it follows that G has a Wehrfritz p-subgroup, W say. Since W is Černikov, we have that W has finite section p-rank, and so the order of every elementary abelian section of W is at most p^k for some positive integer k. Let A, B be subgroups of G such that A is a normal subgroup of B and B/A is a finite elementary abelian p-group. Then, by Lemma 1.29, there exists a p-subgroup Y of B such that $B = YA$. Since W is a Wehrfritz p-subgroup of G, Y is isomorphic to a subgroup of W. Hence $B/A \cong Y/(Y \cap A)$ is isomorphic to a elementary abelian section of W. Therefore $|B/A| \leq p^k$. This means that G has finite section p-rank, contrary to supposition.

The proof of the lemma is complete. □

In hyper-(abelian or finite) groups it is possible to find normal Sylow subgroups looking at S-permutable subgroups; in fact, whenever G has a Sylow p-subgroup S-permutable, then it must be normal in G.

Lemma 3.12. *Let G be a hyper-(abelian or finite) group. If a Sylow p-subgroup S of G is S-permutable in G, then S is normal in G.*

Proof. Suppose that S is not normal. Then there exists a Sylow p-subgroup P of G such that $P \neq S$. Since S is S-permutable in G, SP is a subgroup of G. Since $P \neq S$, $SP \neq S$. Applying [2, Corollary 3.2.7], it follows that $\pi(SP) = \{p\}$. Hence $SP = P$, against supposition. Consequently S is normal in G. □

As an immediate corollary we have the following

Corollary 3.13. *Let G be a hyper-(abelian or finite) group of infinite section p-rank. If every subgroup of G of infinite section p-rank is S-permutable in G, then the Sylow p-subgroups of G are normal in G.*

Proof. By Lemma 3.11, G has a non-Černikov Sylow p-subgroup S. By Lemma 1.19, S has infinite section p-rank. Hence S is S-permutable in G. Applying Lemma 3.12, we obtain that S is normal in G. □

The following corollary is a key result to prove the Theorem 3.4

Lemma 3.14. *Let G be a locally finite group with a normal Sylow p-subgroup S that is not Černikov and $G/C_G(S)$ is not a p-group. If every subgroup of G of infinite section p-rank is S-permutable in G, then S is abelian and every subgroup of S is G-invariant.*

Proof. Let $q \in \pi(G/C_G(S))$ with $q \neq p$ and let Q be a Sylow q-subgroup of G. Let B be any non-Černikov subgroup of S. Then B has infinite section p-rank by Lemma 1.19 and so B is S-permutable in G. In particular, BQ is a subgroup of G and $B^Q \leq \langle B, Q\rangle = BQ$. Since $B^Q \leq S$, it follows that $B^Q \leq S \cap BQ = B(S \cap Q) = B$. This means that every subgroup of S of infinite section p-rank is Q-invariant.

Let g be any element of S. Since S is not Černikov, we can apply [61] to conclude that S has an infinite elementary abelian p-subgroup A which is $\langle g\rangle$-invariant. Without loss of generality we can suppose that A is countable. Then $A = \operatorname{Dr}_{n\in\mathbb{N}} \langle a_n\rangle$. We now construct inductively a family $\{A_n | n \in \mathbb{N}\}$ of finite $\langle g\rangle$-invariant subgroups of A such that $E = \langle A_n | n \in \mathbb{N}\rangle = \operatorname{Dr}_{n\in\mathbb{N}} A_n$.

Since g has finite order, we have that $A_1 = \langle a_1\rangle^{\langle g\rangle}$ is finite. Hence $A = A_1 \times C_1$ for some infinite subgroup C_1 of A and C_1^x has finite index in A for all $x \in \langle g\rangle$. Furthermore, the family $\{C_1^x | x \in \langle g\rangle\}$ is finite. Therefore $D_1 = \mathrm{Core}_{\langle g\rangle}(C_1)$ is a $\langle g\rangle$-invariant infinite elementary abelian p-subgroup of C_1 which has finite index in A. Let $1 \neq d \in D_1$ and write $A_2 = \langle d\rangle^{\langle g\rangle}$. Then A_2 is a finite $\langle g\rangle$-invariant subgroup of D_1. Hence $A_1 \cap A_2 = \{1\}$ and $A = (A_1A_2) \times C_2$ for some infinite subgroup C_2. Therefore A_1 and A_2 satisfies the above property and the construction proceeds. Without loss of generality we can suppose that $\langle g\rangle \cap E = \{1\}$. Indeed, if $\langle g\rangle \cap E = \langle b\rangle$, then there exists a positive integer m such that $\langle b\rangle \leq \operatorname{Dr}_{1\leq n\leq m} A_n$. Then we can replace E by $\operatorname{Dr}_{n>m} A_n$.

Let Σ, Δ be two infinite subsets of $\mathbb{N}$ such that $\Delta \cap \Sigma = \emptyset$ and $\Sigma \cup \Delta = \mathbb{N}$. Set $E_\Sigma = \operatorname{Dr}_{n\in\Sigma} A_n$ and $E_\Delta = \operatorname{Dr}_{n\in\Delta} A_n$. Then E_Σ and E_Δ are infinite $\langle g\rangle$-invariant elementary abelian p-subgroups of E and $\langle g\rangle = \langle g\rangle E_\Sigma \cap \langle g\rangle E_\Delta$. Since $\langle g\rangle E_\Sigma$ and $\langle g\rangle E_\Delta$ have infinite section p-rank, it follows that $\langle g\rangle E_\Sigma$ and $\langle g\rangle E_\Delta$ are Q-invariant. Then $\langle g\rangle = \langle g\rangle E_\Sigma \cap \langle g\rangle E_\Delta$ is Q-invariant. Since g is an arbitrary element of S, we obtain that every subgroup of S is Q-invariant, that is, Q acts on S by conjugation as a group of power automorphisms.

Let $1 \neq yC_G(S) \in G/C_G(S)$ be a p'-element, and let $s \in S$ such that $[s, y] \neq 1$. If $a \in S$, then y induces a power p'-automorphism on $K = \langle s, a\rangle$. Applying [11, Lemma 1.3.4], it follows that K is abelian. Consequently, S is an abelian subgroup of G.

Since every subgroup of S is Q-invariant for each Sylow q-subgroup Q of G and each $q \in \pi(G/C_G(S))$, with $q \neq p$, and S is abelian, we conclude that every subgroup of S is G-invariant. □

In a locally finite group G with all subgroups of infinite section p-rank S-permutable, every countable quotient of G with a normal subgroup of infinite section p-rank is locally nilpotent.

Lemma 3.15. *Let G be a locally finite group. Suppose that G has a normal subgroup L such that L has infinite section p-rank and G/L is countable. If every subgroup of G of infinite section p-rank is S-permutable in G, then G/L is locally nilpotent.*

Proof. Let H/L be a subgroup of G/L. Then H has infinite section p-rank. Therefore H is S-permutable in G. By Lemma 1.29, H/L is S-permutable in G/L. Therefore every subgroup of G/L is S-permutable in G/L and G/L is locally nilpotent by Corollary 3.10. □

It is possible to prove Theorem 3.1 for countable hyper-(abelian or finite) groups.

Proposition 3.16. *Let G be a countable hyper-(abelian or finite) group. Suppose that G has infinite section p-rank. If every subgroup of G of infinite section p-rank is S-permutable in G, then G is locally nilpotent.*

Proof. Assume for a contradiction that the group G is not locally nilpotent. Applying Lemma 3.11 and Corollary 3.13, we conclude that G has a non-Černikov Sylow p-subgroup S which is normal in G. Clearly every subgroup of G containing S has infinite section p-rank and so G/S is locally nilpotent by Corollary 3.10. Since G is countable, it follows that G splits over S by [25, Theorem 2.4.5], that is, G has a p'-subgroup R such that $G = SR$.

Suppose that $G/C_G(S)$ is a p-group, then $R \leq C_G(S)$, and hence $G = S \times R$. Since G/S is locally nilpotent, then R is locally nilpotent and then G is locally nilpotent, against supposition. Therefore $G/C_G(S)$ is not a p-group. Applying Lemma 3.14, it follows that S is abelian and every subgroup of S is G-invariant. Let $A = \Omega_1(S) = \{x \in S | x^p = 1\}$. Since S is not Černikov, A is an infinite elementary abelian p-subgroup of S by Lemma 1.19. Let B and C be infinite subgroups of A such that $A = B \times C$. Then B and C are G-invariant subgroups of S of infinite section p-rank. By Lemma 3.15, G/B and G/C are locally nilpotent. Therefore G is locally nilpotent, and this contradiction proves the proposition. □

We are now able to prove Theorem 3.1.

Proof of Theorem 3.1. Let F be a finite subgroup of G. Since G has infinite section p-rank, G has a non-Černikov Sylow p-subgroup by Lemma 3.11. By Lemma 1.19, G has an infinite elementary abelian countable p-subgroup, B say. Set $K = \langle F, B\rangle$. Then K is countable and it has infinite section p-rank. Since every subgroup of K with infinite section p-rank is S-permutable in G, it follows from Lemma 1.29 that K has all its subgroups of infinite section p-rank S-permutable. By Proposition 3.16, K is locally nilpotent and then F is nilpotent. Consequently, G is locally nilpotent. □

Corollary 3.2 follows directly from Theorem 3.1.

To prove Theorem 3.3 we need some previous results

Proposition 3.17. *Let G be a periodic locally soluble group of infinite rank and let X be a finite subgroup of G. Then there exists an abelian subgroup $A = A_1 \times A_2$ of G of infinite rank such that A_1 and A_2 are X-invariant of infinite rank and, furthermore, $A \cap X = \{1\}$.*

Proof. By [35] there exists an abelian X-invariant subgroup B of G such that B has infinite rank and $B = B_1 \times B_2 \times \cdots$, where each B_i is a minimal normal subgroup of BX and hence each B_i is an elementary abelian p-group, for some prime p. Since X is finite and G is locally finite, it is clear that each B_i is also finite and hence of finite rank. Since X is finite there exists n such that $X \cap (B_n \times B_{n+1} \times \cdots) = \{1\}$. Clearly $A = B_n \times B_{n+1} \times \cdots$ is X-invariant of infinite rank and by renumbering and omitting terms if necessary we may assume that $r(B_i) < r(B_{i+1})$ for $i \geq n$. Then we may write $A = A_1 \times A_2$ where A_1, A_2 are X-invariant of infinite rank. □

It is useful to prove a property on countable locally finite groups with infinite rank and all subgroups of infinite rank S-permutable, since the proof of the main theorem can be restricted to countable groups.

Lemma 3.18. *Let G be a countable locally finite group of infinite rank whose subgroups with infinite rank are S-permutable. If H is a normal subgroup of G with infinite rank, then G/H is locally nilpotent.*

Proof. If K/H is a subgroup of G/H, then K has infinite rank. It follows by [14, Corollary 2.3] that every subgroup of G/H is S-permutable. By our remark in the Introduction, G/H is locally nilpotent. □

Now it is possible to prove Theorem 3.3.

Proof of Theorem 3.3. It is easy to check that it is sufficient to prove the theorem for countable groups, so let G be such a countable group. We prove the result in a series of steps.

Step 1. If G has a p-subgroup T of infinite rank, for some prime p, then every Sylow p-subgroup of G has infinite rank.

Let S be a Sylow p-subgroup of G. In fact we prove that $T \cap S$ has infinite rank. Suppose, to the contrary, that $r(T \cap S)$ is finite. Since T has infinite rank it contains an infinite elementary abelian p-subgroup A. Then $A = (A \cap S) \times A^*$, for some subgroup A^* and since $A \cap S \leq T \cap S$, it follows that A^* has infinite rank also. Of course $S \cap A^* = \{1\}$. Let $x \in A^*$ so that $A^* = \langle x \rangle \times C$, for some subgroup C and we may write $C = U \times V$, with U and V of infinite rank. Clearly $U\langle x \rangle \cap V\langle x \rangle = \langle x \rangle$. Furthermore, $U\langle x \rangle$ and $V\langle x \rangle$ have infinite rank and hence permute with S. Let $y \in S$. If t is an integer, then $x^t y = s_1 u = s_2 v$, with $u \in U\langle x \rangle$, $v \in V\langle x \rangle$ and $s_1, s_2 \in S$. Hence $s_2^{-1} s_1 = vu^{-1} \in S \cap A^* = \{1\}$, so $u = v \in \langle x \rangle$. It follows that $S\langle x \rangle$ is a p-subgroup of G. Since S is a Sylow p-subgroup of G we deduce that $x \in S$, contrary to our choice of x. Consequently, $T \cap S$ has infinite -rank and, in particular, S has infinite rank.

Step 2. If G has a Sylow p-subgroup with infinite rank, for some prime p, then every element of order q belongs to $O_q(G)$, for every prime $q \neq p$.

Let P be a Sylow p-subgroup with infinite rank and let x be an element of order q, for the prime $q \neq p$. Assume, by way of a contradiction, that there exists a Sylow q-subgroup Q of G such that $x \notin Q$. Consider $\bar{P} := \bigcap_{i=1}^{q} P^{x^i}$. By Step 1, $\bar{P}$ has infinite rank. Then, by Proposition 3.17, $\bar{P}$ has an elementary abelian p-subgroup $B = B_1 \times B_2$ such that each B_i is $\langle x \rangle$-invariant of infinite rank and $B \cap \langle x \rangle = \{1\}$. Clearly $B_1\langle x \rangle \cap B_2\langle x \rangle = \langle x \rangle$ and $B_i\langle x \rangle$ has infinite rank for $i = 1, 2$. Hence $(B_i\langle x \rangle)Q = Q(B_i\langle x \rangle)$ for $i = 1, 2$. If $Q \cap B\langle x \rangle$ is trivial, then, as above, $\langle x \rangle$ permutes with Q so $Q\langle x \rangle$ is a q-group and $x \in Q$, a contradiction.

On the other hand suppose that $Q \cap B\langle x \rangle$ is non-trivial. The Sylow q-subgroups of $B\langle x \rangle$ have order q so $Q \cap B\langle x \rangle = \langle b \rangle$ is cyclic of order q. If $\langle b \rangle \leq B_1\langle x \rangle \cap B_2\langle x \rangle$, then $\langle b \rangle \leq \langle x \rangle$, so $\langle x \rangle = \langle b \rangle \leq Q$, contrary to the choice of Q. Therefore, we may assume $\langle b \rangle \nleq B_1\langle x \rangle$ and since $\langle b \rangle$ is cyclic of order q, we have $\langle b \rangle \cap B_1\langle x \rangle = \{1\}$. Then $Q \cap B_1\langle x \rangle \leq Q \cap B\langle x \rangle = \langle b \rangle$, so $Q \cap B_1\langle x \rangle \leq \langle b \rangle \cap B_1\langle x \rangle = \{1\}$. Hence we may replace B by B_1 and obtain the contradiction that $x \in Q$ again.

Consequently, x belongs to every Sylow q-subgroup of G and then $x \in O_q(G)$.

Step 3. Suppose that G has a p-subgroup X of infinite rank, for some prime p. If Q is a Sylow q-subgroup of G, with $q \neq p$, then $Q \triangleleft G$.

Consider the group $G/O_q(G)$. If $O_q(G)$ has infinite rank, then $G/O_q(G)$ is a locally nilpotent q'-group. On the other hand, if $O_q(G)$ has finite rank, then $G/O_q(G)$ has infinite rank and all its subgroups of infinite rank are S-permutable. It contains the p-subgroup $XO_q(G)/O_q(G)$ of infinite rank and, by Step 2, every element of order q in $G/O_q(G)$ lies in $O_q(G/O_q(G))$, which is trivial. Hence, in both cases, $G/O_q(G)$ is a q'-group, so $Q = O_q(G)$ as required.

Step 4. For at most one prime p, the group G has p-subgroups of infinite rank.

Suppose that P is a Sylow p-subgroup of infinite rank and that Q is a Sylow q-subgroup of infinite rank. By Step 3, every Sylow subgroup of G is normal in G, so that G is locally nilpotent.

Step 5. G has a Sylow p-subgroup of infinite rank, for some prime p.

By contradiction, suppose that every Sylow subgroup of G has finite rank, so that G satisfies the minimal condition on p-subgroups for every prime p. Let F be any finite subgroup of G and put $\pi = \pi(F)$. Then π is a finite set and $G/O_{\pi'}(G)$ is a Černikov group ([25, Theorem 3.5.15], and Corollary 1.26). It follows that $O_{\pi'}(G)$ has infinite rank and the factor group $G/O_{\pi'}(G)$ is locally nilpotent by Lemma 3.18. In particular, $F \simeq FO_{\pi'}(G)/O_{\pi'}(G)$ is nilpotent and G is locally nilpotent, a contradiction.

Step 6. Final step.

By Step 5, G has a Sylow p-subgroup of infinite rank, for some prime p. Then, by Step 3, G has a unique Sylow q-subgroup G_q, for any $q \neq p$ and, by Step 4, G_q has finite rank. Thus $R = \operatorname{Dr}_{q \neq p} G_q$ is a normal p'-subgroup of G. Furthermore, since G is countable, it follows by [25, Theorem 2.4.5] that there exists a Sylow p-subgroup P of G such that $G = RP$. In particular, by Step 1, P has infinite rank.

Fix a prime $q \neq p$ and put $Q = G_q$. It is clear that $C_P(Q)$ is normal in PQ, since Q is normal in G. Now Q is Černikov and $P/C_P(Q)$ is a periodic group of automorphisms of Q, so is also Černikov by Theorem 1.9. Hence $C_P(Q)$ has infinite rank and all subgroups of $PQ/C_P(Q)$ are S-permutable. Consequently, $PQ/C_P(Q)$ is locally nilpotent. On the other hand PQ/Q is certainly locally nilpotent, so by Remak's theorem, PQ embeds in $PQ/Q \times PQ/C_P(Q)$. Hence PQ is locally nilpotent. In particular $[P, Q] = \{1\}$ and this hold for every Sylow q-subgroup G_q of G, with $q \neq p$. It follows that $[P, R] = \{1\}$ and $G = P \times R$ is locally nilpotent. This last contradiction completes the proof of the theorem. □

3.3 Groups with permutable subgroups of infinite section rank

In this section it is analysed the structure of locally finite groups whose subgroups of infinite section p-rank are permutable. One of the aim of this section is to prove an extension of Dixon and Karatas' result in the locally finite case and we prove it as a consequence of the main theorem of the current section.

Let's start giving some important properties of locally finite groups with all subgroups with infinite section p-rank permutable.

Lemma 3.19. *Let G be a locally finite group. Suppose that G has infinite section p-rank. If every subgroup of G of infinite section p-rank is permutable in G, then*

1. *G has normal Sylow p-subgroup S which is not Černikov;*
2. *every subgroup of G/S is permutable;*
3. *G/S is locally nilpotent and metabelian.*

Proof. By Lemma 3.11, G has a Sylow p-subgroup S that is not Černikov. Since the section p-rank of S is infinite, S is permutable in G. Then S is an ascendant subgroup of G by [54, Theorem 6.2.10]. It follows that S^G is a p-subgroup of G and then $S = S^G$ is normal in G. Now, let H/S be a subgroup of G/S. Since the section p-rank of S is infinite, H has infinite section p-rank. Therefore it is permutable in G. Then H/S is permutable in G/S. Thus every subgroup of G/S is permutable. By [54, Theorem 2.4.13 and Theorem 2.4.22], G/S is locally nilpotent and metabelian. □

Now we are able to extend Dixon and Karatas' result to locally finite groups with infinite section p-rank, that is the main result of this section.

Proof of Theorem 3.4. By Lemma 3.19, G has normal Sylow p-subgroup S which is not Černikov and every subgroup of G/S is permutable.

Let g, x be elements of S and write $H = \langle g, x\rangle$. Since S is not Černikov, S has an infinite elementary abelian p-subgroup A, which is H-invariant by [61]. Without loss of generality we can suppose that A is countable. Then $A = \operatorname{Dr}_{n\in\mathbb{N}} \langle a_n\rangle$. We can argue as in the proof of Lemma 3.14. Then we determine a family $\{A_n | n \in \mathbb{N}\}$ of finite H-invariant groups of G such that $E = \langle A_n | n \in \mathbb{N}\rangle = \operatorname{Dr}_{n\in\mathbb{N}} A_n$ and $E \cap H = \{1\}$. Let $L = N_G(E)$. Then $H \leq L$, and every subgroup of L of infinite section p-rank is permutable in L. In particular, every subgroup of L containing E is permutable in L and L/E is an Iwasawa group. Since $E \cap H = \{1\}$, it follows that $\langle g\rangle\langle x\rangle = \langle x\rangle\langle g\rangle$. Therefore

g permutes with every cyclic subgroup of S, and hence it permutes with every subgroup of S. Applying [54, Theorem 2.4.14], we obtain that S is nilpotent. By Lemma 3.19, G/S is soluble. Hence G is soluble and so G is locally nilpotent by Theorem 3.1. In particular $G = S \times Q$, where Q is the Sylow p'-subgroup of G. As we proved above, every subgroup of S (respectively, $Q \cong G/S$) is permutable in S (respectively in Q). Therefore G is an Iwasawa group. □

Proof of Corollary 3.6. Suppose that G has infinite section rank. If H is a subgroup of G of infinite section rank, then H has infinite (special) rank. Therefore H is permutable in G. Applying Corollary 3.5, we obtain that G is an Iwasawa group. Assume now that G has finite section rank. Then G has Černikov Sylow subgroups. Let F be a finite subgroup of G. Then $\pi = \pi(F)$ is finite, and $F \cap O_{\pi'}(G) = 1$. Hence $F \cong FO_{\pi'}(G)/O_{\pi'}(G)$. Moreover, $G/O_{\pi'}(G)$ is Černikov by Corollary 1.26 and [25, Theorem 3.5.15]; in particular, $G/O_{\pi'}(G)$ is countable and $O_{\pi'}(G)$ has infinite (special) rank. Let $H/O_{\pi'}(G)$ be a subgroup of $G/O_{\pi'}(G)$. Then H has infinite (special) rank. Therefore H is permutable in G. So $H/O_{\pi'}(G)$ is permutable, and in particular S-permutable in $G/O_{\pi'}(G)$. Therefore every subgroup of $G/O_{\pi'}(G)$ is S-permutable in $G/O_{\pi'}(G)$ and $G/O_{\pi'}(G)$ is locally nilpotent by Lemma 3.10. Then F is nilpotent, and G is locally nilpotent. □

Changing our focus now into Dedekind groups, Evans and Kim proved in [33] that a (generalised) soluble group of infinite rank is Dedekind if its subgroups of infinite rank are normal. The following extension in the locally finite universe for section p-rank follows from Theorem 3.4.

Proof of Theorem 3.7. Applying Lemma 3.19, we obtain that G has normal Sylow p-subgroup S which is not Černikov. By Theorem 3.4, G is locally nilpotent. Then $G = S \times Q$, where Q is the Sylow p'-subgroup of G. Choose elements g, x in S and write $H = \langle g, x \rangle$. Arguing as in Theorem 3.4, we determine a subgroup L of G and an infinite elementary abelian p-subgroup E of L such that E is H-invariant and $E \cap H = \{1\}$. Then every subgroup of L of infinite section p-rank is normal in L. Since the section p-rank of E is infinite, L/E is a Dedekind group. Since $H \cong HE/E$, it follows that $\langle g \rangle$ is $\langle x \rangle$-invariant. This means that G is a Dedekind group. □

CHAPTER 4

On locally finite groups whose subgroups of infinite rank have some permutable property

Contents

4.1 Introduction

In this chapter we take our analysis of the impact of the embedding of the subgroups of infinite section rank on the stucture of a periodic group further. The results are contained in the following article:

Ballester-Bolinches, A.; Camp-Mora, S.; Dixon, M.R.; Ialenti, R.; Spagnuolo, F., **On locally finite groups whose subroups of infinite rank have some permutable property**. Submitted.

Our main aim in Section 4.2 is to prove that a locally finite group with all subgroups of infinite section p-rank semipermutable, has all subgroups semipermutable. However in Section 4.3 a simple example is given of a metabelian group in which every subgroup of infinite rank is S-semipermutable but not all subgroups are S-semipermutable.

4.2 Groups of infinite rank with all subgroups of infinite rank semipermutable

The purpose of this section is to prove that a locally finite group whose subgroups of infinite rank are semipermutable has all subgroups semipermutable. The following result states that in a locally finite group it is enough to look at the cyclic subgroups of prime power order to determine if every subgroup is semipermutable.

Lemma 4.1. *Let G be a locally finite group. Then all subgroups of G are semipermutable if and only if whenever x is a p-element and y is a q-element of G, then $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$, for all distinct primes p and q.*

Proof. If every subgroup of G is semipermutable, the statement is clear, so assume that all cyclic p-subgroups and all cyclic q-subgroups of G permute, where p and q are different prime numbers. Let H and K be subgroups of G such that $\pi(H) \cap \pi(K) = \emptyset$. Suppose that $h \in H$ and $k \in K$. We prove that $\langle h\rangle\langle k\rangle$ is a subgroup of G. Note that

$$\langle h\rangle = \langle h\rangle_{p_1} \times \cdots \times \langle h\rangle_{p_s}$$
$$\langle k\rangle = \langle k\rangle_{q_1} \times \cdots \times \langle k\rangle_{q_t}$$

where $\langle h\rangle_{p_i}$ is the Sylow p_i-subgroup of $\langle h\rangle$ and $\langle k\rangle_{q_j}$ is the Sylow q_j-subgroup of $\langle k\rangle$. By hypothesis, every $\langle h\rangle_{p_i}$ permutes with every $\langle k\rangle_{q_j}$, for $i = 1, \dots s$ and $j = 1, \dots t$, and hence $\langle h\rangle$ permutes with $\langle k\rangle$. It follows that $HK = KH$ and the lemma is proved. □

Next result gives a second restriction: if a locally finite group with all subgroups of infinite section p-rank semipermutable has a p-subgroup S of infinite rank, then all subgroups of S are semipermutable.

Lemma 4.2. *Let G be a locally finite group with infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. If G has a p-subgroup S with infinite rank and an element x with $o(x) = q^\alpha$ such that $q \neq p$ and q is a prime number, then every subgroup of S permutes with $\langle x\rangle$. In particular $S\langle x\rangle$ is a $\{p, q\}$-group.*

Proof. Let $y \in S$. We prove that $\langle y\rangle\langle x\rangle = \langle x\rangle\langle y\rangle$. By Proposition 3.17, S has an abelian subgroup $A = A_1 \times A_2$ such that A_i has infinite rank, $A_i^{\langle y\rangle} = A_i$ for $i = 1, 2$ and $A \cap \langle y\rangle = \{1\}$. Then $A_1\langle y\rangle \cap A_2\langle y\rangle = \langle y\rangle$. Since $A_i\langle y\rangle$ has infinite section p-rank, $\langle x\rangle(A_i\langle y\rangle) = (A_i\langle y\rangle)\langle x\rangle$, for $i = 1, 2$. It follows that for all $i, j \in \mathbb{Z}$, $x^i y^j = a_1 x^m = a_2 x^n$, with $m, n \geq 1$ and $a_i \in A_i\langle y\rangle$. Hence $x^{m-n} = a_1^{-1}a_2 \in \langle x\rangle \cap A\langle y\rangle = \{1\}$. Then $a_1 = a_2 \in \langle y\rangle$ and $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. □

Furthermore, if a locally finite group with the previous hypothesis has a p-subgroup of infinite rank, then all q-subgroups permute with all r-subgroups, with q and r distinct primes different from p.

Lemma 4.3. *Let G be a locally finite group with infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. Let q, r be distinct primes different from p and let S be a p-subgroup of G with infinite section p-rank. If x is a q-element and y is an r-element of G, then $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$.*

Proof. Let A be an abelian subgroup of infinite rank of S such that $A = A_1 \times A_2$ and A_i has infinite section p-rank, for $i = 1, 2$. Thus $A_i\langle x\rangle = \langle x\rangle A_i$ and $A_1\langle x\rangle \cap A_2\langle x\rangle = \langle x\rangle$. Furthermore by Lemma 4.2, $A_i\langle x\rangle$ is a $\{p, q\}$-group with infinite section p-rank, so $(A_i\langle x\rangle)\langle y\rangle = \langle y\rangle(A_i\langle x\rangle)$. It follows that $xy = y^m a_1 = y^n a_2$, with $m, n \geq 1$ and $a_i \in A_i\langle x\rangle$. Therefore $a_1 a_2^{-1} = y^{n-m} \in A\langle x\rangle \cap \langle y\rangle = \{1\}$. So $a_1 = a_2 \in \langle x\rangle$ and $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. □

Finally, next lemma reduces the problem to countable locally finite groups.

Lemma 4.4. *Let G be a locally finite group with infinite section p-rank. If every countable subgroup of G with infinite section p-rank has all subgroups semipermutable, then all subgroups of G are semipermutable.*

Proof. Let x, y be element of G with relatively prime orders. Since G has infinite section p-rank, it follows from Theorem 1.19 that there exists a countable elementary abelian p-subgroup A of infinite rank. Form $H = \langle x, y, A\rangle$, a countable subgroup of G of infinite section p-rank. Then every subgroup of H is semipermutable, by hypothesis, so $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. Hence every subgroup of G is semipermutable, by Lemma 4.1. □

We are now able to prove the main theorem of present section.

Theorem 4.5. *Let G be a locally finite group with infinite section p-rank whose subgroups of infinite section p-rank are semipermutable. Then every subgroup of G is semipermutable.*

Proof. Using Lemma 4.4 and Lemma 1.31 we may suppose that G is a countable group.

By Lemma 3.11, G has a Sylow p-subgroup P which is not Černikov. Let $x, y \in G$ with $o(x) = r^\alpha$, $o(y) = q^\beta$ and $r \neq q$. We prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. By Lemma 4.3 if $q, r \neq p$, then $\langle x\rangle$ and $\langle y\rangle$ permute. So we may suppose that $r = p$ and prove that x belongs to a p-subgroup of infinite rank of G.

Let $\{F_n\}_{n\in\mathbb{N}}$ be a totally ordered local system of finite subgroups of G. We may assume that $x \in F_1$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a totally ordered local system of finite subgroups of

P such that the rank of P_n is at least n for every $n \geq 1$, and consider the finite subgroup $G_n = \langle F_n, P_n \rangle$. Let S_1 be a Sylow p-subgroup of G_1 containing x and notice that the rank of S_1 is at least 1. Let $n \geq 1$ and let S_n be a Sylow p-subgroup of G_n such that $x \in S_n$. Since $G_n \leq G_{n+1}$ there exists a Sylow p-subgroup S_{n+1} of G_{n+1} such that $S_n \leq S_{n+1}$. We note that $rk(S_{n+1}) \geq n+1$. The subgroup $S = \bigcup_{n\geq 1} S_n$ is a p-subgroup of G with infinite rank such that $x \in S$. The result follows by Lemma 4.2. □

The result on groups with infinite section rank is an immediate consequence of Theorem 4.5

Corollary 4.6. *Let G be a locally finite group with infinite section rank whose subgroups of infinite section rank are semipermutable. Then every subgroup of G is semipermutable.*

Otherwise, for (special) rank, things are not so direct.

Theorem 4.7. *Let G be a locally finite group with infinite (special) rank whose subgroups of infinite rank are semipermutable. Then every subgroup of G is semipermutable.*

Proof. It is enough to prove that G has a Sylow p-subgroup of infinite rank, for some prime p, so that the theorem will follow as an application of Theorem 4.5. Suppose, by a contradiction, that every p-subgroup has finite rank, so that G has min-p, for every prime p. By [25, Theorem 3.5.15], G contains a locally soluble normal subgroup S of finite index. Let x and y be respectively a p-element and a q-element of G, with $p \neq q$, and put $\pi = \{p, q\}$. Then $G/O_{\pi'}(S)$ is a Černikov group by Corollary 1.26 and $O_{\pi'}(S)$ has infinite rank. By [35, Theorem 1], $O_{\pi'}(S)$ contains an abelian subgroup $B = B_1 \times B_2$, where B_1 and B_2 are $\langle x \rangle$-invariant subgroups of infinite rank and $\langle x \rangle \cap B = \{1\}$. Therefore, for every $i = 1, 2$, $B_i\langle x \rangle$ is a q'-subgroup of infinite rank of G and $(B_i\langle x \rangle)\langle y \rangle = \langle y \rangle (B_i \langle x \rangle)$. As $B\langle x \rangle \cap \langle y \rangle = \{1\}$, the following equalities hold:

$$\langle x \rangle \langle y \rangle = (B_1\langle x \rangle \cap B_2\langle x \rangle)\, \langle y \rangle = (B_1\langle x \rangle)\langle y \rangle \cap (B_2\langle x \rangle)\langle y \rangle$$

and $\langle x \rangle \langle y \rangle$ is a subgroup of G. The theorem now follows from Lemma 4.1. □

4.3 Groups of infinite rank with all subgroups of infinite rank S- semipermutable

Focus now the attention on S-semipermutability. The following result is the corresponding of Lemma 4.1 for S-semipermutable subgroups and it is easy to establish.

Lemma 4.8. *Let G be a locally finite group and let p, q be distinct primes. Then every subgroup of G is S-semipermutable if and only if every p-element of G permutes with every Sylow q-subgroup of G, for all such pairs of primes in $\pi(G)$.*

For locally finite groups with all subgroups of infinite rank S-semipermutable the best it could be proved is that all subgroups are S-semipermutable only if the group has the minimal condition on p-subgroups for every prime p.

Theorem 4.9. *Let G be a locally finite group of infinite rank with min-p for every p. If every subgroup of infinite rank of G is S-semipermutable, then all subgroups of G are S-semipermutable.*

Proof. Let x be a p-element of G and let Q be a Sylow q-subgroup of G, where p and q are different prime numbers. By [25, Theorem 3.5.15], G has a locally soluble normal subgroup S of finite index in G. Let π be a finite subset of $\pi(S)$ such that $p, q \notin (\pi' \cap \pi(S))$. By Corollary 1.26, $G/O_{\pi'}(S)$ is a Černikov group and hence $O_{\pi'}(S)$ has infinite rank. By Proposition 3.17 there is an abelian subgroup $B = B_1 \times B_2$ in $O_{\pi'}(S)$ such that B_1 and B_2 have infinite rank and both are normalized by x. Then the q'-subgroups $B_i\langle x\rangle$, which have infinite rank, permute with Q and

$$\langle x\rangle Q = (B_1\langle x\rangle)Q \cap (B_2\langle x\rangle)Q$$

is a subgroup of G. In particular, $\langle x\rangle$ is S-semipermutable in G, so every subgroup of G is S-semipermutable, by Lemma 4.8. □

Unfortunately a result analogous to Theorem 4.7 does not hold, as the following example shows.

Proposition 4.10. *There exists a periodic metabelian group G with infinite rank whose subgroups of infinite rank are S-semipermutable but not every subgroup of G is S- semipermutable.*

Proof. For every integer $i \geq 1$, let $T_i = \langle a_i, b_i \mid a_i^3 = b_i^2 = 1, b_i^{-1}a_ib_i = a_i^{-1}\rangle$ be an isomorphic copy of the symmetric group on three letters S_3 and let $T = \mathrm{Dr}_{i\geq 1}T_i$.

Let $P = \mathrm{Dr}_{i\geq 1}\langle b_i\rangle$, let $Q = \langle a_1\rangle \times \langle a_2\rangle$ and consider $G = P \ltimes Q$. Observe that P is an elementary abelian 2-group of infinite rank, so that G is a countable metabelian group of infinite rank.

Let A be a subgroup of G of infinite rank. Since G has a finite normal Sylow 3-subgroup, there are only two possibilities for the set $\pi(A)$: either $\pi(A) = \{2, 3\}$ or

$\pi(A) = \{2\}$. In the first case, A is trivially S-semipermutable in G. In the second case, A permutes with the normal Sylow q-subgroup Q. Then every subgroup of G of infinite rank is S-semipermutable.

Suppose, for a contradiction, that every subgroup of G is S-semipermutable. Let $X = \langle a_1 a_2 \rangle$. By assumption, X is S-semipermutable and then PX is a subgroup of G. Since $X = PX \cap Q$, X is a normal subgroup of PX. However this is a contradiction since the element

$$b_1^{-1} a_1 a_2 b_1 = b_1^{-1} a_1 b_1 a_2 = a_1^2 a_2$$

does not belong to X. □

CHAPTER 5

On semipermutable subgroups of finite groups

Contents

5.1 Introduction

The results presented in this chapter are the contents of the paper:

Ballester-Bolinches, A.; Heineken, H.; Spagnuolo, F., **On semipermutable subgroups of finite groups**. Submitted.

In the following, G always denotes a finite group.

The properties of semipermutability and S-semipermutability on finite groups have been extensively studied in recent years (see, for example, the survey paper [5]). We mention here the paper of Isaacs [37] where some interesting properties of S- semipermutable subgroups were proved. Among other results, he showed that the normal closure of an S-semipermutable nilpotent Hall subgroup is soluble.

The aim of this chapter is to offer some results that are more or less in the same spirit. We study the embedding of subgroups of odd order with all subgroups semipermutable and, in particular, we show that their normal closure is supersoluble.

In the sequel we call a group G *semimodular* if all subgroups of G are semipermutable in G. Note that semimodular groups are exactly the soluble BT-groups by Theorem 1.36.

Our first result confirms the supersolubility of a group which is the product of a normal supersoluble subgroup and a subnormal semimodular subgroup of odd order.

Theorem 5.1. *Let the factorised group $G = NS$ be the product of a normal supersoluble subgroup N and a subnormal semimodular subgroup S of odd order. Then G is supersoluble.*

If the condition S of odd order is removed from Theorem 5.1, then the conclusion is no longer true.

Example 5.1. *Let $G = \langle a_1, a_2, b_1, b_2, c | a_1^2 = a_2^2 = b_1^3 = b_2^3 = c^2 = 1, [a_1, b_2] = [a_2, b_1] = [a_1, a_2] = [b_1, b_2] = 1, a_1^{-1}b_1a_1 = b_1^{-1}, a_2^{-1}b_2a_2 = b_2^{-1}, c^{-1}a_1c = a_2, c^{-1}b_1c = b_2\rangle$ be the regular wreath product $\Sigma_3 \wr C_2$ of the symmetric group of degree 3 with a cyclic group of order 2. Set $N = \langle a_1, a_2, b_1, b_2\rangle$. Then N is a normal subgroup of G which is isomorphic to $\Sigma_3 \times \Sigma_3$. Hence N is supersoluble. Let $A = \langle b_1, b_2, c\rangle$. Observe that A is a subnormal subgroup of G and $B = \langle b_1, b_2\rangle = \langle b_1b_2\rangle \times D$ for some normal subgroup D of A such that $S = D\langle c\rangle$ is isomorphic to Σ_3. Then S is subnormal in $G = NS$, S is a semimodular subgroup of G but G is not supersoluble.*

The next theorem is an application of Theorem 5.1. It shows that the product of a normal supersoluble subgroup and the normal closure of a subnormal semimodular subgroup of odd order of a group is always supersoluble.

Theorem 5.2. *Let G be a group and N a normal supersoluble subgroup of G. If S is a subnormal semimodular subgroup of G of odd order, then NS^G is supersoluble.*

The following corollaries are consequences of Theorem 5.2

Corollary 5.3. *If S is a subnormal semimodular subgroup of odd order of a group G, then S^G is supersoluble.*

Corollary 5.4. *Let N be a normal supersoluble subgroup of a group G and $S_1, \dots, S_k$ be subnormal semimodular subgroups of G of odd order. Then $\langle N, S_1, \dots, S_k\rangle$ is supersoluble.*

Proof. Using induction on k and applying Theorem 5.2, we obtain that $NS_1^G \dots S_k^G$ is supersoluble. Hence $\langle N, S_1, \dots, S_k\rangle$ is supersoluble. □

Corollary 5.5. *Let $S_1, \dots, S_k$ be subnormal semimodular subgroups of a group G of odd order. Then $\langle S_1, \dots, S_k\rangle$ is supersoluble.*

5.2 Proofs

Proof of Theorem 5.1. Assume that the result is false and that G is a counterexample of smallest possible order with least $|G : N| + |S|$. Then $N \neq 1$ and $S \neq 1$. Applying

Lemma 1.38, it follows that G is a product of two subnormal supersoluble subgroups. Hence G is soluble.

Let L be a minimal normal subgroup of G. Then L is an elementary abelian p-group for some prime p. Our goal is to show that G/L satisfies the hypotheses of the theorem. First of all, $G/L = (NL/L)(SL/L)$ is the product of the normal supersoluble subgroup NL/L of G/L and the subnormal odd order subgroup $SL/L \cong S/S \cap L$ which is semimodular by Lemma 1.38. Since G/L satisfies the hypotheses of the theorem, the minimal choice of G yields G/L is supersoluble. Since the class of all supersoluble groups is a saturated formation, it follows that G is a primitive group.

Let $D = \mathrm{Soc}(G) = \mathrm{F}(G)$ be the unique minimal normal subgroup of G. Then $D \leq N$ and $C_G(D) = D$. Since N is supersoluble, it follows that N/D is abelian of exponent dividing $p-1$. In particular, N/D is a p'-group. By [32, Chapter A, Lemma 14.3], D normalises S. Let p be the prime dividing $|D|$. Then $D = O_p(G)$. Let X be a minimal normal subgroup of N contained in D. Since N is supersoluble, it follows that X is cyclic of order p. Since G/D is supersoluble, we conclude that $X \neq D$, and so X is not normal in G. In particular, $D = X^G = X^{NS} = X^S$. Let us denote $Y = XS$. Then Y is a subgroup of G containing D and $Y = DS$. Assume that $Y = G$ then S is normal in G and $G = S$, against our choice of G. Therefore Y is a proper subgroup of G. Clearly $Y = (Y \cap N)S$ satisfies the hypotheses of the theorem. The minimal choice of G implies that Y is supersoluble. Since D is the Fitting subgroup of G and DS is subnormal in G, also D is the Fitting subgroup of Y, and $DS/D \cong S/D \cap S$ is abelian of exponent dividing $p-1$. It follows that $Y/O_{p',p}(Y)$ is abelian of exponent dividing $p-1$. Since $O_{p'}(Y)$ is a normal subgroup of Y, it follows that $O_{p'}(Y) \leq C_G(D) = D$. Hence $O_{p'}(Y) = \{1\}$. In particular, $O_{p',p}(Y) = O_p(Y)$ and $Y/O_p(Y)$ is abelian of exponent dividing $p-1$. Let T be a Hall p' subgroup of S. By hypothesis, S is semimodular so that T permutes with every subgroup of $D \cap S$. Since $D \cap S$ is a normal Sylow p-subgroup of S, it follows that every element of T induces a power automorphism on $D \cap S$. Therefore $[D \cap S, T] = D \cap S$ since S cannot be nilpotent. In particular, $D \cap S$ is the nilpotent residual of S which is just the nilpotent residual of DS. If X were contained in $D \cap S$, then it would be normalised by S and so X would be a normal subgroup of G, contrary to supposition. Let us denote $D \cap S = A$. Therefore $\{1\} \neq XA/A$ is centralised by TA/A. Applying [32, Chapter A, Proposition 2.15], we obtain that $D = XA = [D,T] \times C_D(T)$. Since $D \cap S$ is a maximal subgroup of D contained in $[D,T]$ and $[D,T] \neq D$, it follows that $C_D(T)$ is a normal subgroup of Y of order p.

Since T does not normalise X, there exists a q-element y of T for some prime q such that $y \notin N_G(X)$. Let us assume that y is of minimal order. Then $\langle y^q \rangle \leq \mathrm{N}_G(X)$. Moreover, $y \notin N$. Assume that $N\langle y \rangle$ is a proper subgroup of G. Since $N\langle y \rangle$ satisfies the hypotheses of the theorem, we have that $N\langle y\rangle$ is supersoluble. Moreover S normalises $A\langle y\rangle$ (note that T is abelian). Therefore $N\langle y\rangle$ is a normal subgroup of G. Because N is a proper subgroup of $N\langle y\rangle$, then the minimal choice of (G, N, S) would imply the supersolubility of G. This contradiction yields $G = N\langle y\rangle$. Then

$$D = X^G = X^{\langle y\rangle} = XX^y \dots X^{y^{q-1}}$$

Assume that $X = \langle x\rangle$. Then $x^\alpha = x^{y^q}$ for some $1 \leq \alpha \leq p-1$. Since $\{1\} \neq \mathrm{C}_D(T) \leq \mathrm{C}_D(\langle y\rangle)$, we have that $\alpha = 1$. Therefore the characteristic polynomial of the linear map induced by y on the vector space D is $x^q - 1$. On the other hand, $D = A \times \mathrm{C}_D(T)$. Since y induces a power automorphism in $D \cap S$, it follows that $a^y = a^m$ for some $1 \leq m \leq p-1$ and all $a \in D \cap S$. If $m = 1$, then $y \in \mathrm{C}_G(D) = D$. This contradiction yields $m > 1$. Therefore the matrix of y relative to a basis of D composed of a basis of A and non-trivial element of $\mathrm{C}_D(T)$ has characteristic polynomial $(x-1)(x-m)^{q-1}$ which does not equal to $x^q - 1$ since q is odd. This final contradiction proves the result. □

The following lemma is probably well known. We include a proof for the sake of completeness.

Lemma 5.6. *If S is a subnormal subgroup of a group G, there exist subgroups $S_1, S_2, \dots, S_m$ of G such that $S = S_1 \leq S_2 \leq \dots \leq S_m = S^G$ and, for all $i \in \{1, \dots, m-1\}$, S_i is a normal subgroup of S_{i+1} such that $S_{i+1} = S_i S^{g_i}$ for some $g_i \in G$.*

Proof. We use induction on the defect of S in G. Clearly the lemma holds if S is a normal subgroup of G. Hence we may assume that $S \neq S^G$. Then $T = S^g \neq S$ for some $g \in G$. By induction, there exist subgroups $T_1, T_2, \dots, T_a$ of S^G such that $T = T_1 \leq T_2 \leq \dots \leq T_a = T^{S^G}$ and, for all $i \in \{1, \dots, a-1\}$, T_i is a normal subgroup of T_{i+1} such that $T_{i+1} = T_i T^{g_i}$ for some $g_i \in S^G$. Let $M = S^{S^G}$. Let $S_i = T_i^{g^{-1}}$ for all $i \in \{1, \dots, a-1\}$. Then $S = S_1 \leq S_2 \leq \dots \leq S_a = M \leq MT_1 \leq \dots \leq MT_a$ satisfies the statement of the lemma. If $S^G = MT_a$ we are done. If not, there exists an element $h \in G$ such that $N = S^h$ is not contained in MT_a. Arguing as above, we can construct a chain from S to $S^{S^G} T^{S^G} N^{S^G}$. Since S^G is a product of finitely many subgroups of the form $(S^x)^{S^G}$, the construction of the series of subgroups stated in the lemma can be carried out. □

Proof of Theorem 5.2. Applying Lemma 5.6, we obtain that there exists a series $S = S_1 \leq S_2 \leq \dots \leq S_m = S^G$ and, for all $i \in \{1, \dots, m-1\}$, S_i is a normal subgroup of S_{i+1} such that $S_{i+1} = S_i S^{g_i}$ for some $g_i \in G$. We prove that NS_i is supersoluble for all i by induction on m. If $m = 1$, then $NS_1 = NS$ is supersoluble by Theorem 5.1. Assume that

$m > 1$ and NS_{m-1} is supersoluble. Note that NS_{m-1} is a normal subgroup of NS_m and $S^{g_{m-1}}$ is a subnormal semimodular subgroup of NS_m of odd order such that $NS_m = NS_{m-1}S^{g_{m-1}}$. Applying Theorem 5.1, we have that $NS_m = NS^G$ is supersoluble. $\square$

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