Stability of King's family of iterative methods with memory☆

Beatriz Campos[a], Alicia Cordero[b,*], Juan R. Torregrosa[b], Pura Vindel[a]

[a] *IMAC, Department de Matemàtiques, Universitat Jaume I, Castellón, Spain*
[b] *Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, València, Spain*

Abstract

In the literature there exist many iterative methods with memory for solving nonlinear equations, the most of them designed in the last years. As they use the information of (at least) the two previous iterates to generate the new one, usual techniques of complex dynamics are not useful in this case. In this paper, we present some real multidimensional dynamical tools to afford this task, applied on a very well-known family of iterative schemes; King's class. It is showed that the most of elements of this class present a very stable behavior, visualized in different dynamical planes. However, pathological cases as attracting strange fixed point or periodic orbits can also be found.

Keywords: Nonlinear equations, iterative method with memory, basin of attraction, dynamical plane, stability.

1. Introduction

In the last decades, iterative methods for solving nonlinear equations have proved their usefulness in many branches of Science and Technology. Many of them are designed almost ad-hoc, for solving specific types of problems, like derivative-free schemes, for those problems that does not allow to calculate the derivative of the nonlinear equation to be solved, usually because it does not have an explicit expression, or it is too expensive (in the computational sense of the term) to calculate it.

In this work, we use the dynamical tools presented in [5] on iterative schemes with memory for solving nonlinear equations. The design of this kind of methods has experimented an important growth in the last years, the early works of Traub [15], later developed by Petković et al. [16, 17, 18] and used by other authors (see [19, 21, 22, 23] and references inside), but the understanding of their stability has not been developed. Nevertheless, as the fixed point iteration functions have more than one variable, it is necessary to use some specific dynamical elements joint with some auxiliary functions to facilitate the calculations. Also some dynamical concepts have been adapted to achieve the appropriate numerical sense.

Let us consider the problem of finding a simple zero of a function $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, that is, a solution $\alpha \in I$ of the nonlinear equation $f(x) = 0$. If an iterative method with memory is employed (specifically, one that uses two previous iterations to calculate the following estimation), whose iterative expression is

$$x_{k+1} = g(x_{k-1}, x_k), \quad k \geq 1,$$

where x_0 and x_1 are the initial estimations, a fixed point will be obtained when $x_{k+1} = x_k$, that is, $g(x_{k-1}, x_k) = x_k$. Now, this solution can be obtained as a fixed point of the function $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by means of the fixed-point iteration method

$$\begin{aligned} G(x_{k-1}, x_k) &= (x_k, x_{k+1}), \\ &= (x_k, g(x_{k-1}, x_k)), \quad k = 1, 2, \ldots, \end{aligned}$$

being x_0 and x_1 the initial estimations. So, we will state that (x_{k-1}, x_k) is a fixed point of G if

$$G(x_{k-1}, x_k) = (x_{k-1}, x_k).$$

So, not only $x_{k+1} = x_k$, but also $x_{k-1} = x_k$ by definition of G. Besides, $x^* \in \mathbb{R}^2$ is a k-periodic point if $G^k(x^*) = x^*$ and $G^p(x^*) \neq x^*$, for $p = 1, 2, \ldots, k-1$.

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*Corresponding author
Email addresses: campos@uji.es (Beatriz Campos), acordero@mat.upv.es (Alicia Cordero), jrtorre@mat.upv.es (Juan R. Torregrosa), vindel@uji.es (Pura Vindel)

We will analyze the local convergence of each one of the methods with memory under study. To get this aim, we will use the following result, that can be found in [24].

Theorem 1. *Let ψ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converges to α. If there exist a nonzero constant η and nonnegative numbers t_i, $i = 0, 1, \ldots, m$, such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i}$$

holds, then the R-order of convergence of the iterative method ψ satisfies the inequality

$$O_R(\psi, \alpha) \geq s^*,$$

where s^* is the unique positive root of the equation

$$s^{m+1} - \sum_{i=0}^{m} t_i s^{m-i} = 0.$$

In order to analyze the dynamical behavior of a fixed-point iterative method with memory for nonlinear equations on a polynomial $p(z)$, it is necessary to recall some basic dynamical concepts.

Let us denote by $G(z)$ the vectorial fixed-point function associated to an iterative method with memory on the scalar polynomial $p(z)$. Let us note that the next concepts and results are also valid when the iterative method is applied on a general function $f(z)$.

Definition 1. *Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector function. The orbit of a point $x^* \in \mathbb{R}^2$ is defined as the set of successive images of x^* by the vector function, $\{x^*, G(x^*), \ldots, G^m(x^*), \ldots\}$.*

The dynamical behavior of the orbit of a point of $\mathbb{R}^2$ can be classified depending on its asymptotic behavior. In this way, we will consider that a point $(z, x) \in \mathbb{R}^2$ is a fixed point of G if $G(z, x) = (z, x)$.

Moreover, as the concept of critical point corresponds to any that makes singular the Jacobian matrix associated to fixed point operator, we will state that a point $x_c \in \mathbb{R}^2$ is a critical point of G if $det(G'(x_c)) = 0$. Indeed, if a critical point is not (r_i, r_i), $i = 1, 2$ where r_i, $i = 1, 2$ are the roots of $p(z)$, it will be called free critical point. On the other hand, if a fixed point (z, x) is different from (r_i, r_i), $i = 1, 2$ where r_i, $i = 1, 2$ are the roots of $p(z)$, it is called strange fixed point and its character can be analyzed in the same manner.

We recall a known result in Discrete Dynamics that gives the stability of fixed points for multivariable nonlinear operators.

Theorem 2 ([25], page 558). *Let G from $\mathbb{R}^n$ to $\mathbb{R}^n$ be $\mathcal{C}^2$. Assume x^* is a k-periodic point. Let $\lambda_1, \lambda_1, \ldots, \lambda_n$ be the eigenvalues of $G'(x^*)$.*

a) If all the eigenvalues λ_j have $|\lambda_j| < 1$, then x^* is attracting.

b) If one eigenvalue λ_{j_0} has $|\lambda_{j_0}| > 1$, then x^* is unstable, that is, repelling or saddle.

c) If all the eigenvalues λ_j have $|\lambda_j| > 1$, then x^* is repelling.

In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue λ_i such that $|\lambda_i| < 1$ and an eigenvalue λ_j such that $|\lambda_j| > 1$, the hyperbolic point is called saddle point.

Then, if x^* is an attracting fixed point of the rational function G, its basin of attraction $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \{x_0 \in \mathbb{R}^n : G^m(x_0) \to x^*, m \to \infty\}.$$

The rest of the paper is organized as follows: Section 2 is devoted to the construction of a low-order variant with memory of King's family of iterative methods. The real multidimensional discrete dynamics on this class of schemes is made in Section 3 and some dynamical planes, covering the stable and unstable behavior showed in the previous section, are presented in Section 4. Finally, some conclusions and future works are stated.

2. Design of a modified King's family with memory

It is well known that King's family [14] of iterative methods has fourth-order of convergence, being its iterative expression

$$y_k = x_k - \frac{f(x_)}{f'(x_k)}, \tag{1}$$

$$x_{k+1} = y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f'(x_k)}, \quad k \geq 0, \tag{2}$$

where β is a real parameter. For $\beta = 0$, the well-known Ostrowski's method is obtained.

Let us now to modify this class of methods introducing some accelerating parameters, one per step. Usually, two kind of accelerating parameters are used: a damping parameter in the divided difference used for eliminating the derivatives involved in the iterative expression, or factors of the nonlinear function evaluated at one previous point, that are added to the derivatives of each step, as is the case:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)+af(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f'(x_k)+bf(x_k)}, \quad k \geq 0, \end{aligned} \tag{3}$$

Then, it is necessary to analyze if this modified King's class hold the order of convergence or if, on the contrary, some conditions are needed on parameters a and b to hold it. As this family is a particular case of that analyzed in [20], the following result can be stated.

Theorem 3. *Let α be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open interval D. If x_0 and x_1 are sufficiently close to α, then the order of convergence of class (3) is at least 4 if $b = 2a$, being its error equation*

$$e_{k+1} = (a + c_2)\left(2(-1+\beta)a^2 + (-1+4\beta)ac_2 + (1+2\beta)c_2^2 - c_3\right)e_k^4 + O\left(e_k\right)^5,$$

where $e_k = x_k - \alpha$ and $c_k = \dfrac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k \geq 2$.

This expression of the error equation is a key fact for transforming the iterative family in other one with memory: if we consider $a = -c_2 = -\dfrac{1}{2}\dfrac{f''(\alpha)}{f'(\alpha)}$, the order of the family would be at least five. As it has no sense to use the root of function f, we can estimate it by using the two previous iterations, $a_k = -\dfrac{1}{2}\dfrac{f'[x_k, x_{k-1}]}{f[x_k, x_{k-1}]}$, where $h[\cdot,\cdot]$ is the usual first order divided difference. The final iterative expression of the family, that will be called MKM is, given x_0, x_1 initial estimations,

$$\begin{aligned} a_k &= -\frac{1}{2}\frac{f'[x_k, x_{k-1}]}{f[x_k, x_{k-1}]}, \\ y_k &= x_k - \frac{f(x_k)}{f'(x_k)+a_kf(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f'(x_k)+2a_kf(x_k)}, \quad k = 1, 2, \ldots \end{aligned} \tag{4}$$

being β a real parameter. The local convergence of this class is analyzed in the following result.

Theorem 4. *Let α be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open interval D. Let x_0 and x_1 initial guesses sufficiently close to α. Then, for any value of parameter β, the order of convergence of the family of iterative methods with memory (4) is at least $2 + \sqrt{5}$, being its error equation*

$$e_{k+1} = \left(-c_2^2c_3 + \frac{3}{2}c_3^2\right)e_{k-1}e_k^4 + O_6(e_{k-1}e_k),$$

where $e_{k-1} = x_{k-1} - \alpha$, $e_k = x_k - \alpha$, $c_k = \dfrac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k \geq 2$ and $O_6(e_{k-1}e_k)$ indicates that the sum of the exponents of e_{k-1} and e_k in the rejected terms of the development is at least 6.

Proof: By using Taylor series expansions, $f(x_k)$ and $f(x_{k-1})$ can be expressed as

$$\begin{aligned} f(x_{k-1}) &= f'(\alpha)\left[e_{k-1}+c_2e_{k-1}^2+c_3e_{k-1}^3+c_4e_{k-1}^4+c_5e_{k-1}^5\right]+O(e_{k-1}^6), \\ f(x_k) &= f'(\alpha)\left[e_k+c_2e_k^2+c_3e_k^3+c_4e_k^4+c_5e_k^5\right]+O(e_k^6). \end{aligned}$$

Then, the Taylor development for $f'(x_k)$ and $f'(x_{k-1}$ are

$$\begin{aligned} f'(x_{k-1}) &= f'(\alpha)\left[1+2c_2e_{k-1}+3c_3e_{k-1}^2+4c_4e_{k-1}^3+5c_5e_{k-1}^4\right]+O(e_{k-1}^5), \\ f'(x_k) &= f'(\alpha)\left[1+2c_2e_k+3c_3e_k^2+4c_4e_k^3+5c_5e_k^4\right]+O(e_k^5). \end{aligned}$$

So, algebraic manipulations give

$$\begin{aligned} a_k &= -\frac{1}{2}\frac{f'(x_{k-1})-f'(x_k)}{f(x_{k-1})-f(x_k)} \\ &= \left(-c_2+\left(c_2^2-\frac{3}{2}c_3\right)e_{k-1}+\left(-c_2^3+\frac{5c_2c_3}{2}-2c_4\right)e_{k-1}^2+\left(c_2^4-\frac{7}{2}c_2^2c_3+\frac{3c_3^2}{2}+3c_2c_4-\frac{5c_5}{2}\right)e_{k-1}^3\right) \\ &\quad+\left(\left(c_2^2-\frac{3c_3}{2}\right)-2\left(c_2^3-2c_2c_3+c_4\right)e_{k-1}+\left(3c_2^4-\frac{17}{2}c_2^2c_3+3c_3^2+5c_2c_4-\frac{5c_5}{2}\right)e_{k-1}^2\right)e_k \\ &\quad+\left(\left(-c_2^3+\frac{5c_2c_3}{2}-2c_4\right)+\left(3c_2^4-\frac{17}{2}c_2^2c_3+3c_3^2+5c_2c_4-\frac{5c_5}{2}\right)e_{k-1}\right)e_k^2 \\ &\quad+\left(\left(c_2^4-\frac{7}{2}c_2^2c_3+\frac{3c_3^2}{2}+3c_2c_4-\frac{5c_5}{2}\right)\right)e_k^3+O_4(e_{k-1}e_k) \end{aligned}$$

and

$$\begin{aligned} y_k-\alpha &= \left(\left(c_2^2-\frac{3c_3}{2}\right)e_{k-1}+\left(-c_2^3+\frac{5c_2c_3}{2}-2c_4\right)e_{k-1}^2+\left(c_2^4-\frac{7}{2}c_2^2c_3+\frac{3c_3^2}{2}+3c_2c_4-\frac{5c_5}{2}\right)e_{k-1}^3\right)e_k^2 \\ &\quad+\left(\frac{c_3}{2}-2\left(c_2^3-2c_2c_3+c_4\right)e_{k-1}+\frac{1}{4}\left(8c_2^4-22c_2^2c_3+3c_3^2+20c_2c_4-10c_5\right)e_{k-1}^2\right)e_k^3 \\ &\quad+\left(\left(-\frac{1}{2}c_2c_3+c_4\right)+\frac{1}{2}\left(6c_2^4-19c_2^2c_3+9c_3^2+10c_2c_4-5c_5\right)e_{k-1}\right)e_k^4 \\ &\quad+\left(\frac{1}{4}\left(2c_2^2c_3-3c_3^2-4c_2c_4+6c_5\right)\right)e_k^5+O_6(e_{k-1}e_k). \end{aligned}$$

So, by using the previous developments, it can be stated that

$$\begin{aligned} f(y_k) &= f'(\alpha)\left[\left(\left(c_2^2-\frac{3c_3}{2}\right)e_{k-1}+\left(-c_2^3+\frac{5c_2c_3}{2}-2c_4\right)e_{k-1}^2+\frac{1}{2}\left(2c_2^4-7c_2^2c_3+3c_3^2+6c_2c_4-5c_5\right)e_{k-1}^3\right)e_k^2\right. \\ &\quad+\left(\frac{c_3}{2}-2\left(\left(c_2^3-2c_2c_3+c_4\right)\right)e_{k-1}+\frac{1}{4}\left(8c_2^4-22c_2^2c_3+3c_3^2+20c_2c_4-10c_5\right)e_{k-1}^2\right)e_k^3 \\ &\quad+\left(\left(-\frac{1}{2}c_2c_3+c_4\right)+\frac{1}{2}\left(6c_2^4-19c_2^2c_3+9c_3^2+10c_2c_4-5c_5\right)e_{k-1}\right)e_k^4 \\ &\quad\left.+\left(\frac{1}{4}\left(2c_2^2c_3-3c_3^2-4c_2c_4+6c_5\right)\right)e_k^5\right]+O_6(e_{k-1}e_k). \end{aligned}$$

By replacing all the obtained developments in (4), we have

$$e_{k+1} = \left(-c_2^2c_3+\frac{3}{2}c_3^2\right)e_{k-1}e_k^4-\frac{c_3^2}{2}e_k^5+O_6(e_{k-1}e_k).$$

As the lower term of the error equation is $\left(-c_2^2c_3+\frac{3}{2}c_3^2\right)e_{k-1}e_k^4$, by using Theorem 1 the unique positive root of polynomial p^2-4p-1 gives us the R-order of the method, being in this case $p=2+\sqrt{5}$. □

3. Multidimensional dynamical analysis

As our aim is to analyze the dynamical behavior of MKM on real quadratic polynomials, we will study the fixed point operator associated to the family on $p_1(t)=t^2-1$, $p_2(t)=t^2+1$, $p_3(t)=t^2$, that will be denoted by $K_{1,\beta}(z,x)$, $K_{2,\beta}(z,x)$, $K_{3,\beta}(z,x)$, respectively. Each operator is a function of two variables: the last iteration, x_k (denoted by x), the previous one x_{k-1} denoted by z and one parameter, β.

3.1. Analysis of operator $K_{1,\beta}(z,x)$

The fixed point operator resulting from applying this class on $p_1(t)=t^2-1$ is:

$$K_{1,\beta}(z,x)=\left(x,\frac{z+x(2+xz)}{1+x^2+2xz}-\frac{\left(-1+x^2\right)^2(x+z)\left(-1+z^2\right)\left(\left(1+x^2+2xz\right)^2+\beta\left(-1+x^2\right)\left(-1+z^2\right)\right)}{2(1+xz)\left(1+x^2+2xz\right)^2\left(-1+4x^2+x^4+4x\left(1+x^2\right)z+2\left(1+x^2\right)z^2+\beta\left(-1+x^2\right)\left(-1+z^2\right)\right)}\right).$$

To study the stability of the family we will analyze the asymptotic behavior of the fixed points of $K_{1,\beta}(z,x)$.

Theorem 5. *The fixed points of the operator associated to MKM on quadratic polynomial $p_1(t)$ are:*

a) Points $(1,1)$ and $(-1,-1)$ associated to the roots, being both attracting.

b) The origin $(z,x)=(0,0)$, which is an attracting fixed point for $\frac{1}{3}<\beta\leq\frac{1}{49}\left(31-8\sqrt{2}\right)$; it is repulsive for $\frac{1}{49}\left(31+8\sqrt{2}\right)\leq\beta<1$ and it is a saddle point for the rest of real values of β.

c) Points $(r_i(\beta),r_i(\beta))$ where $r_i(\beta)$ $i\in\{1,2,\ldots,8\}$ are the real roots of polynomial $r(t)=-1+3\beta+16t^2+(86-2\beta)t^4+(104-8\beta)t^6+(51+7\beta)t^8$, whose number varies depending on the range of parameter β:

- *if $\beta<-\frac{51}{7}$, there are four real roots of $r(t)$ such that two of them ($r_1(\beta)$ and $r_4(\beta)$) are repulsive and the other ones ($r_2(\beta)$ and $r_3(\beta)$) are saddle points;*
- *two if $-\frac{51}{7}\leq\beta<\frac{1}{3}$, that are saddle points in $[-\frac{51}{7},0.0645[$ and attractors in $[0.0645,\frac{1}{3}[$;*
- *none if $\beta\geq\frac{1}{3}$.*

Proof: The fixed points are obtained by solving the equation

$$K_{1,\beta}(z,x)=(z,x),$$

that is, $z=x$ and

$$x\left(-1+x^2\right)r(x)=0.$$

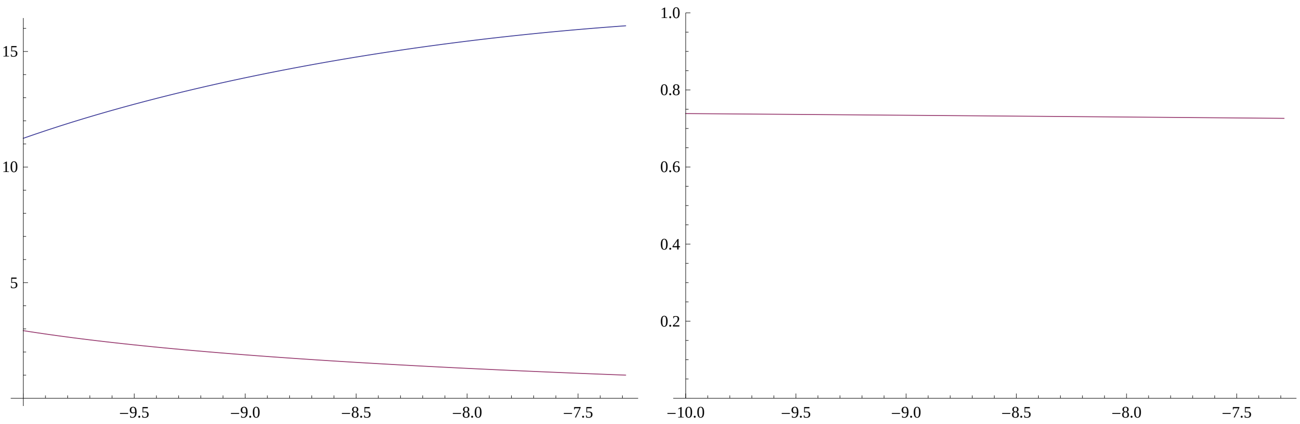
It is clear that the points $(1,1)$ and $(-1,-1)$ satisfy the previous equation and both eigenvalues of the associate Jacobian matrix on them are null, so they are attracting. Obviously, $(0,0)$ is also a fixed point and the eigenvalues of the Jacobian matrix on it are $\lambda_1=\frac{-3+5\beta-\sqrt{17-62\beta+49\beta^2}}{4(-1+\beta)}$ and $\lambda_2=\frac{-3+5\beta+\sqrt{17-62\beta+49\beta^2}}{4(-1+\beta)}$. It can be checked that, being β real, $|\lambda_1|<1$ if and only if $1/3<\beta\leq\frac{1}{49}(31-8\sqrt{2})$ or $\beta>1$, meanwhile $|\lambda_2|<1$ if and only if $1/3<\beta\leq\frac{1}{49}(31-8\sqrt{2})$. So, when parameter β is taken in this interval, the origin is an attracting strange fixed point. By using a similar reasoning, the interval in which $(0,0)$ is repulsive can be found.

So, the rest of strange fixed points will be the roots of the eighth-degree polynomial $r(x)$, denoted by r_i, $i=1,2,\ldots,8$. We will analyze now their stability by studying the absolute value of the eigenvalues of the Jacobian matrix associated to the fixed point operator of the method on these fixed points. It can be checked that, when $\beta<-\frac{51}{7}$, only four roots of $r(x)$, r_1 to r_4, are real and the absolute value of the eigenvalues of the Jacobian matrix associated to the fixed point operator on them coincide and they can be seen in Figure 1. It can be observed that for $r_1(\beta)$ and $r_4(\beta)$ both eigenvalues are always greater than one in absolute value. Moreover, for $r_2(\beta)$ and $r_2(\beta)$, one of the eigenvalues is always lower than one (in absolute value) and the other one remains higher than one; so, by applying Theorem 1, $r_1(\beta)$ and $r_4(\beta)$ are saddle points meanwhile $r_2(\beta)$ and $r_3(\beta)$ are saddle points.

On the other hand, when $-\frac{51}{7}\leq\beta<\frac{1}{3}$, only two of the roots of $r(x)$ are real, $r_1(\beta)$ and $r_2(\beta)$. Both of them have the same stability, as it can be observed in Figure 3. Numerically, it can be checked that, for $\beta\in[-\frac{51}{7},0.0645[$ two roots are saddle points, meanwhile for $\beta\in[0.0645,\frac{1}{3}[$, both are attracting points. Let us remark that in the interval where both points are attracting, $\lambda_1(\beta)=\lambda_2(\beta)$ until $\beta\approx 0.29$, but they are different in the rest of the interval.

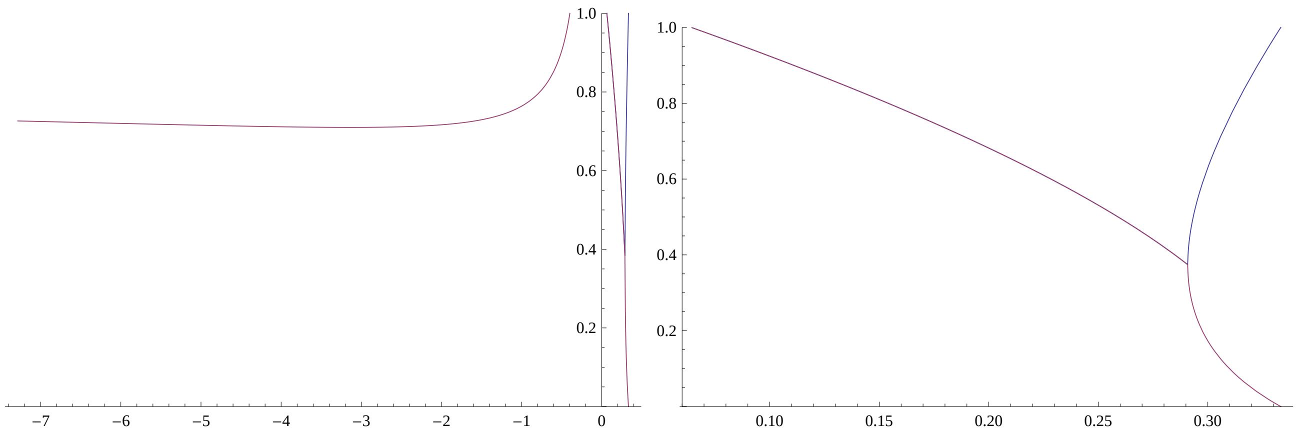
Finally, for $\beta\geq\frac{1}{3}$, all the roots of $r(x)$ are complex numbers. □

By summarizing, any element of the family MKM (corresponding to a value of β) selected in $]-\infty,0.0645[\cup]\frac{1}{49}(31-8\sqrt{2}),+\infty[$ will be free of attracting strange fixed points, although it does not means that it is free of any other anomalies, as it will be seen in Section 4.



(a) Absolute value of the eigenvalues of $K'_{1,\beta}(r_i, r_i)$,$i = 1, 4$ (b) Absolute value of the eigenvalues of $K'_{1,\beta}(r_i, r_i)$,$i = 2, 3$

Figure 1: Stability of some strange fixed points of $K_{1,\beta}(z, x)$ for $\beta < -\frac{51}{7}$



(a) Eigenvalues of $K'_{1,\beta}(r_i, r_i)$, $i = 1, 4$, $-\frac{51}{7} \leq \beta < \frac{1}{3}$ (b) Eigenvalues of $K'_{1,\beta}(r_i, r_i)$, $i = 2, 3$, $0.0645 \leq \beta < \frac{1}{3}$

Figure 2: Stability of some strange fixed points of $K_{1,\beta}(z, x)$ for $-\frac{51}{7} \leq \beta < \frac{1}{3}$

3.2. Study of operator $K_{2,\beta}(z, x)$

By applying MKM family on $p_2(t) = t^2 + 1$ a fixed point operator is obtained, whose expression is

$$K_{2,\beta}(z,x) = \left(x, \frac{-2x - z + x^2 z}{-1 + x^2 + 2xz} - \frac{(1+x^2)^2 (x+z)(1+z^2)\left((-1+x^2+2xz)^2 + b(1+x^2)(1+z^2)\right)}{2(-1+xz)(-1+x^2+2xz)^2(-1+x^4-4xz+4x^3z-2z^2+2x^2(-2+z^2)+b(1+x^2)(1+z^2))} \right).$$

We will calculate the real fixed points of this operator and analyze their asymptotic behavior to analyze the stability of the family.

Theorem 6. *The real fixed points of the operator associated to MKM on quadratic polynomial $p_2(t)$ are:*

a) The origin $(z, x) = (0, 0)$, which is an attracting fixed point for $\frac{1}{3} < \beta \leq \frac{31-8\sqrt{2}}{49}$, it is repulsive if $\frac{31-8\sqrt{2}}{49} \leq \beta < 1$ and saddle in other case;

b) Points $(m_i(\beta), m_i(\beta))$, where $m_i(\beta)$ $i \in \{1, 2, \ldots, 8\}$ are the real roots of polynomial $m(t) = -1 + 3\beta - 16t^2 + (86 - 2\beta)t^4 + (-104 + 8\beta)t^6 + (51 + 7\beta)t^8$,

$$\begin{array}{llll} m_1(\beta) = \sqrt{s_1(\beta)}, & m_2(\beta) = -\sqrt{s_1(\beta)}, & m_3(\beta) = \sqrt{s_2(\beta)}, & m_4(\beta) = -\sqrt{s_2(\beta)}, \\ m_5(\beta) = \sqrt{s_3(\beta)}, & m_6(\beta) = -\sqrt{s_3(\beta)}, & m_7(\beta) = \sqrt{s_4(\beta)}, & m_8(\beta) = -\sqrt{s_4(\beta)}, \end{array}$$

where

$$
\begin{aligned}
s_1(\beta) &= \frac{26-2\beta}{51+7\beta} - \frac{1}{\sqrt{3}} r_2(\beta) - \sqrt{\frac{2}{3}}\sqrt{A(\beta)},\\
s_2(\beta) &= \frac{26-2\beta}{51+7\beta} - \frac{1}{\sqrt{3}} r_2(\beta) + \sqrt{\frac{2}{3}}\sqrt{A(\beta)},\\
s_3(\beta) &= \frac{26-2\beta}{51+7\beta} + \frac{1}{\sqrt{3}} r_2(\beta) - \sqrt{\frac{2}{3}}\sqrt{A(\beta)},\\
s_4(\beta) &= \frac{26-2\beta}{51+7\beta} + \frac{1}{\sqrt{3}} r_2(\beta) + \sqrt{\frac{2}{3}}\sqrt{A(\beta)},
\end{aligned}
$$

being

$$
\begin{aligned}
A(\beta) = & \frac{12(-13+\beta)^2}{(51+7\beta)^2} + \frac{-43+\beta}{51+7\beta} - \frac{2(51+7\beta)\left(7+7\beta+\beta^2\right)}{(51+7\beta)^2 r_1(\beta)} - \frac{2r_1(\beta)}{51+7\beta}\\
& + \frac{3\sqrt{3}\left(5731+3685\beta-751\beta^2+15\beta^3\right)}{(51+7\beta)^3 r_2(\beta)},
\end{aligned}
$$

$$
r_1(\beta) = \left(44+12\beta-30\beta^2+\beta^3+3\sqrt{3}\sqrt{59+\beta-136\beta^2-47\beta^3+28\beta^4-3\beta^5}\right)^{1/3}
$$

and

$$
r_2(\beta) = \sqrt{\frac{12(-13+\beta)^2}{(51+7\beta)^2} + \frac{-43+\beta}{51+7\beta} + \frac{4(51+7\beta)\left(7+7\beta+\beta^2\right)}{(51+7\beta)^2 r_1} + \frac{4r_1(\beta)}{51+7\beta}}.
$$

The number of real roots varies depending on the range of parameter β:

- *If $\beta < -\frac{51}{7}$, there are no real roots of $r(t)$;*
- *$m_2(\beta)$ and $m_4(\beta)$ are real roots if $-\frac{51}{7} \leq \beta < -1.02409$, and they are saddle points;*
- *If $\beta = -1.02409$, $m_1(\beta)$, $m_2(\beta)$, $m_4(\beta)$ and $m_6(\beta)$ are real, being $m_6(\beta)$ repulsive and the rest of fixed points, saddle.*
- *When $-1.02409 \leq \beta < -1$, six strange fixed points are real, $m_1(\beta)$, $m_4(\beta)$, $m_6(\beta)$ and $m_8(\beta)$ being saddle points; $m_2(\beta)$ and $m_3(\beta)$ are repulsive fixed points.*
- *If $\beta = -1$, ± 0.824188 are the only real roots of $m(t)$, and they are saddle points.*
- *For $-1 \leq \beta < 0$ the only real fixed point (among the roots of $m(t)$) is $m_4(\beta)$, being a saddle point.*
- *If $\beta = 0$, none of the roots of $m(t)$ is real.*
- *If $0 \leq \beta < \frac{1}{3}$, $m_2(\beta)$ and $m_4(\beta)$ are the unique real zeros of $m(t)$, and they are saddle.*
- *When $\beta = \frac{1}{3}$, ± 0.503737 are the only real roots of $m(t)$, and they are saddle points.*
- *For $\frac{1}{3} < \beta < 0.620014$, the following behavior is observed on the only real zeros of $m(t)$:*
  - *$m_4(\beta)$ and $m_6(\beta)$ are saddle points for $\frac{1}{3} < \beta < 0.52$, repulsive in $\frac{1}{3} \leq \beta < 0.619818$ and attractive in the rest of the interval;*
  - *$m_2(\beta)$ and $m_8(\beta)$ are saddle points.*
- *If $\beta = 0.620014$, then $m_4(\beta)$ and $m_8(\beta)$ are saddle points and are the only real roots of $m(t)$.*
- *If $\beta > 0.620014$, there not exist real roots of $m(t)$.*

To proof this result, it is needed to solve the equation

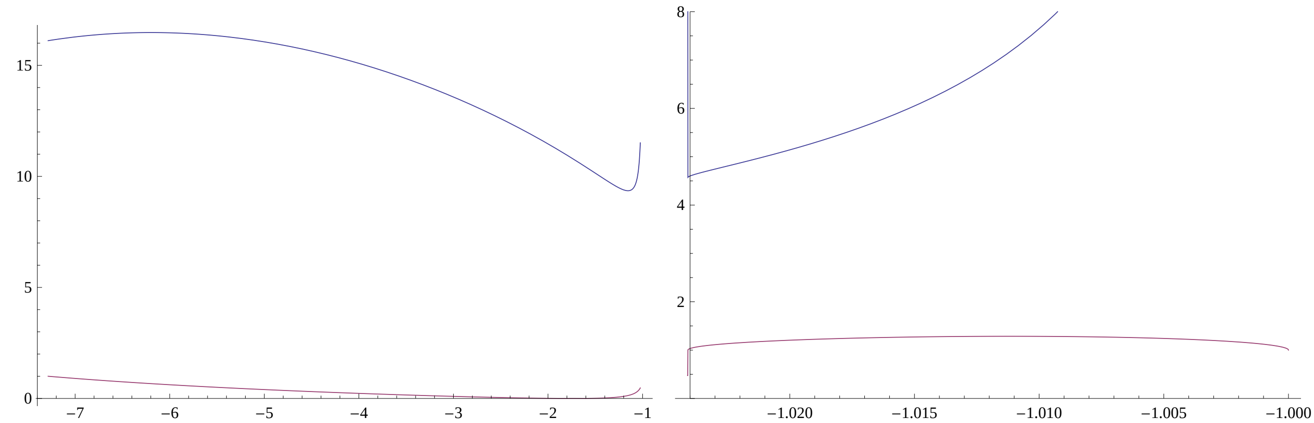
$$K_{2,\beta}\left(z,x\right)=\left(z,x\right),$$

that is, $z=x$ and

$$x\left(1+x^2\right)m(x)=0.$$

In this case, points $(1,1)$ and $(-1,-1)$ satisfy the previous equation and both eigenvalues of the associate Jacobian matrix on them are null, proving that they are attracting. Again, $(0,0)$ is also an strange fixed point (if $\beta\neq 1$); its associated eigenvalues are $\lambda_1=\frac{-3+5\beta-\sqrt{17-62\beta+49\beta^2}}{4(-1+\beta)}$ and $\lambda_2=\frac{-3+5\beta+\sqrt{17-62\beta+49\beta^2}}{4(-1+\beta)}$. It can be checked that, being β real, $|\lambda_1|<1$ if and only if $1/3<\beta\leq\frac{1}{49}(31-8\sqrt{2})$ or $\beta>1$, meanwhile $|\lambda_2|<1$ if and only if $1/3<\beta\leq\frac{1}{49}(31-8\sqrt{2})$. So, when β is taken in this interval, $(0,0)$ is attracting.

Then, other strange fixed points will be the real roots of the eighth-degree polynomial $m(x)$, denoted by $m_i(\beta)$, $i=1,2,\ldots,8$. Their stability is analyzed by calculating the absolute value of the eigenvalues of the Jacobian matrix associated to the fixed point operator of the method on them. Firstly, we calculate the ranges in which the number of real roots changes; after this, the eigenvalues of the Jacobian matrix at this points are calculated and analyzed in order to classify the strange fixed points. In this terms, the thesis of this Theorem are stated. In Figure 3, some of this studies are shown.



(a) $(m_i(\beta),m_i(\beta))$, $i=2,4$ on $-\dfrac{51}{7}<\beta<-1.02409$ (b) $(m_i(\beta),m_i(\beta))$, $i=2,3$ on $-1.02409<\beta<-1$

Figure 3: Absolute value of the eigenvalues of $K_{2,\beta}\left(z,x\right)$ for some strange fixed points

Let us remark that, from information stated in Theorem 6, there exist attracting strange fixed points only in the interval $\frac{1}{3}<\beta\leq\frac{1}{49}(31-8\sqrt{2})$ (where $(0,0)$ is attractive) and $0.619818<\beta<0.620014$, where $m_4(\beta)$ and $m_6(\beta)$ are attracting.

3.3. Analysis of operator $K_{3,\beta}(z,x)$

Finally, let us apply MKM family to $p_3(t)=t^2$ a fixed point operator is obtained depending on the last iteration x and the previous one z. Then,

$$K_{3,\beta}(z,x)=\left(x,\frac{xz\left(x^3+7x^2z+(12+\beta)xz^2+(4+3\beta)z^3\right)}{2(x+2z)^2\left(x^2+4xz+(2+\beta)z^2\right)}\right).$$

In a similar way as in previous sections, we calculate the fixed points of this operator and analyze their stability in the following result.

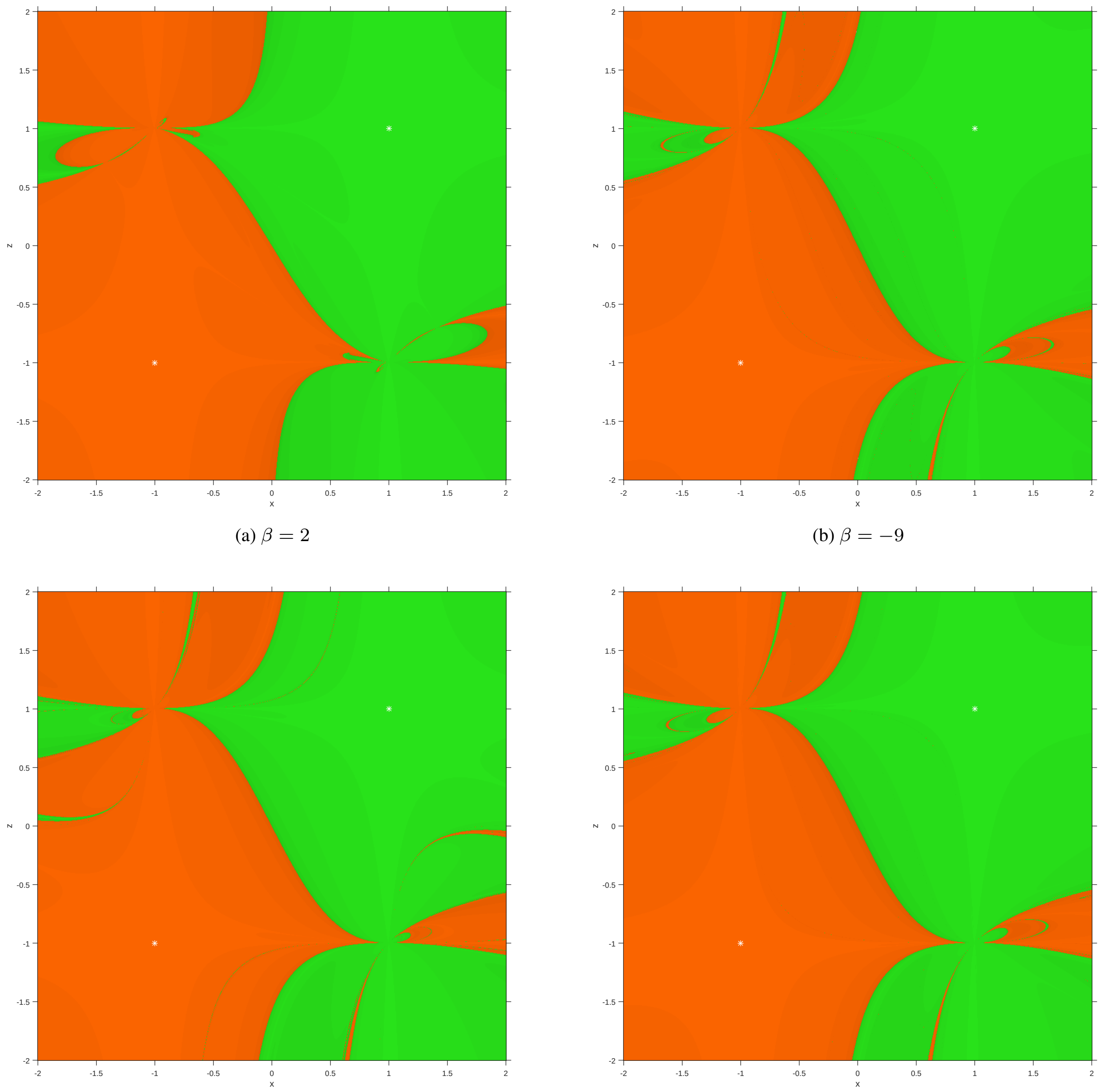
Theorem 7. *The only fixed point of the operator associated to MKM on quadratic polynomial $p_3(t)$ is $(z,x)=(0,0)$, that is attractive.*

Proof: By solving the equation $K_{3,\beta}(z,x)=x$, the only fixed point is obtained to be $(0,0)$. Due to the multiplicity of the root, the Jacobian matrix at this point is not defined. □

4. Dynamical planes

In this section we will represent the real bidimensional dynamics of the fixed point operator associated to each polynomial with simple roots, $p_1(t)$ and $p_2(t)$. To get this, we will use some routines in Matlab that are a slight modification of those presented in [27].

In general, we have used a mesh of 800×800 points that represent the pair (x, z) (that is, (x_k, x_{k-1})) used as initial estimation for the particular member of family MKM under study. This point is plotted in different colors depending on the fixed point it tends to, after 80 iterations. If this maximum number of iterates is reached without converging to any fixed point or periodic orbit, with a tolerance of 10^{-3}, it is drawn in black color.



(a) $\beta = 2$

(b) $\beta = -9$

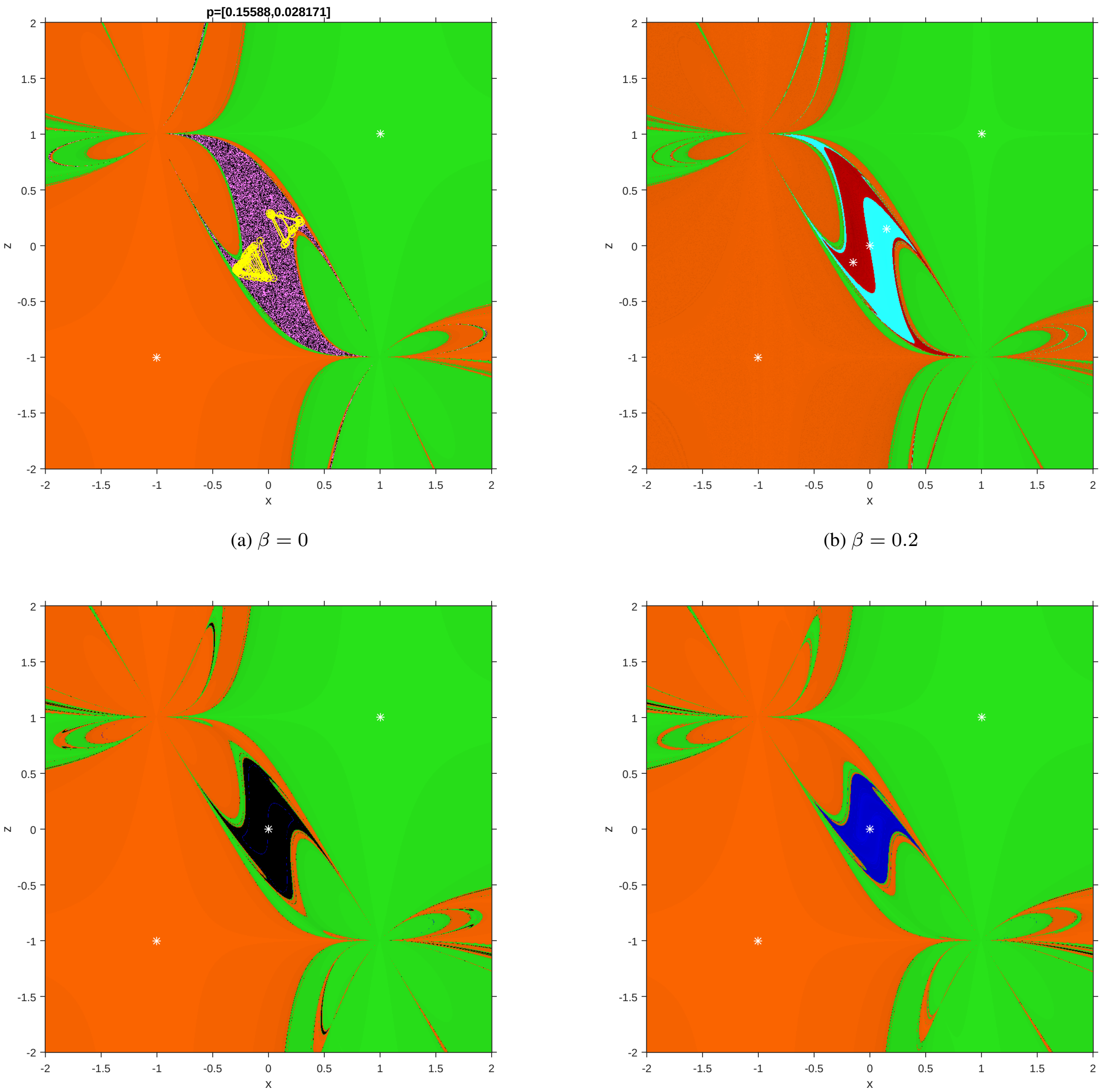
(c) $\beta = 9$

(d) $\beta = -15$

Figure 4: Some stable dynamical planes of $K_{1,\beta}\,(z, x)$

By choosing values of the parameter in the different regions appeared in the previous section, we will show as stable as unstable behavior with different kind of pathologies of the family. In case of $p_1(t)$, it has been proved that there are no attracting strange fixed points for $\beta \in]-\infty, 0.0645[\cup] \frac{1}{49}(31-8\sqrt{2}), +\infty[$. So, taking values of the parameter is this domain, the resulting dynamical planes will show the basins of attraction of fixed points $(-1,-1)$ and $(1,1)$, although other attracting elements, such as periodic orbits, could appear. This is not the case of the dynamical planes corresponding to the values of parameter $\beta = 2$ and $\beta = -9$ that are near the bounds of this region and also $\beta = 9$ and $\beta = -15$, showed in Figure 4.

However, undesired behavior can also appear for selected values of parameter β in the intervals where any of the strange fixed points described in Theorem 5 is attracting. In Figure 5 some of them are shown, starting with the variant of Ostrowski's method with memory that does not have attracting strange fixed points shows two different periodic orbits of period 4 (see Figure 5a); in Figure 5b two attracting roots of $r(t)$ an their respective basins of attraction appear for $\beta = 0.2$; if $\beta = \dfrac{1}{3}$, both attractive point collapse at zero, holding the attractive behavior (see Figure 5c) although this convergence is slower; finally, we show for $\beta = \dfrac{1}{49}(31-8\sqrt{2})$ (Figure 5d) the dynamical plane that corresponds to the biggest value of the parameter that makes the origin attractive.



(a) $\beta = 0$

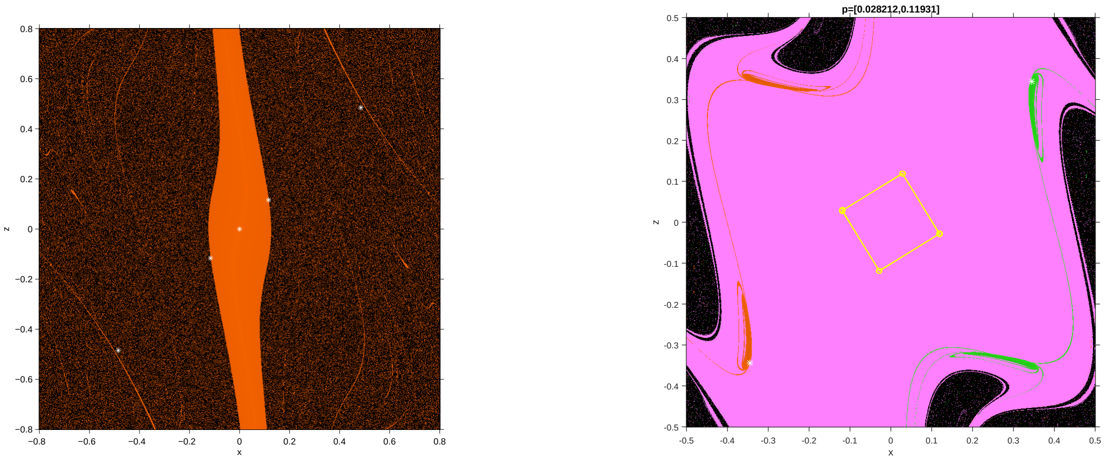
(b) $\beta = 0.2$

(c) $\beta = \dfrac{1}{3}$

(d) $\beta = \dfrac{1}{49}(31-8\sqrt{2})$

Figure 5: Some unstable dynamical planes of $K_{1,\beta}(z,x)$

Respect to the stability of the family when the polynomial is $p_2(t)$, without real roots, it has been stated in Theorem 6 that there exist attracting strange fixed points only in the small region described by $\frac{1}{3} < \beta \leq \frac{1}{49}(31 - 8\sqrt{2})$ (where $(0,0)$ is attractive) and $0.619818 < \beta < 0.620014$, where $m_4(\beta)$ and $m_6(\beta)$ are attracting. In Figure 6, two dynamical planes showing unstable behavior are presented: when $\beta = 0.4$, $(0,0)$ is the only attracting fixed point and if $\beta = 0.6199$, the basins of attraction of $m_4(\beta)$ and $m_6(\beta)$ are showed in orange and green but the greatest basin (pink color) corresponds to an attracting periodic orbit of period 4, whose trajectory is marked with yellow lines in Figure 6b.



(a) $\beta = 0.4$ (b) $\beta = 0.6199$

Figure 6: Some unstable dynamical planes of $K_{2,\beta}(z,x)$

5. Conclusions

In this paper, we have introduced some tools of the dynamical analysis of multivariate real discrete problems to analyze the stability of the fixed points of iterative methods with memory on quadratic polynomials. We have designed a variant of King's family with memory with lower order of convergence that other ones existing in the literature but with possibilities of establishing a bidimensional dynamical analysis. Our statements, based on consistent real multidimensional discrete dynamics results, allow us to select the most stable elements of the class and to find those that present convergence to other points different from the solution of our problem. Further works in this area will lead us in the future to extend this kind of analysis to higher-order methods with memory, although this will imply to work with higher dimension also in the dynamical study.

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