INTERPOLATING SEQUENCES FOR $H^\infty(B_H)$

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ABSTRACT. We prove that under the extended Carleson's condition, a sequence $(x_n) \subset B_H$ is linear interpolating for $H^\infty(B_H)$ for an infinite dimensional Hilbert space H. In particular, we construct the interpolating functions for each sequence and find a bound for the constant of interpolation.

1. INTRODUCTION

Let A be a space of bounded functions defined on X. A sequence (x_n) in X is called interpolating for A if for any sequence $(a_n) \in \ell_\infty$, there exists $f \in A$ such that $f(x_n) = a_n$ for all $n \in \mathbb{N}$. We consider the linear and continuous $R : A \to \ell_\infty$ defined by $R(f) = (f(x_n))$. The sequence (x_n) is interpolating for A if and only if there exists a map $T : \ell_\infty \to A$ such that $R \circ T = Id_{\ell\infty}$. If T is linear, the sequence (x_n) is said to be linear interpolating for A. For any $\alpha = (\alpha_j) \in \ell^\infty$, let $M_\alpha = \inf \{\|f\|_\infty : f(x_j) = \alpha_j,\ j \in \mathbb{N}, f \in A\}$. The constant of interpolation for (x_n) is defined by $M = \sup \{M_\alpha : \alpha \in \ell^\infty, \|\alpha\|_\infty \leq 1\}$.

It is a classical result in function theory that a sequence (z_n) in the open unit disc $\mathbf{D} \subset \mathbb{C}$ is interpolating for H^∞, the space of analytic bounded functions on $\mathbf{D}$, if and only if Carleson's condition holds, i. e.:

$$\text{There is } \delta > 0 \text{ such that } \prod_{k \neq j} \rho(z_k, z_j) \geq \delta \text{ for any } j \in \mathbb{N}, \qquad (1.1)$$

where $\rho(z_k, z_j)$ denotes the pseudohyperbolic distance for points $z_k, z_j \in \mathbf{D}$, given by

$$\rho(z_k, z_j) = \left| \frac{z_k - z_j}{1 - \overline{z_k} z_j} \right|.$$

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Recall the Schwarz-Pick lemma: $\rho(f(z), f(w)) \leq \rho(z, w)$ for any $z, w \in \mathbf{D}$ and $f \in H^\infty$, $\|f\| \leq 1$. If ψ is an automorphism of $\mathbf{D}$, then $\rho(\psi(z), \psi(w)) = \rho(z, w)$.

If we deal with complex Banach spaces E, we denote by B_E its open unit ball. A function $f : B_E \to \mathbb{C}$ is said to be analytic if it is Fréchet differentiable. Denote by $H(B_E)$ the space of analytic functions $f : B_E \to \mathbb{C}$ and by $H^\infty(B_E)$ the space $\{f : B_E \to \mathbb{C} : f \text{ is analytic and bounded }\}$, which becomes a uniform Banach algebra when endowed with the sup-norm $\|f\| = \sup\{|f(x)| : x \in B_E\}$ and it is, obviously, the analogue of the space H^∞ for an arbitrary Banach space.

Sufficient conditions for a sequence to be interpolating for $H^\infty(B_E)$ where given by the authors in [GM]. Bearing in mind the Davie-Gamelin extension of $f \in H^\infty(B_E)$ to $\tilde{f} \in H^\infty(B_{E^{**}})$, the authors proved that a sufficient condition for a sequence $(x_n) \subset B_{E^{**}}$ to be linear interpolating for $H^\infty(B_E)$ is that the sequence of norms $(\|x_n\|)$ is interpolating for H^∞. Examples of sequences which satisfy this condition are, for instance, those which grow exponentially to the unit sphere, which we call the Hayman-Newman condition: $1-\|x_{k+1}\| < c(1-\|x_k\|)$ for some $0 < c < 1$ for any $k \in \mathbb{N}$. Interpolating sequences on $H^\infty(B_E)$ have been very useful to study the spectra of composition operators on spaces of analytic functions (see [GGL], [GLR] and [GM2]).

During all the manuscript, we will deal with a complex Hilbert space$H$. The notion of pseudohyperbolic distance can be carried over to $H^\infty(B_H)$ by considering for any $x, y \in B_H$,

$$\rho_H(x, y) = \sup\{\rho(f(x), f(y)) : f \in H^\infty(B_H), \|f\| \leq 1\}, \qquad (1.2)$$

where $\rho(z, w)$ is the pseudohyperbolic distance in $\mathbf{D}$. B. Berndtsson [B] showed that a sequence (x_n) in the open unit Euclidean ball B_n of $\mathbb{C}^n$ is interpolating for $H^\infty(B_n)$ if the following extended Carleson's condition holds:

$$\text{There is } \delta > 0 \text{ such that} \qquad \prod_{k \neq j} \rho_H(x_j, x_k) \geq \delta \qquad \forall j \in \mathbb{N}. \quad (1.3)$$

As P. Galindo, T. Gamelin and M. Lindström pointed out in [GGL], the result given by Berndtsson can be extended to the case of an infinite dimensional complex Hilbert space H by interpolating on finite subsets of the sequence with uniform bounds and applying a normal families argument.

The aim of this paper is to adapt the proof given by Berndtsson to the infinite dimensional case and prove that under the extended Carleson's condition 1.3, a sequence $(x_n) \subset B_H$ is linear interpolating.

In particular, we will construct the interpolating functions for each sequence and will find a bound for the constant of interpolation.

For our purpose, we will study the automorphisms on B_H and will adapt some results given by B. Berndtsson (see [B]) to the infinite dimensional case.

2. Background

The results of this section and further information about the automorphisms of B_H and the pseudohyperbolic distance on B_H can be found in [GR].

Automorphisms on B_H. Recall that the set of automorphisms on $\mathbf{D}$ is denoted by $Aut(\mathbf{D})$. It is well-known that this set is given by all the mappings $f : \mathbf{D} \to \mathbf{D}$ which are the composition of rotations with Möbius transformations $m_a : \mathbf{D} \longrightarrow \mathbf{D}$ given by

$$m_a(z) = \frac{a - z}{1 - \bar{a}z} \quad \text{for any } a \in \mathbf{D}. \tag{2.1}$$

The analogues of Möbius transformations on H are $\varphi_a : B_H \longrightarrow B_H$, $a \in B_H$, defined according to

$$\varphi_a(x) = (s_a Q_a + P_a)(m_a(x)) \tag{2.2}$$

where $s_a = \sqrt{1 - \|a\|^2}$, $m_a : B_H \longrightarrow B_H$ is the analytic map

$$m_a(x) = \frac{a - x}{1 - \langle x, a\rangle}, \tag{2.3}$$

$P_a : H \longrightarrow H$ is the orthogonal projection along the one-dimensional subspace spanned by a, that is,

$$P_a(x) = \frac{\langle x, a\rangle}{\langle a, a\rangle} a$$

and $Q_a : H \longrightarrow H$, is the orthogonal complement, $Q_a = Id - P_a$. Recall that P_a and Q_a are self-adjoint operators since they are orthogonal projections, so $\langle P_a(x), y\rangle = \langle x, P_a(y)\rangle$ and $\langle Q_a(x), y\rangle = \langle x, Q_a(y)\rangle$ for any $x, y \in H$.

The set of automorphisms on B_H is given by all the mappings $\varphi : B_H \to B_H$ which are the composition of such analogous Möbius transformations with unitary transformations U of H. Recall that unitary transformations U of H are self-maps of H satisfying $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for all $x, y \in H$.

Remarks on the pseudohyperbolic distance. It is clear by the definition that for any $x, y \in B_H$,

$$\rho(f(x), f(y)) \leq \rho_H(x, y) \quad \text{for any } f \in H^\infty(B_H),\ \|f\|_\infty \leq 1, \tag{2.4}$$

It is also well-known that

$$\rho_H(x,y) = \|\varphi_y(x)\| \tag{2.5}$$

so making some calculations we obtain

$$\rho_H(x,y)^2 = 1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2}. \tag{2.6}$$

3. Results

First, we recall Proposition 2.1 in [GLM]:

Proposition 3.1. *Let E be a complex Banach space and $(x_n) \subset B_E$. If there exists $M > 0$ and a sequence of functions $(F_j) \subset H^\infty(B_E)$ satisfying $F_j(x_n) = \delta_{j,n}$ for any $j \in \mathbb{N}$ and $\sum_j |F_j(x)| \leq M$ for all $x \in B_E$, then (x_n) is linear interpolating for $H^\infty(B_E)$.*

We will call (F_n) a sequence of Beurling functions for (x_n). Under conditions of Proposition 3.1, we have that $T : \ell_\infty \to H^\infty(B_E)$ defined by $T((\alpha_n)) = \sum_n \alpha_n F_n$ is a well-defined, linear operator such that $\|T\| \leq M$ and $T((\alpha_n))(x_k) = \alpha_k$ for any $k \in \mathbb{N}$, so (x_n) is linear interpolating. In particular, the constant M is an upper bound for the constant of interpolation.

The following calculations are straightforward and can be found in [GM].

Lemma 3.2. *We have the following statements:*

$$1 - x \leq -\log x \quad \text{for } 0 < x \leq 1. \tag{3.1}$$

$$\Re e\left[\frac{1+\alpha z}{1-\alpha z}\right] = \frac{1-|\alpha|^2|z|^2}{|1-\alpha z|^2} \quad \text{for any } \alpha \in \overline{\mathbf{D}}, z \in \mathbf{D}. \tag{3.2}$$

The following three lemmas are just calculus:

Lemma 3.3. *The function $u^2 \exp(-ut/8)$ is bounded from above by $\min\{1, \frac{256}{e^2 t^2}\}$ for $0 \leq u \leq 1$ and $t > 0$.*

Lemma 3.4. *Let $0 < c_k < 1$ for any $k \in \mathbb{N}$ and suppose that $h(t)$ is a non-increasing function on $(0,\infty)$. Then,*

$$\sum_{j=1}^{n} c_j h\left(\sum_{k\geq j} c_j\right) \leq \int_0^\infty h(t)dt.$$

The following result will be needed to simplify the proof of Theorem 3.11. The proof is just an exercise.

Lemma 3.5. *Let $(a_n) \subset [0,1)$ such that $\lim_n a_n = 1$. Then, (a_n) can be reordered into a non-decreasing sequence (b_n) such that $\lim_n b_n = 1$.*

Now we provide a lemma which includes some calculations related to the automorphisms φ_x.

Lemma 3.6. *Let $x, y \in B_H$ and $\varphi_{-y} : H \longrightarrow H$ the corresponding automorphism defined as in 2.2. Then, we have that*

$$\langle \varphi_y(x), \varphi_y(z) \rangle = 1 - \frac{(1 - \langle x, z \rangle)(1 - \langle y, y \rangle)}{(1 - \langle x, y \rangle)(1 - \langle y, z \rangle)}.$$

Proof. Since for any $x \in B_H$ we have $\varphi_y(x) = (s_y Q_y + P_y)(m_y(x))$, and bearing in mind that P and Q are orthogonal, we obtain that

$$\langle \varphi_y(x), \varphi_y(z) \rangle = \langle (s_y Q_y + P_y)(m_y(x)), (s_y Q_y + P_y)(m_y(z)) \rangle =$$

$$s_y^2 \langle Q_y(m_y(x)), Q_y(m_y(z)) \rangle + \langle P_y(m_y(x)), P_y(m_y(z)) \rangle =$$

$$\frac{(1 - \|y\|^2)\langle Q_y(y - x), Q_y(y - z) \rangle + \langle P_y(y - x), P_y(y - z) \rangle}{(1 - \langle x, y \rangle)\overline{(1 - \langle z, y \rangle)}}$$

by (2.3) just making some calculations. Since we have that $P_a + Q_a = Id_H$ for any $a \in H$, we obtain that

$$\langle \varphi_y(x), \varphi_y(z) \rangle = \frac{\langle y - x, y - z \rangle - \|y\|^2 \langle Q_y(y - x), Q_y(y - z) \rangle}{(1 - \langle x, y \rangle)(1 - \langle y, z \rangle)}.$$

The complement of the orthogonal projection is given by

$$Q_y(x) = x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y,$$

hence $Q_y(y - x) = -Q_y(x)$ and $Q_y(y - z) = -Q_y(z)$. Moreover,

$$\langle -Q_y(x), -Q_y(z) \rangle = \langle Q_y(x), Q_y(z) \rangle = \langle x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y, z - \frac{\langle z, y \rangle}{\langle y, y \rangle} y \rangle =$$

$$\langle x, z \rangle - \frac{1}{\|y\|^2} \langle x, y \rangle \langle y, z \rangle - \frac{1}{\|y\|^2} \langle x, y \rangle \langle y, z \rangle + \frac{1}{\|y\|^2} \langle x, y \rangle \langle y, z \rangle =$$

$$\langle x, z \rangle - \frac{1}{\|y\|^2} \langle x, y \rangle \langle y, z \rangle = \frac{\langle x, z \rangle \langle y, y \rangle - \langle x, y \rangle \langle y, z \rangle}{\|y\|^2}.$$

$$\text{So, } \langle \varphi_y(x), \varphi_y(z) \rangle = \frac{\langle x - y, z - y \rangle - \|y\|^2 \frac{\langle x, z \rangle \langle y, y \rangle - \langle x, y \rangle \langle y, z \rangle}{\|y\|^2}}{(1 - \langle x, y \rangle)(1 - \langle y, z \rangle)} =$$

$$\frac{\langle x - y, z - y \rangle - \langle x, z \rangle \langle y, y \rangle + \langle x, y \rangle \langle y, z \rangle}{(1 - \langle x, y \rangle)(1 - \langle y, z \rangle)}.$$

Adding and subtracting 1 and arranging terms, we obtain that the numerator equals to $(1-\langle x,y\rangle)(1-\langle y,z\rangle)-(1-\langle x,z\rangle)(1-\langle y,y\rangle)$.

Therefore, dividing by the denominator and making calculations, we obtain

$$\langle \varphi_y(x),\varphi_y(z)\rangle = 1-\frac{(1-\langle x,z\rangle)(1-\langle y,y\rangle)}{(1-\langle x,y\rangle)(1-\langle y,z\rangle)},$$

and the lemma is proved. □

Considering $z=x$, we obtain that formulas 2.5 and 2.6 are the same expression for the pseudohyperbolic distance for $x,y\in B_H$.

We will also need some technical lemmas. For the first one, we will need Proposition 5.1.2 in [R], which is stated as follows,

Lemma 3.7. *Let a,b,c be points in the unit ball of a finite dimensional Hilbert space. Then,*

$$|1-\langle a,b\rangle| \leq (\sqrt{|1-\langle a,c\rangle|}+\sqrt{|1-\langle b,c\rangle|})^2$$

Then, we obtain the following lemma,

Lemma 3.8. *Let H be an infinite dimensional complex Hilbert space and $x_1,x_2,x_3\in B_H$. Then,*

$$|1-\langle x_1,x_2\rangle| \leq 2(|1-\langle x_1,x_3\rangle|+|1-\langle x_2,x_3\rangle|)$$

and

$$1-|\langle x_1,x_2\rangle| \leq 2(1-|\langle x_1,x_3\rangle|+1-|\langle x_2,x_3\rangle|).$$

Proof. Let $x_1,x_2,x_3\in B_H$ and set the space $H_1=\text{span}\{x_1,x_2,x_3\}$. We have that H_1 is itself a Hilbert space and we can consider an orthonormal basis $\{e_1,e_2,e_3\}$ of H_1. Consider for $j=1,2,3$ the vectors $y_j=(y_j^1,y_j^2,y_j^3)$ given by the components of x_j in that basis. It is clear that these vectors are in the unit Euclidean ball of $\mathbb{C}^3$ and $\langle x_j,x_k\rangle=\langle y_j,y_k\rangle$, so we apply Lemma 3.7 to deduce

$$|1-\langle x_1,x_2\rangle| \leq (\sqrt{|1-\langle x_1,x_3\rangle|}+\sqrt{|1-\langle x_2,x_3\rangle|})^2 =$$

$$|1-\langle x_1,x_3\rangle|+|1-\langle x_2,x_3\rangle|+2\sqrt{|1-\langle x_1,x_3\rangle|}\sqrt{|1-\langle x_2,x_3\rangle|}.$$

By the arithmetic-geometric means inequality, we have $|1-\langle x_1,x_2\rangle| \leq$

$$|1-\langle x_1,x_3\rangle|+|1-\langle x_2,x_3\rangle|+2\frac{|1-\langle x_1,x_3\rangle|+|1-\langle x_2,x_3\rangle|}{2} =$$

$$2(|1-\langle x_1,x_3\rangle|+|1-\langle x_2,x_3\rangle|).$$

To prove the other result, notice that

$$1-|\langle x_j,x_k\rangle| = \min_{\theta\in[0,2\pi)} |1-e^{i\theta}\langle x_j,x_k\rangle|.$$

We have that $1-|\langle x_1,x_3\rangle| = |1-e^{i\alpha}\langle x_1,x_3\rangle|$ and $1-|\langle x_2,x_3\rangle| = |1-e^{i\beta}\langle x_2,x_3\rangle|$ for some $\alpha,\beta\in[0,2\pi)$. Then, applying the inequality above, we have that

$$\begin{aligned}1-|\langle x_1,x_2\rangle| = 1-|\langle e^{i\alpha}x_1,e^{i\beta}x_2\rangle| \le |1-\langle e^{i\alpha}x_1,e^{i\beta}x_2\rangle| \le \\ 2(|1-e^{i\alpha}\langle x_1,x_3\rangle| + |1-e^{i\beta}\langle x_2,x_3\rangle|) = \\ 2(1-|\langle x_1,x_3\rangle| + 1-|\langle x_2,x_3\rangle|). \quad \square\end{aligned}$$

Then, we can prove the following lemma, which extends Lemma 6 in [B] to the infinite dimensional case. We give the proof for the sake of completeness.

Lemma 3.9. *Let H be a Hilbert space and $x_k, x_j \in B_H$. If $\|x_k\| \ge \|x_j\|$, then*

$$\frac{1-|\langle x_k,x\rangle|^2}{1-|\langle x_k,x_j\rangle|^2} \ge \frac{1}{8}\frac{1-\|x_k\|^2}{1-|\langle x_j,x\rangle|^2} \qquad \text{for any } x\in B_H. \tag{3.3}$$

Proof. By Lemma 3.8, $1-|\langle x_k,x_j\rangle| \le 2(1-|\langle x_k,x\rangle| + 1-|\langle x,x_j\rangle|)$, and we consider two cases depending on $x\in B_H$. If $1-|\langle x_k,x\rangle| \ge 1-|\langle x_j,x\rangle|$, then $1-|\langle x_k,x_j\rangle|^2 \le 8(1-|\langle x_k,x\rangle|)$ so, bearing in mind that $\|x_k\|\ge\|x_j\|$,

$$\frac{1-|\langle x_k,x\rangle|^2}{1-|\langle x_k,x_j\rangle|^2} \ge \frac{1}{8}\frac{1-|\langle x_k,x\rangle|^2}{1-|\langle x_k,x\rangle|} \ge \frac{1}{8} \ge \frac{1}{8}\frac{1-\|x_j\|^2}{1-|\langle x_j,x\rangle|^2} \ge \frac{1}{8}\frac{1-\|x_k\|^2}{1-|\langle x_j,x\rangle|^2}.$$

If $1-|\langle x_k,x\rangle| \le 1-|\langle x_j,x\rangle|$, then $1-|\langle x_k,x_j\rangle|^2 \le 8(1-|\langle x_j,x\rangle|)$ so,

$$\frac{1-|\langle x_k,x\rangle|^2}{1-|\langle x_k,x_j\rangle|^2} \ge \frac{1}{8}\frac{1-|\langle x_k,x\rangle|^2}{1-|\langle x_k,x\rangle|} \ge \frac{1}{8}\frac{1-\|x_k\|^2}{1-|\langle x_k,x\rangle|} \ge \frac{1}{8}\frac{1-\|x_k\|^2}{1-|\langle x_j,x\rangle|^2},$$

so we are done. $\square$

We will also need the following lemma,

Lemma 3.10. *Let $\{x_n\}\subset B_H$ and $\delta>0$ satisfying*

$$\prod_{k\ne j}\rho_H(x_k,x_j) \ge \delta \text{ for all } j\in\mathbb{N}. \tag{3.4}$$

Then, we have that

$$\sum_{k\ne j}^{\infty}(1-\|x_k\|^2) \le 2\log\left(\frac{1}{\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|} \qquad \text{for any } j\in\mathbb{N}, \tag{3.5}$$

and for any $j \in \mathbb{N}$,

$$\sum_{k=1}^{\infty}(1-\|x_k\|^2) \leq \left(1+2\log\frac{1}{\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|}. \tag{3.6}$$

Proof. Taking squares and logarithms in 3.4 we obtain

$$-\sum_{k\neq j}\log\rho_H(x_k,x_j)^2 \leq -2\log\delta = 2\log\frac{1}{\delta}.$$

By (3.1), we have that $1-\rho_H(x_k,x_j)^2 \leq -\log\rho_H(x_k,x_j)^2$ for any $k \neq j$, so bearing in mind (2.6), we obtain

$$\sum_{k\neq j}\frac{(1-\|x_k\|^2)(1-\|x_j\|^2)}{|1-\langle x_k,x_j\rangle|^2} \leq 2\log\frac{1}{\delta}.$$

In consequence,

$$\sum_{k\neq j}^{\infty}(1-\|x_k\|^2) = \sum_{k\neq j}^{\infty}\frac{(1-\|x_k\|^2)(1-\|x_j\|^2)}{|1-\langle x_k,x_j\rangle|^2}\frac{|1-\langle x_k,x_j\rangle|^2}{1-\|x_j\|^2} \leq$$

$$2\left(\log\frac{1}{\delta}\right)\frac{(1+\|x_j\|)^2}{1-\|x_j\|^2} = 2\left(\log\frac{1}{\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|}$$

and the lemma is proved. □

Now we are ready to prove the result for complex Hilbert spaces. In addition, we will provide an upper estimate for the constant of interpolation depending only on δ.

Theorem 3.11. *Let H be a Hilbert space and (x_n) a sequence in B_H. Suppose that there exists $\delta > 0$ such that (x_n) satisfies the generalized Carleson condition (1.3) for δ. Then, there exists a sequence of Beurling functions (F_n) for (x_n). In particular, the sequence (x_n) is interpolating for $H^\infty(B_H)$ and the constant of interpolation is bounded by*

$$\frac{128(1+2\log\frac{1}{\delta})}{e\delta}.$$

Proof. Define, for any $k,j \in \mathbb{N}$, $k \neq j$, the analytic function $g_{k,j} : H \longrightarrow \mathbb{C}$ given by $g_{k,j}(x) = \langle\varphi_{x_k}(x),\varphi_{x_k}(x_j)\rangle$. For each $j \in \mathbb{N}$ we define the function $B_j : B_H \longrightarrow \mathbb{C}$ by $B_j(x) = \prod_{k\neq j} g_{k,j}(x)$. First we check that the infinite product converges uniformly on $rB_H = \{x \in B_H : \|x\| \leq r\}$ for fixed $0 < r < 1$. Let $x \in rB_H$. We have, by Lemma 3.6, that

$$1-g_{k,j}(x) = 1-\langle\varphi_{x_k}(x),\varphi_{x_k}(x_j)\rangle = \frac{(1-\langle x,x_j\rangle)(1-\langle x_k,x_k\rangle)}{(1-\langle x,x_k\rangle)(1-\langle x_k,x_j\rangle)}.$$

Since $|1-\langle x,x_j\rangle|\leq 1+r$, $|1-\langle x,x_k\rangle|\geq 1-r$ and $|1-\langle x_k,x_j\rangle|\geq 1-\|x_k\|\|x_j\|\geq 1-\|x_j\|$, we have that

$$|1-g_{k,j}(x)|\leq \frac{1+r}{1-r}\frac{1-\|x_k\|^2}{1-\|x_j\|},$$

so for any $j\in\mathbb{N}$, the series $\sum_{k\neq j}|1-g_{k,j}(x)|$ is uniformly convergent on rB_H by Lemma 3.10. In particular, the infinite product $\prod_{k\neq j}g_{k,j}(x)$ converges uniformly on compact sets, so $B_j\in H(B_H)$. In addition, notice that for $x\in B_H$, $|B_j(x)|=\prod_{k\neq j}|g_{k,j}(x)|=\prod_{k\neq j}|\langle\varphi_{x_k}(x),\varphi_{x_k}(x_j)\rangle|\leq \prod_{k\neq j}\|\varphi_{x_k}(x)\|\|\varphi_{x_k}(x_j)\|\leq 1$, so $\|B_j\|_\infty\leq 1$ and we obtain that $B_j\in H^\infty(B_H)$. It is clear that $B_j(x_k)=0$ for $k\neq j$ since $\varphi_{x_k}(x_k)=0$ and, according to 2.5, we have that

$$|B_j(x_j)|=\prod_{k\neq j}|g_{k,j}(x_j)|=\prod_{k\neq j}|\langle\varphi_{x_k}(x_j),\varphi_{x_k}(x_j)\rangle|=$$

$$\prod_{k\neq j}\|\varphi_{x_k}(x_j)\|^2=\prod_{k\neq j}\rho(x_k,x_j)^2\geq\delta^2.$$

Consider the functions $q_j, A_j\in H(B_H)$ for any $k\in\mathbb{N}$ defined by

$$q_j(x)=\left(\frac{1-\|x_j\|^2}{1-\langle x,x_j\rangle}\right)^2,$$

$$A_j(x)=\sum_{\{k:\|x_k\|\geq\|x_j\|\}}\frac{(1-\|x_k\|^2)(1-\|x_j\|^2)}{1-|\langle x_k,x_j\rangle|^2}\frac{1+\langle x_k,x\rangle}{1-\langle x_k,x\rangle}.$$

The function q_j is clearly analytic and bounded. By Lemma 3.5, we will consider that the sequence $(\|x_n\|)$ is non-decreasing, so $\{k:\|x_k\|\geq\|x_j\|\}=\{k:k\geq j\}$. Notice also that for $0<r<1$ and $x\in rB_H$ we have that $|A_j(x)|\leq$

$$\sum_{k\geq j}\frac{(1-\|x_k\|^2)(1-\|x_j\|^2)}{1-\|x_j\|^2}\frac{1+r}{1-r}\leq\frac{1+r}{1-r}\sum_{k\geq j}(1-\|x_k\|^2)$$

so by Lemma 3.10, the series converges uniformly on rB_H and hence $A_j\in H(B_H)$. Moreover, $\exp(-A_j)$ belongs to $H^\infty(B_H)$ since $|\exp(-A_j)|=\exp(-\Re e\ A_j)$ and using formula 3.2, we have

$$\Re e\ A_j(x)=\sum_{k\geq j}\frac{(1-\|x_k\|^2)(1-\|x_j\|^2)(1-|\langle x_k,x\rangle|^2)}{(1-|\langle x_k,x_j\rangle|^2)(|1-\langle x_k,x\rangle|^2)}>0.$$

Consider $C_\delta=1/(1+2\log 1/\delta)$ and for any $j\in\mathbb{N}$, the analytic function $F_j:B_H\longrightarrow\mathbb{C}$ defined by

$$F_j(x)=\frac{B_j(x)}{B_j(x_j)}q_j(x)^2\exp(-C_\delta(A_j(x)-A_j(x_j))).$$

It is clear that $F_j(x_j) = 1$ and $F_j(x_k) = 0$ for any $k \neq j$. We claim that there exists $M > 0$ such that $\sum_{j=1}^{\infty} |F_j(x)| \leq M$ for any $x \in B_H$. Indeed, by (3.2) that

$$\Re e\ A_j(x) = \sum_{k \geq j} \frac{(1 - \|x_k\|^2)(1 - \|x_j\|^2)(1 - |\langle x_k, x\rangle|^2)}{(1 - |\langle x_k, x_j\rangle|^2)(|1 - \langle x_k, x\rangle|^2)}.$$

In particular, for $x = x_j$, we obtain

$$\Re e\ A_j(x_j) = \sum_{k \geq j} \frac{(1 - \|x_k\|^2)(1 - \|x_j\|^2)(1 - |\langle x_k, x_j\rangle|^2)}{(1 - |\langle x_k, x_j\rangle|^2)(|1 - \langle x_k, x_j\rangle|^2)}.$$

Using formula (2.6) we obtain that

$$\Re e\ A_j(x_j) = \sum_{k \geq j} (1 - \rho_H(x_k, x_j)^2) = 1 + \sum_{k > j} (1 - \rho_H(x_k, x_j)^2)$$

and by (3.1), we have that $\Re e\ A_j(x_j) \leq$

$$1 - \sum_{\{k : \|x_k\| > \|x_j\|\}} \log \rho_H(x_k, x_j)^2 \leq 1 - \sum_{k \neq j} \log \rho_H(x_k, x_j)^2 \leq 1 + 2 \log \frac{1}{\delta}.$$

Moreover, to estimate $\Re e\ A_j(x)$ from below we use Lemma 3.9 and we obtain that

$$\Re e\ A_j(x) \geq \frac{1}{8} \frac{1 - \|x_j\|^2}{1 - |\langle x_j, x\rangle|^2} \sum_{k \geq j} \frac{(1 - \|x_k\|^2)^2}{|1 - \langle x_k, x\rangle|^2}.$$

We define for any $k \in \mathbb{N}$,

$$b_k(x) = \frac{1 - \|x_k\|^2}{1 - |\langle x_k, x\rangle|^2}$$

so

$$\Re e\ A_j(x) \geq \frac{1}{8}\ b_j(x) \sum_{k \geq j} |q_k(x)|. \tag{3.7}$$

It is clear that $1 - |\langle x_j, x\rangle|^2 = (1 + |\langle x_j, x\rangle|)(1 - |\langle x_j, x\rangle|) \leq 2|1 - \langle x_j, x\rangle|$, so

$$|q_j(x)| = \left| \frac{1 - \|x_j\|^2}{1 - \langle x, x_j\rangle} \right|^2 \leq 4 \left(\frac{1 - \|x_j\|^2}{1 - |\langle x, x_j\rangle|^2} \right)^2 = 4 b_j(x)^2.$$

Using that $|B_j(x_j)| \geq \delta$, $|B_j(x)| \leq 1$, the bound for $\Re e\ A_j(x_j)$ and 3.7, we obtain

$$|F_j(x)| \leq \frac{4}{\delta}|q_j(x)|b_j(x)^2 \exp\left(-C_\delta\left(\frac{1}{8}b_j(x)\sum_{k\geq j}|q_k(x)| - \frac{1}{C_\delta}\right)\right) \leq$$
$$\frac{4e}{\delta}|q_j(x)|b_j(x)^2 \exp\left(-\frac{1}{8}C_\delta b_j(x)\sum_{k\geq j}|q_k(x)|\right).$$

Since $0 \leq b_k(x) \leq 1$, we consider $u = b_j(x)$ and $t = C_\delta \sum_{k\geq j}|q_k(x)| > 0$ and apply Lemma 3.3 to conclude that

$$|F_j(x)| \leq \frac{4e}{\delta C_\delta}C_\delta|q_j(x)|h\left(C_\delta\sum_{k\geq j}|q_k(x)|\right),$$

where $h(t) = \min\{1, 256/e^2t^2\}$. Hence, summing on j, we obtain

$$\sum_{j=1}^{\infty}|F_j(x)| \leq \frac{4e}{\delta C_\delta}\sum_{j=1}^{\infty}C_\delta|q_j(x)|h\left(\sum_{k\geq j}C_\delta|q_k(x)|\right),$$

and applying Lemma 3.4 , we obtain that

$$\sum_{j=1}^{\infty}|F_j(x)| \leq \frac{4e}{\delta C_\delta}\int_0^\infty h(t)dt = \frac{4e.32}{e^2\delta C_\delta} = \frac{128}{e\delta C_\delta}.$$

Hence , by Proposition 3.1, we conclude that (x_n) is linear interpolating. □

Given $(x_n) \subset B_H$ satisfying the extended Carleson's condition and any $(\alpha_n) \in \ell_\infty$, the function $f(x) = \sum_{j=1}^\infty \alpha_j F_j(x)$, where F_j is defined as in Theorem 3.11, is well-defined and interpolates the values α_n in the points x_n for any $n \in \mathbb{N}$.

Notice also that the function

$$f(\delta) = \frac{1}{\delta C_\delta^2} = \frac{1 + 2\log 1/\delta}{\delta}$$

is non-increasing for $0 < \delta \leq 1$. Since $\lim_{\delta\to 1} f(\delta) = 1$, un upper bound for the constant of interpolation is close to $\frac{128}{e}$ if we deal with sequences satisfying the extended Carleson's condition with δ close to 1. Can the number $\frac{128}{e}$ be decreased?

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