On the Connectedness of Rational Arithmetic Discrete Hyperplanes

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Abstract. While connected arithmetic discrete lines are entirely characterized by their arithmetic thickness, only partial results exist for arithmetic discrete hyperplanes in any dimension. In the present paper, we focus on 0-connected rational arithmetic discrete planes in $\mathbb{Z}^3$. Thanks to an arithmetic reduction on a given integer vector $\mathbf{n}$, we provide an algorithm which computes the thickness of the thinnest 0-connected arithmetic plane with normal vector $\mathbf{n}$.

1 Introduction

In [1], J.-P. Reveillès initiated a new approach of linear discrete objets and introduced arithmetic discrete lines as sets of pairs of integers satisfying a double Diophantine inequality : the *arithmetic discrete line* with *normal vector* $\mathbf{n} \in \mathbb{R}^2$, *translation parameter* $\mu \in \mathbb{R}$ and *thickness* $w \in \mathbb{R}$ is the set $\mathbf{D}(\mathbf{n}, \mu, w) = \left\{\mathbf{x} \in \mathbb{Z}^2,\ 0 \leq \mathbf{n} \cdot \mathbf{x} + \mu < w\right\}$, where $\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + n_2 x_2$ is the usual Euclidean scalar product of $\mathbf{n}$ and $\mathbf{x}$. Geometrically, an arithmetic discrete line can be viewed as a set of integer points of the plane $\mathbb{R}^2$ included in a band delimited by two parallel Euclidean lines (see Fig. 1). The thickness parameter w plays a key role in the topology of the arithmetic discrete lines: given $\mathbf{n} \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$, the thinnest 0-connected (resp. 1-connected) arithmetic discrete line among the ones with normal vector $\mathbf{n}$ and translation parameter μ is the arithmetic discrete line $\mathbf{D}(\mathbf{n}, \mu, \|\mathbf{n}\|_\infty)$ (resp. $\mathbf{D}(\mathbf{n}, \mu, \|\mathbf{n}\|_1)$) (see Section 2 for the definition of the 0-connectedness and 1-connectedness) [1].

The definition of arithmetic discrete lines extends naturally in dimension 3 to the *arithmetic discrete planes* and in any dimension $d \geq 2$ to the *arithmetic discrete hyperplanes* [2]. It is thus natural to try to exhibit a similar relation between the κ-connectedness of an arithmetic discrete hyperplane and its thickness. In fact, the 2-dimensional case is somewhat confusing since a 0-connected (resp. 1-connected) arithmetic discrete line is also 1-separating (resp. 1-separating) in $\mathbb{Z}^2$ (see Section 2).

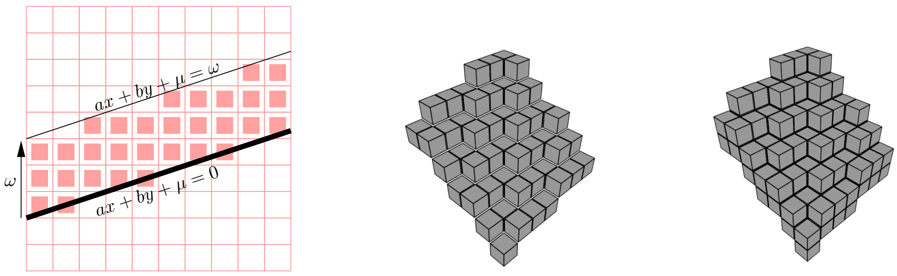


Fig. 1. From left to right: an arithmetic discrete line - a naive discrete plane - a standard discrete plane

In the particular case of rational arithmetic discrete hyperplanes (remember that an arithmetic discrete hyperplane is *rational* if its normal vector $\mathbf{n} \in \mathbb{R}^d$ is colinear to an integer vector, or equivalently, if the $\mathbb{Q}$-vector space spanned by $\{n_1, \ldots, n_d\}$ is of dimension 1), several approaches have been attempted [2,3,4] although none of them provides an explicit formula to compute the thickness of the thinnest 0-connected rational arithmetic discrete hyperplane with any given normal vector.

In [4], V. Brimkov and R. Barneva partially solved this request for rational arithmetic discrete planes whose the normal vector $\mathbf{n} \in \mathbb{Z}^2$ satisfies particular conditions (for instance when $|n_1| + 2|n_2| \leq |n_3|$) and provided an algorithm for the entire problem. Unfortunately, their algorithm seems to incorrect and does not generally return the right thickness (see Section 4).

In [3], Y. Gérard investigated a problem close to the one we are interested in in the present paper: given an arithmetic discrete hyperplane $\mathbf{P}(\mathbf{n}, \mu, w)$ and $\kappa \in \{0, \ldots, d-1\}$, is $\mathbf{P}(\mathbf{n}, \mu, w)$ κ-connected ? In other words, given the graph $\mathbf{G}(\mathbf{n}, \mu, w)$ whose vertices are the points of $\mathbf{P}(\mathbf{n}, \mu, w)$ and whose edges are the pairs $\{\mathbf{x}, \mathbf{y}\}$ of κ-adjacent points of $\mathbf{P}(\mathbf{n}, \mu, w)$, does $\mathbf{G}(\mathbf{n}, \mu, w)$ admit a unique connected component ? The main difficulty of this problem is the possibly infiniteness of $\mathbf{G}(\mathbf{n}, \mu, w)$. Assuming $\dim_{\mathbb{Q}}\{n_1, \ldots, n_d\} = 1$, one reduces $\mathbf{G}(\mathbf{n}, \mu, w)$ to a finite graph by quotienting $\mathbf{G}(\mathbf{n}, \mu, w)$ iteratively by a subgroup of rank 1 of the lattice of periods of $\mathbf{P}(\mathbf{n}, \mu, w)$. Since $\mathbf{G}(\mathbf{n}, \mu, w)$ is injectively projectable in $\mathbb{Z}^d$, then, with at most d such quotienting processes, one reduces $\mathbf{G}(\mathbf{n}, \mu, w)$ to a finite graph with the same connectedness as $\mathbf{G}(\mathbf{n}, \mu, w)$.

In the present paper, we deal with the determination of the thickness of the thinnest 0-connected rational arithmetic discrete plane with a given normal vector $\mathbf{n}$. For this purpose, we give a short and elementary algorithm which takes a vector $\mathbf{n} \in \mathbb{Z}^3$ as entry and returns the thickness w of the thinnest 0-connected arithmetic discrete plane with normal vector $\mathbf{n}$. While Y. Gérard, V. Brimkov and R. Barneva's approaches need to determine a connected component, our algorithm is *entirely* arithmetic and does not need to consider any connectivity graph.

Here is the sketch of the present paper. Section 2 is devoted to the basic notions useful for the remaining. In Section 3, we investigate the notions of κ-connectedness and κ-separatingness and state a first comparison between their characterization in the case of rational arithmetic discrete lines and rational arithmetic discrete hyperplanes. In Section 4, we focus on V. Brimkov and R. Barneva's investigation [4]. After having recalled some of their results, we exhibit a counter example of the algorithm they proposed. In Section 5, we introduce an arithmetic reduction on the integer vectors preserving the 0-connectedness of arithmetic discrete planes. We end this section by designing an elementary and quite short algorithm which computes the minimal thickness by iterating this arithmetic reduction.

2 Basic Notions

The aim of this section is to introduce the basic notions and definitions we use throughout the present paper.

Let d be an integer equal or greater than 2 and let $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$ denote the canonical basis of the Euclidean vector space $\mathbb{R}^d$. Let us call a *discrete set* any subset of the *discrete space* $\mathbb{Z}^d$. In the following, for the sake of clarity, we denote by $(x_1, \ldots, x_d)$ the point (resp. vector) $\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e_i} \in \mathbb{R}^d$. An integer point $\mathbf{x} \in \mathbb{Z}^d$ is called a *voxel* (resp. a *pixel* if $d = 2$). A subset of $\mathbb{Z}^d$ is called a *discrete set*.

Let $\kappa \in \{0, \ldots, d-1\}$. Two voxels $\mathbf{x} \in \mathbb{Z}^d$ and $\mathbf{x'} \in \mathbb{Z}^d$ are said to be *κ-adjacent* if $\|\mathbf{x} - \mathbf{x'}\|_\infty = 1$ and $\|\mathbf{x} - \mathbf{x'}\|_1 \leq d - \kappa$. In other words, $\mathbf{x} \in \mathbb{Z}^d$ and $\mathbf{x'} \in \mathbb{Z}^d$ are *κ-adjacent* if they are distinct, the differences of their coordinates are at most 1 and $\mathbf{x}$ and $\mathbf{x'}$ have at most $d-\kappa$ different coordinates (resp. at least κ identical components). A *κ-path* is a (finite or infinite) sequence of consecutive κ-adjacent voxels. If $(\gamma_i)_{1 \leq i \leq n}$ is a finite κ-path, then we say that γ *links* the voxel γ_1 to the voxel γ_n. A subset $E \subseteq \mathbb{Z}^d$ is said *κ-connected* if, for each pair of voxels $(\mathbf{x}, \mathbf{x'}) \in E^2$, there exists a κ-path in E linking $\mathbf{x}$ to $\mathbf{x'}$. Given a discrete set $E \subseteq \mathbb{Z}^d$ and given $\kappa \in \{0, \ldots, d-1\}$, one says that E is *κ-separating* in $\mathbb{Z}^d$ if its complement in $\mathbb{Z}^d$ has (at least) two κ-connected components.

In [1], J.-P. Reveillès introduced the arithmetic discrete line as a set of integer points satisfying a double Diophantine inequality. This definition extends in a natural way to higher dimensions:

Definition 1 (Arithmetic discrete hyperplane [1,2]). *The* arithmetic discrete hyperplane *with* normal vector $\mathbf{n} \in \mathbb{Z}^d$*,* translation parameter $\mu \in \mathbb{Z}$ *and* thickness $w \in \mathbb{Z}$ *is the discrete set* $\mathbf{P}(\mathbf{n}, \mu, w)$ *defined by:*

$$\mathbf{P}(\mathbf{n}, \mu, w) = \left\{ \mathbf{x} \in \mathbb{Z}^d,\ 0 \leq \mathbf{n} \cdot \mathbf{x} + \mu < w \right\}, \tag{1}$$

where $\mathbf{n} \cdot \mathbf{x}$ *denotes the usual Euclidean scalar product in* $\mathbb{R}^d$*. If* $w = \|\mathbf{n}\|_\infty$ *(resp.* $w = \|\mathbf{n}\|_1$*) then* $\mathbf{P}(\mathbf{n}, \mu, w)$ *is said* naive *(resp.* standard*). If* $d = 2$ *the arithmetic discrete hyperplane* $\mathbf{P}(\mathbf{n}, \mu, w)$ *is called an* arithmetic discrete line *and is denoted by* $\mathbf{D}(\mathbf{n}, \mu, w)$*. If* $d = 3$ *the arithmetic discrete hyperplane* $\mathbf{P}(\mathbf{n}, \mu, w)$ *is called an* arithmetic discrete plane*.*

Remark 1. Throughout the present paper, when $\mathbf{P}(\mathbf{n}, \mu, w)$ is a rational arithmetic hyperplane, we assume, with no loss of generality, that $\gcd\{n_1, \dots, n_d\} = 1$, $\mu \in \mathbb{Z}$ and $w \in \mathbb{Z}$. Moreover, since the isometry group of the unit cube $[-0.5, 0.5]^d$ acts on the set of arithmetic discrete hyperplanes and since any isometry of $[-0.5, 0.5]^d$ preserves the κ-connectedness of any arithmetic discrete hyperplane, whatever $\kappa \in \{0, \dots, d-1\}$, then in the following, except when explicitly mentioned, we suppose the normal vector $\mathbf{n} \in \mathbb{Z}^d$ to satisfy $0 \le n_1 \le \cdots \le n_d$.

In Section 3, we recall some partial results on the connectedness of arithmetic discrete lines and give a first extension of them to arithmetic discrete hyperplanes.

3 κ-Connected Arithmetic Discrete Lines *vs.* κ-Separating Arithmetic Discrete Hyperplanes

Let us first deal with the case $d = 2$. In [1], J.-P. Reveillès showed how the κ-connectedness of an arithmetic discrete line depends only on its normal vector and its thickness:

Theorem 1 ([1]). *Let $\mathbf{D}(\mathbf{n}, \mu, w)$ be the arithmetic discrete line with normal vector $\mathbf{n} \in \mathbb{Z}^2$, translation parameter $\mu \in \mathbb{Z}$ and thickness $w \in \mathbb{Z}$. Then $\mathbf{D}(\mathbf{n}, \mu, w)$ is 0-connected (resp. 1-connected) if and only if $w \ge \|\mathbf{n}\|_\infty$ (resp. $w \ge \|\mathbf{n}\|_1$).*

It becomes natural to try to extend Theorem 1 to higher dimensions, that is, given $\mathbf{n} \in \mathbb{Z}^d$, $\mu \in \mathbb{Z}$ and $\kappa \in \{0, \dots, d-1\}$, to try to characterize the thickness of the thinnest κ-connected arithmetic discrete hyperplane with normal vector $\mathbf{n}$ and translation parameter μ.

Let us give a helpful reduction of our problem: if $\mu \in \mathbb{Z}$ and $\mathbf{n} \in \mathbb{Z}^d$, then the κ-connectedness (resp. κ-separatingness in $\mathbb{Z}^d$) of $\mathbf{P}(\mathbf{n}, \mu, w)$, whatever $d \ge 2$ and $\kappa \in \{0, \dots, d-1\}$, does not depend on the translation parameter μ. Indeed, it is a direct consequence of the following lemma:

Lemma 1. *Let $\mathbf{P}(\mathbf{n}, \mu, w)$ be an arithmetic discrete hyperplane with $d \ge 2$, $\mu \in \mathbb{Z}$ and $\mathbf{n} \in \mathbb{Z}^d$. For all $\mu' \in \mathbb{Z}$, there exists a vector $\boldsymbol{\alpha} \in \mathbb{Z}^d$ such that $\mathbf{P}(\mathbf{n}, \mu, w) = \mathbf{P}(\mathbf{n}, \mu', w) + \boldsymbol{\alpha}$.*

Proof. It obviously follows form Bezout's Lemma applied on the coordinates of $\mathbf{n}$. □

From now on, we consider only rational arithmetic discrete hyperplanes with a null translation parameter. Thanks to Lemma 1, in the determination of the thickness of the thinnest arithmetical discrete hyperplane with a given rational normal vector, this assumption is not restrictive. From now on, in order to simplify the notation, we denote by $\mathbf{P}(\mathbf{n}, w)$ the arithmetic discrete hyperplane with normal vector $\mathbf{n}$, translation parameter 0 and thickness w.

Definition 2 **(κ-Connecting thickness).** *Let $\mathbf{n} \in \mathbb{Z}^d$ and $\kappa \in \{0, \ldots, d-1\}$. The thickness w_κ of the thinnest κ-connected arithmetic discrete hyperplane with normal vector $\mathbf{n}$ is called the* κ-connecting thickness *of* $\mathbf{n}$.

Let us now investigate the κ-connectedness of arithmetic discrete planes ($d = 3$). It is not difficult to exhibit a 0-connected arithmetic discrete plane $\mathbf{P}(\mathbf{n}, w)$ thinner than the naive one, that is, satisfying $w < \|\mathbf{n}\|_\infty$ (see Fig. 2). Similarly, one easily finds a 2-connected arithmetic discrete plane $\mathbf{P}(\mathbf{n}, w)$ thinner than the standard one, that is, with $w < \|\mathbf{n}\|_1$.

(a) A 0-connected arithmetic discrete plane thinner than the naive one

(b) A 1-connected arithmetic discrete plane thinner than the standard one

Fig. 2. Connected arithmetic discrete planes

Nevertheless, although Theorem 1 does not seem to extend naturally to higher dimensions, it admits a quite nice generalization of it concerning the κ-separating arithmetic discrete hyperplane. For the sake of clarity, we introduce the following notation, providing a norm on $\mathbb{R}^d$:
NOTATION. — Let $\mathbf{x} \in \mathbb{R}^d$ and let σ be a permutation over the set $\{1, \ldots, d\}$ such that, for all $i \in \{1, \ldots, d-1\}$, $|x_{\sigma(i)}| \leq |x_{\sigma(i+1)}|$. For all $\kappa \in \{0, \ldots, d-1\}$, we denote by $]\mathbf{x}[_\kappa$ the following number:

$$]\mathbf{x}[_\kappa = \sum_{i=d-\kappa}^{d} |x_{\sigma(i)}|.$$

In other words, $]\mathbf{x}[_\kappa$ is equal to the sum of the $(\kappa + 1)$ greatest absolute values of the coordinates of $\mathbf{x}$.

One checks that, for each $\kappa \in \{0, \ldots, d-1\}$, the map $] \cdot [_\kappa : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a norm on $\mathbb{R}^d$. Moreover, one has $] \cdot [_0 = \| \cdot \|_\infty$ and $] \cdot [_{d-1} = \| \cdot \|_1$.

In the particular case of $d = 2$, for $\kappa \in \{0, 1\}$, the κ-connected arithmetic discrete lines are exactly the $(2 - (\kappa + 1))$-separating ones in $\mathbb{Z}^2$ and Theorem 1 is reformulated as follows:

Theorem 2 ([1]). *Let* $\mathbf{D}(\mathbf{n}, w)$ *be the arithmetic discrete line with normal vector* $\mathbf{n} \in \mathbb{Z}^2$ *and thickness* $w \in \mathbb{Z}$. *Let* $\kappa \in \{0, 1\}$. *Then,* $\mathbf{D}(\mathbf{n}, w)$ *is* $(1 - \kappa)$*-separating in* $\mathbb{Z}^2$ *if and only if* $w \geq]\mathbf{n}[_\kappa$.

In fact, as previously mentioned, the κ-separatingness of an arithmetic discrete hyperplane $\mathbf{P}(\mathbf{n}, w)$, whatever the dimension d, is entirely characterized by $]\mathbf{n}[_\kappa$. Indeed, Theorem 2 extends in the most natural way to every dimension:

Theorem 3 ([2]). *Let* $\mathbf{P}(\mathbf{n}, w)$ *be the arithmetic discrete hyperplane with normal vector* $\mathbf{n} \in \mathbb{Z}^d$ *and thickness* $w \in \mathbb{Z}$. *Let* $\kappa \in \{0, \ldots, d-1\}$. *The arithmetic discrete hyperplane* $\mathbf{P}(\mathbf{n}, w)$ *is* $(d-\kappa-1)$*-separating in* $\mathbb{Z}^d$ *if and only if* $w \geq]\mathbf{n}[_\kappa$.

4 V. Brimkov and R. Barneva's Investigation : an Algorithmic Approach [4]

In [4], V. Brimkov and R. Barneva investigated 0-connected rational arithmetic discrete planes. They explicitly provided the 0-connecting thickness of some vectors $\mathbf{n} \in \mathbb{Z}^3$ and an algorithm for computing it in the general case. In the present section, we exhibit a counter-example to this algorithm and deduce that it does not always return the correct output.

Let $\mathbf{P}(\mathbf{n}, w)$ be a rational arithmetic discrete plane. It is well known that if $w \geq \|\mathbf{n}\|_\infty$ then $\mathbf{P}(\mathbf{n}, w)$ is 0-connected (see [2] Cor. 10 p. 307). Hence, if w_0 is the 0-connecting thickness of $\mathbf{n}$, then $w_0 \leq \|\mathbf{n}\|_\infty$. In [4], V. Brimkov and R. Barneva reduced the determination of w_0 to the determination of the 0-connectedness of a subset of $\mathbb{Z}^2$ as follows:

Theorem 4 ([4]). *Let* $\mathbf{P}(\mathbf{n}, w)$ *be a rational arithmetic discrete plane with* $\|\mathbf{n}\|_\infty = |v_3|$ *and* $w \leq \|\mathbf{n}\|_\infty$. *The arithmetic dscrete plane* $\mathbf{P}(\mathbf{n}, w)$ *is* 0*-connected in* $\mathbb{Z}^3$ *if and only if the set* $\{\mathbf{x} \in \mathbb{Z}^2,\ v_1x_1 + v_2x_2 \mod v_3 \in [0, w[\}$ *is* 0*-connected in* $\mathbb{Z}^2$.

Remark 2. Let us remember that, thanks to Remark 1, the condition $\|\mathbf{n}\|_\infty = |v_3|$ in Theorem 4 is not restrictive. Up to an isometry, one can similarly treat the cases $\|\mathbf{n}\|_\infty = |v_1|$ and $\|\mathbf{n}\|_\infty = |v_2|$.

For the sake of clarity, we introduce the following notation:
NOTATION. — Let $\mathbf{P}(\mathbf{n}, w)$ be an arithmetic discrete plane with $\|\mathbf{n}\|_\infty = |v_3|$ and $w \leq \|\mathbf{n}\|_\infty$. We denote by $\mathbf{\Pi}(\mathbf{n}, w)$ the set $\{\mathbf{x} \in \mathbb{Z}^2,\ v_1x_1 + v_2x_2 \mod v_3 \in [0, w[\}$. In what follows, since $\mathbf{\Pi}(\mathbf{n}, w)$ can be indexed by (a subset of) $\mathbb{Z}^2$, we call $\mathbf{\Pi}(\mathbf{n}, w)$ the *array of remainders* of $\mathbf{P}(\mathbf{n}, w)$. For $\mathbf{x} \in \mathbb{Z}^2$, the number $v_1x_1 + v_2x_2 \mod v_3$ is called the *remainder* of $\mathbf{x}$. Let us notice that this denomination is not exactly the one used in [4,5], but is equivalent in the way we use it.

With this notation, from Theorem 4, it follows:

Corollary 1. *Let* $\mathbf{P}(\mathbf{n}, w)$ *be a rational arithmetic discrete plane with* $\|\mathbf{n}\|_\infty = |v_3|$ *and* $w \leq \|\mathbf{n}\|_\infty$. *The arithmetic dscrete plane* $\mathbf{P}(\mathbf{n}, w)$ *is* 0*-connected in* $\mathbb{Z}^3$ *if and only if the set* $\mathbb{Z}^2 \setminus \mathbf{\Pi}(\mathbf{n}, w)$ *is not* 0*-separating in* $\mathbb{Z}^2$.

Before describing V. Brimkov and R. Barneva's algorithm, let us introduce a notation:

NOTATION. — Let $\mathbf{n} \in \mathbb{Z}^3$ such that $0 \leq n_1 \leq n_2 \leq n_3$ and $\gcd\{n_1, n_2, n_3\} = 1$. We denote by $\mathbf{\Gamma}(\mathbf{n})$ the set of 1-paths in $\mathbf{\Pi}(\mathbf{n}, \|\mathbf{n}\|_\infty)$ linking two points of maximal remainder, that is, $n_3 - 1$. For a 1-path $\gamma \in \mathbf{\Gamma}(\mathbf{n})$, we denote by

$$\min(\gamma) = \min\{n_1 i_1 + n_2 i_2 \mod n_3, (i_1, i_2) \in \gamma\}.$$

In other words, $\min(\gamma)$ is the smallest remainder reached in γ.

In [4], V. Brimkov and R. Barneva stated:

Theorem 5 ([4]). *Let* $\mathbf{n} \in \mathbb{Z}^3$ *such that* $0 \leq n_1 \leq n_2 \leq n_3$ *and* $\gcd\{n_1, n_2, n_3\} = 1$. *Let* $w_0 \in \mathbb{Z}$ *be the* 0*-connecting thickness of* $\mathbf{n}$. *Then* $w_0 = \max\{\min(\gamma) \in \Gamma(\mathbf{n})\} + 1$.

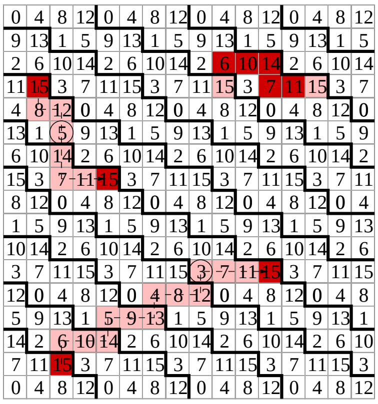
Given a vector $\mathbf{n} \in \mathbb{Z}^3$ satisfying $0 \leq n_1 \leq n_2 \leq n_3$ and $\gcd\{n_1, n_2, n_3\} = 1$, the problem of determining w_0 can thus be reduced to the following one: how to compute $\max\{\min(\gamma) \in \Gamma(\mathbf{n})\}$ in a reasonable time ? V. Brimkov and R. Barneva assumed that only exclusively down-right or up-right searches (with additional conditions) in $\mathbf{\Pi}(\mathbf{n}, \|\mathbf{n}\|_\infty)$ are necessary to compute w_0 (see [4]). This assertion is false and here is a counter-example:

Example 1. Let $\mathbf{n} = (4, 7, 16)$. Let w_0 be the 0-connecting thickness of $\mathbf{n}$. In Figure 3(a), both light paths are computed by V. Brimkov and R. Barneva's algorithm. Minimal remainders of each one are respectively 3 and 5, and the algorithm returns $w_0 = \max\{3, 5\} + 1 = 6$. In Figure 3(b), one sees that $\mathbf{\Pi}(\mathbf{n}, 6)$ is not 0-connected, and by Theorem 4, so is $\mathbf{P}(\mathbf{n}, 6)$. In fact, the correct 0-connecting thickness for the vector $\mathbf{n}$ is 7 as shown in Figure 3(c). This value is obtained with the dark grey path in Figure 3(a), which cannot be computed using exclusively up-right or down-right searches.

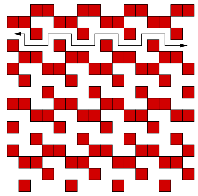
5 Arithmetic Reduction of an Arithmetic Discrete Plane

We have seen in Section 4, that V. Brimkov and R. Barneva's algorithm needs a graph traversal for computing the 0-connecting thickness of a given integer vector. Similarly, Y. Gérard proposed an algorithm, based on a graph traversal too, testing whether a given rational arithmetic discrete hyperplane is κ-connected. In the present section, we propose a reduction acting on the normal vector and the arithmetic thickness of an arithmetic discrete plane $\mathbf{P}(\mathbf{n}, w)$ which returns an arithmetic discrete plane $\mathbf{P}(\mathbf{n'}, w')$ with the same 0-connectedness as $\mathbf{P}(\mathbf{n}, w)$ and such that $|n'_1| < |n_1|$. By iterating this reduction, we obtain in a finite time an arithmetic discrete plane $\mathbf{P}(\mathbf{n'}, w')$ with a zero coordinate. The 0-connecting thickness (see Definition 2) of such a vector is easy to determine:

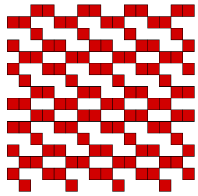
Lemma 2. *Let* $\mathbf{P}(\mathbf{n}, w)$ *be a rational arithmetic discrete plane. Let us suppose there exists* $i \in \{1, 2, 3\}$ *such that* $n_i = 0$. *Then,* $\mathbf{P}(\mathbf{n}, w)$ *is* 0*-connected if and only if* $w \geq \|\mathbf{n}\|_\infty$. *In other words, the* 0*-connecting thickness of* $\mathbf{n}$ *is* $\|\mathbf{n}\|_\infty$.



(a) 1-connected paths in the 2-dimensional representation.



(b) Array of remainders $\mathbf{\Pi}(\mathbf{n}, 6)$.



(c) Array of remainders $\mathbf{\Pi}(\mathbf{n}, 7)$.

Fig. 3. Computation of V. Brimkov and R. Barneva's algorithm [4] on the vector $\mathbf{n} = (4, 7, 16)$

Proof. It is well known that, if $w \geq \|\mathbf{n}\|_\infty$ then $\mathbf{P}(\mathbf{n}, w)$ is 0-connected [2]. Conversely, let us suppose, with no loss of generality, that $n_1 = 0$ and $0 \leq n_2 \leq n_3$. Let $\mathbf{x} \in \mathbb{Z}^2$ such that $n_1x_1 + n_2x_2 = n_2x_2 \equiv n_3 - 1 \bmod n_3$ (remember we assume $\gcd\{n_1, n_2, n_3\} = 1$). Then, for all $k \in \mathbb{Z}$, $(x_1 + k)n_1 + x_2n_2 = x_1n_1 + x_2n_2 \equiv n_3 - 1 \bmod n_3$. Hence, for all $k \in \mathbb{Z}$, $(x_1 + k, x_2) \in \mathbf{\Pi}(\mathbf{n}, w)$ and $\mathbf{\Pi}(\mathbf{n}, w)$ is not 0-connected. The result follows from Theorem 4. □

Remember that, thanks to Theorem 4, one can reduce the determination of the 0-connectedness of the arithmetic discrete plane $\mathbf{P}(\mathbf{n}, w)$ to the one of $\mathbf{\Pi}(\mathbf{n}, w) = \left\{\mathbf{x} \in \mathbb{Z}^2,\ n_1x_1 + n_2x_2 \bmod n_3 \in [0, w[\right\}$ with $\mathbf{n} \in \mathbb{Z}^3$ and $n_3 = \|\mathbf{n}\|_\infty$. Moreover, a direct consequence of Theorem 4 is:

Lemma 3 (Symmetry Lemma [4]). *Let $\Omega : \mathbb{N}^3 \to \mathbb{N}$ be the function mapping each vector of $\mathbb{N}^3$ to its 0-connecting thickness. For all $\mathbf{n} \in \mathbb{Z}^3$, if $0 \leq n_1, n_2 \leq n_3$, then $\Omega(n_1, n_2, n_3) = \Omega(n_3 - n_1, n_2, n_3) = \Omega(n_1, n_3 - n_2, n_3) = \Omega(n_3 - n_1, n_3 - n_2, n_3)$.*

> Given a vector $\mathbf{n} \in \mathbb{Z}^3$, thanks to Lemma 3 and to the action of the isometry group of the cube on the set of arithmetic discrete planes, one suppose **with no loss of generality** and in order to compute the 0-connecting thickness of $\mathbf{n}$ that $0 \leq 2n_1 \leq 2n_2 \leq n_3$.

Let us now state the main theorem of the present section:

Theorem 6 (Arithmetic reduction). *Let $\mathbf{n} \in \mathbb{Z}^3$ such that $0 \leq 2n_1 \leq 2n_2 \leq n_3$ and let $w \in \mathbb{Z}$. Let $(q, r) \in \mathbb{N}^2$ be the unique pair of integers such that*

$n_2 = qn_1 + r$ *and* $r \in [0, n_1[$. *Let* $\mathbf{n'} = M \cdot \mathbf{n}$ *with*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ 1-q & -1 & 1 \end{pmatrix},$$

and let $w' = w - (n_2 - n_1)$. *Then, the arithmetic discrete plane* $\mathbf{P}(\mathbf{n}, w)$ *is* 0*-connected if and only if so is the arithmetic discrete plane* $\mathbf{P}(\mathbf{n'}, w')$.

In order to prove Theorem 6, let us introduce in some sense the *dual* notion of the κ-connecting thickness of a vector:

Definition 3 (κ-separating thickness). *Let* $\mathbf{n} \in \mathbb{Z}^3$ *and let* $\kappa \in \{0, 1\}$. *The* κ-separating thickness $\overline{w}_\kappa$ *of* $\mathbf{n}$ *is the thickness of the thinnest* κ*-separating* $\mathbf{\Pi}(\mathbf{n}, w)$, *with* $w \in \mathbb{Z}$.

An easy computation directly gives:

Lemma 4. *Let* $\mathbf{n} \in \mathbb{Z}^3$ *such that* $0 \leq n_1, n_2 \leq n_3$ *and* $\gcd\{n_1, n_2, n_3\} = 1$. *Let* w_0 *(resp.* $\overline{w}_0$*) be the* 0*-connecting thickness (resp.* 0*-separating thickness) of* $\mathbf{n}$. *Then* $w_0 + \overline{w}_0 = n_3 + 1$.

Proof. Let $w \in \mathbb{N}$. Then

$$\begin{aligned} \mathbb{Z} \setminus \mathbf{\Pi}(\mathbf{n}, w) &= \left\{(x_1, x_2) \in \mathbb{Z}^2, n_1 x_1 + n_2 x_2 \bmod n_3 \in [w, n_3[\right\} \\ &= \left\{(x_1, x_2) \in \mathbb{Z}^2, n_1 x_1 + n_2 x_2 - w \bmod n_3 \in [0, n_3 - w[\right\} \end{aligned}$$

Let $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$ such that $n_1\alpha_1 + n_2\alpha_2 \equiv -w \bmod n_3$. Thus, $\mathbb{Z} \setminus \mathbf{\Pi}(\mathbf{n}, w) + (\alpha_1, \alpha_2) = \mathbf{\Pi}(\mathbf{n}, n_3 - w)$ and $\mathbf{\Pi}(\mathbf{n}, w)$ is 0-connected if and only if $\mathbf{\Pi}(\mathbf{n}, n_3 - w)$ is not 0-separating. Since $\mathbf{\Pi}(\mathbf{n}, w_0)$ (resp. $\mathbf{\Pi}(\mathbf{n}, w_0 - 1)$) is 0-connected (resp. is not 0-connected), then $\mathbf{\Pi}(\mathbf{n}, n_3 - w_0)$ (resp. $\mathbf{\Pi}(\mathbf{n}, n_3 - w_0 + 1)$) is not 0-separating (resp. is 0-separating). Hence $\overline{w}_0 = n_3 - w_0 + 1$ and the result follows. □

Since the κ-connectedness and the κ-separatingness of a rational arithmetic discrete plane do not depend on the translation parameter, an easy computation gives the equivalent reformulation of Theorem 6:

Theorem 7 (Arithmetic reduction). *Let* $\mathbf{n} \in \mathbb{Z}^3$ *such that* $0 \leq 2n_1 \leq 2n_2 \leq n_3$ *and let* $w \in \mathbb{Z}$. *Let* $(q, r) \in \mathbb{N}^2$ *be the unique pair of integers such that* $n_2 = qn_1 + r$ *and* $r \in [0, n_1[$. *Let* $\mathbf{n'} = M \cdot \mathbf{n}$ *with*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ 1-q & -1 & 1 \end{pmatrix},$$

and let $w' = w - qn_1$. *Then,* $\mathbf{\Pi}(\mathbf{n}, w)$ *is* 0*-separating if and only if so is* $\mathbf{\Pi}(\mathbf{n'}, w')$.

Proof (sketch). For clarity, let us first introduce a quite natural notation. One naturally represents a 1-path γ in $\Pi(\mathbf{n}, w)$ as a triple (A, u, B) with:

i) $A \in [0, w[$ (resp. $B \in [0, w[$) is the starting (resp. the ending) remainder of

4	8	12	0	4	8	12	0	4
13	1	5	9	13	1	5	9	13
6	10	14	2	6	10	14	2	6
15	3	7	11	15	3	7	11	15
8	12	0	4	8	12	0	4	8
1	5	9	13	1	5	9	13	1

Fig. 4. A 1-path corresponding to the triple $(1, [-n_2, n_1, n_1 - n_3, -(n_2 - n_3), -n_2, n_1, n_1, n_1 - n_3, n_1], 4)$

the 1-path γ.
ii) $u \in \{\pm n_1, \pm n_2, \pm(n_1 - n_3), \pm(n_2 - n_3)\}^k$ is a finite sequence of *movements* between A and B (see Figure 4) (the integer $k \in \mathbb{N}$ is called the *length* of u). Let us notice that the *movements* $\pm(n_1 - n_3)$ and $\pm(n_2 - n_3)$ corresponds to horizontal (resp. vertical) movements in $\Pi(\mathbf{n}, w)$ with a change of height in $\mathbf{P}(\mathbf{n}, w)$. Such a change is represented by a thick line in the array of remainders (see Figure 4).

Conversely, let (A, u, B) be a triple with $(A, B) \in [0, w[^2$ and $u \in \{\pm n_1, \pm n_2, \pm(n_1 - n_3), \pm(n_2 - n_3)\}^k$, with $k \in \mathbb{N}$, then (A, u, B) is a 1-path in $\Pi(\mathbf{n}, w)$ if and only if, for all $j \in \{0, \dots, k\}$, $A + \sum_{i=1}^{j} u_k \in [0, w[$.

The aim of this proof is to show that $\Pi(\mathbf{n}, w)$ admits an infinite 1-path if and only if so does $\Pi(\mathbf{n'}, w')$.

Let us first prove that each pair of two 1-adjacent pixels in $\Pi(\mathbf{n'}, w')$ can be *expanded* into a 1-path in $\Pi(\mathbf{n}, w)$.

i) Let (A, n'_1, B) represent a pair of two 1-adjacent pixels in $\Pi(\mathbf{n'}, w')$. Then $0 \leq A < w' = w - qn_1 \leq w$, $0 \leq B < w' = w - qn_1 \leq w$ and $(A, n_1, B) = (A, n'_1, B)$ is a 1-path in $\Pi(\mathbf{n}, w)$.

ii) Let $(A, n'_1 - n'_3, B)$ represent a pair of two 1-adjacent pixels in $\Pi(\mathbf{n'}, w')$. Since $n'_1 - n'_3 = qn_1 + n_2 - n_3$ and $A < w - qn_1$, then $0 \leq A + qn_1 < w$ and $(A, \underbrace{n_1, \dots, n_1}_{q}, n_2 - n_3, B)$ is a 1-path in $\mathrm{Pi}(\mathbf{n}, w)$.

The other cases, namely (A, n'_2, B) and $(A, n'_2 - n'_3, B)$, are obtained in the same way. For summarize, see Figure 5 for a correspondance between a 1-path in $\Pi(\mathbf{n'}, w')$ and a 1-path in $\Pi(\mathbf{n}, w)$. Conversely, if $\Pi(\mathbf{n}, w)$ admits an infinite 1-

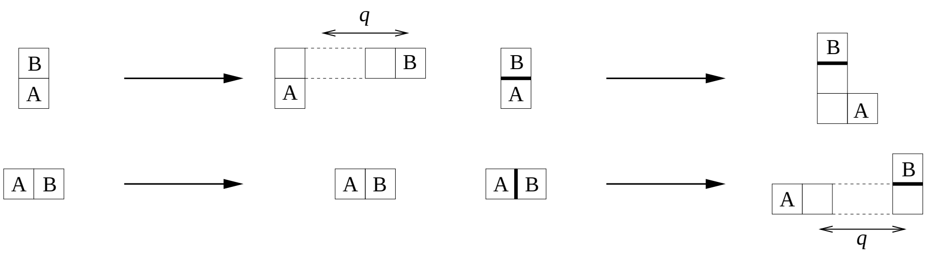


Fig. 5. Transformation of 1-paths in $\Pi(\mathbf{n'}, w')$ into 1-paths in $\Pi(\mathbf{n}, w)$

path, then by a similar recoding of it, one obtains an infinite 1-path in $\Pi(\mathbf{n'}, w')$. The complete proof if this theorem will appear in a forthcoming long version of the present paper. □

6 Algorithm

In the present section, we design an algorithm which computes the 0-connecting thickness of a given integer vector $\mathbf{n} \in \mathbb{Z}^3$. It iterates the reduction introduced in Theorem 6 until 0-connecting thickness becomes easy to determine.

The arithmetic reduction mentioned above only preserves 0-connectedness between the arithmetic discrete plane with normal vector $\mathbf{n}$ and its image under some conditions on $\mathbf{n}$. Nevertheless changing the components of a vector according to the symmetry lemma 3 or sorting them do not change the associated 0-connecting thickness. It is then possible to find from any vector $\mathbf{n}$ a vector $\mathbf{n'}$ with the same 0-connecting thickness meeting the requirement of Theorem 6. A step consisting of application of symmetry lemma, sorting, and the arithmetic reduction can be repeated and turns the vector $\mathbf{n}$ into another vector $\mathbf{n'}$ such that $n_1 \leq n'_1$, $n_2 < n'_2$ and $n_3 < n'_3$. Consequently, after a finite number of iteration, we always obtain a vector with a zero component for which the 0-connecting thickness is easy to determine.

Algorithm 1 follows from those considerations. It always terminates since the stopping condition, that is a vector with a zero component, is always reached in a finite number of iteration.

Algorithm 1 Determination of the 0-connecting thickness.

Input : $\mathbf{n} \in \mathbb{N}^3$.
Output : $w_0 \in \mathbb{Z}$, the 0-connecting thickness of $\mathbf{n}$.

```
ω ← 0
while n2 ≠ 0 do
    {Symmetry and ordering}
    n1 ← min(n1, n3 − n1)
    n2 ← min(n2, n3 − n2)
    t ← min(n1, n2)
    n2 ← max(n1, n2)
    n1 ← t
    {Reduction}
    q ← ⌊n2/n1⌋
    ω ← ω + (n2 − n1)
    n3 ← n3 − (n2 + (q − 1)n1)
    n2 ← n2 − q n1
end while
return ω + n3
```

7 Conclusion and Perspectives

In the present paper, we presented an algorithm computing the 0-connecting thickness of any integer vector. The main difference between this algorithm and the ones already known [4,3] is that it does not need a graph traversal and only computes basic reductions on an integer vector.

In a forthcoming work, we plan to investigate the case of non-rational arithmetic planes. Since the reduction of Theorem 6 does not depend on the nature of the input vector (integer or not), we hope to extend this approach to any vector $\mathbf{n} \in \mathbb{R}^3$.

Other interesting investigations should be, on the one hand, the computation of κ-connected thickness for $\kappa \in \{1, 2\}$ and, on the other, the extension of this work to arithmetic discrete hyperplanes in any dimension.

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