Explicit Bézier control net of a PDE surface

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Abstract

The PDE under study here is a general fourth-order linear elliptic Partial Differential Equation. Having prescribed the boundary control points, we provide the explicit expression of the whole control net of the associated PDE Bézier surface.

In other words, we obtain the explicit expressions of the interior control points as linear combinations of free boundary control points. The set of scalar coefficients of these combinations works like a mould for PDE surfaces. Thus, once this mould has been computed for a given degree, real-time manipulation of the resulting surfaces becomes possible by modifying the prescribed information.

Keywords: Partial Differential Equation, PDE surface, Surface Generation, Tensor product Bézier surface, Biharmonic surface, Explicit solution.

1. Introduction

Techniques that allow interactive design are always welcome for computer-aided design. Therefore, the purpose of this paper is to describe a method of surface generation for PDE surfaces in the Bézier language of CAGD. A Bézier surface is defined by

$$\overrightarrow{\mathbf{x}}(u,v) = \sum_{i,j=0}^{n} B_i^n(u)\, B_j^n(v)\, \mathbf{P}_{i,j}$$

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where $B_i^n(t)$ are the Bernstein basis polynomials $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$, and $\mathbf{P}_{i,j}$ are the control points. Consequently, a Bézier surface is determined by a net of control points. Our aim is therefore to offer the explicit expression of a PDE Bézier surface control net.

PDE surfaces are used in geometric modelling and computer graphics to create smooth surfaces conforming to a given boundary configuration. Such surfaces enable the designer to model complex shapes in an easy and predictable fashion, without the need to enter the control points one by one, and to avoid consequent irregularities in the resulting surface. Their areas of application include computer-aided design, interactive design, parametric design, computer animation, computer-aided physical analysis and design optimization. Tools that enable differential equations to be solved are also a valuable aid in both the academic world and the engineering industry; in fact, there are companies such as *Explicit Solutions* that specialize in providing researchers with explicit solutions to ordinary differential equations.

We have developed a wide programme of research about PDE surfaces from the CAGD point of view. Our first steps were [12] and [13], where it was shown that, surprisingly, a unique harmonic Bézier surface is determined by prescribing only two boundary curves (not the four boundaries) of a tensor product Bézier surface. The second step was to extend this result to the biharmonic case ([14]) and to a class of more general 4-th order PDE ([15]). In these cases it was shown that by prescribing the four boundaries a unique Bézier surface satisfying the corresponding PDE was determined. Furthermore, the tetraharmonic case was studied in [9], whereas the triangular cases were studied in [1], [2], [3], [4] and [5]. Note that in [11] and [16] some of the formulas that appear in the cited papers were cleaned of mistakes and typos and some of the results were extended.

In all these previous steps, the algorithms to compute the Bézier surface from prescribed boundary control points were based on the recursive resolution of a more or less complicated auxiliary system of linear equations whose unknowns were the coefficients of the polynomial surface in the power basis. A new step was paper [7], where we were able to solve those auxiliary systems of linear equations explicitly. We obtained explicit polynomial PDE surfaces but again in the power basis of polynomials. Therefore, translation from the power basis to the Bernstein basis was still needed. This is the goal we achieve here. Up until now a change in the prescribed control points meant having to start the computation of the PDE surface from the beginning.

The importance of our results in this work lies in the fact that control points have geometric meaning, while the power basis coefficients do not. Now we obtain the explicit expressions of the interior control points as linear combinations of prescribed points. Once the set of scalars of these linear combinations has been computed, PDE surfaces can be generated easily. This is why we say these scalars act like a pattern or a mould for PDE surfaces.

In our last work, [6], we introduced a new point of view and considered harmonicity in a more theoretical way. Instead of working with a polynomial surface of a particular degree, we took the approach involving the use of generating functions. We constructed a set of harmonic generating functions whose derivatives are the harmonic Bézier functions that have as their control points the scalars on the linear combinations we are looking for.

Unfortunately, generating functions are not easily obtained, not even for the biharmonic case. Hence, here we obtain the explicit expressions of interior control points of a quite general PDE surface in terms of boundary control points, but the scalars of the linear combinations in these expressions are not deduced from any generating function. Biharmonic surfaces are a particular case of the PDE surfaces in our study.

Now, for a given degree and only once, we compute the set of scalar coefficients, $\alpha, \beta, \gamma, \delta, \xi, \tau, \eta, \sigma$, of the linear combination, which is a set for any interior control point.

$$\begin{aligned}\mathbf{P}_{k,\ell} = &\sum_{w=1}^{n-1} \alpha_{k,\ell,w}\ \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} \beta_{k,\ell,w}\ \mathbf{P}_{n,w} + \sum_{w=1}^{n-1} \gamma_{k,\ell,w}\ \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} \delta_{k,\ell,w}\ \mathbf{P}_{w,n} \\ &+\xi_{k,\ell}\ \mathbf{P}_{0,0} + \tau_{k,\ell}\ \mathbf{P}_{0,n} + \eta_{k,\ell}\ \mathbf{P}_{n,0} + \sigma_{k,\ell}\ \mathbf{P}_{n,n}.\end{aligned} \tag{1}$$

This set of scalars works like a pattern and allows real-time manipulation of the resulting surfaces. Thus, to compute the control point $\mathbf{P}_{k,\ell}$, we only have to multiply the matrix of scalar coefficients by the matrix of boundary control points entry by entry.

$$
\begin{array}{ccccc}
\xi_{k,\ell} & \alpha_{k,\ell,1} & \cdots & \alpha_{k,\ell,n-1} & \tau_{k,\ell} \\
\gamma_{k,\ell,1} & & & & \delta_{k,\ell,1} \\
\gamma_{k,\ell,2} & & & & \delta_{k,\ell,2} \\
\vdots & & & & \vdots \\
\gamma_{k,\ell,n-2} & & & & \delta_{k,\ell,n-2} \\
\gamma_{k,\ell,n-1} & & & & \delta_{k,\ell,n-1} \\
\eta_{k,\ell} & \beta_{k,\ell,1} & \cdots & \beta_{k,\ell,n-1} & \sigma_{k,\ell}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \cdots & \mathbf{P}_{0,n-1} & \mathbf{P}_{0,n} \\
\mathbf{P}_{1,0} & & & & \mathbf{P}_{1,n} \\
\mathbf{P}_{2,0} & & & & \mathbf{P}_{2,n} \\
\vdots & & & & \vdots \\
\mathbf{P}_{n-2,0} & & & & \mathbf{P}_{n-2,n} \\
\mathbf{P}_{n-1,0} & & & & \mathbf{P}_{n-1,n} \\
\mathbf{P}_{n,0} & \mathbf{P}_{n,1} & \cdots & \mathbf{P}_{n,n-1} & \mathbf{P}_{n,n}
\end{array}
$$

The goal we achieve here is to work with the Bézier basis. We compute the Bézier surface control points, which have geometric meaning, while the power basis coefficients do not. In order to show how interactive design could be per-

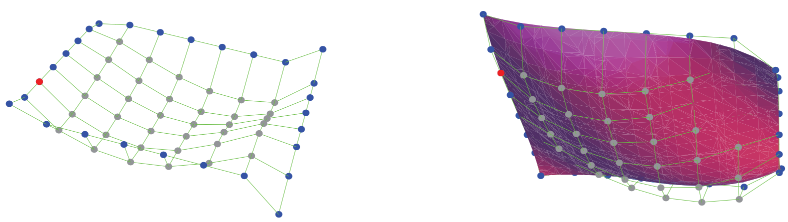

Figure 1: Working with the Bézier control net allows direct control over the shape of the surface, which is the advantage of the Bézier basis over the power basis of polynomials.

formed, we have implemented our method in `Mathematica` and it is available online at Wolfram Demonstrations Project, `http://demonstrations.wolfram.com`.

In the second section of this paper we will recall the fourth-order linear elliptic Partial Differential Equation we introduced and studied in [7]. This PDE is the Euler-Lagrange equation of a quadratic functional defined by a norm, and therefore we can state that, in addition, the PDE surface minimizes the associated functional. In the third section we give the explicit control net formulas.

2. A Fourth-Order Linear Elliptic PDE

A tensor product Bézier surface can be written both in the power and the Bernstein basis:

$$\overrightarrow{\mathbf{x}}(u,v) = \sum_{k,\ell=0}^{n} \frac{\mathbf{a}_{k,\ell}}{k!\ell!} u^k v^\ell = \sum_{i,j=0}^{n} B_i^n(u)\, B_j^n(v)\, \mathbf{P}_{i,j}$$

where $B_i^n(t) = \binom{n}{i} t^i\,(1-t)^{n-i}$ are the Bernstein polynomials. As we have said earlier, here our aim is to find the explicit control net $\mathbf{P}_{i,j}$ of the PDE surface that satisfies the general equation

$$\rho^2 \overrightarrow{\mathbf{x}}_{uuuu} + 2\rho\cos t\, \overrightarrow{\mathbf{x}}_{uuuv} + (1+\rho^2)\overrightarrow{\mathbf{x}}_{uuvv} + 2\rho\cos t\, \overrightarrow{\mathbf{x}}_{uvvv} + \overrightarrow{\mathbf{x}}_{vvvv} = 0 \quad (2)$$

for $0 \le t \le 2\pi$.

We introduced this equation in [7]. It is the Euler-Lagrange equation of a kind of quadratic functional defined by a norm, and so, in addition, we can state that the PDE surface minimizes the associated functional. Now we are looking for the control net.

In the cited paper we gave the polynomial form of its explicit solution, that is, the coefficients $\mathbf{a}_{k,\ell}$ of the PDE surface in terms of the power basis of polynomials. First, in Lemma 1 in [7], we gave the explicit solution after prescribing the boundary curve $\overrightarrow{\mathbf{x}}\,(u,0)$ and its first three transversal partial derivatives.

Lemma 1. *(Lemma 1, [7]) The solution to the linear system,*

$$0 = \rho^2 \mathbf{a}_{k+4,\ell} + 2\rho\cos t\ \mathbf{a}_{k+3,\ell+1} + (1+\rho^2)\,\mathbf{a}_{k+2,\ell+2} + 2\rho\cos t\ \mathbf{a}_{k+1,\ell+3} + \mathbf{a}_{k,\ell+4}, \quad (3)$$

for all $k,\ell \in \mathbb{N}$ in terms of the first four columns of coefficients is given by,

$$\mathbf{a}_{k,\ell} = \overline{A}_\ell\ \mathbf{a}_{k+\ell,0} + \overline{B}_\ell\ \mathbf{a}_{k+\ell-1,1} + \overline{C}_\ell\ \mathbf{a}_{k+\ell-2,2} + \overline{D}_\ell\ \mathbf{a}_{k+\ell-3,3}, \quad \ell > 3, \quad (4)$$

where,

$$\overline{A}_n = -\rho^2 \overline{D}_{n-1}, \quad \overline{B}_n = \sin\frac{n\pi}{2} + \overline{D}_n, \quad \overline{C}_n = -\cos\frac{n\pi}{2} - \rho^2 \overline{D}_{n-1},$$
$$\overline{D}_n = \frac{(\rho^2 - 1)\sin\frac{n\pi}{2} - 2\rho\cos t\cos\frac{n\pi}{2} + (-\rho)^{n-1}\csc t\,(\rho^2\sin(n-2)t + \sin nt)}{1 + \rho^4 + 2\rho^2\cos 2t}. \tag{5}$$

At this point, the procedure we adopted was to interchange the third and fourth columns of coefficients $\mathbf{a}_{k,\ell}$ with the first two rows, thus solving the linear system,

$$\mathbf{a}_{0,k+\ell} = \overline{A}_{k+\ell}\ \mathbf{a}_{k+\ell,0} + \overline{B}_{k+\ell}\ \mathbf{a}_{k+\ell-1,1} + \overline{C}_{k+\ell}\ \mathbf{a}_{k+\ell-2,2} + \overline{D}_{k+\ell}\ \mathbf{a}_{k+\ell-3,3},$$

$$\mathbf{a}_{1,k+\ell-1} = \overline{A}_{k+\ell-1}\ \mathbf{a}_{k+\ell,0} + \overline{B}_{k+\ell-1}\ \mathbf{a}_{k+\ell-1,1} + \overline{C}_{k+\ell-1}\ \mathbf{a}_{k+\ell-2,2} + \overline{D}_{k+\ell-1}\ \mathbf{a}_{k+\ell-3,3},$$

with $\mathbf{a}_{k+\ell-2,2}, \mathbf{a}_{k+\ell-3,3}$ as unknowns and then substituting the solution into Eq.(4).

Proposition 2. *(Proposition 1, [7]) The solution to the linear system in Eq.(3) for all $k, \ell \in \mathbb{N}$ in terms of the first two rows and the first two columns of coefficients is given by,*

$$\mathbf{a}_{k,\ell} = A_{k,\ell}\,\mathbf{a}_{k+\ell,0} + B_{k,\ell}\,\mathbf{a}_{k+\ell-1,1} + C_{k,\ell}\,\mathbf{a}_{1,k+\ell-1} + D_{k,\ell}\,\mathbf{a}_{0,k+\ell} \tag{6}$$

for all $k, \ell > 1$, where,

$$\begin{aligned}
A_{k,\ell} &= \tfrac{1}{M_{k+\ell}(C,D)}\left(\overline{A}_\ell\ M_{k+\ell}(C,D) - \overline{C}_\ell\ M_{k+\ell}(A,D) + \overline{D}_\ell\ M_{k+\ell}(A,C)\right),\\
B_{k,\ell} &= \tfrac{1}{M_{k+\ell}(C,D)}\left(\overline{B}_\ell\ M_{k+\ell}(C,D) - \overline{C}_\ell\ M_{k+\ell}(B,D) + \overline{D}_\ell\ M_{k+\ell}(B,C)\right),\\
C_{k,\ell} &= \tfrac{1}{M_{k+\ell}(C,D)}\left(-\overline{C}_\ell\ \overline{D}_{k+\ell} + \overline{D}_\ell\ \overline{C}_{k+\ell}\right),\\
D_{k,\ell} &= \tfrac{1}{M_{k+\ell}(C,D)}\left(\overline{C}_\ell\ \overline{D}_{k+\ell-1} - \overline{D}_\ell\ \overline{C}_{k+\ell-1}\right),
\end{aligned}$$

where, $\overline{A}_\ell, \overline{B}_\ell, \overline{C}_\ell, \overline{D}_\ell$ are defined in Eq.(5) and

$$M_n(X,Y) = \det\begin{pmatrix} \overline{X}_n & \overline{Y}_n \\ \overline{X}_{n-1} & \overline{Y}_{n-1} \end{pmatrix}.$$

3. Bézier Solutions to Fourth-Order Linear Elliptic PDEs

As we have already stated in the introduction, our aim is to work with the Bézier basis instead of with the power basis coefficients. PDE surfaces are advantageous in CAGD, since they enable the designer to model smooth surfaces conforming to a given boundary without the need to enter the whole control net. Our purpose here is to obtain the PDE surface control net by prescribing only the boundary control points. We want to know a set of scalar coefficients, $\alpha, \beta, \gamma, \delta, \xi, \tau, \eta, \sigma$, for any control point; see Eq.(1.)

In the following subsection we can see that a PDE Bézier surface associated to our PDE in Eq.(1) has some symmetries in the scalar coefficients, which in general we have denoted with Greek letters. This will lead us later to a change of notation.

3.1. The control net symmetries

Suppose that $\mathbf{P}_{k,\ell}$ are the control points of a tensor product Bézier PDE surface $\overrightarrow{\mathbf{x}}(u,v)$ satisfying Eq.(1).

If we define $\mathbf{Q}_{k,\ell} = \mathbf{P}_{n-k,n-\ell}$, we obtain the control net of the Bézier surface $\overrightarrow{\mathbf{y}}(u,v) = \overrightarrow{\mathbf{x}}(1-u,1-v)$ that also fulfils the PDE. Therefore, if the control net $\mathbf{Q}_{k,\ell}$ satisfies Eq.(1), then we have that

$$\begin{aligned}\mathbf{Q}_{k,\ell} = &\sum_{w=1}^{n-1} \alpha_{k,\ell,w}\ \mathbf{Q}_{0,w} + \sum_{w=1}^{n-1} \beta_{k,\ell,w}\ \mathbf{Q}_{n,w} + \sum_{w=1}^{n-1} \gamma_{k,\ell,w}\ \mathbf{Q}_{w,0} + \sum_{w=1}^{n-1} \delta_{k,\ell,w}\ \mathbf{Q}_{w,n}\\ &+\xi_{k,\ell}\ \mathbf{Q}_{0,0} + \tau_{k,\ell}\ \mathbf{Q}_{0,n} + \eta_{k,\ell}\ \mathbf{Q}_{n,0} + \sigma_{k,\ell}\ \mathbf{Q}_{n,n}.\end{aligned}$$

that is,

$$\begin{aligned}\mathbf{P}_{n-k,n-\ell} = &\sum_{w=1}^{n-1} \alpha_{k,\ell,w}\ \mathbf{P}_{n,n-w} + \sum_{w=1}^{n-1} \beta_{k,\ell,w}\ \mathbf{P}_{0,n-w}\\ &+ \sum_{w=1}^{n-1} \gamma_{k,\ell,w}\ \mathbf{P}_{n-w,n} + \sum_{w=1}^{n-1} \delta_{k,\ell,w}\ \mathbf{P}_{n-w,0} \qquad (7)\\ &+\xi_{k,\ell}\ \mathbf{P}_{n,n} + \tau_{k,\ell}\ \mathbf{P}_{n,0} + \eta_{k,\ell}\ \mathbf{P}_{0,n} + \sigma_{k,\ell}\ \mathbf{P}_{0,0}.\end{aligned}$$

But according to Eq.(1), we have that

$$\begin{aligned}
\mathbf{P}_{n-k,n-\ell} = & \sum_{w=1}^{n-1} \alpha_{n-k,n-\ell,w}\ \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} \beta_{n-k,n-\ell,w}\ \mathbf{P}_{n,w} \\
& + \sum_{w=1}^{n-1} \gamma_{n-k,n-\ell,w}\ \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} \delta_{n-k,n-\ell,w}\ \mathbf{P}_{w,n} \\
& + \xi_{n-k,n-\ell}\ \mathbf{P}_{0,0} + \tau_{n-k,n-\ell}\ \mathbf{P}_{0,n} \\
& + \eta_{n-k,n-\ell}\ \mathbf{P}_{n,0} + \sigma_{n-k,n-\ell}\ \mathbf{P}_{n,n}.
\end{aligned} \tag{8}$$

Therefore by comparing Eqs. (8) and (7) we obtain

$$\begin{aligned}
&\alpha_{k,\ell,w} = \beta_{n-k,n-\ell,n-w} \quad && \gamma_{k,\ell,w} = \delta_{n-k,n-\ell,n-w} \\
&\xi_{k,\ell} = \sigma_{n-k,n-\ell} && \tau_{k,\ell} = \eta_{n-k,n-\ell}
\end{aligned} \tag{9}$$

In the following subsection we will state a proposition in which we will denote the remaining four different coefficients, $\beta_{k,\ell,w}$ and $\delta_{k,\ell,w}$, by $\{z^i_{k,\ell,w}\}_{i=1}^2$ and $\eta_{k,\ell}$, $\sigma_{k,\ell}$ by $\{z^i_{k,\ell}\}_{i=3}^4$, respectively. Let us remark that the coefficients that appear with corner control points in Eq.(1) do not depend on w, and we added this variable in order to give a unique definition of $z^i_{k,\ell,w}$. Therefore, we should have in mind that

$$z^i_{k,\ell} = z^i_{k,\ell,w} \quad \text{when} \quad i = 3,4 \quad \forall w,$$

whereas $z^i_{k,\ell,w}$ for $i = 1,2$ of course do depend on w.

A control point will be defined from now on as follows,

$$\begin{aligned}
\mathbf{P}_{k,\ell} = & \sum_{w=1}^{n-1} z^1_{n-k,n-\ell,n-w}\ \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} z^1_{k,\ell,w}\ \mathbf{P}_{n,w} \\
& + \sum_{w=1}^{n-1} z^2_{n-k,n-\ell,n-w}\ \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} z^2_{k,\ell,w}\ \mathbf{P}_{w,n} \\
& + z^3_{n-k,n-\ell}\ \mathbf{P}_{0,0} + z^4_{n-k,n-\ell}\ \mathbf{P}_{0,n} + z^4_{k,\ell}\ \mathbf{P}_{n,0} + z^3_{k,\ell}\ \mathbf{P}_{n,n}.
\end{aligned}$$

3.2. The control net

Before presenting the control net result we must outline several lemmas. First, we recall the basis conversion in the following Lemma.

Lemma 3. *Let us consider a polynomial surface, $\overrightarrow{\mathbf{x}}(u,v)$, in terms of the power basis of polynomials and in its Bézier form,*

$$\overrightarrow{\mathbf{x}}(u,v)=\sum_{k,\ell=0}^{n}\frac{\mathbf{a}_{i,j}}{i!j!}u^iv^j=\sum_{k,\ell=0}^{n}B_k^n(u)\,B_\ell^n(v)\,\mathbf{P}_{k,\ell}.$$

The coefficients $\mathbf{a}_{i,j}$ are related with the control points $\mathbf{P}_{k,\ell}$ through the following equations,

$$\mathbf{a}_{i,j}=i!j!\binom{n}{i}\binom{n}{j}\Delta^i\mathbf{P}_{0,0},\ \textit{with}\ \Delta^{i,j}\mathbf{P}_{0,0}=\sum_{s,t=0}^{i,j}\binom{i}{s}\binom{j}{t}(-1)^{i+j-s-t}\mathbf{P}_{s,t}$$

and conversely

$$\mathbf{P}_{k,\ell}=\sum_{s=0}^{k}\sum_{t=0}^{\ell}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{\mathbf{a}_{s,t}}{s!t!}.$$

In the following lemma we give the explicit solution to a quite general matrix system, whose coefficient matrices depend on $B_{k,\ell}$, $C_{k,\ell}$, which we defined in Proposition 2. This result will be needed to prove our next Proposition.

Lemma 4. *The matrix system*

$$\left.\begin{aligned}\boldsymbol{p}&=\widetilde{M}\mathbf{x}+\widetilde{N}\mathbf{y}\\ \boldsymbol{q}&=M\mathbf{x}+N\mathbf{y}\end{aligned}\right\}$$

where $M=(b_{\ell,m})$, $N=(c_{\ell,m})$, and $\widetilde{M}=(\widetilde{b}_{\ell,m})=(b_{m-\ell+1,m})$, $\widetilde{N}=(\widetilde{c}_{\ell,m})=(c_{m-\ell+1,m})$ are $(n-1)\times(n-1)$ matrices with

$$b_{\ell,m}=\frac{B_{m-\ell+1,\ell}}{(m-\ell+1)!\ell!},\quad c_{\ell,m}=\frac{C_{m-\ell+1,\ell}}{(m-\ell+1)!\ell!}\qquad \ell,m=2,...,n,$$

has a unique solution.

Proof. *First of all, note that $\widetilde{M}$ is a regular matrix. Since it is an upper triangular matrix whose diagonal elements are $\frac{1}{j!}$ for $j=2,...,n$, we have $|\widetilde{M}|=\prod_{j=2}^{n}\frac{1}{j!}$, and so we can compute $\mathbf{x}$ in the first equation,*

$$\mathbf{x}=\widetilde{M}^{-1}(\boldsymbol{p}-\widetilde{N}\mathbf{y}).$$

Now if we substitute $\mathbf{x}$ in the second equation, we obtain

$$\mathbf{y}=A^{-1}(\boldsymbol{q}-M\widetilde{M}^{-1}\boldsymbol{p}).$$

This can always be done, since $A = N - M\widetilde{M}^{-1}\widetilde{N}$ is an upper triangular matrix too, with the same diagonal as $\widetilde{M}$ and is therefore regular.

$\square$

We can now give the explicit formula of each interior control point of a PDE Bézier surface satisfying Eq.(1).

Proposition 5. *The control net of a Bézier PDE surface that satisfies*

$$\rho^2 \overrightarrow{\mathbf{x}}_{uuuu} + 2\rho\cos t\,\overrightarrow{\mathbf{x}}_{uuuv} + (1+\rho^2)\overrightarrow{\mathbf{x}}_{uuvv} + 2\rho\cos t\,\overrightarrow{\mathbf{x}}_{uvvv} + \overrightarrow{\mathbf{x}}_{vvvv} = 0$$

can be determined in terms of its boundary control points,

$$\begin{aligned}\mathbf{P}_{k,\ell} = &\sum_{w=1}^{n-1} z^1_{n-k,n-\ell,n-w}\ \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} z^1_{k,\ell,w}\ \mathbf{P}_{n,w} \\ &+ \sum_{w=1}^{n-1} z^2_{n-k,n-\ell,n-w}\ \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} z^2_{k,\ell,w}\ \mathbf{P}_{w,n} \\ &+ z^3_{n-k,n-\ell}\ \mathbf{P}_{0,0} + z^4_{n-k,n-\ell}\ \mathbf{P}_{0,n} + z^4_{k,\ell}\ \mathbf{P}_{n,0} + z^3_{k,\ell}\ \mathbf{P}_{n,n},\end{aligned} \tag{10}$$

where

$$z^i_{k,\ell,w} = \delta^4_i \sum_{s=0}^{k} \frac{\binom{k}{s}\binom{\ell}{n-s}}{\binom{n}{n-s}} A_{s,n-s} + \sum_{s,t=1}^{k,\ell} \frac{\binom{k}{s}\binom{\ell}{t}}{s!\,t!\,\binom{n}{s}\binom{n}{t}} \left(B_{s,t}\, x^{4,w}_{s+t-1} + C_{s,t}\, y^{4,w}_{s+t-1}\right) \tag{11}$$

where δ^j_i is the Kronecker delta, and $\{x^{i,w}_m, y^{i,w}_m\}_{i=1}^4$ are the derivatives of $\{\mathbf{a}_{m,1}, \mathbf{a}_{1,m}\}$ with respect to $\mathbf{P}_{n,w}$, $\mathbf{P}_{w,n}$, $\mathbf{P}_{n,n}$, $\mathbf{P}_{n,0}$ respectively.

The value of $x^{i,w}_1 = y^{i,w}_1$ is given by

$$x^{i,w}_1 = y^{i,w}_1 = -\sum_{k=2}^{n} \frac{x^{i,w}_k}{k!}$$

except for $i = 1, w = 1$ and for the case $i = 3, \forall w$, which is given by

$$x^{1,1}_1 = y^{1,1}_1 = -\sum_{\ell=2}^{n} \frac{y^{1,1}_\ell}{\ell!} \qquad x^{3,w}_1 = y^{3,w}_1 = -\sum_{\ell=2}^{n} \frac{y^{3,w}_\ell}{\ell!}.$$

The value of $\{x^{i,w}_m, y^{i,w}_m\}_{i=1}^4$ for $m = 2, \ldots, n$ is given by Lemma 4 with the constant terms,

$i=1$	$\boldsymbol{p}=\boldsymbol{0}$	$\boldsymbol{q}=\boldsymbol{q}^w$
$i=2$	$\boldsymbol{p}=\boldsymbol{q}^w$	$\boldsymbol{q}=\boldsymbol{0}$
$i=3$	$\boldsymbol{p}=\boldsymbol{Q}^1$	$\boldsymbol{q}=\boldsymbol{q}^0+\boldsymbol{q}^2$
$i=4$	$\boldsymbol{p}=\boldsymbol{q}^n$	$\boldsymbol{q}=\boldsymbol{q}^n$

where $\boldsymbol{q}^w = \{q_\ell^{n,w}\}_{\ell=2}^n$, *with* $q_\ell^{n,w} = \binom{n}{\ell}\binom{\ell}{w}(-1)^{\ell-w}$ *and* $\boldsymbol{Q}^1 = \{-\binom{n}{\ell}A_{\ell,n-\ell}\}_{\ell=2}^n$, $\boldsymbol{Q}^2 = \{-\binom{n}{\ell}A_{n-\ell,\ell}\}_{\ell=2}^n$. $A_{k,\ell}, B_{k,\ell}, C_{k,\ell}$ *are defined in Proposition 2.*

Proof. Having in mind Lemma 3 and Proposition 2, we have that a control point can be written in terms of coefficients $\mathbf{a}_{s,t}$ at the first two rows and columns,

$$\begin{aligned}\mathbf{P}_{k,\ell} &= \sum_{s=0}^{k}\sum_{t=0}^{\ell}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{\mathbf{a}_{s,t}}{s!t!}\\ &= \sum_{t,s=0}^{\ell,k}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{A_{s,t}\,\partial\mathbf{a}_{s+t,0}+B_{s,t}\,\partial\mathbf{a}_{s+t-1,1}+C_{s,t}\,\partial\mathbf{a}_{1,s+t-1}+D_{s,t}\,\partial\mathbf{a}_{0,s+t}}{s!t!}.\end{aligned}$$

Moreover, since coefficients $\mathbf{a}_{s,t}$ depend on the control points too, Lemma 3, we can compute $\{z_{k,\ell,w}^i\}_{i=1}^4$ by taking derivatives of Eq.(10). For example,

$$\begin{aligned}z_{k,\ell,w}^1 &= \frac{\partial\mathbf{P}_{k,\ell}}{\partial\mathbf{P}_{n,w}} = \frac{\partial}{\partial\mathbf{P}_{n,w}}\sum_{t=0}^{\ell}\sum_{s=0}^{k}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{\mathbf{a}_{s,t}}{s!t!}\\ &= \sum_{t,s=1}^{\ell,k}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{1}{s!t!}\left(B_{s,t}\,\frac{\partial\mathbf{a}_{s+t-1,1}}{\partial\mathbf{P}_{n,w}}+C_{s,t}\,\frac{\partial\mathbf{a}_{1,s+t-1}}{\partial\mathbf{P}_{n,w}}\right)\\ &= \sum_{t,s=1}^{\ell,k}\frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{1}{s!t!}\left(B_{s,t}\,x_{s+t-1}^{1,w}+C_{s,t}\,y_{s+t-1}^{1,w}\right).\end{aligned}$$

And the same would be done for $\{z_{k,\ell,w}^i\}_{i=2}^4$.

Now we must compute the derivatives of $\mathbf{a}_{m,1}$ and $\mathbf{a}_{1,m}$. We introduce the notation $\{x_m^{i,w}, y_m^{i,w}\}_{i=1}^4$ for the derivatives of $\{\mathbf{a}_{m,1}, \mathbf{a}_{1,m}\}$ with respect to $\mathbf{P}_{n,w}$, $\mathbf{P}_{w,n}$, $\mathbf{P}_{n,n}$ and $\mathbf{P}_{n,0}$ respectively. Nevertheless, let us remark that for $i=3,4$ we have no dependence on w, for example, $x_m^{3,w} = \frac{\partial\mathbf{a}_{m,1}}{\partial\mathbf{P}_{n,n}}$ and $y_m^{3,w} = \frac{\partial\mathbf{a}_{1,m}}{\partial\mathbf{P}_{n,n}}$ for all w.

In order to determine these derivatives, $\{x_m^{i,w}, y_m^{i,w}\}_{i=1}^4$, let us start with $\{x_m^{1,w}, y_m^{1,w}\}$. We study the boundary curves, which are given data. The boundary curves $\overrightarrow{\mathbf{x}}(0,v)$ and $\overrightarrow{\mathbf{x}}(u,0)$ only depend on control points in the first row and column, and from Lemma 3, we have

$$\mathbf{a}_{0,j} = j!\binom{n}{j}\Delta^{0j}\mathbf{P}_{0,0}, \qquad \mathbf{a}_{i,0} = i!\binom{n}{i}\Delta^{i0}\mathbf{P}_{0,0}. \tag{12}$$

On the other hand, we have the boundary curves

$$\begin{aligned}\overrightarrow{\mathbf{x}}(u,1) &= \sum_{k=0}^{n}\binom{n}{k}\Delta^{k0}\mathbf{P}_{0,n}\,u^k = \sum_{\ell=0}^{n}\frac{\mathbf{a}_{0,\ell}}{\ell!} + \sum_{\ell=0}^{n}\frac{\mathbf{a}_{1,\ell}}{\ell!}\,u + \sum_{k=2,\ell=0}^{n}\frac{\mathbf{a}_{k,\ell}}{k!\ell!}\,u^k\\ \overrightarrow{\mathbf{x}}(1,v) &= \sum_{\ell=0}^{n}\binom{n}{\ell}\Delta^{0\ell}\mathbf{P}_{n,0}\,v^\ell = \sum_{k=0}^{n}\frac{\mathbf{a}_{k,0}}{k!} + \sum_{k=0}^{n}\frac{\mathbf{a}_{k,1}}{k!}\,v + \sum_{k=0,\ell=2}^{n}\frac{\mathbf{a}_{k,\ell}}{k!\ell!}\,v^\ell.\end{aligned} \tag{13}$$

If we consider Eq.(13) for $k,\ell = 2,...,n$

$$\left.\begin{aligned}\tbinom{n}{k}\Delta^{k0}\mathbf{P}_{0,n} &= \textstyle\sum_{\ell=0}^{n}\frac{\mathbf{a}_{k,\ell}}{k!\ell!}\\ \tbinom{n}{\ell}\Delta^{0\ell}\mathbf{P}_{n,0} &= \textstyle\sum_{k=0}^{n}\frac{\mathbf{a}_{k,\ell}}{k!\ell!}\end{aligned}\right\} \tag{14}$$

and take derivatives with respect to $\mathbf{P}_{n,w}, \mathbf{P}_{w,n}, \mathbf{P}_{n,n}$ and $\mathbf{P}_{n,0}$, we obtain matrix systems involving $\mathbf{x}^{i,w} = \{x_m^{i,w}\}_{m=2}^n$, $\mathbf{y}^{i,w} = \{y_m^{i,w}\}_{m=2}^n$ for $i = 1,2,3,4$, respectively, as we will see.

To illustrate this, let us take derivatives, for example, with respect to $\mathbf{P}_{n,w}$ in some detail. Thus, we take derivatives in Eq.(14) having in mind again Eq.(6) in Proposition 2, then

$$\left.\begin{aligned}0 &= \textstyle\sum_{\ell=0}^{n}\frac{1}{k!\ell!}\left(B_{k,\ell}\,\frac{\partial\mathbf{a}_{k+\ell-1,1}}{\partial\mathbf{P}_{n,w}} + C_{k,\ell}\,\frac{\partial\mathbf{a}_{1,k+\ell-1}}{\partial\mathbf{P}_{n,w}}\right)\\ \tbinom{n}{\ell}\tbinom{\ell}{w}(-1)^{\ell-w} &= \textstyle\sum_{k=0}^{n}\frac{1}{k!\ell!}\left(B_{k,\ell}\,\frac{\partial\mathbf{a}_{k+\ell-1,1}}{\partial\mathbf{P}_{n,w}} + C_{k,\ell}\,\frac{\partial\mathbf{a}_{1,k+\ell-1}}{\partial\mathbf{P}_{n,w}}\right)\end{aligned}\right\}$$

that is,

$$\left.\begin{aligned}0 &= \textstyle\sum_{m=2}^{n}\left(b_{m-k+1,m}\,x_m^{1,w} + c_{m-k+1,m}\,y_m^{1,w}\right)\\ \tbinom{n}{\ell}\tbinom{\ell}{w}(-1)^{\ell-w} &= \textstyle\sum_{m=2}^{n}\left(b_{\ell,m}\,x_m^{1,w} + c_{\ell,m}\,y_m^{1,w}\right)\end{aligned}\right\}$$

where,

$$b_{\ell,m} = \frac{B_{m-\ell+1,\ell}}{(m-\ell+1)!\ell!}, \quad c_{\ell,m} = \frac{C_{m-\ell+1,\ell}}{(m-\ell+1)!\ell!}.$$

As we have said, it is a matrix system like the one we solved in Lemma 4, with the unknowns $\mathbf{x}^{1,w} = \{x_m^{1,w}\}_{m=2}^n$ and $\mathbf{y}^{1,w} = \{y_m^{1,w}\}_{m=2}^n$ and with the constant terms $\mathbf{p} = 0$, $\mathbf{q} = q^w = \{q_\ell^{n,w}\}_{\ell=2}^n$, with $q_\ell^{n,w} = \binom{n}{\ell}\binom{\ell}{w}(-1)^{\ell-w}$.

Analogously, we would take the derivatives in Eq.(14) with respect to $\mathbf{P}_{w,n}$, $\mathbf{P}_{n,n}$ and $\mathbf{P}_{n,0}$ to obtain matrix systems as defined in Lemma 4 with constant terms,

$\mathbf{p} = \mathbf{q}^w$	$\mathbf{q} = \mathbf{0}$
$\mathbf{p} = \mathbf{q}^n$	$\mathbf{q} = \mathbf{q}^n$
$\mathbf{p} = \mathbf{Q}^1$	$\mathbf{q} = \mathbf{q}^0 + \mathbf{Q}^2$

respectively, being $\mathbf{Q}^1 = \{Q_\ell^n\}_{\ell=2}^n$, $\mathbf{Q}^2 = \{Q_{n-\ell}^n\}_{\ell=2}^n$ with $Q_\ell^n = -\binom{n}{\ell} A_{\ell,n-\ell}$.

At this point we have just computed $\{x_m^{i,w}, y_m^{i,w}\}_{m=2}^n$. We will now compute $x_1^{i,w} = y_1^{i,w}$, which is the derivative of $\mathbf{a}_{1,1}$ with respect to $\mathbf{P}_{n,w}, \mathbf{P}_{w,n}, \mathbf{P}_{n,n}$ and $\mathbf{P}_{n,0}$. We consider the system in Eq.(13) with $k = 1, \ell = 1$, respectively,

$$\left.\begin{aligned} n\left(\mathbf{P}_{1,n} - \mathbf{P}_{0,n}\right) &= \textstyle\sum_{\ell=0}^n \frac{\mathbf{a}_{1,\ell}}{\ell!} \\ n\left(\mathbf{P}_{n,1} - \mathbf{P}_{n,0}\right) &= \textstyle\sum_{k=0}^n \frac{\mathbf{a}_{k,1}}{k!} \end{aligned}\right\}.$$

If we take the derivatives above, we get a pair of formulas with which to obtain any element in $\{\frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,w}}, \frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{w,n}}, \frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,n}}, \frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,0}}\}$. For example, if we take derivatives with respect to $\mathbf{P}_{n,n}$, we find that,

$$\left.\begin{aligned} \tfrac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,n}} &= -\textstyle\sum_{\ell=2}^n \frac{y_\ell^{4,w}}{\ell!} \\ \tfrac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,n}} &= -\textstyle\sum_{k=2}^n \frac{x_k^{4,w}}{k!} \end{aligned}\right\}.$$

In sum

$$x_1^{i,w} = y_1^{i,w} = -\sum_{k=2}^n \frac{x_k^{i,w}}{k!}$$

except for $i = 1, w = 1$ and for the case $i = 3, \forall w$. These derivatives are given by

$$x_1^{1,1} = y_1^{1,1} = \frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,1}} = -\sum_{\ell=2}^n \frac{y_\ell^{1,1}}{\ell!} \qquad x_1^{3,w} = y_1^{3,w} = \frac{\partial \mathbf{a}_{1,1}}{\partial \mathbf{P}_{n,0}} = -\sum_{\ell=2}^n \frac{y_\ell^{3,w}}{\ell!}.$$

$\square$

From all the previous discussion, the algorithm for generating a PDE surface Bézier control net explicitly can be formulated as follows.

Define $\overline{A}_n$, $\overline{B}_n$, $\overline{C}_n$, and $\overline{D}_n$ by Eq. (5).

Define $A_{k,\ell}$, $B_{k,\ell}$ and $C_{k,\ell}$ by Proposition 2.

Define the matrices M, N, $\widetilde{N}$, $\widetilde{N}$, A and the vectors $\mathbf{x}$ and $\mathbf{y}$ by Lemma 5.

Define the constant terms $\mathbf{p}$ and $\mathbf{q}$ by Proposition 6.

Define $x_m^{i,w}$ and $y_m^{i,w}$ by Proposition 6.

Define $z^i_{k,\ell,w}$ by Eq. (11).

Compute the control points $\mathbf{P}_{k,\ell}$ by Eq. (10).

We have finally achieved our purpose: We now have the explicit expression of any interior control point as a linear combination of boundary control points. Nevertheless, it would be desirable to obtain a generating function, as we did in [6] for the harmonic case. If a generating function is known, the linear combinations of boundary control points that describe the whole control net are obtained more easily from this function. Unfortunately, for the time being we are unable to do that, not even for the biharmonic case.

Therefore, as could be expected, the linear combinations of boundary control points that describe a harmonic control net imply a lower computational cost in comparison to the PDE surfaces satisfying the general fourth-order PDE we compute here. In any case, as we have said before, given the degree of the Bézier surface the set of scalar factors in the linear combinations that describe interior control points only needs to be calculated once: one pattern for any given degree. Once we have this pattern associated to the given degree, a change in the boundary control points can be made freely without significantly increasing our computational cost.

In the following subsections we will consider the modified biharmonic and the biharmonic equation, which are particular cases of our general fourth-order PDE.

3.3. The modified biharmonic case.

In [7] we can find the polynomial form of the explicit solution to the modified biharmonic equation,

$$\overrightarrow{\mathbf{x}}_{uuuu} + 2\alpha^2 \overrightarrow{\mathbf{x}}_{uuvv} + \alpha^4 \overrightarrow{\mathbf{x}}_{vvvv} = 0, \tag{15}$$

where $\alpha \in \mathbb{R}$. For $k, \ell > 1$,

$$\mathbf{a}_{k,\ell} = \frac{1}{\left[\frac{k}{2}\right] + \left[\frac{\ell}{2}\right]} \left((-\frac{1}{\alpha^2})^{\left[\frac{\ell}{2}\right]} \left[\frac{k}{2}\right] \mathbf{a}_{k+2\left[\frac{\ell}{2}\right], \ell \bmod 2} + (-\alpha^2)^{\left[\frac{k}{2}\right]} \left[\frac{\ell}{2}\right] \mathbf{a}_{k \bmod 2, 2\left[\frac{k}{2}\right]+\ell} \right). \tag{16}$$

This expression of the power basis coefficients of the PDE surface, $\mathbf{a}_{k,\ell}$, simplifies our work to find the Bézier solution of the modified biharmonic case. Moreover, in this special case we find some more symmetry relations. As before, we have that

$$\begin{array}{ll} \alpha_{k,\ell,w} = z^1_{n-k,n-\ell,n-w} & \gamma_{k,\ell,w} = z^2_{n-k,n-\ell,n-w} \\ \xi^n_{k,\ell} = z^4_{n-k,n-\ell} & \tau^n_{k,\ell} = z^3_{n-k,n-\ell} \end{array}$$

But, in addition, if we suppose that $\mathbf{P}_{k,\ell}$ are the control points of a Bézier PDE surface, $\overrightarrow{\mathbf{x}}(u,v)$, satisfying Eq.(15) and we define $\mathbf{Q}_{k,\ell} = \mathbf{P}_{n-k,\ell}$, we get the control net of the surface $\overrightarrow{\mathbf{y}}(u,v) = \overrightarrow{\mathbf{x}}(1-u,v)$ that also fulfils the modified biharmonic equation. Thus, following the same reasoning as before, we obtain some more symmetrical coefficients,

$$z^3_{k,\ell} = z^4_{k,n-\ell}.$$

Therefore, in sum, we would only have to compute three different coefficients,

$$\begin{aligned} \mathbf{P}_{k,\ell} = & \sum_{w=1}^{n-1} z^1_{n-k,n-\ell,n-w} \; \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} z^1_{k,\ell,w} \; \mathbf{P}_{n,w} \\ & + \sum_{w=1}^{n-1} z^2_{n-k,n-\ell,n-w} \; \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} z^2_{k,\ell,w} \; \mathbf{P}_{w,n} \\ & + z^4_{n-k,n-\ell} \; \mathbf{P}_{0,0} + z^4_{n-k,\ell} \; \mathbf{P}_{0,n} + z^4_{k,n-\ell} \; \mathbf{P}_{n,0} + z^4_{k,\ell} \; \mathbf{P}_{n,n}. \end{aligned}$$

In order to compute $z^i_{k,\ell,w}$ by means of Eq.(11), we find the matrix system we solved in Lemma 4 again, but now we can give an easier formula of the matrices involved since

$$B_{s,t} = \frac{(-\frac{1}{\alpha^2})^{\left[\frac{t}{2}\right]} \left[\frac{s}{2}\right] t \bmod 2}{\left[\frac{s}{2}\right] + \left[\frac{t}{2}\right]} \qquad C_{s,t} = \frac{(-\alpha^2)^{\left[\frac{s}{2}\right]} \left[\frac{t}{2}\right] s \bmod 2}{\left[\frac{s}{2}\right] + \left[\frac{t}{2}\right]}$$

for $s, t = 2, ..., n$ and with $B_{1,1} = 0$ and $C_{1,1} = 1$.

At `http://demonstrations.wolfram.com/AlphaBiharmonicBezierSurfaces` we have implemented a Bézier $\alpha-$biharmonic surfaces generator (see Figure 2). There, the reader can compute his or her own examples. Some resulting biharmonic surfaces are shown in Figure 3.

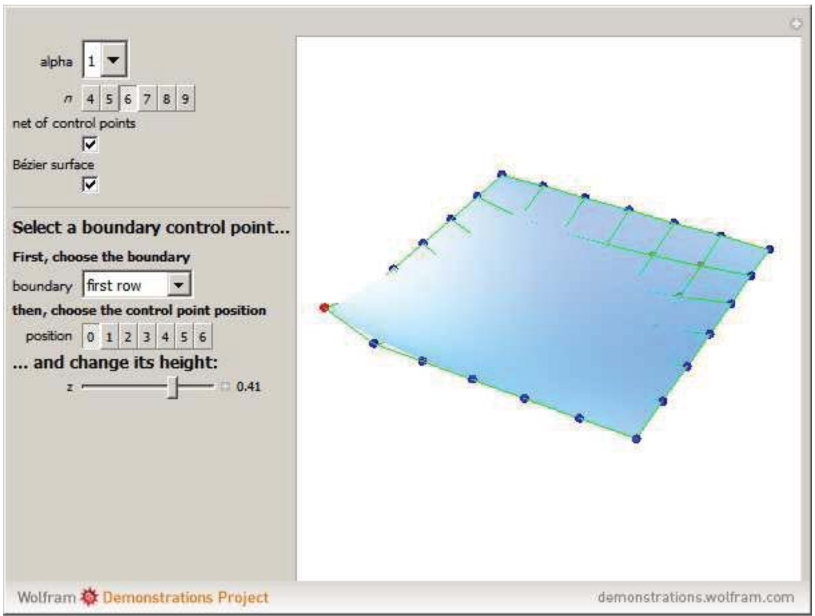

Figure 2: A Bézier surface generator for the modified biharmonic case can be found at http://demonstrations.wolfram.com/AlphaBiharmonicBezierSurfaces/.

3.3.1. The Biharmonic case.

This case is a particular case of the previous one with $\alpha = 1$. In [14] the existence of the solution was proved and in [7] we gave it explicitly in terms of the power basis. For $k, \ell > 1$,

$$\mathbf{a}_{k,\ell} = \frac{1}{\left[\frac{k}{2}\right] + \left[\frac{\ell}{2}\right]} \left((-1)^{\left[\frac{\ell}{2}\right]} \left[\frac{k}{2}\right] \mathbf{a}_{k+2\left[\frac{\ell}{2}\right], \ell \bmod 2} + (-1)^{\left[\frac{k}{2}\right]} \left[\frac{\ell}{2}\right] \mathbf{a}_{k \bmod 2, 2\left[\frac{k}{2}\right]+\ell} \right).$$

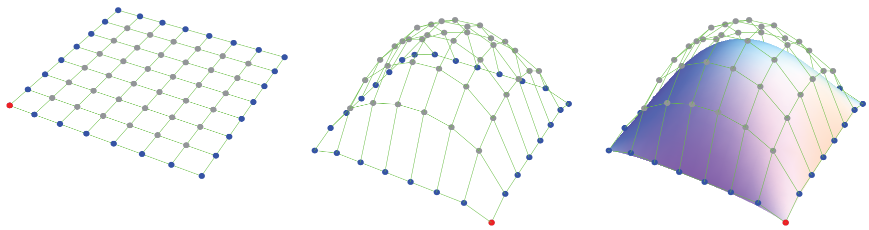

Figure 3: These are some biharmonic examples obtained thanks to the surface generator mentioned above. Once the initialization has been evaluated for a given degree, although the general biharmonic case takes a longer time than the harmonic, interactive design is possible.

As in the modified biharmonic case, we can add some symmetry properties. In fact it will be enough to compute two coefficients,

$$\begin{aligned}\mathbf{P}_{k,\ell} = & \sum_{w=1}^{n-1} z^1_{n-k,n-\ell,n-w}\ \mathbf{P}_{0,w} + \sum_{w=1}^{n-1} z^1_{k,\ell,w}\ \mathbf{P}_{n,w} \\ & + \sum_{w=1}^{n-1} z^1_{n-\ell,n-k,n-w}\ \mathbf{P}_{w,0} + \sum_{w=1}^{n-1} z^1_{\ell,k,w}\ \mathbf{P}_{w,n} \\ & + z^4_{n-k,n-\ell}\ \mathbf{P}_{0,0} + z^4_{n-k,\ell}\ \mathbf{P}_{0,n} + z^4_{k,n-\ell}\ \mathbf{P}_{n,0} + z^4_{k,\ell}\ \mathbf{P}_{n,n}.\end{aligned}$$

4. Conclusion

In this paper we show how elliptic PDEs can be used as an intuitive surface generation and manipulation tool.

Our previous work in this field consists in, first, finding out how many control points are free for prescription in order to determine a PDE surface and, second, obtaining explicit PDE surfaces in the power basis of polynomials. Now, our third step is to provide explicit PDE surfaces in Bézier formulation.

We have compared computational times in order to show the advantages of obtaining an explicit PDE surface in its Bézier form, although improving computational cost was not our main reason for solving this problem. In essence our motivation comes from a theoretical point of view: explicit formulas are wanted as a Bézier characterization of PDE surfaces and as a direct method to avoid the three-step algorithm involving the standard basis of polynomials:

1. Compute the standard basis coefficients, $\mathbf{a}_{k,\ell}$, prescribed by the given boundary control points.
2. Determine those $\mathbf{a}_{k,\ell}$ still unknown with the explicit solution in the power basis of polynomials we gave in [7] and which we recalled in Proposition 2 (*Explicit Power Basis Formulas*) or by directly solving the linear system associated to the PDE (*No explicit solution*).
3. Come back to the Bézier basis.

Our goal here was to gain explicit knowledge of the scalars that characterize a PDE surface control net, and then avoid the change of basis.

Obviously, since we are solving general fourth-order PDEs, the expressions we obtain are not easy to read. Nevertheless, we hope they could help to solve other problems, such us finding a generating function at least for the easier case of biharmonic surfaces, as we did in [6] for harmonic surfaces.

Reduction of computational cost is an additional advantage. The following table shows a comparison between methods for computing our PDE surfaces. We have compared execution times, on a personal computer, of our `Mathematica 8.0` programs for computing a PDE surface, corresponding to three equivalent methods. In the first column of the following table we show the computational time with our method described in this paper, *Explicit PDE Bézier surfaces.* Furthermore, there are two methods following the three-step algorithm involving the standard basis of polynomials: first with the use of the explicit solution in Proposition 2 (*Explicit Power Basis Formulas*) and the second one substituting this step by directly solving the linear system associated to the PDE (*No explicit solution*). Let us remark that we did not consider solving the linear system associated to the PDE in Bézier basis directly because the times required are extremely long.

n	*Explicit PDE Bézier surface*	*Explicit Power Basis Formulas*	*No explicit solution*
4	1.079	4.578	0.328
5	1.656	7.281	2.172
6	3.672	15.	13.344
7	10.5	28.625	20.609
8	36.422	53.812	61.719
9	120.75	109.891	1881.13
10	404.234	270.844	14737.8

We can see that our new method improves computation times for degrees smaller than 9, which is good since Bézier surfaces of small degrees are the most commonly used in CAGD. When more complex surfaces are required, B-Spline surfaces are generally used instead of increasing the degree of a Bézier surface.

In any case, all of these methods allow interactive design equivalently since, for a given degree, all of them lead us to the expression of inner control points as a linear combination of boundary control points. Once these expressions have been computed for a given degree, with the corresponding speed afforded by the method, the PDE surface can be obtained in real time and freely changing the boundary.

Smooth surfaces conforming to a given boundary can be modelled taking advantage of working with PDE surfaces, without the need to enter the whole control net, but only the boundary control points.

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